ON BALANCE OF ENSTROPHY PRODUCTION AND ITS DISSIPATION

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Does the Tennekes & Lumley balance hold only

 # in the mean?*

 # at sufficiently large Reynolds numbers?*

 # in statistically stationary turbulence?

*See fig 6.6 in Tsinober 2001, Kluwer

DNS:

B. GALANTI
N. SANDHAM

Experiments:

G. GULITSKII,
M. KHOLMYANSKY
S. YORISH

T E N N E K E S &

L U M L E Y 1 9 7 2

The equation for $\overline{\omega_i\omega_i}$ **for statistically stationary turbulent shear flow**

$$U_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\,\overline{\omega_i\omega_i}\right) = -\overline{u_j\omega_i}\,\frac{\partial \Omega_i}{\partial x_j} - \frac{1}{2}\frac{\partial}{\partial x_j}\left(\overline{u_j\omega_i\omega_i}\right) + \overline{\omega_i\omega_j s_{ij}} + \overline{\omega_i\omega_j}\,S_{ij}$$

$$+\,\Omega_j\,\overline{\omega_i s_{ij}} + \nu \frac{\partial^2}{\partial x_j \partial x_j}\left(\tfrac{1}{2}\,\overline{\omega_i\omega_i}\right) - \nu\,\overline{\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j}}\,. \qquad (3.3.38)$$

at sufficiently high Reynolds numbers the turbulent vorticity budget (3.3.38) may be approximated as (Taylor, 1938)

pp. 86-91

$$\overline{\omega_i\omega_j s_{ij}} = \nu\,\overline{\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j}}\,. \qquad (3.3.62)$$

The budget of mean-square vorticity fluctuations is thus approximately independent of the structure of the mean flow. Turbulent vorticity fluctuations, unlike turbulent velocity fluctuations, do not need the continued presence of a source term associated with the mean flow field. Of course, in the absence of a source of energy, turbulent vorticity fluctuations will decay, too. **Also, the rate of change of** $\overline{\omega_i\omega_i}$, as represented by (3.3.59), is small compared to the rate at which turbulent vortex stretching occurs.

TENNEKES AND LUMLEY BALANCE(1972, P.91):

$$\overline{\omega_i \omega_j s_{ij}} = \nu \overline{\frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j}}$$

Enstrophy production is approximately balanced in the mean by viscous terms at *sufficiently high* Reynolds numbers* in statistically stationary turbulent shear flow → Three questions:

Is it only/just in the mean?

Does this happen only at sufficiently large Reynolds numbers?

Is this balance violated in statistically non-stationary turbulence?

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*In this sense - but not only in this - turbulence is not slightly viscous at whatever large Reynolds number. In this context the question: what happens with enstrophy/strain production as $\nu \to 0$ is of special interest

Self-Amplification of Vorticity and Strain

$$(½)D\omega^2/Dt = \omega_i\omega_j s_{ij} + \nu\omega_i\Delta\omega_i + \varepsilon_{ijk}\omega_i\,\partial F_k/\partial x_j$$

$$(½)Ds^2/Dt = - s_{ij}s_{jk}s_{ki} - (1/4)\omega_i\omega_j s_{ij} - s_{ij}\partial^2 p/\partial x_i\partial x_j + \nu s_{ij}\Delta s_{ij} + s_{ij}F_{ij}$$

The property of self amplification of vorticity and strain is responsible for the fact the neither enstrophy ω^2 nor the total strain s^2 are inviscid invariants as is the kinetic energy u^2

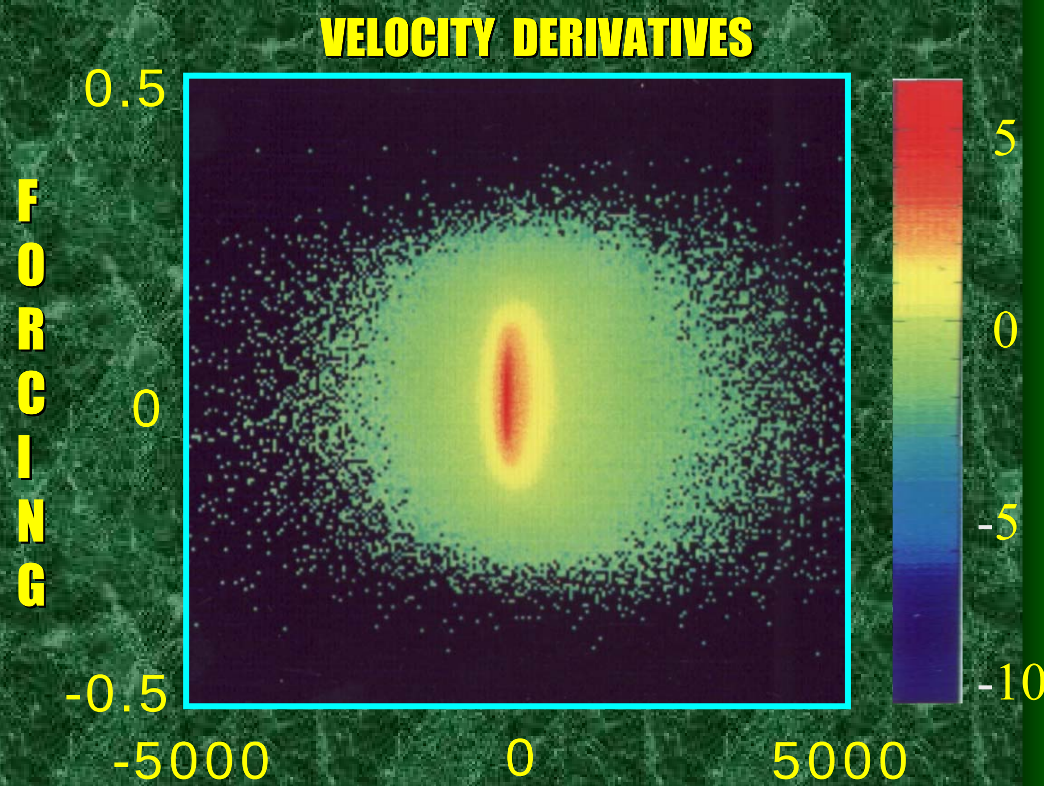
SELF-AMPLIFICATION OF VORTICITY AND STRAIN

SELF-RANDOMIZATION/INTRINSIC STOCHASTICITY: NO SOURCE OF RANDOMNESS IS NEEDED, THE FORCING CAN BE CONSTANT IN TIME

AT THE LEVEL OF VELOCITY DERIVATIVES: VORTICITY AND STRAIN (DISSIPATION) THE EXTERNAL FORCING IS IRRELEVANT

Three cases:

