Introduction to Higher Form Symmetries and the Higher Group Structure

Master Thesis

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Abstract

Symmetry lies at the heart of theoretical physics, whereas higher form symmetry is the most recent and powerful generalization of it. Building from the definition of topological symmetry operators, insertions and linking number etc., the formalism of p-form symmetry is introduced. We specifically look at how higher form symmetries manifest themselves in general discussions of unitary gauge theory, including spontaneous symmetry breaking and anomaly cancellation. Higher group is an in particular interesting structure arising from the mixing of higher form symmetries of different degrees. We will introduce the nested structure via discussions on 2-groups and 3-groups. The underlying mathematical structure, the category theory, is also briefly covered as a natural introduction to the concept of higher charges. The non-invertible nature of higher form symmetries is revealed by clarifying its connection to symmetry categories.

Contents

Chapter 1

Introduction

Higher form global symmetry has been formulated, celebrated and explored intensively in recent years[4][12][2]. It can be very naturally applied to various quantum field theories, gauge theories in particular. Higher form symmetry has proven to be a novel but extremely powerful perspective to generalize the role symmetry arguments play in theoretical particle physics. Attempts have also been made to apply concepts of higher form symmetries to quantum gravity theories. As global symmetries, the 't Hooft anomalies of higher form symmetries provide new constraints on dynamics of a theory along renormalization group flow[10].

In chapter 2, starting from Noether's theorem as an example of 0-form symmetry[1], we will introduce what a general p-form symmetry is. The main generalization made is stated as identifying topological symmetry operators as the fundamental component of higher form symmetries. We will go through why such operators respect a group structure[4] and how to determine which objects are charged under the action of them via discussions of linking number[14]. Throughout the following chapters, coupling the original theory to background gauge fields will be used as a generally helpful method.

In chapter 3, we will explore how higher form symmetries manifest themselves in the context of unitary gauge field, in particular in Maxwell's theory, the existence of anomalies and the phenomenon of spontaneous higher form symmetry breaking. Anomaly inflow will be introduced as a way to probe anomalies by uplifting the dimension. Some preliminaries in gauge theories such as Dirac monopole and theta term are presented too[7].

In chapter 4, the arising higher group structure when higher form symmetries of different degrees mix is introduced and explored using a few specific examples[4]. It is noted specifically how anomaly is exactly cancelled due to the modified transformation of background gauge fields in a higher group.

In chapter 5, attempts are made to clarify why higher form symmetries are also known as categorical symmetries[12]. Higher group as previously introduced will be integrated into the formalism of category theory. The definition of higher charges and why they appear in higher form symmetries are provided as well in this chapter[12]. We will look at a particular simple example of non-invertible symmetries as suggested by the categorical structure at the end of this chapter.

In chapter 6, a brief discussion why higher form symmetry is a promising topic is presented. A mathematical appendix on group cohomology and how it is related with simplicial cohomology[9] is attached to the end.

Chapter 2

Topological Operators and p-Form Symmetries

In action formalism, each specific expression of action exhibits various symmetries of the physical systems described, ie. the identification of a set of field transformations under which the action remains invariant. Local symmetry encodes the most fundamental internal degrees of freedom of gauge bosons that mediate interactions between elementary particles. However, physical phenomena often arise from asymmetry, or more accurately symmetries being broken. Symmetries can be explicitly broken or spontaneously broken due to non-trivial transformations of vacua under symmetry group generators. Some classical symmetries get preserved after quantization via path integral and thus are described as anomaly-free. It can be seen later in Chapter 4 that gauge anomalies and global anomalies can be associated to non-invariance under infinitesimal loops and under parallel transport in the configuration space, correspondingly[10]. It is evident from the success of Standard Model that symmetry operations described by group theory lie at the heart of theoretical physics, inspired by which physicists have dedicated to generalizing the concept of symmetry to explore the rich mathematical structure underneath. Higher form symmetry, also known as generalized symmetry for apparent reasons, is a fairly new concept that has spacetime topology organically integrated in its formalism. Topological operators U_Σ are defined on a given submanifold Σ in spacetime M and are invariant under small continuous deformations of it. Higher form symmetry refers to any symmetry generated by possible topological operators allowed by the physics theory and spacetime topology. Sources are no longer restricted to electric current analogues that is a vector field, ie. smooth 1-form field on M, but welcome any physical differential forms with suitable dynamical restrictions. We will develop the story of symmetries by connecting ordinary symmetries in QFT with a general p-form symmetry in the following sections.

2.1 0-Form Symmetry

2.1.1 Noether's Theorem

Noether's Theorem is central to the ordinary notion of symmetries in field theories. It provides a systematic way of finding the respected symmetry and the corresponding current conservation equation. Noether pointed out that the existence of continuous symmetries is always accompanied by conserved charges at the classical level[1]. We will walk through a brief review of the original argument as follows.

We start by assuming that the Lagrangian is a function of fields and their first derivatives only, $\mathcal{L}(\varphi(x), \partial_\mu \varphi(x))$. Substituting the Euler-Lagrange equation into the scalar-like transformation of $\mathcal{L}$, we have

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi(x)} \delta \varphi(x) \right) \tag{2.1}$$

Taking the simplest case of invariant Lagrangian, the conserved current can be always identified to be

$$\partial_\mu j^\mu = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi(x)} \Delta \varphi(x) = 0 \tag{2.2}$$

Re-expressing this in the language of differential forms, it can be stated that

$$d * j_1 = 0 \tag{2.3}$$

ie. $j_1 = j_\mu dx^\mu$ is a co-closed 1-form. d is the exterior map taking rank-r forms to rank-$(r+1)$ forms. The Hodge star appears to take care of upstairs indices[9]. Integrating the time component of j_1 over the spatial coordinates and dropping surface terms as usual, the conserved charge Q can be written as

$$Q = \int_{\Sigma_{d-1}} *j_1 = \int_{\Sigma_{d-1}} j^\mu(x) \hat{n}_\mu d^{d-1} x \tag{2.4}$$

, where $\hat{n}_\mu$ is a conventional normal vector and Σ_{d-1} is the space extended by all space-like coordinates here, but can be generalized to be any closed spacetime submanifold with codimension= 1, using Euclidean signature. Given that current conservation is valid as an operator expression, the integrated Ward identity of local operator $O(\varphi)$ gives[5]

$$\int d^d x \varepsilon(x) \left\{ -\langle \partial_\mu j^\mu O(\varphi) \rangle + \langle \frac{\delta O(\varphi)}{\delta \varphi(x)} \delta \varphi(x) \rangle \right\} = 0 \tag{2.5}$$

Say G is the symmetry group and $O_R(\varphi)$ transforms under representation $R(g)$, the Ward identity is then translated to

$$\partial_\mu j^\mu O_R(x) = \delta^{(d)}(x-y) R(T^a) O_R(y) \tag{2.6}$$

, where the delta function arises in the contact term and spacetime dependence of $O(\varphi)$ is made explicit for this reason. When the local operator at interest is a current operator, we see that

$$\partial_\mu j_a^\mu(x) j_b^\nu(y) = f_{ab}^c j_c^\nu(x) \delta^{(d)}(x-y) \tag{2.7}$$

, where f_{ab}^c is the structure constant of group G.

2.1.2 Topological Operators and 0-Form Symmetry

An unitary operator parametrised by λ to implement the transformation of G in a quantum theory is

$$\hat{U}_g = e^{i\lambda\hat{Q}} \tag{2.8}$$

Such symmetry operators furnish a $G-$action on the Hilbert space and are thus naturally labelled by group elements $g \in G$. In other words, from(2.7), we can write down the action of $\hat{U}_{g_1}$ on $\hat{U}_{g_2}$[4],

$$\hat{U}_{g_1} \cdot \hat{U}_{g_2} = \hat{U}_{g_1 g_2} \tag{2.9}$$

Discrete symmetries also have well-defined unitary operators[2] although it is unclear what the corresponding charges represent. Although Hodge dual is involved in the definition, operators $\hat{U}_g$ are in fact metric independent and thus topological. To see this, we pick a neighbouring submanifold Σ'_{d-1} that can be obtained from the original Σ via small continuous deformations without crossing any non-trivial operators, ie. $\partial_\mu j^\mu = 0$ holds throughout the deformation.

$$\hat{U}_g(\Sigma) \cdot \hat{U}_{g^{-1}}(\Sigma') = exp(i\lambda \int_{\hat{\Sigma}} \partial_\mu j^\mu d^d x) \tag{2.10}$$

Using Stoke's Theorem and $\hat{\Sigma}$ which is defined to be the d-dimensional manifold bounded by the union of Σ and $\Sigma'' \cong \Sigma'$ except with $\hat{n}''_\mu = -\hat{n}'_\mu$, the RHS is clearly the identity operator. Based on (2.9), we have the invertibility statement of symmetry operators inherited from the group structure,

$$\hat{U}_g(\Sigma) \cdot \hat{U}_{g^{-1}}(\Sigma) = \mathbb{1} \tag{2.11}$$

Comparing with (2.10), it is evident that an equivalence relation can be established between symmetry operators defined with respect to homeomorphic submanifolds, as expected[4]. Conservation of currents implies topological invariance of symmetry operators. The charge operator acts on physical objects via quantum commutators. For example, taking two immediately neighbouring time slices with a non-trivial point operator in between, it is imaginable that after small continuous deformation of these time slices, the point operator will travel through one time slice and be enclosed to give its charge. Now to see explicitly how $\hat{U}_g$ acts on point operators $O_R(x)$ in a QFT in the regular adjoint way so that quantum probability is preserved, we should look at the case when the mentioned deformation crosses point x. The contact term in (2.6) will be invoked.

$$\begin{aligned}\hat{U}_g(\Sigma)\cdot O_R(x)\cdot\hat{U}_{g^{-1}}(\Sigma' or \Sigma) &= exp(i\lambda\int_{\hat{\Sigma}}\partial_\mu j^\mu d^d y)O_R(x)\\ &= \sum_{n=0}^{\infty}\frac{(i\lambda^a R(T^a))^n}{n!}(\int_{\hat{\Sigma}}\delta^{(n)}(x-y)d^d y)^n O_R(x)\\ &= R(g)\cdot O_R(x)\end{aligned} \tag{2.12}$$

The d-dimensional delta function integration is a well-defined unity as $\hat{\Sigma}$ is d-dimensional. We see that we can pass by charged point operators at the price of transforming it under the corresponding representation. In a $U(1)$ theory, a phase depending on the charge of the point particle(operator) will be picked up. Guass's law in electromagnetism is a prominent example[7].

