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A brief introduction to symmetry and supersymmetry

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Declaration

I hereby declare that this essay is independently completed by myself and all of the cited references are carefully examined.

Zheng Miao

Acknowledgement

I am grateful for this opportunity to learn new knowledge, to make new friends and to live a new life. Thanks for my advisors Kelly and Torben.

Also, send my most sincere love to my parents, my family, my friends and my girlfriend Yinuo Shi. Every steps that I took seems futile without you.

The world is my oyster.

Abstract

Symmetry is a core concept when dealing with many physical systems. In this essay, the mathematical structures of the geometric symmetry, the internal symmetry and the supersymmetry are analysed. Seeking the mathematical patterns of the fundamental particles are crucial, therefore, this essay is a gentle introduction to the topic of supersymmetry. Starting from the historic development, then followed by the analysis of the algebras between the most general symmetries for the S-matrix. The properties of superalgebras, supermultiplets and their interpretations are introduced at the end. Different types of supersymmetry are discussed throughout the essay.

1 Introduction

Supersymmetric theories of nature are a set of mathematical structures where internal symmetry and spacetime symmetry (Poincare group) for usual quantum field theory are extended to a new type of symmetry $\rightarrow$ supersymmetry [1, 2]. The correspondence of the transformations between bosons and fermions represents the elegance and importance of such theories. Moreover, it was initially realised by implementing the anti-commutators between the generators of the above symmetry groups [3, 4]. In addition, the language of supersymmetry is used in flat spacetime where geodesics are straight lines, whereas the combination between the supersymmetry and the gravity can be applied to a curved geometry setting in general [5]. Despite that the viewpoint of considering supergravity as a potential candidate of the unifying field theory which avoids the horrible ultraviolet problems has changed, supergravity, as the name of itself points out that the unification of four fundamental forces including gravity can be expected from the perspective of regarding supergravity as a low-energy effective field theory at present [6].

Quantum states which live on the Hilbert space can be generalised in the basis of $|a, s, p\rangle$ where each of the three letters demonstrate different types of particles, spins and the associate 4-momentum respectively [7]. One of the crucial internal symmetry among the elementary particles, the $SU(3)_f$ flavor symmetry, was established to solve the interactions between quarks of the same spin s [8]. Therefore, a potentially larger symmetry structure which has inherent flavor symmetry and the transformations between different spins s could be suggested as in the $SU(6)$ group [7, 8, 9]. However, this $SU(6)$ structure is not the ultimate answer of the hardrons (under the relativistic conditions) as well as the S-matrix, according to Coleman and Mandula (C-M theory hereafter) in 1967, such that one has to construct a certain type of symmetry group which contains the isomorphism structure merging the spacetime symmetry group (Poincare group) and an internal symmetry one spontaneously [7, 8, 10, 11].

The historic development of supersymmetry was not only directly led by the discoveries of Coleman and Mandula who were digging for the fundamental symmetries of the S-matrix. Nonetheless, it was also inspired from seeking the conformal invariant pattern for fermionic fields in string theory.

Proposition 1.1. The invariant conformal transformation is viewed as follows [8],

$$\Sigma^{\pm} \to F^{\pm}(\Sigma^{\pm})$$

where Σ is the spatial parameter, T stands for the time parameter paring with the spatial ones Σ.
$\Sigma^{\pm}$ is therefore the light-cone coordinates in the form of $\Sigma^{\pm} = \frac{T \pm \Sigma}{\sqrt{2}}$ and $F^{\pm}$ represents two independent functions arbitrarily [8].

With two additional d-dimensional fermion field doublets $\Psi_1^a(\Sigma, T)$ and $\Psi_2^a(\Sigma, T)$, an action for $\frac{1}{2}$-integer spin particles can be extended as [8, 12]:

$$\mathcal{S}[X, \Psi] = \int d\Sigma^+ \int d\Sigma^- (\mathcal{T} \frac{\partial X^a}{\partial \Sigma^+} \frac{\partial X_a}{\partial \Sigma^-} + i\Psi_2^a \frac{\partial}{\partial \Sigma^+} \Psi_{2a} + i\Psi_1^a \frac{\partial}{\partial \Sigma^-} \Psi_{1a}) \tag{1}$$

where X is the bosonic field and $\mathcal{T}$ represents the string tension which distinguishes the time parameter T. The first term in the above brackets is the bosonic term in string theory, and the rest of the expressions are the extended fermionic actions in which two fermionic fields are involved in the system obeying the Pauli's exclusion theorem for fermions.

Hence, the fermionic fields $\Psi_1^a(\Sigma, T)$ and $\Psi_2^a(\Sigma, T)$, in order to keep the conformal transformation invariance as stated in the Theorem 1.1, the invariant conformal transformation upon the fermionic fields can be demonstrated respectively as [8, 12]:

$$\Psi_1^a(\Sigma,T) \to (\frac{dF^+}{d\Sigma^+})^{-\frac{1}{2}}\Psi_1^a(\Sigma,T) \tag{2}$$

$$\Psi_2^a(\Sigma,T) \to (\frac{dF^+}{d\Sigma^+})^{-\frac{1}{2}}\Psi_2^a(\Sigma,T) \tag{3}$$

Indeed, the fermionic fields Ψ_1^a and Ψ_2^a hold for $(2+d)$-dimensional symmetric invariance with two conformal ones plus the Lorentz invariance.

Noticeably, the interchange between the bosons and fermions can be obtained from a special kind of symmetry $\to$ supersymmetry, which could be derived from the above theory [8, 12]. The minuscule symmetry between interchanging those two fields are displayed as follows:

$$\delta\Psi_2^a(\Sigma^+,\Sigma^-) = i\mathcal{T}\Theta_2(\Sigma^-)\frac{\partial}{\partial\Sigma^-}X^a(\Sigma^+,\Sigma^-) \tag{4}$$

$$\delta\Psi_1^a(\Sigma^+,\Sigma^-) = i\mathcal{T}\Theta_1(\Sigma^+)\frac{\partial}{\partial\Sigma^+}X^a(\Sigma^+,\Sigma^-) \tag{5}$$

$$\delta X^a(\Sigma^+,\Sigma^-) = \Theta_2(\Sigma^-)\Psi_2^a(\Sigma^+,\Sigma^-) + \Theta_1(\Sigma^+)\Psi_1^a(\Sigma^+,\Sigma^-) \tag{6}$$

where Θ_1 and Θ_2 are the infinitesimal functions of the fermionic fields, similar to the Grassmann variables which are anti-commuting with themselves [8].

Since both the action $\mathcal{S}$ in the above theory and the light-cone coordinates $\Sigma^\pm$ suggest that this is a 2-dimensional supersymmetry in which the symmetry transformation between different quantum states could result in the change of the spin s, which differs from the flavor symmetry, the 4-dimensional field theory satisfying the interchange between fermions and bosons is therefore theoretically possible.

One of the earliest supersymmetric model in 4-dimensions can be traced back to 1970s, involving one Majorana field Φ and four bosonic fields A, B, F and G. The minuscule transformations of the above fields are written as [2, 8, 13, 14, 15]:

$$\delta A = \bar{\alpha}\Phi \tag{7}$$

$$\delta B = -i\bar{\alpha}\gamma_5\Phi \tag{8}$$

$$\delta \Phi = \partial_\mu(A + i\gamma_5 B)\gamma^\mu\alpha + (F - i\gamma_5 G)\alpha \tag{9}$$

$$\delta F = \bar{\alpha}\gamma^\mu\partial_\mu\Phi \tag{10}$$

$$\delta G = -i\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\Phi \tag{11}$$

Therefore, a general Lagrangian density can be built up upon the above fields and symmetry transformations in the following form [2, 3, 8, 13, 14, 15]:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_f + \mathcal{L}_m + \mathcal{L}_g \\ &= -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{1}{2}\bar{\Phi}\gamma^\mu\partial_\mu\Phi + \frac{1}{2}(F^2 + G^2) \\ &+ m[FA + GB - \frac{1}{2}\bar{\Phi}\Phi] + g[F(A^2 + B^2) + 2GAB - \bar{\Phi}(A + i\gamma_5 B)\Phi]\end{aligned} \tag{12}$$

where the Lagrangian density $\mathcal{L}$ is constructed by the free Lagrangian $\mathcal{L}_f = -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{1}{2}\bar{\Phi}\gamma^\mu\partial_\mu\Phi + \frac{1}{2}(F^2 + G^2)$ showing on the second line of eq(12), followed by two invariants $\mathcal{L}_m$, $\mathcal{L}_g$ added together on the last line of eq(12).

One potential invariant $\mathcal{L}_\lambda = \lambda F$ can be further considered as a shift and therefore can be neglected [3].

