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Department of Physics

Generalized Geometry in Type II Supergravity: A Review

Author:
Yaocheng Li

Supervisor:
Professor Daniel Waldram

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Abstract

This paper reviews the reformulation of Type II supergravity theory as a generalized geometry in mathematical details. The generalized tangent bundle is studied as an exact Courant algebroid. Generalized differential geometry is studied on the Courant algebroid extended by the density bundle $\det T^*$, with $O(d,d) \times \mathbb{R}^+$ structure. The mathematical structures of the generalized geometric objects are reviewed. These includes generalized version of the Levi-Civita connection. With the generalized geometric tools developed, the Type II theories are finally formulated in a $Spin(9,1) \times Spin(1,9)$-covariant form as a generalized analogue of the Einstein theory.

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Chapter 1

Introduction

In the physical point of view, the generalized geometry is the geometry for supergravities, which are commonly regarded as low-energy limits of the string theory. As a geometric theory, generalized geometry studies the generalized structures [1, 2, 3], which are extensions of the structure of tangent bundles. For Type II supergravity theories, the generalized structure is $E \simeq T \oplus T^*$ [4], where local diffeomorphism symmetry is extended with NS-NS gauge symmetry, and structural group becomes $O(d,d)$. It turns out that differential geometry with generalized structures are in close relation with supergravity theories.

In this dissertation we reviews the mathematical structures of the generalized geometry for Type II supergravities, that is, $d = 10$ IIA and IIB supergravity. The Type II theories will be reformulated as the theory in generalized geometric formalism. Such theory will have manifested local $Spin(9,1) \times Spin(1,9)$ symmetry, analogous to ordinary gravity theories. Generalized geometric objects are usually constructed in analogues of conventional ones. Particular interesting object is the generalized Levi-Civita connection, which leads to generalized curvatures that build up the Type II theory resembling the Einstein gravity. The generalized version of Type II theories also admit supersymmetry. To be more specific, to the leading order of the fermions, it captures both bosonic and fermionic actions, equations of motion, and the supersymmetry variations in simple forms by virtue of the generalized connection.

This review follows canonically [4], with more details in the derivations. Mathematics are introduced more extensively, but proofs are all presented. The description of the Type II supergravity theories is in democratic formalism [5]. It provides a unified treatment for IIA and IIB theories, and is more natural in connection with string theory.

The structure of this paper is as follow. We start with the bosonic symmetries of the Type II theory, which largely motivate the construction of generalized tangent bundle. The generalized tangent bundle is an exact Courant algebroid [6], and its basic structure and the general properties are studies in Chapter 2. Then we develop the generalized geometry on the Courant algebroid. We will introduce density $\det T^*$ to describe the dilaton in Type II theories, and this leads to $O(d,d) \times \mathbb{R}^+$ structure. The generalized analogues are introduced, for example, tensors, Lie derivative, connections, and torsion. Tensorial curvature arises after reviewing the generalized version of the metric and Levi-Civita connnetion. The corresponding structure is $O(p,q) \times O(q,p)$. Finally, after completing the Type II supergravities with fermions, we establish the full generalized geometric formalism.

Finally, some comments on the notations; cf. D. If P is a fiber bundle, we may use P denotes both manifold $p : P \to M$, and sections of p over open subsets. For example, $\omega \in \Omega^2$ denotes a two-form defined over some open set, while global sections will be emphasized by $\omega \in \Gamma\Omega^2$. The same convention applies to $C^\infty(G)$, which denotes the smooth G-valued functions on M. In the case $G = \mathbb{R}$, we simply write C^∞. T, T^* by default are tangent and cotangent bundles. Usually, morphisms are well-defined for both local and global sections.

Chapter 2

Generalized Tangent Bundle

Generalized geometry [3] is the study of structures on a *generalized tangent bundle* E. It exhibits a close relation to supergravity theories. In the case of Type II supergravity, we are interested in $E \simeq T \oplus T^*$, which admits many structures making it a *Courant Algebroid*. Before studying its properties, it will be motivating to review the symmetries in Type II supergravities that lead to a generalized structure. We will be in democratic formalism [5]; conventions are summarized in A.

2.1 Bosonic Symmetries in Type II Supergravity

2.1.1 The Bosonic Sector

Type II fields in bosonic sector are

• $g_{\mu\nu}$	NS-NS	space-time metric
• $B_{\mu\nu}$	NS-NS	two-form potential
• ϕ	NS-NS	dilaton
• $A^{(n)}_{\mu_1 \dots \mu_n}$	R-R	n-form potenitials

where n odd for type IIA and even for type IIB. Explicitly,

$$
\begin{aligned}
&\text{IIA:} \quad \{g_{\mu\nu}, B_{\mu\nu}, \phi, A^{(1)}_{\mu_1}, A^{(3)}_{\mu_1\mu_2\mu_3}, A^{(5)}_{\mu_1\dots\mu_5}, A^{(7)}_{\mu_1\dots\mu_7}, A^{(9)}_{\mu_1\dots\mu_9}\} \\
&\text{IIB:} \quad \{g_{\mu\nu}, B_{\mu\nu}, \phi, A^{(0)}, A^{(2)}_{\mu_1\mu_2}, A^{(4)}_{\mu_1\dots\mu_4}, A^{(6)}_{\mu_1\dots\mu_6}, A^{(8)}_{\mu_1\dots\mu_8}\}
\end{aligned} .
$$

The bosonic pseudo-action has form

$$S_B = \frac{1}{2\kappa^2}\int\sqrt{-g}\left[e^{-2\phi}\left(\mathcal{R}+4(\partial\phi)^2-\frac{1}{12}H^2\right)-\frac{1}{4}\sum_n\frac{1}{n!}\left(F_{(B)}^{(n)}\right)^2\right], \tag{2.1}$$

where $H = dB$ and $F_{(B)}^{(n)}$ are the n-form R-R field strengths. Type II theories have gauge symmetries: B and $A^{(n)}$ are locally defined gauge fields. It is convenient to define the sum

$$A = \sum_{n \text{ even or odd}} A^{(n)}, \quad A \in \Lambda^{\bullet}T^*. \tag{2.2}$$

The gauge transformations take form

$$B \mapsto B + d\lambda, \qquad A \mapsto A + d\hat{\Lambda} - A\wedge d\hat{\Lambda} - m\hat{\Lambda}, \tag{2.3}$$

where $\lambda \in \Omega^1$ and $\hat{\Lambda} = \sum_n \hat{\Lambda}^{(n)} \in \Omega^{\bullet}$, n even for IIA, odd for IIB, are local forms. m is the constant mass parameter for IIA; $m = 0$ for IIB. We are using "A-basis"; cf. (A.8). The field strengths are gauge invariant and globally defined as

$$\begin{aligned} H &= dB, \\ F_{(B)} &= \sum_n F_{(B)}^{(n)} = e^B\wedge(dA+m), \end{aligned} \tag{2.4}$$

where $e^B = 1 + B + \frac{1}{2}B\wedge B + \dots$. The action is pseudo, because we have to impose by hand the self-duality relation

$$F_{(B)}^{(n)} = (-1)^{[n/2]} * F_{(B)}^{(10-n)}, \tag{2.5}$$

where $[n]$ denotes the integer part, $*$ is the Hodge dual operator. This does not follow from the equations of motion:

$$\begin{aligned} 0 &= \mathcal{R}_{\mu\nu} - \frac{1}{4}H_{\mu\alpha\beta}H_{\nu}{}^{\alpha\beta} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}e^{2\phi}\sum_n\frac{1}{(n-1)!}F_{(B)}^{(n)}{}_{\mu\alpha_1\dots\alpha_{n-1}}F_{(B)}^{(n)}{}_{\nu}{}^{\alpha_1\dots\alpha_{n-1}}, \\ 0 &= \nabla^\alpha\left(e^{-2\phi}H_{\alpha\mu\nu}\right) - \frac{1}{2}\sum_n\frac{1}{(n-2)!}F_{(B)}^{(n)}{}_{\mu\nu\alpha_1\dots\alpha_{n-2}}F_{(B)}^{(n-2)\alpha_1\dots\alpha_{n-2}}, \\ 0 &= \nabla^2\phi - (\nabla\phi)^2 + \frac{1}{4}\mathcal{R} - \frac{1}{48}H^2, \\ 0 &= dF_{(B)} - H\wedge F_{(B)}, \end{aligned} \tag{2.6}$$

which correspond to variations $\delta g_{\mu\nu}, \delta B_{\mu\nu}, \delta\phi$ and the Bianchi identity for $F_{(B)}$ by its definition (2.4), respectively, and ∇ is the Levi-Civita connectoin. Note that these equations are put in the form retrieving string β-functions [7] after setting R-R fields to zero.

We will introduce the fermions with supersymmetry in 4. For now we shall focus on the NS-NS gauge symmetries, and see how it "generalizes" infinitesimal diffeomorphisms, i.e. vector fields.

2.1.2 Gauge Symmetry

NS-NS sector has a gauge symmetry associated with B indicated in (2.3). As an analogue, in electrodynamics, the global structure the Maxwell A_μ is defined by connections on $U(1)$-principal bundles over the spacetime M, and a gauge is described by a local section. Similarly, associated to two-form potentials B is a much larger space define over M, called *gerbe* [2, 1, 8, 9]. It can be considered as a receptacle for the topological obstruction to construct a desired fiber bundle due to the degree 3 cohomology. The relevant gerbes will have gauge group $U(1)$, since B is abelian and the string theory implies that flux H is quantized [10].

The field B encodes a connective structure along with a curving on a $U(1)$-gerbe. Given a good covering $\{U_i\}$ over M, the field B is defined by two-forms $B_{(i)} \in \Omega^2(U_i)$ patched on $U_i \cap U_j$ via

$$B_{(i)} = B_{(j)} - d\Lambda_{(ij)}, \tag{2.7}$$

where one-forms $\Lambda_{(ij)}$ satisfy

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = d\Lambda_{(ijk)}, \tag{2.8}$$

on $U_i \cap U_j \cap U_k$, where $\Lambda_{(ijk)} \in C^\infty(U(1))$. Fix a covering $\{U_i\}$, $U(1)$-valued functions $\{\Lambda_{(ijk)}\}$ defines the *gerbe*[1], one-forms $\{\Lambda_{(ij)}\}$ with (2.8) defines the *connective structure*, and local two-forms $\{B_{(i)}\}$ defines the *curving*. Globally defined three-form field strength $H = dB$ is the 3-*curvature*. A data of B field (2.7) including both the connective structure and curving often refers to the "connection" on the gerbe $\{\Lambda_{(ijk)}\}$. The relation between the notion presented here and the general one is summarized in C.

Note that by (A.9), the "A-basis" has a similar patching over $\{U_i\}$. In terms of A defined

[1]The functions $g_{ijk} = exp\big(i\Lambda_{(ijk)}\big)$ are required to satisfy 2-cocycle condition $g_{jkl}g_{ikl}^{-1}g_{ijl}g_{ijk}^{-1} = 1$ on $U_i \cap U_j \cap U_k \cap U_l$.

in (2.2),

$$A_{(i)} = e^{d\Lambda_{(ij)}} \wedge A_{(j)} + d\hat{\Lambda}_{(ij)}, \tag{2.9}$$

where $\hat{\Lambda}_{(ij)}$ is a sum of even forms in IIA, odd in IIB.

Over $\{U_i\}$, the local gauge symmetry takes form[2]

$$B_{(i)} \mapsto B_{(i)} - d\lambda_{(i)}, \qquad A_{(i)} \mapsto e^{d\lambda_{(i)}} \wedge A_{(i)}. \tag{2.10}$$

The compatibility condition of the connective structure requires $d\lambda_{(i)} = d\lambda_{(j)}$ on $U_i \cap U_j$. Thus $\{d\lambda_{(i)}\}$ is equivalent to a globally defined two-form $\omega \in \Omega^2(M)$, such that $d\omega = 0$, $\omega|_{U_i} = d\lambda_{(i)}$. In another word, the group of gauge symmetries is the abelian group of closed two-forms $\Omega^2_{\text{closed}}(M)$.

2.1.3 NS-NS Bosonic Symmetries

As for all gravity theories, supergravity has *diffeomorphism invariance*. The group is denoted by Diff(M). This is an infinite-dimensional Lie group, whose (infinite-dimensional) Lie algebra is the space of vector fields $T(M)$, where the Lie bracket is the ordinary Lie bracket between vector fields [11].

The combination of the two symmetries in NS-NS sector, Diff(M) and $\Omega^2_{\text{closed}}(M)$, forms the total symmetry group G_{NS}. The gauge transofrmations and diffeomorphisms do not commute, so the group only has semi-direct product structure $G_{\text{NS}} = \text{Diff}(M) \ltimes \Omega^2_{closed}(M)$. The fibered structure in terms of exact squence of groups is

$$1 \longrightarrow \Omega^2_{\text{closed}}(M) \longrightarrow G_{\text{NS}} \longrightarrow \text{Diff}(M) \longrightarrow 1. \tag{2.11}$$

Note that we can identify $\Omega^2_{\text{closed}}(M)$ with its tangent spaces by linearity, so the tangent space of group G_{NS} at the identity is $\Gamma(T \oplus \Omega^2_{\text{closed}})$. However, the group structure, thus its Lie algebra, is not as trivial as its fibered structure. For example, $v+\omega \in \Gamma(T \oplus \Omega^2_{\text{closed}})$ will not lead to the general combined symmetry one would expect. We shall see this now.

Let v be a vector field and $\{\lambda_{(i)}\}$ a set of one-forms defined over $\{U_i\}$ and consider a

[2]there is a choice of sign in the gauge transform in order to match the generalized geometry conventions.

general variation combining both diffeomorphism and gauge transformation

$$\delta_{v+\lambda} g = \mathcal{L}_v g, \qquad \delta_{v+\lambda}\phi = \mathcal{L}_v \phi, \qquad \delta_{v+\lambda} B_{(i)} = \mathcal{L}_v B_{(i)} - d\lambda_{(i)}. \tag{2.12}$$

The variation of a curving $\{B_{(i)}\}$ on a gerbe is a globally defined 2-form on M [8]; this is analogous to the fact that variation of a connection is an one-form. Therefore the patching (2.7) implies that $\{\lambda_{(i)}\}$ has to be patched as

$$d\lambda_{(i)} = d\lambda_{(j)} - \mathcal{L}_v d\Lambda_{(ij)}. \tag{2.13}$$

In particular, $\{d\lambda_{(i)}\}$ in a general variation (2.12) does not define an element in $\Omega^2_{\text{closed}}(M)$. Thus we can hardly identify the structure of the infinitesimal variations in the representation $\Gamma(T \oplus \Omega^2_{\text{closed}})$. This could be a motivation for an alternative description of the symmetries.

We further note that since $\lambda_{(i)} + df_{(i)}$ defines the same gauge transformation to $\lambda_{(i)}$, and using the celebrated Cartan formula $\mathcal{L}_v = i_v d + d i_v$, patching (2.13) is equivalent to

$$\lambda_{(i)} = \lambda_{(j)} - i_v d\Lambda_{(ij)} \tag{2.14}$$

on $U_i \cap U_j$. We are ready to construct the generalized tangent bundle.

2.2 Courant Algebroid

Each $U(1)$-gerbe with a connective structure defines a *Courant algebroid*, hence generalized tangent spaces. This is similar to the fact that each principal bundle defines an Atiyah algebroid $A = TP/G \simeq T \oplus \operatorname{ad} P$ [12], and a connection there plays the role of the curving $\{B_{(i)}\}$ here, which we will turn to in 2.2.3. In fact, Courant Algebroid can be viewed as a higher-degree generalization of Lie algebroid [3].

2.2.1 The Constructions

We have seen the general infinitesimal variation of the bosonic fields in (2.12) by a combination of vector and local one-forms; they are expected to be our *generalized tangents*, as an additional "one-form direction" at each point.

Let M be a manifold, T, T^* its tangent and cotangent bundle respectively. Given a connective structure on a $U(1)$-gerbe over M, there is an open cover $\{U_i\}$ such that B

is patched as in (2.7). On each patch U_i, we extend the vectors $v_{(i)} \in TU_i$ by one-forms $\lambda_{(i)} \in T^*U_i$ as

$$0 \longrightarrow T^*U_i \longrightarrow EU_i := TU_i \oplus T^*U_i \longrightarrow TU_i \longrightarrow 0, \tag{2.15}$$

so $V_{(i)} = v_{(i)} + \lambda_{(i)}$ defines a section of EU_i. We then patch EU_i according to (2.14)

$$v_{(i)} + \lambda_{(i)} = v_{(j)} + \left(\lambda_{(j)} - i_{v_{(j)}} d\Lambda_{(ij)}\right) \tag{2.16}$$

on $U_i \cap U_j$. Sequences (2.15) descend to the exact sequence of vector bundles

$$0 \longrightarrow T^* \longrightarrow E \longrightarrow T \longrightarrow 0, \tag{2.17}$$

where the vector bundle E is the *generalized tangent bundle.*

This is well-defined. Indeed, by definition, $v_{(i)} = v|_{U_i}$ for some $v \in \Gamma T$, while one-forms do not have a global amalgamation. The connective structure (2.8) implies the cocycle condition (B.1) by

$$i_v\left(d\Lambda_{(ij)} + d\Lambda_{(jk)} + d\Lambda_{(ki)}\right) = i_v\left(d^2\Lambda_{(ijk)}\right) = 0.$$

Remark 2.1. Note that the sequence (2.15) furnishes local coordinates for E, namely those of $TU_i \oplus T^*U_i$. Any other choice of $\{U_i\}$ with gauges over each neighbourhood will only lead to a different coordinates.

Remark 2.2. However, in the definition of E we have used the connective structure (2.8). Given another connective structure $\{\Lambda'_{(ij)}\}$, their difference $\{\Lambda'_{(ij)} - \Lambda_{(ij)}\}$ satisfies the cocycle condition (B.1), hence defines a Ω^1-bundle, as an affine bundle of all principal $U(1)$-connections. This implies E will be twisted by such a Ω^1-bundle. However, this "twist" is more or less a shearing: all so-constructed generalized tangent bundles are isomorphic to $T \oplus T^*$ [3], even as a Courant algebroid. This symmetry will be reviewed in 2.2.3.

Remark 2.3. The construction of E does not depend on the specific content about $\{B_{(i)}\}$ for a given connective structure. We will return to them in 2.2.3 too.

Importantly, E admits canonically a $O(d,d)$ metric $\langle \cdot, \cdot \rangle$ defined by the quadratic form

$$\langle v + \lambda, v + \lambda \rangle = i_v \lambda. \tag{2.18}$$

[3] T^* means the same thing as Ω^1

This is well-defined since $i_{v_{(i)}}\lambda_{(i)} = i_{v_{(j)}}\lambda_{(j)} - i_v^2 d\Lambda_{(ij)} = i_{v_{(j)}}\lambda_{(j)}$. Corresponding symmetric form $\langle\cdot,\cdot\rangle : E\otimes E \to C^\infty(\mathbb{R})$ is

$$\langle V, W\rangle = \frac{1}{2}(\langle V+W, V+W\rangle - \langle V,V\rangle - \langle W,W\rangle). \tag{2.19}$$

This metric yields an $O(d,d)$-structure F for vector bundle E as (B.7).

The split signature of $\langle\cdot,\cdot\rangle$ has some interesting consequences. By the classification of spinors [13], $Spin(d,d)$ has two real-type (Majorana) spin representations dual to each other if d odd, self-dual of d even. Reality of the spin representations allows the standard exterior algebra models for complex spinors [14] in real scalar. They turn out to be differential forms in this case.

Explicitly, on each coordinate neighbourhood U_i, the forms $S_i := \Lambda^\bullet T^*U_i$ can be made into a Clifford module by $\Gamma_i : EU_i \otimes S_i \to S_i$,

$$\Gamma(v+\lambda)\Psi = i_v\Psi + \lambda\wedge\Psi, \tag{2.20}$$

so for $V, W \in EU_i$,

$$\{\Gamma(V), \Gamma(W)\} = 2\langle V, W\rangle, \tag{2.21}$$

since $i_v \circ \lambda \wedge + \lambda \wedge \circ i_v = i_v\lambda$ on $\Omega^\bullet$. As $\langle\cdot,\cdot\rangle$ is compatible among patches, S_i over U_i can be patched by Clifford Module isomorphisms

$$\Psi_{(i)} = e^{d\Lambda_{(ij)}} \wedge \Psi_{(j)} \tag{2.22}$$

on U_{ij}, and thus define a $Spin(d,d)$ spinor bundle (S,Γ) over M. Truly,

$$\begin{aligned}\Gamma\big((v_{(i)}+\lambda_{(i)}\big)\Psi_{(i)} &= i_v\Psi_{(i)} + \lambda_{(i)}\wedge\Psi_{(i)} \\ &= i_v e^{d\Lambda_{(ij)}}\wedge\Psi_{(j)} + e^{d\Lambda_{(ij)}}\wedge i_v\Psi_{(j)} + (\lambda_{(j)} - i_v d\Lambda_{(ij)})e^{d\Lambda_{(ij)}}\wedge\Psi_{(j)} \\ &= e^{d\Lambda_{(ij)}}\wedge\big(i_v\Psi_{(j)} + \lambda_{(j)}\Psi_{(j)}\big) \\ &= e^{d\Lambda_{(ij)}}\wedge\Gamma\big((v_{(j)}+\lambda_{(j)}\big)\Psi_{(j)}.\end{aligned}$$

This defines the Clifford module S, and it becomes $Spin(d,d)$-spin bundle if we tensor a line bundle $S\otimes(\det T^*)^{-1/2}$. (d,d) signature always have even dimension, so S decomposes into chiral spinors $S\otimes(\det T^*)^{-1/2} = S^+\otimes(\det T^*)^{-1/2} \oplus S^-\otimes(\det T^*)^{-1/2}$, defining two $Spin(d,d)$ spinor bundles. Locally, $S_i = S_i^+ \oplus S_i^-$ corresponds to decomposition into even and odd degree forms $\Lambda^\bullet T^*U_i = (\Lambda^\bullet T^*U_i)^0 \oplus (\Lambda^\bullet T^*U_i)^1$. We will see, up to an isomorphism, this is also true globally.

The $Spin(d,d)$-invariant spinor pairing also extends to S since transition functions are morphisms. It is given by the Mukai pairing [3]: $(\cdot,\cdot) : S^+ \otimes S^- \to \Lambda^d T^*$ for d odd, $(\cdot,\cdot) : S^\pm \otimes S^\pm \to \Lambda^d T^*$ for d even as

$$(\Psi, \Psi') = \sum_n (-1)^{[(n+1)/2]} \Psi^{(d-n)} \wedge \Psi'^{(n)}, \tag{2.23}$$

so $(\cdot,\cdot) : S^+ \otimes (\det T^*)^{-1/2} \otimes S^- \otimes (\det T^*)^{-1/2} \to C^\infty$.

Finally, the differential operator $d : \Lambda^\bullet T^* \to \Lambda^\bullet T^*$ also induces a well-defined map on S,

$$\begin{aligned} d : S^\pm &\longrightarrow S^\mp, \\ \Psi &\longmapsto d\Psi, \end{aligned} \tag{2.24}$$

because $d\Psi_{(i)} = d\big(e^{d\Lambda_{(ij)}} \wedge \Psi_{(j)}\big) = e^{d\Lambda_{(ij)}} \wedge d\Psi_{(j)}$.

2.2.2 Derived Bracket

Less obvious canonical structure on E is a bracket.

Note that Γ defines an embedding into the graded Lie algebra $\Gamma : E \to (\mathrm{End}(S), [\cdot,\cdot])$, where $[\cdot,\cdot]$ is the graded commutator, and the grading is given by the chirality. The image of E under Γ will not be closed under $[\cdot,\cdot]$, but will be closed under a *derived bracket* [15]. This bracket $[\cdot,\cdot]_d : \mathrm{End}(S) \otimes_{\mathbb{R}} \mathrm{End}(S) \to \mathrm{End}(S)$ is defined by

$$[X, Y]_d = [[X, d], Y], \tag{2.25}$$

where $d \in \mathrm{End}(S)$ is just defined in (2.24). It is no longer a graded Lie bracket, but will be a *graded Leibniz algebra*, or *Loday algebra* [15], on $\mathrm{End}(S)$. Induced bilinear map on E will satisfy the Jacobi identity, but is not antisymmetric.

