Averaged Vorticity Alignment Dynamics

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Collaborators and main references for this talk

JD Gibbon, DDH, RM Kerr & I Roulstone,

Quaternions and particle dynamics in the Euler fluid equations.

Nonlinearity **19** (2006) 1969-1983

JD Gibbon & DDH,

Lagrangian particle paths and ortho-normal quaternion frames.

http://arxiv.org/abs/nlin.CD/0607020

Lagrangian analysis of alignment dynamics

http://arxiv.org/abs/nlin.CD/0608009

BJ Geurts, DDH and A Kuczaj,

Direct and large-eddy simulation of rotating turbulence.

In preparation.

Notation: The 3D incompressible Euler fluid

The equations for **Eulerian fluid velocity** $\boldsymbol{u}$ in 3D are

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p\,, \quad \text{with} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0$$

Taking the curl yields the **vorticity stretching equation** ($\boldsymbol{\varpi} = \operatorname{curl}\boldsymbol{u}$)

$$\frac{D\boldsymbol{\varpi}}{Dt} = \boldsymbol{\varpi}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\varpi}$$

The **vortex stretching vector** is $S\boldsymbol{\varpi}$ with $S = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$, or

$$S_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

and preservation of $\operatorname{div}\boldsymbol{u} = 0$ determines the **pressure** p as

$$-\Delta p = u_{i,j}u_{j,i} =: |\nabla\boldsymbol{u}|^2 = \operatorname{Tr} S^2 - \tfrac{1}{2}\,\varpi^2\,,$$

aka **Okubo-Weiss formula**

The 3D Rotating, Stratified, Compressible Euler fluid

The equations for **Eulerian fluid velocity** $\boldsymbol{u}$ and density ρ in 3D are

$$\frac{D\boldsymbol{u}}{Dt} - \underbrace{\boldsymbol{u} \times 2\boldsymbol{\Omega}}_{\text{Coriolis}} =: \boldsymbol{\mathcal{F}} = \underbrace{-\rho^{-1}\nabla p - g\nabla z}_{\text{Pressure \& Gravity}} \quad \text{with} \quad \frac{D\rho^{-1}}{Dt} = \rho^{-1}\text{div}\,\boldsymbol{u}$$

Taking the curl yields the equation for **total vorticity**

$$\boxed{\textbf{Total Vorticity} \quad \boldsymbol{\omega} := \rho^{-1}(\text{curl}\,\boldsymbol{u} + 2\boldsymbol{\Omega})}$$

$$\frac{D\boldsymbol{\omega}}{Dt} = \underbrace{\boldsymbol{\omega}\cdot\nabla\boldsymbol{u} + \boldsymbol{\mathcal{W}}}_{\text{Vortex stretching}} \quad \text{with} \quad \boxed{\boldsymbol{\mathcal{W}} := \rho^{-1}\text{curl}\,\boldsymbol{\mathcal{F}}}$$

The extra vortex stretching vector vanishes ($\boldsymbol{\mathcal{W}} \equiv 0$) for

(1) Barotropic compressible fluids ($\boldsymbol{\mathcal{F}} = -\nabla(h(\rho) + gz)$) and

(2) Incompressible Euler fluids ($\boldsymbol{\mathcal{F}} = -\nabla p$)

Dynamics of Vorticity vs Relative Alignment

1) Even when $\boxed{\boldsymbol{\mathcal{W}} \equiv 0}$ the gradient $\nabla \boldsymbol{\mathcal{F}}$ still matters in the evolution of the total vorticity stretching term $(\boldsymbol{\omega} \cdot \nabla \boldsymbol{u})$ with $\boldsymbol{\omega} := \rho^{-1}(\mathrm{curl}\,\boldsymbol{u} + 2\boldsymbol{\Omega})$.

2) The orientation of total vorticity arises from **1st time derivative** $D\boldsymbol{\omega}/Dt$

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}) \quad \text{when} \quad \boldsymbol{\mathcal{W}} \equiv 0$$

3) The **alignment** $(\boldsymbol{\omega} \cdot \nabla \boldsymbol{u})$ is governed by the **2nd time derivative** $D^2\boldsymbol{\omega}/Dt^2$

$$\frac{D^2\boldsymbol{\omega}}{Dt^2} = \frac{D(\boldsymbol{\omega} \cdot \nabla \boldsymbol{u})}{Dt} = \boldsymbol{\omega} \cdot \nabla \big(\boldsymbol{\mathcal{F}} + \boldsymbol{u} \times 2\boldsymbol{\Omega} \big) \quad \text{when} \quad \boldsymbol{\mathcal{W}} \equiv 0$$

4) We focus on the dynamics of relative alignment or vortex stretching $(\boldsymbol{\omega} \cdot \nabla \boldsymbol{u})$, in addressing questions such as, **"How long does the vorticity stay aligned?"**

Ertel's Theorem (1942)

Theorem: (Ertel 1942) *If $\boldsymbol{\omega}$ satisfies the $3D$ vortex stretching equation, then an arbitrary differentiable function μ satisfies*

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla\mu) = \boldsymbol{\omega}\cdot\nabla\left(\frac{D\mu}{Dt}\right) .$$

Proof: In characteristic (Lie-derivative) form, vorticity stretching is

$$\frac{D}{Dt}\Big(\boldsymbol{\omega}\cdot\frac{\partial}{\partial\mathbf{x}}\Big) = \Big(\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega}\cdot\nabla\boldsymbol{u}\Big)\cdot\frac{\partial}{\partial\mathbf{x}} = 0 \quad\text{along}\quad \frac{d\mathbf{x}}{dt} = \boldsymbol{u}(\mathbf{x},t)$$

So $\boldsymbol{\omega}\cdot\frac{\partial}{\partial\mathbf{x}}(t) = \boldsymbol{\omega}\cdot\frac{\partial}{\partial\mathbf{x}}(0)$ (Cauchy 1859) and the derivatives commute

$$\left[\frac{D}{Dt}\,,\,\boldsymbol{\omega}\cdot\nabla\right] = 0$$

Hence, Ertel's theorem follows.

Corollary: $D\mu/Dt = 0$ implies $D(\boldsymbol{\omega}\cdot\nabla\mu)/Dt = 0$ (e.g. PV in GFD).

Some Ertel references

- Ertel; *Ein Neuer Hydrodynamischer Wirbelsatz*, Met. Z. **59**, 271-281, (1942).
- Hoskins, McIntyre, & Robertson; *On the use & significance of isentropic potential vorticity maps*, Quart. J. Roy. Met. Soc., **111**, 877-946, (1985).
- Ohkitani; *Eigenvalue problems in 3D Euler flows*, Phys. Fluids, **A5**, 2570, (1993).
- Viudez; *On the relation between Beltrami's material vorticity and Rossby-Ertel's Potential*, J. Atmos. Sci. (2001).

