Generalised Geometry and Supergravity

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"What can be said at all can be said clearly, and what we cannot talk about we must pass over in silence"

– Ludwig Wittgenstein

Ai miei genitori

To my parents

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Abstract

This dissertation reviews the topic of generalised geometry with a focus on applications to supergravity. Relevant concepts in complex "ordinary" differential geometry are presented through the elegant language of fibre bundles, G-structures, and intrinsic torsion - a framing best suited to the extension of these ideas to their "generalised" counterparts. A brief overview of key equations, symmetries and fluxless solutions of the relevant supergravity theories is given. Consideration of supersymmetric flux backgrounds leads to constraints which motivate the compact geometric description that follows. The complete framework of $O(d,d) \times \mathbb{R}^+$ generalised geometry is assembled piece by piece, with each object constructed in a manner that directly mirrors general relativity; allowing the NSNS sector of type II supergravity to be recast as an analogue of Einstein gravity. The utility of this toolset is further illustrated by discussing flux compactification in terms of generalised Calabi-Yau manifolds.

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Introduction

1

Many of the major historical developments of physics, from Newton's theory of gravitation to Maxwell's formulation of electromagnetism, have been connected by a single common thread: *unification* - the merging of previously disjoint elements into a single consistent description. One notable example is The Standard Model, an experimentally verifiable theory which successfully captures three of the four fundamental interactions. Built within the framework of perturbative quantum field theory, it is able to yield finite and thus physically meaningful predictions only via *renormalisation*, whereby ultraviolet divergences are cancelled by the addition of a finite set of counterterms. This is in stark contrast with the non-renormalisable nature of Einstein's general relativity, a fact that presents an immediate obstacle to any attempt to incorporate gravity into the picture.

The search for a way around this issue eventually led to string theory, in which point particles are replaced by one-dimensional extended objects such that the scattering amplitudes are no longer plagued with short-distance singularities. First conceived by Nambu, Susskind, and Nielsen [1, 2, 3] as a method to explain hadronic phenomena, it emerged as a quantum gravity candidate once Schwarz, Scherk, and Yoneya [4, 5] began to interpret the massless spin 2 state of the closed string spectrum as a graviton. Its first iteration contained only bosonic particles and suffered from instabilities caused by the presence of tachyons, so was unsuitable as a potential theory of everything. Both of these problems were solved by imposing *supersymmetry*, an enlargement of the Poincaré group that creates a mapping between bosons and fermions. So-called "superstring" theory was categorised into five distinct versions by Green, Gross, and others [6, 7] until, following a conjecture by Witten [8], they were all found to be limits of an unknown beast called M-theory.

An unavoidable property of these models is the existence of extra dimensions: conformal invariance on the world-sheet is preserved during quantisation only if the spacetime is ten-dimensional. It is thus natural to split this into the product of an internal manifold and an external spacetime through a *compactification* mechanism, with an aim to extract four-dimensional physics from a low-energy limit of string theory called supergravity. This is achieved by dimensional reduction, a procedure initially developed by Kaluza and Klein [9, 10] involving the formal contraction of the characteristic length scale of the compact space to zero. The reduced theory typically inherits an infinite tower of massive modes of which it becomes necessary to take a finite subset such that the removed heavy states cannot be sourced by those that remain. These *consistent truncations* are of particular interest because uplifting low-dimensional solutions produces exact solutions of the high-dimensional theory.

Outline of this Dissertation

The central focus of this work is *generalised geometry*, which was originally discovered by mathematicians Hitchen and Gualtieri [11, 12] as a framework encompassing both complex and symplectic structures as well as unifying several apparently-distinct aspects of their geometry. These two objects, along with fibre bundles, G-structures, and intrinsic torsion, are introduced in Chapter 2 and will be core to much of what follows. Soon after its formulation, generalised geometry attracted the attention of physicists, such as Hull and Waldram, who started to notice its scope as a means to help understand supergravity [13, 14]. In order to emphasise the suggestive features that led to these realisations, Chapter 3 offers a brief overview of the key equations and symmetries of type II supergravity theories. An instructive example attempt at finding simple supersymmetric solutions will illustrate the limitations of the "standard" apparatus and motivate the pursuit of a more sophisticated description.

The basic premise of this new toolset is the extension of the tangent bundle TM to a larger space, the *generalised tangent bundle* E, that is isomorphic to the direct sum $TM \oplus T^*M$. Chapter 4 details this procedure and formalises the generalisations of the Lie derivative, metric, and other important structures from ordinary geometry. It will become apparent that this construction is suited to representing the symmetries of type II supergravity, incorporating into a single geometric object invariance under both diffeomorphisms and gauge transformations. Following this trail to its conclusion leads to a beautiful result: the geometric reformulation of the NSNS sector of type II supergravity, found by Waldram and others in [15]. Chapter 5 is devoted to deriving this surprisingly natural consequence of considering the generalised counterparts of the Levi-Civita connection and curvature, and proceeding as in general relativity by defining a direct analogue of the Einstein-Hilbert action.