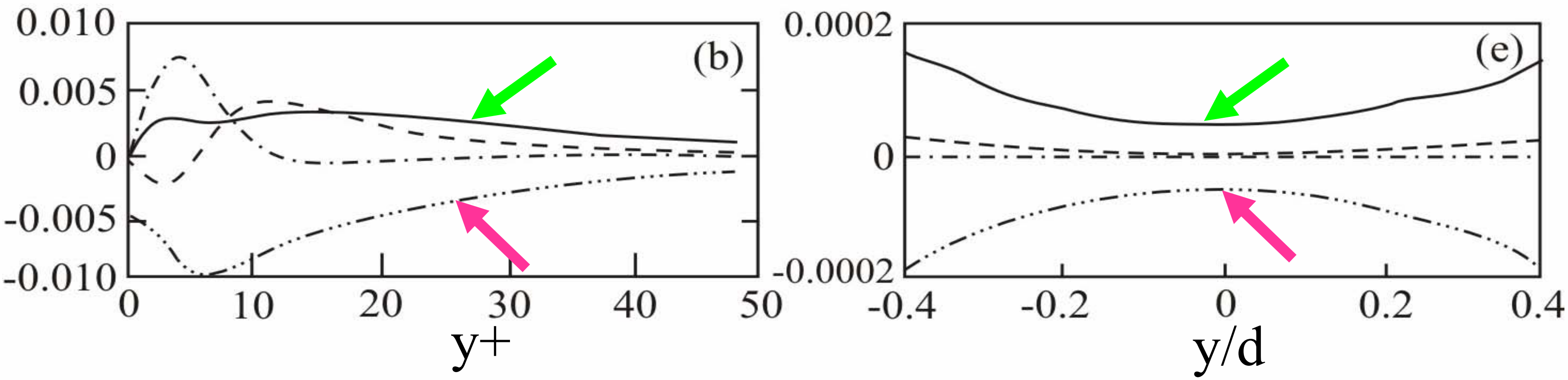
1. DNS in a periodic box, $Re_\lambda=10^2$
2. DNS in a channel flow, Re=5600
3. Atmospheric SL, $Re_\lambda=10^4$; $Re=10^8$



DNS in a Channel Flow

Re=5600

Enstrophy balance



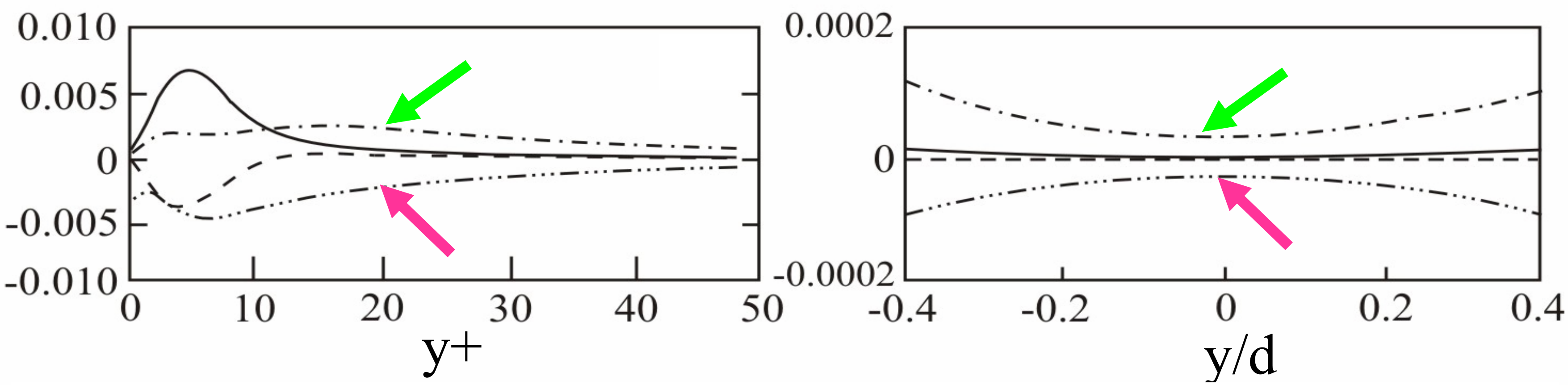
$$\frac{1}{2}\frac{D_U\langle\omega^2\rangle}{Dt} \approx \langle\omega_i\omega_j s_{ij}\rangle + \langle\omega_i\omega_j\rangle S_{ij} + \langle\omega_i s_{ij}\rangle\Omega_j + \langle\nu\omega_i\nabla^2\omega_i\rangle$$

The equation for the mean enstrophy of the turbulent fluctuations $\frac{1}{2}\langle\omega^2\rangle$

$$\frac{1}{2}\frac{D_U\langle\omega^2\rangle}{Dt} = -\langle u_j\omega_i\rangle\frac{\partial\Omega_i}{\partial x_j} - \frac{1}{2}\frac{\partial}{\partial x_j}\{\langle u_j\omega_i\omega_i\rangle\} + \langle\omega_i\omega_j s_{ij}\rangle +$$

$$+\langle\omega_i\omega_j\rangle S_{ij} + \langle\omega_i s_{ij}\rangle\Omega_j + \nu\langle\omega_i\nabla^2\omega_i\rangle.;$$

Strain balance



$$\frac{D_U \frac{1}{2}\langle s_{ij}s_{ij}\rangle}{Dt} \approx -2\langle s_{ij}s_{ik}\rangle S_{kj} - \frac{1}{2}\langle \omega_i s_{ij}\rangle \Omega_j - \langle s_{ij}s_{jk}s_{ki}\rangle - \frac{1}{4}\langle \omega_i\omega_j s_{ij}\rangle + \nu s_{ij}\nabla^2 s_{ij}$$

The equation for the mean total strain of the turbulent fluctuations $\langle s_{ij}s_{ij}\rangle$

$$\frac{D_U \frac{1}{2}\langle s_{ij}s_{ij}\rangle}{Dt} = -\langle u_k s_{ij}\rangle \frac{\partial S_{ij}}{\partial x_k} - \frac{1}{2}\frac{\partial}{\partial x_k}\{\langle u_k s_{ij}s_{ij}\rangle\} - 2\langle s_{ij}s_{ik}\rangle S_{kj}$$

$$-\frac{1}{2}\langle \omega_i s_{ij}\rangle \Omega_j - \langle s_{ij}s_{jk}s_{ki}\rangle - \frac{1}{4}\langle \omega_i\omega_j s_{ij}\rangle - \langle s_{ij}\frac{\partial^2 p}{\partial x_i \partial x_j}\rangle + \nu\langle s_{ij}\nabla^2 s_{ij}\rangle$$

Atmospheric Surface Layer

$Re_{\lambda}=10^4$; $Re=10^8$

The equation for the mean enstrophy of the turbulent fluctuations $\frac{1}{2}\langle\omega^2\rangle$

$$\frac{1}{2}\frac{D_U\langle\omega^2\rangle}{Dt} = -\langle u_j\omega_i\rangle\frac{\partial\Omega_i}{\partial x_j} - \frac{1}{2}\frac{\partial}{\partial x_j}\left\{\langle u_j\omega_i\omega_i\rangle\right\} + \langle\omega_i\omega_j s_{ij}\rangle +$$