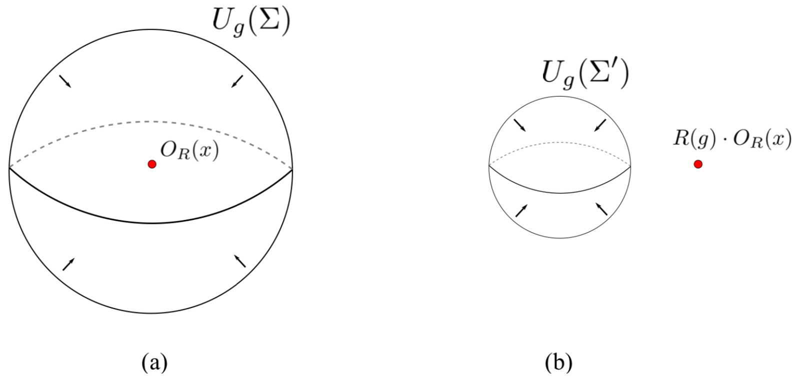

Figure 2.1: In d=3 spacetime, acting 2-dimensional symmetry operators on charged point operator $O_R(x)$ by enclosing it and deforming submanifold Σ to Σ' renders the point operator now transformed. A graphical presentation of (2.12).[4]

2.2 1-Form Global Symmetries

2.2.1 $U(1)^{(1)}$ Theory

Intrinsically defined point operators in a QFT are 0-dimensional and we have seen how that leads to a co-closed 1-form current when symmetries are present. The integration of these charged point operators on a 1-dimensional submanifold will give 1-dimensional line operators. In fact, 1-form symmetry is what directly acts on line operators in a theory. The simplest example will be $U(1)^{(1)}$ which is both continuous and Abelian. We will explore the feasibility of a non-Abelian higher form group further in the next section via discussions of linking.

A natural candidate for currents of such a symmetry will be a conserved 2-form field, ie.

$$d * J_2 = 0 \tag{2.13}$$

To construct a well-defined integral of current $*J_2$, a closed submanifold with codimension 2, Σ_{d-2} should be used. Symmetry operators $U_g(\Sigma_{d-2})$ can be defined similarly as in the 0-form symmetry case.

$$U_g(\Sigma_{d-2}) = exp(i\lambda \int_{\Sigma_{d-2}} *J_2) \tag{2.14}$$

Here the parameter λ has to respect the periodicity of $u(1)$ to be consistent. This is once again a topological operator that inherits the group structure[4]. Since $G^{(1)}$ is taken to be $U(1)$ here, the Ward identity as (2.7) reduces to

$$\partial_\mu J^{\mu\nu}(x) J^{\alpha\beta} = 0 \tag{2.15}$$

Hence,

$$\hat{U}_{g_1} \cdot \hat{U}_{g_2} = \hat{U}_{g_1+g_2} \tag{2.16}$$

Due to the Abelian nature, structure constants and higher order terms in BCH formula drop out[4]. Performing the same type of deformation to submanifold Σ_{d-2}, we see that

$$\hat{U}_g(\Sigma_{d-2}) \cdot \hat{U}_{g^{-1}}(\Sigma'_{d-2}) = exp(i\lambda \int_{\hat{\Sigma}} \partial_\mu j^\mu d^d x) = \mathbb{1} \tag{2.17}$$

, where the $(d-1)-$dimensional $\hat{\Sigma}$ is constructed in the same way as previously stated. Invoking Stoke's theorem, we prove its topological invariance. What is the action of such symmetry operators on non-trivial line operators? Ward identity for (Ward) Line operators $L_q(\gamma)$ is

$$d * J_2(x) L_q(\gamma) = q\delta^{(d-1)}(x \in \gamma) L_q(\gamma) \tag{2.18}$$

, where we note γ defines the line itself. Irregardless of the dimension of operator $L_q(\gamma)$, LHS is a differential form of rank $d-1$. This leads to a generalized Dirac delta being a $(d-1)$-form for consistency. Now the distribution should be located on a 1-dimensional submanifold rather than a spacetime point and only a line operator provides a natural label for such a submanifold. In case of a time-like line operator, it can be thought of the world line generated by an infinitely massive static point particle with charge q in the theory[2]. Line operators are often termed as defects or insertion, the reason being time-like line operators changes the Hamiltonian and thus the theory, however, space-like lines solely operate on the phase space. This difference is made implicit in an Euclidean signature, but can always be traced back using Wick rotation.

2.2.2 Coupling to Background Gauge Field

To probe the introduced higher form symmetries even more, we can couple the original theory to a background gauge field correspondingly. It being in the background signifies the fact that it has no dynamical degrees of freedom, while satisfying the classical equations of motion and controlling the coupled symmetry mechanically. Complications such as gauge fixing and ghost fields can be avoided as long as the gauge field is kept static[7]. We can build our case from 0-form symmetry. The background gauge field that couples to a 1-form dual current will also be a 1-form field, A_1. An extra term

$$i \int A_\mu(x) j^\mu(x) d^d x = i \int A_1 \wedge * j_1 \tag{2.19}$$

will contribute to the total action in the regular way. Given a 0-form $\lambda(x)$, the background gauge transformations are

$$\begin{aligned} \delta_\lambda A_1 &= d\lambda(x), \\ \delta_\lambda S &= i \int \partial_\mu \lambda(x) j^\mu(x) d^d(x) \\ &= -i \int \lambda(x) \partial_\mu j^\mu(x) d^d x \\ &= -i \int \lambda(x) d * j_1 = 0 \end{aligned} \tag{2.20}$$

, where we note the exterior derivative enjoys a graded Leibniz rule[9],

$$d(\mu_k \wedge \nu_j) = d\mu_k \wedge \nu_j + (-1)^k \mu_k \wedge d\nu_j \tag{2.21}$$

Conservation of current is directly translated into background gauge invariance, and we will see this is a generally correct statement. The same procedure applies to 1-form symmetry and potentially higher form symmetries. A 2-form current and thus a 2-form background gauge field is relevant for 1-form symmetry as discussed. The coupling term in the action will be simply

$$i \int B_2 \wedge * J_2 \tag{2.22}$$

The set of background gauge transformations are

$$\begin{aligned} \delta_\Lambda B_2 &= d\Lambda_1, \\ \delta_\Lambda S &= i \int d\Lambda_1 \wedge * J_2 = i \int \Lambda_1 \wedge d * J_2 = 0 \end{aligned} \tag{2.23}$$

Tuning the background gauge corresponds to actions of different symmetry operators. Gauge parameters are transformed by λ_0 on one side and kept invariant on the other side of Σ upon the action of symmetry operator $U_{g_0}(\Sigma)$, assuming the submanifold is orientable[4]. Since gauge field itself is an element in the Lie algebra, taking derivative of the step function, we see the gauge configuration should be a Dirac delta function parametrized by λ_0[7], which, consistent with the topological nature of symmetry operators, is also distinguished up to homotopies.

2.3 p-Form Global Symmetries

2.3.1 Linking Number

In d=3 spacetime, a spherical surface encloses a point but not a line. Mathematically, this ”enclosing” relationship is characterized by non-vanishing linking number[4][14]. This provides a systematic way to identify the dimension of operators that directly transform under a higher form symmetry. It is defined as the graded sum of intersection points. Given two submanifolds M and N of dimensions m and n of a manifold L of dimension l. M and N intersect with each other transversally if

$$T_pM \oplus T_pN \subseteq T_pL \quad \text{for} \quad \forall p \in M \cap N \tag{2.24}$$

T_pM is the tangent space of manifold M at the intersection point p. The $(d-1)$-form Dirac delta $\delta^{(d-1)}(x \in \gamma)$ in (2.18) returns 1 after integrating on a manifold that intersects with γ transversally only. Now if the manifold L is a closed and orientable manifold with $l = m + n + 1$ and we are allowed to write submanifolds as boundaries of higher dimensional manifolds. In particular, denote $O \subset L$ with dimension $n + 1$ with $\partial O = N$. The intersection points of manifolds O and M are collected in the set $\{p_i\}$. The condition of transversal intersection is saturated here since

$$\begin{aligned} m + n + 1 &= l, \\ T_pM \oplus T_pN &= T_pL \end{aligned} \tag{2.25}$$

The linking number of manifolds M and N is now defined to be

$$Link(M_m, N_{l-m-1}) = \sum_i sign(p_i) \tag{2.26}$$

, where $sign(p_i)$ is determined by whether or not the induced orientation on manifold L at point p_i differs from its original orientation[9]. Hence, manifolds link with each other if there exists intersection that is not cancelled out by opposite orientation in the space of one dimension higher. The important message here is that dimensions of linking manifolds are restricted. Linking is a topological invariant and symmetry operators thus only act directly on operators that are linked with the definition spacetime submanifold. Note linking number has very important applications in knot theory[14].

