Unipotent representations & symplectic duality.

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1) Concept of unipotent representation.
2) Unipotent ideals & canonical quantizations
3) Special unipotent ideals & symplectic duality.

One highlight of this story: a problem inspired by very classical Quantum Physics (classification of unitary irreps of s/simple Lie groups) turns out to be related to a very recent development inspired by Quantum Physics: symplectic duality a.k.a. 3D Mirror symmetry.

1) **Problem:** Given a Lie group G classify its unitary irreps (in Hilbert spaces)

Guess (Orbit method): these should be related to orbits of $G \curvearrowright \mathfrak{g}^*$

e.g., for G nilpotent, by Kirillov (61, 62): unitary G-irreps $\xleftrightarrow{\sim}$ G-orbits in $\mathfrak{g}^*$ (symplectic manifolds w. Hamiltonian G-action).

Physics motivation - geometric quantization

$$\begin{tikzcd} \{\text{unitary irreps}\} \arrow[r, bend left, "\text{q. classical limit}"] & \{G\text{-orbits in } \mathfrak{g}^*\} \arrow[l, bend left, "\text{quantization}"] \end{tikzcd}$$

This picture suggests that, in general, we also need to include equivariant covers of G-orbits in $\mathfrak{g}^*$: if G/H is a G-orbit, then by its equivariant cover we mean $G/\underline{H}$ w. $H^\circ \subset \underline{H} \subset H$.

Question: Can one extend Orbit method to semisimple Lie groups G?

In this talk we'll care about complex groups, e.g. $SL_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$, E_8...

One can completely describe the orbits of $G \curvearrowright \mathfrak{g}^*$ (some kind of JNF theorem) & their equivariant covers. The classification of unitary irreps is also known in some cases (e.g. complex classical groups) but is often very complicated and doesn't have any clear structure – in particular, a connection to orbits of $G \curvearrowright \mathfrak{g}^* \simeq \mathfrak{g}$ is unclear.

Question: can we relate <u>nilpotent</u> G-orbits (& their covers) to some unitary irreps – a partial Orbit method. This (previously undefined) class of irreps is called unipotent

Example (nilpotent orbits & their covers)

$G = Sp_{2n}$,

$$\{\text{nilpotent orbits in } \mathfrak{g}\} \xleftrightarrow[\sim]{\text{Jordan type}} \{\text{partitions of } 2n \text{, where each odd part occurs w. even multiplicity.}\}$$

$$\mathbb{O}_\lambda \longleftrightarrow \lambda$$

$$\pi_1(\mathbb{O}_\lambda) = (\mathbb{Z}/2\mathbb{Z})^{\#\text{ different even parts}}$$

e.g. $\lambda = (4,4,3,3,2,1,1) \rightsquigarrow \pi_1 \simeq (\mathbb{Z}/2\mathbb{Z})^2$

$\rightsquigarrow$ classification of covers, they correspond to subgroups in π_1.

2) Unipotent ideals & canonical quantizations.

2.1) Harish-Chandra bimodules.

Let G be a complex semisimple Lie group. Harish-Chandra bimodules are algebraic counterparts of reps of G.

The universal enveloping algebra $U(\mathfrak{g})$ is an associative algebra. It's basis is formed by ordered monomials in a basis of $\mathfrak{g}$, commutation relations come from $\mathfrak{g}$. A representation of $U(\mathfrak{g})$ is the same thing as a representation of $\mathfrak{g}$.

Definition: A Harish-Chandra (shortly, HC) $U(\mathfrak{g})$-bimodule is a $U(\mathfrak{g})$-bimodule (vector space w. commuting left & right actions of $U(\mathfrak{g})$) $\mathcal{B}$ s.t.

- it's finitely generated.
- every $b \in \mathcal{B}$ lies in finite dim'l subspace stable under adjoint $\mathfrak{g}$-action ($x.b = xb - bx$)

Example: $U(\mathfrak{g})$, any 2-sided ideal $I \subset U(\mathfrak{g})$, $U(\mathfrak{g})/I$ are HC bimodules.

A reason to care about HC bimodules:

Theorem (Harish-Chandra) There's a 1-1 correspondence between:

- Unitary irreps of G.
- Irreducible HC $\mathcal{U}(\mathfrak{g})$-bimodules that are <u>unitarizable</u>: have a positive definite scalar product w. suitable invariant properties (this is hard to check!)