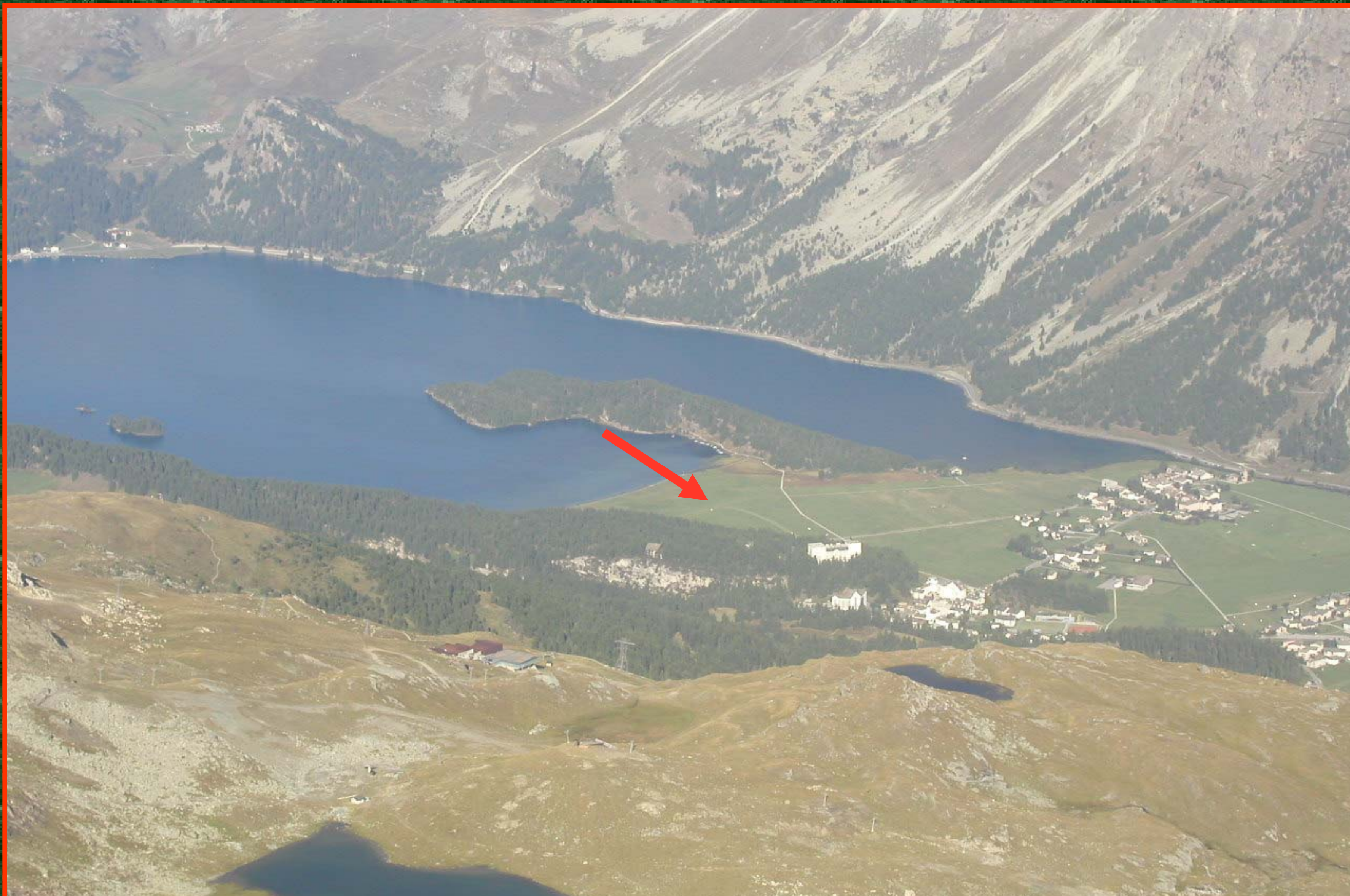
$$+\langle\omega_i\omega_j\rangle S_{ij} + \langle\omega_i s_{ij}\rangle\Omega_j + \underline{\nu\langle\omega_i\nabla^2\omega_i\rangle}$$

The equation for the mean total strain of the turbulent fluctuations $\langle s_{ij}s_{ij}\rangle$

$$\frac{D_U\frac{1}{2}\langle s_{ij}s_{ij}\rangle}{Dt} = -\langle u_k s_{ij}\rangle\frac{\partial S_{ij}}{\partial x_k} - \frac{1}{2}\frac{\partial}{\partial x_k}\left\{\langle u_k s_{ij}s_{ij}\rangle\right\} - 2\langle s_{ij}s_{ik}\rangle S_{kj}$$

$$-\frac{1}{2}\langle\omega_i s_{ij}\rangle\Omega_j - \langle s_{ij}s_{jk}s_{ki}\rangle - \frac{1}{4}\langle\omega_i\omega_j s_{ij}\rangle - \underline{\langle s_{ij}\frac{\partial^2 p}{\partial x_i\partial x_j}\rangle} + \underline{\nu\langle s_{ij}\nabla^2 s_{ij}\rangle}$$

The Maria Sils Site, Switzerland



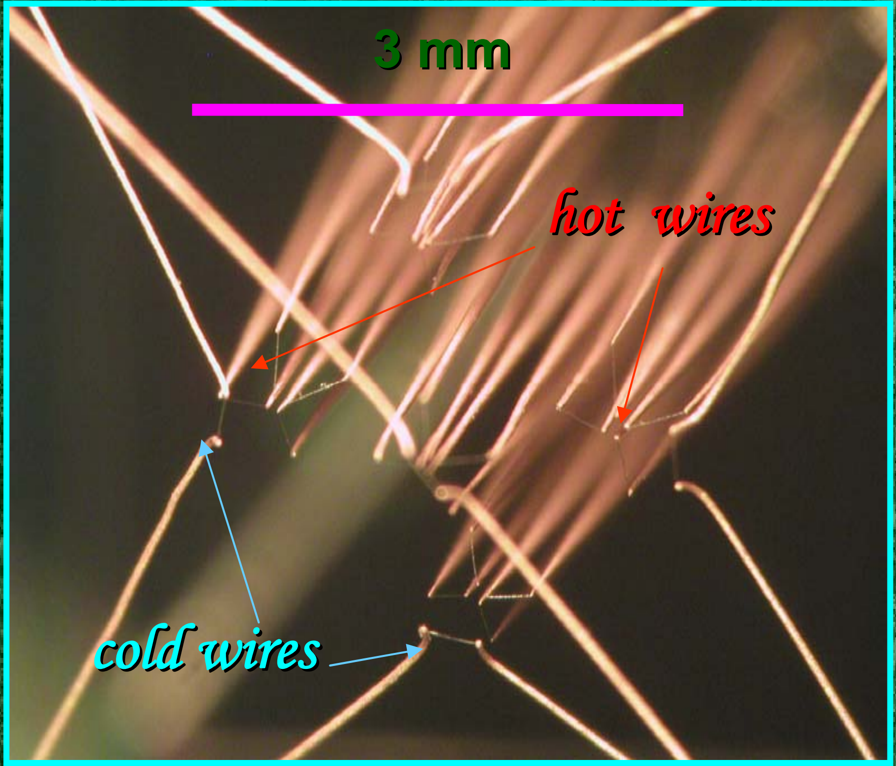
FIELD EXPERIMENT SUMMER 2004 *SILS MARIA, SWITZERLAND*

The calibration unit at 3 m
in the field

Height 1850 m
Experiment was performed in collaboration of Institute of Hydromechanics and Water Resources Management, ETH Zurich

The Israeli team

THE PROBE



Manganin is used as a material for the sensor prongs instead of tungsten because the temperature coefficient of the electrical resistance of manganin is 400 times smaller than that of tungsten.