Proposition 2.1. On the image of E in $\mathrm{End}(S)$,

$$[v + \lambda, w + \xi]_d = [v, w] + \mathcal{L}_v \xi - i_w d\lambda, \tag{2.26}$$

where we identify E with its image, $v + \lambda, w + \xi \in E$, and $[v, w]$ is the Lie bracket between vector fields v, w.

Proof. Let's first check that the expression (2.26) is well-defined on E. On U_{ij},

$$\begin{aligned}[v,w]+\mathcal{L}_v\xi_{(i)}-i_w d\lambda_{(i)} &= [v,w]+\mathcal{L}_v(\xi_{(j)}-i_w d\Lambda_{(ij)}) \\ &\quad - i_w d(\lambda_{(j)}-i_v d\Lambda_{(ij)}) \\ &= [v,w]+\mathcal{L}_v\xi_{(j)}-i_w d\lambda_{(j)} \\ &\quad - \big(\mathcal{L}_v i_w d\Lambda_{(ij)} - i_w d(i_v d\Lambda_{(ij)})\big),\end{aligned}$$

where, by $\mathcal{L}_v = i_v d + d i_v$, $i_{[v,w]} = \mathcal{L}_v i_w - i_w \mathcal{L}_v$,

$$\begin{aligned}\mathcal{L}_v i_w d\Lambda_{(ij)} - i_w d i_v d\Lambda_{(ij)} &= i_{[v,w]} d\Lambda_{(ij)} + i_w \mathcal{L}_v d\Lambda_{(ij)} \\ &\quad - i_w \mathcal{L}_v d\Lambda_{(ij)} + i_w i_v d^2\Lambda_{(ij)} \\ &= i_{[v,w]} d\Lambda_{(ij)},\end{aligned}$$

so

$$[v,w]+\mathcal{L}_v\xi_{(i)}-i_w d\lambda_{(i)} = [v,w]+\big(\mathcal{L}_v\xi_{(j)}-i_w d\lambda_{(j)}-i_{[v,w]}d\Lambda_{(ij)}\big).$$

Therefore, it suffices to consider the local product structure (2.15).

Next, recall the definition (2.20), bracket (2.25) between forms and vectors can be easily obtained using Cartan formulae and $[\xi\wedge, d] = d\xi\wedge$ as

$$\begin{aligned}[v,w]_d &= [[i_v,d],i_w] = [\mathcal{L}_v,i_w] = i_{[v,w]} = \Gamma([v,w]), \\ [v,\xi\wedge]_d &= \mathcal{L}_v\circ\xi\wedge - \xi\wedge\circ\mathcal{L}_v = \mathcal{L}_v\xi\wedge = \Gamma(\mathcal{L}_v\xi), \\ [\lambda\wedge,v]_d &= [[\lambda\wedge,d],i_v] = [d\lambda\wedge,i_v] = -i_v d\lambda\wedge = \Gamma(-i_v d\lambda), \\ [\lambda\wedge,\xi\wedge]_d &= [d\lambda\wedge,\xi\wedge] = 0,\end{aligned}$$

where $[\cdot,\cdot]$ is the graded commutator in $\mathrm{End}(S)$. (2.26) follows from linearity. ∎

Proposition 2.2. $[\cdot,\cdot]_d$ satisfies the Jacobi identity,

$$[U,[V,W]_d]_d = [[U,V]_d,W]_d + [V,[U,W]_d]_d, \tag{2.27}$$

where $U, V, W \in E$ sections.

Proof. This can be obtained from (2.26) by a direct computation. However, there is a more elegant way using the property of derived brackets. The Jacobi identity (2.27) essentially follows from that of $(\mathrm{End}(S), [\cdot,\cdot])$

$$\begin{aligned}
[U,[V,W]_d]_d &= [[U,d],[[V,d],W]] \\
&= [[[U,d],[V,d]],W] + [[V,d],[[U,d],W]] && \text{(graded Jacobi identity)} \\
&= [[[[U,d],V],d],W] + [V,[U,W]_d]_d && \text{(nilpotency of } [\cdot,d]) \\
&= [[U,V]_d,W]_d + [V,[U,W]_d]_d.
\end{aligned}$$

■

Remark 2.4. It is not hard to see that $[\cdot,\cdot]_d$ also satisfy the Leibniz rule

$$[v+\lambda, f(w+\xi)]_d = f[v+\lambda, w+\xi]_d + (vf)(w+\xi), \tag{2.28}$$

where f is a smooth function.

We shall from now on omit subscript "d", and $(E[\cdot,\cdot])$ will always refer to the bracket induced by the derived bracket. It is often called *Dorfman bracket*. From the propositions above, Dorfman bracket is almost a Lie bracket except that it is not antisymmetric,

$$[v+\lambda, v+\lambda] = \mathcal{L}_v\lambda - i_v d\lambda = d i_v\lambda = d\langle v+\lambda, v+\lambda\rangle, \tag{2.29}$$

which is related to the canonical pairing. Let's summarize the canonical structures on the generalized tangent bundle E and their properties.

Definition 2.1. [6] A *Courant algebroid* E is an extension of real vector bundles

$$0 \longrightarrow T^* \xrightarrow{\rho^*} E \xrightarrow{\rho} T \longrightarrow 0, \tag{2.30}$$

where T, T^* denote the tangent and cotangent bundles over M, together with a metric $\langle\cdot,\cdot\rangle$ of split signature, such that $\langle\rho^*\lambda, V\rangle = \frac{1}{2}\lambda(\rho(V))$, and a bilinear bracket $[\cdot,\cdot]$ on sections of E. They are required to satisfy

i). $[U,[V,W]] = [[U,V],W] + [V,[U,W]]$, (Jacobi Identity)

ii). $[V,fW] = f[V,W] + (\rho(V)f)W$, (Leibniz Rule)

iii). $\rho(V)\langle U,W\rangle = \langle [V,U],W\rangle + \langle U,[V,W]\rangle$, (Invariance of $\langle\cdot,\cdot\rangle$)

iv). $[V,V] = \rho^* d\langle V,V\rangle$. (Antisymmetry Anomaly)

We are interested in the case where (2.30) is exact for the specific reason.

Proposition 2.3. The generalized tangent bundle (2.17) is an exact Courant algebroid.

Proof. It remains to show the property iii). By linearity it suffices to prove the the case $U = W$. Let $V = v + \lambda, W = w + \xi$, by (2.26)

$$\rho(V)\langle W, W\rangle = \mathcal{L}_v(i_w\xi) = i_{[v,w]}\xi + i_w\mathcal{L}_v\xi = 2(\langle [V,W], \xi\rangle + \langle [V,W], w\rangle) = 2\langle [V,W], W\rangle .$$

∎

Remark 2.5. The antisymmetry anomaly implies that

$$[V, W] = -[W, V] + 2d\langle V, W\rangle , \tag{2.31}$$

where we always use *anchor* and *coanchor* ρ, ρ^* to identify generalized tangents with vectors and one-forms.

The antisymmetrization of the Dorfman bracket is the *Courant bracket*,

$$[\![V, W]\!] := \frac{1}{2}([V, W] - [W, V]). \tag{2.32}$$

Note that

$$[\![V, W]\!] = [V, W] - d\langle V, W\rangle = -[W, V] + d\langle V, W\rangle , \tag{2.33}$$

and for exact Courant algebroid

$$[\![v + \lambda, w + \xi]\!] = [v, w] + \mathcal{L}_v\xi - \mathcal{L}_w\lambda - \frac{1}{2}d(i_v\xi - i_w\lambda). \tag{2.34}$$

2.2.3 Splittings

Let E be the generalized tangent bundle (2.17). We now present the promised isomorphism $E \simeq T \oplus T^*$. Every vector bundle underlying an exact Courant algebroid has such a structure.

An isomorphism $E \simeq T \oplus T^*$ compatible with the exact sequence (2.30) is equivalent to the existence of a splitting. Recall that a *splitting* means a map $s : T \to E$ such that $\rho \circ s = 1$, and the corresponding isomorphism is defined by $E \simeq s(T) \oplus \ker\rho \simeq T \oplus T^*$. We say a splitting s *isotropic* if $\langle s(T), s(T)\rangle = 0$.

Theorem 2.1. There is an one-to-one correspondence between isotropic splittings $s : T \to E$ and curvings of a connection B.

To be more specific, let $\{U_i\}$ a good cover, $\{\Lambda_{(ij)}\}$ the connective strucuture that determines E. Then there is an one-to-one correspondence between s and a collection of two-forms $\{B_{(i)}\}$ patched as (2.7).

Proof. Given a curving $\{B_{(i)}\}$, and a vector v, on each coordinate neighborhood U_i, the splitting is defined by $s(v)_{(i)} = v + i_v B_{(i)}$, which is well-deinfed on E because $s(v)_{(i)} = v + i_v B_{(j)} - i_v d\Lambda_{(ij)} = s(v)_{(j)} - i_v d\Lambda_{(ij)}$ on U_{ij}. Isotroy follows from $\langle s(v), s(v)\rangle = i_v^2 B_{(i)} = 0$ by linearity.

Conversely, every isotropic splitting s defines on each U_i a bilinear map $\chi_{(i)}$ by $i_v\chi_{(i)} := s(v)_{(i)} - v$, so $i_v\chi_{(i)} = s(v)_{(j)} - i_v d\Lambda_{(ij)} - v = i_v\big(\chi_{(j)} - d\Lambda_{(ij)}\big)$, having patching (2.7) by definition. And $\chi_{(i)}$ is a two-form since s is isotropic: $\chi_{(i)}(v,v) = 2\,\langle \chi_{(i)}(v), v\rangle = \langle s(v), s(v)\rangle - \langle v, v\rangle = 0$. ∎

Remark 2.6. We can write out explicitly the isomorphism $E \simeq T \oplus T^*$. Let $\{\hat{e}_a\}$ be a basis for T, and $\{e^a\}$ its dual in T^*. Then by theorem 2.1, the generalized tangent bundle E has basis

$$\hat{E}_A = \begin{cases} \hat{E}_a = \hat{e}_a + i_{\hat{e}_a} B & \text{for } A = a \\ E^a = e^a & \text{for } A = a + d \end{cases}. \tag{2.35}$$

Then the explicit isomorphism is given by

$$\begin{aligned} E &\xrightarrow{\sim} T \oplus T^*, \\ V &\longmapsto V_{(B)}, \end{aligned} \tag{2.36}$$

where if $V = v^a\hat{E}_a + \lambda_a E^a \in E$, then $V_{(B)} = v^a\hat{e}_a + \lambda_a e^a \in T \oplus T^*$. (2.35) is an additional structural on the frame bundle, which leads to a reduction; we will come to that in the next chapter.

Remark 2.7. The isomorphism of frames implies the isomorphism of the spinors $S \simeq \Lambda^\bullet T^*$ given by

$$\begin{aligned} S &\xrightarrow{\sim} \Lambda^\bullet T^*, \\ \Psi &\longmapsto \Psi_{(B)}, \end{aligned} \tag{2.37}$$

where $\Psi_{(B)}|_{U_i} = e^{B_{(i)}} \wedge \Psi_{(i)}$. It is easy to see that it is well-defined. Indeed, it is a Clifford morphism since

$$\begin{aligned}\Gamma_{(B)}(\hat{e}_a)\Psi_{(B)} &= i_{\hat{e}_a}(e^{B_{(i)}} \wedge \Psi_{(i)}) \\ &= i_{\hat{e}_a}B_{(i)} \wedge e^{B_{(i)}} \wedge \Psi_{(i)} + e^{B_{(i)}} \wedge i_{\hat{e}_a}\Psi_{(i)} \\ &= e^{B_{(i)}} \wedge \Gamma(\hat{E}_a)\Psi_{(i)}, \\ \Gamma_{(B)}(e^a)\Psi_{(B)} &= e^a \wedge e^{B_{(i)}} \wedge \Psi_{(i)} = e^{B_{(i)}} \wedge \Gamma(E^a)\Psi_{(i)}.\end{aligned}$$

Now we note that $T \oplus T^*$ is also an exact Courant bracket, with structures defined in the same manner without a patching. It is natural to compare the two Courant algebroid structures: the canonical one on $T \oplus T^*$, and the one induce by the splitting. By definition of the splitting (2.36), the metric $\langle \cdot, \cdot \rangle$ is preserved. However, as can be seen from the proof of proposition 2.1, the Dorfman bracket on $T \oplus T^*$ will be twisted; i.e. the bracket of E in terms of sections of $T \oplus T^*$ is

$$[v + \lambda, w + \xi]_{(B)} = [v, w] + \mathcal{L}_v \xi - i_w d\lambda + i_v i_w H, \tag{2.38}$$

where $H = dB$ is the 3-curvature.

In fact, $E \simeq T \oplus T^*$ is a Courant algebroid isomorphism if and only if $H = 0$; i.e. when the gerbe is flat [3]. In particular, when B is shifted by a closed two-form, $B \mapsto B + \omega, \omega \in \Omega^2_{\text{closed}}$, the corresponding two splittings are isomorphic Courant algebroids. The description of a Courant algebroid is independent of coordinates, so there is also diffeomorphism invariance. We can conclude a notable result: the symmetry of a Courant algebroid is the same as the NS-NS sector symmetry $G_{\text{NS}} = \text{Diff}(M) \ltimes \Omega^2_{closed}(M)$ in Type II supergravities.

Recall that two connective structures will have two generalized tangent bundles related by an Ω^1-torsor. This corresponds to shifts by exact two-forms, so resulting the same Courant algebroid E.

By (2.38), the 3-curvature of a connection B is also given by

$$i_v i_w H(u) = \langle [s(v), s(w)], u \rangle \,, \tag{2.39}$$

where s is the splitting defined by B. While the connective structure (2.8) is a symmetry, the curving $\{B_{(i)}\}$ will contribute to H, thus twist the Dorfman bracket. Since every curving is dfferent by a two-form, the cohomology class $[H]/2\pi \in H^3(M, \mathbb{R})$ does not depend on any splitting. Actually, $[H]$ classifies the isomorphism class of Courant algebroid [16].

2.2.4 Relation to Gerbes

We may summarize that the isomorphism class of the constructed Courant algebroid is completely determined by a gerbe, and the connections B correspond to isotropic splittings.

Isomorphism classes of exact Courant algebroids are $H^3(M, \mathbb{R})$. It is known that every 3-curvature H of an $U(1)$-gerbes is in the class $[H]/2\pi \in H^3(M, \mathbb{Z})$, and isomorphism classes of $U(1)$-gerbes are in one-to-one correspondence with $H^3(M, \mathbb{Z})$ [8]. So the characteristic class (2.39) of the Courant algebroid determined by an $U(1)$-gerbe is in $H^3(M, \mathbb{Z})$. Conversely, if an exact Courant algebroid given a splitting s, and $[H]/2\pi \in H^3(M, \mathbb{Z})$, where H is defined by (2.39), it is isomorphic a generalized tangent bundle we have constructed [3].

Chapter 3

Generalized Geometry

Generalized tangent bundle admits many analogous constructions from the ordinary geometry, which are reviewed in B.2. These generalized objects turn out to furnish a geometric framework for the Type II supergravities. We study the generalized geometric objects in this chapter, and establish the connection to Type II theories after introducing the fermionic sector in the next chapter. Techniques in the treatment of vector bundles are reviewed in B.1.

3.1 Generalized Differential Geometry

3.1.1 $O(d,d) \times \mathbb{R}^+$ structure

To describe dilaton, we will need to study the weights on tensors. Recall that, on a d-dimensional manifold M, the *weight* of a tensor is given by the representation $\det T^* := \Lambda^d T^*$. It is a real line bundle. The p-tensor product $(\det T^*)^p$ has *weight* p. Every $f \in \mathrm{Diff}(M)$ induces a map

$$\det T^* \longrightarrow \det T^*, \quad \Phi \longmapsto (\det df)\Phi. \tag{3.1}$$

The Lie derivative on the tensor representation is

$$\mathcal{L}_v : (\det T^*)^p \longrightarrow (\det T^*)^p, \quad \Phi \longmapsto \big(v^\mu \partial_\mu + p\partial_\mu v^\mu\big)\Phi, \tag{3.2}$$

since $\mathcal{L}_{fv}(\Phi) = \mathcal{L}_v(f\Phi)$ for top forms $\Phi \in \det T^*$.

Let E be the generalized tangent bundle (2.17), we may introduce the first weighted tensor, obtained by twisting the generalized tangent bundle by the line bundle $\det T^*$,

$$\tilde{E} := \det T^* \otimes E. \tag{3.3}$$

The canonical pairing (2.18) of E induces a reduction of its structural bundle. In the notation of B.1, let $F_{\tilde{E}}$ be the frame bundle of $\tilde{E}$, then the reduction $\tilde{F} \subset F_{\tilde{E}}$ takes form

$$\tilde{F} = \left\{ (\hat{E}_A(x)) \in F_{\tilde{E}} : x \in M, \langle \hat{E}_A, \hat{E}_B \rangle = \Phi^2 \eta_{AB} \right\}, \qquad \eta = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.4}$$

where $\Phi \in \det T^*$ non-vanishing by definition of $F_{\tilde{E}}$. It is immediate that this defines an $O(d,d) \times \mathbb{R}^+$-principal bundle. $\tilde{F}$ is called the *generalized structure bundle*, and frames $\{\hat{E}_A\} \in \tilde{F}$ are called *conformal basis*. The topology of $\tilde{F}$ encodes both the topology of T and of the B-field gerbe [4], as true for the $O(d,d)$-bundle F of the Courant algebroid E; cf. 2.2.4. Note that specifying a global non-vanishing conformal factor $\Phi \in \Gamma(\det T^*)$ leads to a reduction of structure bundle $F \hookrightarrow \tilde{F}$, by $\hat{E}_A \mapsto \Phi \otimes \hat{E}_A$.

One particular conformal basis is the coordinate basis. Locally it is induced by the inclusion $FU_i \hookrightarrow \tilde{F}U_i$, where $V \in \tilde{E}U_i$ has representation $V = v^\mu(\partial/\partial x^\mu) + \lambda_\mu dx^\mu$. Then the generalized basis is $\{\hat{E}_A\} = \{\partial/\partial x^\mu\} \cup \{dx^\mu\}$. It will be convenient to use *index M* to refer to this frame, on E or $\tilde{E}$. Then the components of V will be

$$V^M = \begin{cases} v^\mu & \text{for } M = \mu \\ \lambda_\mu & \text{for } M = \mu + d \end{cases}. \tag{3.5}$$

This local structure in fact extends to a global one, by the splitting (2.35) of E. There will be a two-step reduction of the structure bundle $\tilde{F}$:

$$GL(d,\mathbb{R}) \ltimes \mathbb{R}^{d(d-1)/2} =: G_{\text{split}} \subset G_{\text{split}} \times \mathbb{R}^+ \subset O(d,d) \times \mathbb{R}^+. \tag{3.6}$$

For $\{\hat{e}_a\}, \{e^a\}$ basis for T, T^* in duality, the G_{split}-structure is defined by the *split frames* for $\tilde{E}$ as

$$\begin{gathered} \hat{E}_A = \begin{cases} \hat{E}_a = (\det e) \otimes (\hat{e}_a + i_{\hat{e}_a} B) & \text{for } A = a \\ E^a = (\det e) \otimes e^a & \text{for } A = a + d \end{cases}, \\ \langle \hat{E}_A, \hat{E}_B \rangle = (\det e)^2 \eta_{AB}, \end{gathered} \tag{3.7}$$

and $G_{\text{split}} \times \mathbb{R}^+$-structure is defined by the *conformal split frames* as

$$\hat{E}_A = \begin{cases} \hat{E}_a = e^{-2\phi}(\det e) \otimes (\hat{e}_a + i_{\hat{e}_a} B) & \text{for } A = a \\ E^a = e^{-2\phi}(\det e) \otimes e^a & \text{for } A = a + d \end{cases}, \qquad (3.8)$$

$$\langle \hat{E}_A, \hat{E}_B \rangle = (e^{-2\phi} \det e)^2 \eta_{AB},$$

where $\phi \in C^\infty$. The transition function $M \in C^\infty(G_{\text{split}} \times \mathbb{R}^+)$ between two conformal split frames takes form

$$M = \sigma(\det A)^{-1} \otimes \begin{pmatrix} 1 & \\ \omega & 1 \end{pmatrix} \cdot \begin{pmatrix} A & \\ & (A^{-1})^T \end{pmatrix}, \qquad (3.9)$$

where $\sigma \in C^\infty(\mathbb{R}^+)$, $A \in C^\infty(GL(d,\mathbb{R}))$, and $\omega \in \Omega^2$. σ is a rescaling, A transforms coordinates $\hat{e}_a \mapsto \hat{e}_a (A^{-1})^b{}_a$, and ω transforms the splitting $B \mapsto B + \omega$, where ω needs to be closed to preserve Dorfman bracket. This confirms $G_{\text{split}} = GL(d,\mathbb{R}) \ltimes \mathbb{R}^{d(d-1)/2} \subset O(d,d) \times \mathbb{R}^+$.

The point of these frames is that fix a bosonic pair (B, ϕ), (3.7) or (3.8) determines a splitting $\tilde{E} \simeq \det T^* \otimes (T \oplus T^*)$ by

$$\begin{aligned} \tilde{E} &\xrightarrow{\sim} \det T^* \otimes (T \oplus T^*), \\ v^a \hat{E}_a + \lambda_a E^a &\longmapsto (\det e) \otimes (v^a \hat{e}_a + \lambda_a e^a). \end{aligned} \qquad (3.10)$$

Note that in coordinate frames, these are exactly $V = v + \lambda \longmapsto V_{(B)} = (v + \lambda - i_v B)$ and $V \longmapsto V_{(B,\phi)} = e^{2\phi}(v + \lambda - i_v B)$.

We finally comment that the reduction (3.6) of the structure bundle $\tilde{F}$ implies that patching of $\tilde{E}$ requires only transitions function in the subgroup $C^\infty(G_{\text{split}}) \subset C^\infty(O(d,d) \times \mathbb{R}^+)$.

3.1.2 Generalized Tensors

We may now establish the generalized tensor calculus, including weights.

Consider vector bundles associated to the generalized structure bundle (3.4) by the $O(d,d) \times \mathbb{R}^+$-representations of form

$$E^{\otimes n}_{(p)} = (\det T^*)^p \otimes E \otimes \cdots \otimes E, \qquad (3.11)$$

then we call a section of $E_{(p)}^{\otimes n}$ a *generalized tensor* of *weight* p. Recall $E^* \simeq E$ by the canonical pairing, so it suffice to label only one type of rank. $O(d,d)\times\mathbb{R}^+$-structure also admits spin representations since E has (S,Γ) defined by (2.22). It is given by

$$S_{(p)}^{\pm} = (\det T^*)^{p-1/2}\otimes S^{\pm}, \tag{3.12}$$

so sections of $S_{(p)}^{\pm}$ is called *generalized spinors* of *weight* p. The Mukai pairing (2.23) extends to $S_{(p)}^{\pm}$, as a map $(\cdot,\cdot): S_{(p)}^{\pm}\otimes S_{(p)}^{\pm} \to (\det T^*)^{2p}$. Note that $S_{(1/2)}^{\pm} = S^{\pm}$. Similar to (3.10), the map $\Psi \mapsto \Psi_{(B)}$ defined by (2.37) can be extended to the isomorphism

$$\begin{aligned} (B,\phi) : S_{(p)}^{\pm} &\xrightarrow{\sim} (\det T^*)^{p-1/2}\otimes \Lambda^{\text{even/odd}}T^*, \\ \Psi &\longmapsto \Psi_{(B,\phi)} = e^{(2p-1)\phi}e^{B}\wedge\Psi. \end{aligned} \tag{3.13}$$

The Dorfman bracket on E is a derivative in the sense of 2.1. This derivative can be generalized to tensors. For $V\in E$, we define the *Dorfman derivative* L_V as follow. On $W\in E$, it is the Dorfman bracket

$$L_V W = [V,W]. \tag{3.14}$$

On $\Phi \in (\det T^*)^p$,

$$L_V\Phi = \mathcal{L}_{\rho(V)}\Phi, \tag{3.15}$$

where ρ is the anchor. Then L_V extends to any $E_{(p)}^{\otimes n}$ by the differential of the tensor representation (3.11).