The equations of motion for the scalar and pseudo-scalar fields F and G can be found below respectively [2, 3, 8], which are useful to simplify the expression of the Lagrangian density above.

$$F + mA + g(A^2 - B^2) = 0 \tag{13}$$

$$G + mB + 2gAB = 0 \tag{14}$$

Hence, the same Lagrangian density can be re-expressed as [2, 3, 8, 14, 15]:

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2}\partial_\mu A \partial^\mu A - \frac{1}{2}\partial_\mu B \partial^\mu B - \frac{1}{2}\bar{\Phi}\gamma^\mu \partial_\mu \Phi + \frac{1}{2}(F^2 + G^2) \\
&- \frac{1}{2}(A^2 + B^2) - \frac{1}{2}m\bar{\Phi}\Phi \\
&- gmA(A^2 + B^2) - \frac{1}{2}g^2(A^2 + B^2) - G\Phi(A + i\gamma_5 B)\Phi
\end{aligned} \tag{15}$$

The above Lagrangian density, indicated as eq(15), provides the relationships of the masses in different fields and many interactions which are the result of supersymmetric invariance [3, 8].

Overall, the history of supersymmetry is developed closely to the symmetric transformations between different quantum states for both bosons and fermions. In addition to the $SU(3)_f$ internal symmetry, the transformations between the different values of the spin s can be considered as a result of the supersymmetric actions. Thus, some field theories which satisfied supersymmetry in 4-dimension can be then generated systematically. In short, the following sections of this essay will focus on the mathematical construction and the physical applications of the supersymmetry theory.

2 Mathematical Preliminaries

2.1 Lorentz Group

Define the $n \times n$ identity matrix in the form of $\mathbf{I}_n$, and the matrix elements of $\mathbf{I}_{n,m}$ is:

$$\mathbf{I}_{n,m} = \begin{bmatrix} \mathbf{I}_n & 0 \\ 0 & -\mathbf{I}_m \end{bmatrix} \tag{16}$$

Then, the Minkowski metric tensor can be also written as $\boldsymbol{\eta} = \mathbf{I}_{1,3}$ (The metric signature for the 4-dimensional spacetime is $(+, -, -, -)$ if not mentioned afterwards).

Definition 2.1. A Lorentz group can be defined as [2, 5, 16]

$$O(t, x) = \{\mathbf{M} \in GL(n, \mathbb{R}) : M^T \mathbf{I}_{t,x} M = \mathbf{I}_{t,x}\} \tag{17}$$

where the term $\mathbf{I}_{t,x}$ in the 4-dimensional spacetime is the metric tensor $\boldsymbol{\eta}$. As for higher dimensions of the $(n+1)$-spacetime, the 3-dimensional space-like parameter x can be replaced by n where the structure of the group $O(t, n)$ remains the same as $O(t, x)$.

Proposition 2.2. The Lorentz transformation of the metric tensor preserves the form and notation of itself. The covariant form of the expression is:

$$\Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \eta_{\mu\nu} = \eta_{\alpha\beta} \tag{18}$$

where $\Lambda^{\mu}{}_{\nu}$ is the matrix representation of the form of Lorentz transformation group.

Notably, the indices labelling in Greek letters in the eq(18) are Lorentz indices which are different from the matrix element indices.

Proposition 2.3. Because of the constraints $det\Lambda = \pm 1$ and $|\Lambda^0{}_0| \geq 1$, there are four connected elements disconnected in the Lorentz group $O(t, n)$ as indicated in the following form [5, 16, 17]:

$$O(t, n) = L = L_+^\uparrow \oplus L_+^\downarrow \oplus L_-^\uparrow \oplus L_-^\downarrow \tag{19}$$

where $\uparrow$ and $\downarrow$ stand for the positive and negative sign of the time parameter, which also defines the orthochronous and non-orthochronous Lorentz transformation respectively [5]. Moreover, the subscript plus or minus sign indicates the determinant being 1 or -1. The notation of $O(t, n)$ rather than $O(t, x)$ can be regarded as the generalisation from 4-dimensional spacetime to n-dimensional pseudo-Euclidean basis [5].

Therefore, some important subgroups of the Lorentz group $O(t, n)$ can be determined such that [5, 16, 17]:

$$SO(t, n) = L_+^\uparrow \oplus L_+^\downarrow \tag{20}$$

$$SO(t, n)_+ = L_+^\uparrow \tag{21}$$

where the orthochronous Lorentz group $SO(t, n)$ is a special orthogonal group which is defined to obey the condition that $det\Lambda = 1$.

The proper orthochronous Lorentz group is therefore $SO(t, n)_+$, satisfying those two constraints and mapping to the light-cone aligning with the direction of the time flows [5].

Definition 2.4. The Lie algebra of the Lorentz group $O(1,3)$ for a 4-dimensional spacetime is defined to be [5]:

$$o(1,3) = \{m \in \mathbf{M}_{4\times 4}(\mathbb{R}) : m^T = -\eta m \eta\} \tag{22}$$

Proof 2.5. Using the exponential mapping for the Lie algebras, one can write the Lorentz transformation matrix Λ explicitly as [5]:

$$O(1,3) \ni \Lambda(\tau) = e^{(\tau m)} \tag{23}$$

Then, because of the fact that the Lie algebra $o(1,3)$ satisfies the Lorentz transformation which preserves the metric, the following equality can be obtained:

$$\Lambda(\tau)^T \eta \Lambda(\tau) = \eta \tag{24}$$

Differentiating both sides of the eq(24), it can be shown as [5]:

$$\frac{d}{dt}[(e^{\tau m})^T \eta (e^{\tau m})]|_{t\to 0} = 0 \tag{25}$$

$$\Rightarrow [\frac{d}{dt}(e^{\tau m})^T]\eta(e^{\tau m})|_{t\to 0} + (e^{\tau m})^T \eta[\frac{d}{dt}(e^{\tau m})]|_{t\to 0} = 0 \tag{26}$$

Hence,

$$m^T \eta + \eta m = 0 \iff m^T = -\eta m \eta, \forall m \in o(1,3) \tag{27}$$

□

2.2 Spinor Algebra 1

Moreover, if the inner product of the vector components can be written as in the same signature as the matrix $\mathbf{I}_{n,m}$ is defined, such that $< x, x >= (x^1)^2 + (x^2)^2 + ... + (x^n)^2 - (x^{n+1})^2 - ... - (x^{n+m})^2 = \eta(x,x)$, then the Clifford group $\Gamma_{n,m}$ can be defined based on the Clifford algebras $Cl(n,m)$ as follows [16]:

Definition 2.6. A real Clifford algebra $Cl(n,m)$ (Clifford algebra hereafter), relies hugely on the metric signature represented in the above inner product with n positive inputs and m negative ones, can be defined to satisfy the following properties apart from the associativity [16, 18]:

$$\{e_a, e_b\} = e_a e_b + e_b e_a = 2\mathbf{I}_{n,m} = \begin{cases} +1, & a = b = 1, ..., n. \\ 0, & a \neq b. \\ -1, & a = b = m, ..., n+m. \end{cases} \tag{28}$$

Also, one possible orthonormal basis of the Clifford algebra is [16]:

$$\{1, e_{n1}, e_{n1}e_{n2}, e_{n1}e_{n2}e_{n3}, ..., e_{n1}e_{n2}e_{n3}...e_{n+m-1}, e_{n1}e_{n2}e_{n3}...e_{n+m-1}e_{n+m}\}$$

In the case of the scalar product satisfies $\eta(x,y) = 0$, where there is no quadratic form in its expression, the Grassmann algebra can therefore be obtained [16].

Proposition 2.7. The dimension of the Clifford algebra is 2^{n+m}, if the metric is denoted to be the same as the matrix $\mathbf{I}_{n,m}$ [16, 18].

Definition 2.8. Therefore, the Clifford group can be defined as [16]:

$$\Gamma_{n,m} \subset \{x \in Cl(n,m) : \forall \nu \in E, x\nu x^{-1} \in E\} \tag{29}$$

where E is a $(n+m)$-dimensional vector space over the real field.

After introducing the Clifford group, there are a few subgroups to be classified which have significant relations to the Lorentz groups $\rightarrow$ orthogonal group and its subgroups as introduced above in eq(20) and eq(21).

Definition 2.9. Define one of the subgroups of the Clifford group $Cl(n,m) \supset Pin(n,m)$, and then [16, 17]:

$$Pin(n,m) = \{x \in \Gamma_{n,m} : |n(x)| = 1\} \tag{30}$$

where $n(x)$ is the norm of element x in the Clifford group $Cl(n,m)$. Since x_1, $x_2 \in \Gamma_{n,m}$, the norm of the product of x_1 and x_2 can be re-written as $n(xy) = n(x)n(y)$ [16].

Other subgroups of the Clifford group can be expressed as follows [16, 17, 19]:

$$Spin(n,m) = Pin(n,m) \cap Cl(n,m)_{even} \tag{31}$$

$$Spin_{+}(n,m) = \{x \in Spin(n,m) : n(x) = 1\} \tag{32}$$

where $Cl(n,m)_{even}$ is the even subalgebra of the Clifford algebra $Cl(n,m)$ satisfying the condition that $n + m =$ even [16].

Proposition 2.10. The double covering map for a general group G, to be acted on the space X evenly, can be written as $\rho : X \rightarrow X/G$ [20]. Subsequently, the subgroup of the Clifford group $Spin_{+}(n,m) \subset Cl(n,m)$ is the double cover of the subgroup of the orthogonal group $SO_{+}(n,m) \subset O(n,m)$ [16, 17].

Before proving the above proposition, a double covering relationships in the Lie group theory can be simply understood in the following way that the homomorphic (isomorphic when bijective) structures of different Lie groups remain the same except for the mappings. The bijective mapping pattern is therefore replaced by a 2-1 surjective structure.