The tip of the probe with prongs made of manganin

The equation for the mean enstrophy of the turbulent fluctuations $\frac{1}{2}\langle\omega^2\rangle$

$$\frac{1}{2}\frac{D_U\langle\omega^2\rangle}{Dt} = -\langle u_j\omega_i\rangle\frac{\partial\Omega_i}{\partial x_j} - \frac{1}{2}\frac{\partial}{\partial x_j}\{\langle u_j\omega_i\omega_i\rangle\} + \langle\omega_i\omega_j s_{ij}\rangle +$$

$$+\langle\omega_i\omega_j\rangle S_{ij} + \langle\omega_i s_{ij}\rangle\Omega_j + \underline{\nu\langle\omega_i\nabla^2\omega_i\rangle}$$

The equation for the mean total strain of the turbulent fluctuations $\langle s_{ij}s_{ij}\rangle$

$$\frac{D_U\frac{1}{2}\langle s_{ij}s_{ij}\rangle}{Dt} = -\langle u_k s_{ij}\rangle\frac{\partial S_{ij}}{\partial x_k} - \frac{1}{2}\frac{\partial}{\partial x_k}\{\langle u_k s_{ij}s_{ij}\rangle\} - 2\langle s_{ij}s_{ik}\rangle S_{kj}$$

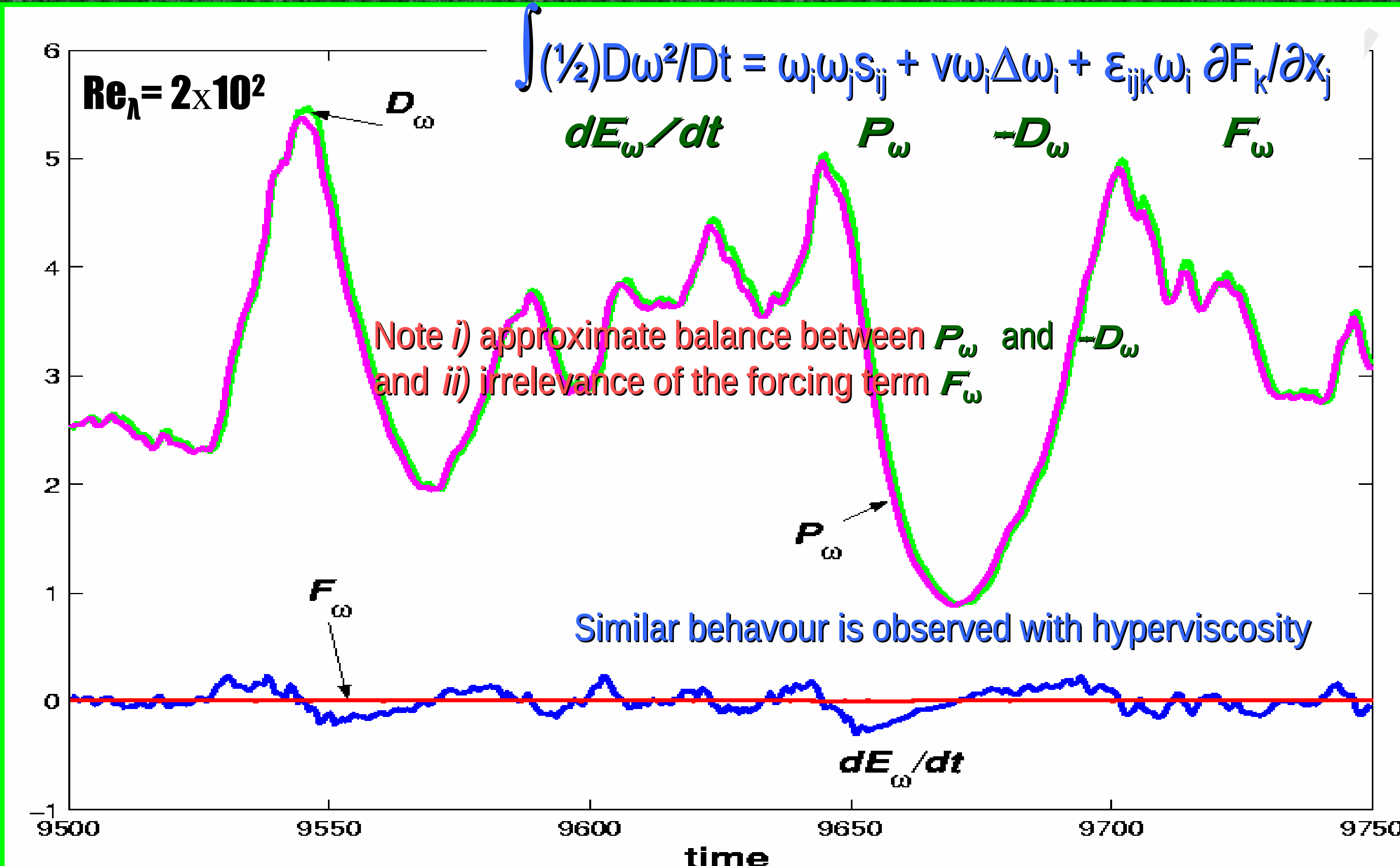
$$-\frac{1}{2}\langle\omega_i s_{ij}\rangle\Omega_j - \langle s_{ij}s_{jk}s_{ki}\rangle - \frac{1}{4}\langle\omega_i\omega_j s_{ij}\rangle - \underline{\langle s_{ij}\frac{\partial^2 p}{\partial x_i\partial x_j}\rangle} + \underline{\nu\langle s_{ij}\nabla^2 s_{ij}\rangle}$$

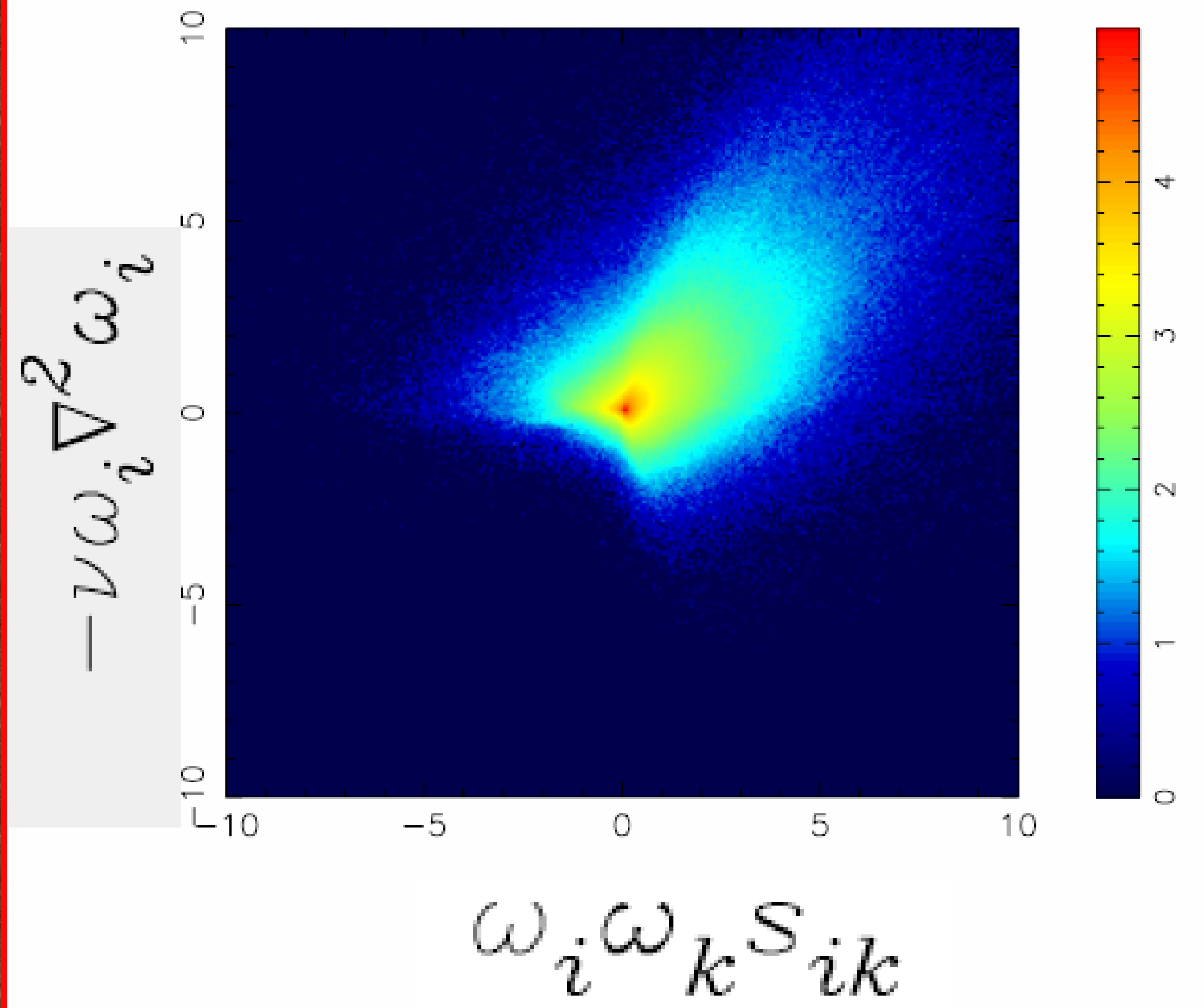
DNS in a periodic box $Re_\lambda \sim 10^2$

