Unbalanced vortex stretching in non-stationary turbulence

Charles Speziale's work on unbalanced vortex stretching revisited

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introduction: ideal turbulence and non-ideal turbulence

− ideal turbulence: constant flux steady state (Kolmogorov theory)

− generalization: time-dependent flux (local Kolmogorov theory)

− non-ideal turbulence: something else?

Models assume that the Kolmogorov theory is a permanent feature of all or of some scales of motion.

We will focus on the consequence that at high Reynolds number, large scales are independent of viscosity.

* $\kappa_d^4 = \epsilon/\nu^3$: $\nu \to 0$ implies $\kappa_d \to \infty$ while ϵ is constant.

* Viscosity does not appear in models except at low Re.

Hypothetical Re dependence of C_K, skewness, ... is a different issue

example: the two-equation model

$$\dot{k} = P - \epsilon \quad \dot{\epsilon} = \frac{\epsilon}{k}\left(C_{\epsilon 1}P - C_{\epsilon 2}\epsilon\right)$$

where k is kinetic energy of turbulent fluctuations, ϵ is the dissipation rate, and P is the (known) production.

– Any 'property' of turbulence can be estimated dimensionally:

$$L = C_L \frac{k^{3/2}}{\epsilon} \qquad T = C_T \frac{k}{\epsilon}$$

– A possible conceptual improvement on mixing length models.

basic principle/belief/dogma

High Reynolds number 'fixed point' independent of κ_d, ν, Re.

− Speziale and Bernard contradicted this assumption by proposing
a Reynolds-number dependent model for high Reynolds turbulence.

− It allowed *unbalanced vortex stretching* (to be explained) in

∗ self-similar homogeneous decay

∗ self-similar growth of homogeneous shear flow.

present viewpoint:

– Although the Kolmogorov theory does not describe a *permanent feature* of the small scales, it does describe an *attractor*. Thus:

* Unbalanced vortex stretching is possible in transients,

* but Kolmogorov phenomenology is recovered in self–similar flows.

analytical formulation

The *statistical theory of turbulence* derives (unclosed) spectral evolution equation for homogeneous isotropic turbulence

$$\dot{E}(\kappa, t) = P(\kappa, t) - T(\kappa, t) - 2\nu\kappa^2 E(\kappa, t)$$

E is the energy spectrum, P is the (known) production spectrum, and T is the nonlinear transfer.

moments of the evolution equation

The integrated quantities

$$\int_0^\infty d\kappa \ E(\kappa, t) \ = \ k \text{ kinetic energy}$$

$$\int_0^\infty d\kappa \ P(\kappa, t) \ = \ P \text{ production}$$

$$\int_0^\infty d\kappa \ T(\kappa, t) \ = \ 0 \text{ (energy conservation)}$$

$$\int_0^\infty d\kappa \ 2\nu\kappa^2 E(\kappa, t) \ = \ \epsilon \text{ dissipation}$$

satisfy the energy balance

$$\dot{k} = P - \epsilon$$

exact dissipation rate equation

Take another moment: the exact equation for ϵ is

$$\int_0^\infty d\kappa \ 2\nu\kappa^2 \ \dot{E}(\kappa,t) = \int_0^\infty d\kappa \ 2\nu\kappa^2 \left[P(\kappa,t) - T(\kappa,t) - 2\nu\kappa^2 E(\kappa,t) \right]$$

Note

$$\int_0^\infty d\kappa \ 2\nu\kappa^2 E(\kappa,t) = \epsilon \text{ dissipation: } \nu\langle\omega_p\omega_p\rangle$$

$$\int_0^\infty d\kappa \ 2\nu\kappa^2 P(\kappa,t) \approx 0$$

$$\int_0^\infty d\kappa \ 2\nu\kappa^2 T(\kappa,t) = S \text{ vortex stretching: } \nu\langle\omega_i\omega_j s_{ij}\rangle$$

$$\int_0^\infty d\kappa \ 4\nu^2\kappa^4 E(\kappa,t) = G \text{ 'palinstrophy:' } \nu^2\langle\omega_{i,j}\omega_{i,j}\rangle$$

Therefore,

$$\boxed{\dot{\epsilon} = S - G}$$

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problems posed by this equation

− In modeling, we expect to be near a Kolmogorov state so that
$E(\kappa) \sim \epsilon^{2/3}\kappa^{-5/3}; \kappa_0 = \epsilon/k^{3/2} \ll \kappa \ll \kappa_d = (\epsilon/\nu^3)^{1/4}$

− **ideal turbulence heuristic**: limit $\kappa_d \to \infty$ should be finite.

* Of course $\int_0^{\kappa_d} \nu\kappa^2 E(\kappa)d\kappa \sim \nu\kappa_d^{4/3} \sim \epsilon^{1/3} \sim \kappa_d^0$ is finite.

* but $G = \int_0^{\kappa_d} \nu^2\kappa^4 E(\kappa)d\kappa \sim \nu^2\kappa_d^{10/3} \sim \kappa_d^{2/3}$ diverges with κ_d.

* Also $S = -\int_0^{\kappa_d} \nu\kappa^2 T(\kappa)d\kappa \sim \int_0^{\kappa_d} 2\nu\kappa\epsilon d\kappa \sim \kappa_d^{2/3}$ diverges.

Tennekes-Lumley analysis

– Elementary arguments suggest $S \sim Re^{1/2}(\sim \nu^{-1/2} \sim \kappa_d^{2/3})$

– Then Re-independence of $\dot{\epsilon}$ requires also $G \sim Re^{1/2}$.