Proof 2.11. Firstly, if an element $y \in Spin_+(n,m)$, then the negative valued $-y \in Spin_+(n,m)$ is true [16]. Secondly, the kernel group of the homomorphic relationship between the $Spin_+(n,m)$ group and the $SO_+(n,m)$ group is $\mathbb{Z}_2$ [20]. Lastly, use the path-lifting characteristic of the covering mappings to construct a smooth curve from -1 to 1 which subsequently proves the double-cover relationship between the $Spin_+(n,m)$ and the $SO_+(n,m)$ groups [16, 20]. Introduce a path $s(\theta)$ in the following form:

$$s(\theta) = e^{(2\theta e_a e_b)} = \cos 2\theta + \sin 2\theta e_a e_b \tag{33}$$

take θ in the range of $[0, \frac{\pi}{2}]$ to complete the path which indicates that $Spin_+(n,m)$ is a double-cover of $SO_+(n,m)$ group.

$\square$

Similarly, the double cover group relations can be extended to $Spin(n,m)$ as of $SO(n,m)$ and $Pin(n,m)$ as of $O(n,m)$ respectively using the same technique of proof as shown above [20]. More importantly, for a simply-connected double covering group $Spin(n)$ ($n > 3$ to satisfy the simply-connected property), it then can be treated as a universal covering group of $SO(n)$ ($SO(n) = SO(n,0)$, such that $min(n,m) \leq 1$) [16, 19, 20].

Proposition 2.12. The exact sequence of the n-dimensional ($n > 3$) spin group $Spin(n)$ is [19, 20, 21]:

$$1 \to \{-1, +1\} \to Spin(n) \to SO(n) \to 1$$

where 1 represents the type of groups contains single element [21].

The topological aspects of the Lorentz group $\to$ the orthogonal group $O(n, m)$ and its subgroup $SO(n, m)$ are displayed as follows.

Proposition 2.13. The homeomorphic structure of the $O(n, m)$ group is $O(n) \times O(m) \times \mathbb{R}^{nm}$ [21]; The homeomorphic structure of the $SO(n, m)$ group is $S(O(n) \times O(m)) \times \mathbb{R}^{n,m}$ [21].
The dimension of the above topological spaces is $\frac{(n+m)(n+m-1)}{2}$ [21].

2.3 Lorentz Algebra

For the restricted (proper orthochronous) Lorentz group $SO_+(n, m)$ ($SO_+(1, 3)$ in the conventionally 4-dimensional spacetime), the corresponding Lorentz algebra can be considered as the properties of the small perturbations near its identity [5].

Proposition 2.14.

$$\Lambda = \mathbf{1}_{4\times 4} + \omega \tag{34}$$

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \tag{35}$$

$$\Lambda^{\mu}{}_{\nu} = [e^{-\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}}]^{\mu}{}_{\nu} \tag{36}$$

$$\omega^{\mu}{}_{\nu} = -\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}{}^{\mu}{}_{\nu} \tag{37}$$

$$M_{\rho\sigma}{}^{\mu}{}_{\nu} = i(\eta_{\sigma\nu}\delta_{\rho}{}^{\mu} - \eta_{\rho\nu}\delta_{\sigma}{}^{\mu}) \tag{38}$$

Using Proposition 2.2. eq(18), the invariance of the metric tensor indicates the antisymmetric property of the small perturbation term ω as shown in eq(35). Also, the matrix $M_{\rho\sigma}{}^{\mu}{}_{\nu}$ is the generator of the Lorentz group as described in eq(37) specifically. Therefore, since the antisymmetric property of the perturbation matrix ω, to avoid the whole quantity being zero, the generator must be antisymmetric as well. (The product of symmetric tensor and antisymmetric tensor is zero) The explicit form of the Lorentz generator is demonstrated as in eq(38). It is logical to apply the result in eq(22) in the proper orthochronous Lorentz group such that the generators M form a basis of the Lorentz algebra $o(1,3)$ [5]. Another property of the generators of the Lorentz group is Hermitian. Since the transpose of a pair of matrix indices μ, ν interchange, the minus sign obtained by the complex conjugate of i cancels out. Then, the commutation relations of the generators M can be demonstrated as follow [5]:

Proposition 2.15.

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}) \tag{39}$$

2.4 Poincare Group and its Algebra

The Lorentz invariance pluses the translation invariance equals to the Poincare invariance as in $ISO(1,3) = SO(1,3) \rtimes \mathbb{R}_4$. The Poincare group is isometric [5, 22]. Denote the Poincare group as P, the product of two consistent Poincare transformations can be determined by [1, 2, 5, 8, 17]:

Proposition 2.16.

$$(\Lambda_i, a_i) \cdot (\Lambda_j, a_j) = (\Lambda_i \Lambda_j, \Lambda_i a_i + a_j) \tag{40}$$

The a_i expressed above is the translation constant. The group (Λ, a) can be decomposed into four components in its 5×5 matrix form [17].

Proposition 2.17.

$$(\Lambda, a) = \begin{bmatrix} \Lambda^{\mu}{}_{\nu} & a^{\mu} \\ 0 & 1 \end{bmatrix} \tag{41}$$

The resultant Poincare group P is the mixture of Lorentz group L and the translation group $\mathbb{T}$ which form a semi-direct product in 4-D [5]. Similarly, the Poincare group has 4 components as well as the Lorentz group, denoted as [5]:

Proposition 2.18.

$$P = P_+^{\uparrow} \oplus P_+^{\downarrow} \oplus P_-^{\uparrow} \oplus P_-^{\downarrow} \tag{42}$$

Notably, the identity of the Poincare group is not trivially the identity matrix, but a blended matrix in the form of containing all the diagonal elements as 1, except for the last diagonal term as 0. So that, the identity in Poincare group is $P_{identity} = diag(1,1,1,1,0)$ [5].

Define the representation of the proper orthochronous Poincare group $P_+^\uparrow$ as $g(\Lambda, a)$, then the infinitesimal version of the representation matrix can be expressed as [2, 5]:

Proposition 2.19.

$$g(\Lambda, a) = \mathbb{1} - \frac{i}{2}\omega^{\rho\sigma} M_{\rho\sigma}{}^{\mu}{}_{\nu} + i a_\mu P^\mu \tag{43}$$

The above ω, a are all infinitesimal. Using the definition of the Lie algebra given in eq(24), the nature of the generators M and P can be determined as a tensor and a vector respectively [5]. The commutation relation between the different generators P_i and P_j is 0 because the commutation reality between different translations is satisfied universally in presumption. Finally, the commutation relations between the different generator M and P can be obtained as expanding the Lorentz matrix around identity. To sum up [2, 5, 17],

Proposition 2.20.

$$[P_\mu, P_\nu] = 0 \tag{44}$$

$$[M_{\mu\nu}, P_\chi] = -i(\eta_{\mu\chi} P_\nu - \eta_{\nu\chi} P_\mu) \tag{45}$$

According to the boost and rotation generators in the Lorentz transformation, the group $SO(n,m)$ is related to the total angular momentum [2, 5]. However, as stated in the previous section, the double covering identity can also be obtained if the group $SO(n,m)$ is half-integer-valued [2]. The construction of Casimir operator is based on decomposing the Lie algebra $so(1,3;\mathbb{K}) \sim sl(2,\mathbb{C}) \times sl(2,\mathbb{C})$ [2, 5]. Consider the simplest representations as follow [2, 5]:

Proposition 2.21. 1. $(0,0)$-representation: total spin $= 0$; transforms as a 1-D scalar[2, 5].
2. $(\frac{1}{2},0)$-representation: total spin $= \frac{1}{2}$; transforms as a 2-D left-handed spinor[2, 5].
3. $(0,\frac{1}{2})$-representation: total spin $= \frac{1}{2}$; transforms as a 2-D right-handed spinor[2, 5].

Consider a tensor of $(2,0)$-type, the decomposition of the tensor $T^{\mu\nu}$ is [2],

$$\begin{aligned} T^{\mu\nu} \in (\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2}) &= (\frac{1}{2},0) \otimes (0,\frac{1}{2}) \otimes (0,\frac{1}{2}) \otimes (\frac{1}{2},0) \\ &= [(1,0) \oplus (0,0)] \otimes [(0,1) \oplus (0,0)] \\ &= (1,1) \oplus (1,0) \oplus (0,1) \oplus (0,0) \end{aligned}$$

which suggests that the tensor of $(2,0)$-type can be constructed by the a scalar (spin-0) field, a spin-1 field and a spin-2 field [2]. Furthermore, every spinor representation can be generated using direct products and direct sums of the left and right-handed spinor fields [2, 5]. Notably, the Dirac spinor can be represented according to the direct sum of the left and right-handed spinor representations as $(\frac{1}{2},0) \oplus (0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$ [2, 5].