Now returning to the promised discussion of whether or not non-Abelian higher form symmetry exists, we first note that in order to have a non-Abelian group, the exchange of order of symmetry operators should have

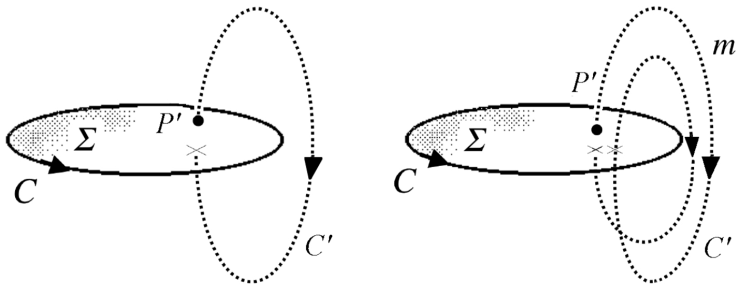

Figure 2.2: A graphic example of two circles C and C' linking with each other without intersecting. Manifold Σ fills circle C. $Link(C,C') = 1$ in (a) and $Link(C,C') = 2$ in (b). The number of intersection points is subject to different choices of Σ filling, however, the linking number remains invariant as mentioned.[11]

a non-trivial result, ie. at least one dimension can be utilized to have a well-defined order like time[4]. Given a $G^{(1)}$, we know that in d-dimension spacetime, a line operator links to a $(d-2)$-dimensional submanifold from previous analysis and the linking constraint. There is a total of 2 dimensions remaining available along which smooth deformations can be done to exchange the order of symmetry operators.

2.3.2 Generalizing to p-Form

For a p-form symmetry, p-dimensional charged operators link to $(d-q-1)$-dimensional symmetry operators, more available dimensions come along with higher form symmetries. So we see that it is obvious that all higher form symmetries(excluding 0-form symmetries) have to be of an Abelian nature[4]. This of course greatly simplifies our discussions. As expected, a p-form symmetry will generate a $(p+1)$-form current. The corresponding symmetry operators will be defined on spacetime submanifolds of codimension= $p+1$. 0-form and 1-form symmetries can very naturally be integrated into the formalism of p-form symmetry. The current conservation reads

$$d * J_{(p+1)} = 0 \tag{2.27}$$

The background gauge field transforms by

$$\delta B_(p+1) = d\Lambda_p \tag{2.28}$$

, under which the coupling term

$$S_{bf} = i \int B_{p+1} \wedge *J_{p+1} \tag{2.29}$$

remains invariant. Λ_p is the usual p-form gauge parameter. The topological symmetry operator that inherited the group structure can be again written explicitly as

$$U_g(\Sigma_{d-p-1}) = exp(i\lambda \int_\Sigma *J_{p+1}) \tag{2.30}$$

Here λ also generates an Abelian group. The Ward identity at this degree is

$$d * J_{p+1}(x) W_q(\Gamma_p) = q\delta^{(d-p)}(x \in \Gamma_p) W_q(\Gamma_p) \tag{2.31}$$

This follows from the Abelian statement

$$d * J_{p+1}(x) * J_{p+1}(y) = 0 \tag{2.32}$$

, which is a variation of (2.7) with vanishing RHS. We see in quantum theory p-dimensional $W_q(\Gamma_p)$ is the directly charged operator. In fact, the transformation done to $W_q(\Gamma_p)$ is dependent on $Link(\Sigma_{d-p-1}, \Gamma_p)$ as well[4].

2.3.3 Periodic Scalar Field

Periodicity often occurs in field theories when a particular group is gauged out and the remaining components of the theory have to respect the quotient group[7]. It turns out a simple model of a periodic scalar field exhibits higher form symmetries of mixed degrees. Make the identification of

$$\varphi \cong \varphi + 2\pi f \tag{2.33}$$

on top of the action

$$S = \frac{1}{2} \int d\varphi \wedge *d\varphi \tag{2.34}$$

to implement the periodicity. Current conservation relations are encoded here in equation of motion of $d\varphi$ and Bianchi identity, as follows,

$$\begin{aligned} d * d\varphi &= 0 \quad & d(d\varphi) &= 0 \\ j_3 &= *d\varphi \quad & J_1 &= d\varphi \end{aligned} \tag{2.35}$$

The EOM implies the respected 0-form shift symmetry, ie. φ describes a massless boson. The Bianchi identity clearly reveals a conserved 3-form current $J^{\mu\nu\alpha} = \varepsilon^{\mu\nu\alpha\beta}\partial_\beta\varphi$ after dualization and thus a 2-form symmetry[4]. Integrating this on a non-contractible loop, we will obtain a non-trivial charge operator as the generator, according to Poincaré's Lemma[9].

Activating background gauge fields as before, the coupling contributions are

$$i(\int A_1 \wedge j_3 + \int C_3 \wedge \frac{J_1}{2\pi}) \tag{2.36}$$

Naturally, A_1 and C_3 are background gauge fields implementing 0- and 2-form symmetries. Charged objects are

$$\begin{aligned} &Point : O(x) = \exp^{ia\varphi(x)} \\ &2 - Cycle : W(\Gamma_2) \end{aligned} \tag{2.37}$$

They link to the following symmetry operators in d=4 spacetime.

$$\begin{aligned} U(\Sigma_3) &= exp(i\lambda \int_{\Sigma_3} *d\varphi) \\ Z(\tilde{\Sigma}_1) &= exp(i\lambda \int_{\tilde{\Sigma}_1} \frac{d\varphi}{2\pi}) \end{aligned} \tag{2.38}$$

The interpretation of the 0-form symmetry charge is straightforward. It is the integer point charge of field φ which has a shift symmetry of 2π. The 2-form charge on the other hand can be thought of the integer winding degree of freedom of φ around Γ_2[4]. The extra factor of 2π in (2.36) is to justify the integer valued 2-form charges. Physically, a periodic scalar field can be used to describe massless Goldstone boson arising from spontaneously broken symmetry, or the dynamical coefficient of the topological theta term in unitary gauge theory promoted to a field. Mixing of higher form symmetries is not necessarily trivial and can lead to a structure called higher group[6], which will be addressed in later sections.

Previous discussions about higher form symmetries stem from Noether's Theorem, which excluded important discrete symmetries like parity and charge conjugation etc. In fact, discrete gauge groups can manifest as higher form symmetries as well[2], without resorting to current operators, but we will focus on continuous higher form symmetries here.

Chapter 3

Symmetries and Anomalies in Unitary Gauge Theory

In this section, we will explore how higher form symmetries manifest themselves in the context of $U(1)$ and non-Abelian $SU(N)$ gauge symmetries in various ways. Coupling to suitable background fields will once again prove to be powerful for discussions of spontaneous symmetry breaking of higher form symmetries and anomalies. We will build up from Maxwell's Theory, p-form electromagnetism to Axion Yang-Mills Theory.

3.1 Gauge Theory Background

Firstly, let us cover some background details about gauge theories. Electric charges are quantized and magnetic monopoles are allowed by an Abelian gauge theory like U(1)[7]. This is the gist of Dirac quantization condition. A topological term that corresponds to a total derivative in the classical action can be added to Maxwell's action. Turning theta term on or not leads to different physical phenomena.

3.1.1 Dirac Monopole

Originally gauge potentials were thought of as purely mathematical constructs for more elegant expressions of equations of motion. However, Aharovnov-Bohm effect stated that even without the presence of electromagnetic fields themselves, an electrically charged particle will pick up a phase determined by the gauge potential field $\mathbf{A}(x)$ in the background, after being slowly transported along a closed loop path C[7]. Such a phase difference is an observable in quantum theories. The wave function will become

$$\psi' = e^{ie\alpha}\psi \qquad \alpha = \oint_C \mathbf{A} \cdot dx \tag{3.1}$$

If the same electric charge is placed in the field configuration emanating from a magnetic monopole of magnetic charge g, the phase difference can be written as

$$\alpha = \int_S dS \cdot B \tag{3.2}$$

using Maxwell's equations and $\delta S = C$. See Figure 3.1. The monopole condition is

$$\int_{S^2} dS \cdot B = g \tag{3.3}$$

Phase α is then in fact $\alpha = \frac{\Omega g}{4\pi}$, where Ω is the solid angle extended by S. It

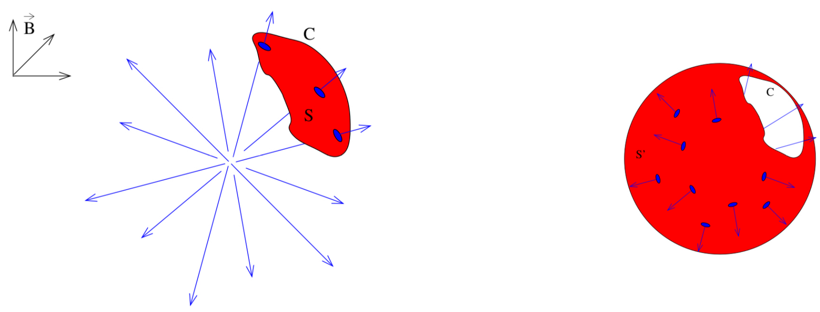

Figure 3.1: Diverging magnetic field lines piercing through area S enclosed by C. There are two different ways of choosing suitable areas in 4-dimensional spacetime, S and S'[13].

is easy to see that the solid angle extended by the second choice in the figure above will be $\Omega' = 4\pi - \Omega$[13]. S' here will also have a different orientation from the original one. The following equality called the Dirac quantization condition has to be satisfied

$$eg = 2\pi n \quad with \quad n \in \mathbb{Z} \tag{3.4}$$

so that after exponentiation the two phase differences are equivalent. It is obvious from this that quantized electric charges is consistent with the existence of magentic monopoles as mentioned. Particles that carry both electric and magnetic charges are called dyons and are characterized by the pair (e, g)[13].