* In fact, much more is required: $S - G \sim Re^0$ $(*)$.

* Certainly true in a steady state (Batchelor's skewness relation).

– But what can justify $(*)$ in an unsteady problem?

– **No theory answers this question** (except 'our theory,' to be explained.)

− **remark** −

− Doering has shown that enstrophy production is at most $O(Re^3)$.

− The bound can be realized by a suitable initial condition.

− Steady state: enstrophy production is $O\left(\frac{1}{\nu}Re^{1/2}\right) = O(Re^{3/2})$.

− Steady state turbulence organizes itself so that enstrophy production is *much less* than what the Navier-Stokes equations permit.

This is the consequence of a statistical hypothesis, not of NSE alone.

unbalanced vortex stretching?

– Why hasn't the Tennekes-Lumley balance been justified?

– Perhaps because it does not occur.

– Imbalance $\dot{\epsilon} = S - G = O(Re^{1/2})$ causes explosive growth of ϵ.

– Speziale and Bernard explored this possibility through models containing an *unbalanced vortex stretching* term

$$\dot{\epsilon} \propto Re^{1/2}\frac{\epsilon^2}{k} = \sqrt{\frac{\epsilon}{\nu}}\epsilon$$

present approach: study time-dependent isotropic turbulence; some possible problems

— One steady state to another: step change, gradual change ....

— *However*, like laminar boundary layers, steady state turbulence is more common in theory than in practice.

— Can time-dependent but self-similar states exhibit Tennekes-Lumley balance? Self-similarity fixes relations between all scales of motion.

— Consider a problem in which turbulence evolves from a steady state to a state of self-similar growth.

transient unbalanced vortex stretching
example 1: ramp flow

$-$ Steady state isotropic turbulence is driven by forcing with linearly increasing amplitude at a fixed length scale (Rubinstein, Clark, Livescu, Luo; *JoT*).

$-$ Turbulence evolves from a Kolmogorov steady state to a state of self-similar asymptotic growth with $\epsilon(t) \sim P(t) \sim t$ and $k(t) \sim \epsilon(t)^{2/3} \sim t^{2/3}$. (in fact, $\epsilon(t) \sim t - at^{-1/3}$)

$-$ Studied problem by closure theory, single-point models, DNS.

spectral closure model

(Rubinstein and Clark, TCFD 2004)

$$\dot{E}(\kappa, t) = P(\kappa, t) - \frac{\partial}{\partial \kappa} \mathcal{F}[E(\kappa, t)] - 2\nu \kappa^2 E(\kappa, t)$$

where

$$\mathcal{F} = \mathsf{C} \left\{ \int_0^\kappa d\mu \ \mu^2 E(\mu) \int_\kappa^\infty dp \ E(p)\theta(p) - \int_0^\kappa d\mu \ \mu^4 \int_\kappa^\infty dp \ E^2(p)\theta(p)p^{-2} \right\}$$

with evolution equation

$$\dot{\theta}(\kappa) = 1 - \eta(\kappa)\theta(\kappa) \quad \eta(\kappa) = \theta(\kappa) \int_0^\kappa d\mu \ \mu^2 E(\mu)$$

A simplified test-field model, enhanced Heisenberg model; sign of $\mathcal{F}$ is indefinite; consistent with equipartition. Compare Canuto-Dubovikov model.

two-equation model

$$\dot{k} = P - \epsilon \qquad \dot{\epsilon} = \frac{3\,\epsilon}{2\,k}(P - \epsilon)$$

– Consistent with steady state and self-similar growth.

– Interesting because it assumes the Tennekes-Lumley balance.

– Preliminary studies by multiple equation models (Schiestel).

**transient unbalanced vortex stretching
example 1: ramp flow results**

– Closure calculations (Tim Clark) show imbalance between S and G during adjustment phase, with recovery of the T-L balance during asymptotic growth.

– We also observe explosive growth of ϵ immediately before re-establishment of the balance.