This can be stated more explicitly in terms of local coordinate frames. We need to first bring the partial derivative operator to the generalized setting. This can be done by the adjoint map of the anchor $(\rho)^*: T^* \to E^*, v^*\mapsto v^*\circ\rho$, and yields in a frame $\{E^M\}$ in E^* dual to the coordinate frame

$$\partial_M = \begin{cases} \partial_\mu & \text{for } M=\mu \\ 0 & \text{for } M=\mu+d \end{cases}, \tag{3.16}$$

so if $V = v+\lambda$, $V^N\partial_N = v^\mu\partial_\mu$, and index is raised by $E^* \xrightarrow{\langle\cdot,\cdot\rangle^{-1}} E$ as

$$\partial^M = \begin{cases} 0 & \text{for } M=\mu \\ 2\partial_\mu & \text{for } M=\mu+d \end{cases}. \tag{3.17}$$

This compares with the map induced by coanchor ρ^* by

$$\partial^M f = 2(\rho^* df)^M. \tag{3.18}$$

Then by (2.26) and (B.12), (3.14) says

$$L_V W^M = V^N \partial_N W^M + (\partial^M V^N - \partial^N V^M) W_N, \tag{3.19}$$

as

$$\partial_M V^N = (\partial_\mu v^\nu + \partial_\mu \lambda_\nu),$$
$$\partial^M V_N - \partial_N V^M = \partial_\mu v^\nu - \partial_\nu v^\mu + (\partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu).$$

Note that this implies for $V = v + \lambda$ the Dorfman derivative L_V extends the (minus) adjoint action of $a^\mu{}_\nu := \partial_\nu v^\mu \in \mathfrak{gl}(d.\mathbb{R}) \otimes C^\infty$ in the Lie derivative $\mathcal{L}_{\rho(V)}$ (B.12) by the two form $\omega = d\lambda$ via the element in $\mathfrak{o}(d,d) \otimes C^\infty$

$$\begin{pmatrix} a & 0 \\ -\omega & -a^T \end{pmatrix}. \tag{3.20}$$

Additionally, by (3.9), this element is in $\mathfrak{g}_{\text{split}} \otimes C^\infty$. Next, by (3.2), (3.15) becomes

$$L_V \Phi = V^N \partial_N \Phi + p(\partial_N V^N)\Phi, \tag{3.21}$$

which corresponds to the (minus) adjoint action of Lie $\mathbb{R}^+ = \mathbb{R}$ element

$$-p \operatorname{Tr} a. \tag{3.22}$$

With these notions, we can write out the Dorfman derivative of any generalized tensor via their adjoint actions by $\mathfrak{o}(d,d) \oplus \mathbb{R}$. For $\alpha \in E_{(p)}^{\otimes n}$, we have

$$\begin{aligned} L_V \alpha^{M_1 \dots M_n} = V^N \partial_N \alpha^{M_1 \dots M_n} + (\partial^{M_1} V^N - \partial^N V^{M_1}) \alpha_N{}^{M_2 \dots M_n} + \dots \\ \dots + (\partial^{M_n} V^N - \partial^N V^{M_n}) \alpha^{M_1 \dots M_{n-1}}{}_N + p(\partial_N V^N) \alpha^{M_1 \dots M_n}, \end{aligned} \tag{3.23}$$

where the indices are lowered by the canonical $\langle \cdot, \cdot \rangle$. The argument using representation immediately applies to spinors. If $\Psi \in S^\pm_{(p)}$, we define

$$L_V \Psi = V^N \partial_N \Psi + \frac{1}{4}(\partial^M V^N - \partial^N V^M)\Gamma_{MN}\Psi + p(\partial_N V^N)\Psi, \tag{3.24}$$

where $\Gamma_{MN} = \frac{1}{2}[\Gamma_M, \Gamma_N]$, and $\frac{1}{4}[\Gamma_M, \Gamma_N]$ is the embedding of $\mathfrak{o}(d,d)$ into the Clifford algebra.

Finally, we note that we have seen the operator (3.16) before, namely, in (2.24) as the graded derivation on $S_{(1/2)}$, since

$$d\Psi = \frac{1}{2}\Gamma^M \partial_M \Psi, \tag{3.25}$$

where the factor $\frac{1}{2}$ comes from raising the index M by metric η in form (3.4).

3.1.3 Generalized Connections

In analogue with (B.9), we can define covariant differentiation in the direction of generalized tangents.

Definition 3.1. [6] A *generalized connection* on a vector bundle $\mathcal{V}$ over M is an $\mathbb{R}$-linear morphism of vector bundles

$$D : \mathcal{V} \longrightarrow E \otimes \mathcal{V}, \tag{3.26}$$

satisfying the Leibniz rule

$$D(fV) = (2\rho^* df) \otimes V + fDV, \tag{3.27}$$

for $V \in \mathcal{V}, f \in C^\infty$.

Remark 3.1. Again the conventional factor of 2 in front of the coanchor comes from raising the index. Given a splitting (not necessarily isotropic) $s : T \to E$, (3.27) implies that D decomposes as $D = s(\chi) + 2\rho^*(\nabla)$, where $\nabla : \mathcal{V} \to T^* \otimes \mathcal{V}$ is an ordinary connection on $\mathcal{V}$, and $\chi \in T \otimes \mathrm{End}(\mathcal{V})$ is an endomorphism valued vector field.

In the case $\mathcal{V} = E$, we are interested in the connections that is *compatible* with the $O(d,d)$-structure; that is, a generalized connection such that

$$d\langle V, W\rangle = \langle DV, W\rangle + \langle V, DW\rangle, \tag{3.28}$$

where $\langle \cdot, \cdot\rangle$ is the canonical pairing on E. Similar to the conventional differential geometry, this implies, in frame indices (any frame), D takes form

$$D_M W^A = \partial_M W^A + \Omega_M{}^A{}_B W^B, \tag{3.29}$$

where $\Omega \in E^* \otimes \mathfrak{o}(E)$, so

$$\Omega_{MAB} = -\Omega_{MBA}. \tag{3.30}$$

This extends to the case $\mathcal{V} = \tilde{E}$ by an exterior tensor product of representations, so

$$D_M W^A = \partial_M W^A + \tilde{\Omega}_M{}^A{}_B W^B, \tag{3.31}$$

where $\tilde{\Omega} \in E^* \otimes (\mathfrak{o}(E) \oplus \mathfrak{gl}(\det T^*))$ and

$$\tilde{\Omega}_M{}^A{}_B = \Omega_M{}^A{}_B - \Lambda_M \delta^A_B, \tag{3.32}$$

for an $O(d,d) \times \mathbb{R}^+$-compatible generalized connection D.

Given a connection $D : \tilde{E} \to E \otimes \tilde{E}$, exact same treatment for the Dorfman derivative applies to extend D to tensors $E^{\otimes n}_{(p)}$. If D compatible, it also extends to spinors $S^{\pm}_{(p)}$. Namely,

$$\begin{aligned} D_M \alpha^{A_1\dots A_n} &= \partial_M \alpha^{A_1\dots A_n} + \Omega_M{}^{A_1}{}_B \alpha^{BA_2\dots A_n} + \dots \\ &\qquad \dots + \Omega_M{}^{A_n}{}_B \alpha^{A_1\dots A_{n-1}B} - p\Lambda_M \alpha^{A_1\dots A_n}, \\ D_M \Psi &= \left(\partial_M + \frac{1}{4}\Omega_M{}^{AB}\Gamma_{AB} - p\Lambda_M\right)\Psi. \end{aligned} \tag{3.33}$$

As hinted by remark 3.1, a conventional connection ∇ with a splitting induces a generalized connection by setting $\chi = 0$. This has a refinement.

Example 3.1. Fix a bosonic pair (B, ϕ), an affine connection ∇ on T induces a compatible generalizd connection D on $\tilde{E}$ as follows. Note first that (B, ϕ) determines a set of conformal splitting frames of form (3.8). Let $\hat{s} : \tilde{E} \simeq \det T^* \otimes (T \oplus T^*)$ be the splitting defined by these frames as in (3.10), then ∇ extends to $\tilde{E}$ by

$$\tilde{E} \xrightarrow{\hat{s}} \det T^* \otimes (T \oplus T^*) \xrightarrow{\nabla} T^* \otimes \det T^* \otimes (T \oplus T^*) \xrightarrow{1\otimes\hat{s}^{-1}} T^* \otimes \tilde{E}, \tag{3.34}$$

denoted again by ∇. Now $2\rho^*\nabla : \tilde{E} \to E \otimes \tilde{E}$ is manifestly a compatible generalized connection. We may denote this connection by $D^{\nabla}_{(B,\phi)}$ or simply D^{∇}.

Again, we may work out this explicitly in covariant indices. If $W \in \tilde{E}$, then in a conformal split frame $\{\hat{E}_A\}$ defined by (B, ϕ)

$$W = W^A \hat{E}_A = w^a \hat{E}_a + \xi_a E^a.$$

By the splitting $\hat{s}$, w^a, ξ_a are identified as components of weighted tensors in $\det T^* \otimes T$ and $\det T^* \otimes T^*$, so we have the expression

$$D^{\nabla}_M(w^a + \xi_a) = \begin{cases} \nabla_\mu w^a + \nabla_\mu \xi_a & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases}, \tag{3.35}$$

with

$$\Omega_M{}^A{}_B = \begin{cases} \omega_\mu{}^a{}_b + \omega_\mu{}^b{}_a & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases},$$
$$\Lambda_M = \begin{cases} \omega_\mu{}^a{}_a & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases}, \tag{3.36}$$

where $\omega_\mu{}^a{}_b$ are the components of the connection ∇ in frames $\{\hat{e}_a\}, \{e^a\}$ associated to $\{\hat{E}_A\}$.

3.1.4 Gerneralized Torsion

When a generalized connection D is defined on E, it is natural to compare the connection derivative with the Dorfman bracket. This is the notion of torsion. In generalized geometry, it is defined in analogue with the ordinary torsion (B.13).

Definition 3.2. [6] The *generalized torsion* $\mathrm{T} \in \Gamma(\Lambda^2 E \otimes E)$ of a generalized connection D on E is defined by

$$\mathrm{T}(V, W, U) = \langle D_V W - D_W V - [\![V, W]\!], U\rangle + \frac{1}{2}(\langle D_U V, W\rangle - \langle D_U W, V\rangle), \tag{3.37}$$

where $[\![\cdot, \cdot]\!]$ is the Courant bracket.

Note that Dorfman bracket is antisymmetrized to compare with $D_V W - D_W V$, and the last two terms are needed for torsion T to be a tensor.

Proposition 3.1. T is well-defined, and if D is compatible with $\langle\cdot, \cdot\rangle$, then $\mathrm{T} \in \Gamma(\Lambda^3 E)$.

Proof. The linearity in U is trivial. Since by definition T is antisymmetric in V, W, it suffices to check for V. By definition of the connection D,

$$D_{fV} W = f D_V W, \quad D_W(fV) = f D_W V + \rho(W) f V, \quad D_U(fV) = f D_U V + \rho(U) f V.$$

By (2.31), (2.33), and Leibniz rule,

$$\begin{aligned}
[\![fV,W]\!] &= -[W,fV] + d\langle fV,W\rangle \\
&= -f[W,V] - \rho(W)fV + df\langle V,W\rangle + fd\langle V,W\rangle \\
&= f[\![V,W]\!] - \rho(W)fV + df\langle V,W\rangle .
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{T}(fV,W,U) &= f\mathrm{T}(V,W,U) + \langle -\rho(W)fV + \rho(W)fV - df\langle V,W\rangle, U\rangle + \frac{1}{2}\langle \rho(U)fV,W\rangle \\
&= f\mathrm{T}(V,W,U) + \langle V,W\rangle\left(\frac{1}{2}\rho(U)f - \langle df,U\rangle\right) \\
&= f\mathrm{T}(V,W,U).
\end{aligned}$$

Now assume (3.28), then (2.33) implies

$$[\![V,W]\!] = [V,W] - \langle DV,W\rangle - \langle V,DW\rangle = -[W,V] + \langle DV,W\rangle + \langle V,DW\rangle . \tag{3.38}$$

Then with 2.1. (iii)

$$\begin{aligned}
\mathrm{T}(V,W,U) &= \langle D_VW - D_WV - [V,W],U\rangle \\
&\quad + \frac{1}{2}(\langle D_UV,W\rangle + \langle V,D_UW\rangle) + \frac{1}{2}(\langle D_UV,W\rangle - \langle D_UW,V\rangle) \\
&= \langle D_VW - D_WV,U\rangle + \langle D_UV,W\rangle - \langle [V,W],U\rangle \\
&= \langle D_VW - D_WV,U\rangle + \langle D_UV,W\rangle - (-\langle W,[V,U]\rangle + \rho(V)\langle W,U\rangle) \\
&= \langle D_VW - D_WV,U\rangle + \langle D_UV,W\rangle + \langle [V,U],W\rangle - 2\langle d\langle W,U\rangle, V\rangle \\
&= \langle D_VW - D_WV,U\rangle + \langle D_UV,W\rangle - 2\langle \langle DW,U\rangle + \langle W,DU\rangle, V\rangle + \langle [V,U],W\rangle \\
&= \langle D_VW - D_WV,U\rangle + \langle D_UV,W\rangle - \langle D_VW,U\rangle - \langle W,D_VU\rangle + \langle [V,U],W\rangle \\
&= -(\langle D_VU - D_UV,W\rangle + \langle D_WV,U\rangle - \langle [V,U],W\rangle) \\
&= -\mathrm{T}(V,U,W).
\end{aligned}$$

This suffices. ■

Remark 3.2. From the proof we actually find a more concise expression for T when D is compatible. Namely,

$$\mathrm{T}(V,W,U) = \langle D_UV,W\rangle + \langle D_VW - D_WV - [V,W],U\rangle . \tag{3.39}$$

One will find it familiar if we express $\langle D_UV,W\rangle + \langle D_VW - D_WV,U\rangle$ in coordinate basis and switch the positions of the first and second terms, as

$$U_N\big(V_M D^M W^N + (D^N V^M - D^M V^N)W_M\big).$$

Indeed, this is the form of the Dorfman derivative on E. This allows us to write an analogue to (B.16)

$$\mathrm{T}(V)W = L^D_V W - L_V W, \tag{3.40}$$

where $W \in E$, L^D_V denotes (3.19) with D in place of ∂, and $\mathrm{T}(V) \in \Lambda^2 E \simeq \mathfrak{o}(E)$ acts on E by adjoint action.

Remark 3.3. We can view (3.40) from another perspective. When D is compatible, the action of D on spinors $S_{(1/2)} = S$ analogous to (3.25) is well-defined

$$\not{D}\Psi = \frac{1}{2}\Gamma^M D_M \Psi. \tag{3.41}$$

A formal definition of "derived bracket"

$$[V, W]_{\not{D}} = [[V, \not{D}], W]$$

will give you L^D_V. But we stress that this is *not* a derived bracket since in general $\not{D}^2 \neq 0$ but will be a "curvature". By the uniqueness of the operator d and (3.40), $\mathrm{T} = 0$ if and only if $\not{D} = d$.

We may extend the notion of generalized torsion to connections $D : \tilde{E} \to E \otimes \tilde{E}$ on $\tilde{E}$ by either representation argument or remarks 3.2, 3.3. Using (3.40), the generalized torsion $\mathrm{T} \in \Gamma(\operatorname{ad}\tilde{F} \otimes E)$, where $\operatorname{ad}\tilde{F} = \mathfrak{o}(E) \oplus \mathfrak{gl}(\det T^*) \simeq \Lambda^2 E \oplus \mathbb{R}$, and is defined by the adjoint action on $\alpha \in E^{\otimes n}_{(p)}$

$$\mathrm{T}(V) \cdot \alpha = L^D_V \alpha - L_V \alpha. \tag{3.42}$$

Then by proposition 3.1 we have $\mathrm{T} \in \Gamma(\Lambda^3 E \oplus E)$, so T decomposes as

$$\mathrm{T} = \mathrm{T}_1 - \mathrm{T}_2, \tag{3.43}$$

where $\mathrm{T}_1 \in \Lambda^3 E$ and $\mathrm{T}_2 \in E$.

If $\{\hat{E}_A\}$ is a general conformal frame, then $\{\Phi^{-1}\hat{E}_A\}$ is an orthonormal basis for E. We use this orthonormal basis to derive the components of $\mathrm{T} \in \Gamma(\Lambda^3 E \oplus E)$ for a compatible connection D on $\tilde{E}$ with components as in (3.31). Let T has index structure $\mathrm{T}(\hat{E}_A)^M{}_N =$

$\mathrm{T}^{M}{}_{AN}$. Then by (3.39),

$$\begin{aligned}\mathrm{T}_{ABC} &= \mathrm{T}(\Phi^{-1}\hat{E}_B, \Phi^{-1}\hat{E}_C, \Phi^{-1}\hat{E}_A) \\ &= \tilde{\Omega}_{ACB} + \tilde{\Omega}_D{}^D{}_B\eta_{AC} + \tilde{\Omega}_{BAC} - \tilde{\Omega}_{CAB} \\ &\quad + \Phi\partial_B\Phi^{-1}\eta_{AC} - \langle[\Phi^{-1}\hat{E}_B, \Phi^{-1}\hat{E}_C], \Phi^{-1}\hat{E}_A\rangle \\ &= -3\tilde{\Omega}_{[ABC]} + \tilde{\Omega}_D{}^D{}_B\eta_{AC} - \Phi^{-2}\langle[\Phi^{-1}\hat{E}_B, \hat{E}_C], \hat{E}_A\rangle\,,\end{aligned} \tag{3.44}$$

where $\Phi\partial_B\Phi^{-1}\eta_{AC}$ comes from term $D_B(\Phi^{-1}V^B) = \partial_B\Phi^{-1} + \Phi^{-1}D_B(V^B)$ in the extended action of D on $\tilde{E}$.

Example 3.2. We continue the example 3.1 by calculating the generalized torsion for the connections $D^{\nabla}_{(B,\phi)}$. (B,ϕ) determines a splitting, then we may use coordinate basis $\{\hat{e}_\mu, e^\mu\}$ to define the conformal split frame (3.8) with $\Phi = e^{-2\phi}\det e$ so that the orthonormal basis $\{\Phi^{-1}\hat{E}_A\}$ takes form (2.35). Then by the Leiniz rule

$$[\Phi^{-1}\hat{E}_B, \hat{E}_C] = \Phi\Big([\Phi^{-1}\hat{E}_B, \Phi^{-1}\hat{E}_C]\Big) + \big(L_{\Phi^{-1}\hat{E}_B}\Phi\big)\Phi^{-1}\hat{E}_C, \tag{3.45}$$

and

$$\begin{aligned}\langle[\Phi^{-1}\hat{E}_B, \Phi^{-1}\hat{E}_C], \Phi^{-1}\hat{E}_A\rangle &= \langle[\hat{e}_\mu + i_{\hat{e}_\mu}B + e^\mu, \hat{e}_\nu + i_{\hat{e}_\nu}B + e^\nu], \hat{e}_\sigma + i_{\hat{e}_\sigma}B + e^\sigma\rangle \\ &= \langle\mathcal{L}_{\hat{e}_\mu}(i_{\hat{e}_\nu}B) - i_{\hat{e}_\nu}d\big(i_{\hat{e}_\mu}B\big), \hat{e}_\sigma + i_{\hat{e}_\sigma}B + e^\sigma\rangle \\ &= \langle i_{\hat{e}_\nu}i_{\hat{e}_\mu}dB, \hat{e}_\sigma + i_{\hat{e}_\sigma}B + e^\sigma\rangle \\ &= \frac{1}{2}i_{\hat{e}_\sigma}i_{\hat{e}_\nu}i_{\hat{e}_\mu}dB\end{aligned}$$

$$\Phi^{-1}\big(L_{\Phi^{-1}\hat{E}_B}\Phi\big)\,\langle\Phi^{-1}\hat{E}_C, \Phi^{-1}\hat{E}_A\rangle = -(2\partial_B\phi)\eta_{CA}$$

the last term in (3.44) reduces to

$$\Phi^{-2}\langle[\Phi^{-1}\hat{E}_B, \hat{E}_C], \hat{E}_A\rangle = \frac{1}{2}H_{BCA} - (2\partial_B\phi)\eta_{CA},$$

where $H = dB$ and the embedding $\Lambda^\bullet T^* \hookrightarrow \Lambda^\bullet E^*$ is induced by the map defining (3.16). In the case ∇ is torsion free, the generalized torsion of D^∇ becomes

$$\mathrm{T}_{ABC} = -\frac{1}{2}H_{BCA} + (2\partial_B\phi)\eta_{CA}. \tag{3.46}$$

After raising indices with $\langle\cdot,\cdot\rangle^{-1}$, we have $\mathrm{T} \in \Gamma(\Lambda^3 E \oplus E)$ and the decompostion (3.43) take form

$$\mathrm{T}_1 = -4H, \qquad \mathrm{T}_2 = -4d\phi, \tag{3.47}$$

where factors are from η, and importantly, similar to (3.18), after raising indices with coefficients added in front, H and $d\phi$ now denote the embedding with respect to the coanchor $T^* \overset{\rho^*}{\hookrightarrow} E$.

3.1.5 A Curvature Operator

One may expect a generalized version of the curvature (B.17) of the form

$$R(V,W,U) = [D_V, D_W]U - D_{[\![V,W]\!]}U. \tag{3.48}$$

However, this cannot be made tensorial as for generalized torsion since the fact relation

$$[\![fV, gW]\!] = fg[\![V,W]\!] + f\rho(V)gW - g\rho(W)fV + \frac{1}{2}\langle V, W\rangle (gdf - fdg) \tag{3.49}$$

differs from that of Lie bracket by an antisymmetric anomaly implies

$$R(fV, gW, hU) = fghR(fV, gW, hU) - \frac{1}{2}h\langle V, W\rangle D_{(gdf-fdg)}U. \tag{3.50}$$

So (3.48) is linear only for $\langle V, W\rangle = 0$. This requires an additional structure: if $C_1, C_2 \subset E$ orthogonal subbundles, then $R \in \Gamma(C_1 \otimes C_2 \otimes \mathfrak{o}(E))$ is a tensor.

The last component in the NS-NS bosonic fields, metric g, will supply such a structure as we are going to see.

3.2 Generalized Levi-Civita Connection

3.2.1 Generalized Metric

Note that in (3.10) the bosonic doublet (B, ϕ) determines a twisted conformal split. We now make an extension by describing the NS-NS triplet (B, ϕ, g) in the generalized setting.

Definition 3.3. [6] A *generalized metric* on $\tilde{E}$ is an $O(p,q) \times O(q,p)$-structure $\tilde{G} \subset \tilde{F}$.

This definition makes sense because of this.

Theorem 3.1. There is an one-to-one correspondence between the triplet (B, ϕ, g) and generalized metrics on $\tilde{E}$.