More about rotating Euler fluids

Applying $\frac{D}{Dt}$ time derivative to $\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = \boldsymbol{\mathcal{W}} := \rho^{-1}\text{curl}\,\boldsymbol{\mathcal{F}}$ yields

$$\frac{D^2\boldsymbol{\omega}}{Dt^2} = \boldsymbol{\omega}\cdot\nabla\Big(\boldsymbol{\mathcal{F}} + \underbrace{\boldsymbol{u}\times 2\boldsymbol{\Omega}}_{\text{Coriolis}}\Big) + \Big(\underbrace{\frac{D\boldsymbol{\mathcal{W}}}{Dt} - \boldsymbol{\mathcal{W}}\cdot\nabla\boldsymbol{u}}_{\boldsymbol{\mathcal{W}}\text{ is also stretched}}\Big)$$

For incompressible and barotropic rotating Euler flows $\boxed{\boldsymbol{\mathcal{W}}\equiv 0}$ and

$$\frac{D\boldsymbol{\omega}}{Dt} = \underbrace{\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}}_{\text{Alignment \#1}} \quad \& \quad \frac{D^2\boldsymbol{\omega}}{Dt^2} = \frac{D(\boldsymbol{\omega}\cdot\nabla\boldsymbol{u})}{Dt} = \underbrace{\boldsymbol{\omega}\cdot\nabla\big(\boldsymbol{\mathcal{F}} + \boldsymbol{u}\times 2\boldsymbol{\Omega}\big)}_{\text{Alignment \#2}}$$

Alignment Dynamics #1 & #2

Alignment #1 of $\boldsymbol{\omega}$ with gradient $\nabla\boldsymbol{u}$ is vortex stretching, which drives $\boldsymbol{\omega}$

Alignment #2 of $\boldsymbol{\omega}$ with gradient ∇(total force) drives $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}$

Alignment #2 is Ohkitani's relation for $\boldsymbol{\Omega} = 0$ & $\boldsymbol{\mathcal{F}} = -\nabla p$ (incompress)

Rotation affects density dynamics in barotropic fluids

The Vortex Stretching and Ohkitani relations are the 1st & 2nd time-derivatives

Vortex Stretching: $$\frac{D\boldsymbol{\omega}}{Dt} = \underbrace{\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}}_{\textbf{Alignment \#1}} \qquad \boxed{\boldsymbol{\omega} := \rho^{-1}(\text{curl}\,\boldsymbol{u} + 2\boldsymbol{\Omega})}$$

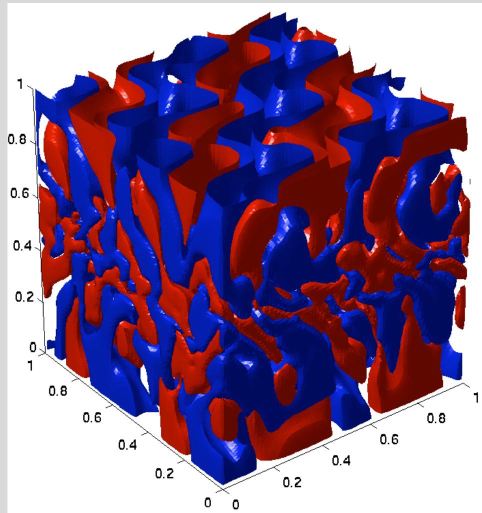
Ohkitani relation: $$\frac{D^2\boldsymbol{\omega}}{Dt^2} = \frac{D(\boldsymbol{\omega}\cdot\nabla\boldsymbol{u})}{Dt} = \underbrace{\boldsymbol{\omega}\cdot\nabla\big(\boldsymbol{\mathcal{F}} + \boldsymbol{u}\times 2\boldsymbol{\Omega}\big)}_{\textbf{Alignment \#2}}$$

Finally, for barotropic $\boldsymbol{\mathcal{F}} = \nabla(h(\rho) + gz)$ we need **density dynamics**

$$\frac{D\rho^{-1}}{Dt} = \rho^{-1}\text{div}\,\boldsymbol{u}$$

Coriolis force affects the 2nd-time-derivative of specific volume ρ^{-1},

$$\frac{D^2\rho^{-1}}{Dt^2} = \rho^{-1}\Big((\text{div}\,\boldsymbol{u})^2 - |\nabla\boldsymbol{u}|^2 + \text{div}\,\underbrace{(\boldsymbol{\mathcal{F}} + \boldsymbol{u}\times 2\boldsymbol{\Omega})}_{\textbf{Rotation in } D^2\rho^{-1}/Dt^2}\Big)$$



Coriolis effects in compressible rotating shear layers

B. J. Geurts, DDH and A. K. Kuczaj, ECT11 2006

DNS show that **rotation can modulate turbulence** at low Mach number.

In rotating shear layers, 2D and 3D tendencies compete, thereby causing

(1) Formation of **columns of vorticity that oscillate** and

(2) **Nonmonotonic decay rate** of kinetic energy

Dynamics of alignment of total vorticity with axis of rotation $\Omega = \Omega_0 \hat{\boldsymbol{z}}$ in compressible shear layers

Use Ertel's theorem to find evolution of $\omega_z := \hat{\boldsymbol{z}} \cdot \boldsymbol{\omega}$ with $\boldsymbol{\omega} = \text{curl}\, \boldsymbol{u} + 2\boldsymbol{\Omega}$

$$\frac{D\omega_z}{Dt} - \boldsymbol{\omega} \cdot \nabla u_z = \mathcal{W}_z := \rho^{-1} \hat{\boldsymbol{z}} \cdot \text{curl}\, \boldsymbol{\mathcal{F}}$$

and hence

$$\frac{D^2\omega_z}{Dt^2} = \boldsymbol{\omega} \cdot \nabla \mathcal{F}_z + \left(\frac{D\mathcal{W}_z}{Dt} - \boldsymbol{\mathcal{W}} \cdot \nabla u_z \right)$$

where

$$\boldsymbol{\mathcal{F}} := -2\Omega_0 \hat{\boldsymbol{z}} \times \boldsymbol{u} - (RoM^{-2})\rho^{-1}\nabla p - \nu \Delta \boldsymbol{u} \quad \text{and} \quad \mathcal{F}_z = -(RoM^{-2})\rho^{-1}\frac{\partial p}{\partial z} - \nu \Delta u_z$$

These alignment evolution equations govern the transition to columnar motion demanded by the **Taylor-Proudman theorem**.

The mechanism for transition to columnarity is likely to involve vortex merger along the axis of rotation.

Outline for the rest of the talk
(Retreat back to Euler's equations)

1. Use **Ohkitani's relation** to derive **vorticity frame dynamics and alignment dynamics** for Euler's equations.

2. Use **Ertel's theorem** to derive **Lagrangian dynamics** of the Frenet-Serret curvature and torsion of vortex lines

3. Represent total vorticity alignments as **quaternionic products** denoted $\circledast$

$$\hat{\boldsymbol{\omega}} \cdot \nabla \boldsymbol{u} =: S\hat{\boldsymbol{\omega}} \; = \; \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi} \times \hat{\boldsymbol{\omega}} = [\alpha,\, \boldsymbol{\chi}] \circledast [0,\, \hat{\boldsymbol{\omega}}]$$

$$\hat{\boldsymbol{\omega}} \cdot \nabla \big(\boldsymbol{\mathcal{F}} + \boldsymbol{u} \times 2\boldsymbol{\Omega} \big) =: P\hat{\boldsymbol{\omega}} \; = \; \alpha_p\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}_p \times \hat{\boldsymbol{\omega}} = [\alpha_p,\, \boldsymbol{\chi}_p] \circledast [0,\, \hat{\boldsymbol{\omega}}]$$

4. Derive **dynamics of quaternions** $\zeta = [\alpha,\, \boldsymbol{\chi}]$ driven by $\boldsymbol{\zeta}_p = [\alpha_p,\, \boldsymbol{\chi}_p]$

$$\frac{D\boldsymbol{\zeta}}{Dt} + \boldsymbol{\zeta} \circledast \boldsymbol{\zeta} + \boldsymbol{\zeta}_p = 0 \quad \textbf{(Ricatti equation)}$$

5. Apply this structure to Lagrangian-averaged models of rotating fluid turbulence

Define vorticity growth rate (α) and swing rate (χ)