– DNS (Daniel Livescu) confirmed explosive growth phase, but the calculations could not be continued to self-similar growth.
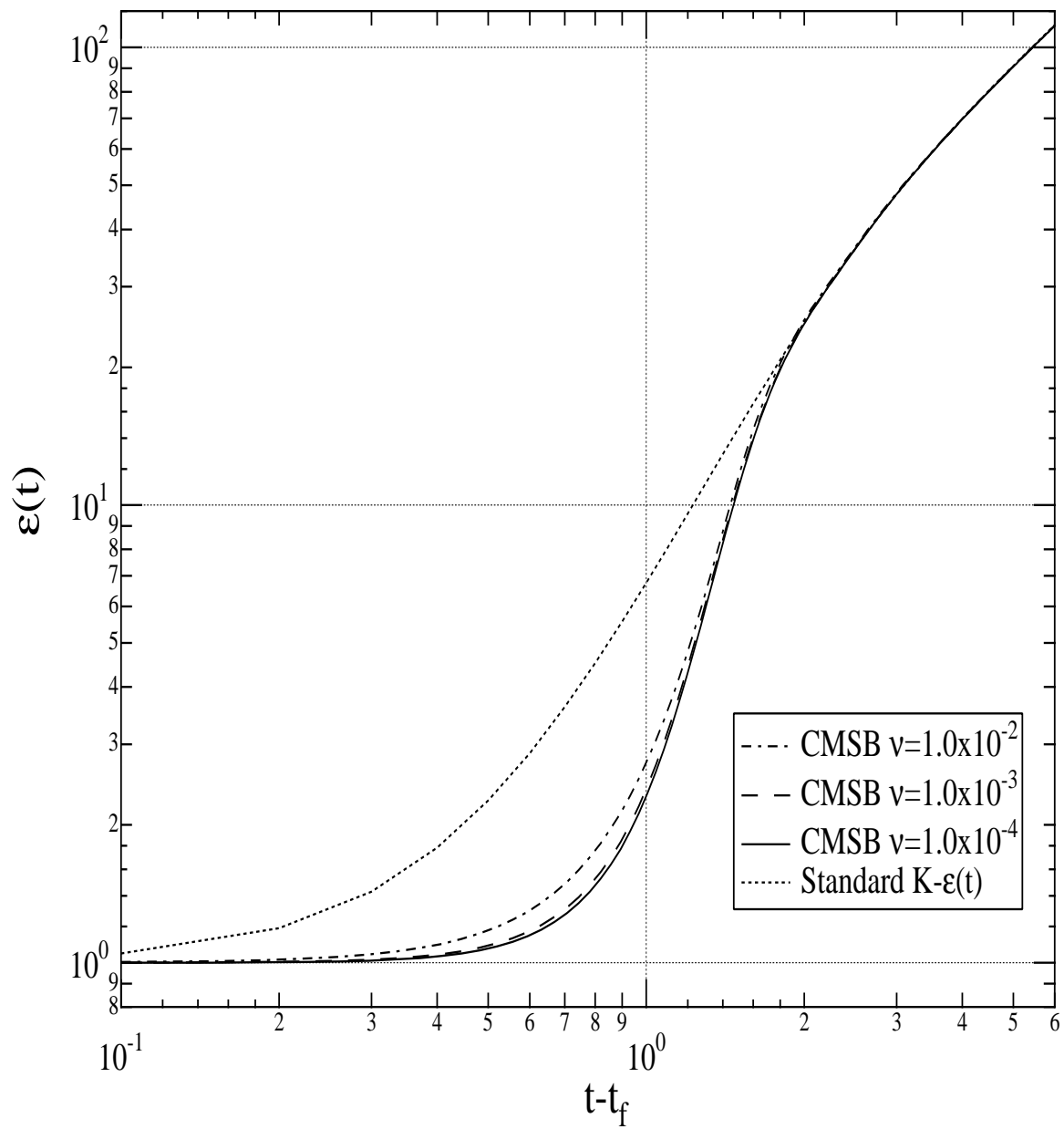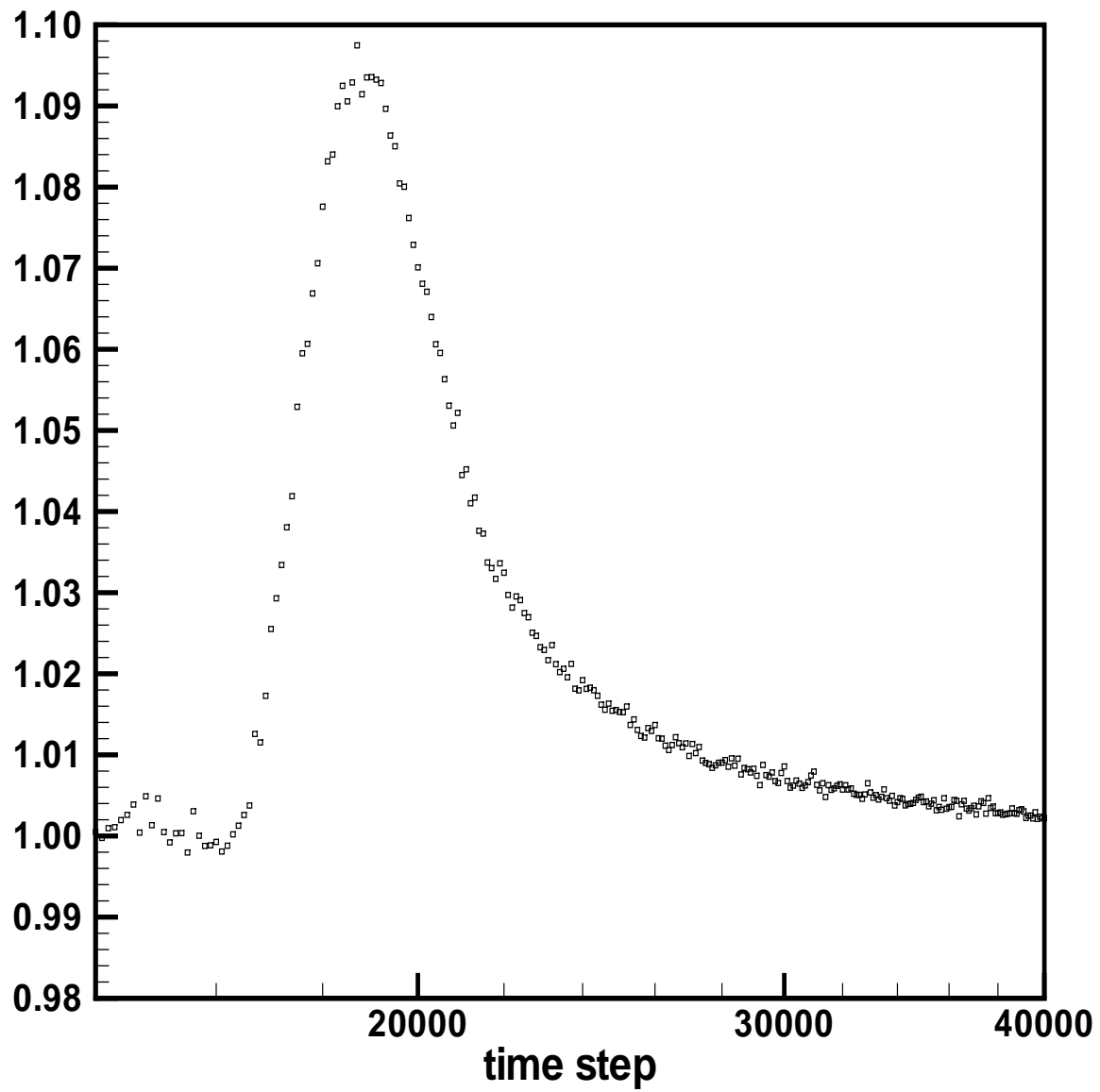