Proof. Let $\tilde{G}$ be a generalized metric on $\tilde{E}$. Firstly, $\tilde{G}$ determines a non-vanishing $\Phi \in \Gamma(\det T^*)$. Locally, if $\{\hat{E}_A\} \in \tilde{G}U_i$, then by definition (3.4), $\langle \hat{E}_A, \hat{E}_B \rangle = \Phi_i^2 \otimes \eta_{AB}$ for some $\Phi_i \in \det T^* U_i$. As $\tilde{G}$ is globally defined, and η is $O(d,d)$-invariant, Φ_i is forced to satisfy the cocycle condition (B.1) with respect to $\mathbb{R}^+$, thus descends to a global non-vanishing section $\Phi \in \Gamma(\det T^*)$. Then Φ defines an isomorphism of vector bundles

$$\begin{aligned} (\Phi) : E &\xrightarrow{\sim} \tilde{E}, \\ V &\longmapsto \Phi \otimes V. \end{aligned} \tag{3.51}$$

This induces an $O(p,q) \times O(q,p)$-structure on E, and we may identify $\tilde{G}$ with this structure on E. The group structure of $O(p,q) \times O(q,p) \in O(d,d)$ implies an orthogonal splitting

$$E = C_+ \oplus C_-, \tag{3.52}$$

such that

$$\tilde{G} \simeq \{\{\hat{E}_A\} : \langle \hat{E}_A, \hat{E}_B \rangle = \Phi^2 \eta_{AB}\}, \tag{3.53}$$

where

$$\hat{E}_A = \begin{cases} \hat{E}_a^+ \in C_+ & \text{for } A = a \\ \hat{E}_{\bar{a}}^- \in C_- & \text{for } A = \bar{a} + d \end{cases}, \quad \eta_{AB} = \begin{pmatrix} \eta_{ab} & \\ & -\eta_{\bar{a}\bar{b}} \end{pmatrix}, \tag{3.54}$$

and $\eta_{ab}, \eta_{\bar{a}\bar{b}}$ are flat metrics of signature (p,q). Each subspace has dimension $d = p + q$, and endowed with metrics defined by

$$g_+ = \Phi^{-2} \eta_{ab} E^{+a} \otimes E^{+b} \in (C_+ \otimes C_+)^*, \quad g_- = \Phi^{-2} \eta_{\bar{a}\bar{b}} E^{-\bar{a}} \otimes E^{-\bar{b}} \in (C_- \otimes C_-)^*, \tag{3.55}$$

where $\{E^{\pm}\}$ denote the dual basis. Thus

$$\Phi^{-2} \langle \cdot, \cdot \rangle = g_+ \oplus (-g_-). \tag{3.56}$$

Then the restrictions of anchor ρ on $C_\pm$ are isomorphic, and their inverses

$$\hat{s}_\pm : T \xrightarrow{\sim} C_\pm \subset E, \tag{3.57}$$

which are (importantly, non-isotropic) splittings of the exact sequence (2.30), induce the metric

$$g = \hat{s}_+^* g_+ = \hat{s}_-^* g_- \tag{3.58}$$

of signature (p, q) on T. The pull-back metrics must be equal by the $O(d, d)$-structure. By definition, $\hat{s}_\pm : T \to E$ are isometric and anti-isometric, respectively, up to a conformal factor Φ.

Now, we have the volume form $\sqrt{-g} \in \Gamma(\det T^*)$, so there exits $\phi \in \Gamma(C^\infty(\mathbb{R}))$ that defines a transition function in $\det T^*$ by

$$\Phi = e^{-2\phi}\sqrt{-g}. \tag{3.59}$$

Let $s : T \to E$ defined by

$$s = \frac{1}{2}(\hat{s}_+ + \hat{s}_-), \tag{3.60}$$

then s is an isotropic splitting by definition (3.55), and by theorem 2.1 is given by a B-field.

Conversely, given a bosonic triplet (B, ϕ, g), the $O(p, q) \times O(q, p)$-structure is directly given by the frames

$$\begin{aligned} \hat{E}_a^+ &= \Phi\big(\hat{e}_a^+ + e_a^+ + i_{\hat{e}_a^+} B\big), \\ \hat{E}_{\bar{a}}^- &= \Phi\Big(\hat{e}_{\bar{a}}^- - e_{\bar{a}}^- + i_{\hat{e}_{\bar{a}}^-} B\Big), \end{aligned} \tag{3.61}$$

where Φ is given by (3.59), $\{\hat{e}_a^+\}, \{\hat{e}_{\bar{a}}^-\}$, with dual $\{e^{+a}\}, \{e^{-\bar{a}}\}$ respectively, are orthonormal frames for the metric g, so

$$\begin{aligned} g = \eta_{ab} e^{+a} \otimes e^{+b} = \eta_{\bar{a}\bar{b}} e^{-\bar{a}} \otimes e^{-\bar{b}}, \\ g(\hat{e}_a^+, \hat{e}_b^+) = g(\hat{e}_a^-, \hat{e}_b^-) = \eta_{ab}, \end{aligned} \tag{3.62}$$

and the indices in (3.61) are lowered by $\eta_{ab}, \eta_{\bar{a}\bar{b}}$. It is easy to check that (3.61) satisfies (3.53) and (3.54). ■

Remark 3.4. Note that in (3.53),(3.54), we used a different form of η from that used for $\tilde{F}$ in (3.4). Two forms of η are related by a bundle isomorphism defined by a constant $\mathrm{GL}(2d, \mathbb{R})$-transition function. The structures on E in definition 2.1 are independent of this choice. It will be convenient to stick to this structure bundle when $O(p, q) \times O(q, p)$-

structure is introduced, as we shall do so from now on. For example, they are related to g-orthonormal bases by (3.61).

Remark 3.5. As noted in the proof, $\hat{s}_\pm$ are not isotropic, but (conformally) orthogonal and anti-orthogonal. In particular, (3.61) is in contrast with isotropic conformal split frames (3.8). The point is that metric on T is g, while metric on E is the canonical pairing $\Phi^{-2}\langle\cdot,\cdot\rangle$. The generalized metric $\tilde{G}$ encodes (B,ϕ,g) onto the canonical structures of $\tilde{E}$ by bundle reduction.

We may single out the conformal density (3.59) and define an $O(p,q)\times O(q,p)$-invariant tensor, which may also be called *generalized metric* $G\in E\otimes E$

$$G=\Phi^{-2}\Big(\eta^{ab}\hat{E}^+_a\otimes\hat{E}^+_b+\eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}}\otimes\hat{E}^-_{\bar{b}}\Big). \tag{3.63}$$

By (3.61) and (3.62), G in coordinate frames has form

$$G_{MN}=\frac{1}{2}\begin{pmatrix} g-Bg^{-1}B & -Bg^{-1}\\ g^{-1}B & g^{-1}\end{pmatrix}_{MN}. \tag{3.64}$$

The $O(p,q)\times O(q,p)$-invariant pair (G,Φ) has the same information as $\tilde{G}$, and parametrize the coset space $(O(d,d)\times\mathbb{R}^+)/O(p,q)\times O(q,p)$.

3.2.2 $O(p,q)\times O(q,p)$-Structure

We will need to work with components of full $O(p,q)\times O(q,p)$-covariance, so it will be useful to get into more details. Theorem 3.1 tells us that the $O(p,q)\times O(q,p)$-structure $\tilde{G}$ consists of conformal frames $\{\hat{E}^+_a\}\cup\{\hat{E}^-_{\bar{a}}\}$ of form (3.61) determined by a bosonic triplet (B,ϕ,g). The pull-backs $\hat{s}^*_\pm\tilde{G}$ along the splittings (3.57) reduce to the $O(p,q)$-bundle over M, defined by g, by projections onto the first and second factors.

There are four pairings, $\{g,g_+,g_-,\Phi^{-2}\langle\cdot,\cdot\rangle\}$ on T,$C_\pm$, and E respectively. They are related by (3.56),(3.58), and allow us to raise and lower indices. On $C_\pm$ we use $g_\pm$, and on E we continue to use $\Phi^{-2}\langle\cdot,\cdot\rangle$. This means η_{ab} for $a,b,c,\dots$, $\eta_{\bar{a}\bar{b}}$ for $\bar{a},\bar{b},\bar{c},\dots$, and η_{AB} for $A,B,C,\dots$. Then the image of the coframes $\{E^\pm\}$ in $C_\pm$ are $\{\hat{E}^{+a}\},\{\hat{E}^{-\bar{a}}\}$, satisfying

$$\begin{aligned} g_+(\hat{E}^{+a},\hat{E}^+_b)&=\delta^a{}_b, & g_-(\hat{E}^{-\bar{a}},\hat{E}^-_{\bar{b}})&=\delta^{\bar{a}}{}_{\bar{b}},\\ \langle\hat{E}^{+a},\hat{E}^+_b\rangle&=\Phi^2\delta^a{}_b, & \langle\hat{E}^{-\bar{a}},\hat{E}^-_{\bar{b}}\rangle&=-\Phi^2\delta^{\bar{a}}{}_{\bar{b}}, \end{aligned} \tag{3.65}$$

and

$$\hat{E}^A = \begin{cases} \hat{E}^{+a} & \text{for } A = a \\ -\hat{E}^{-\bar{a}} & \text{for } A = \bar{a} + d \end{cases} \in \tilde{E}, \qquad \langle \hat{E}^A, \hat{E}_B \rangle = \Phi^2 \delta^A{}_B. \tag{3.66}$$

With these basis, we identify the space with its dual, so raising and lowering indices are now considered as an automorphism in $\tilde{E}$ or E.

We omit Φ by the identification $\hat{E}^{\pm} \sim \Phi^{-1}\hat{E}^{\pm}$ for now. Note then the duals $E^{\pm} \sim \Phi E^{\pm}$. By (3.61) and covariance, one easily finds the following. The anchor ρ takes form

$$\begin{aligned} \rho : E &\longrightarrow T, \\ V^a \hat{E}^+_a + V^{\bar{a}} \hat{E}^-_{\bar{a}} &\longmapsto V^a \hat{e}^+_a + V^{\bar{a}} \hat{e}^-_{\bar{a}}, \end{aligned} \tag{3.67}$$

with restrictions

$$\begin{aligned} \rho_{\pm} : C_{\pm} &\xrightarrow{\sim} T, \\ V^a \hat{E}^+_a &\longmapsto V^a \hat{e}^+_a, \\ V^{\bar{a}} \hat{E}^-_{\bar{a}} &\longmapsto V^{\bar{a}} \hat{e}^-_{\bar{a}}, \end{aligned} \tag{3.68}$$

whose inverses are splitting (3.57). And coanchor ρ^*

$$\begin{aligned} \rho^* : T^* &\longrightarrow E, \\ \lambda_a e^{+a} = \lambda_{\bar{a}} e^{-\bar{a}} &\longmapsto \frac{1}{2}\Big(\lambda_a \hat{E}^{+a} - \lambda_{\bar{a}} \hat{E}^{-\bar{a}}\Big), \end{aligned} \tag{3.69}$$

where again an extra $\frac{1}{2}$ from raising index, while the embedding for the operator (3.16) take form

$$\begin{aligned} T^* &\xrightarrow{2\rho^*} E \xrightarrow{\langle \cdot,\cdot \rangle} E^*, \\ \lambda_a e^{+a} = \lambda_{\bar{a}} e^{-\bar{a}} &\longmapsto \lambda_a \hat{E}^{+a} - \lambda_{\bar{a}} \hat{E}^{-\bar{a}} \longmapsto \lambda_a E^{+a} - \lambda_{\bar{a}} E^{-\bar{a}}. \end{aligned} \tag{3.70}$$

Note that $\tilde{G}$-frames are free of factor 2 in raising indices. This is by the virtue of the imposed compatibility between metric self-dualities summarized in the commutative diagram

$$\begin{array}{ccccc} C_+{}^* & \underset{(\hat{s}_+)^*}{\overset{(\rho_+)^*}{\leftrightarrows}} & T^* & \underset{(\hat{s}_-)^*}{\overset{(\rho_-)^*}{\rightleftarrows}} & C_-{}^* \\ g_+^{-1} \downarrow \uparrow g_+ & & g^{-1} \downarrow \uparrow g & & g_-^{-1} \downarrow \uparrow g_- \\ C_+ & \underset{\hat{s}_+}{\overset{\rho_+}{\rightleftarrows}} & T & \underset{\hat{s}_-}{\overset{\rho_-}{\leftrightarrows}} & C_- \end{array}, \tag{3.71}$$

where if we identify $C_{\pm} \simeq C_{\pm}{}^*$ by $g_{\pm}$, then $\rho^* = \frac{1}{2}((\rho_+)^* + (\rho_-)^*) = \frac{1}{2}(\rho)^*$, so the embedding $2\rho^* : T^* \to E$ is precisely the adjoint map of anchor $(\rho)^* : T^* \to E^*$ raised by $(g_\pm)^{-1}$, which is compatible with $\pm\langle\cdot,\cdot\rangle$, as shown in (3.67), (3.69), and (3.70); cf. (3.18).

We note that embeddings (3.69) and (3.70) extend to tensors $(T^*)^{\otimes n} \to E_{(p)}^{\otimes n}$ in a natural way. And by $\rho^* = \frac{1}{2}((\rho_+)^* + (\rho_-)^*)$ we obtain $O(p,q) \times O(q,p)$-covariant components, where by (3.68)

$$\begin{aligned} (\rho_\pm)^* : T^* &\xrightarrow{\sim} C_\pm, \\ \lambda^+_a e^{+a} &\longmapsto \lambda^+_a \hat{E}^{+a}, \\ \lambda^-_{\bar a} e^{-\bar a} &\longmapsto \lambda^-_{\bar a} \hat{E}^{-\bar a}. \end{aligned} \tag{3.72}$$

Then we have covariant components for forms by the following identification

$$\Lambda^n T^* \xhookrightarrow{\Lambda^n\rho^*} \Lambda^n E \xrightarrow{\sim} \Lambda^n C_+ \oplus (\Lambda^{n-1}C_+ \otimes C_-) \oplus \cdots \oplus (C_+ \otimes \Lambda^{n-1}C_-) \oplus \Lambda^n C_- \ . \tag{3.73}$$

One last $O(p,q)\times O(q,p)$-covariant object to mention is the $Spin(p,q)\times Spin(q,p)$ spinors. In expectation to describe supergravity, where notions of spinors are needed in supersymmetry, we may further assume (M,g) have spin structure to include $Spin(p,q)$-spinors. In this case, the generalized metric (B,ϕ,g) will then have $Spin(p,q)\times Spin(q,p)$-structure. Then subbundles $(C_\pm, g_\pm)$ with metrics (3.55) have spinor bundles $(S(C_\pm),\gamma_\pm)$, respectively. Let $\gamma^a, \gamma^{\bar a}$ denote the $\tilde G$-frame components of $\gamma_\pm$, then

$$\frac{1}{2}\{\gamma^a,\gamma^a\} = \eta^{ab}, \qquad \frac{1}{2}\{\gamma^{\bar a},\gamma^{\bar a}\} = \eta^{\bar a\bar b}. \tag{3.74}$$

Note that we also have (2.21) on $Spin(d,d)$-spinors $S^\pm_{(p)}$ for $(E,\langle\cdot,\cdot\rangle)$, where in $\tilde G$-frames

$$\frac{1}{2}\{\Gamma^A,\Gamma^B\} = \eta^{AB}, \qquad \eta^{AB} = \begin{pmatrix} \eta^{ab} & \\ & -\eta^{\bar a\bar b} \end{pmatrix}. \tag{3.75}$$

They are related by

$$S_{(1/2)} \simeq S(C_+) \otimes S(C_-), \tag{3.76}$$

where isomorphism takes form

$$\Gamma^A = \begin{cases} \gamma^a \otimes 1 & \text{for } A = a \\ \gamma^{(d)} \otimes \gamma^{\bar a}\gamma^{(d)} & \text{for } A = \bar a + d \end{cases}, \tag{3.77}$$

where $\gamma^{(d)}$ is the top gamma in $\mathrm{Cliff}(C_\pm)$. This isomorphism is established essentially because the orthogonal decomposition (3.56) induces the Clifford isomorphism

$$\mathrm{Cliff}(E, \langle\cdot,\cdot\rangle) \simeq \mathrm{Cliff}(C_+, g_+) \otimes \mathrm{Cliff}(C_-, -g_-).$$

Then the chiralities on $S(C_\pm)$ correspond to two chirality operators $\Gamma^{(\pm)}$ defined on $S_{(1/2)}$, given by

$$\Gamma^{(+)} = \frac{1}{d!}\epsilon^{a_1\ldots a_d}\Gamma_{a_1}\ldots\Gamma_{a_d}, \quad \Gamma^{(-)} = \frac{1}{d!}\epsilon^{\bar{a}_1\ldots\bar{a}_d}\Gamma_{\bar{a}_1}\ldots\Gamma_{\bar{a}_d}. \tag{3.78}$$

In (B) splitting (3.13) for $S_{(p)}$, these are related to the Hodge dual $*$ of forms with respect to g:

$$\Gamma^{(+)}\Psi^{(n)}_{(B)} = (-1)^{[n/2]} * \Psi^{(n)}_{(B)}, \quad \Gamma^{(-)}\Psi^{(n)}_{(B)} = (-1)^{d}(-1)^{[(n+1)/2]} * \Psi^{(n)}_{(B)}, \tag{3.79}$$

where $[n/2]$ arises from the transposition of indices, and $[(n+1)/2] = [n/2] - n$. This can be checked by noting in (B) splitting, the action of $\tilde{G}$-frames is

$$\Gamma(\hat{E}^{\pm}_{a/\bar{a}}) \cdot \Psi_{(B)} = i_{\hat{e}^{\pm}_{a/\bar{a}}}\Psi_{(B)} \pm e^{\pm}_{a/\bar{a}} \wedge \Psi_{(B)}. \tag{3.80}$$

3.2.3 Torsion-free Metric Connections

With basic tools sorted, we finally come to the generalized Levi-Civita connections.

Definition 3.4. A generalized connection D on $\tilde{E}$ is said to be *compatible* with the generalized metric $\tilde{G} \in \tilde{F}$ if

$$DG = 0, \quad D\Phi = 0, \tag{3.81}$$

where (G, Φ) is defined in (3.59) and (3.63).

It is equivalent to say D acts within $\tilde{G}$. Specifically, note that the compatibility implies that the connection D can be identified as

$$D : E \xrightarrow{(\Phi)} \tilde{E} \xrightarrow{D} E \otimes \tilde{E} \xrightarrow{1\otimes(\Phi)^{-1}} E \otimes E, \tag{3.82}$$

a generalized connection respecting metric G on E.

In frames, if $w^a_+ \hat{E}^+_a + w^{\bar{a}}_- \hat{E}^-_{\bar{a}} \in \tilde{E}$,

$$D_M(w^a_+ + w^{\bar{a}}_-) = \partial_M(w^a_+ + w^{\bar{a}}_-) + \Omega_M{}^a{}_b w^b_+ + \Omega_M{}^{\bar{a}}{}_{\bar{b}} w^{\bar{b}}_-, \tag{3.83}$$

it is the requirement to satisfy

$$\Omega_{Mab} = -\Omega_{Mba}, \quad \Omega_{M\bar{a}\bar{b}} = -\Omega_{M\bar{b}\bar{a}}. \tag{3.84}$$

Theorem 3.2. Given a generalized metric $\tilde{G}$, there exists a torsion-free, compatible generalized connection D on $\tilde{E}$.

Proof. Let ∇ be the Levi-Civita connection for the metric g. Similar to example 3.1 (3.34), there is a sequence defining $D^{\nabla}_{(B,\phi,g)}$

$$\tilde{E} \xrightarrow{\hat{s}} T \oplus T \xrightarrow{\nabla} T^* \otimes (T \oplus T) \xrightarrow{(\rho)^* \otimes \hat{s}^{-1}} E \otimes \tilde{E}, \tag{3.85}$$

where $\hat{s}$ is defined using (3.51) and (3.68) by

$$\hat{s} : \tilde{E} \xrightarrow{(\Phi)^{-1}} E \xrightarrow{\rho_+ \oplus \rho_-} T \oplus T. \tag{3.86}$$

Note that by definition, $\hat{s}^{-1} = \hat{s}_+ \oplus \hat{s}_-$. In $\tilde{G}$-frames,

$$D^{\nabla}_M(w^a_+ + w^{\bar{a}}_-) = \begin{cases} \nabla_\mu w^a_+ + \nabla_\mu w^{\bar{a}}_- & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases}, \tag{3.87}$$

where on the right hand side $w^a_+, w^{\bar{a}}_-$ are identified as components of vectors in T with basis $\{\hat{e}^+_a\}, \{\hat{e}^-_{\bar{a}}\}$, by $\hat{s}$. Obviously this is manifestly $O(p,q) \times O(q,p)$-compatible.

Since ∇ is Levi-Civita, the volume form density is invariant $\nabla\sqrt{-g} = 0$, which tells us that $D^{\nabla}_{(B,\phi,g)}$ coincides with $D^{\nabla(g)}_{(B,\phi)}$ in example 3.1 with $\Lambda_M = 0$. Therefore $D^{\nabla}_{(B,\phi,g)}$ has the generalized torsion T^{∇} as (3.47) in example 3.2.

Analogous to the ordinary geometry, by remark 3.1, given another generalized connection D on $\tilde{E}$, it differs from D^{∇} by a tensor $\Sigma \in \Gamma(E \otimes \mathrm{End}(\tilde{E}))$ as

$$D = D^{\nabla} + \Sigma. \tag{3.88}$$

If D is compatible, this implies $\Sigma \in \Gamma(E \otimes \mathfrak{o}(\tilde{E})) \simeq \Gamma(E \otimes (\mathfrak{o}(p,q) \oplus \mathfrak{o}(q,p)))$; that is, in orthonormal frames $\{\Phi^{-1}\hat{E}_A\}$ on E where $\{\hat{E}_A\}$ are $\tilde{G}$-frames

$$\begin{gathered}\Sigma_{M\ \bar{b}}^{\ a} = \Sigma_{M\ b}^{\ \bar{a}} = 0,\\ \Sigma_{Mab} = -\Sigma_{Mba}, \quad \Sigma_{M\bar{a}\bar{b}} = -\Sigma_{M\bar{b}\bar{a}}.\end{gathered} \tag{3.89}$$

Now by (3.44) and (3.47), the generalized torsion $\mathrm{T} \in \Gamma(\Lambda^3 E \oplus E)$ of D has components

$$(\mathrm{T}_1)_{ABC} = -4H_{ABC} - 3\Sigma_{[ABC]}, \quad (\mathrm{T}_2)_A = -4d\phi_A - \Sigma_{C\ A}^{\ C}, \tag{3.90}$$

where $H, d\phi$ are embedded into $\Lambda^3 E \oplus E$ by $\Lambda^3\rho^* \oplus \rho^*$ as commented after (3.47). Then we obtain a torsion-free compatible D if we set Σ to let (3.90) vanish. ■

Remark 3.6. Despite the existence, such a connection is *not* completely determined by (B, ϕ, g) in contrast with the conventional Levi-Civita connection. Indeed, let's work out the solution set of $\Sigma \in \Gamma(E \otimes \mathfrak{o}(p,q) \oplus E \otimes \mathfrak{o}(q,p))$ that yields torsion-free compatible connections D in (3.88).