Euler's equations imply material rates of change of $|\boldsymbol{\omega}|$ and $\hat{\boldsymbol{\omega}}$ given by

$$\frac{D\boldsymbol{\omega}}{Dt} = S\boldsymbol{\omega} \quad \text{with} \quad S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}} = (S\hat{\boldsymbol{\omega}})_{||} + (S\hat{\boldsymbol{\omega}})_{\perp}$$

- *The* *scalar* $\alpha = \hat{\boldsymbol{\omega}}\cdot S\hat{\boldsymbol{\omega}}$ *is the* *vorticity growth rate*

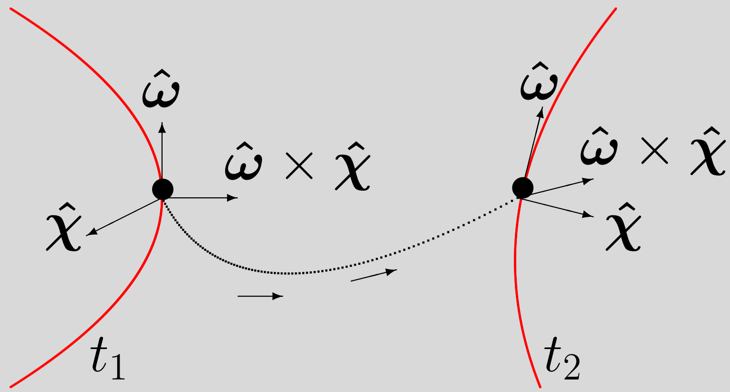
$$\frac{D|\boldsymbol{\omega}|}{Dt} = \alpha\,|\boldsymbol{\omega}| \qquad \begin{array}{ll} \alpha > 0 & \text{stretching} \\ \alpha < 0 & \text{shrinking} \end{array}$$

- *The* *3-vector* $\boldsymbol{\chi} = \hat{\boldsymbol{\omega}}\times S\hat{\boldsymbol{\omega}}$ *is the* *vorticity swing rate*

$$\frac{D\hat{\boldsymbol{\omega}}}{Dt} = \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}}\,, \qquad \hat{\boldsymbol{\omega}}\times\frac{D\hat{\boldsymbol{\omega}}}{Dt} = \boldsymbol{\chi} \quad \text{(frequency)}$$

Remark: If $\boldsymbol{\omega}$ aligns with an eigenvector $S\hat{\boldsymbol{\omega}} = \lambda\,\hat{\boldsymbol{\omega}}$, then $\boldsymbol{\chi} = 0$.
For such alignment, the vorticity direction is **frozen** *into the flow.*

Lagrangian frame dynamics: tracking the orientation of vorticity following a fluid particle



The figure shows a vortex line at two times t_1 & t_2, the Lagrangian trajectory of one of its vortex line elements, and the orientations of the orthonormal frame $\{\hat{\omega},\, \hat{\chi},\, (\hat{\omega} \times \hat{\chi})\}$ attached to it at the two times.

The Trouble with pressure in Alignment Dynamics

- **Vorticity is driven by $S-$Alignment (Vorticity stretching)**

$$\frac{D\boldsymbol{\omega}}{Dt} = S\boldsymbol{\omega} = \alpha\,\boldsymbol{\omega} + \boldsymbol{\chi} \times \boldsymbol{\omega}$$

- $S-$**Alignment is driven by $P-$Alignment (Ohkitani)**

$$\frac{DS\boldsymbol{\omega}}{Dt} = -\,P\boldsymbol{\omega} = -\alpha_p\,\boldsymbol{\omega} - \boldsymbol{\chi}_p \times \boldsymbol{\omega}$$

The same form holds for incompressible Euler fluids and

$$\boldsymbol{u} = \mathrm{curl}^{-1}\boldsymbol{\omega}, \quad \mathrm{tr}\,P = \Delta p = -\,|\nabla\boldsymbol{u}|^2.$$

- **Incompressibility summons the pressure Hessian P.**

- **The pressure solve is spatially nonlocal, not evolutionary.**

- **Dropping pressure would mean** $\dfrac{D^2\boldsymbol{\omega}}{Dt^2} = \dfrac{DS\boldsymbol{\omega}}{Dt} = 0$.
  This would freeze alignment $S\boldsymbol{\omega} = \lambda\,\boldsymbol{\omega}$ **with** $\dfrac{D\lambda}{Dt} = -\,\lambda\alpha$

$[\,||,\ \perp]$ Alignment variables $[\alpha,\ \boldsymbol{\chi}]$ & $[\alpha_p,\ \boldsymbol{\chi}_p]$

$S\hat{\boldsymbol{\omega}}$ lies in the $(\hat{\boldsymbol{\omega}},\ \hat{\boldsymbol{\omega}}\times\hat{\boldsymbol{\chi}})$ plane and $P\hat{\boldsymbol{\omega}}$ in the $(\hat{\boldsymbol{\omega}},\ \hat{\boldsymbol{\omega}}\times\hat{\boldsymbol{\chi}}_p)$ plane

$$S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}}\,, \qquad P\hat{\boldsymbol{\omega}} = \alpha_p\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}_p\times\hat{\boldsymbol{\omega}}$$

where $(\alpha,\ \boldsymbol{\chi})$ & $(\alpha_p,\ \boldsymbol{\chi}_p)$ define $S\hat{\boldsymbol{\omega}}$ & $P\hat{\boldsymbol{\omega}}$ as stretched & rotated $\hat{\boldsymbol{\omega}}$

$$\alpha = \hat{\boldsymbol{\omega}}\cdot S\hat{\boldsymbol{\omega}}\,, \quad \boldsymbol{\chi} = \hat{\boldsymbol{\omega}}\times S\hat{\boldsymbol{\omega}}\,,$$

$$\alpha_p = \hat{\boldsymbol{\omega}}\cdot P\hat{\boldsymbol{\omega}}\,, \quad \boldsymbol{\chi}_p = \hat{\boldsymbol{\omega}}\times P\hat{\boldsymbol{\omega}} =: -c_1\hat{\boldsymbol{\chi}}\times\hat{\boldsymbol{\omega}} - c_2\hat{\boldsymbol{\chi}}$$

Evolution of vorticity alignment

For the Euler fluid, we have

$$\frac{D\boldsymbol{\omega}}{Dt} = S\boldsymbol{\omega} \quad \& \quad \frac{D^2\boldsymbol{\omega}}{Dt^2} = -\,P\boldsymbol{\omega}$$

$$\text{where} \quad S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}} \quad \text{and} \quad P\hat{\boldsymbol{\omega}} = \alpha_p\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}_p\times\hat{\boldsymbol{\omega}}$$

As we know, P-alignment drives S-alignment. That is,

$$\frac{DS\boldsymbol{\omega}}{Dt} = -\,P\boldsymbol{\omega} \quad \text{or} \quad \frac{D}{Dt}(\alpha\,\boldsymbol{\omega} + \boldsymbol{\chi}\times\boldsymbol{\omega}) = -\,(\alpha_p\,\boldsymbol{\omega} + \boldsymbol{\chi}_p\times\boldsymbol{\omega})$$

A direct calculation shows that P-parameters $[\alpha_p,\,\boldsymbol{\chi}_p]$ drive S-parameters $[\alpha,\,\boldsymbol{\chi}]$ in the following **alignment-parameter dynamics**

$$\boxed{\frac{D\alpha}{Dt} + \alpha^2 - \chi^2 = -\,\alpha_p \quad \text{and} \quad \frac{D\boldsymbol{\chi}}{Dt} + 2\alpha\boldsymbol{\chi} = -\,\boldsymbol{\chi}_p}$$

Later we'll interpret alignment-parameter dynamics as one quaternionic equation.