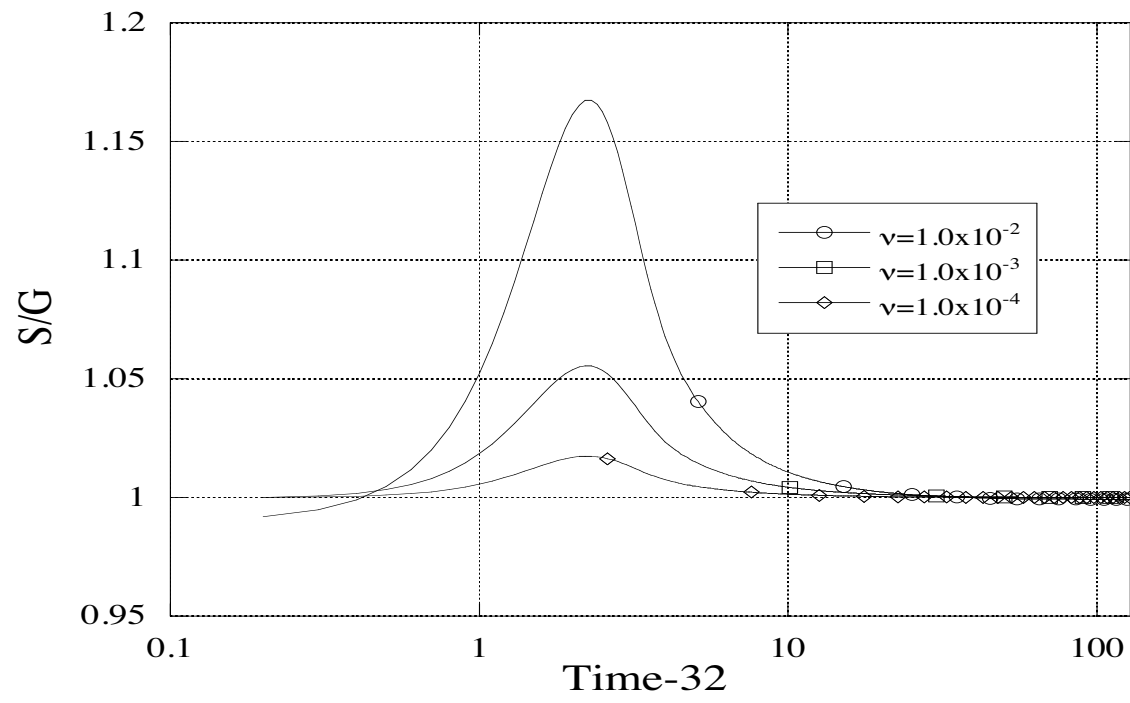
sanity check - energy evolution
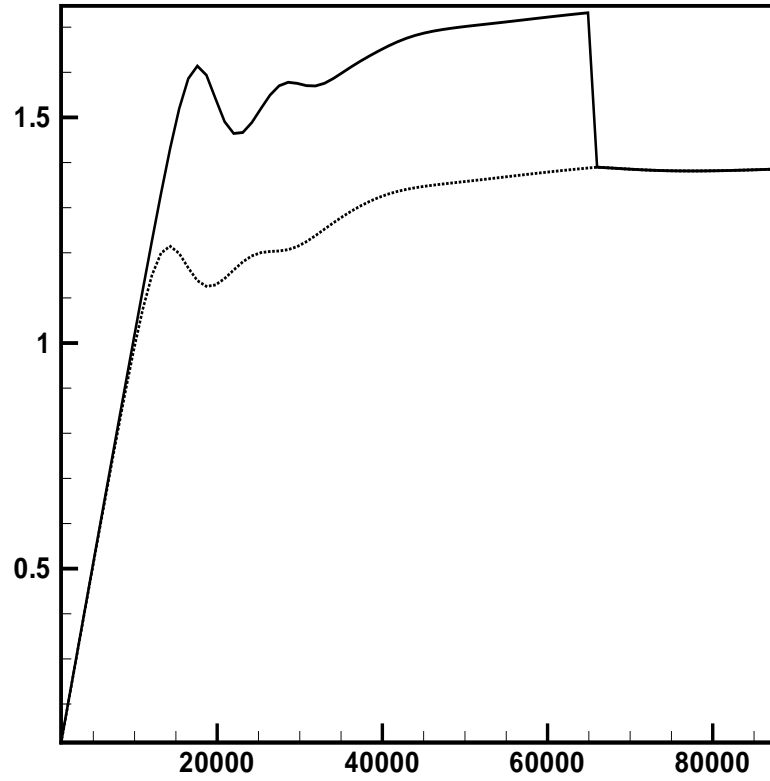
Figure 1: Kinetic energy computed by CMSB-Smagorinsky model (solid) compared to resolved energy (dotted).