2.5 Spinor Algebra 2

To introduce the Weyl spinor representations, several concepts should be considered in prior. Define M_i as an arbitrary element of the group $SL(2,\mathbb{C})$, which can form a mapping between the group $SL(2,\mathbb{C})$ and the automorphism group $\mathbb{A}$, then the representation g should satisfy the following equations [5],

Proposition 2.22.

$$g(\mathbb{1}_{\mathbb{SL}(2,\mathbb{C})}) = \mathbb{1}_{\mathbb{A}} \tag{46}$$

$$g(M_i) \cdot g(M_j) = g(M_i \cdot M_j) \tag{47}$$

Therefore, the arbitrary element ψ_i in $\mathbb{A}$ can be structured as a linear combination of its magnitude and the basis vector [5]:

Proposition 2.23.

$$\psi = \Sigma_{i=1}^{dim\mathbb{A}} \psi_i \hat{e}_i \tag{48}$$

$$\Rightarrow g(M)\psi = \Sigma_{i=1}^{dim\mathbb{A}} [\Sigma_{j=1}^{dim\mathbb{A}} g_i{}^j(M)\psi_j]\hat{e}_i \tag{49}$$

The equivalent representations of g can be defined as satisfying the following property: $g^i(M) = Ug^j(M)U^{-1}$ [5]. So the inequivalent type of the representations are actually the classification of the left and right-handed Weyl spinor.

1. For the left-handed Weyl spinor [5],

$$\psi'_a = M_a{}^b \psi_b$$
$$(a, b = 1, 2)$$
$$\forall M \in SL(2, \mathbb{C})$$

where M is the representation mapping to itself in the special linear group $SL(2, \mathbb{C})$.

2. For the right-handed Weyl spinor [5],

$$\bar{\psi}'_{\dot{a}} = (M)^*{}_{\dot{a}}{}^{\dot{b}} \bar{\psi}_{\dot{b}}$$
$$(\dot{a}, \dot{b} = 1, 2)$$
$$\forall M \in SL(2, \mathbb{C})$$

where M^* the complex conjugate of M is applied as a representation mapping to itself again in the special linear group $SL(2,\mathbb{C})$. The representations M and its conjugation M^* are not equivalent for which M^* acts on the dual space $\mathbb{A}^*$ opposed to $\mathbb{A}$ [5].

The transformations between the differently dotted spinors can be obtained as follow [5]:

Proposition 2.24.

$$\psi'_a = M_a{}^b\psi_b = (M^{-1T})^a{}_b\psi^b \tag{50}$$

$$\bar{\psi}'_{\dot{a}} = (M)^*{}_{\dot{a}}{}^{\dot{b}}\bar{\psi}_{\dot{b}} = (M)^{*-1T}{}_{\dot{a}}{}^{\dot{b}}\bar{\psi}^{\dot{b}} \tag{51}$$

Similar to the process of lowering/raising the indices for the Lorentz tensors, a spinor metric tensor ϵ is to lower or raise the indices in spinors [5]. Accordingly, as shown in the Figure 1, the transformation of the spinors with different handed representations and different spinor indices can be generalised by the metric for spinors. The relations between the differently dotted and differently handed Weyl spinors can be therefore illustrated in Figure 2, where the vector spaces and its corresponding dual spaces are given for each type of Weyl spinors. Notably, only two kinds of Weyl spinor are not equivalent algebraically, due to the property of the special linear group $SL(2,\mathbb{C})$ [2, 5].

Proposition 2.25.

$$\epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{52}$$

The metric tensor ϵ is antisymmetric and the Grassmann variables can be constructed based upon this. Below are two figures illustrating that the metric tensor is the bridge between spinors and their corresponding spaces.

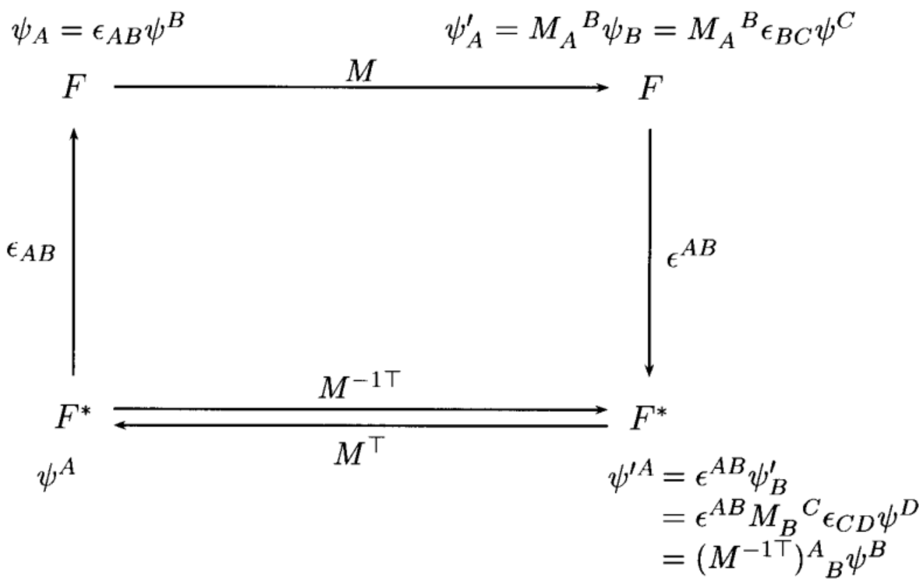

Figure 1: Mapping relations between the vector space to the dual space for Weyl spinors[5]

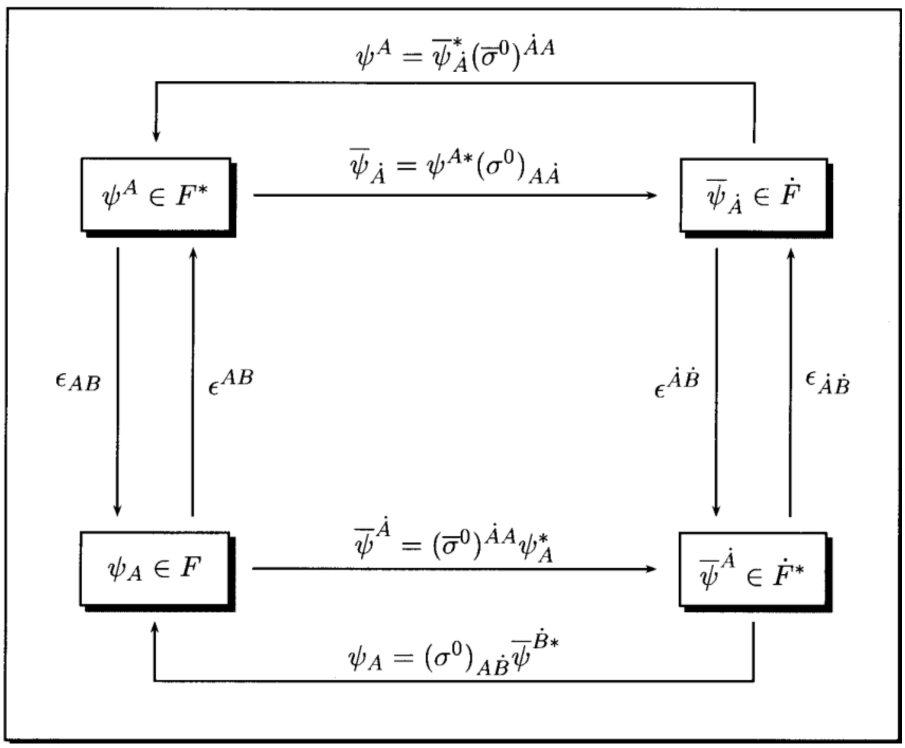

Figure 2: The space and transformation relations for differently dotted and differently handed Weyl spinors[5]

Two useful invariant expressions in the composition of the spinor fields can be obtained as [5]:

$$(\psi\chi) = \psi^a \chi_a \tag{53}$$

$$(\bar{\psi}\bar{\chi}) = \bar{\psi}_{\dot{a}} \bar{\chi}^{\dot{a}} \tag{54}$$

where a and $\dot{a}$ vary from 1 to 2 as integer coefficients, $\psi \in \mathbb{A}^*$ is the spinor in the dual space of $\mathbb{A}$ where χ acts on. (In the above figures, the vector space and its dual space are noted as $\mathbb{F}$ and $\mathbb{F}^*$, to avoid any confusions, the notation I have used for the rest of the essay is $\mathbb{A}$)

Definition 2.26. Grassmann variables is defined based on the properties of the spinor algebra as [5, 23]:

$$\{\psi_a, \psi^b\} = \{\psi_a, \psi_b\} = \{\psi^a, \psi^b\} = 0 \tag{55}$$

$$\{\bar{\chi}_{\dot{a}}, \bar{\chi}^{\dot{b}}\} = \{\bar{\chi}_{\dot{a}}, \bar{\chi}_{\dot{b}}\} = \{\bar{\chi}^{\dot{a}}, \bar{\chi}^{\dot{b}}\} = 0 \tag{56}$$

$$\{\psi_a, \bar{\chi}_{\dot{b}}\} = 0 \tag{57}$$

This anti-commuting relations are the direct result of the spin-statistic theorem where bosons with integer-valued spins commute with each other and fermions with half-integer-valued spins anticommute with each other [2, 5, 23, 24].