Now what kind of gauge field configuration will lead to a magnetic monopole? Just like the Coulomb field of a static electric charge is not well defined at the origin, here we will be looking at the manifold $R^3 / \{0\}$[13]. In spherical polar coordinates, the following gauge field patches are related by a gauge transformation and will produce the monopole magentic field as expected.

$$\begin{aligned} A^N &= \frac{g}{4\pi r}\frac{1-cos\theta}{sin\theta} \\ A^S &= -\frac{g}{4\pi r}\frac{1+cos\theta}{sin\theta} \\ A^N - A^S &= \frac{1}{rsin\theta}\partial_\varphi(\frac{g\varphi}{2\pi}) \end{aligned} \tag{3.5}$$

A^N and A^S are not well-defined either at the north pole or the south pole, reminiscent of stereographic projection[9]. The difference between A^N and A^S is manifestly a total derivative. However, in order for the term inside brackets to be a valid gauge transformation on the wavefunction, it has to be effectively single-valued on this chart after exponentiation. This condition beautifully coincides with Dirac quantization condition.

3.1.2 Axion Field

The well-known Maxwell action is

$$S_{Maxwell} = \int d^4x(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}) \tag{3.6}$$

, to which another perfectly gauge invariant term quadratic term we can add is

$$S_\theta = \frac{\theta e^2}{4\pi^2}\int d^4x\frac{1}{4} * F^{\mu\nu}F_{\mu\nu} \tag{3.7}$$

$*F^{\mu\nu}$ is the dual field strength, corresponding to taking the Hodge dual in the language of differential forms. θ is a dimensionless parameter at this point. Writing out the epsilon symbol explicitly, the theta term can be translated to[13]

$$S_\theta = \frac{\theta e^2}{8\pi^2}\int d^4x\partial_\mu(\varepsilon^{\mu\nu\alpha\beta}A_\mu\partial_\alpha A_\beta) \tag{3.8}$$

This integral is independent from the metric just as that in the symmetry operators. We know the introduction of an innocuous total derivative does not affect the classical equation of motion. However, as advertised once $\theta(x)$ admits spacetime variance, the following deformed Maxwell equations for axion eletrodynamics[13]

$$\begin{aligned} \nabla\cdot\mathbf{E} &= -\frac{\alpha}{\pi}\nabla\theta\cdot\mathbf{B} \\ -\frac{\partial\mathbf{E}}{\partial t} + \nabla\times\mathbf{B} &= \frac{\alpha}{\pi}(\dot{\theta}\mathbf{B} + \nabla\theta\times\mathbf{E}) \end{aligned} \tag{3.9}$$

are obtained. The Bianchi identity remains the same as

$$dF_2 = 0 \tag{3.10}$$

Specific values of θ can lead to topological magneto-electric effect of topological insulator[13].

3.2 4d U(1) Maxwell

3.2.1 $U(1)^{(1)} \times U(1)^{(1)}$

In this subsection, rewriting the Maxwell action in the language of differential forms,

$$S = \frac{1}{2g^2} \int F \wedge *F \tag{3.11}$$

We can write down the equation of motion and Bianchi identity as for the action of periodic scalar field.

$$d * F = 0 \quad dF = 0 \tag{3.12}$$

The conservation of two 2-form currents are stated.

$$dJ_2^e = d * (\frac{1}{g^2} F) = 0 \quad dJ_2^m = d * (\frac{1}{2\pi} * F) = 0 \tag{3.13}$$

Hence, Maxwell's theory has a $U(1)_e^{(1)} \times U(1)_m^{(1)}$ structure, corresponding to the electric and magnetic 1-form symmetries. The symmetry operator for $U(1)_e^{(1)}$ can be written as

$$U_g^{(e)}(\Sigma_{d-2}) = exp(i\alpha \int_{\Sigma_{d-2}} *F) \quad g = e^{i\alpha} \in U(1) \tag{3.14}$$

Following the established formalism, the objects charged under the 1-form electric symmetry are the Wilson line operators[4]

$$W_q(L) = exp(i2\pi q \int_L A) \tag{3.15}$$

L is a line in the d=4 spacetime that this operator is defined on. Current acts on the Wilson line operator as[4]

$$(d * F)W_q(L) = q\delta^{d-1}(L)W_q(L) \tag{3.16}$$

$W_q(L)$ is labelled by the $U(1)$ representation it transforms under. The 1-form charge of a line operator coincides with its electric charge. This is in other words Gauss's law under disguise. Using the generalized definition of Dirac delta distribution that only returns 1 when integrated on a manifold that intersects transversally with L as previously introduced, inserting a Wilson's line into the path integral will look like[4]

$$\int [dA] exp(iq \int_{M_4} \delta^3(L) \wedge A + \frac{1}{2g^2} F \wedge *F) \tag{3.17}$$

The integrated equation of motion becomes

$$\int_{\Sigma_2} *F = qg^2 Link(\Sigma_2, L) \tag{3.18}$$

Recall the relation between linking number and transversal intersection. For $U(1)^{(1)}_m$, the symmetry operator reads

$$U^{(m)}_g(\Sigma_{d-2}) = exp(i\alpha \int_{\Sigma_{d-2}} \frac{F}{2\pi}) \tag{3.19}$$

Objects carrying 1-form magnetic charge are called 't Hooft line operators. To write its expression directly is slightly more complicated, but using the dual field strength $\tilde{A}$ with $*F = d\tilde{A}$, it can be written in the exact way as in (3.16),

$$T_m(\gamma) = exp(im \int_\gamma \tilde{A}) \tag{3.20}$$

We can also impose the field configuration of Dirac monopole solution as boundary conditions locally along the line γ and thus avoid $\tilde{A}$[4]. It is not difficult to see that in d=3 spacetime, Maxwell's theory exhibits a 0-form magnetic symmetry which demands the existence of a monopole operator. Hence, 't Hooft line operator is sometimes referred to as monopole operators.

3.2.2 Coupling to Background Gauge Fields and Anomaly

Turning on background gauge fields, the action becomes

$$S = \frac{1}{2g^2} \int (F - B^e_2) \wedge *(F - B^e_2) + \frac{i}{2\pi} \int B^m_2 \wedge F \tag{3.21}$$

The 2-form background gauge fields are labelled by the symmetries they are in charge of. Apart from the second term and standard coupling in the first term, a local counter term of the form[4]

$$S = \frac{1}{2g^2} B^e_2 \wedge B^e_2 \tag{3.22}$$

, which involves no dynamics, is added to the action to preserve the background gauge invariance. Performing the magnetic gauge transformation $B^{m\prime}_2 = B^m_2 + d\Lambda_m$, the dynamical gauge field and the electric background gauge field remain intact so the change of the action amounts to[4]

$$\delta S = i2\pi \int \frac{d\Lambda_m}{2\pi} \wedge \frac{F}{2\pi} \tag{3.23}$$

This is of no concern as we note

$$\frac{d\Lambda_m}{2\pi}, \quad \frac{F}{2\pi} \in \mathbf{H}^2(M_4, \mathbf{Z}) \tag{3.24}$$

Elements from the homology class are called cochain elements[9], and they return integers after integrated. The wedge product is in fact the discretized version, a cup product $\smile$[2]. This will be presented with more details in the mathematical appendices. It is sufficient to note that in this case

$$\delta S \in i2\pi\mathbf{Z} \tag{3.25}$$

The partition function is effectively unchanged and thus the magnetic background gauge invariance is protected. Moving on to electric gauge transformation

$$\begin{aligned} B_2^{e\prime} &= B_2^e + d\Lambda_e \\ A' &= A + \Lambda_e \\ F' &= F + d\Lambda_e \end{aligned} \tag{3.26}$$

Here evidently Λ_e is no longer flat. Current $*J_2^e$ acts on the Wilson line via 1-form electric symmetry and shifts the electric dynamical gauge field by Λ_e as its conjugate momentum in Lagrangian formalism[4]. The action transforms under (3.26) by

$$\delta S = \frac{i}{2\pi}\int B_2^m \wedge d\Lambda^e \tag{3.27}$$

1-form electric symmetry seems to be no longer respected. However, given a different local counter term[4],

$$S = -\frac{i}{2\pi}\int B_2^m \wedge B_2^e \tag{3.28}$$

, it is not difficult to see that the action

$$S = \frac{1}{2g^2}\int (F - B_2^e) \wedge *(F - B_2^e) + \frac{i}{2\pi}\int B_2^m \wedge (F - B_2^e) \tag{3.29}$$

remains invariant under the electric background gauge transformation. However, under the magnetic gauge transformation, the following non-trivial transformation occurs

$$\delta S = \frac{i}{2\pi}\int -d\Lambda^m \wedge B_2^e \tag{3.30}$$

This implies that this $U(1)^{(1)^e} \times U(1)^{(1)^m}$ theory has a mixed 't Hooft anomaly[4], a global anomaly that gets preserved through RG flow and thus very useful for UV behaviour determination.

We can always introduce a static background connection B_p for a theory equipped with a global symmetry G, ordinary or higher form. The anomalous phase $\mathcal{A}[\lambda_{d-1}, B_p]$ is defined to be[4]

$$Z[B_p + d\lambda_{p-1}] = exp(i\int \mathcal{A}[\lambda_{p-1}, B_p])Z[B_p] \tag{3.31}$$

ie. the non-trivial transformation of partition function under background gauge transformation. If $\mathcal{A}[0, B_p] = 0$, such an anomaly is a 't Hooft anomaly. Given M as the manifold of all B, the theory has a non-anomalous global symmetry G if the partition function is a well-defined 0-form field on the quotient manifold M/G. Invariance under infinitesimal loop of M/G, ie. vanishing curvature, implies absence of local anomalies, whereas invariance under parallel transport along regular loops corresponds to absence of global anomalies[10]. In the example above, we see that although independently suitable local counter terms can be chosen to cancel out the anomaly, it is impossible to keep both 1-form symmetries non-anomalous. A dimensional lift often proves powerful in anomaly discussion[10]. The original anomalous d-dimensional theory can be coupled to a theory in $(d+1)$-dimension with an opposite anomaly built in. If we can identify

$$\hat{\mathcal{A}}[B_p + d\Lambda_{p-1}] = d\mathcal{A}[B_p, \Lambda_{p-1}] \tag{3.32}$$

, then the new partition function

$$\hat{\mathbf{Z}}[B_p] = \mathbf{Z}[B_p]exp(-i\int_{M_{d+1}} \hat{\mathcal{A}}[B_p]) \tag{3.33}$$

, using again Stoke's theorem, will be non-anomalous as designed. The higher dimensional phase above is called a symmetry protected phase corresponding to a topological quantum field theory[4]. The anomaly inflow in our case can be written in d=5 spacetime as

$$S_{d=5} = \frac{i}{2\pi}\int_{M_5} B_e \wedge dB_m \tag{3.34}$$

In the case where $\delta M_5 = \emptyset$, (3.34) is clearly invariant under $B_{2e,m}' = B_{2e,m} + d\Lambda_{e,m}$. Interestingly, when $\delta M_5 = M_4$ with M_4 being our physical spacetime manifold, we see that this cancels out the calculated anomaly

$$\delta S_{d=5} = \frac{i}{2\pi}\int_{M_5} d\Lambda_e \wedge dB_m = \frac{i}{2\pi}\int_{M_4} \Lambda_e \wedge dB_m \tag{3.35}$$

, noting that the exterior derivative is nilpotent[9].