time step

S/G

1.2

1.15

1.1

1.05

1

0.95

0.1   1   10   100

Time-32

$\nu=1.0 \times 10^{-2}$
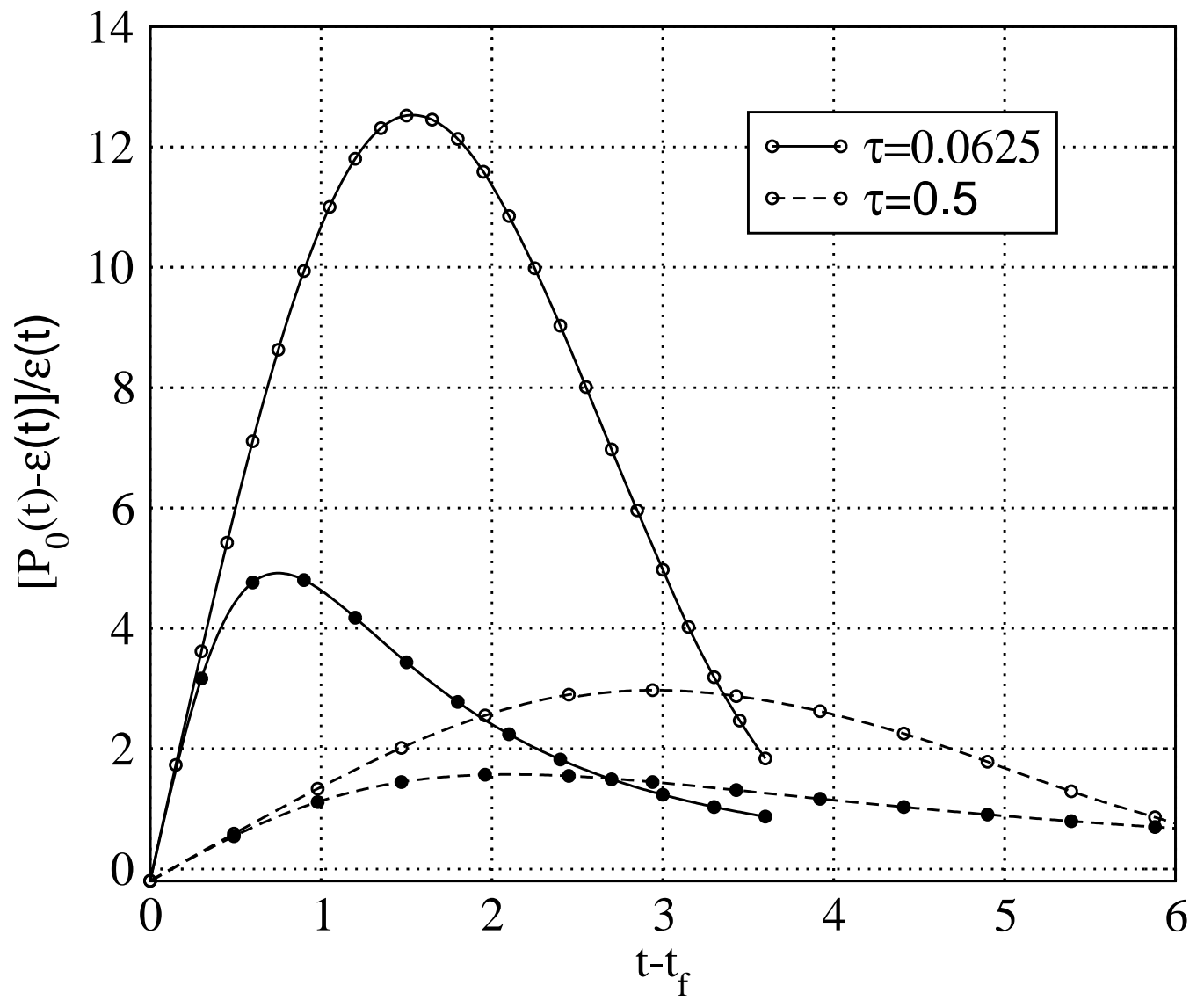$\nu=1.0 \times 10^{-3}$
$\nu=1.0 \times 10^{-4}$

Figure 1. The ratio of production to dissipation in DNS (open symbols) and in the two-equation model (closed symbols) at different ramp rates.

main conclusions

– Initial transient exhibits

* unbalanced vortex stretching with $S/G > 1$.

* rapid growth of ϵ on rapid time scales of small scales of motion

– **Recovery of balance** $S/G \approx 1$ during self-similar growth.

example 2 (tentative): growing length scale

– Constant amplitude force at linearly growing scale.