Proposition 2.27. The Grassmann variables have the following calculation properties based upon the metric tensor ϵ[2, 5]:

$$\theta^a \theta^b = -\frac{1}{2}\epsilon^{ab}(\theta\theta) \tag{58}$$

$$\theta_a \theta_b = \frac{1}{2}\epsilon_{ab}(\theta\theta) \tag{59}$$

$$\bar{\theta}^{\dot{a}} \bar{\theta}^{\dot{b}} = \frac{1}{2}\epsilon^{\dot{a}\dot{b}}(\bar{\theta}\bar{\theta}) \tag{60}$$

$$\bar{\theta}_{\dot{a}} \bar{\theta}_{\dot{b}} = -\frac{1}{2}\epsilon_{\dot{a}\dot{b}}(\bar{\theta}\bar{\theta}) \tag{61}$$

$$(\theta\theta)\theta^a = 0 \tag{62}$$

After introducing the anticommutation relations of the Grassmann variables, the different types of spinors from the special linear group (Lie group) $SL(2, \mathbb{C})$ and the algebraic properties of the proper orthochronous Lorentz group (Lie group) $L_+^\uparrow$, the relationship between those two groups is led by a universal cover such that [2, 5]:

Proposition 2.28.

$$L_+^\uparrow \cong SL(2, \mathbb{C})/\mathbb{Z}_2 \tag{63}$$

Similar to the Lorentz generator in the restricted Lorentz group $L_+^\uparrow$, the generator for its universal-cover group has the following identities [5]:

Proposition 2.29.

$$Tr[\sigma^{\mu\nu}\sigma^{\rho\sigma}] = \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} \tag{64}$$

$$Tr[\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma}] = \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} \tag{65}$$

Definition 2.30. The Dirac 4-spinor can be defined as [2, 5]:

$$\Psi = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{66}$$

where $\phi \in \mathbb{A}$ and $\bar{\psi} \in \dot{\mathbb{A}}^*$ are given.

In the γ-matrices representation, the Weyl spinor can be expressed as in the rules for Clifford algebra [5].

$$\{\gamma_W^\mu, \gamma_W^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_{4\times 4} \tag{67}$$

where $\mu = 0, 1, 2, 3$.

$$\gamma_W^5 = i\gamma_W^0\gamma_W^1\gamma_W^2\gamma_W^3 \tag{68}$$

Definition 2.31. In the Weyl basis [2, 5, 24],

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix} \tag{69}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{70}$$

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix} \tag{71}$$

The Dirac representation can also be concluded as follow [2, 5, 24]:

Definition 2.32. In the canonical basis for the Dirac spinors[2, 5, 24],

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix} \tag{72}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \bar{\sigma}^i & 0 \end{pmatrix} \tag{73}$$

$$\gamma^5 = \begin{pmatrix} 0 & \sigma^0 \\ \bar{\sigma}^0 & 0 \end{pmatrix} \tag{74}$$

Finally, the Majorana representation can be obtained by [2, 5, 24]:

Definition 2.33. In the Majorana representation [2, 5, 24],

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ -\bar{\sigma}^2 & 0 \end{pmatrix} \tag{75}$$

$$\gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \tag{76}$$

$$\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ -\bar{\sigma}^2 & 0 \end{pmatrix} \tag{77}$$

$$\gamma^2 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix} \tag{78}$$

$$\gamma^2 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \tag{79}$$

3 Supergroups, Superalgebras and Their Representations

3.1 Commutation and anticommutation rules

As stated in the introduction, Coleman and Mandula tried to determine the symmetry of the scattering matrix which led to the realization of the supersymmetry to some extent. Now, the algebraic structure of supersymmetry can be accordingly defined as the concepts of Lie superalgebra are involved [1, 2, 5]. To specify the difference between the Poincare algebra and the Poincare superalgebra, one has to consider an additional Majorana term in the commutation and anticommutation relations as followed [1, 2, 5, 8]:

Proposition 3.1.

$$\{Q_a, \bar{Q}_b\} = 2(\gamma^\mu)_{ab} P_\mu \tag{80}$$

$$[Q_a, P_\mu] = 0 \tag{81}$$

$$[Q_a, M^{\mu\nu}] = (\sigma^{4\mu\nu})_{ab} Q_b \tag{82}$$

$$\{Q_a, Q_b\} = -2(\gamma^\mu C)_{ab} P_\mu \tag{83}$$

where Q_a is the Majorana spinor-charge, labelling from 1 to 4 [1, 2, 5, 8]. Also, $\sigma^{4\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ and $\bar{Q}_a = (Q^\dagger \gamma_0)_a$ [1, 2, 5, 8].

3.2 Lie superalgebra

Definition 3.2. To define a Lie superalgebra, for simplicity, taking the $mod2$-graded vector spaces(because of the bosonic and fermionic fields), which is to consider the composition laws of such two vector subspaces, then [2, 5]:

$$L_0 \oplus L_1 = L \tag{84}$$

and the bilinear relation [2, 5],

$$[.,.\} : L \times L \to L \tag{85}$$

the generators with different grades have the following property [2, 5],

$$[t_a, t_b\} = iC^c_{ab} t_c \tag{86}$$

where the grades for different generators are assigned by the vector spaces they belong to [2, 5]. The subscript values 0 and 1 for L is the corresponding values of the grade for each generators within its vector space. The dimensions for different subspaces are D_0 and D_1 respectively [2, 3].

The notation $[.,.\}$ represents both the antisymmetric product as in $[.,.]$ and the symmetric product as in $\{.,.\}$ respectively, depending on whether the elements of the bosonic generator exists or not [2].

There are three properties for the Lie superalgebra which should be noticed as demonstrated below:

Proposition 3.3. 1. Composition law of the grades (for the elements in the subspaces)[2, 5]

$$\forall t_i \in L_i \ \ (i,j=0,1) \quad [t_i, t_j\} \in L_{(i+j)mod2} \tag{87}$$

2. S-ACR property[2, 5]

$$\forall t_i \in L_i \ \ t_j \in L_j \ \ (i,j=0,1) \quad [t_i, t_j\} = -(-1)^{ij}[t_j, t_i\} \tag{88}$$

3. S-JAC property[2, 5]

$$[t_k, [t_i, t_j\}\}(-1)^{kj} + [t_i, [t_j, t_k\}\}(-1)^{ik} + [t_j, [t_k, t_i\}\}(-1)^{ji} = 0 \tag{89}$$

Since in the supersymmetry, the relationships between the bosons and fermions have been imposed an additional symmetry. The above two vector subspaces L_0 and L_1, building up the complete vector space L, are the spaces for the boson-generators and fermion-generators respectively [2]. Hence, the results of the bilinear products between different generators differ from the number of the bosonic and fermionic generators contained in the commutation (or anticommutation) relations [2]. The results read off as the grading property and the s-ACR property above suggested, such that [2, 5]:

$$[L_0, L_0\} \subset L_0 \tag{90}$$

$$[L_0, L_1\} \subset L_1 \tag{91}$$

$$[L_1, L_1\} \subset L_0 \tag{92}$$

$$\Rightarrow C^k_{ji} = -(-1)^{ij} C^k_{ij} \quad (k=(i+j)mod2) \tag{93}$$

The s-Jacobi identity can be achieved naturally as eq(101) and eq(102) in the adjoint representation below [2],

Definition 3.4. Define the adjoint representation as [2],

$$ad_{t_i} : t_j \to ad_{t_i}(t_j) = [t_i, t_j\} \tag{94}$$

$$\Rightarrow [ad_{t_i}, ad_{t_j}\}(t_k) - (-1)^{ij}[ad_{t_j}, ad_{t_i}\}(t_k) = ad_{[t_i,t_j\}}(t_k) \tag{95}$$

$$\Rightarrow [ad_{t_i}, ad_{t_j}\} = ad_{[t_i,t_j\}} = iC_{ij}^k ad_{t_k} \tag{96}$$

In Hilbert space, only when two operators are all in the fermionic space L_1, an anticommutation relation would be obtained [2]. Otherwise, the commutator would be obtained for the presence of any bosonic operator [2].

$$[t_i, t_j\} = t_i t_j - (-1)^{ij} t_j t_i \tag{97}$$

For Hermitian generators, the structure constant has the following relation [2],

$$\bar{C}_{ij}^k = -C_{ji}^k \tag{98}$$

Denote bosonic generators as T_i and fermionic generators as Q_α satisfying the above three properties of the Lie superalgebras. Then, the interchanging relations between the bosonic fields and the fermionic fields can be concluded as in the following commutation/anticommutation relations [2]:

$$[T_i, Q_\alpha] = iC_{i\alpha}^\beta Q_\beta; \quad \{Q_\alpha, Q_\beta\} = iC_{\alpha\beta}^i T_i \tag{99}$$

The only remaining structure constants are therefore listed below correspondingly [2].

$$C_{ij}^{k} = -C_{ji}^{k}, \quad C_{i\alpha}^{\beta} = -C_{\alpha i}^{\beta}, \quad C_{\alpha\beta}^{i} = -C_{\beta\alpha}^{i} \tag{100}$$

Proposition 3.5. The supersymmetric-Jacobi identity in the adjoint representation are:

1. 3 generators in terms of 2 bosonic ones + 1 fermionic one[2]

$$C_{j\alpha}^{\beta} C_{i\beta}^{\gamma} + C_{ij}^{k} C_{\alpha k}^{\gamma} + C_{\alpha i}^{\beta} C_{j\beta}^{\gamma} = 0 \tag{101}$$

where $(C_i)_{\alpha}^{\beta}$ is the matrix element of the structure constant.