We first simplify the tensor equations (3.90) by $O(p,q) \times O(q,p)$-compatibility. We continue to use the frames introduced in the proof. By (3.68),(3.69), if frames $\{\hat{E}^A\}$ of $\tilde{E}$ are defined in (3.66), the components of $H, d\phi$ under embedding (3.73) with respect to the induced frames are then

$$\begin{gathered}\rho^* = \frac{1}{2}((\rho_+)^* + (\rho_-)^*), \qquad \Lambda^3\rho^* = \frac{1}{8}\Lambda^3((\rho_+)^* + (\rho_-)^*),\\ d\phi \in C_+ \oplus C_-, \qquad H \in (\Lambda^3 C_+) \oplus (\Lambda^2 C_+ \otimes C_-) \oplus (C_+ \otimes \Lambda^2 C_-) \oplus (\Lambda^3 C_-),\end{gathered}$$

$$d\phi_A = \begin{cases} \frac{1}{2}\partial_a\phi & \text{for } A = a \\ \frac{1}{2}\partial_{\bar{a}}\phi & \text{for } A = \bar{a}+d \end{cases}, \quad H_{ABC} = \begin{cases} \frac{1}{8}H_{abc} & \text{for } (A,B,C) = (a,b,c) \\ \frac{1}{8}H_{ab\bar{c}} & \text{for } (A,B,C) = (a,b,\bar{c}+d) \\ \frac{1}{8}H_{a\bar{b}\bar{c}} & \text{for } (A,B,C) = (a,\bar{b}+d,\bar{c}+d) \\ \frac{1}{8}H_{\bar{a}\bar{b}\bar{c}} & \text{for } (A,B,C) = (\bar{a}+d,\bar{b}+d,\bar{c}+d) \end{cases}, \tag{3.91}$$

where right hand sides are components in $\Lambda^\bullet T^*$. Together with (3.89) and (3.90), we obtain the tensor equations

$$\begin{gathered}\Sigma_{[abc]} = -\frac{1}{6}H_{abc}, \quad \Sigma_{\bar{a}bc} = -\frac{1}{2}H_{\bar{a}bc}, \quad \Sigma_{a\ b}^{\ a} = -2\partial_b\phi,\\ \Sigma_{[\bar{a}\bar{b}\bar{c}]} = +\frac{1}{6}H_{\bar{a}\bar{b}\bar{c}}, \quad \Sigma_{a\bar{b}\bar{c}} = +\frac{1}{2}H_{a\bar{b}\bar{c}}, \quad \Sigma_{\bar{a}\ \bar{b}}^{\ \bar{a}} = -2\partial_{\bar{b}}\phi,\\ \Sigma_{abc} + \Sigma_{acb} = \Sigma_{\bar{a}\bar{b}\bar{c}} + \Sigma_{\bar{a}\bar{c}\bar{b}} = 0,\end{gathered} \tag{3.92}$$

where the signs arised when lowering the middle index of Σ because of the convention in (3.66). This is a system of inhomogeneous linear equations, having C^∞-linear space of solutions. The general solution takes form

$$\begin{aligned} D_{\bar a} w_+^b &= \nabla_{\bar a} w_+^b - \frac{1}{2} H_{\bar a}{}^b{}_c w_+^c, \\ D_a w_-^{\bar b} &= \nabla_a w_-^{\bar b} + \frac{1}{2} H_a{}^{\bar b}{}_{\bar c} w_-^{\bar c}, \\ D_a w_+^b &= \nabla_a w_+^b - \frac{1}{6} H_a{}^b{}_c w_+^c - \frac{2}{d-1}\left(\delta_a{}^b \partial_c \phi - \eta_{ac}\partial^b\phi\right) w_+^c + A^{+}_a{}^b{}_c w_+^c, \\ D_{\bar a} w_-^{\bar b} &= \nabla_{\bar a} w_-^{\bar b} + \frac{1}{6} H_{\bar a}{}^{\bar b}{}_{\bar c} w_-^{\bar c} - \frac{2}{d-1}\left(\delta_{\bar a}{}^{\bar b} \partial_{\bar c} \phi - \eta_{\bar a\bar c}\partial^{\bar b}\phi\right) w_-^{\bar c} + A^{-}_{\bar a}{}^{\bar b}{}_{\bar c} w_-^{\bar c}, \end{aligned} \tag{3.93}$$

where $A^\pm \in \Gamma(C_\pm \otimes (\Lambda^2 C_\pm))$ satisfying

$$\begin{aligned} A^+_{[abc]} &= 0, \quad A^{+}_a{}^a{}_b = 0, \\ A^-_{[\bar a\bar b\bar c]} &= 0, \quad A^{-}_{\bar a}{}^{\bar a}{}_{\bar b} = 0 \end{aligned} \tag{3.94}$$

parametrize the affine space of torsion-free compatible generalized connections. Note that indices $a, b, c, \bar a, \bar b, \bar c$ are raised by $\eta^{ab}, \eta^{\bar a\bar b}$ in $O(p,q) \times O(q,p)$-covariance.

3.2.4 Unique Operators

We expect generalized objects having applications to supergravity theories to be completely determined by field configuration (B, ϕ, g). So we seek for the objects that depend only on the solution class parametrized by $A^\pm$. For further constructions, we note some typical operators that are class-function.

From (3.93) there are the immediate ones

$$\begin{aligned} D^+ &: C_+ \to C_- \otimes C_+, \qquad D_{\bar a} w_+^b = \nabla_{\bar a} w_+^b - \frac{1}{2} H_{\bar a}{}^b{}_c w_+^c; \\ D^- &: C_- \to C_+ \otimes C_-, \qquad D_a w_-^{\bar b} = \nabla_a w_-^{\bar b} + \frac{1}{2} H_a{}^{\bar b}{}_{\bar c} w_-^{\bar c}. \end{aligned} \tag{3.95}$$

By tracing, we note

$$\begin{aligned} D^+\cdot &: C_+ \to C^\infty, \qquad D_a w_+^a = \nabla_a w_+^a - 2(\partial_a\phi) w_+^a; \\ D^-\cdot &: C_- \to C^\infty, \qquad D_{\bar a} w_-^{\bar a} = \nabla_{\bar a} w_-^{\bar a} - 2(\partial_{\bar a}\phi) w_-^{\bar a}; \end{aligned} \tag{3.96}$$

are class-functions because of (3.94).

Less trivial are the ones on spinors. For $\epsilon^{\pm} \in S(C_{\pm})$,

$$\begin{aligned}
D^{+}: S(C_{+}) \to C_{-} \otimes S(C_{+}), &\qquad D_{\bar{a}}\epsilon^{+} = \left(\nabla_{\bar{a}} - \frac{1}{8}H_{\bar{a}bc}\gamma^{bc}\right)\epsilon^{+};\\
D^{-}: S(C_{-}) \to C_{+} \otimes S(C_{-}), &\qquad D_{a}\epsilon^{-} = \left(\nabla_{a} + \frac{1}{8}H_{a\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}\right)\epsilon^{-};\\
\not{D}^{+}: S(C_{+}) \to S(C_{+}), &\qquad \gamma^{a}D_{a}\epsilon^{+} = \left(\gamma^{a}\nabla_{a} - \frac{1}{24}H_{abc}\gamma^{abc} - \gamma^{a}\partial_{a}\phi\right)\epsilon^{+};\\
\not{D}^{-}: S(C_{-}) \to S(C_{-}), &\qquad \gamma^{\bar{a}}D_{\bar{a}}\epsilon^{-} = \left(\gamma^{\bar{a}}\nabla_{\bar{a}} + \frac{1}{24}H_{\bar{a}\bar{b}\bar{c}}\gamma^{\bar{a}\bar{b}\bar{c}} - \gamma^{\bar{a}}\partial_{\bar{a}}\phi\right)\epsilon^{+};
\end{aligned} \tag{3.97}$$

are independent of $A^{\pm}$, where actions of compatible D on $S(C_{\pm})$ are given by identifying $\tilde{E}$ with E as

$$D_{M}\epsilon^{+} = \left(\partial_{M} + \frac{1}{4}\Omega_{Mab}\gamma^{ab}\right)\epsilon^{+}, \quad D_{M}\epsilon^{-} = \left(\partial_{M} + \frac{1}{4}\Omega_{M\bar{a}\bar{b}}\gamma^{\bar{a}\bar{b}}\right)\epsilon^{-}.$$

In the last two expressions in (3.97), $A^{\pm}$ are removed because of the identity $\gamma^{a}\gamma^{bc} = \gamma^{abc} + \eta^{ab}\gamma^{c} - \eta^{ac}\gamma^{b}$, where $\gamma^{abc}A^{+}_{abc} = \gamma^{abc}A^{+}_{[abc]} = 0$, $(\eta^{ab}\gamma^{c} - \eta^{ac}\gamma^{b})A^{+}_{abc} = 2\eta^{ab}\gamma^{c}A^{+}_{abc} = 2\gamma^{c}A^{+a}_{a\ \ c} = 0$, and similar for A^{-}. This is a very useful identity, let's note down its generalizations

$$\begin{aligned}
\gamma^{a}\gamma^{b_1\ldots b_n} &= \gamma^{ab_1\ldots b_n} + n\cdot\eta^{a[b_1}\gamma^{b_2}\ldots\gamma^{b_n]},\\
\gamma^{b_1\ldots b_n}\gamma^{a} &= \gamma^{b_1\ldots b_n a} + n\cdot\gamma^{[b_1}\ldots\gamma^{b_{n-1}}\eta^{b_n]a}.
\end{aligned} \tag{3.98}$$

The proof is simply a combinatorics.

3.2.5 Generalized Curvatures

The subbundles $C_{\pm}$ are orthogonal, so the operator (3.48) is tensorial on $C_{\pm} \otimes C_{\mp}$, and has components $\left(R_{a\bar{b}\ \ d}^{\ \ c}, R_{a\bar{b}\ \ \bar{d}}^{\ \ \bar{c}}\right), \left(R_{\bar{a}b\ \ d}^{\ \ c}, R_{\bar{a}b\ \ \bar{d}}^{\ \ \bar{c}}\right)$, but is by no means unique. However, further structures in ordinary geometry do have certain uniquely determined generalized analogues. First, note that we have the expression of Ricci tensor in form (B.19) when connection is torsion-free.

Definition 3.5. The *generalized Ricci tensor* of a torsion-free compatible connection D is the tensor $R \in \Gamma(C_{+} \otimes C_{-})$ satisfying

$$R_{a\bar{b}}w^{a}_{+} = [D_{a}, D_{\bar{b}}]w^{a}_{+}, \tag{3.99}$$

for all $w_{+} \in C_{+}$.

Remark 3.7. The index structure implies that R is uniquely determined by (B, ϕ, g). The generalized Ricci tensor can also be equivalently defined by $R \in \Gamma(C_- \otimes C_+)$

$$R \cdot w_- = [D^- \cdot, D^+] w_- . \tag{3.100}$$

Note that $O(p,q) \times O(q,p)$-structure does not distinguish $C_\pm$, and this symmetry implies the equivalence.

Theorem 3.3. In an aligned $\tilde{G}$-frame, where $\hat{e}^+_a = \hat{e}^-_a$ in the definition (3.61), R has components

$$R_{ab} = \mathcal{R}_{ab} - \frac{1}{4} H_{acd} H_b{}^{cd} + 2\nabla_a \nabla_b \phi - \frac{1}{2} e^{2\phi} \nabla^c \big(e^{-2\phi} H_{cab} \big). \tag{3.101}$$

Remark 3.8. Components of arbitrary unaligned frames can be obtained by $O(p,q) \times O(q,p)$-covariant transformations.

Proof. Let the frames aligned, so $\hat{E}^{-\bar{b}} \sim \hat{E}^{+b}$. By (3.95) and (3.96),

$$\begin{aligned} D_a D_b w_+^a &= \nabla_a \left(\nabla_b w_+^a - \frac{1}{2} H_b{}^a{}_c w_+^c \right) - 2(\partial_a \phi) \left(\nabla_b w_+^a - \frac{1}{2} H_b{}^a{}_c w_+^c \right) \\ &\quad - \frac{1}{2} H_a{}^d{}_b \left(\nabla_d w_+^a - \frac{1}{2} H_d{}^a{}_c w_+^c \right) \\ &= \nabla_a \nabla_b w_+^a - \frac{1}{2} \nabla_a H_b{}^a{}_c w_+^c - 2(\partial_a \phi) \nabla_b w_+^a + \frac{1}{2} (\partial_a \phi) H_b{}^a{}_c w_+^c + \frac{1}{4} H_a{}^d{}_b H_d{}^a{}_c w_+^c , \end{aligned} \tag{3.102}$$

and since covariant derivative commutes with tracing,

$$\begin{aligned} D_b D_a w_+^a &= \nabla_b (\nabla_a w_+^a - 2(\partial_a \phi) w_+^a) \\ &= \nabla_b \nabla_a w_+^a - 2\nabla_b (\partial_a \phi) w_+^a - 2(\partial_a \phi) \nabla_b w_+^a , \end{aligned} \tag{3.103}$$

so

$$\begin{aligned} R_{ab} w_+^a &= [D_a, D_b] w_+^a \\ &= D_a D_b w_+^a - D_b D_a w_+^a \\ &= [\nabla_a, \nabla_b] w_+^a - \frac{1}{4} H_{acd} H_b{}^{cd} w_+^a + 2\nabla_b \nabla_a \phi w_+^a - \frac{1}{2} \nabla^c H_{cab} w_+^a + \frac{1}{2} (\partial^c \phi) H_{cab} w_+^a \\ &= \mathcal{R}_{ab} w_+^a - \frac{1}{4} H_{acd} H_b{}^{cd} w_+^a + 2\nabla_b \nabla_a \phi w_+^a - \frac{1}{2} e^{2\phi} \nabla^c \big(e^{-2\phi} H_{cab} \big) w_+^a , \end{aligned} \tag{3.104}$$

by (B.19) and symmetry of $\mathcal{R}_{ab}$. ■

Remark 3.9. If the frames are aligned using tensor $R_{\bar{a}b}$ in the alternative definition (3.100), we will have a sign flip in the antisymmetric part

$$R_{ab} = \mathcal{R}_{ab} - \frac{1}{4}H_{acd}H_b{}^{cd} + 2\nabla_a\nabla_b\phi + \frac{1}{2}e^{2\phi}\nabla^c\big(e^{-2\phi}H_{cab}\big). \tag{3.105}$$

This can be seen by observing the unique operators and consider a sign flip on H.

No contractions can be made for generalized Ricci tensor. However, note that by the virtue of $Spin(p,q)\times Spin(q,p)$-covariance, the generalized Ricci tensor can also be defined by its action on spinors by operators (3.97) as

$$\frac{1}{2}R_{a\bar{b}}\gamma^a\epsilon^+ = [\gamma^a D_a, D_{\bar{b}}]\epsilon^+, \qquad \frac{1}{2}R_{\bar{a}b}\gamma^{\bar{a}}\epsilon^- = [\gamma^{\bar{a}} D_{\bar{a}}, D_b]\epsilon^-. \tag{3.106}$$

This definition works for ordinary geometry, and by Clifford algebra, the conventional Ricci scalar of a Levi-Civita connection, where $\mathcal{R}_{ab}$ is symmetric, has expression

$$\begin{aligned}
-\frac{1}{4}\mathcal{R}\epsilon &= -\frac{1}{4}\mathcal{R}_{ab}g^{ab}\epsilon = -\frac{1}{4}\mathcal{R}_{ab}\gamma^a\gamma^b\epsilon = -\frac{1}{2}\gamma^a[\gamma^b\nabla_b, \nabla_a]\epsilon \\
&= -\frac{1}{2}\big(\gamma^a\gamma^b\nabla_b\nabla_a - \gamma^a\nabla_a\gamma^b\nabla_b\big)\epsilon = -\frac{1}{2}\big(2\nabla^a\nabla_a - 2\gamma^a\nabla_a\gamma^b\nabla_b\big)\epsilon \\
&= \big(\gamma^a\nabla_a\gamma^b\nabla_b - \nabla^a\nabla_a\big)\epsilon.
\end{aligned} \tag{3.107}$$

Therefore we obtain a sensible definition of the curvature scalar in generalized geometry.

Definition 3.6. The *generalized curvature scalar* of a torsion-free compatible connection D is the scalar S satisfying

$$-\frac{1}{4}S\epsilon^+ = \Big(\gamma^a D_a\gamma^b D_b - D^{\bar{a}}D_{\bar{a}}\Big)\epsilon^+, \tag{3.108}$$

for all $\epsilon^+ \in S(C_+)$.

Remark 3.10. It is obvious uniquely determined by (B,ϕ,g). Again, we have the equivalent definition

$$-\frac{1}{4}S\epsilon^- = \Big((\not{D}^-)^2 - D^-\cdot D^-\Big)\epsilon^-, \tag{3.109}$$

for all $\epsilon^- \in S(C_-)$.

Theorem 3.4. The generalized curvature scalar S is well-defined, and it has expression

$$S = \mathcal{R} + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{12}H^2. \tag{3.110}$$

Proof. It suffices to work in aligned frames by covariance. By (3.97),

$$
\begin{aligned}
D^a D_a \epsilon^+ &= \left(\nabla^a - \frac{1}{8} H^a{}_{bc}\gamma^{bc} - 2(\partial^a \phi)\right)\left(\nabla_a - \frac{1}{8} H_{ade}\gamma^{de}\right)\epsilon^+ \\
&= \nabla^a \nabla_a \epsilon^+ - \frac{1}{8}\nabla^a\big(H_{abc}\gamma^{bc}\epsilon^+\big) \\
&\quad - \frac{1}{8} H^a{}_{bc}\gamma^{bc}\nabla_a \epsilon^+ + \frac{1}{64} H^a{}_{bc}\gamma^{bc} H_{ade}\gamma^{de}\epsilon^+ \\
&\quad - 2(\partial^a \phi)\nabla_a \epsilon^+ + \frac{1}{4}(\partial^a\phi) H_{ade}\gamma^{de}\epsilon^+ \\
&= \nabla^a \nabla_a \epsilon^+ - \frac{1}{8}\gamma^{bc}\nabla^a H_{abc}\epsilon^+ \\
&\quad - \frac{1}{4}\gamma^{bc} H^a{}_{bc}\nabla_a\epsilon^+ + \frac{1}{64}\gamma^{bc}\gamma^{de} H^a{}_{bc} H_{ade}\epsilon^+ \\
&\quad - 2(\partial^a \phi)\nabla_a \epsilon^+ + \frac{1}{4}(\partial^a\phi) H_{ade}\gamma^{de}\epsilon^+,
\end{aligned}
\tag{3.111}
$$

since $\nabla\gamma = 0$. Also

$$
\begin{aligned}
\gamma^a D_a \gamma^b D_b \epsilon^+ &= \left(\gamma^a\nabla_a - \frac{1}{24} H_{def}\gamma^{def} - \gamma^a\partial_a\phi\right)\left(\gamma^b\nabla_b - \frac{1}{24} H_{hlk}\gamma^{hlk} - \gamma^b\partial_b\phi\right)\epsilon^+ \\
&= \gamma^a\nabla_a\gamma^b\nabla_b\epsilon^+ - \frac{1}{24}\gamma^a\nabla_a\big(H_{hlk}\gamma^{hlk}\epsilon^+\big) - \gamma^a\nabla_a\big(\gamma^b(\partial_b\phi)\epsilon^+\big) \\
&\quad - \frac{1}{24} H_{def}\gamma^{def}\gamma^b\nabla_b\epsilon^+ + \frac{1}{24\times 24} H_{def}\gamma^{def} H_{hlk}\gamma^{hlk}\epsilon^+ + \frac{1}{24} H_{def}\gamma^{def}\gamma^b(\partial_b\phi)\epsilon^+ \\
&\quad - \gamma^a(\partial_a\phi)\gamma^b\nabla_b\epsilon^+ + \frac{1}{24}\gamma^a(\partial_a\phi) H_{hlk}\gamma^{hlk}\epsilon^+ + \gamma^a(\partial_a\phi)\gamma^b(\partial_b\phi)\epsilon^+ \\
&= \gamma^a\nabla_a\gamma^b\nabla_b\epsilon^+ \\
&\quad - \frac{1}{24}\gamma^a\gamma^{hlk}\nabla_a H_{hlk}\epsilon^+ - \frac{1}{24}\gamma^a\gamma^{hlk} H_{hlk}\nabla_a\epsilon^+ \\
&\quad - \gamma^a\gamma^b(\nabla_a\nabla_b\phi)\epsilon^+ - \gamma^a\gamma^b(\partial_b\phi)\nabla_a\epsilon^+ \\
&\quad - \frac{1}{24}\gamma^{def}\gamma^b H_{def}\nabla_b\epsilon^+ + \frac{1}{24\times 24}\gamma^{def}\gamma^{hlk} H_{def} H_{hlk}\epsilon^+ + \frac{1}{24}\gamma^{def}\gamma^b H_{def}(\partial_b\phi)\epsilon^+ \\
&\quad - \gamma^a\gamma^b(\partial_a\phi)\nabla_b\epsilon^+ + \frac{1}{24}\gamma^a\gamma^{hlk}(\partial_a\phi) H_{hlk}\epsilon^+ + \gamma^a\gamma^b(\partial_a\phi)(\partial_b\phi)\epsilon^+.
\end{aligned}
\tag{3.112}
$$

Then

$$\begin{aligned}(\gamma^a D_a\gamma^b D_b - D^a D_a)\epsilon^+ &= (\gamma^a\nabla_a\gamma^b\nabla_b - \nabla^a\nabla_a)\epsilon^+\\ &+\left(-\frac{1}{24}\gamma^a\gamma^{hlk}\nabla_a H_{hlk} + \frac{1}{8}\gamma^{bc}\nabla^a H_{abc}\right)\epsilon^+\\ &+\left(-\frac{1}{24}\gamma^a\gamma^{hlk}H_{hlk} - \frac{1}{24}\gamma^{def}\gamma^a H_{def} + \frac{1}{4}\gamma^{bc}H^a{}_{bc}\right)\nabla_a\epsilon^+\\ &+\left(-\gamma^a\gamma^b(\nabla_a\nabla_b\phi) + \gamma^a\gamma^b\partial_a\phi\partial_b\phi\right)\epsilon^+\\ &+\left(-\gamma^a\gamma^b(\partial_b\phi) - \gamma^b\gamma^a(\partial_b\phi) + 2\partial^a\phi\right)\nabla_a\epsilon^+\\ &+\left(\frac{1}{24\times 24}\gamma^{def}\gamma^{hlk}H_{def}H_{hlk} - \frac{1}{64}\gamma^{bc}\gamma^{de}H^a{}_{bc}H_{ade}\right)\epsilon^+\\ &+\left(\frac{1}{24}\gamma^{def}\gamma^b H_{def}(\partial_b\phi) + \frac{1}{24}\gamma^a\gamma^{hlk}(\partial_a\phi)H_{hlk} - \frac{1}{4}(\partial^a\phi)H_{ade}\gamma^{de}\right)\epsilon^+.\end{aligned} \tag{3.113}$$

Let's exam the expressions in each line.

$$(\gamma^a\nabla_a\gamma^b\nabla_b - \nabla^a\nabla_a)\epsilon^+ = -\frac{1}{4}\mathcal{R}\epsilon^+, \tag{3.114}$$

by (3.107). Now use (3.98) we have identity

$$\begin{aligned}\gamma^a\gamma^{hlk} &= \gamma^{ahlk} + 3\cdot\eta^{a[h}\gamma^l\gamma^{k]}\\ &= \gamma^{ahlk} + \eta^{ah}\gamma^{lk} - \eta^{al}\gamma^{hk} + \eta^{ak}\gamma^{hl},\end{aligned} \tag{3.115}$$

which reduces the term

$$\begin{aligned}\frac{1}{24}\gamma^a\gamma^{hlk}\nabla_a H_{hlk} &= \frac{1}{24}\left(\gamma^{ahlk}\nabla_a H_{hlk} + \gamma^{lk}\nabla^a H_{alk} - \gamma^{hk}\nabla^a H_{hak} + \gamma^{hl}\nabla^a H_{hla}\right)\\ &= \frac{1}{24}\left(\gamma^{ahlk}\nabla_a H_{hlk}\right) + \frac{1}{8}\gamma^{bc}\nabla^a H_{abc},\end{aligned} \tag{3.116}$$

by antisymmetry. But

$$\gamma^{ahlk}\nabla_a H_{hlk} = \gamma^{ahlk}\nabla_{[a}H_{hlk]} = \gamma^{ahlk}dH_{ahlk} = 0, \tag{3.117}$$

because ∇ Levi-Civita and H closed. Thus

$$\left(-\frac{1}{24}\gamma^a\gamma^{hlk}\nabla_a H_{hlk} + \frac{1}{8}\gamma^{bc}\nabla^a H_{abc}\right)\epsilon^+ = \left(-\frac{1}{8}\gamma^{bc}\nabla^a H_{abc} + \frac{1}{8}\gamma^{bc}\nabla^a H_{abc}\right)\epsilon^+ = 0. \tag{3.118}$$