– Difficult even with closure. Self-similarity was not achieved.
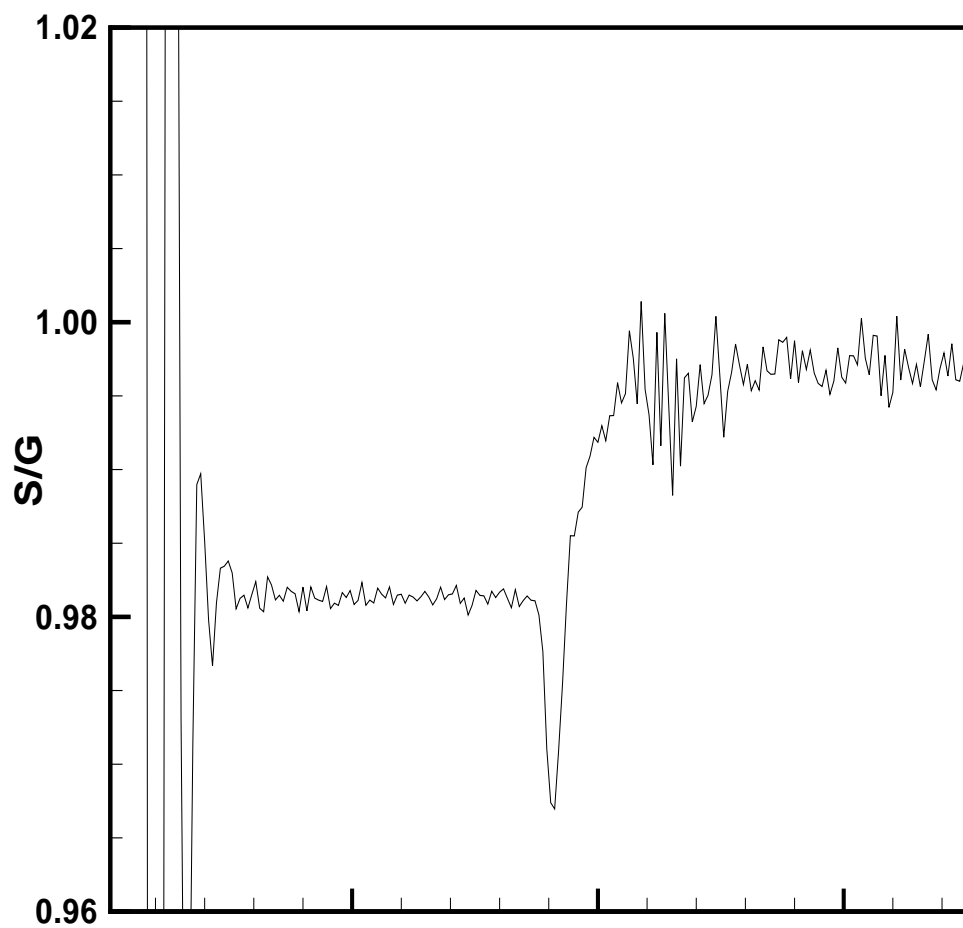
* Instantaneous *decrease* of S/G, followed by recovery.

Figure 1: ratio S/G in forcing with linearly increasing forcing scale

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transient unbalanced vortex stretching
example 3: periodically forced turbulence

– Steady state turbulence is perturbed by a force with periodically varying amplitude (Lohse, van de Water, von der Heydt).

– The problem is characterized by the phase averages $\tilde{k}$ and $\tilde{\epsilon}$ and the phase lags with respect to production, ϕ_k and ϕ_ϵ.