2. 3 generators in terms of 2 fermionic ones + 1 bosonic one[2]

$$C_{\beta\gamma}^{i} C_{\alpha i}^{\delta} + C_{\alpha\beta}^{i} C_{\gamma i}^{\delta} + C_{\gamma\alpha}^{i} C_{\beta i}^{\delta} = 0 \tag{102}$$

The adjoint representation for any group has the following property such that the structure constants satisfy [2]:

$$[C_i, C_j] = -C_{ij}^{k} C_k \tag{103}$$

Notably, the structure constant for the vector space L determines the nature of the algebra. In this case, the composition of two symmetric components in the fermionic subspace L_1 is still a symmetric term, unfulfilling the anti-symmetric property for the Lie algebras in general[2]. Then, the fermionic subspace L_0 cannot form a Lie algebra, whereas, the bosonic subspace L_0 forms one [2]. Thus, L does not form a Lie algebra because the antisymmetric property cannot be satisfied without restrictions.

As stated above, the subspace of the bosonic generators L_0 can form a Lie algebra with its antisymmetric properties fulfilled. Assume $L_0 = su(2)$, the representations read off [2],

$$(C_i)_\alpha^\beta = \frac{i}{2}(\sigma_i)_\alpha^\beta \tag{104}$$

$$\Rightarrow [T_i, Q_\alpha] = -\frac{1}{2}(\sigma_i Q)_\alpha \quad \{Q_\alpha, Q_\beta\} = i(\sigma_1\sigma_2)_{\alpha\beta}T_i \quad [T_i, T_j] = i\epsilon_{ijk}T_k \tag{105}$$

Subsequently, the generators can also be explicitly expressed in matrix form such that [2],

$$T_i = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\sigma_i \end{pmatrix} \quad Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad Q_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{106}$$

The interchange between the bosonic fields and the fermionic fields can be expressed mathematically in terms of the algebra of the above generators [2].

$$M = \begin{pmatrix} 0 & B \\ \epsilon B^T & D \end{pmatrix} \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{107}$$

where M is the Lie superalgebra of $su(2)$ in matrix form. D is a Hermitian 2×2 matrix connecting the fermionic subspace L_1 with fermionic subspace L_1, and B is a 2×1 matrix connecting the bosonic subspace L_0 from the fermionic subspace L_1 [2].

In short,

$$\begin{pmatrix} 0 & B \\ \epsilon B^T & 0 \end{pmatrix} \in L_1 \quad \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \in L_0 \tag{108}$$

The Lie superalgebra of $su(2)$ can therefore be concluded in the following commutation/anticommutation relations [2]:

Proposition 3.6.

$$[M_1, M_2\} = \begin{pmatrix} 0 & B_3 \\ \epsilon B_3^T & D_3 \end{pmatrix} \tag{109}$$

$$B_3 = B_{[1}D_{2]} \tag{110}$$

$$D_3 = [D_1, D_2] + \epsilon(B_{[1}^T B_{2]}) \tag{111}$$

where the property $\epsilon D = -D^T \epsilon$ is applied [2].

In addition, the s-Jacobi identity cannot be satisfied if the dimension of the Lie group becomes 3 [2].

3.3 $N = 1$ supersymmetry

As mentioned in the introduction, the C-M theory states that the maximally symmetric constitutions in the scattering matrix consists of the translation generators P_μ, Lorentz generator $M_{\mu\nu}$ and the internal symmetry generator (as well as a Lie group generator) B_l [2, 5, 7, 25].

The Lie superalgebra is the mathematical tool to extend the symmetry for the scattering matrix in C-M theory, in particular, the anticommutator is applied as well as the commonly used commutator for the generators of both the geometric symmetry groups and the internal symmetry groups [2, 5, 25].

As provided in Proposition 3.1, the commutation/anticommutation relations for the generators of the Poincare group ($M_{\mu\nu}$ and P_μ), a compact Lie group (B_l) and the Majorana spinors (Q_α) can be generalised in terms of the Weyl basis and its conjugate Q_A and $\bar{Q}_{\dot{A}}$ as [1, 2, 5, 8, 14, 15, 17]:

Proposition 3.7.

$$\{Q_A, Q_B\} = 0 \tag{112}$$

$$\{Q_A, \bar{Q}_{\dot{B}}\} = 2(\sigma^\mu)_{A\dot{B}} P_\mu \tag{113}$$

$$[Q_A, P_\mu] = 0 \tag{114}$$

$$[M_{\mu\nu}, Q_A] = \frac{1}{2}(\sigma_{\mu\nu})_A^B Q_B \tag{115}$$

$$[Q_A, B_l] = i S_l Q_A \tag{116}$$

$$[B_l, B_m] = i c_{lm}^k B_k \tag{117}$$

where S is a Hermitian matrix as the representation of the internal symmetry, c_{lm}^k is the structure constant of the corresponding lie group [5].

According to the Haag-Lopuszanski-Sohnius theorem (H-L-S theory hereafter) [25], as an extension of the C-M theory, the most general symmetry in the scattering matrix consist of a Lie group of any compact kind plus the Lie superalgebra relations concluded in the Proposition 3.7 [5, 25]. Furthermore, the commutators between the internal generators and the Poincare generators are zero[5].

Consider eq(116), the ($N = 1$, $N = 2$, ...) supersymmetry is governed by the dimensions of the representations of the S_l matrix [5]. In other words, the number i of the spinor charges $Q_{i\alpha}$ determines the dimensions of the supersymmetry [1].

The Casimir operators defined in the superalgebra commute with three types of the generators above, including the Lorentz generators, the translation generators and the Majorana charge [2, 5]. There are two different explicit expressions of the Casimir operators:

Proposition 3.8. The first Casimir operator is [2, 5],

$$[P^2, Q_a] = [P^2, P_\mu] = [P^2, M_{\mu\nu}] = 0 \tag{118}$$

$$P^2 = P^\mu P_\mu \tag{119}$$

The second Casimir operator is [2, 5],

$$[C^2, Q_a] = [C^2, P_\mu] = [C^2, M_{\mu\nu}] = 0 \tag{120}$$

where C^2 is defined as [2, 5],

$$C^2 = C^{\mu\nu}C_{\mu\nu} = (B^\mu P^\nu - B^\nu P^\mu)(B_\mu P_\nu - B_\nu P_\mu) \tag{121}$$

$$B_\mu = W_\mu + \frac{1}{4}X_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma + \frac{1}{4}(\frac{1}{2}\bar{Q}\gamma_\mu\gamma^5 Q) \tag{122}$$

In analogy to the quantum harmonic oscillator, the creation operator $\bar{Q}$ and the annihilation operator Q in supersymmetry accordingly govern the bosonic and fermionic super-multiplets which can be subsequently obtained after seeking the invariant Casimir operators [5, 14, 15].

Proposition 3.9.

$$\{Q_A, \bar{Q}_{\dot{B}}\} = 2(\sigma^\mu)_{A\dot{B}}P_\mu \quad (\sigma^\mu)_{A\dot{B}} = (1, \sigma^i) \tag{123}$$

The Hamiltonian operator can therefore be determined in supersymmetry as it is constructed by the creation/raising and annihilation/lowering operators [15]. As the time component of the momentum operator, the explicit form of the Hamiltonian is [15]:

Proposition 3.10.

$$H = P^0 = \frac{1}{4}(Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2) \tag{124}$$

In phenomenology, if supersymmetry is imposed in the standard model, then the operation of a spinor charge operator acting on a bosonic field will result in a fermionic field [15]. The connection between the bosons and fermions is therefore realised such that a group of boson and the corresponding fermion (or vice versa) is generalised as a super-partner pair [15]. The spinor charge is therefore the bridge to the super-partner particles which is inherited in the supersymmetric laws in cooperation with the geometric symmetry.

Without considering the supersymmetry spontaneous breaking pattern, the masses between the usual standard model particles and their super-partners should be the same, since the first Casimir operator P^2 commutes with the spinor charge and its conjugate [15]. The gauge transformation should also be the same accordingly, because the commutation relation between the spinor charges and the gauge operators is the same as the Casimir operator [15].

The number of the bosonic states is the same as the number of fermionic states [2, 5, 15]. Therefore, the number of bosons and fermions is identical even though the super-partner particles are involved [5, 15]. To see this, a new operator counting the numbers of fermion should be introduced such that [15],

Proposition 3.11.

$$(-1)^F \left| bosonic \ \ states \right\rangle = +1 \left| bosonic \ \ states \right\rangle \tag{125}$$

$$(-1)^F \left| fermionic \ \ states \right\rangle = +1 \left| fermionic \ \ states \right\rangle \tag{126}$$

$$\Rightarrow \{(-1)^F, Q_\alpha\} = 0 \tag{127}$$

Proof 3.12. The proof of the equal number theorem between the bosons and the fermions can be obtained by taking the trace of the following identity [15],

$$
\begin{aligned}
\Sigma_i \langle i| (-1)^F P^0 |i\rangle &= \frac{1}{4}(\Sigma_i \langle i| (-1)^F QQ^\dagger |i\rangle + \Sigma_i \langle i| (-1)^F Q^\dagger Q |i\rangle) \\
&= \frac{1}{4}(\Sigma_i \langle i| (-1)^F QQ^\dagger |i\rangle + \Sigma_{ij} \langle j| Q |i\rangle \langle i| (-1)^F Q^\dagger |j\rangle) \\
&= \frac{1}{4}(\Sigma_i \langle i| (-1)^F QQ^\dagger |i\rangle + \Sigma_i \langle j| Q(-1)^F Q^\dagger |j\rangle) \\
&= \frac{1}{4}(\Sigma_i \langle i| (-1)^F QQ^\dagger |i\rangle - \Sigma_i \langle j| (-1)^F QQ^\dagger |j\rangle) \\
&= 0
\end{aligned}
$$

where $P^0 \neq 0$. $Q^\dagger = \bar{Q}$.