3.2.3 Spontaneous Symmetry Breaking

Spontaneous symmetry breaking in general occurs when a charged operator shows non-vanshing vev, ie.

$$< \hat{O}(x) > \neq 0 \tag{3.36}$$

When a continuous ordinary symmetry G spontaneously breaks down to H, a 0-rank Goldstone boson field arises to describe the variation of potential along massless directions. Coupling this broken symmetry with a background gauge field, the Goldstone field gets shifted by the background gauge parameter. When a p-form symmetry gets broken, each broken generator corresponds to a massless "Goldstone" boson field of rank $p-1$[4]. Following the same logic, whichever operator charged under the higher form symmetry at interest $W_q(\Gamma_p)$ should have a non-zero vev when spontaneous breaking happens.

$$< W(\Gamma_p) >\sim e^{-F(\Gamma_p)} \tag{3.37}$$

, where $F(\Gamma_p)$ is the scale function of the spontaneous symmetry breaking. p is the dimension of the spacetime submanifold this operator is defined on. A simple divergence test can be used to determine whether or not a symmetry spontaneously breaks. Only when the following limit exists do spontaneous symmetry breaking takes place[4].

$$lim_{Vol(\Gamma_p)\to\infty} Re[\frac{F(\Gamma_p)}{Vol(\Gamma_p)}] = finite \quad const. \tag{3.38}$$

For example, the volume of a Wilson loop will be its perimeter rather than area. This is closely related with the area law in confinement. Given the existence of such a limit, a local counter term can always be identified to define a physical renormalized operator that has a finite vev[4].

$$\hat{W}(\Gamma_p) = exp(-\int_{\Gamma_p} dV) \times W(\Gamma_p) \tag{3.39}$$

In light of (3.37), it can be easily seen that if the above limit does not exist, no such local counter term dependent on Γ_p itself only can be found and there will be no spontaneous symmetry breaking.

The masslessness of photon can be most perfectly explained through its effectively Goldstone nature with broken 1-form electric/magnetic symmetry[8]. Inspired by ordinary Goldstone theorem, the following quadratic term will appear in the broken phase effective Lagrangian[4]

$$\mathcal{L}_{eff} = -\frac{1}{2g^2}(dA_1 + B_2)^2 \tag{3.40}$$

Here, A_1 is the generalized Goldstone mode and B_2 is the 2-form background gauge field as introduced before in Maxwell's Theory. Turning off the background to zero, we recover the Maxwell term for photon gauge field A_1, which will naturally be massless.

3.3 Anomaly Inflow in p-Form Electromagnetism

In this subsection, we will apply the method of anomaly inflow to p-form electrodynamics. Firstly, the action can be written down as

$$S = \frac{1}{2g^2}\int F_{p+1} \wedge *F_{p+1} \quad F_{p+1} = dA_p \tag{3.41}$$

A_p is the p-form $U(1)$ gauge field. The $U(1)$ gauge transformation is

$$A'_p = A_p + d\Lambda_{p-1} \tag{3.42}$$

The equation of motion and Bianchi identity are

$$d * dA_p = 0 \quad dF_{p+1} = 0 \tag{3.43}$$

The two conserved currents for electric and magnetic symmetries are, as before,

$$* J_e = \frac{i}{g^2} * F_{p+1} \quad * J_m = \frac{F_{p+1}}{2\pi} \tag{3.44}$$

The electric one also shifts the dynamical gauge field by[4]

$$A'_p = A_p + \alpha\Lambda_p \tag{3.45}$$

, where the gauge parameter respects the periodicity of $U(1)$. Naturally, the dynamical field strength is not invariant but transforms by $\Lambda^{(e)}_{p+1} = \alpha d\Lambda_p$, note the label did not include the magnetic one. The action becomes

$$S = \int \frac{1}{g^2}(F_{p+1} - B^{(e)}_{p+1}) \wedge *(F_{p+1} - B^{(e)}_{p+1}) + \frac{i}{2\pi} F_{p+1} \wedge B^{(m)}_{d-p-1} \tag{3.46}$$

after including background coupling. The electric background gauge field transforms by an element of $H^{p+1}(M_d, \mathbf{Z})$ and thus the shift in the action is

$$\delta_e S = i \int \Lambda^{(e)}_{p+1} \wedge \frac{B^{(m)}_{d-p-1}}{2\pi} \tag{3.47}$$

We can clearly see this expression of a mixed higher form anomaly is consistent with previous discussion when $p = 1$[4]. The d=5 anomaly inflow to cancel this our will be then

$$\mathcal{I} = \frac{i}{2\pi} \int B^{(e)}_{p+1} \wedge dB^{(m)}_{d-p-1} \tag{3.48}$$

The impossibility of quantizing both electric higher form symmetry and magnetic higher form symmetry holds generally.

Chapter 4

Higher Group and Anomaly Cancellation

The higher group structure develops when higher form symmetries of different degrees of a system intertwine with each other[4]. Symmetries operating on forms of different ranks are not completely independent from each other. After all they are connected by the geometry of the spacetime they share. When p-form gauge transformation shifts not only the corresponding $(p+1)$-form background gauge field, but also B_{q+1} that gauges q-form symmetry. The higher group is labelled by its component of the highest degree. An h-group contains an $(h-1)$-form symmetry to the highest. This most perfectly fits into the structure of h-category[12], from which higher form symmetry inherits its name as categorical symmetry. This, and the non-invertibility it implies, will be explored in more details in the next section.

4.1 Nested Structure and Hierarchy Constraint

When different symmetries are present, following the symmetry breaking pattern and thus determining energy scales that each symmetry manifests itself can be very useful for understanding how these symmetries connect with each other and the dynamics. There is nothing stopping a field theory from being equipped with all the following global symmetries coordinated by increasing degree[4],

$$G^{(p_1+1)} < G^{(p_2+1)} < ... < G^{(p_n+1)} \tag{4.1}$$

Each one of them can be coupled with a background gauge field as before. However, the mixing happens when the following canonical transformation

$$\delta_i A_{p_i+1} = d\Lambda_{p_i} \tag{4.2}$$

simply does not suffice to keep background gauge invariance. Extra transformations need to be done. They are in general of the form[4]

$$\delta_i A_{p_i+1} = d\Lambda_{p_i} + \sum_{j\leq i} \Lambda_{p_j} \wedge \alpha^{(i)}_{p_i+1-p_j}(\left\{A_{p_j+1}\right\}) + ... \tag{4.3}$$

Note that $\alpha^{(i)}_{p_i+1-p_j}$ can only depend on background gauge fields that couple to higher form symmetries of lower degree than p_i[4]. Structure that is defined on a higher rank will not affect the transformation of background gauge fields of a lower rank. It is possible that non-linear terms that does not depend on dynamical gauge fields might also appear on the RHS.

The spontaneous breaking pattern of the higher group has a nested structure[4], as, for example, in the broken phase of the top form symmetry , ie. failure to preserve the top form symmetry will affect every higher form symmetry down below and hence breaks the consistency of the entire $p_n + 1$-group. Now moving on to the case when p_i-form symmetry is in the broken phase, it is not difficult to arrive at the conclusion that all p_j-form symmetries with $j < i$ will break as a consequence based on (5.3). This can be summarized in the nested/ladder structure of a higher group[4],

$$\mathcal{G}^{(p_n+1)}_{p_n} \subset ... \subset \mathcal{G}^{(p_n+1)}_{p_2} \subset \mathcal{G}^{(p_n+1)} \tag{4.4}$$

The notation $\mathcal{G}^{(p_n+1)}_{p_i}$ is the $(p_n + 1)$-higher group with degrees lower than p_i truncated. In other words, along with the energy flow, symmetries consecutively break as

$$\mathcal{G}^{(p_n+1)} \longrightarrow \mathcal{G}^{(p_n+1)}_{p_i} \longrightarrow \mathcal{G}^{(p_n+1)}_{p_{j>i}} \longrightarrow \mathcal{G}^{(p_n+1)}_{p_{k>j}} \longrightarrow ... \tag{4.5}$$

The mixing of higher form symmetries is closely related with non-trivial group extension characterized by exact sequences. In the case of ordinary symmetries, the following exact sequence encodes the fact that group G is an extension of group H by the normal Abelian subgroup A[2].