DNS in a periodic box $Re_\lambda \sim 102$

STATIONARY. The equations are solved using a standard pseudospectral method for the space variables and a second-order Adams–Bashforth finite difference scheme for time stepping. Resolutions range from 32^3 to $\mathbf{256}^3$ uniformly distributed grid points, according to the Reynolds number. Several versions of forcing were used—all in large scales. The first one is a deterministic forcing with a force corresponding to the ABC flow,[5] $\mathbf{f}=f\{A \sin z+C \cos y, B \sin x+A \cos z, C \sin y+B \cos x\}$, $A=B=C$. This forcing, denoted in the sequel as ABC, is strongly helical, curl $\mathbf{f} \| \mathbf{f}$, and therefore along with kinetic energy such a forcing makes an input of helicity into the flow. The second kind of forcing corresponds to a force in the form $\mathbf{f}=f\{A \cos z \cos y, B \cos x \cos z, C \cos y \cos x\}$, $A=B=C$. This forcing, denoted in the sequel as NH, is nonhelical, $\mathbf{f} \cdot \text{curl}\, \mathbf{f}=0$. Computations were also made with the random versions (RABC and RNH) of the above-mentioned forcings, in which the A, B, C coefficients were random functions in time.

TEMPORAL EVOLUTION OF SPATIAL INTEGRALS IN THE ENSTROPHY BALANCE EQUATION





$-\nu\omega_i\nabla^2\omega_i$
10
5
0
−5
−10
−10
−5
0
5
10
$\omega_i\omega_k s_{ik}$
0
1
2
3
4

Does the T & L balance hold only/just in the mean?

Does the T & L balance hold only at sufficiently large Reynolds numbers?

Does the T & L balance hold only in statistically stationary turbulence?

PERIODICALLY FORCED TURBULENCE

The simulation parameters:

Resolution 128^3

Forcing: ABC multiplied by $(1 + A_t \cos \Omega t)$ with $A_t = 0.5$ and $\Omega = 0, 6$ and 30

Velocity field parameters:

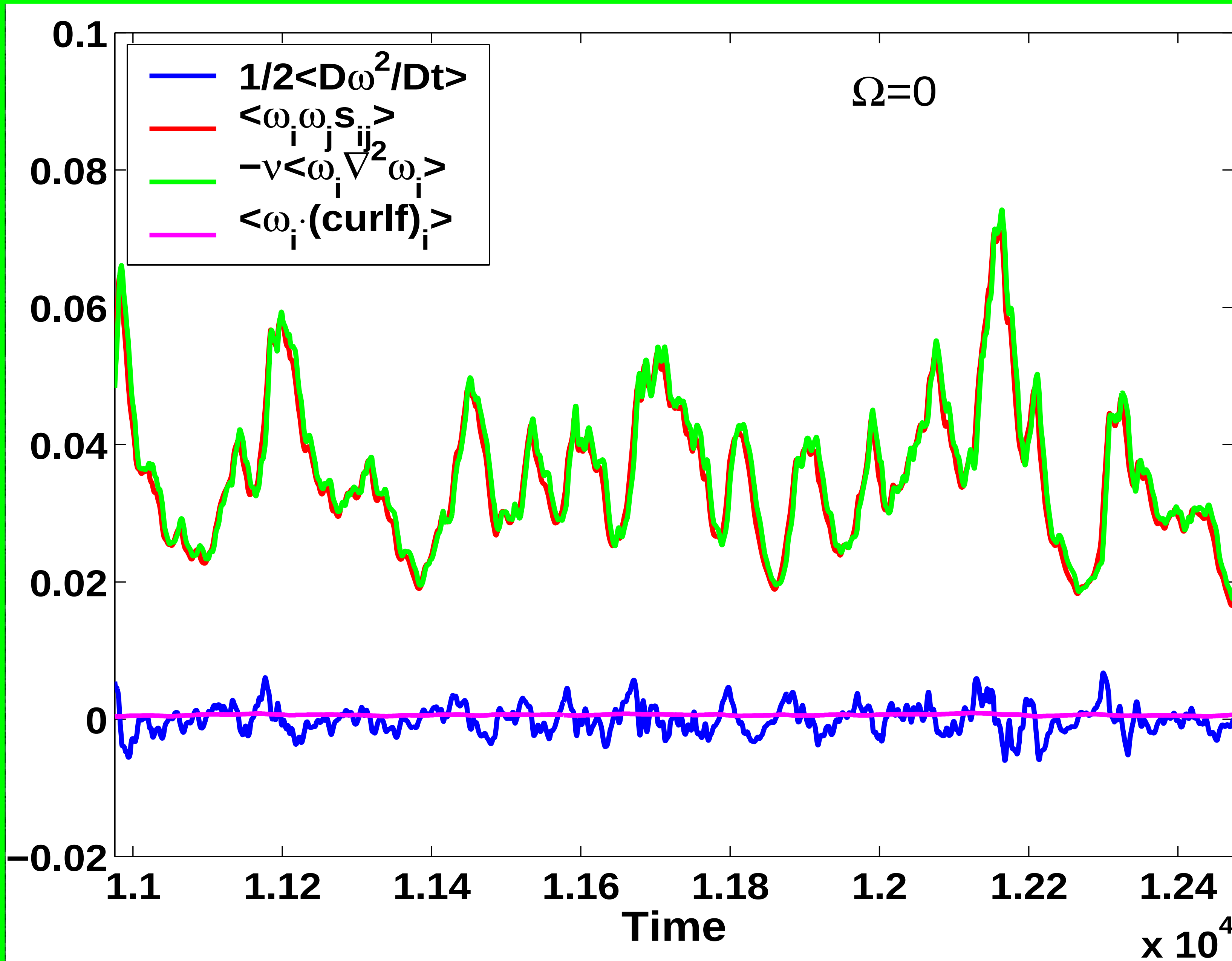
Eddy turn over time 50

Taylor Reynolds number $Re_\lambda = 50$

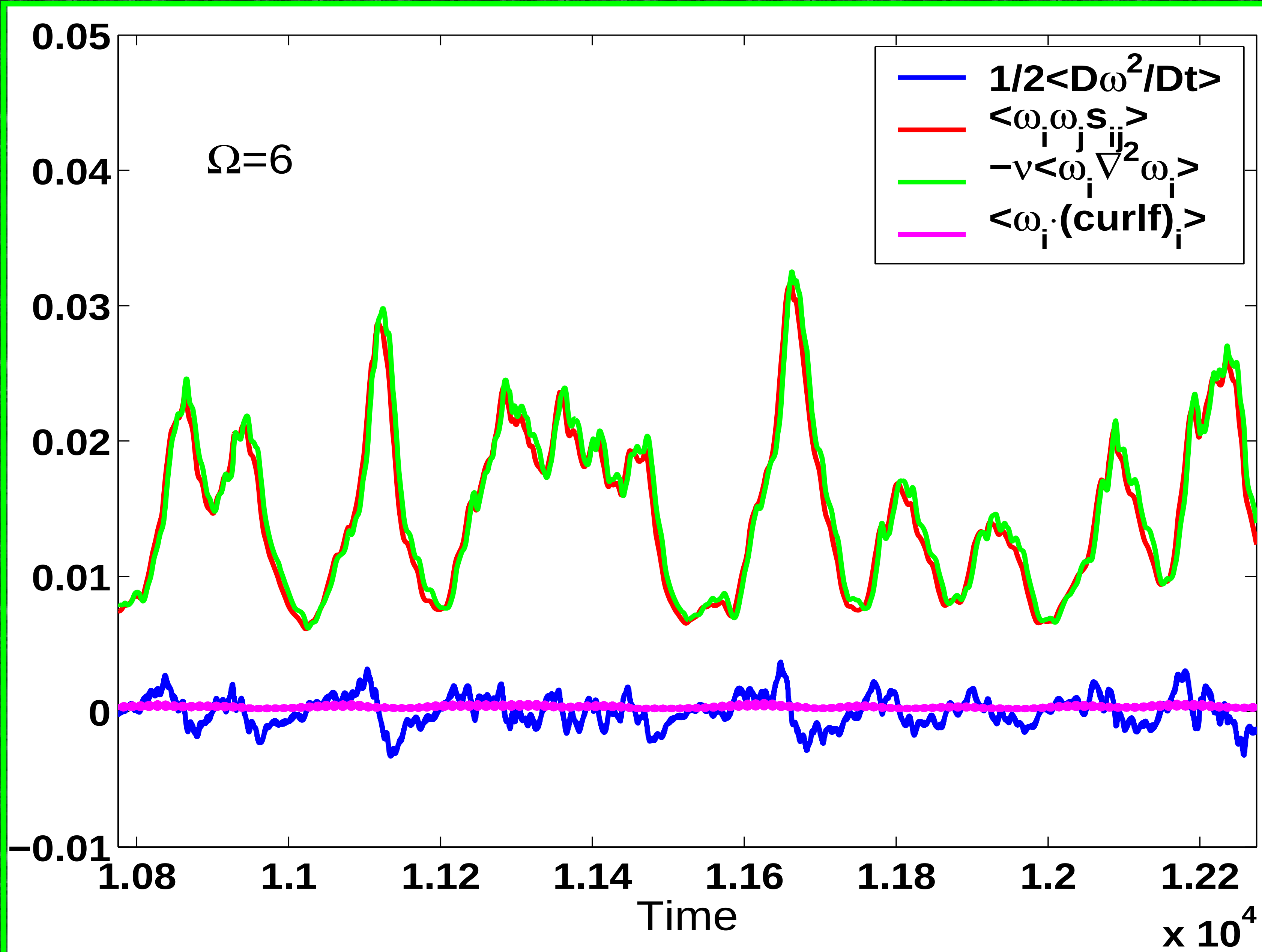
From now on $\langle \ldots \rangle$ means $\int \ldots dV$

Enstrophy balance

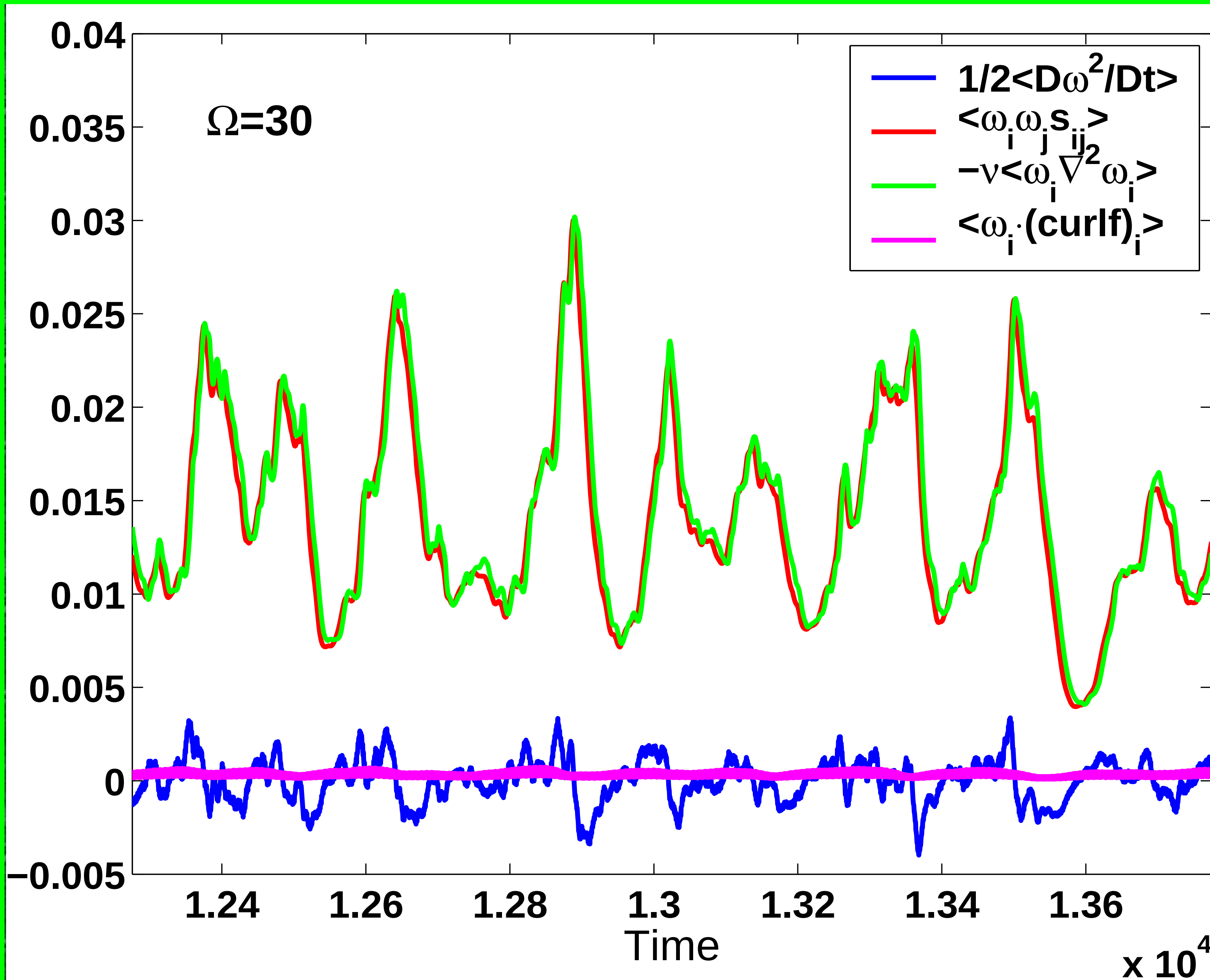
$$(½)D\omega^2/Dt = \omega_i\omega_j S_{ij} + \nu\omega_i\Delta\omega_i + \varepsilon_{ijk}\omega_i\, \partial F_k/\partial x_j$$