– EDQNM calculations (Wouter Bos, Ecole Centrale Lyon) show that $\tilde{\epsilon}$ and ϕ_ϵ depend strongly on Reynolds number.
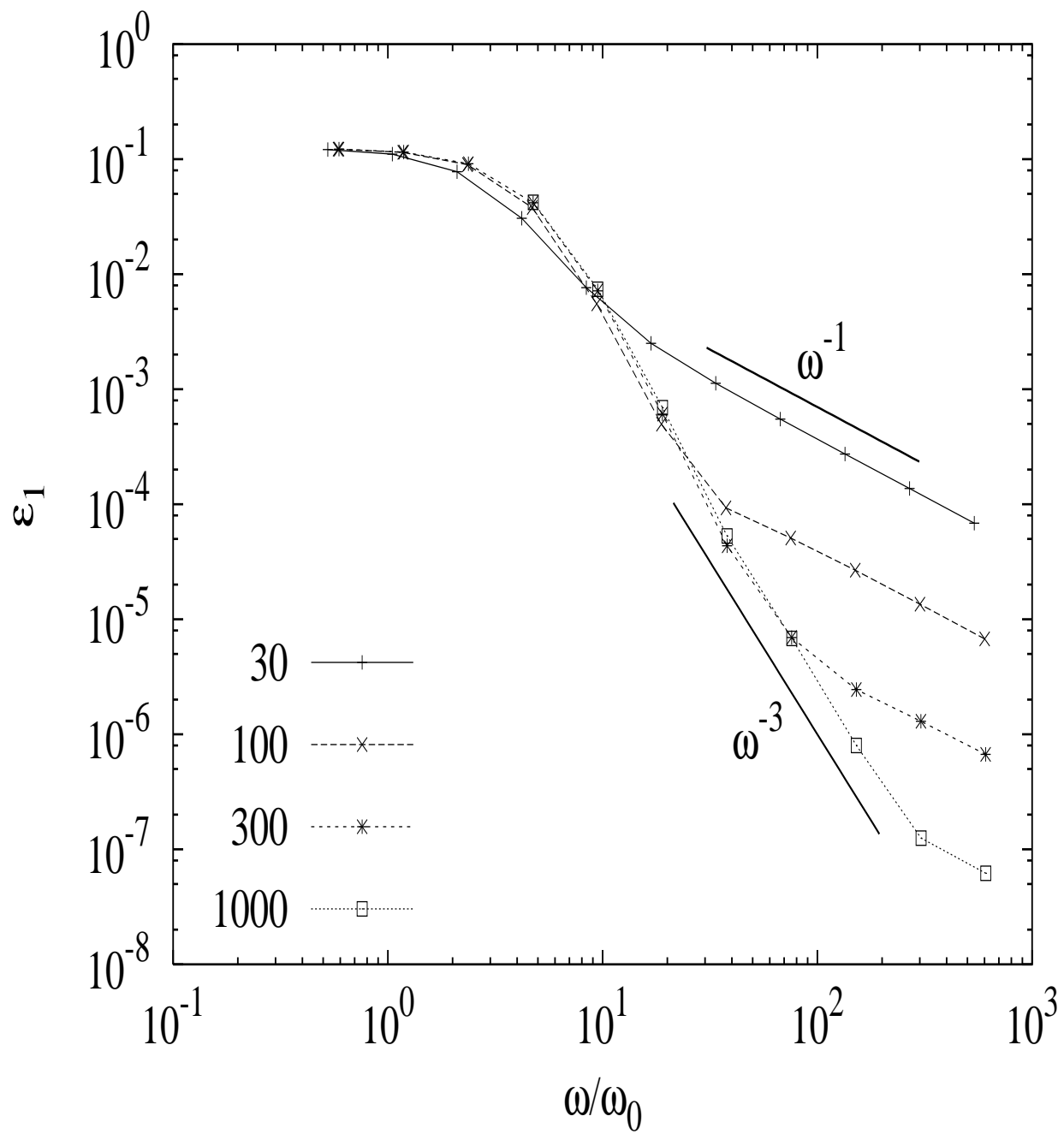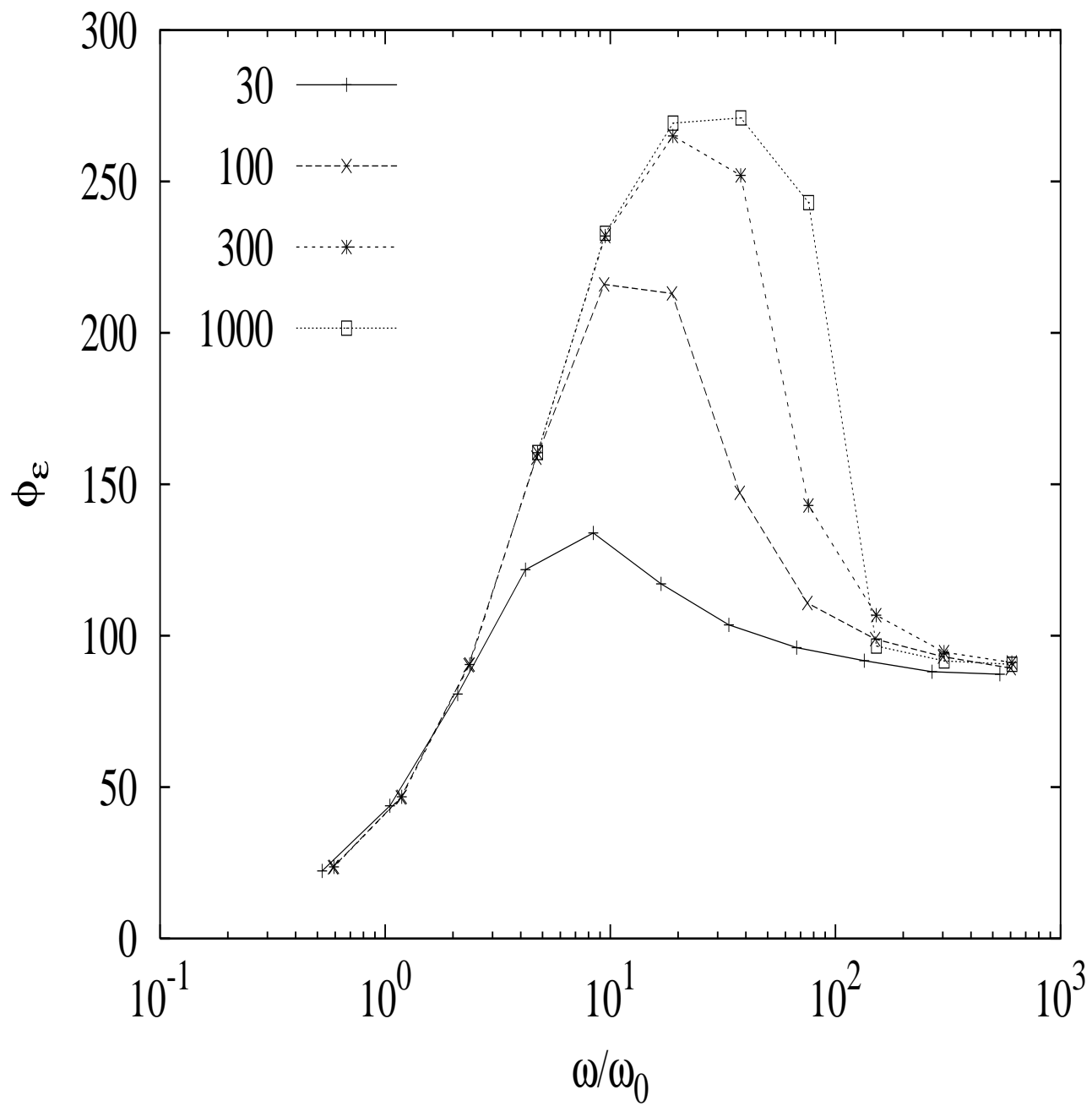
transient unbalanced vortex stretching
example 3: periodically forced turbulence results

– At fixed Re, in the limit $\omega \to \infty$, $\tilde{\epsilon} \sim \omega^{-1}$:

* frozen turbulence limit; oscillations do not reach small scales

– But at fixed ω, in the limit $Re \to \infty$, $\tilde{\epsilon} \sim \omega^{-3}$:

– No convergence of ϕ_ϵ as $Re \to \infty$.

summary: unbalanced vortex stretching and Speziale-Bernard revisited

− The Speziale-Bernard model replaces the usual hypothesis $\dot{\epsilon} = f(k, \epsilon, \nabla U)$ by $\dot{\epsilon} = f(k, \epsilon, \nabla U, Re)$ in order to model *unbalanced vortex stretching*.