$$1 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 1 \tag{4.6}$$

The arrows represent a series of group homomorphisms that map previous images to the consecutive kernels. The essence of such an extension is captured by the existence of a group homomorphism

$$\psi : H \longrightarrow Aut(A) \tag{4.7}$$

, here the automorphism implies that there exists a unique pair of elements in groups A and H that under the group product of G produces a given element $g \in G$. At the level of sets, elements in G can always be written as the Cartesian product of sets A and H, $G = A \times H$. All possible group extensions G are classified by a 2-group-cohomology element

$$\omega \in \mathcal{H}^2(H, A) \tag{4.8}$$

$\mathcal{H}^2(H, A)$ are the equivalence classes of co-cycles that are not co-boundaries in the set of cochain elements

$$\mathcal{C}^2(H, A) = \{functions : H^2 \longrightarrow A\} \tag{4.9}$$

ω is referred to as twisting since the group product can be expressed as

$$(h_1, a_1) \cdot (h_2, a_2) = (h_1 h_2, a_1 + a_2 + \omega(h_1, h_2)) \tag{4.10}$$

It quantifies how twisted the group extension is compared to a direct group product. We see from above that group H is not closed under the given group product and thus not simply a subgroup of the group extension G. Group H itself thus cannot be a symmetry of the theory if the normal subgroup A is not. This gives us sufficient information to write down the hierarchy constraint of energy scales of effective emergent symmetries as follows[4],

$$E_A \geq E_H \tag{4.11}$$

Note this relation only holds true in the sense that

$$E_A \ll E_H \tag{4.12}$$

is not permitted. Similarly, the hierarchy constraint for energy scales of different degrees of emergent symmetries in a higher group $\mathcal{G}^{(p_n+1)}$ is

$$... \geq E_{p_i+1} \geq E_{p_i} \geq E_{p_i-1} \geq ... \tag{4.13}$$

This is evidently consistent with the nested structure of the higher group as discussed. Important information about UV completion or IR structure can be inferred based on the nested structure of higher groups and such hierarchy constraints.

4.2 2-Groups

Two ordinary symmetries can mix to form a 1-group. For example, when 2 discrete 0-form symmetries $\mathcal{Z}_2^0$ mix, two types of group extensions are allowed

$$\begin{aligned} &Klein \quad group: \quad \mathcal{Z}_2 \times \mathcal{Z}_2 \\ &Cyclic \quad group: \quad \mathcal{Z}_4 \end{aligned} \tag{4.14}$$

This is consistent with the statement that these are the only two inequivalent group multiplication tables for Abelian groups with 4 elements, from finite group theory. In fact, this is based on the group cohomology class being a discrete group $\mathcal{Z}_2$, ie.

$$\mathcal{H}^2(\mathcal{Z}_2, \mathcal{Z}_2) = \mathcal{Z}_2 \tag{4.15}$$

A 2-group structure arises when 0-form global symmetries mix with 1-form global symmetries or when different 1-form global symmetries mix with each other[6]. Here is an example of a continuous 2-group made of a $U(1)$ ordinary symmetry and a $U(1)$ 1-form symmetry in d=4 spacetime. Denote background fields that gauge the 0-form and 1-form global symmetries as A_1 and B_2, the curvature of these connections are F_2 and H_3 correspondingly. From the general expression of background gauge field transformations in higher groups, we can write down[4]

$$\delta A_1 = d\lambda_0 \quad \delta B_2 = d\Lambda_1 - \frac{\kappa}{2\pi}\lambda_0 F_2 \tag{4.16}$$

This particular set of transformation typically arises from the mixing of an anomalous ordinary symmetry and a 1-form global symmetry[6]. The $(d+2)$-form anomaly polynomial coupling the 1-form symmetry current with the anomalous 0-form symmetry field strengths as usual is

$$\mathcal{I}_6 = i\kappa \int \frac{F_2 \wedge F_2 \wedge *j_2}{4\pi} \tag{4.17}$$

The anomalous phase of the partition function is then[4]

$$\delta_{\lambda_0}\mathbf{Z} = exp(\frac{i\kappa}{2\pi}\int \lambda_0 F \wedge *J_2)\mathbf{Z} \tag{4.18}$$

It is obvious that the extra term in the transformation of B_2

$$\delta_{extra} B_2 = -\frac{\kappa}{2\pi}\lambda_0 F_2 \tag{4.19}$$

exactly cancels the anomaly of the 0-form symmetry via the coupling contribution to the action

$$S_{coupling} = i\int B_2 \wedge *J_2 \tag{4.20}$$

Hence, the 2-group symmetry itself is not anomalous.

4.2.1 2-group Structure in QED

Taking the $U(1)$ connection in QED to be a_g with dynamical field strength f_g, when 4 Weyl fermions, specifically 2 set of chiral fermions of distinctive flavours $\psi_i^{\pm}$ with $i = 1, 2$, are coupled to the photon gauge field, there exists one anomalous 0-form global symmetry, the flavour symmetry $U(1)_f$ and of course a 1-form magnetic dual symmetry as pointed out previously[4]. In addition to gauge charges, fermions $\psi_i^{\pm}$ carry distinctive flavour charges

± 1 and $\pm q$ correspondingly. The expression of a general 6-form anomaly polynomial[10][4]

$$\mathcal{I}_6 = \frac{1}{(2\pi)^2}\int [\frac{\kappa_{g^3}}{3!} f_g \wedge f_g \wedge f_g + \frac{\kappa_{g^2 f}}{2!} f_g \wedge f_g \wedge f_2 + \frac{\kappa_{gf^2}}{2!} f_g \wedge F_2 \wedge F_2 + \frac{\kappa_{f^3}}{3!} F_2 \wedge F_2 \wedge F_2] \tag{4.21}$$

, as these are the only 4 inequivalent combinations of currents. The factorials in the prefactors are to take care of repetitive permutations of each potential anomaly. By summing over the charge contribution of each chiral fermion as usual and noting that[10]

$$j_{gauge} = j_{right} - j_{left} \quad j_{global} = j_{right} + j_{left} \tag{4.22}$$

, the only existing anomaly has a coefficient of

$$\kappa_{gf^2} = (+1)\cdot(+1)^2 + (+1)\cdot(-1)^2 + (-1)\cdot(+q)^2 + (-1)\cdot(-q)^2 = 2(1-q^2) \tag{4.23}$$

All other coefficients in (4.21) can be calculated in the same way and shown to vanish. This is again a ′t Hooft anomaly. Substituting the expression of the coefficient κ_{gf^2} and the dual magnetic global 2-form current

$$* J_2 = \frac{1}{2\pi} f_g \tag{4.24}$$

into (4.21), we can write the anomalous phase as[4]

$$\delta_{\lambda_0} Z = exp(\frac{2(1-q^2)i}{2\pi}\int \lambda_0 F_2 \wedge *J_2) Z \tag{4.25}$$

Given that the 2-form magnetic background gauge field B_2 transforms as

$$\delta B_2 = d\Lambda_1 - \frac{2(1-q^2)}{2\pi}\lambda_0 F_2 \tag{4.26}$$

, the anomaly can be removed due to the coupling contribution in the action as before,

$$S_{coupling} = i\int B_2 \wedge *J_2 \tag{4.27}$$

Hence, again as expected, the 2-group global symmetry is preserved at the quantum level.

4.3 3-Groups

Naturally, a 3-group is in general made of the mix of 0-form, 1-form and 2-form symmetries according to the definition of higher groups[4]. The simplest case will be to consider $U(1)^{(0)}$, $U(1)^{(1)}$ and $U(1)^{(2)}$ with corresponding background gauge fields A_1, B_2 and C_3. This is well justified since we have shown that all higher form symmetries are Abelian. The general background gauge transformation permitted by a 3-group structure is[4]

$$\begin{aligned}
&\delta A_1 = d\lambda_0 \\
&\delta B_2 = d\Lambda_1 + k\lambda_0 F_1 \\
&\delta C_3 = d\Lambda_2 + \alpha d\lambda_0 \wedge F_1 + \beta_1 \Lambda_1 \wedge B_2 + \beta_2 \Lambda_1 \wedge d\Lambda_1 + \gamma_1 \Lambda_1 \wedge F_2 + ...
\end{aligned} \tag{4.28}$$

, where suitable combinations of all terms dependent purely on background gauge fields of lower ranks than C_3 should appear in the RHS of the transformation of C_3.

4.3.1 4d Axion SU(N)

Yang-Mills field theories can also be coupled to an axion field as in the $U(1)$ case. A physical example that exhibits such a 3-group structure is Axion-Yang-Mills in d=4 spacetime. The action is

$$S = \frac{1}{2}\int da \wedge *da + \frac{1}{g^2}\int Tr[F \wedge *F] - \frac{i}{8\pi^2 f_a}\int aTr[F \wedge F] \tag{4.29}$$

, where the dynamical axion field is a periodic scalar field as introduced previously. We can make the identification

$$a \sim a + 2\pi f_a \tag{4.30}$$

, where f_a is of 0-rank. There is a 2-form global symmetry $U(1)^{(2)}$ arising from the axion dependent terms in the action, and it corresponds to the winding degree of freedom of the axion field. Writing out the background gauge field coupling explicitly,[4]

$$\begin{aligned}
S = &\frac{1}{2}\int da \wedge *da + \frac{i}{2\pi f_a}\int a \quad dC_3 + \frac{1}{g^2}\int Tr[(F - B_2) \wedge *(F - B_2)] \\
&- \frac{i}{8\pi^2 f_a}\int aTr[(F - B_2) \wedge (F - B_2)] + \int \psi Tr[F - B_2]
\end{aligned} \tag{4.31}$$

Note the second term here is obtained by integration by parts and thus has the opposite sign. The coefficient ψ in the last term will be promoted to be a dynamical complex scalar field in potential discussions of spontaneous symmetry breaking, which will be omitted here. It can be shown that the instanton number related with the topological term is no longer an integer[4], which spoils the periodicity of axion field a as a consequence. This nontrivial transformation of the action can be cancelled given the following 3-group transformations of background gauge fields[4]

$$\begin{aligned}
&B_2' = B_2 + d\Lambda_1 \\
&C_3' = C_3 + d\Lambda_2 - \frac{N(N-1)}{2\pi}\Lambda_1 \wedge B_2 - \frac{N(N-1)}{4\pi}\Lambda_1 \wedge d\Lambda_1
\end{aligned} \tag{4.32}$$

The instanton number is denoted as N here. Hence, this is in fact a 3-group symmetry with a trivial 0-form symmetry, or say without the participation of any ordinary symmetry. The constraint on emergent symmetry energy scale is

$$E_2 \geq E_1 \tag{4.33}$$

, where the subscript denotes the degree of the emergent symmetries. Tracking along the RG group flow, this hierarchy constraint can be used for determination of the UV completion theory of axion Yang-Mills.