Again by (3.115), and another application of (3.98)

$$\gamma^{def}\gamma^{b} = \gamma^{defb} + \gamma^{ef}\eta^{db} - \gamma^{df}\eta^{eb} + \gamma^{de}\eta^{fb}, \tag{3.119}$$

we have reductions

$$\begin{aligned} \frac{1}{24}\gamma^{a}\gamma^{hlk}H_{hlk} &= \frac{1}{24}\gamma^{ahlk}H_{hlk} + \frac{1}{8}\gamma^{bc}H^{a}{}_{bc}, \\ \frac{1}{24}\gamma^{def}\gamma^{a}H_{def} &= -\frac{1}{24}\gamma^{adef}H_{def} + \frac{1}{8}\gamma^{bc}H^{a}{}_{bc}, \end{aligned} \tag{3.120}$$

so

$$\begin{aligned} &\left(-\frac{1}{24}\gamma^{a}\gamma^{hlk}H_{hlk} - \frac{1}{24}\gamma^{def}\gamma^{a}H_{def} + \frac{1}{4}\gamma^{bc}H^{a}{}_{bc}\right)\nabla_{a}\epsilon^{+} \\ &= \left(-2\times\frac{1}{8}\gamma^{bc}H^{a}{}_{bc} + \frac{1}{4}\gamma^{bc}H^{a}{}_{bc}\right)\nabla_{a}\epsilon^{+} = 0. \end{aligned} \tag{3.121}$$

A simple application of Clifford algebra yields

$$\begin{aligned} -\gamma^{a}\gamma^{b}(\nabla_{a}\nabla_{b}\phi) + \gamma^{a}\gamma^{b}\partial_{a}\phi\partial_{b}\phi &= -\frac{1}{2}\{\gamma^{a},\gamma^{b}\}(\nabla_{a}\nabla_{b}\phi) + \frac{1}{2}\{\gamma^{a},\gamma^{b}\}\partial_{a}\phi\partial_{b}\phi \\ &= -\left(\nabla^{2}\phi\right) + (\partial\phi)^{2}. \end{aligned} \tag{3.122}$$

Similarly,

$$\left(-\gamma^{a}\gamma^{b}(\partial_{b}\phi) - \gamma^{b}\gamma^{a}(\partial_{b}\phi) + 2\partial^{a}\phi\right)\nabla_{a}\epsilon^{+} = (-2\partial^{a}\phi + 2\partial^{a}\phi)\nabla_{a}\epsilon^{+} = 0. \tag{3.123}$$

Note that

$$\gamma^{hlk} = \frac{1}{3}\left(\gamma^{h}\gamma^{lk} - \gamma^{l}\gamma^{hk} + \gamma^{k}\gamma^{hl}\right),$$

and again by three application of (3.119)

$$\begin{aligned} \frac{1}{24\times 24}\gamma^{def}\gamma^{hlk}H_{def}H_{hlk} &= \frac{1}{24\times 24}\cdot\frac{1}{3}\cdot 3\left(\gamma^{defh}\gamma^{lk}H_{def}H_{hlk} + 3\cdot\gamma^{ef}\gamma^{lk}H^{a}{}_{ef}H_{alk}\right) \\ &= \frac{1}{24\times 24}\gamma^{defh}\gamma^{lk}H_{def}H_{hlk} + \frac{1}{24\times 8}\gamma^{ef}\gamma^{lk}H^{a}{}_{ef}H_{alk}. \end{aligned} \tag{3.124}$$

Here

$$\gamma^{defh}\gamma^{lk}H_{def}H_{hlk} = \gamma^{defh}\gamma^{lk}H_{[def}H_{hlk]} = \gamma^{defh}\gamma^{lk}(H\wedge H)_{defhlk} = 0, \tag{3.125}$$

since $H \wedge H = 0$ for being three form and abelian. Then

$$\begin{aligned}&\left(\frac{1}{24\times 24}\gamma^{def}\gamma^{hlk}H_{def}H_{hlk} - \frac{1}{64}\gamma^{bc}\gamma^{de}H^{a}{}_{bc}H_{ade}\right)\epsilon^{+}\\ &= \left(\frac{1}{24\times 8}\gamma^{ef}\gamma^{lk}H^{a}{}_{ef}H_{alk} - \frac{1}{64}\gamma^{bc}\gamma^{de}H^{a}{}_{bc}H_{ade}\right)\epsilon^{+}\\ &= \left(-\frac{2}{3}\cdot\frac{1}{64}\gamma^{bc}\gamma^{de}H^{a}{}_{bc}H_{ade}\right)\epsilon^{+}.\end{aligned} \tag{3.126}$$

Similar procedure using

$$\gamma^{de} = \frac{1}{2}\big(\gamma^{d}\gamma^{e} - \gamma^{e}\gamma^{d}\big), \qquad \gamma^{bc}\gamma^{d} = \gamma^{bcd} + \gamma^{b}\eta^{cd} - \gamma^{c}\eta^{bd}, \tag{3.127}$$

yields

$$\begin{aligned}\left(-\frac{2}{3}\cdot\frac{1}{64}\gamma^{bc}\gamma^{de}H^{a}{}_{bc}H_{ade}\right)\epsilon^{+} &= -\frac{2}{3}\cdot\frac{1}{64}\cdot\frac{1}{2}2\cdot\big(\gamma^{bcd}\gamma^{e}H^{a}{}_{bc}H_{ade} - 2\cdot\gamma^{c}\gamma^{e}H^{ab}{}_{c}H_{abe}\big)\\ &= \frac{2}{3}\cdot\frac{1}{64}\cdot 2\big(\gamma^{c}\gamma^{e}H^{ab}{}_{c}H_{abe}\big)\\ &= \frac{2}{3}\cdot\frac{1}{64}\cdot 2\big(\gamma^{ce}H^{ab}{}_{c}H_{abe} + \eta^{ce}H^{ab}{}_{c}H_{abe}\big)\\ &= \frac{1}{48}\big(H^{abc}H_{abc}\big)\\ &= \frac{1}{48}H^{2}.\end{aligned} \tag{3.128}$$

Finally, similar to the third line (3.121), apply again (3.115) and (3.119), we obtain

$$\frac{1}{24}\gamma^{def}\gamma^{b}H_{def}(\partial_{b}\phi) + \frac{1}{24}\gamma^{a}\gamma^{hlk}(\partial_{a}\phi)H_{hlk} = \frac{1}{24}\cdot 2\cdot 3(\partial^{a}\phi)H_{ade}\gamma^{de} = \frac{1}{4}(\partial^{a}\phi)H_{ade}\gamma^{de}, \tag{3.129}$$

so

$$\left(\frac{1}{24}\gamma^{def}\gamma^{b}H_{def}(\partial_{b}\phi) + \frac{1}{24}\gamma^{a}\gamma^{hlk}(\partial_{a}\phi)H_{hlk} - \frac{1}{4}(\partial^{a}\phi)H_{ade}\gamma^{de}\right)\epsilon^{+} = 0. \tag{3.130}$$

Gathering non-vanishing terms and by definition (3.108), upon multiplying factor -4 we have the expression (3.110) as desired. ■

Remark 3.11. The alternative definition (3.109) will yield exact the same expression.

Chapter 4

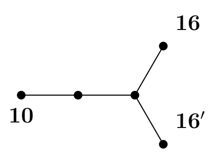
$O(9,1) \times O(1,9)$ Generalized Gravity

We are now ready to formulate the Type II theories in terms of generalized geometry. It will be a gravity theory with local $O(9,1) \times O(1,9)$ symmetry. The expressions of generalized curvatures in (3.101) and (3.110) may readily look familiar. Yet, before starting to establish the formalism, we first introduce a more complete picture of the Type II theories than we did in 2.1. As we have noted, not only the bosonic sector but also the fermionic sector to the leading order will fit into this formalism. Furthermore, the supersymmetry between the two sectors also admit generalized notions. Let's briefly review these degrees of freedom. Again, we are in democratic formalism [5] and basic conventions are listed in A.

4.1 d=10 Type II Supergravity

4.1.1 Supersymmetry

Type II theories have local $d = 10, N = 2$ supersymmetry. The classification of spinors [13] tells us that for $d = 10 \equiv 2$ MOD 8, signature $9 - 1 \equiv 0$ MOD 8, we have two irreducible real (Majorana) semi-spin representations of $Spin(9,1)$ dual to each other, i.e. different chirality. They are **16** and **16′** as in the Dynkin diagram



, (4.1)

where **10** is the fundamental representation for vectors. The local supersymmetries take form

$$\text{IIA: } M^{10|(1,1)}, \quad \text{IIB: } M^{10|(2,0)}, \tag{4.2}$$

which means the 32 supercharges are in spinorial representations

$$\text{IIA: } \mathbf{16'}\oplus\mathbf{16}, \quad \text{IIB: } \mathbf{16}\oplus\mathbf{16}, \tag{4.3}$$

so IIA is non-chiral and IIB is chiral. On a spinor manifold, we can build associated spinor bundles $S^-\oplus S^+$ and $2S^+$ then the supersymmetry variations are parameterizaed by sections $\epsilon\in S^\pm\oplus S^+$, which has decomposition

$$\begin{aligned} &\text{IIA: } \epsilon=\epsilon^++\epsilon^-, && \gamma^{(10)}\epsilon^\pm=\mp\epsilon^\pm,\\ &\text{IIB: } \epsilon=\begin{pmatrix}\epsilon^+\\ \epsilon^-\end{pmatrix}, && \gamma^{(10)}\epsilon^\pm=\epsilon^\pm, \end{aligned} \tag{4.4}$$

where in IIA we group the semi-spin representations into the irreducible Clifford Module, and $\gamma^{(10)}$ is the top gamma for the spin structure. The supersymmetry algebra is constructed by the symmetric map $S^\pm\otimes S^\pm\to V, \epsilon\otimes\epsilon'\mapsto\bar{\epsilon}^\pm\gamma\epsilon'^\pm$.

4.1.2 Fermionic Degrees of Freedom

Fermions are in R-NS, NS-R sectors. The fields are

$$\{\psi_\mu,\lambda\}, \tag{4.5}$$

where ψ_μ are gravitini, λ are dilatini. Similar to supercharges, they come in pair, non-chiral in IIA, chiral in IIB. The corresponding representations for the spinor bundles are

$$\begin{aligned} &\text{IIA: } \{\mathbf{10}*(\mathbf{16'}\oplus\mathbf{16}),\mathbf{16}\oplus\mathbf{16'})\},\\ &\text{IIB: } \{\mathbf{10}*(\mathbf{16}\oplus\mathbf{16}),\mathbf{16'}\oplus\mathbf{16'})\}, \end{aligned} \tag{4.6}$$

where $*$ is the Cartan composite, so $\mathbf{10}*\mathbf{16},\mathbf{10}*\mathbf{16'}$ are the irreducible summands in $\mathbf{10}\otimes\mathbf{16},\mathbf{10}\otimes\mathbf{16'}$ with the largest highest integral weights, respectively; thus the index structure in (4.5). Note that ψ_μ,λ are in opposite chirality, and we also have the decomposition

$$\begin{aligned} \text{IIA:} \quad & \begin{aligned} \psi_\mu &= \psi_\mu^+ + \psi_\mu^-, \quad \gamma^{(10)}\psi_\mu^\pm = \mp\psi_\mu^\pm, \\ \lambda &= \lambda^+ + \lambda^-, \quad \gamma^{(10)}\lambda^\pm = \pm\lambda^\pm, \end{aligned} \\ \text{IIB:} \quad & \begin{aligned} \psi_\mu &= \begin{pmatrix} \psi_\mu^+ \\ \psi_\mu^- \end{pmatrix}, \quad \gamma^{(10)}\psi_\mu^\pm = \psi_\mu^\pm, \\ \lambda &= \begin{pmatrix} \lambda^+ \\ \lambda^- \end{pmatrix}, \quad \gamma^{(10)}\lambda^\pm = -\lambda^\pm. \end{aligned} \end{aligned} \tag{4.7}$$

To fit into generalized geometry, we consider instead of $\lambda^\pm$ the fields

$$\rho^\pm := \gamma^\mu \psi_\mu^\pm - \lambda^\pm. \tag{4.8}$$

Since the map $\gamma : V \otimes S^\pm \to S^\mp$,

$$\text{IIA} \;: \rho^\pm \in S^\pm, \quad \text{IIB} \;: \rho^\pm \in S^-. \tag{4.9}$$

4.1.3 Equations of Motions and SUSY Variations

The fermionic actoin, up to quadratic order in fermions, is [4]

$$\begin{aligned} S_F = -\frac{1}{2\kappa^2}\int \sqrt{-g}\Big[& \\ & e^{-2\phi}\Big(2\bar{\psi}^+ \cdot \nabla\!\!\!/\, \psi^+ - 4\bar{\psi}^+ \cdot \nabla\rho^+ - 2\bar{\rho}^+ \nabla\!\!\!/\, \rho^+ \\ & \quad +2\bar{\psi}^- \cdot \nabla\!\!\!/\, \psi^- - 4\bar{\psi}^- \cdot \nabla\rho^- - 2\bar{\rho}^- \nabla\!\!\!/\, \rho^- \\ & \quad -\frac{1}{2}\bar{\psi}^+ \cdot H\!\!\!/\, \psi^+ - H(\bar{\psi}^+ \gamma \psi^+) - \frac{1}{2}H(\bar{\rho}^+ \gamma^{(2)} \psi^+) + \frac{1}{2}\bar{\rho}^+ H\!\!\!/\, \rho^+ \\ & \quad +\frac{1}{2}\bar{\psi}^- \cdot H\!\!\!/\, \psi^- + H(\bar{\psi}^- \gamma \psi^-) + \frac{1}{2}H(\bar{\rho}^- \gamma^{(2)} \psi^-) - \frac{1}{2}\bar{\rho}^- H\!\!\!/\, \rho^-\Big) \\ & \quad -\frac{1}{4}e^{-\phi}\Big(\bar{\psi}_\mu^+ \gamma^\nu F\!\!\!/_{(B)} \gamma^\mu \psi_\nu^- + \bar{\rho}^+ F\!\!\!/_{(B)} \rho^-\Big)\Big], \end{aligned} \tag{4.10}$$

where ∇ is the Levi-Civita connection. The equations of motion, up to linear order of fermions, are

$$\begin{aligned} \frac{1}{16}e^\phi \sum_n (\pm 1)^{[(n+1)/2]} \gamma^\nu F\!\!\!/^{(n)}_{(B)} \gamma_\mu \psi_\nu^\mp = & \left(\nabla\!\!\!/ \mp \frac{1}{24} H\!\!\!/ - \partial\!\!\!/ \phi\right)\psi_\mu^\pm \pm \frac{1}{2}\gamma^\nu H_{\nu\mu}{}^\lambda \psi_\lambda^\pm \\ & - \left(\nabla_\mu \mp \frac{1}{8} H_{\mu\nu\lambda}\gamma^{\nu\lambda}\right)\rho^\pm, \end{aligned} \tag{4.11}$$

$$\begin{aligned}\frac{1}{16}e^{\phi}\sum_n(\pm1)^{[(n+1)/2]}\not{F}^{(n)}_{(B)}\rho^{\mp} = &\left(\nabla_\mu \mp \frac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda} - 2\partial_\mu\phi\right)\psi^{\pm\mu}\\ &-\left(\not{\nabla} \mp \frac{1}{24}\not{H} - \not{\partial}\phi\right)\rho^{\pm},\end{aligned} \tag{4.12}$$

which correspond to variations $\delta\psi^{\pm}_{\mu}, \delta\rho^{\pm}$ of the action (4.10), respectively. We also include here the bosonic pseudo-action (2.1) and equations of motion (2.6) for completeness:

$$S_B = \frac{1}{2\kappa^2}\int\sqrt{-g}\left[e^{-2\phi}\left(\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{12}H^2\right) - \frac{1}{4}\sum_n\frac{1}{n!}\left(F^{(n)}_{(B)}\right)^2\right], \tag{4.13}$$

and

$$\begin{aligned}
0 &= \mathcal{R}_{\mu\nu} - \frac{1}{4}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}e^{2\phi}\sum_n\frac{1}{(n-1)!}F^{(n)}_{(B)\mu\alpha_1\ldots\alpha_{n-1}}F^{(n)}_{(B)\nu}{}^{\alpha_1\ldots\alpha_{n-1}},\\
0 &= \nabla^\alpha\left(e^{-2\phi}H_{\alpha\mu\nu}\right) - \frac{1}{2}\sum_n\frac{1}{(n-2)!}F^{(n)}_{(B)\mu\nu\alpha_1\ldots\alpha_{n-2}}F^{(n-2)\alpha_1\ldots\alpha_{n-2}}_{(B)},\\
0 &= \nabla^2\phi - (\nabla\phi)^2 + \frac{1}{4}\mathcal{R} - \frac{1}{48}H^2,\\
0 &= dF_{(B)} - H\wedge F_{(B)}.
\end{aligned} \tag{4.14}$$

Now we have the type II supergravity theories invariant under the supersymmetry transformations [4]

$$\begin{aligned}
\delta e^a_\mu &= \bar{\epsilon}^+\gamma^a\psi^+_\mu + \bar{\epsilon}^-\gamma^a\psi^-_\mu,\\
\delta B_{\mu\nu} &= 2\bar{\epsilon}^+\gamma_{[\mu}\psi^+_{\nu]} - 2\bar{\epsilon}^-\gamma_{[\mu}\psi^-_{\nu]},\\
\delta\phi - \frac{1}{4}\delta\log(-g) &= -\frac{1}{2}\bar{\epsilon}^+\rho^+ - \frac{1}{2}\bar{\epsilon}^-\rho^-,\\
\left(e^B\wedge\delta A\right)^{(n)}_{\mu_1\ldots\mu_n} &= \frac{1}{2}e^{-\phi}\left(\bar{\psi}^+_\nu\gamma_{\mu_1\ldots\mu_n}\gamma^\nu\epsilon^- - \bar{\epsilon}^+\gamma_{\mu_1\ldots\mu_n}\rho^-\right)\\
&\quad\mp\frac{1}{2}e^{-\phi}\left(\bar{\epsilon}^+\gamma^\nu\gamma_{\mu_1\ldots\mu_n}\psi^-_\nu + \bar{\rho}^+\gamma_{\mu_1\ldots\mu_n}\epsilon^-\right),
\end{aligned} \tag{4.15}$$

for bosons, where e_μ is an g-orthonormal frame and the sign in the last line is minus for IIA, plus for IIB, and

$$\begin{aligned}
\delta\psi^{\pm}_\mu &= \left(\nabla_\mu \mp \frac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda}\right)\epsilon^{\pm} + \frac{1}{16}e^{\phi}\sum_n(\pm1)^{[(n+1)/2]}\not{F}^{(n)}_{(B)}\gamma_\mu\epsilon^{\mp},\\
\delta\rho^{\pm} &= \left(\not{\nabla} \mp \frac{1}{24}\not{H} - \not{\partial}\phi\right)\epsilon^{\pm},
\end{aligned} \tag{4.16}$$

for fermions.

4.2 Generalized Geometry Formalism

Now set us into the generalized geometry studied in Chapter 2, 3, and we start establishing the isomorphism between supergravities and generalized gravities. Here we will be interested in the case $(p, q) = (9, 1)$.

4.2.1 Supergtavity Fields

NSNS Sector

By theorem 3.1, we immediately have the correspondence between bosonic triplet and the generalized metric

$$(B, \phi, g) \simeq (G, \Phi) \sim \frac{O(d,d)}{O(9,1) \times O(1,9)} \times \mathbb{R}^+. \tag{4.17}$$

The generalized tangent bundle E indeed encodes the NS-NS gauge symmetry.

Proposition 4.1. The variation

$$\delta_V G = L_V G, \qquad \delta_V \Phi = L_V \Phi, \tag{4.18}$$

for $V \in E$, is equivalent to the infinitesimal NS-NS bosonic variation (2.12).

Proof. In coordinate frames, generalized tensor G has form (3.64) while the Dorfman derivative takes form of adjoint actions by matrix (3.20). Let $a\cdot$ denote the adjoint action on the entries, then for $V = v + \lambda$, the direct computation yields

$$\begin{aligned} L_V G &= V^A \partial_A G - a \cdot \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} d\lambda g^{-1}B - B_{\mu\beta} d\lambda_{\nu\alpha} g^{-1\beta\alpha} & d\lambda g^{-1} \\ d\lambda g^{-1} & 0 \end{pmatrix} \\ &= \mathcal{L}_v \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} d\lambda g^{-1}B + Bg^{-1}d\lambda & d\lambda g^{-1} \\ -g^{-1}d\lambda & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} g' - B'g'^{-1}B' & -B'g'^{-1} \\ g'^{-1}B' & g'^{-1} \end{pmatrix}, \end{aligned}$$

where $g' = \mathcal{L}_v g$, $g'^{-1} = \mathcal{L}_v g^{-1}$, and $B' = B - d\lambda$ are exactly variations of g, B in (2.12). For ϕ, it is by definition (3.15) that $L_V \Phi = \mathcal{L}_v \Phi$. ■

Remark 4.1. Note that

$$[\delta_V,\delta_W]=\delta_{[\![V,W]\!]}, \tag{4.19}$$

which can be seen by the Jacobi's identity when action is restricted on E, and the general case follows. Therefore the variation by the generalized tangents is an algebra homomorphism from the Courant bracket on E to the Lie bracket of $\mathfrak{g}_{\mathrm{NS}}$, the Lie algebra of $\mathrm{Diff}(M)\ltimes\Omega^2_{closed}(M)$. Thus the Courant bracket generalizes the ordinary Lie algebra, and yields the notion of Courant algebroid in studying the symmetries.

Fermions

The two components of fermions now fit into the $Spin(9,1)\times Spin(1,9)$-covariance in the generalized geometry. Recall the $Spin(9,1)$-spinor bundles $S(C_\pm)$ defined in 3.2.2 with gamma matrices (3.74), and let $S^\pm(C_+),S^\pm(C_-)$ be the further decomposition $\mathbf{16}\oplus\mathbf{16}'$ under the chirality. Now by (4.4), (4.7), and (4.9), we have the $Spin(9,1)\times Spin(1,9)$-covariant definition of fermions in Type II theories, as generalized objects

$$\begin{gathered}\text{IIA}\\ \begin{aligned}\psi^+_{\bar a}&\in\Gamma\big(C_-\otimes S^-(C_+)\big), & \psi^-_{a}&\in\Gamma\big(C_+\otimes S^+(C_-)\big),\\ \rho^+&\in\Gamma\big(S^+(C_+)\big), & \rho^-&\in\Gamma\big(S^-(C_-)\big),\\ \epsilon^+&\in\Gamma\big(S^-(C_+)\big), & \epsilon^-&\in\Gamma\big(S^+(C_-)\big),\end{aligned}\\ \text{IIB}\\ \begin{aligned}\psi^\pm_{\bar a/a}&\in\Gamma\big(C_\mp\otimes S^+(C_\pm)\big),\\ \rho^\pm&\in\Gamma\big(S^-(C_\pm)\big),\\ \epsilon^\pm&\in\Gamma\big(S^+(C_\pm)\big),\end{aligned}\end{gathered} \tag{4.20}$$

RR Fields

Consider the locally defined polyform field strengths $F_{(i)}=dA_{(i)}$. The patching (2.9) of local A potentials implies that $\{F_{(i)}\}$ has patching (2.22), and therefore we find the generalized objects being generalized spinors

$$\begin{aligned}\text{IIA:}\quad & F\in\Gamma(S^+_{(1/2)}),\\ \text{IIB:}\quad & F\in\Gamma(S^-_{(1/2)}).\end{aligned} \tag{4.21}$$

Recall that $S^{\pm}_{(1/2)} = S^{\pm}$ is the weight-half generalized spinors of $(E, \langle\cdot,\cdot\rangle)$, and the $\pm$ chirality corresponds exactly to even and odd forms. In fact, T-duality suggests that R-R field strengths have $Spin(10,10)$-trasnformatations [4, 17]. Comparing with the definition (2.4), the splitting (3.13) implies the image $F_{(B)}$ in $\Lambda^\bullet T^*$ of F under B-splitting is precisely the $F^{(n)}_{(B)}$ formulated in the Type II supergravities.