$[\,||,\,\perp]$ decomposition $\Leftrightarrow$ quaternionic multiplication

Seek alignment-parameter dynamics of growth rate (α) and swing rate $(\boldsymbol{\chi})$ in

$$S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}} = (S\hat{\boldsymbol{\omega}})_{||} + (S\hat{\boldsymbol{\omega}})_{\perp}$$

for a **combined** scalar and vector quantity denoted

$$\boldsymbol{\zeta} = [\alpha,\,\boldsymbol{\chi}] = [\hat{\boldsymbol{\omega}}\cdot S\hat{\boldsymbol{\omega}},\,\hat{\boldsymbol{\omega}}\times S\hat{\boldsymbol{\omega}}]$$

Rewrite $[\,||,\,\perp]$ decomposition of $S\hat{\boldsymbol{\omega}}$ as **quaternionic multiplication** $\circledast$

$$[0,\,S\hat{\boldsymbol{\omega}}] = [\alpha,\,\boldsymbol{\chi}]\circledast[0,\,\hat{\boldsymbol{\omega}}]$$

where the product $\circledast : \mathfrak{Q}\times\mathfrak{Q}\to\mathfrak{Q}$ is defined in components by

$$\boxed{\mathfrak{p}\circledast\mathfrak{q} = \Big[pq - \boldsymbol{p}\cdot\boldsymbol{q}\,,\,p\boldsymbol{q} + \boldsymbol{p}q + \boldsymbol{p}\times\boldsymbol{q}\Big] \quad \text{for} \quad \mathfrak{p} = [p,\,\boldsymbol{p}]\,, \quad \mathfrak{q} = [q,\,\boldsymbol{q}]}$$

Check the $\mathfrak{p}\circledast\mathfrak{q}$ multiplication with $\mathfrak{p} = [\alpha,\,\boldsymbol{\chi}]$ and $\mathfrak{q} = [0,\,\hat{\boldsymbol{\omega}}]$

$$[\alpha,\,\boldsymbol{\chi}]\circledast[0,\,\hat{\boldsymbol{\omega}}] = [\alpha 0 - \boldsymbol{\chi}\cdot\hat{\boldsymbol{\omega}},\,\alpha\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}0 + \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}}] = [0,\,\alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}}] = [0,\,S\hat{\boldsymbol{\omega}}]$$

Quaternionic form of Euler's equations (Gibbon, 2002)

Define velocity & pressure quats $\mathcal{U}$ & $\boldsymbol{\Pi}$ and the 4-derivative $\boldsymbol{\nabla}$ as

$$\mathcal{U} = [0, \boldsymbol{u}] \qquad \boldsymbol{\Pi} = [p, 0] \qquad \boldsymbol{\nabla} = [0, \nabla]$$

Then Euler's fluid equation is written in quaternionic form as

$$\boxed{\frac{D\mathcal{U}}{Dt} = -\boldsymbol{\nabla} \circledast \boldsymbol{\Pi}}$$

The **vorticity quat** $\boldsymbol{\Omega}$ is formed from

$$\boldsymbol{\nabla} \circledast \mathcal{U} = [-\mathrm{div}\, \boldsymbol{u}\,,\ \mathrm{curl}\, \boldsymbol{u}] = [0, \boldsymbol{\omega}] =: \boldsymbol{\Omega}$$

Operating with $\boldsymbol{\nabla}\circledast$ on Euler's equation above produces

$$[\Delta p,\ 0] = \Big[-\ |\nabla \boldsymbol{u}|^2\,, \quad \underbrace{\frac{D\boldsymbol{\omega}}{Dt} - S\boldsymbol{\omega}}_{\text{Vortex stretching}} \quad \Big]$$

Identifying terms yields $\Delta p = -\,|\nabla \boldsymbol{u}|^2$ and Euler's vorticity equation.

Theorem: *The vorticity quat* $\boldsymbol{\Omega}(\boldsymbol{x}\,,t) = [0,\,\boldsymbol{\omega}]$ *satisfies*

$$\frac{D\boldsymbol{\Omega}}{Dt} = \boldsymbol{\zeta} \circledast \boldsymbol{\Omega} \qquad \text{(Frozen-in quat field)}$$

$$\frac{D^2\boldsymbol{\Omega}}{Dt^2} + \boldsymbol{\zeta}_p \circledast \boldsymbol{\Omega} = 0 \qquad \text{(Ohkitani's relation)}$$

where $\boldsymbol{\zeta} = [\alpha,\,\boldsymbol{\chi}]$ **and** $\boldsymbol{\zeta}_p = [\alpha_p,\,\boldsymbol{\chi}_p]$**.**

Consequently, the growth & swing rate quat $\boldsymbol{\zeta}(\boldsymbol{x}\,,t) = [\alpha,\,\boldsymbol{\chi}]$ *satisfies*

$$\boxed{\frac{D\boldsymbol{\zeta}}{Dt} + \boldsymbol{\zeta} \circledast \boldsymbol{\zeta} + \boldsymbol{\zeta}_p = 0}$$

Remark: The $\boldsymbol{\zeta}$-equation is a Ricatti equation driven by $\boldsymbol{\zeta}_p$ which, in turn, depends on the other variables through the pressure Hessian P.

The growth/swing rate quat $\boldsymbol{\zeta}(\boldsymbol{x}\,,t) = [\alpha,\,\boldsymbol{\chi}]$ evolves by quadratic nonlinearity and is driven by the $P-$alignment quat $\boldsymbol{\zeta}_p = [\alpha_p,\,\boldsymbol{\chi}_p]$.

Proof:

$$\frac{D\boldsymbol{\Omega}}{Dt} = [0,\ \underbrace{\alpha\,\boldsymbol{\omega} + \boldsymbol{\chi}\times\boldsymbol{\omega}}_{S\boldsymbol{\omega}}\,] = [\alpha\,,\boldsymbol{\chi}]\circledast[0,\ \boldsymbol{\omega}] = \boldsymbol{\zeta}\circledast\boldsymbol{\Omega}\,.$$

$$P\boldsymbol{\omega} = \alpha_p\,\boldsymbol{\omega} + \boldsymbol{\chi}_p\times\boldsymbol{\omega} \qquad \Rightarrow \qquad [0,\ P\boldsymbol{\omega}] = \boldsymbol{\zeta}_p\circledast\boldsymbol{\Omega}$$

Use Ertel's Theorem to express Ohkitani's relation as

$$\frac{D^2\boldsymbol{\Omega}}{Dt^2} = \frac{D}{Dt}[0,\ S\boldsymbol{\omega}] = -[0,\ P\boldsymbol{\omega}] = -\,\boldsymbol{\zeta}_p\circledast\boldsymbol{\Omega}$$

Compare this relation with $D^2\boldsymbol{\Omega}/Dt^2 = D/Dt(\boldsymbol{\zeta}\circledast\boldsymbol{\Omega})$ to find

$$0 = \frac{D\boldsymbol{\zeta}}{Dt}\circledast\boldsymbol{\Omega} + \boldsymbol{\zeta}\circledast(\boldsymbol{\zeta}\circledast\boldsymbol{\Omega}) + \boldsymbol{\zeta}_p\circledast\boldsymbol{\Omega}$$