Chapter 5

Categorical Structure and Higher Charges

As mentioned in the last section, higher group can be most naturally integrated into the structure of an n-category, apart from being viewed as a non-trivial group extension[12]. We will present a brief introduction of a mathematical category and how that is related with higher charges and non-invertible symmetries in general.

5.1 n-Category

Symmetry operators have been proved to obey group multiplication law and thus are invertible by exploiting their topological nature. However, most generally, higher form symmetries are known as categorical symmetries and are not necessarily manifestly invertible. As the name suggests, as how 0-form symmetries fit into group theory and mixing of degrees higher group[2], categories are what underlies the complete story of higher form symmetries. A category is a set of objects that is equipped with maps between these objects, ie. morphisms. When it is a map between objects that have the same structure then we can denote the collection of all possible homomorphisms between object a and b[12][3] for example as

$$hom(a,b) = \{\text{All homomorphisms between elements a and b}\} \tag{5.1}$$

These morphisms respect a composition rule $\cdot$ defined as

$$\cdot : hom(a,b) \times hom(b,c) \longrightarrow hom(a,c) \tag{5.2}$$

This product is associative as group products and also has the identity elements, the identity morphism id_x[3]

$$id_x : x \mapsto x \quad id_b \cdot f = f \cdot id_a = f \quad \forall f \in hom(a,b) \tag{5.3}$$

Are objects equipped with any group product like binary structure? A functor, a structure preserving map between different categories, will potentially induce a fusion product between objects in a category. It is a map F between categories C and D satisfying the following properties[3]

$$\begin{aligned} &F : C \longrightarrow D \\ &\forall x \in object(C), \quad F(x) \in object(D) \\ &\forall f \in hom_C(x,y), \quad F(f) \in hom_D(F(x),F(y)) \end{aligned} \tag{5.4}$$

A covariant functor also respects the the morphism composition and existence of identity map that comes builtin with the category. In other words,

$$\begin{aligned} &\forall x \in object(C): \quad F(id_x) = id_{F(x)} \\ &\forall f \in hom_C(x,y), \quad g \in hom_C(y,z): \\ &F(g \circ f) = F(g) \circ F(f) \end{aligned} \tag{5.5}$$

A tensor category is a category C as introduced with a "monoidal" or tensorial structure. It can be equipped with a bifunctor called fusion as follows[3]

$$\otimes : C \times C \longrightarrow C \tag{5.6}$$

Here $C \times C$ is the product category. It is not difficult to see that this induces a binary product on objects in the category called "fusion". A bifunctor also has to respect the intrinsic structure of a category such as associativity of morphism composition and the existence of identity maps like the functor. What the above describes is a 1-category. Generalizing to an n-category, we see that it has $n+1$ levels with a ladder structure

$$\begin{aligned} &\cdot\, 0 - morphism: \quad \text{objects} \\ &\cdot\, 1 - morphism: \text{between objects} \\ &\cdot\, 2 - morphism: \text{between 1-morphisms} \\ &... \\ &\cdot\, n - morphism: \text{between (n-1)-morphisms} \end{aligned} \tag{5.7}$$

Each level of morphism has composition rules that satisfy properties listed in (5.3), however, there may exist multiple equivalently justifiable composition rules at higher level. For a theory in d=m spacetime that admits topological symmetry operators, a (d-1)-category called the symmetry category $C_{\mathcal{T}}$ with symmetry operators as its objects can be constructed as the following[3]. The zeroth level is

$$\begin{aligned} &\cdot\, \text{objects}: \quad D_{m-1}(\Sigma_{m-1}) \\ &\text{bifunctor/fusion}: \quad D^{(1)}_{m-1} \otimes D^{(2)}_{m-1} = D^{(12)}_{m-1} \end{aligned} \tag{5.8}$$

The superscript labels different symmetry operators at the same degree. The group multiplication law of symmetry operators are in general a statement of fusion algebra in category theory. The first level is

$$\text{1-morphisms} : D^{(1)}_{d-1}, \quad D^{(2)}_{d-1} \longrightarrow D^{(1,2)}_{d-2} \tag{5.9}$$

Symmetry operators with codimension = 2 that are defined at the intersective spacetime submanifold of symmetry operators with codimension = 1 constitute the first level of the $(m-1)$- category. This corresponds to higher form symmetries of a higher degree. 1-morphisms can be composed in two different ways. Following the same convention, denote $D^{(1,2)}_{m-2}$, $D^{(2,3)}_{m-2}$ as the 1-morphisms arising from intersections of objects $D^{(1)}_{m-1}$ and $D^{(2)}_{m-1}$, with $D^{(2)}_{m-1}$ and $D^{(3)}_{m-1}$ correspondingly[12]. Using morphism composition rule, $D^{(2,3)}_{m-2} \cdot D^{(1,2)}_{m-2}$ is the 1-morphism from object $D^{(1)}_{m-1}$ to object $D^{(3)}_{m-1}$. Another valid fusion rule for 1-morphisms makes use of fusion algebra of objects induced by the bifunctor. Denoting $D^{(13)}_{m-1}$ and $D^{(24)}_{m-1}$ as $(D^{(1)}_{m-1}, D^{(3)}_{m-1})$ and $(D^{(2)}_{m-1}, D^{(4)}_{m-1})$ fused together pairwise[12],

$$D^{(1,2)}_{m-2} \otimes D^{(3,4)}_{m-2} = D^{(13,24}_{m-2} \tag{5.10}$$

The second level will be 2-morphisms which are defined at the intersection of 1-morphisms, ie. symmetry operators with codimension = 3 that are defined at the intersective spacetime submanifold of symmetry operators with codimension = 2. Similarly for 2-morphisms, there exists one fusion rule

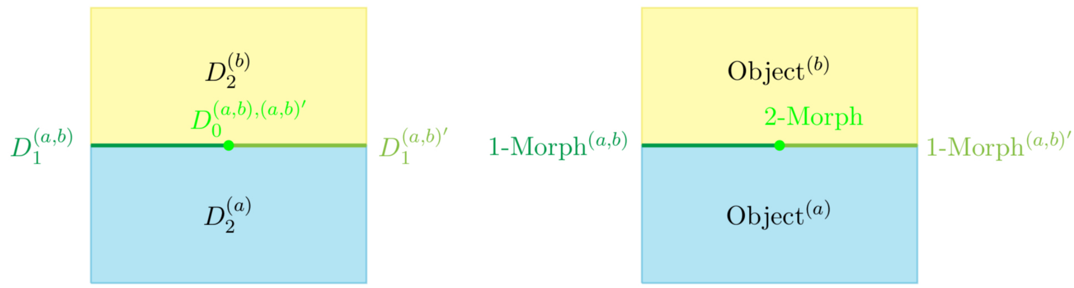

Figure 5.1: This figure shows the ladder structure of a category, going from objects, 1-morphisms to 2-morphisms. Symmetry operators are essentially morphisms of morphisms, whereas ordinary symmetry operators are objects in the theory. They are labelled by the dimension of spacetime submanifold they are defined on here[12].

based directly on morphism composition without resorting to the bifunctor structure. There are other two effective fusion rules that differ depending on how objects and morphisms are stacked together and fused. The inference that there exists n+1 fusion rules for n-morphisms does hold in general.

5.2 Higher Charges

5.2.1 p-Charge

When identifying the dimension of charged operator under a certain higher form symmetry, linking number is used determine which object is *directly* charged under a symmetry operator of a fixed codimension. However, it is possible that objects that are not *linked* to the symmetry operators are indirectly charged by inheriting non-trivial charges of a lower-form symmetry. This is essentially what higher charges are.

q-charges are the generalized/higher form charges carried by q-dimensional operators[3]. The word *direct* used implicitly previously refers to the fact that the degree of the charge and that of the higher form symmetry coincide, ie. operators linked to symmetry topological operators carry p-charges of a p-form symmetry. We will see that in fact all operators with dimensions higher than the degree of the higher form symmetry at interest are charged. For example, an ordinary, 0-form, symmetry can act on extended operators that are not point operators by definition[3]. A prominent example of 0-form symmetry in QED will be charge conjugation, despite being discrete. Maxwell's theory in d=4 spacetime evidently has a charge conjugation ordinary symmetry $\mathbb{Z}_2^{(0)}$.

$$A_1 \longrightarrow -A_1 \tag{5.11}$$

, and of course as discussed the product 1-form symmetry. Wilson lines carry 1-electric-charge while the dual 't Hooft lines carry 1-magnetic-charge directly. The point operator that carries 0-charge of $\mathbb{Z}_2^{(0)}$ is the gauge invariant field strength operator

$$F(x)_2 \longrightarrow -F(x)_2 \tag{5.12}$$

, ie. $F(x)_2$ has an 0-charge -1. Since we pointed out that the 1-charge carried by Wilson lines is equal to its electric gauge charge, it is obvious that Wilson lines have a non-trivial 1-charge under this 0-form symmetry of -1 as well. Analogously, the 1-magnetic-monopole-charge carried by't Hooft lines picks up the same 1-charge under actions of charge conjugation. More interestingly, the symmetry operators themselves also inherit 2-charges of the conjugation symmetry[3], the expressions of which make this apparent.