* Not a low Re model, where Re effect disappears when $Re \to \infty$.

− They suggested Re dependence in self-similar decay and homogeneous shear. 'It has been illustrated above that the occurrence of equilibrium states with residual vortex stretching can have profound consequences for the prediction of turbulent flows.'

− We beg to differ: in closure calculations Re dependence disappears when(ever?) the turbulence becomes self-similar.

− Speziale later modified this view:

'...there is some validity to the belief that, in time, vortex stretching will come into balance with the leading order part of the viscous destruction term.' (ICASE/LaRC workshop 1991)

− Accordingly, Speziale proposed a linear relaxation model $\dot{S} - \dot{G} \propto -(S - G)$.

* It cannot explain the observation in ramp flow: initially $S = G$, imbalance develops under transient forcing, then $S \approx G$ is restored during self-similar evolution.

unbalanced vortex stretching: present view

− Unbalanced vortex stretching is possible in any transient problem.

− Small scales are not in 'equilibrium.' subgrid modeling?

− Transient unbalanced vortex stretching makes finite dimensional modeling impossible: the detailed dynamics of S and G become relevant and involves moments of the spectrum of all orders.
cf. rarefied gas dynamics

− But the Tennekes-Lumley balance exists in all *slowly varying* and *self-similar* ('equilibrium') flows.

Tennekes-Lumley balance in slow evolution

(Woodruff and Rubinstein, JFM 2006)

– In a spectral closure model,

$$\dot{E}(\kappa, t) = P(\kappa, t) - \frac{\partial}{\partial \kappa} \mathcal{F}(\kappa, t) - 2\nu\kappa^2 E(\kappa, t)$$

where $\mathcal{F} = \mathcal{F}[E(\kappa, t)]$ is some *functional* of E.

– Consider small, slow perturbations of a Kolmogorov steady state

$$0 = P_0(\kappa) - \frac{\partial}{\partial \kappa} \mathcal{F}_0 - 2\nu\kappa^2 E_0(\kappa)$$

− Small slow perturbations δE satisfy

$$\dot{E}_0(\kappa, t) = \delta P(\kappa, t) - \frac{\partial}{\partial \kappa}\mathcal{L}[\delta E(\kappa, t)] - 2\nu\kappa^2\delta E(\kappa, t) \;\; (*)$$

where $\mathcal{L}$ is the energy transfer linearized about the steady state.

− Parametrize $E_0(\kappa, t) = E_0(\kappa; \epsilon(t), L(t))$ by slowly varying dissipation rate ϵ and length scale L. local 'Kolmogorovian'

− Substitute in (*).

− (*) can be solved if compatibility equations are satisfied (two for the Heisenberg model - a *two-equation model* for $\epsilon(t)$ and $L(t)$).

– Formally, $\delta E \sim \kappa^{-7/3}$ by dimensional analysis or power counting using time scale $\dot{\epsilon}/\epsilon$ (Yoshizawa):

$$\dot{\epsilon}\epsilon^{-1/3}\kappa^{-5/3} \sim \epsilon^{1/3}\kappa^{-4/3}\kappa^2\delta E; \ \delta E \sim \underbrace{\left[\frac{\dot{\epsilon}}{\epsilon}\frac{1}{\epsilon^{1/3}\kappa^{2/3}}\right]}_{\text{'Knudsen number'}} \epsilon^{2/3}\kappa^{-5/3}$$

– $S = S_0 + \delta S$ and $G = G_0 + \delta G$.

– $S_0 = G_0$: cancellation of leading order $Re^{1/2}$ divergence.

– The scaling $\delta E \sim \kappa^{-7/3}$ implies $\delta S, \delta G \sim Re^0$: **QED**

summary

1. Introduce slow spectral evolution *ansatz*
$E(\kappa, t) = E_0(\kappa; \epsilon(t), L(t)) + E_1(\kappa, t).$

2. Compatibility conditions to find E_1 imply

– existence of a two-equation model

– Yoshizawa's correction $E_1 \sim (\dot{\epsilon}/\epsilon)\kappa^{-7/3}.$

– Tennekes-Lumley balance

3. Unbalanced vortex stretching when $E(\kappa, t) \neq E_0(\kappa; \epsilon(t), L(t)).$

CONCLUSIONS

Unbalanced vortex stretching (Re-dependence in at high Reynolds number) is possible; connected to transient failure of Kolmogorov relation $\epsilon = C(u')^3/L$.

It is a feature of transient evolution, but not of self-similar, or slowly varying evolution because the Kolmogorov theory is an attractor for the small scales of motion.

It would seem to limit the applicability of transport and subgrid models that assume 'ideal turbulence' at small scales.

Modeling poses interesting theoretical questions.