The Ricatti equation for ζ follows, because $\circledast$ is associative. ■

Quaternion alignment dynamics in components

The alignment equation for quats $\boldsymbol{\zeta} = [\alpha,\, \boldsymbol{\chi}]$ with $\boldsymbol{\zeta}_p = [\alpha_p,\, \boldsymbol{\chi}_p]$ is

$$\boxed{\frac{D\boldsymbol{\zeta}}{Dt} + \boldsymbol{\zeta} \circledast \boldsymbol{\zeta} + \boldsymbol{\zeta}_p = 0}$$

Recall the components of the quat multiplication rule

$$\mathfrak{p} \circledast \mathfrak{q} = [pq - \boldsymbol{p}\cdot\boldsymbol{q}\,,\; p\boldsymbol{q} + q\boldsymbol{p} + \boldsymbol{p}\times\boldsymbol{q}]$$

So $\boldsymbol{\zeta} \circledast \boldsymbol{\zeta} = [\alpha^2 - \chi^2, 2\alpha\boldsymbol{\chi}]$ in components & the alignment variables α, $\boldsymbol{\chi}$ are driven by α_p, $\boldsymbol{\chi}_p$ according to **alignment-parameter dynamics**

$$\boxed{\frac{D\alpha}{Dt} + \alpha^2 - \chi^2 + \alpha_p = 0 \quad \text{and} \quad \frac{D\boldsymbol{\chi}}{Dt} + 2\alpha\boldsymbol{\chi} + \boldsymbol{\chi}_p = 0}$$

where $\quad S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}} \quad$ and $\quad P\hat{\boldsymbol{\omega}} = \alpha_p\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}_p\times\hat{\boldsymbol{\omega}}$

$$\frac{D\boldsymbol{\omega}}{Dt} = S\boldsymbol{\omega} \quad \& \quad \boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}\,, \qquad \frac{DS\boldsymbol{\omega}}{Dt} = -\,P\boldsymbol{\omega} \quad \& \quad \mathrm{tr}\,P = -\,|\nabla\boldsymbol{u}|^2$$

Ricatti equations also arise in velocity-gradient dynamics.

Alignment dynamics in polar coordinates

In polar coordinates given by the stretching rate along $\hat{\boldsymbol{\omega}}$ as the radius $r = (\alpha^2+\chi^2)^{1/2} = |S\hat{\boldsymbol{\omega}}|$ and the angle $\theta = \tan^{-1}\chi/\alpha$ of rotation about the comoving $\hat{\boldsymbol{\chi}}$ axis from $\hat{\boldsymbol{\omega}}$ to $S\hat{\boldsymbol{\omega}}$, the alignment dynamics derived from

$$\frac{DS\boldsymbol{\omega}}{Dt} = -\,P\boldsymbol{\omega}$$

becomes, upon using

$$S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \chi\,\hat{\boldsymbol{\chi}}\times\hat{\boldsymbol{\omega}} = r(\cos\theta\,\hat{\boldsymbol{\omega}} + \sin\theta\,\hat{\boldsymbol{\chi}}\times\hat{\boldsymbol{\omega}})\,,$$

the 2×2 system in polar coordinates,

$$\frac{D}{Dt}\frac{\sin\theta}{r} + \cos 2\theta \;=\; \frac{\alpha_p}{r^2}$$

$$\frac{D}{Dt}\frac{\cos\theta}{r} - \sin 2\theta \;=\; \frac{\hat{\boldsymbol{\chi}}\cdot\boldsymbol{\chi}_p}{r^2}$$

where one recalls that $\hat{\boldsymbol{\chi}}\cdot\boldsymbol{\chi}_p = -c_2$ and $\theta = 0$ **is perfect alignment**.

A simple solution: the Burgers vortex

The most elementary Burgers vortex solution is (with $\gamma_0 = const$)

$$\boldsymbol{u} = (-\tfrac{1}{2}\gamma_0 x + \psi_y,\, -\tfrac{1}{2}\gamma_0 y - \psi_x,\, z\gamma_0) \qquad \Rightarrow \qquad \boldsymbol{\omega} = (0,\, 0,\, \omega_3)$$

$$\omega_3(r,t) = e^{\gamma_0 t}\omega_0\left(r\, e^{\frac{1}{2}\gamma_0 t}\right) \qquad \text{(note exponential growth)}$$

Thus, for the Burgers vortex one computes

$$\alpha = \gamma_0\,, \qquad \boldsymbol{\chi} = 0\,, \qquad \alpha_p = -\,\gamma_0^2$$

$$\boldsymbol{\zeta} = [\gamma_0,\, 0] \qquad\qquad \boldsymbol{\zeta}_p = -[\gamma_0^2,\, 0]$$

Conclusions: Burgers tubes/sheets are scalar objects: they don't swing.

(In fact, they are steady solutions of the ζ-equation.)

When tubes & sheets bend then $\boldsymbol{\chi} \neq 0$ and $\boldsymbol{\zeta}$ becomes a full quat driven by $\boldsymbol{\zeta}_p$ which is coupled back through the pressure Hessian P.

When do $[\alpha, \chi]$ quat equations arise in fluids?

- The Euler equation $\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{\mathcal{F}}$ with vorticity $\boldsymbol{\varpi} := \text{curl}\,\boldsymbol{u}$ implies

$$\frac{D\boldsymbol{\varpi}}{Dt} = \boldsymbol{\varpi}\cdot\nabla\boldsymbol{u} + \boldsymbol{\mathcal{W}} \quad \text{with} \quad \boldsymbol{\mathcal{W}} := \frac{1}{\rho}\,\text{curl}\,\boldsymbol{\mathcal{F}}$$

- These produce an Ertel Theorem and Ohkitani relation

$$\Big[\frac{D}{Dt}\,,\,\boldsymbol{\varpi}\cdot\nabla\Big] = \boldsymbol{\mathcal{W}}\,, \quad \text{so} \quad \frac{D^2\boldsymbol{\varpi}}{Dt^2} = \boldsymbol{\varpi}\cdot\nabla\boldsymbol{\mathcal{F}} + \Big(\frac{D\boldsymbol{\mathcal{W}}}{Dt} - \boldsymbol{\mathcal{W}}\cdot\nabla\boldsymbol{u}\Big)$$

- If $\boldsymbol{\mathcal{W}} = 0$, then $\boldsymbol{\varpi}\cdot\nabla$ is a Frozen-in Vector Field

- In turn the frozen-in property produces orthonormal Frame Dynamics for $\widehat{\boldsymbol{\varpi}}$, whose alignment parameters will satisfy Quaternion equations.

- Other examples:
  (1) Rotating fluid flow (both incompressible and barotropic)
  (2) Lagrangian Averaged Euler-alpha (LAE$-\alpha$) equations
  (3) A surprise lurks in barotropic fluid dynamics – oscillations!

Lagrangian Averaged Euler-alpha (LAE$-\alpha$) model

Lagrangian averaging preserves Kelvin's circulation theorem, which leads to a frozen-in vector field and thereby produces Ertel's theorem.