1/2<Dω²/Dt>
<ω_i ω_j s_ij>
−ν<ω_i ∇² ω_i>
<ω_i·(curlf)_i>
Ω=0
0.1
0.08
0.06
0.04
0.02
0
−0.02
1.1
1.12
1.14
1.16
1.18
1.2
1.22
1.24
Time
x 10⁴



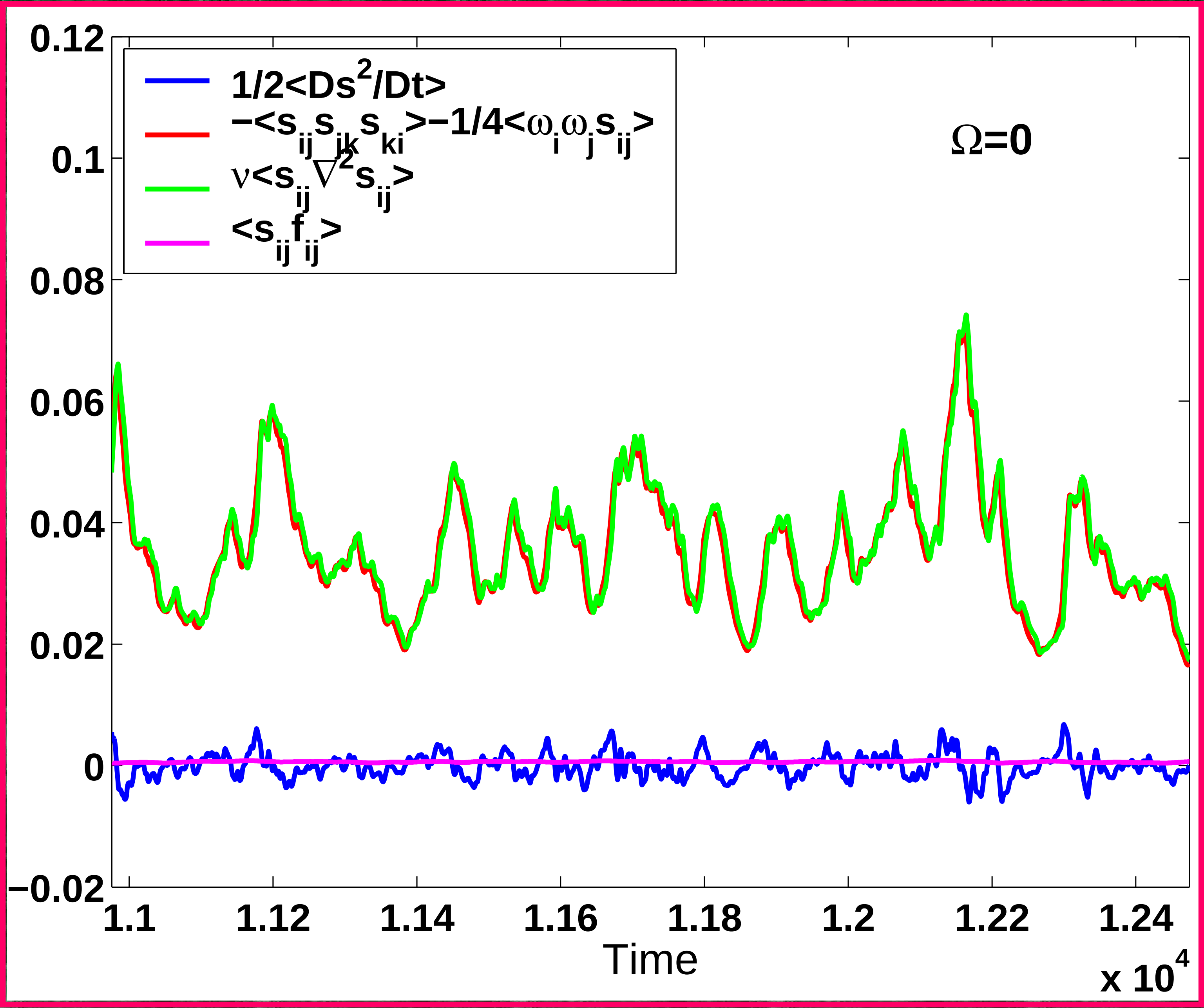
0.05
0.04
0.03
0.02
0.01
0
−0.01
1.08
1.1
1.12
1.14
1.16
1.18
1.2
1.22
Time
x 10^4
Ω=6
1/2<Dω^2/Dt>
<ω_i ω_j s_ij>
−ν<ω_i ∇^2 ω_i>
<ω_i·(curlf)_i>



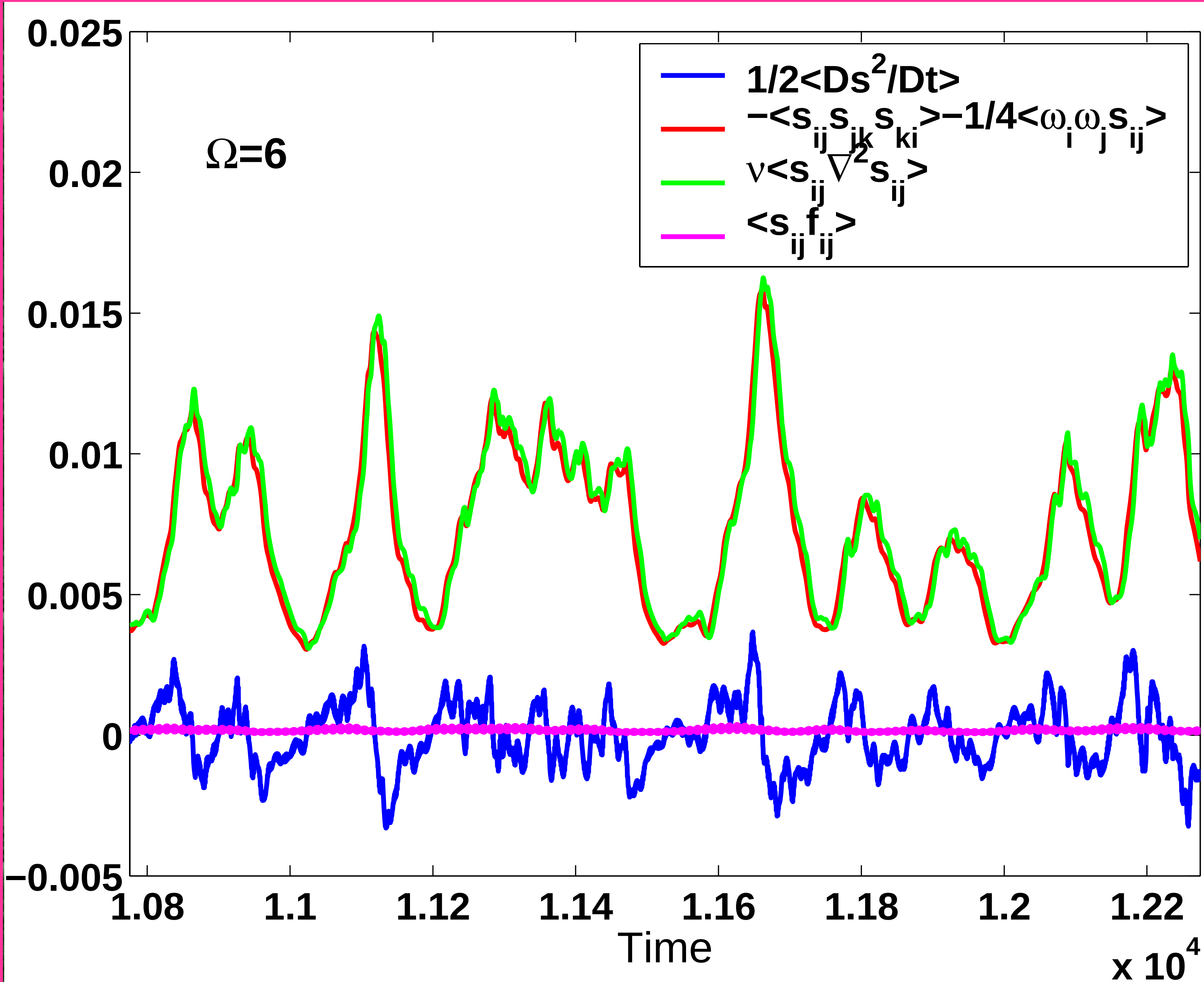
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
−0.005
Ω=30
1/2<Dω²/Dt>
<ω_i ω_j s_ij>
−ν<ω_i ∇² ω_i>
<ω_i·(curlf)_i>
1.24
1.26
1.28
1.3
1.32
1.34
1.36
Time
x 10⁴