5.2.2 Higher Representation

From this simple example, we see that q-charges of a $\mathbf{G^{(0)}}$ are $(q+1)$-representations of group $G^{(0)}$. We will omit the full regular definition of an ordinary group representation, but introduce what a higher representation is using the language of categories. The finite dimensional representation ρ

of a group $G^{(0)}$ is a map[3]

$$\rho: \quad G^{(0)} \longrightarrow End(V) \tag{5.13}$$

, where $End(V)$ is the set of all endomorphisms of finite dimensional vector space V. The regular homomorphism condition on representations applies. Now regarding V as a linear category Vec[3], the objects and morphisms of which are vector spaces and linear maps between them, we note that a group can also be absorbed into the formalism of category theory as $\mathcal{C}_{G^{(\prime)}}$, which only contains one object, ie. the group itself. The group elements are distinguished within the group from each other by manifesting as morphisms of the category, since each group element can be considered to be an endomorphism of the group via its natural group action[3] . Group product in this case provides a way to compose different morphisms of this category. A regular representation as in (5.13) can be seen as a functor

$$\rho: \quad \mathcal{C}_{G^{(\prime)}} \longrightarrow Vec \tag{5.14}$$

Consistency between definitions in different contexts is conveniently verifiable[3] . Now generalizing this newly found definition, we have a $(q+1)$-representation of the group $G^{(0)}$ as again the functor between $(q+1)$-categories[3]

$$\rho^{(q+1)}: \quad \mathcal{C}^{(q+1)}_{G^{(0)}} \longrightarrow (q+1)-Vec \tag{5.15}$$

Recall $(q+1)$-categories have $q+2$ levels of morphisms of morphisms. Specifically, $\mathcal{C}^{(q+1)}_{G^{(0)}}$ contains only one object, the origin of its classifying space at choice. 1-morphisms of this $(q+1)$-category are all possible endomorphisms of this point, which are more inspiringly loops reutrning to the origin in the classifying space, ie. first homotopy group of the classifying space[3] . 2-morphisms are 2-dimensional homotopies between these loops in the classifying space, and the list goes on. $(q+1)-Vec$ is potentially more straightforward in the sense that the maps are all required to be linear. A $(q+1)-Vec$ is a $(q+1)$-category with $(q-1)$-category as objects and equipped with fusion algebra as discussed. Finally, the higher representation of higher groups is reincarnated as a functor as well[3] . A $(q+1)$-representation of a p-group $\mathbb{G}^{(p)}$ is the functor

$$\rho^{(q+1)}: \quad \mathcal{C}^{(q+1)}_{G^{(p)}} \longrightarrow (q+1)-Vec \tag{5.16}$$

The image category remains to be the $(q+1)-Vec$, as expected by its nature of being a higher representation. However, the source category is constructed now from the classifying space of the p-group $\mathbb{G}^{(p)}$[3] . If we define $\Omega(\mathcal{C})$ to be the r-category containing all endomorphisms of the identity object of $\mathcal{C}$, a category of one degree higher, then roughly speaking, each degree r of the higher group $\mathbb{G}^{(p)}$ consists of identity objects of $\Omega^{r+1}(\mathcal{C}^{(q+1)}_{\mathbb{G}^{(p)}})$

at level $r+1$[3] . Moving back to the concept of higher charges, we can most accurately arrive at the conclusion that q-charges of a p-form symmetry are $(q+1)$-representations of the accompanying $(p+1)$-group. After further generalization, the following statement holds true in general, q-charges of symmetry operations in a p-group are $(q+1)$-representations of this higher group. Thus we see that higher charges are deemed to be in effect due to the existence of underlying higher representations from the categorical structure of higher form symmetries, consistent with our intuition.

5.3 A Toy Model of Non-Invertible Symmetry

The arising category structure evidently allows for the loss of invertibility. We can see how this appears physically in the simplest example that has already been explored, Maxwell's theory in d=4 spacetime. 1-form symmetry operators with codimension= 2 in this theory carrying gauge parameters that respect the periodicity of $U(1)$ can be fused together by simple addition

$$U^{\alpha} \otimes U^{\beta} = U^{\alpha+\beta} \tag{5.17}$$

At this stage, being fused is in reality being multiplied using the group product. The additive law is simply an indication of this 1-form symmetry being Abelian, as all higher form symmetries are proven to be. However, they are also endowed with 2-charges under the charge conjugation symmetry. Gauging this discrete 0-form symmetry results in a new gauge group $O(2)$ and calls for operators that are gauge invariant under it. After simple calculations, we can see the surviving symmetry operators are $U^{\alpha=0}$, $U^{\alpha=\pi}$ and $U^{\alpha,+} = U^{\alpha} \oplus U^{-\alpha}$, with $\alpha \in (0,\pi)$ due to its periodicity. The fusion algebra is then

$$\begin{aligned} D^{\alpha,+} \otimes D^{\alpha=0} &= D^{\alpha,+} \\ D^{\alpha,+} \otimes D^{\alpha=pi} &= D^{\alpha+\pi,+} \\ D^{\alpha,+} \otimes D^{\beta,+} &= D^{\alpha+\beta,+} \oplus D^{\alpha+\beta,-} \end{aligned} \tag{5.18}$$

We see in the last line, given that the gauge parameters are not equal, the two symmetry operators correspond to irreducible representations in the new gauge theory, and thus we arrived at non-invertible composition of symmetry operations, ie. fusion. This is often how non-invertible symmetries relevant to us arise, gauging automorphisms of the original gauge group. In fact, we require the automorphisms to be a special class within all possible automorphisms called outer automorphisms of the group, which are with the automorphisms induced by natural group actions of each group element, ie. inner automorphism, quotient out. Additionally for example, axion electrodynamics also exhibits non-invertible symmetries[4].

Chapter 6

Summary

Higher form symmetry has rich structure to be explored. It is intimately connected with the geometry of spacetime and specifically its differential structure. Symmetry operators acting on different ranks of differential form fields on the spacetime make up higher form symmetries of different degrees. The mixing pattern of different degrees is best summarized in higher group, which is a natural consequence of the symmetry category that accommodates higher form symmetries of different degrees that a theory is equipped with. Applying formalism of higher form symmetries to quantum field theory or even quantum gravity should have extremely promising results.

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Appendix A

Appendix

A.1 Group Cohomology and Simplicial Cohomology

We will give an informal introduction of group cohomology and simplicial cohomology here in this appendix. As mentioned earlier, twisting classifies all possible group extensions, and it is said to be an element of the 2-group-cohomology. Let us first define what n-cochains $C^n(H, A)$ are[9]. Using the same notation earlier, group G is the extension of group H by the Abelian normal subgroup A. Let A be a H-module, ie.

$$\exists \quad \psi : \quad H \times A \longrightarrow A \tag{A.1}$$

, or equivalently

$$\exists \quad \psi : \quad H \longrightarrow Aut(A) \tag{A.2}$$

n-cochains are then

$$C^n(H, A) = \{\text{functions} : H^n \longrightarrow A\} = \psi(h, a_1) + \psi(h, a_2) + ... + \psi(h, a_n) \tag{A.3}$$

We can define a coboundary operator δ^{n+1}[9]

$$\begin{aligned}
&\delta^{n+1} : \quad C^n(G, A) \longrightarrow C^{n+1}(G, A) \\
&(\delta^{n+1}\omega)(g_1, ..., g_{n+1}) = g_1 \cdot \omega(g_2, ..., g_{n+1}) + \sum_{r=1}^{n} (-1)^r \omega(g_1, ..., g_{r-1}, g_r g - r + 1, ...) \\
&+ (-1)^{n+1}\omega(g_1, ..., g_n)
\end{aligned} \tag{A.4}$$

Here, ω is a generalization of the previous twisting element. It can be shown that[9]

$$\delta^{n+1} \circ \delta^n = 0 \tag{A.5}$$

Hence, we named it the coboundary operator. Using this, we can write down a short exact sequence,

$$... \overset{\delta^3}{\longleftarrow} C^2 \overset{\delta^2}{\longleftarrow} C^1 \overset{\delta^1}{\longleftarrow} C^0 \tag{A.6}$$

Now the group cohomology is

$$\mathbf{H}^n(H, A) = \frac{\mathbf{Z}^n(H, A)}{\mathbf{B}^n(H, A)} \tag{A.7}$$

, where, consistent with usual notations in de Rahm cohomology[9],

$$\begin{aligned} \mathbf{Z}^n(H, A) &= Ker(\delta^{n+1}) \quad \text{n-cocycle} \\ \mathbf{B}^n(H, A) &= Im(\delta^{n+1}) \quad \text{n-coboundary} \end{aligned} \tag{A.8}$$

Note the special case $\mathbf{B}^0(H, A) = 0$. We will see a very similar structure for simplicial cohomology. Firstly, an n-simplex is defined to be the smallest convex set in $\mathbb{R}^{n+1}$ containing $(n+1)$ independent points.

$$\sigma_n = [P_0, ..., P_n] \tag{A.9}$$

Triangulation which is essential to the construct of simplicial cohomology is defined to be the pair (Δ, f) with f being the hemeomorphism $f: \quad |\Delta|$ for any topological space X, where $|\Delta|$ is the collection of simplices[9]. We can again define boundary operators

$$\partial_r : \quad C_r(\Delta) \longrightarrow C_{r-1}(\Delta) \tag{A.10}$$

Again the nilpotency condition is satisifed[9] to justify the name

$$\partial_r \circ \partial_{r+1} = 0 \tag{A.11}$$

With $C_r(\Delta) = \mathbb{Z}\{\sigma_r\}$, we can write down the short exact sequence

$$0 \overset{id}{\rightarrow} C_n(\Delta) \overset{\partial_n}{\rightarrow} C_{n-1}(\Delta) \overset{\partial_{n-1}}{\longrightarrow} C_{n-2}(\Delta)... \tag{A.12}$$

In conclusion, extensions of groups have a cohomology, and by approximating group manifolds using simplices increasingly accurately, we complete the group short exact sequence. In other words, geometry and group theory go hand in hand.

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