The LAE$-\alpha$ motion equation is

$$\frac{D\boldsymbol{v}}{Dt} + \nabla \boldsymbol{u}^T \cdot \boldsymbol{v} = -\nabla p \quad \text{for} \quad \boldsymbol{v} = \boldsymbol{u} - \alpha^2 \Delta \boldsymbol{u} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = 0$$

or, in Kelvin circulation form,

$$\frac{D}{Dt}(\boldsymbol{v} \cdot d\boldsymbol{x}) = -dp \quad \text{along} \quad \frac{D\boldsymbol{x}}{Dt} = \boldsymbol{u}$$

Stokes-ing (or taking d) and $\nabla \cdot \boldsymbol{u} = 0$ yield a Frozen-in Vector Field

$$\frac{D\boldsymbol{\varpi}}{Dt} = \boldsymbol{\varpi} \cdot \nabla \boldsymbol{u} \quad \text{for} \quad \boldsymbol{\varpi} = \nabla \times \boldsymbol{v}$$

Ertel Theorem & Ohkitani relation for LAE$-\alpha$

The LAE$-\alpha$ motion equation may also be written using $\boldsymbol{u} = G * \boldsymbol{v}$ as

$$\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{\mathcal{F}} = -\,G * (\nabla p + 4\alpha^2 \nabla \cdot \Omega S)$$

where $2\Omega = \nabla \boldsymbol{u} - \nabla \boldsymbol{u}^T$ and $G* = (1-\alpha^2\Delta)^{-1}$ denotes convolution with the Greens function for the Helmholtz operator.

The Ertel Theorem and Ohkitani relation for LAE$-\alpha$ are then

$$\left[\frac{D}{Dt}\,,\,\boldsymbol{\varpi}\cdot\nabla\right] = 0\,, \quad \text{and} \quad \frac{D}{Dt}(\boldsymbol{\varpi}\cdot\nabla\boldsymbol{u}) = \frac{D^2\boldsymbol{\varpi}}{Dt^2} = \boldsymbol{\varpi}\cdot\nabla\boldsymbol{\mathcal{F}}$$

where $\boldsymbol{\varpi} = \nabla\times\boldsymbol{v}$ and $\boldsymbol{v} = (1-\alpha^2\Delta)\boldsymbol{u}$

The rest (Dynamics of Vorticity Frames and Quaternionic Alignment Parameters) follows the pattern of Euler fluids.

Velocity gradients obey tensor Ricatti equations

Hessian P drives a tensor Ricatti equation for velocity gradients M

For velocity gradient ($\mathsf{M} = \nabla \boldsymbol{u}$) and pressure Hessian ($\mathsf{P} = \nabla\nabla p$) Euler implies

$$\frac{D\mathsf{M}}{Dt} + \mathsf{M}^2 = -\mathsf{P}\,, \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 = \operatorname{tr} \mathsf{M} \quad \Rightarrow \quad \operatorname{tr} \mathsf{P} = -\operatorname{tr}(\mathsf{M}^2)$$

We also found a Ricatti equation for the alignment-parameter dynamics

$$\frac{D\alpha}{Dt} + \alpha^2 - \chi^2 = -\alpha_p \quad \text{and} \quad \frac{D\boldsymbol{\chi}}{Dt} + 2\alpha\boldsymbol{\chi} = -\boldsymbol{\chi}_p$$

where $\quad S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi} \times \hat{\boldsymbol{\omega}} \quad$ and $\quad P\hat{\boldsymbol{\omega}} = \alpha_p\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}_p \times \hat{\boldsymbol{\omega}}$

The alignment equation for quats $\boldsymbol{\zeta} = [\alpha,\, \boldsymbol{\chi}]$ with $\boldsymbol{\zeta}_p = [\alpha_p,\, \boldsymbol{\chi}_p]$ is

$$\frac{D\boldsymbol{\zeta}}{Dt} + \boldsymbol{\zeta} \circledast \boldsymbol{\zeta} = -\boldsymbol{\zeta}_p$$

for which determining the Hessian P was a limiting factor.

Algebraic pressure closures may enhance quat equations

$$\frac{D\mathsf{M}}{Dt} + \mathsf{M}^2 = -\mathsf{P}\,, \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 = \operatorname{tr}\mathsf{M} \quad \Rightarrow \quad \operatorname{tr}\mathsf{P} = -\operatorname{tr}(\mathsf{M}^2)$$

Coarse-grained Lagrangian averaging requires closure for P.

The Restricted Euler & Tetrad model imposes algebraic pressure

$$\left[\!\left[\mathsf{P} = -\frac{\mathsf{G}}{\operatorname{tr}\mathsf{G}}\operatorname{tr}(\mathsf{M}^2) \right]\!\right], \qquad \text{then they model the evolution of } \mathsf{G} = \mathsf{G}^T\,.$$

The most general closure for the dynamics of G of this type is

$$\mathsf{P} = -\left[\sum_{\beta=1}^{N} c_\beta \frac{\mathsf{G}_\beta}{\operatorname{tr}\mathsf{G}_\beta}\right]\operatorname{tr}(\mathsf{M}^2)\,, \qquad \text{with} \qquad \sum_{\beta=1}^{N} c_\beta = 1\,, \quad \mathsf{G}_\beta = \mathsf{G}_\beta^T$$

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Restricted Euler closure as a Riemannian metric

Under the Lagrangian flow map $\phi_t : X \to x(t)$, P transforms as a metric

$$ds^2 = \mathsf{P}(t)\, dx(t) \otimes dx(t) = \phi_t \circ \Big(\mathsf{P}(0)\, dX \otimes dX \Big) \,.$$

A **Riemannian metric** in the Lagrangian reference configuration transforms as[1]

$$\mathsf{G}(t) = \mathsf{D}^{-1}(t)\mathsf{G}(0)\mathsf{D}^{-1}(t)\,,$$

where $\mathsf{D}(X,t) = \partial x/\partial X$ with $\det(\mathsf{D}) = 1$ & $[\,\mathsf{D}^{-1}(t)dx(t)\,]^{\cdot} = [\,dx(0)\,]^{\cdot} = 0$.
Set P proportional to the frozen-in metric $\mathsf{G}(t)$ as

$$\mathsf{P} = -\,\frac{\mathsf{G}(t)}{\mathrm{tr}\,\mathsf{G}(t)}\mathrm{tr}\,(\mathsf{M}^2)\,,$$

(1) $\mathsf{G}(t) = \mathsf{Id}$ recovers the restricted Euler equations of Vieillefosse, Cantwell, etc.
(2) $\mathsf{G}(0) = \mathsf{Id}$ recovers the mean flow part of the Chertkov et al. tetrad model.

[1] The metric $\mathsf{G}(t)$ is called the **Finger tensor** in nonlinear elasticity.

Alignment issues in rotating turbulence

- Dynamics of Euler velocity gradients (sym and antisym parts)
- Pressure Hessian
- Lagrangian orthonormal frames
- Ricatti equations
- Extra vortex stretching / forcing
- Dynamics of transition to Taylor-Proudman alignment?
- Approaches
  - Lagrangian coarse-graining / averaging
  - Algebraic pressure closures
- Next steps?

Thank you!

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Figure 1: The PDF of $\cos(\theta)$ for DNS of NSeqns. The solid, dotted and the dash-dotted lines refer to maximum, middle and minimum eigenvalue, respectively.