The (B,ϕ,g)-structure allows us to do further factorization (3.76), with additional chiralities (3.78). Using the the spinor norms on $S(C_\pm)$, which is namely the intertwiners $C, \bar{C} : S(C_\pm) \to S^*(C_\pm)$ defined in (A.3), we may introduce operators associated to $F \in S_{(1/2)}$,

$$\begin{aligned} F_\# &= F\bar{C} : S(C_-) \to S(C_+), \\ F^T_\# &= CF : S(C_+) \to S(C_-), \end{aligned} \tag{4.22}$$

The index structure looks like $(F_\#)^a{}_{\bar{a}} = F^{a\bar{b}}\bar{C}_{\bar{b}\bar{a}}$ and $(F^T_\#)^{\bar{a}}{}_a = C_{ab}F^{b\bar{a}}$, so there involves a transpose in the latter case. We explicitly define such an operator. As usual, (B,ϕ,g) defines a split

$$S_{(1/2)} \xrightarrow{\sim} \Lambda^\bullet T^*, \qquad F \mapsto F_{(B,\phi)} := e^\phi e^B \wedge F, \tag{4.23}$$

and let the frames aligned $e^+ = e^- = e$, so $S(C_+) \simeq S(C_-)$, and then we define the map using $\Lambda^\bullet T^* \simeq \mathrm{Cliff}(9,1;\mathbb{R})$

$$F_\# = \not{F}_{(B,\phi)} := \sum_n \frac{1}{n!} F^{(n)}_{(B,\phi)\,a_1\ldots a_n} \gamma^{a_1\ldots a_n}. \tag{4.24}$$

It extends to a map in (4.22) by $Spin(9,1) \times Spin(1,9)$-covariance. Using the identity $\mathrm{Tr}(\gamma^{a_1\ldots a_n}\gamma_{b_1\ldots b_n}) = (-1)^{[n/2]}\,\mathrm{Tr}(I)\cdot\delta^{a_1}_{b_1}\ldots\delta^{a_n}_{b_n}$, we have the relation

$$F = e^{-\phi}e^{-B} \wedge \frac{1}{32}\sum_n \frac{1}{n!}\Big[(-1)^{[n/2]}\,\mathrm{Tr}\left(\gamma_{(n)}F_\#\right)\Big], \tag{4.25}$$

where $\gamma_{(n)} = \gamma_{a_1\ldots a_n}e^{a_1}\ldots e^{a_n}$, and 32 comes from the dimension of the spinor space. $A_\#$ can also be defined for R-R fields A, but only locally because its gauge symmetry is not in $O(d,d)$-formalism.

Finally, the self-duality conditions (2.5) become the chirality conditions

$$\Gamma^{(-)}F = -F, \tag{4.26}$$

where $\Gamma^{(-)}$ is defined in (3.79). Indeed, for $d = 10$, $(-1)^{[(10-n)/2]} = (-1)^{[(n+1)/2]+1}$.

4.2.2 Supersymmetry

Bosons

Note that the generalized metric $(B,\phi,g)\simeq(G,\Phi)\simeq\tilde{G}$ encodes its structure on frames (3.61).

Definition 4.1. We define the $Spin(9,1)\times Spin(1,9)$-covariant generalized SUSY transformations $\underline{\delta}$ as

$$\begin{aligned} \underline{\delta}\hat{E}^{+}_{a} &= (\delta\log\Phi)\hat{E}^{+}_{a} - (\delta\Lambda_{a\bar{b}})\hat{E}^{-\bar{b}}, \\ \underline{\delta}\hat{E}^{-}_{\bar{a}} &= (\delta\log\Phi)\hat{E}^{-}_{\bar{a}} - (\delta\Lambda_{b\bar{a}})\hat{E}^{+b}, \\ \underline{\delta}A_{\#} &= 16\big(\gamma^{a}\epsilon^{+}\bar{\psi}^{-}_{a} - \rho^{+}\bar{\epsilon}^{-}\big) \mp 16\big(\psi^{+}_{\bar{a}}\bar{\epsilon}^{-}\gamma^{\bar{a}} + \epsilon^{+}\bar{\rho}^{-}\big), \end{aligned} \tag{4.27}$$

where

$$\delta\log\Phi = \bar{\epsilon}^{+}\rho^{+} + \bar{\epsilon}^{-}\rho^{-}, \qquad \delta\Lambda_{a\bar{a}} = \bar{\epsilon}^{+}\gamma_{a}\psi^{+}_{\bar{a}} + \bar{\epsilon}^{-}\gamma_{\bar{a}}\psi^{-}_{a}, \tag{4.28}$$

and in the expression for $\underline{\delta}A_{\#}$, the upper sign is for IIA, lower for IIB.

Proposition 4.2. $\underline{\delta}$ reproduces the supersymmetry (4.15).

Remark 4.2. Note that for $Spin(9,1)\times Spin(1,9)$-covariance in generalized geometry, the variation of a single orthonormal frame is extended to a pair of $O(9,1)\times O(1,9)$-frames.

Proof. Note first that

$$\delta\log\Phi = -2\delta\phi + \frac{1}{2}\delta\log(-g) \tag{4.29}$$

is consistent with the third line of (4.15). By (3.61), the variations of $\tilde{G}$-frames in (4.27) and the form of parameter (4.28) imply the variation δB for aligned frames, and the variations of a pair of g-orthonormal frames $\{e^{\pm}\}$

$$\underline{\delta}e^{+a}_{\mu} = \bar{\epsilon}^{+}\gamma_{\mu}\psi^{+a} + \bar{\epsilon}^{-}\gamma^{a}\psi^{-}_{\mu}, \qquad \underline{\delta}e^{-\bar{a}}_{\mu} = \bar{\epsilon}^{+}\gamma^{\bar{a}}\psi^{+}_{\mu} + \bar{\epsilon}^{-}\gamma_{\mu}\psi^{-\bar{a}}, \tag{4.30}$$

which by (3.62) both yield the SUSY variation of g

$$\delta g_{\mu\nu} = 2\bar{\epsilon}^{+}\gamma_{(\mu}\psi^{+}_{\nu)} + 2\bar{\epsilon}^{-}\gamma_{(\mu}\psi^{-}_{\nu)}. \tag{4.31}$$

When the two frames are aligned $e^{+a} = e^{-a} = e^a$, these variations only differ from (4.15) by Lorentz transformations

$$\underline{\delta} e_\mu^{+a} = \delta e_\mu^{+a} - \Lambda^{+a}{}_b e_\mu^{+b}, \qquad \underline{\delta} e_\mu^{-a} = \delta e_\mu^{-a} - \Lambda^{-a}{}_b e_\mu^{-b}, \tag{4.32}$$

where $\Lambda^{\pm a}{}_b = \bar{\epsilon}^\pm \gamma^a \psi_b^\pm - \bar{\epsilon}^\pm \gamma_b \psi^{\pm a}$.

For R-R potentials, we compare (4.25) with the generalized variation in aligned frames. Note that by (A.3) and (A.5), together with identity

$$\begin{aligned} \bar{\epsilon}\gamma_{(n)}\gamma\chi &= \epsilon^T C \gamma_{(n)} C^{-1} C \gamma C^{-1} C \chi \\ &= (-1)^{[(n+1)/2]+1} \left(\chi^T C \gamma \gamma_{(n)} \epsilon\right)^T \\ &= (-1)^{[(n+1)/2]+1} \bar{\chi} \gamma \gamma_{(n)} \epsilon, \end{aligned} \tag{4.33}$$

where there is an extra minus because of supersymmetry, we can rewrite δA in (4.15) as

$$\begin{aligned} \left(e^B \wedge \delta A\right)^{(n)}_{\mu_1 \dots \mu_n} &= \frac{1}{2} e^{-\phi} (-1)^{[(n+1)/2]} \Big[\left(-\bar{\epsilon}^- \gamma^\nu \gamma_{\mu_1 \dots \mu_n} \psi_\nu^+ - \bar{\rho}^- \gamma_{\mu_1 \dots \mu_n} \epsilon^+\right) \\ &\quad \mp \left(-\bar{\psi}_\nu^- \gamma_{\mu_1 \dots \mu_n} \gamma^\nu \epsilon^+ + \bar{\epsilon}^- \gamma_{\mu_1 \dots \mu_n} \rho^+\right) \Big] \\ &= \frac{1}{2} e^{-\phi} (-1)^n (-1)^{[n/2]} \operatorname{Tr} \Big[\left(\gamma_{\mu_1 \dots \mu_n} \psi_\nu^+ \bar{\epsilon}^- \gamma^\nu + \gamma_{\mu_1 \dots \mu_n} \epsilon^+ \bar{\rho}^-\right) \\ &\quad \mp \left(\gamma_{\mu_1 \dots \mu_n} \gamma^\nu \epsilon^+ \bar{\psi}_\nu^- - \gamma_{\mu_1 \dots \mu_n} \rho^+ \bar{\epsilon}^-\right) \Big], \end{aligned} \tag{4.34}$$

where in the last equation the supersymmetry applies again. Now by (4.25) we find this is consistent with the covariant form

$$\underline{\delta} A_\#^{(n)} = (-1)^n 16 \Big[(\psi_{\bar{a}}^+ \bar{\epsilon}^- \gamma^{\bar{a}} + \epsilon^+ \bar{\rho}^-) \mp (\gamma^a \epsilon^+ \bar{\psi}_a^- - \rho^+ \bar{\epsilon}^-) \Big]. \tag{4.35}$$

Noting that $(-1)^n$ and $\mp$ in the middle are -1 with upper sign for IIA, $+1$ lower sign for IIB, respectively, we are done. ■

Fermions

Definition 4.2. We continue to define the $Spin(9,1) \times Spin(1,9)$-covariant generalized SUSY transformations $\underline{\delta}$

$$\begin{aligned} \underline{\delta} \psi_{\bar{a}}^+ &= D_{\bar{a}} \epsilon^+ + \frac{1}{16} F_\# \gamma_{\bar{a}} \epsilon^-, \qquad & \underline{\delta} \rho^+ &= \gamma^a D_a \epsilon^+, \\ \underline{\delta} \psi_a^- &= D_a \epsilon^- + \frac{1}{16} F_\#^T \gamma_a \epsilon^+, \qquad & \underline{\delta} \rho^- &= \gamma^{\bar{a}} D_{\bar{a}} \epsilon^-. \end{aligned} \tag{4.36}$$

Proposition 4.3. $\underline{\delta}$ reproduces the supersymmetry (4.16).

Proof. This is a direct observation of the definition of unique operators (3.97) and the definition of $F_\#, F_\#^T$ in (4.24). In particular, from (4.22) we have

$$F_\#^T = (CF_\# C^{-1})^T, \tag{4.37}$$

and by (A.5) one can see the sign $(-1)^{[(n+1)/2]}$ is contained in $F_\#^T$. ■

4.2.3 Equations of Motion

We finalize the formalism with $Spin(9,1)\times Spin(1,9)$-generalized gravity theories.

Theorem 4.1. The generalized gravity of Type II supergravity theories has bosonic pseudo-action

$$S_B = \frac{1}{2\kappa^2}\int \left(\Phi S + \frac{1}{4}(F, \Gamma^{(-)}F)\right), \tag{4.38}$$

where S is the generalized curvature scalar, $(\cdot,\cdot)$ is the Mukai pairing defined in (2.23), with equations of motion

$$\begin{aligned} 0 &= R_{a\bar{b}} - \frac{1}{8}\Phi^{-1}(F, \Gamma_{a\bar{b}}F),\\ 0 &= S,\\ 0 &= dF, \end{aligned} \tag{4.39}$$

where Φ^{-1} identifies $\tilde{E}$ and E, and d is defined in (2.24) and (3.25). They are local $O(9,1)\times O(1,9)$-covariant forms of (4.13) and (4.14).

Proof. The first statment follows from theorem 3.4 and definitions. Let's verify the expression

$$\begin{aligned} (F, \Gamma^{(-)}F) &= -(F,F)\\ &= -(\sum_n (-1)^{[(10-n)/2]} * F^{(n)}, F)\\ &= -\sum_n (-1)^{[(n+1)/2]+1}(-1)^{[(n+1)/2]} * F^{(n)}\wedge F^{(n)}\\ &= -\sum_n (F^{(n)})^2, \end{aligned} \tag{4.40}$$

where we denote $F_{(B)}$ as F, and the self-duality were used twice.[1]

Note that (3.110) is different from (4.13) by an integration by part

$$\begin{aligned}\int e^{-2\phi}\sqrt{-g}(4\nabla^2\phi - 4(\partial\phi)^2) &= \int e^{-2\phi}\sqrt{-g}(8(\nabla\phi)^2 - 4(\partial\phi)^2)\\ &= \int e^{-2\phi}\sqrt{-g}(4(\partial\phi)^2).\end{aligned} \tag{4.41}$$

(3.110) also proves the $\delta\phi$ equation of motion, and Bianchi identity is just by definition. The first equation of motion encodes both $\delta g_{\mu\nu}$ and $\delta B_{\mu\nu}$ bosonic equations of motion as symmetric and antisymmetric parts in aligned frames. The NS-NS sector is handled by the theorem 3.3.

For R-R fields, let $\{\hat{e}_a^+\}, \{\hat{e}_{\bar{a}}^-\}$ be a pair of g-orthonormal bases, by the action of the Clifford algebra (3.80) in $\tilde{G}$-frames under (B)-split, we find

$$\Gamma_{a\bar{b}} = \frac{1}{2}[\Gamma_a, \Gamma_{\bar{b}}] = \Gamma_a\Gamma_{\bar{b}}, \tag{4.42}$$

since $\{\Gamma_a, \Gamma_{\bar{b}}\} = 0$, and

$$\begin{aligned}\Gamma_a\Gamma_{\bar{b}}F &= (i_{\hat{e}_a^+} + e_a^+\wedge)(i_{\hat{e}_{\bar{b}}^-} - e_{\bar{b}}^-\wedge)F\\ &= F_{\bar{b}a} + e_{\bar{b}}^- \wedge F_a + e_a^+ \wedge F_{\bar{b}} - e_a^+ \wedge e_{\bar{b}}^- \wedge F,\end{aligned} \tag{4.43}$$

where we denote F_a for $i_{\hat{e}_a^+}F$ and similar for $\bar{b}$, and we used $\{i_{\hat{e}_a^+}, e_{\bar{b}}^-\} = 0$ by $Spin(9,1)\times Spin(1,9)$-covariance. Then by self-duality

$$\begin{aligned}(F, \Gamma_a\Gamma_{\bar{b}}F) &= (\sum_n (-1)^{[(10-n)/2]} * F^{(n)}, \Gamma_a\Gamma_{\bar{b}}F)\\ &= -\sum_n \Big(* F^{(n)} \wedge F_{\bar{b}a}^{(n+2)}\Big)\\ &\quad - \sum_n \Big(* F^{(n)} \wedge e_{\bar{b}}^- \wedge F_a^{(n)}\Big)\\ &\quad - \sum_n \Big(* F^{(n)} \wedge e_a^+ \wedge F_{\bar{b}}^{(n)}\Big)\\ &\quad + \sum_n \Big(* F^{(n)} \wedge e_a^+ \wedge e_{\bar{b}}^- \wedge F^{(n-2)}\Big)\\ &= \sum_n \Big(F_{\bar{b}a}^{(n)} \cdot F^{(n-2)} + F_a^{(n)} \cdot F_{\bar{b}}^{(n)} + F_{\bar{b}}^{(n)} \cdot F_a^{(n)} - F_{a\bar{b}}^{(n)} \cdot F^{(n-2)}\Big),\end{aligned} \tag{4.44}$$

[1] Note that our convention (A.1) on Hodege dual is $*F \wedge F = -F^2$, where the last sign comes from the Minkowski signature.

which in aligned frames becomes

$$(F,\Gamma_a\Gamma_b F) = 2\sum_n \Big(F_a^{(n)}\cdot F_b^{(n)} - F_{ab}^{(n)}\cdot F^{(n-2)}\Big). \tag{4.45}$$

Noting that Φ^{-1} provides a factor of $e^{2\phi}$ (which also pulls a density into a tensor) and the constant coefficient gives the correct normalization compare to (3.101), we obtain the R-R terms in $\delta g, \delta B$ equations of motion. ■

Theorem 4.2. The Type II generalized gravity has the fermionic action

$$\begin{aligned} S_F = -\frac{1}{2\kappa^2}\int 2\Phi\Big[& \operatorname{Tr}\bar\psi^+ \not{D}^+\psi^+ + \operatorname{Tr}\bar\psi^-\not{D}^-\psi^- \\ & +2\bar\rho^+ D^+\cdot\psi^+ + 2\bar\rho^- D^-\cdot\psi^- \\ & -\bar\rho^+\not{D}^+\rho^+ - \bar\rho^-\not{D}^-\rho^- \\ & -\frac{1}{8}\big(\bar\rho^+ F_\#\rho^- + \bar\psi^+_{\bar a}\gamma^a F_\#\gamma^{\bar a}\psi^-_a\big)\Big], \end{aligned} \tag{4.46}$$

with equations of motion

$$\begin{aligned} \frac{1}{16}\gamma^b F_\#\gamma_{\bar a}\psi^-_b &= \not{D}^+\psi^+_{\bar a} - D^+_{\bar a}\rho^+, \\ \frac{1}{16}\gamma^{\bar b} F^T_\#\gamma_a\psi^-_{\bar b} &= \not{D}^-\psi^-_a - D^-_a\rho^-, \\ -\frac{1}{16}F_\#\rho^- &= \not{D}^+\rho^+ - D^+\cdot\psi^+, \\ -\frac{1}{16}F^T_\#\rho^+ &= \not{D}^-\rho^- - D^-\cdot\psi^-, \end{aligned} \tag{4.47}$$

where the unique generalized operators are defined in (3.97). These generalized objects provide the $Spin(9,1)\times Spin(1,9)$-covariance for fermionic sectors (4.10) and (4.11), (4.12).

The proof is a trivial practice to rewrite fermionic sectors in terms of generalized objects.

Theorem 4.3. The generalized SUSY variations (4.36) of the fermionic equations of motion recovers the bosonic ones (4.39).

Proof. Note that the first and last two equations arise the equivalent variations by the symmetry of $Spin(9,1)\times Spin(1,9)$-compatibility, so it suffices to consider one of them from each group.

Varying the first equation yields

$$\frac{1}{16}\gamma^a F_\# \gamma_{\bar{a}}\left(D_a\epsilon^- + \frac{1}{16}F_\#^T\gamma_a\epsilon^+\right) = \gamma^a D_a\left(D_{\bar{a}}\epsilon^+ + \frac{1}{16}F_\#\gamma_{\bar{a}}\epsilon^-\right) - D_{\bar{a}}\left(\gamma^a D_a\epsilon^+\right)$$
$$\frac{1}{16}\cdot\frac{1}{16}\gamma^a F_\#\gamma_{\bar{a}}F_\#^T\gamma_a\epsilon^+ = \gamma^a D_a D_{\bar{a}}\epsilon^+ - D_{\bar{a}}\gamma^a D_a\epsilon^+ + \frac{1}{16}(\gamma^a D_a F_\#)\gamma_{\bar{a}}\epsilon^-,$$

so we have

$$\begin{aligned}\left(\frac{1}{2}R_{a\bar{a}} - \frac{1}{16^2}\gamma_a F_\#\gamma_{\bar{a}}F_\#^T\right)\gamma^a\epsilon^+ &= 0,\\ \frac{1}{16}(\gamma^a D_a F_\#)\gamma_{\bar{a}}\epsilon^- &= 0,\end{aligned} \tag{4.48}$$

where the second equation is immediately $dF = 0$ by (3.25) and the fact that D is torsion-free. Trace of the first equation in particular tells us that

$$8R_{a\bar{a}} - \frac{1}{16^2}\operatorname{Tr}(\gamma_a F_\#\gamma_{\bar{a}}F_\#^T) = 0, \tag{4.49}$$

where a factor of 16 is the dimension of the chiral spinor ϵ^+. By definition (4.24), similar to (4.25), on chiral spinors, the intertwiners $C, \bar{C}$ are related to the Mukai pairing by factors of 16 in the isomorphism (3.76). Then by (3.77), under identifications by (B, ϕ, g), we have

$$\begin{aligned}(F, \Gamma_{a\bar{a}}F) &= (F, \gamma_a\gamma^{(10)}F\gamma^{(10)}\gamma_{\bar{a}})\\ &= \frac{1}{16^2}\operatorname{Tr}(C\gamma_a\gamma^{(10)}F\gamma^{(10)}\gamma_{\bar{a}}\bar{C}F^T)\\ &= \frac{1}{16^2}\operatorname{Tr}(C\gamma_a C^{-1}C\gamma^{(10)}F\gamma^{(10)}\gamma_{\bar{a}}F_\#)\\ &= \frac{1}{16^2}\operatorname{Tr}(C\gamma_a\gamma^{(10)}C^{-1}F_\#^T\gamma^{(10)}\gamma_{\bar{a}}F_\#)\\ &= \frac{1}{16^2}\operatorname{Tr}((\gamma_a\gamma^{(10)})^T F_\#\gamma_{\bar{a}}\gamma^{(10)}F_\#^T)\\ &= \frac{1}{16^2}\operatorname{Tr}(\gamma^{(10)}\gamma_a F_\#\gamma_{\bar{a}}\gamma^{(10)}F_\#^T).\end{aligned}$$

By (4.20), the chiralities imply the $\gamma^{(10)}$ acts as identities, and so (4.49) becomes

$$8R_{a\bar{a}} - \Phi^{-1}(F, \Gamma_{a\bar{a}}F) = 0. \tag{4.50}$$

Finally, the variation of the third equation of motion is

$$-\frac{1}{16}F_{\#}\gamma^{\bar{a}}D_{\bar{a}}\epsilon^- = \gamma^a D_a \gamma^a D_a \epsilon^+ - D^{\bar{a}}\left(D_{\bar{a}}\epsilon^+ + \frac{1}{16}F_{\#}\gamma_{\bar{a}}\epsilon^-\right)$$
$$0 = \left((\not{D}^+)^2 - D^+\cdot D^+\right)\epsilon^+ + \frac{1}{16}D^{\bar{a}}F_{\#}\gamma_{\bar{a}}\epsilon^-,$$

which are

$$S = 0, \quad dF = 0. \tag{4.51}$$

■

Chapter 5

Summary

We thus finished reviewing the generalized geometry for Type II supergravity theories. From the fact that the generalized tangent bundles have the same symmetry structure as NS-NS sector in Type II theory and by developing generalized geometric objects on it we finally lead to an amazing reformulation of Type II supergravities by encoding them in $Spin(9,1) \times Spin(1,9)$-structures. This mathematics is beautiful. As presented in theorems 4.1 and 4.2, supergravities are just Einstein theory with generalized geometry. All generalized objects are analogous to conventional ones, such as Levi-Civita connection, torsion and curvatures, in which sense we constructed a generalized gravity theory for Type II theories with local $Spin(9,1) \times Spin(1,9)$ covariance. Notable difference to the conventional geometry is the non-tensorial curvature and non-uniqueness of the Levi-Civita connection. But all relevant operators turn out to be covariant and unique.

As mentioned at the beginning, this review follows the paradigmatic reference [4], but there is one minor discrepancy found in deriving the mathematics, which is indicated in (4.1), where the generalized version of the equation of motion for $g_{\mu\nu}, B_{\mu\nu}$ has a factor $-1/8$ instead of $1/16$ in front of the R-R field term. This could be a consequence of certain convention. The derivations and proofs of them are fully presented here, one can feel free to check this discrepancy with [4].

There are extended generalized structure with $E_{d(d)} \times \mathbb{R}$ in place of $O(d,d) \times \mathbb{R}^+$, describing eleven-dimensional supergravity restricted to d-dimension [18], sequel to the main reference [4]. It is interesting that supergravity theories, not merely Type II, admit generalized geometric formalisms with different generalized structural group. This general relationship between generalized geometries and supergravities, which are low-energy limit of string theories, may be the probe of certain larger symmetries in full string theory that taking generalized objects as sensible limits, making the story more interesting. This is one of the future directions of my journey in generalized geometry.