Strain balance

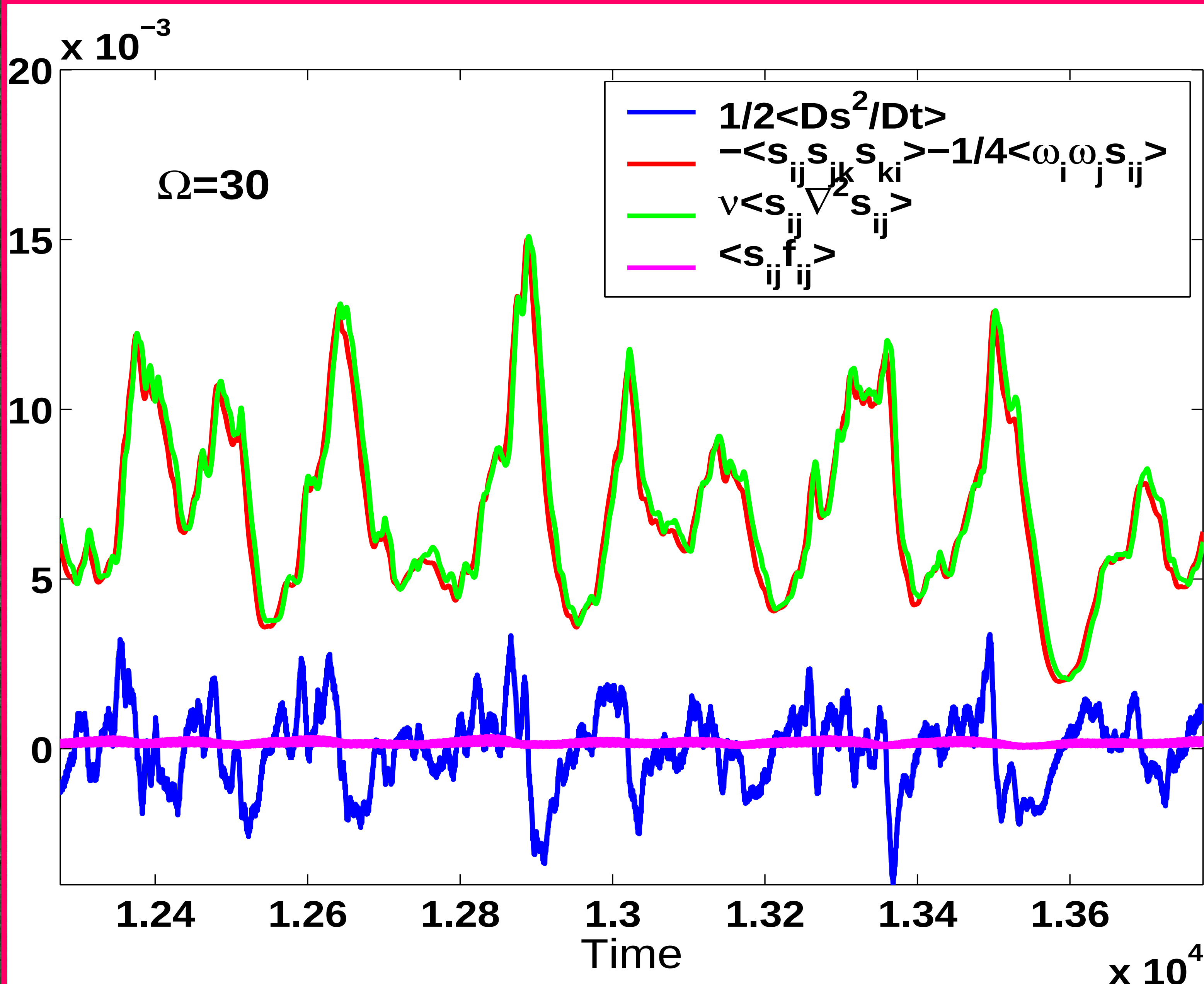
$$(\tfrac{1}{2})Ds^2/Dt = -\, s_{ij}s_{jk}s_{ki} - (1/4)\omega_i\omega_j s_{ij} - s_{ij}\partial^2 p/\partial x_i \partial x_j + \nu s_{ij}\Delta s_{ij} + s_{ij}F_{ij}$$



0.12
0.1
0.08
0.06
0.04
0.02
0
−0.02
1.1
1.12
1.14
1.16
1.18
1.2
1.22
1.24
Time
x 10^4
1/2<Ds^2/Dt>
−<s_ij s_jk s_ki>−1/4<ω_i ω_j s_ij>
ν<s_ij ∇^2 s_ij>
<s_ij f_ij>
Ω=0



0.025
0.02
0.015
0.01
0.005
0
−0.005
1.08
1.1
1.12
1.14
1.16
1.18
1.2
1.22
Time
x 10^4
Ω=6
1/2<Ds^2/Dt>
−<s_ij s_jk s_ki>−1/4<ω_i ω_j s_ij>
ν<s_ij ∇^2 s_ij>
<s_ij f_ij>


x 10^−3
20
15
10
5
0
Ω=30
1/2<Ds^2/Dt>
−<s_ij s_jk s_ki>−1/4<ω_i ω_j s_ij>
ν<s_ij ∇^2 s_ij>
<s_ij f_ij>
1.24
1.26
1.28
1.3
1.32
1.34
1.36
Time
x 10^4

CONCLUDING

The T&L enstrophy balance holds **not** only

in the mean
at sufficiently large Reynolds numbers
in statistically stationary turbulence

A similar balance holds for the total strain $s_{ik}s_{ik}$