Goal: From a G-equiv't cover $\tilde{\mathbb{O}}$ of nilpotent orbit $\mathbb{O} \subset \mathfrak{g}^*$ produce a 2-sided ideal $I_{\tilde{\mathbb{O}}} \subset \mathcal{U}(\mathfrak{g})$, maximal w.r.t. inclusion.

An irreducible HC bimodule $\mathcal{B}$ is called unipotent if it's annihilated by $I_{\tilde{\mathbb{O}}}$ (for some $\tilde{\mathbb{O}}$) on the left & on the right.

2.2) Canonical quantizations.

$\tilde{\mathbb{O}}$ is a symplectic algebraic variety $\rightsquigarrow$ Poisson bracket on the algebra $\mathbb{C}[\tilde{\mathbb{O}}]$ of polynomial functions on $\tilde{\mathbb{O}}$.

$\mathbb{O}$ is nilpotent $\Rightarrow$ $\mathbb{C}^\times$-stable for dilation action $\mathbb{C}^\times \curvearrowright \mathfrak{g}^* \rightsquigarrow$ lift $\mathbb{C}^\times \curvearrowright \tilde{\mathbb{O}} \rightsquigarrow$ grading on $\mathbb{C}[\tilde{\mathbb{O}}]$

Can talk about formal (deformation) quantizations of $\mathbb{C}[\tilde{\mathbb{O}}]$, $\mathbb{C}[[\hbar]]$-algebras. Such a quantization $\mathcal{A}_\hbar$ is called graded if $\mathbb{C}^\times \curvearrowright \mathbb{C}[\tilde{\mathbb{O}}]$ lifts to $\mathbb{C}^\times \curvearrowright \mathcal{A}_\hbar$ by $\mathbb{C}$-algebra automorphisms w. $t.\hbar = t\hbar$. In this case can specialize $\hbar=1$, getting a $\mathbb{C}$-algebra, $\mathcal{A}$.

Fact (I.L. 2016) Graded quantizations of $\mathbb{C}[\tilde{\mathbb{O}}]$ are classified by points of a finite dimensional vector space (depending on $\tilde{\mathbb{O}}$).

Definition: The canonical quantization, $\mathcal{A}_0$, of $\mathbb{C}[\tilde{\mathbb{O}}]$ is the one corresponding to parameter 0.

Example: i) $\mathfrak{g} = \mathfrak{sl}_2$, $\mathbb{O} = G.\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbb{C}[\mathbb{O}] = \dfrac{\mathbb{C}[e,h,f]}{(h^2+4fe)}$

Quantizations $\mathcal{A}$ depend on one parameter $z \in \mathbb{C}$

$\mathcal{A}(z) := \mathcal{U}(\mathfrak{sl}_2)/(C-z)$, where C is the Casimir $C = h^2 + 2h + 4fe$. The canonical quantization corresponds to $z = -1$ (so that the polynomial $h^2 + 2h - z$ has repeated roots)

ii) $\mathfrak{g} = \mathfrak{sl}_2$, $\tilde{\mathbb{O}} = \mathbb{C}^2 \setminus \{0\}$ is a 2-fold cover of $\mathbb{O}$. $\mathbb{C}[\tilde{\mathbb{O}}] = \mathbb{C}[x,y]$ and the only quantization is the 1st Weyl algebra $\mathcal{A} = \mathbb{C}\langle x,y \rangle / (yx - xy = 1)$. It's canonical.

2.3) Unipotent ideals.

Now let $\mathcal{A}$ be any quantization (w. $\hbar=1$) of $\mathbb{C}[\tilde{\mathbb{O}}]$. The action of $G \curvearrowright \mathbb{C}[\tilde{\mathbb{O}}]$ uniquely lifts to $\mathcal{A}$ & classical comoment map $\mathfrak{g} \to \mathbb{C}[\tilde{\mathbb{O}}]$ (pullback under $\tilde{\mathbb{O}} \to \mathfrak{g}^*$) lifts to quantum comoment map $\mathfrak{g} \to \mathcal{A}$ $\rightsquigarrow$ algebra homomorphism $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{A}$.

Definition: The unipotent ideal, $I_{\tilde{\mathbb{O}}}$, associated to $\tilde{\mathbb{O}}$, is $\ker[\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{A}_0]$.