Appendix A

Conventions

The conventions follow [4], recorded here for completeness. The metric has the mostly plus signiture $(- + \cdots +)$. Indices $\mu, \nu, \lambda, \ldots$ for the spacetime coordinate; $a, b, c, \ldots$ for the tangent space; $A, B, C, \ldots$ for the generalized coordinate. Symmetrization of indices is of weight one. On forms,

$$\begin{aligned} \omega^{(k)} &= \frac{1}{k!}\omega_{\mu_1\ldots\mu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} \\ \omega^{(k)} \wedge \eta^{(l)} &= \frac{1}{(k+l)!}\left(\frac{(k+l)!}{k!l!}\omega_{[\mu_1\ldots\mu_k}\eta_{\mu_{k+1}\ldots\mu_{k+l}]}\right) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{k+l}} \\ *\omega^{(k)} &= \frac{1}{(10-k)!}\left(\frac{1}{k!}\sqrt{-g}\epsilon_{\mu_1\ldots\mu_{10-k}\nu_1\ldots\nu_k}\omega^{\nu_1\ldots\nu_k}\right) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{10-k}} \end{aligned} \tag{A.1}$$

where $\epsilon_{01\ldots9} = -\epsilon^{01\ldots9} = 1$.

Gamma matrices $\gamma : T^* \otimes S \to S$ and $\gamma^\bullet : \Lambda^\bullet T^* \otimes S \to S$ by

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \qquad \gamma^{\mu_1\ldots\mu_k} = \gamma^{[\mu_1}\ldots\gamma^{\mu_k]}. \tag{A.2}$$

Use the antisymmetric transpose intertwiner $C : S \to S^*$

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^T, \qquad C^T = -C, \tag{A.3}$$

so

$$C\gamma^\mu = (C\gamma^\mu)^T, \tag{A.4}$$

and $C\gamma : S \otimes S \to T$ corresponds to a symmetric morphism, which is unique up to a real factor if S irreducible [13]. Thus $C \in (S \otimes S)^*$ is the unique pairing on each irreducible component up to a real factor. The Majorana conjugate then defined by $\bar{\epsilon} = C^*(\epsilon) = \epsilon^T C$. The useful identities are

$$\begin{aligned} C\gamma^{\mu_1\cdots\mu_n}C^{-1} &= (-1)^{[(n+1)/2]}(\gamma^{\mu_1\cdots\mu_n})^T, \\ \bar{\epsilon}\gamma^{\mu_1\cdots\mu_n}\chi &= (-1)^{[(n+1)/2]}\bar{\chi}\gamma^{\mu_1\cdots\mu_n}\epsilon. \end{aligned} \tag{A.5}$$

The top gamma $\gamma^{(10)} \in \mathrm{Cliff}(T)$ by

$$\gamma^{(10)} = \gamma^0\gamma^1\cdots\gamma^9 = \frac{1}{10!}\epsilon_{\mu_1\ldots\mu_{10}}\gamma^{\mu_1\cdots\mu_{10}}. \tag{A.6}$$

Dirac slash with weight one for $\Psi \in \Lambda^\bullet T^*$ by

$$\not{\Psi} = \gamma^\bullet(\Psi) = \sum_k \frac{1}{k!}\Psi_{\mu_1\ldots\mu_k}\gamma^{\mu_1\cdots\mu_k}. \tag{A.7}$$

R-R potentials in "C-basis" $\{C^{(n)}\}$ have gauge transformations, parameterized by a set of p-forms $\{\Lambda^{(p)}\}$, of form [5]

$$C^{(p)} \mapsto C^{(p)} + d\Lambda^{(p-1)} + H \wedge \Lambda^{(p-3)}, \tag{A.8}$$

where H is defined in (2.4). This is p-form electrodynamics when $H = 0$. $C^{(n)}$ couples with $D(p-1)$ brane in string theory, making branes charged. Define the sum C as in (2.2), "A-basis" used in 2.1.1 is related to "C-basis" by

$$A = e^{-B} \wedge C. \tag{A.9}$$

Note that $dC^{(p)}$ is not gauge invariant if $H \neq 0$, the gauge invariant field strength is given by (2.4). The R-R gauge symmetry is much larger than p-form electrodynamics because of Chern-Simons terms in Type II theories [19].

Appendix B

Differential Geometry

Generalized geometry involves analysis on vector bundles and their tensors, and develops many constructions analogous to the ordinary differential geometry. This appendix reviews those basic elements in the scope of convential geometry. A general reference to fiber bundles in physics, [20].

B.1 Vector Bundle

Definition B.1. A *vector bundle* $\mathcal{V}$ of rank d is a fiber bundle $\pi : \mathcal{V} \to M$ whose fiber $\pi^{-1}(x)$ is a vector space for each $x \in M$, and whose local trivializations $\phi_U : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$ are fiber-wise linear isomorphisms.

Tangent bundle T is a vector bundle over M with fiber $\mathbb{R}^d$. The generalized tangent bundles E and $\tilde{E}$ is a vector bundle with fiber $\mathbb{R}^{2d}$.

A data of covering $\{U_i\}$, functions $\pi_i : U_i \times \mathbb{R}^d \to U_i$, and fiber-wise linear *transition functions* $\phi_{ij} = \phi_{ji}^{-1} : \pi_i^{-1}(U_i \cap U_j) \xrightarrow{\sim} \pi_j^{-1}(U_i \cap U_j)$ define a vector bundle over M with trivializations $\phi_{U_j} \circ \phi_{U_i}^{-1} = \phi_{ij}$ if and only if it satisfies the *cocycle condition*

$$\phi_{ij} = \phi_{ik}\phi_{kj} \tag{B.1}$$

on $U_i \cap U_j \cap U_k$. (B.1) provides an equivalence relation for gluing locally defined bundles. The set of data satisfying the cocycle condition is called a *descent datum.*

Rank d vector bundle has *structural group* $GL(d, \mathbb{R})$, meaning that isomorphisms defined by transition functions $\phi_\alpha \circ \phi_\beta^{-1}|_{\{x\}\times\mathbb{R}^n}$, $x \in U_\alpha \cap U_\beta$, values in $GL(d, \mathbb{R})$. Collecting all the ordered bases $(e_a(x))$ of vector spaces $\pi^{-1}(x)$ over $x \in M$ yields a principal fiber bundle

$$F = \left\{(e_a(x)) : x \in M \text{ and } e_a(x) \text{ a basis of } \pi^{-1}(x)\right\} \tag{B.2}$$

called *frame bundle*. Its smooth structure and right group action is induced by coordinates $\left\{x^a, (g_{\alpha\beta})^b{}_c\right\}$. An object defined on F that is invariant under the diffeomorphic action of $GL(d, \mathbb{R})$ will be called *equivariant* (or *covariant*). Such an object will be "frame-independent" in the sense that a section of F transforms to any other points by equivariance.

Given a group representation $\rho : GL(d, \mathbb{R}) \to GL(V)$ of the structural group, the *associated vector bundle* of the principal bundle F with respect to ρ is a vector bundle with fiber V defined by

$$\mathcal{V}_\rho = (F \times V)/GL(d, \mathbb{R}), \tag{B.3}$$

which is a quotient by the left action $g \cdot (f, v) = (f \cdot g^{-1}, \rho(g)v)$. The action of $g \in C^\infty(GL(d, \mathbb{R}))$ on F induces one on $\mathcal{V}_\rho$

$$\begin{aligned} g : \mathcal{V}_\rho &\longrightarrow \mathcal{V}_\rho, \\ (f, v) \cdot GL(d, \mathbb{R}) &\longmapsto (f \cdot g^{-1}, v) \cdot GL(d, \mathbb{R}). \end{aligned} \tag{B.4}$$

A map $L : \mathcal{V}_\rho \to \mathcal{V}_\sigma$ between associated bundles is called *covariant* if it intertwines the representations: $\sigma(g) \cdot L(v) = L(\rho(g) \cdot v)$. The infinitesimal version of the covariance is in terms of adjoint actions (the differential), $d\sigma(A) \cdot L(v) = L(d\rho(A) \cdot v)$, where $A \in \mathfrak{gl}(d.\mathbb{R}) \otimes \mathcal{O}_M$, and $\mathcal{O}_M$ the smooth functions defined on M. In the language of D, $\mathfrak{gl}(d.\mathbb{R}) \otimes \mathcal{O}_M$ is the adjoint bundle.

As an important example, T, the tangent bundle over d-dimensional manifold M, has $GL(d, \mathbb{R})$-bundle of bases

$$F_M = \{(\hat{e}_a(x)) : x \in M \text{ and } \hat{e}_a(x) \text{ a basis of } T_x\} \tag{B.5}$$

and the group action of $(A^{-1})^b{}_a \in GL(d, \mathbb{R})$ reads

$$\hat{e}_a \mapsto \hat{e}'_a = \hat{e}_b (A^{-1})^b{}_a, \qquad v^a \mapsto v'^a = A^a{}_b v^b \tag{B.6}$$

for any local section $v = v^a \hat{e}_a$ of T.

T is associated to F_M by the defining representation $\rho_{\text{def}} = id$. The vector bundle associated to F_M by the dual representation $\rho^*_{\text{def}} : A \mapsto \rho_{\text{def}}(A^{-1})^T$, is the *cotangent* *bundle* T^*. The associated bundle with respect to the tensor representation $\rho_{p,q} := (\rho_{\text{def}})^{\otimes p} \otimes (\rho^*_{\text{def}})^{\otimes q}$ is the *tensor bundle* $\mathscr{T}^p_q$, whose sections are *tensors of* (p,q)*-type*.

Tensors $\mathscr{T}^p_q$ have fiber of dimension d^{p+q}, by (B.2) there is a structural group $GL(d^{p+q},\mathbb{R})$, which is of much larger frame bundle $F_{|\mathscr{T}^p_q|}$. Whenever there exists a subbundle P of a principal bundle F whose structural group G is a subgroup of $GL(d,\mathbb{R})$, we say the structural group $GL(d,\mathbb{R})$ is *reducible* to G, and P is called a G*-structure*. $\mathscr{T}^p_q$ admits a $GL(d,\mathbb{R})$-structure by construction.

The existence of a metric tensor $g \in \Gamma\mathcal{S}(T^* \otimes T^*)$ reduces $GL(d,\mathbb{R})$ to $O(p,q)$, where (p,q) is the signiture of g. An explicit $O(p,q)$-structure $P \subset F_M$ is given by

$$P = \left\{(\hat{e}_a) \in F_M; g(\hat{e}_a,\hat{e}_b) = \eta^{(p,q)}_{ab}\right\}, \tag{B.7}$$

where in $\eta^{(p,q)}_{ab}$, p is the number of $+1$'s. At each $x \in M$, all metrics at x define the coset space

$$g|_x \in GL(d,\mathbb{R})/O(p,q). \tag{B.8}$$

G-structure on F_M can impose topological conditions on M as it restricts the transition of tangents. For examples, when $G = SL(d,\mathbb{R})$, M is necessarily orientable; if d even and $G = GL(d/2,\mathbb{C})$, M admits almost complex structure[4].

A *connection* $\nabla : \mathcal{V} \to T^* \otimes \mathcal{V}$ is an $\mathbb{R}$-linear morphism of vector bundles obeying the Leibniz rule

$$\nabla(fv) = df \otimes v + f\nabla v. \tag{B.9}$$

Note that its codomain has tensor covariance. A connection ∇ is called *affine* if it is defined on $\mathcal{V} = T$. Every connection induces a principal connection 1-form $\omega \in \mathfrak{gl}(d.\mathbb{R}) \otimes \Omega^1_F$ on the principal bundle F, whose pullback along a section of F gives the index form

$$\nabla_\mu v^a = \partial_\mu v^a + \omega_\mu{}^a{}_b v^b. \tag{B.10}$$

A principal connection induces a connection for each associated vector bundle by its representation of the structural group. A connection ∇ is *compatible* with a G-structure $P \subset F$ if the corresponding connection of principal bundle F reduces to one on P.

B.2 Tensors in Differential Geometry

Let's complete this appendix by listing objects in ordinary differential geometry having generalized analogues, for references.

The *Lie derivative* $\mathcal{L}_v$ encodes the infinitesimal diffeomorphism generated by vector v. On a vector field w in T,

$$\mathcal{L}_v w = -\mathcal{L}_w v = [v, w], \tag{B.11}$$

where $[\cdot, \cdot]$ denotes the Lie bracket. This forms the (infinite-dimensional) Lie algebra of the diffeomorphism group of M, Diff(M). On a tensor field $\alpha \in \mathscr{T}^p_q$, in coordinates

$$\begin{aligned}\mathcal{L}_v \alpha^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} =& v^\mu \partial_\mu \alpha^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} \\ &- (\partial_\mu v^{\mu_1})\alpha^{\mu\mu_2\cdots\mu_p}{}_{\nu_1\cdots\nu_q} - (\partial_\mu v^{\mu_p})\alpha^{\mu_1\cdots\mu_{p-1}\mu}{}_{\nu_1\cdots\nu_q} \\ &+ (\partial_{\nu_1} v^\mu)\alpha^{\mu_1\cdots\mu_p}{}_{\mu\nu_2\cdots\nu_q} + \left(\partial_{\nu_q} v^\mu\right)\alpha^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_{q-1}\mu}.\end{aligned} \tag{B.12}$$

Note that the second and third lines are in the (minus) adjoint action of the tensor representations $d\rho_{p,q} : \mathfrak{gl}(d.\mathbb{R}) \to \mathrm{End}(V_{\rho_{p,q}})$ by Lie algebra $a^\mu{}_\nu = \partial_\nu v^\mu \in \mathfrak{gl}(d.\mathbb{R}) \otimes \mathcal{O}_M$. The first term is due to the translation of points in M.

Given a general affine connection ∇ on T, the *torsion* $\mathcal{T} \in \Gamma(T \otimes \Lambda^2 T^*)$ of ∇ is defined by comparing with the Lie bracket on T

$$\mathcal{T}(v, w) = \nabla_v w - \nabla_w v - [v, w]. \tag{B.13}$$

In coordinates,

$$\mathcal{T}^\mu_{\nu\lambda} = \omega_\nu{}^\mu{}_\lambda - \omega_\lambda{}^\mu{}_\nu, \tag{B.14}$$

where ω is in (B.10) with coordinate basis. Tensorially, in a general basis

$$\mathcal{T}^a_{bc} = \omega_b{}^a{}_c - \omega_c{}^a{}_b + [\hat{e}_b, \hat{e}_c]^a. \tag{B.15}$$

Also, torsion compares the Lie derivative with ∇,

$$(i_v \mathcal{T})\alpha = \mathcal{L}^\nabla_v \alpha - \mathcal{L}_v \alpha, \tag{B.16}$$

where i_v is the interior product, $\mathcal{L}^\nabla_v$ is (B.12) with ∇ in place of ∂, and $i_v\mathcal{T}$, in indices form $(i_v\mathcal{T})^\mu{}_\nu = v^\lambda \mathcal{T}^\mu_{\lambda\nu}$, acts on a tensor field α by regarding it as a section in $\mathfrak{gl}(d.\mathbb{R})$

adjoint bundle. This form of definition will be applied in the generalized geometry.

The *Riemann tensor* $\mathcal{R} \in \Gamma(\Lambda^2 T^* \otimes T \otimes T^*)$ of a connection ∇ is given by

$$\mathcal{R}(u,v)w = [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w, \qquad \mathcal{R}_{\mu\nu}{}^{\lambda}{}_{\rho}v^{\rho} = [\nabla_\mu, \nabla_\nu]v^\lambda - \mathcal{T}^{\rho}_{\mu\nu}\,\nabla_\rho v^\lambda, \tag{B.17}$$

where $[\nabla_\mu, \nabla_\nu]v^\lambda = \nabla_\mu(\nabla_\nu v^\lambda) - \nabla_\nu(\nabla_\mu v^\lambda)$, the second ∇ acts on the rank two tensors by tensor representations. It measures the failure of ∇ being a Lie algebra homomorphism.

The *Ricci tensor* is obtained by tracing $\mathcal{R}$

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\sigma\mu}{}^{\sigma}{}_{\nu}. \tag{B.18}$$

In particular, if ∇ is torsion-free, $\mathcal{T} = 0$, (B.17) and (B.18) implies that

$$\mathcal{R}_{ba}v^a = [\nabla_a, \nabla_b]v^a. \tag{B.19}$$

If ∇ is the Levi-Civita connection, it defines a symmetric tensor, and in Riemannian geometry it in turn defines a quadratic form on T measuring mean sectional curvature of planes containing the vector.

Furthur tracing yields the *Ricci scalar*

$$\mathcal{R} = g_{\mu\nu}\mathcal{R}^{\mu\nu}, \tag{B.20}$$

again, having some notion of mean sectional curvatures.

Appendix C

B-field Gerbe

This appendix is a complement to the introduction of gerbe in 2.1.2. For more details about gerbes [2, 1, 8, 9].

In short, $U(1)$-*gerbe* is a space fibered over the manifold M whose fibers are groupoids locally isomorphic to the groupoid of all principal $U(1)$-bundles connected by bundle isomorphisms. Let $\{U_i\}$ be a good cover, transitions of two bundles P_i, P_j over two open sets U_i, U_j are given by an isomorphism of bundles $u_{ij} = u_{ji}^{-1} : P_i|_{U_{ij}} \simeq P_j|_{U_{ij}}$ between their pull-backs on the intersection $U_{ij} = U_i \cap U_j$. Over every three open sets $U_{ijk} = U_i \cap U_j \cap U_k$, the composition $g_{ijk} = u_{ij}u_{jk}u_{ki} : P_i|_{U_{ijk}} \simeq P_i|_{U_{ijk}}$, being $U(1)$-bundle isomorphism, is a $U(1)$-valued transition function, and can be written

$$g_{ijk} = \exp\left(i\Lambda_{(ijk)}\right). \tag{C.1}$$

A *connective structure* assigns covariantly a bundle of connections one-forms $Co(P_i)$ over each local principal bundles P_i in fibers. A *curving* assigns covariantly a curvature two-form $K(\nabla_i)$ to each object $\nabla_i \in Co(P_i)$. A gauge over U_i involves a choice of bundle P_i in the isomorphism class, and a choice of $\nabla_i \in Co(P_i)$. Fix a gauge for each U_i, we can define an one-form over U_{ij} as a difference of connections,

$$i\Lambda_{(ij)} = (u_{ji})_*(\nabla_j) - \nabla_i, \tag{C.2}$$

and by gauge transformation of connections

$$i\Lambda_{(ij)} + i\Lambda_{(jk)} + i\Lambda_{(ki)} = (u_{ij}u_{jk}u_{ki})_*\nabla_i - \nabla_i = g_{ijk}^{-1}dg_{ijk}. \tag{C.3}$$

$dK(\nabla_i)$ will turn out to be independent of ∇_i, globally defined, and is called 3-*curvature* of the connective structure with curving K. We can see that this description is parallel to the description in 2.1.2: given a gauge choice $\{(U_i, P_i, \nabla_i)\}$, $B_{(i)} = K(\nabla_i)$ and $H = dB$ is the 3-curvature.

Appendix D

Sheaves

Finite dimensional differential geometry possess another equivalent description, namely, *structural sheaf* [21]. This appendix shed some light on it, to explain the ideas behind certain notations used.

For any open set $U \subset \mathbb{R}^n$, there corresponds a ring of smooth functions defined on U, $C^\infty(U)$; for any two open sets $V \subset U \subset \mathbb{R}$, restriction defines a ring homomorphism $C^\infty(U) \to C^\infty(V), f \mapsto f|_V$; fix an open set U, for any covering $U = \bigcup\{U_i\}_{i\in I}$, and a collection of functions $f_i \in C^\infty(U_i), f_j \in C^\infty(U_j)$ that coincide on the intersections $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}, i, j \in I$, a unique function $h \in C^\infty(U)$ can be defined by $h|_U = f, h|_V = g$. This defines a *sheaf of ring* over $\mathbb{R}^n$. Denote the collection of open sets in $\mathbb{R}^n$ as $\mathcal{O}(\mathbb{R}^n)$; i.e. the topology of $\mathbb{R}^n$. With these properties, C^∞ can be considered as a map that sends each $U \in \mathcal{O}(\mathbb{R}^n)$ the ring $C^\infty(U)$, and each inclusion $V \subset U$ the resctriction map. Together with the gluing property, C^∞ actually characterizes the smooth structure over $\mathbb{R}^n$ [13].

Let M be an n-dimensional manifold and let $\mathcal{O}_M$ denote the function defined similarly as above, with smooth functions in M in place of $\mathbb{R}^n$. Since a manifold is modeled on $\mathbb{R}^n$, its not hard to see that $\mathcal{O}_M$ and C^∞ are *locally isomorphic*; to be more specific, for every $m \in M$, there exists a coordinate neighbourhood (U, x) at m, $\mathcal{O}_M(U) \simeq C^\infty(x(U))$. The two functors $\mathcal{O}_M, C^\infty$ are not equal, just as M is in general only *locally* Euclidean. Indeed, $\mathcal{O}_M$ characterizes the smooth structure on M. This can actually be seen from the fact that all elementray building blocks in ordindary differential geometries are defined using smooth functions. Functor $\mathcal{O}_M$ deserves to be called the *structural sheaf* of M.

Therefore a manifold M can be completely defined as a topological space with a ring-valued functor $\mathcal{O}_M$ locally isomorphic to C^∞. So manifolds are *ringed spaces*. Abstractly, given a functor $\mathcal{F}$ like $\mathcal{O}_M$, the element of $\mathcal{F}(U)$ are natrually called *sections* over U. Sections of $\mathcal{O}_M$ are, and supposed to be, called *functions* on M. All kind of notions in differential geometry can be defined using $\mathcal{O}_M$. For example, a smooth map between manifolds, differential, vector bundle, forms, connection, and curvature. One refers to [13] and [21] for furthur interests.

Now some remarks on my notation can be made. The sections can be considered as an evaluation of sheaves. For example, the sections of tangent bundle $\Gamma(TM)$ can be thought equivalently as $T(M)$, where T is the locally free $\mathcal{O}_M$-module representing the tangent bundle. The description of a bundle $TM \to M$ and a functor T are interchangeable; this is done by *functor of points* [13]. Most of the sheaves in differential geometry, vector bundles in particular, are representable by suitable fiber bundles.

While the two descriptions are equivalent, notations can be made concise having both pictures in mind. For instance, the functor T is one less letter from TM. For sections, $T(U)$ (or simply TU) is less cumbersome than $\Gamma(TU)$. When dealing with morphisms of sections, it is better viewed in sheaf language. For example, morphism in sheaves of vector spaces $\nabla : \mathcal{V} \to T \otimes \mathcal{V}$ instead of $\nabla : \Gamma(\mathcal{V}) \to \Gamma(TM \otimes \mathcal{V})$, where the former refers to sections over all open sets. However, Γ is still useful by defining it as the *global section functor* $\Gamma(\mathcal{F}) = \mathcal{F}(X)$, omitting the reference to the base space. It will be handy to use $f \in \mathcal{F}$ to denote $f \in \mathcal{F}(U)$, meaning a section of $\mathcal{F}$ without specifying an explicit U.

It should be noted that T_x could be confusing by refering to both *germs* of sheaf T at x and the *fiber* of bundle T at x. But $\mathcal{O}_M$ is a *local ring*, $\mathcal{O}_x$ projects to $\mathbb{R}$ fiber in a unique manner, namely, its *value* at x. Given enough local information, the two are again equivalent.

Sheaf language also takes some advantages when dealing with algebraic structures, since sheaves directly value in them. For example, the direct sum and tensor product of two vector bundles are simply those of $\mathcal{O}_M$-modules in pure algebraic language. Also the exact sequence

$$0 \longrightarrow T^* \longrightarrow E \longrightarrow T \longrightarrow 0 \tag{D.1}$$

makes sense in terms of sheaves: the arrows are natural transformations. The patching (2.16) defines an exact sequence of presheaves, and after taking the associated sheaf, the locally free $\mathcal{O}_M$-module E is representable by the generalized tangent bundle, denoted by E again.

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