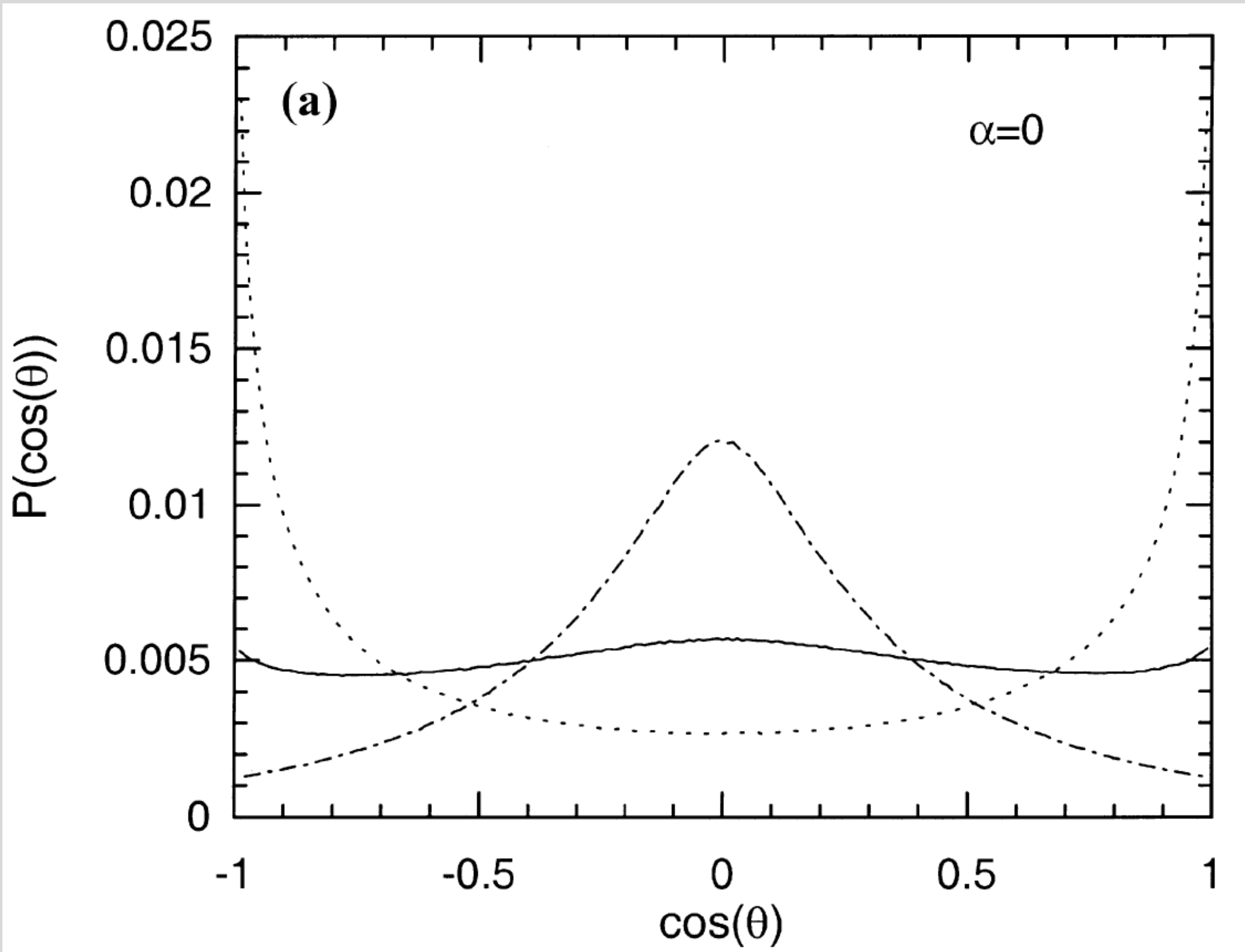


Figure 2: The PDF of $\cos(\theta)$ for DNS with $\alpha = 0$ (a) and $1/32$ (b). The solid, dotted and the dash-dotted lines refer to maximum, middle and minimum eigenvalue, respectively.

78 S. Chen et al. / Physica D 133 (1999) 66–83

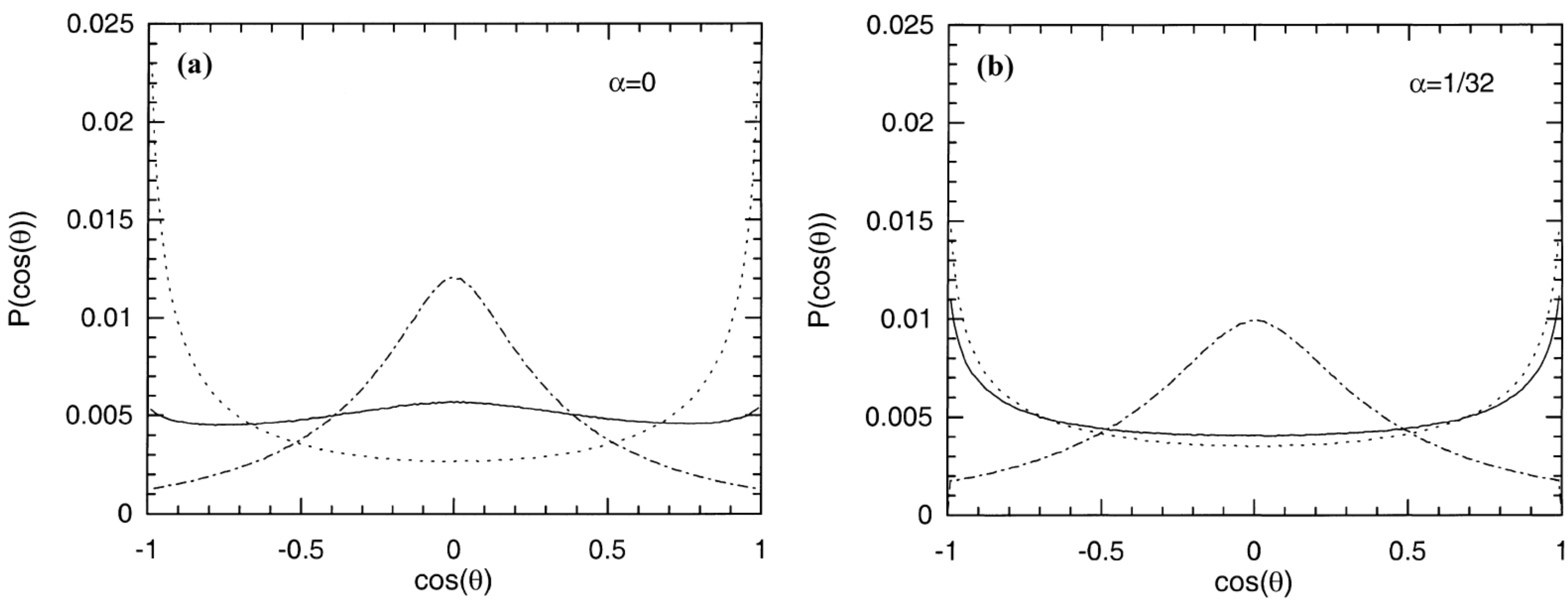


Figure 3: Snapshot of the vertical component of vorticity $\omega_z = \omega \cdot \hat{\boldsymbol{z}}$ in the developed regime at $t = 90$ for a nonrotating flow ($Ro = \infty$). Red ($\omega_z > 0$), Blue ($\omega_z > 0$). This is ***turbulent mixing***.