Examples: i) $\mathbb{O} = G.\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow$

$\mathcal{U}(\mathfrak{g}) \twoheadrightarrow \mathcal{A}_0 = \mathcal{U}(\mathfrak{g})/(C+1) \rightsquigarrow I_{\mathbb{O}} = (C+1)$.

ii) $\tilde{\mathbb{O}} \rightsquigarrow \mathcal{A}_0 = \mathbb{C}\langle x,y\rangle/(yx-xy=1)$, $\mathfrak{g} \longrightarrow \mathcal{A}_0$: $e \mapsto \frac{1}{2}x^2$, $h \mapsto \frac{1}{2}(xy+yx)$, $f \mapsto -\frac{1}{2}y^2$. A direct computation shows $C \mapsto -\frac{3}{4}$. So

$I_{\tilde{\mathbb{O}}} = (C+\frac{3}{4})$.

In general we can:

- Classify unipotent ideals: say when two covers, $\tilde{\mathbb{O}}_1, \tilde{\mathbb{O}}_2$ give the same ideal. The answer is geometric.

- Compute "infinitesimal characters" & prove unipotent ideals are maximal.

- Classify bimodules annihilated by $I_{\tilde{\mathbb{O}}}$: they are in bijection with irreps of a certain finite group that is recovered from $\tilde{\mathbb{O}}$ geometrically.

- For $G = SL_n, SO_n, Sp_{2n}$: prove that unipotent bimodules are unitarizable.

3) Special unipotent ideals & symplectic duality.

3.1) Special unipotent ideals.

There's a classical construction of some unipotent ideals due to Barbasch-Vogan (85) of very different nature.

Let $\mathfrak{g}^\vee$ be Langlands dual Lie algebra (e.g. $\mathfrak{g}=\mathfrak{so}_{2n+1} \leftrightarrow \mathfrak{g}^\vee=\mathfrak{sp}_{2n}$), $\mathbb{O}^\vee \subset \mathfrak{g}^\vee$ nilpotent orbit. Can include $e^\vee \in \mathbb{O}^\vee$ into $\mathfrak{sl}_2$-triple $(e^\vee, h^\vee, f^\vee)$. Then can conjugate so that $h^\vee$ is a dominant element in Cartan $\mathfrak{h}^\vee \subset \mathfrak{g}^\vee \rightsquigarrow$ can view $h^\vee$ as a weight for $\mathfrak{g}$ b/c $\mathfrak{h}^\vee = \mathfrak{h}^*$. Let $\rho := \frac{1}{2}\sum_{\alpha>0}\alpha$, where the summation is over the positive roots.

Def'n: The ==special unipotent ideal== $I(\mathbb{O}^\vee)$ is the annihilator of the irreducible module w. highest wt. $\frac{1}{2}h^\vee - \rho$.

Example: • $\mathfrak{g} = \mathfrak{g}^\vee = \mathfrak{sl}_2$, $\mathbb{O}^\vee = \{0\} \Rightarrow e^\vee = 0,\ h^\vee = 0,\ \rho = 1$

so $I(\mathbb{O}^\vee) =$ annihilator of $\Delta(-1) = I_{\mathbb{O}}$, $\mathbb{O} = G.\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

• $I_{\widetilde{\mathbb{O}}}$ doesn't arise via the BV construction.

Thm: All special unipotent ideals are unipotent in our sense. Moreover, there's a map

$\tilde{d}$: {nilpotent orbits in $\mathfrak{g}^\vee$} $\longrightarrow$

{covers of nilpotent orbits in $\mathfrak{g}$}

s.t $I_{\tilde{d}(\mathbb{O}^\vee)} = I(\mathbb{O}^\vee)$.

Example: $\mathfrak{g} = \mathfrak{sp}_4 = \mathfrak{g}^\vee$

$\mathbb{O}^\vee$	$\tilde{d}(\mathbb{O}^\vee)$
$\{0\}$	(4)
$(2,1,1)$	$(2,2)$
$(2,2)$	2-fold cover of $(2,2)$
(4)	$\{0\}$

3.2) Symplectic duality.

Here's a more conceptual explanation of $\tilde{d}$. Let $S^\vee$ denote the transversal (Slodowy) slice to $\mathbb{O}^\vee$ in $\mathcal{N}^\vee$. This is a singular symplectic variety. It should have a "dual variety". We expect that this is Spec $\mathbb{C}[\tilde{d}(\mathbb{O}^\vee)]$:

- this generalizes existing expectations
- it satisfies expected properties (e.g. the weight $h^\vee/2$ can be seen from the so called deformed Hikita conjecture).

Missing:
- Why should the dual of $S^\vee$ cover an orbit in $\mathfrak{g}^*$?
- What are duals of orbits/covers that do not arise as $\tilde{d}(\mathbb{O}^\vee)$, e.g. $\mathbb{C}^2\setminus\{0\}$ for SL_2?