Tracing out the above operator is also a technique used in discovering the index theorem in supersymmetry [5].

After defining the Hamiltonian and the number operators, the vacuum state with supersymmetry can be correspondingly derived as below [2, 5, 14, 15].

Proposition 3.13. In the absence of the supersymmetric spontaneously breaking, the supersymmetric vacuum is annihilated by the supersymmetric lowering operator [2, 5, 15]:

$$Q_\alpha |0\rangle = 0 \tag{128}$$

$$\Rightarrow \langle 0| H |0\rangle = 0 \tag{129}$$

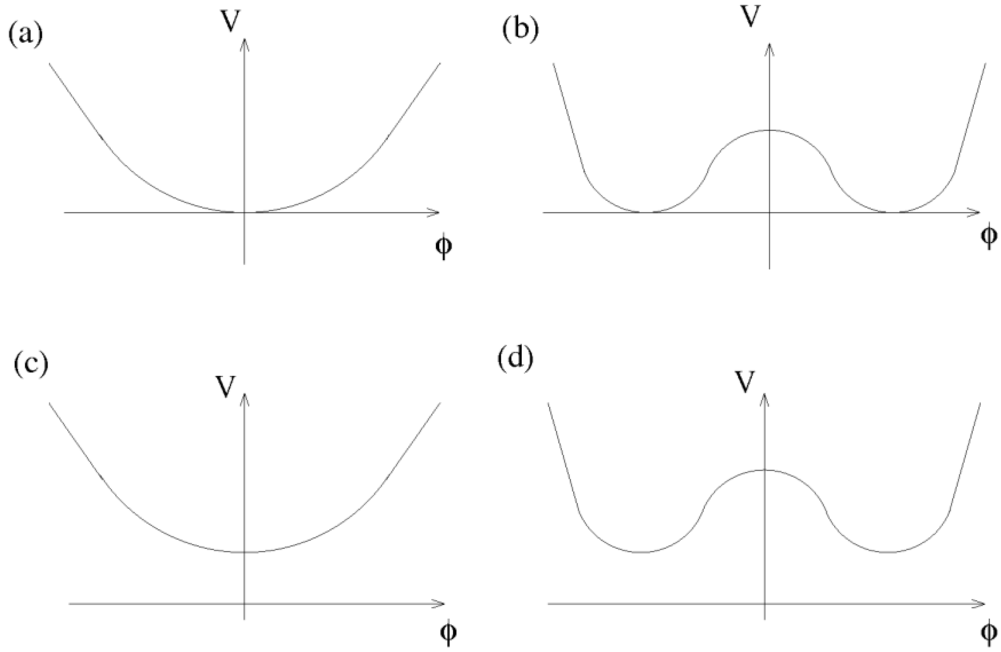

Figure 3: Supersymmetric spontaneously breaking [15]

Four probable supersymmetric symmetry broken patterns are illustrated above [15]. The figure 3 illustrates the cases where the supersymmetry and the internal symmetry are both not broken, only the internal symmetry is broken, only the supersymmetry is broken and the supersymmetry as well as the internal symmetry are all broken respectively [15].

The supersymmetric states, super-multiplets, are directly determined by the mass, spin and the spin in one direction [2, 5, 15]. A single super-multiplet quantum state can be represented as $|m, s, s_3\rangle$ [2, 5, 15].

The spinor charge and its conjugate anticommute with each other, giving a rest frame with the particle rest mass m, the momentum 4-vector is $p_\mu = (m, 0, 0, 0)$ [2, 5, 14, 15]. The only non-vanishing term is $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2m\delta_{\alpha\dot{\alpha}}$ [1, 2, 5, 8, 14, 15]. Then, the supersymmetric algebra stated in the Proposition 3.7 are simplified and regenerated to the Clifford algebra as demonstrated in the Definition 2.6 [15].

Since the quantum states of the supersymmetric particles are defined above in terms of several fundamental variables, then if the supersymmetric spontaneously breaking is not involved, the vacuum state with supersymmetry is generalised as $|\Omega_s\rangle$, which is often denoted as the "Clifford Vacuum" because of its algebraic property [2, 5, 15].

The supersymmetric vacuum is therefore zero after the annihilating operator acts on itself [2, 5, 15].

Proposition 3.14.

$$|\Omega_s\rangle = Q_1 Q_2 |m, s', s'_3\rangle \tag{130}$$

$$Q_1 |\Omega_s\rangle = Q_2 |\Omega_s\rangle = 0 \tag{131}$$

The complete super-multiplet quantum states as in analogy to the Fock's space in the quantum field theory can be therefore constructed in the following way [15]. Noticeably, the anticommutation relations of the spinor charge limit the ways of applying creation operators to the supersymmetric vacuum so that only the following remaining states survived [15].

$$\begin{aligned} &|\Omega_s\rangle \\ Q_1^\dagger |\Omega_s\rangle &, Q_2^\dagger |\Omega_s\rangle \\ Q_1^\dagger Q_2^\dagger &|\Omega_s\rangle \end{aligned}$$

Since the quantum number that differs the supersymmetric vacuum states is the spin as encoded in the subscripts s, suppose $s = 0$ in initial, the massive super-multiplets below indicates one Majorana fermionic field and one complex scalar field respectively [5, 15].

$$
\begin{array}{rc}
state & s_3 \\
\left|\Omega_0\right\rangle & 0 \\
Q_1^\dagger \left|\Omega_0\right\rangle, Q_2^\dagger \left|\Omega_0\right\rangle & \pm\dfrac{1}{2} \\
Q_1^\dagger Q_2^\dagger \left|\Omega_0\right\rangle & 0
\end{array}
$$

Then, suppose $s = \frac{1}{2}$, the massive super-multiplets now represent one real scalar field, one spin-1 vector field and two Majorana spinor fields [15].

$$
\begin{array}{rr}
state & s_3 \\
\left|\Omega_{\frac{1}{2}}\right\rangle & \pm\dfrac{1}{2} \\
Q_1^\dagger \left|\Omega_{\frac{1}{2}}\right\rangle, Q_2^\dagger \left|\Omega_{\frac{1}{2}}\right\rangle & 0, 1, 0, -1 \\
Q_1^\dagger Q_2^\dagger \left|\Omega_{\frac{1}{2}}\right\rangle & \pm\dfrac{1}{2}
\end{array}
$$

Thus, using the same idea, for a larger number of the spin in a specific supersymmetric vacuum, the resultant field interpretations would have super-multiplets with a larger spin value which is greater than 1 [15].

The massive cases are discussed above, determining the different field interpretations with different spins. For massless super-multiplets, the quantum states are specified in $|E, \lambda\rangle$ such that the energy E and the helicity λ of each particle form the complete basis [2, 5, 14, 15]. Choose the rest frame as in $p_\mu = (E, 0, 0, E)$ [5, 15]. Then the anticommutators between the spinor charges became the following algebraic relations [1, 2, 5, 8, 14, 15]:

Proposition 3.15. For massless super-multiplets [1, 2, 5, 8, 14, 15],

$$\{Q_1, Q_1^\dagger\} = 4E \tag{132}$$

$$\{Q_2, Q_2^\dagger\} = 0 \tag{133}$$

$$\{Q_\alpha, Q_\beta\} = 0 \tag{134}$$

$$\{Q_{\dot\alpha}^\dagger, Q_{\dot\beta}^\dagger\} = 0 \tag{135}$$

Because all the terms above vanished except for one creation operator, the vacuum state can be expressed in the following way [15]:

Proposition 3.16.

$$|\Omega_\lambda\rangle = Q_1 |E, \lambda'\rangle \tag{136}$$

$$Q_1 |\Omega_\lambda\rangle = 0 \tag{137}$$

Similarly, the super-multiplets only have two possible states for the massless particles [15],

$$\begin{array}{rc} \textit{state} & \lambda \\ |\Omega_\lambda\rangle & \lambda \\ Q_1^\dagger |\Omega_\lambda\rangle & \lambda + \dfrac{1}{2} \end{array}$$

To satisfy the C-P-T combined symmetries, the following conjugated super-multiplets also exist [15],

$$\begin{array}{cc} state & \lambda \\ \left|\Omega_{-\lambda-\frac{1}{2}}\right\rangle & -\lambda-\dfrac{1}{2} \\ Q_1^{\dagger}\left|\Omega_{-\lambda-\frac{1}{2}}\right\rangle & -\lambda \end{array}$$