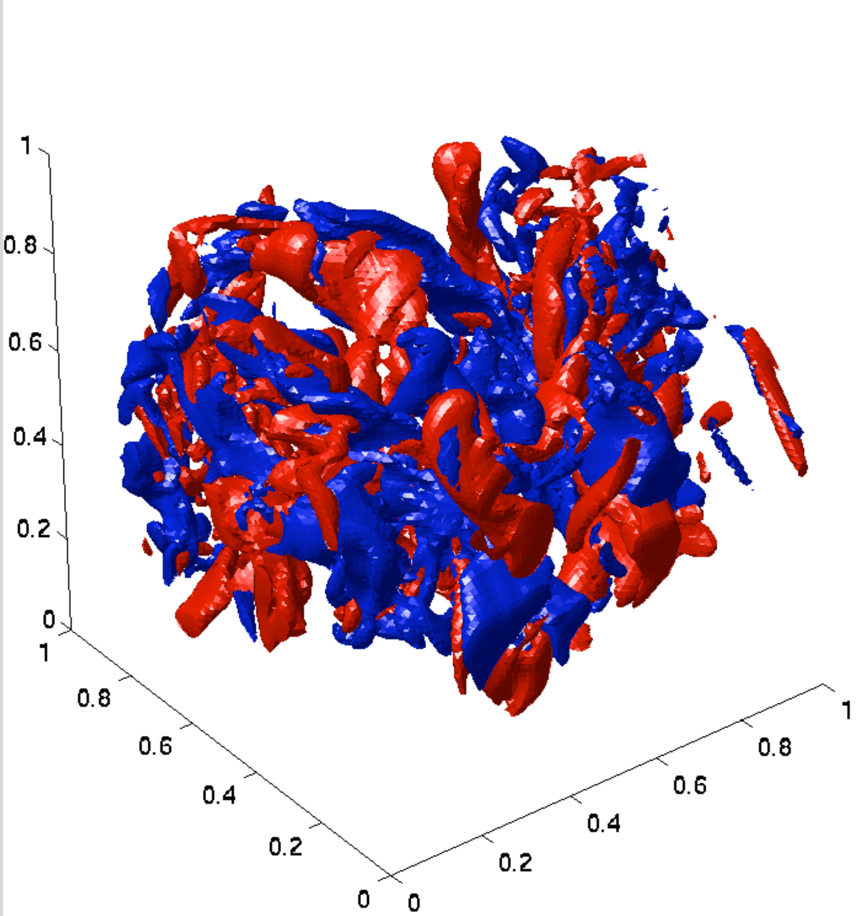
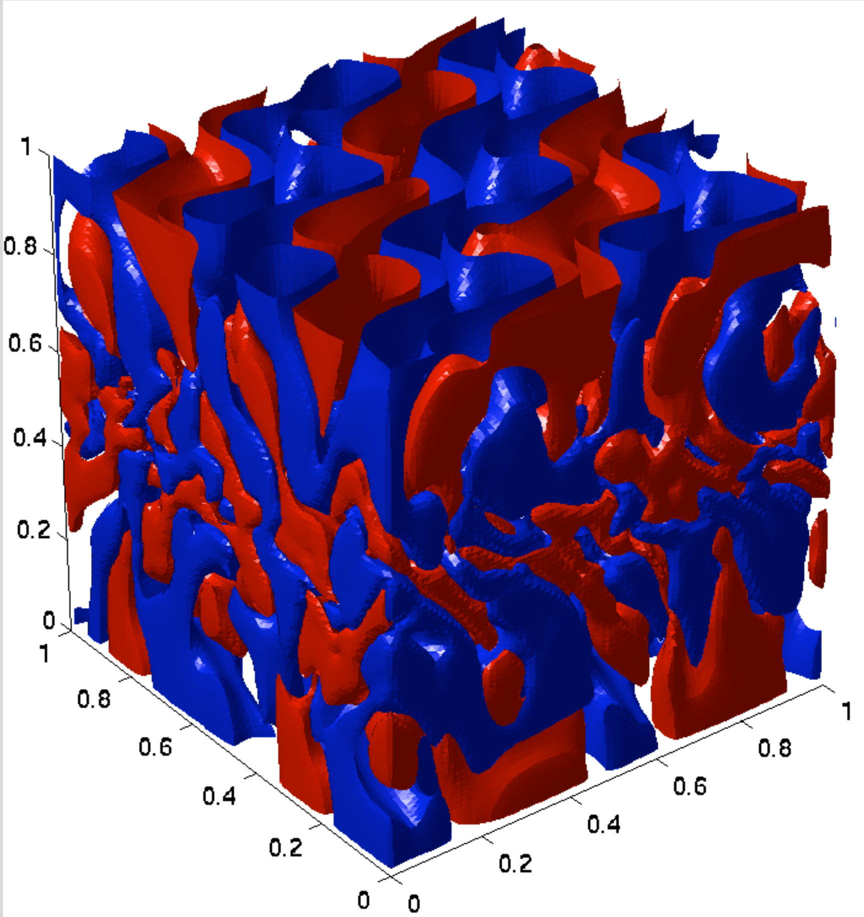


Figure 4: Snapshot of the vertical component of vorticity $\omega_z = \omega \cdot \hat{\boldsymbol{z}}$ in the developed regime at $t = 90$ for a rotating flow at $Ro = 1/10$. Red ($\omega_z > 0$), Blue ($\omega_z > 0$). ***The flow is nearly columnar***.



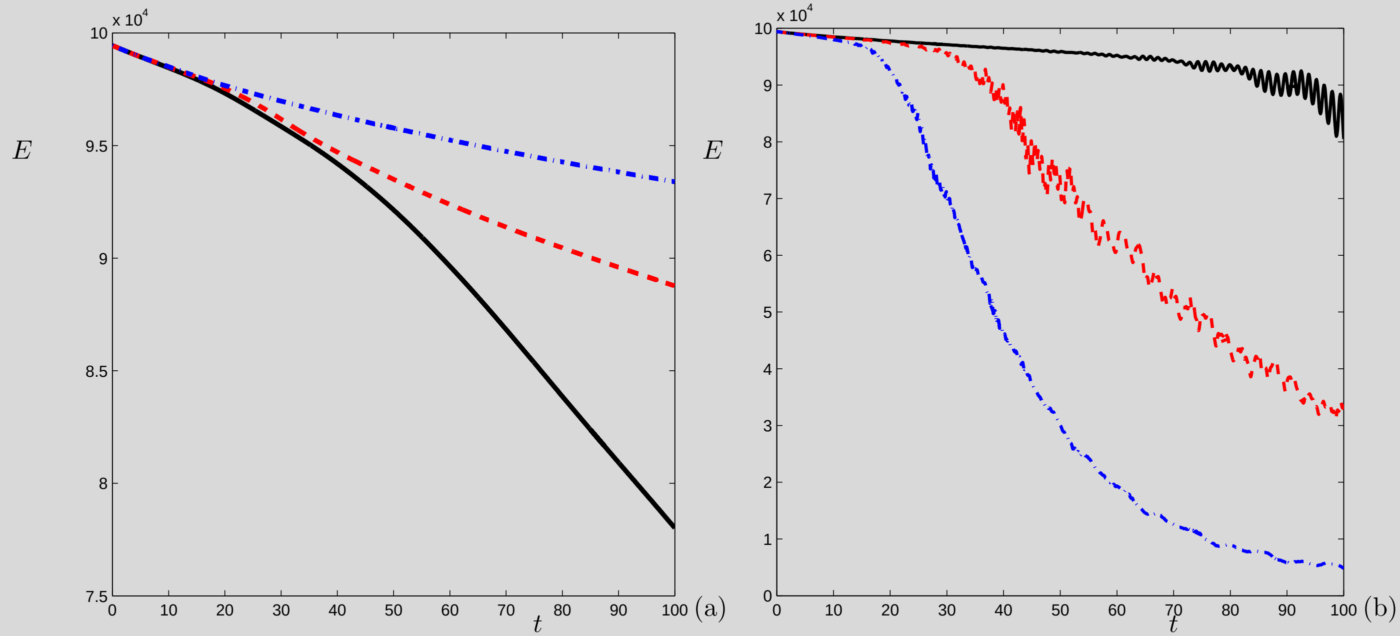


Figure 5: Decay of kinetic energy at low (a) and high (b) rotation rates. In (a) $Ro = \infty$ (solid), $Ro = 10$ (dashed), $Ro = 5$ (dash-dotted); in (b) $Ro = 1$ (solid), $Ro = 0.5$ (dashed) and $Ro = 0.2$ (dash-dotted).