The super-multiplets are therefore containing different values of the helicity in the massless particles case [15]. For instance, the original states and the conjugated states when $\lambda = 0$ and $\lambda = \frac{1}{2}$ are obtained similarly in accordance to the steps above [15].

$$\begin{array}{cc} original \ \ state & \lambda = 0 \\ \left|\Omega_o\right\rangle & 0 \\ Q_1^{\dagger}\left|\Omega_o\right\rangle & \dfrac{1}{2} \end{array}$$

$$\begin{array}{cc} conjugated \ \ state & \lambda = 0 \\ \left|\Omega_{-\frac{1}{2}}\right\rangle & -\dfrac{1}{2} \\ Q_1^{\dagger}\left|\Omega_{-\frac{1}{2}}\right\rangle & 0 \end{array}$$

$$\begin{array}{cc} original \ \ state & \lambda = \dfrac{1}{2} \\ \left|\Omega_{\frac{1}{2}}\right\rangle & \dfrac{1}{2} \\ Q_1^{\dagger}\left|\Omega_{\frac{1}{2}}\right\rangle & 1 \end{array}$$

$$\begin{aligned} conjugated\ \ state \ \ &\lambda = 0 \\ |\Omega_{-1}\rangle \qquad &-1 \\ Q_1^\dagger |\Omega_{-1}\rangle \qquad &-\frac{1}{2} \end{aligned}$$

Therefore, when the helicity is selected to be $\lambda = 0$, the interpretation of the supersymmetric states are constructed by one Weyl fermionic field plus one complex scalar field which differs from the $s = 0$ massive super-multiplets case by the nature of the fermions involved [15]. Moreover, when the helicity is chosen to be $\lambda = \frac{1}{2}$, the super-multiplets consist of one Weyl fermionic field plus one spin-1 gauge boson which appeared to be massless as opposed to massive case stated before [15].

The official names of the super-partners between bosons and fermions can be generalised below [1, 2, 5, 8, 14, 15],

Proposition 3.17.

$$\begin{aligned} fermion &\Rightarrow sfermion \\ quark &\Rightarrow squark \\ gauge\ \ boson &\Rightarrow gaugino \\ gluon &\Rightarrow gluino \end{aligned}$$

4 Extended Supersymmetry

4.1 Superalgebras and the Super-multiplets

For $N > 1$ supersymmetry, the subtlety is the number of the spinor charges varies to N. Then, the following algebra for the extended supersymmetry is generalised as [1, 2, 5, 8, 14, 15].

Proposition 4.1.

$$\{Q^a_\alpha, Q^\dagger_{\dot{\alpha}b}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^a_b \tag{138}$$

$$\{Q^a_\alpha, Q^b_\beta\} = 0 \tag{139}$$

$$\{Q^\dagger_{\dot{\alpha}a}, Q^\dagger_{\dot{\beta}b}\} = 0 \tag{140}$$

where $a, b = 1, ..., N$ [15].

The massless super-multiplets in the extended supersymmetry has the same conditions as in the $N = 1$ supersymmetry such that [2, 5, 15]:

Proposition 4.2.

$$\{Q^a_1, Q^\dagger_{1b}\} = 4E\delta^a_b \tag{141}$$

$$\{Q^a_2, Q^\dagger_{2b}\} = 0 \tag{142}$$

The table for the super-multiplet states are concluded below [15],

state	*helicity*	*degeneracy*
$\lvert\Omega_\lambda\rangle$	λ	1
$Q^\dagger_{1a}\lvert\Omega_\lambda\rangle$	$\lambda+\frac{1}{2}$	N
$Q^\dagger_{1a}Q^\dagger_{1b}\lvert\Omega_\lambda\rangle$	$\lambda+1$	$\frac{N(N-1)}{2}$
...	...	...

Limiting the values of the helicity for the massless particles and the values of spin for the massive particles in the range of $|\lambda| \leq 1$ as well as $\left|\lambda + \frac{N}{2}\right| \leq 1$ [15]. Then, the number of the spinor charges is required as $N \leq 4$ [15].

Take the $N = 2$ supersymmetry to start with, the super-multiplets states are therefore [15],

state	*helicity*	*degeneracy*
$\lvert\Omega_{-1}\rangle$	-1	1
$Q^\dagger\lvert\Omega_{-1}\rangle$	$-\frac{1}{2}$	2
$Q^\dagger Q^\dagger\lvert\Omega_{-1}\rangle$	0	1

conjugated state	*helicity*	*degeneracy*
$\lvert\Omega_0\rangle$	0	1
$Q^\dagger\lvert\Omega_0\rangle$	$\frac{1}{2}$	2
$Q^\dagger Q^\dagger\lvert\Omega_0\rangle$	1	1

Noticeably, the above extended super-multiplet is based on the $N = 1$ super-multiplets with corresponding helicities $\lambda = 0$ and $\lambda = \frac{1}{2}$ [15].

The hyper-super-multiplets are defined with the helicity $\lambda = -\frac{1}{2}$ [15].

state	*helicity*	*degeneracy*
$\left\|\Omega_{-\frac{1}{2}}\right\rangle$	$-\dfrac{1}{2}$	1
$Q^\dagger \left\|\Omega_{-\frac{1}{2}}\right\rangle$	0	2
$Q^\dagger Q^\dagger \left\|\Omega_{-\frac{1}{2}}\right\rangle$	$\dfrac{1}{2}$	1

The first entry of the above hyper-multiplets table corresponds to χ_α, the second entry of the the states is ϕ and the last state above is labelled as $\psi^{\dagger\dot{\alpha}}$ [15]. χ_α and $\psi^{\dagger\dot{\alpha}}$ are fermionic fields where the gauge transformation should be invariant in terms of $\psi^\alpha\chi_\alpha$ [15].

4.2 Central Charges

The algebra for the central charges is developed based on the extended super-symmetric algebra such that the anticommuting components are not merely zero [5, 14, 15, 25].

Proposition 4.3.

$$\{Q^a_\alpha, Q^\dagger_{\dot{\alpha}b}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^a_b \tag{143}$$

$$\{Q^a_\alpha, Q^b_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta} Z^{ab} \tag{144}$$

$$\{Q^\dagger_{\dot{\alpha}a}, Q^\dagger_{\dot{\beta}b}\} = 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}} Z^*_{ab} \tag{145}$$

where $\epsilon = i\sigma^2$ [15].

For a special case when $N = 2$, the above relations reduce to the following ones [15, 25]:

Proposition 4.4.

$$\{Q^a_\alpha, Q^\dagger_{\dot{\alpha}b}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^a_b \tag{146}$$

$$\{Q^a_\alpha, Q^b_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab} Z \tag{147}$$

$$\{Q^\dagger_{\dot{\alpha}a}, Q^\dagger_{\dot{\beta}b}\} = 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{ab} Z \tag{148}$$

where $\epsilon = i\sigma^2$ [15].

Then, define the following operators to simplify the anticommutators [15],

$$A_\alpha = \frac{1}{2}[Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger] \tag{149}$$

$$B_\alpha = \frac{1}{2}[Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger] \tag{150}$$

$$\Rightarrow \{A_\alpha, A^\dagger_\beta\} = \delta_{\alpha\beta}(M + \sqrt{2}Z) \tag{151}$$

$$\Rightarrow \{B_\alpha, B^\dagger_\beta\} = \delta_{\alpha\beta}(M - \sqrt{2}Z) \tag{152}$$

The variables on the right hand side of the above simplified anticommutators are the super-multiplet mass M and the central charge Z which form a complete basis for the quantum states of the super-multiplets as $|M, Z\rangle$ [15].

Short and long super-multiplets are discussed in the context of existence of the central charges [15].

For the $s = 0$ case [15],

$state$	$2j+1$
$\|\Omega_0\rangle$	0
$A^\dagger \|\Omega_0\rangle$	2
$(A^\dagger)^2 \|\Omega_0\rangle$	1

For the $s = \frac{1}{2}$ case [15],

$state$	$2j+1$
$\left\|\Omega_{\frac{1}{2}}\right\rangle$	2
$A^\dagger \left\|\Omega_{\frac{1}{2}}\right\rangle$	$1+3$
$(A^\dagger)^2 \left\|\Omega_{\frac{1}{2}}\right\rangle$	2

The above two tables of super-multiplets appeared to be the same as the corresponding $N = 1$ massive super-multiplets states [15]. Moreover, the quantities of the super-multiplets are also the same as the corresponding $N = 2$ massless super-multiplets [15]. Each short-super-multiplets has the state quantities four times less than the long-super-multiplets in the above example [15].

5 Conclusion

In this essay, the topics of the historic motivations for the development of the supersymmetry, the geometric symmetry, the internal symmetry and ultimately the supersymmetry are discussed respectively. The pavement to the supersymmetry is constructed by the mathematical building blocks. There are some aspects of the advanced topics that is not included in the essay, such as the physical examples of the Anti-de-Sitter superalgebra, the field theory incorporate with the supersymmetry as well as the supersymmetric theory in a curved spacetime, etc. The author shall endeavour to continue learning the fundamentals of those topics. The mathematical realisations of some useful theorems are worth reading in the future.

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