Having been presented with the main components of this formalism, the reader will then be properly equipped to explore another of its significant applications: flux compactifications. It is considered desirable for string backgrounds to be supersymmetric since arguments used to resolve the gauge hierarchy problem rely on this symmetry remaining unbroken at the Planck scale. These supersymmetry-preserving conditions place stringent differential and topological constraints on the geometry. In the simple class of solutions for which the metric is the only non-zero field, the internal manifold is immediately seen to be necessarily Calabi-Yau [16]. The low-energy effective theories that result from this case possess unobserved - and so undesirable - massless scalar fields called moduli. These can be avoided by instead turning on some of the other fields, known as *fluxes*, which can then provide a mass-generating scalar potential. The price paid for this is a back-reaction that further restricts the geometry, modifying and obscuring the internal manifolds. Fortunately, in the language of generalised geometry, such situations are clarified immensely. In works by Graña et al. [17, 18], it was shown that flux compactifications of type II supergravity are described in terms of generalised Calabi-Yau structures. This is explicitly demonstrated in Chapter 6.

This dissertation is aimed at a reader with a solid foundational knowledge of theoretical physics. In particular, familiarity with differential geometry, general relativity and supersymmetry is assumed. Most sections of this work contain derivations with more steps and detail than what is seen in the referenced papers - it is hoped that these extra calculations performed by the author will help to make the subject more accessible to newcomers.

Differential Geometry Preliminaries

2

Before diving into the exciting physics, it is worth considering some important mathematical notions that will help to translate the preservation of supersymmetry into precise differential and topological conditions. The relevant concepts in complex ordinary differential geometry are presented through the elegant language of fibre bundles, G-structures, and intrinsic torsion. It is only from within this framing that it becomes clear how to define extensions of these objects to their generalised counterparts. The main references for this chapter are [19], [20], [21], [22], [23], and [24].

2.1 Fibre Bundles and G-Structures

In trying to model complex physical scenarios, it is often necessary to combine simple mathematical spaces together in order to build more interesting structures. The most straightforward instances of this are just products of manifolds, but many other constructions cannot be understood in these terms. One such example is the famous Möbius strip, which has a "twist" that renders it globally distinct from the cylinder despite locally having the geometry $S^1 \times [0,1]$. The concept of bundles provides a precise toolset to discuss objects of this sort, generalising the trivial case of product manifolds.

Definition. A *fibre bundle* is the quintuple (E, π, M, F, G), where:

- The *total space* E, *base space* M, and *fibre* F are differentiable manifolds.

- The *projection* $\pi : E \to M$ is a continuous surjective map. For any $p \in M$, the *fibre at* p, defined as the preimage $F_p := \pi^{-1}(p)$, is isomorphic to F.

- A *section* $\sigma : M \to E$ of the fibre bundle is a smooth map satisfying $\pi \circ \sigma = \mathbb{1}$. The space of all sections is $\Gamma(M, F)$.

- For a given atlas $\{U_i\}$ covering M, the *local trivialisation* is a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$.

- The *structure group* G is a Lie group with a left action on F. The special case where G and F are identical is referred to as a *principal bundle*.

- Fixing p in $\phi_i(p, f)$ gives the diffeomorphism $\phi_{i,p}(f) : F \to F_p$. For $U_i \cap U_j \neq \emptyset$, $t_{ij}(p) \coloneqq \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$ an element of G. These smooth maps $t_{ij} : U_i \cap U_j \to G$ are known as the *transition functions*.

It is intuitive to picture the fibre at $p \in M$ as a set of points in E attached to p. The projection π then takes these points in F_p to p, while the section sends each $p \in M$ to a corresponding point in F_p. Bearing the above description of Möbius strip in mind, the naming of ϕ_i as the "local trivialisation" is natural since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ onto the direct product $U_i \times F$.

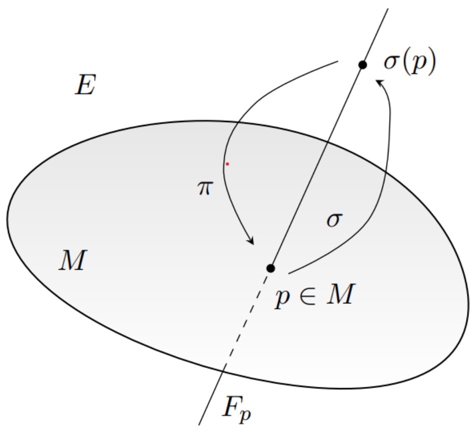

Figure 2.1: Schematic representation of a fibre bundle, courtesy of [25]. The action of the projection π and section σ is represented visually by the arrows.

Example. For any smooth d-dimensional manifold M, one can consider the space

$$TM \coloneqq \sqcup_{p \in M} T_p M, \tag{2.1.1}$$

where each element $u = (p, v) \in TM$ is specified by a point $p \in M$ and a vector $v \in T_p M$ at that point. Coupling this with the projection

$$\begin{aligned} \pi : \quad & TM \to M \\ & (p, v) \to p \end{aligned} \tag{2.1.2}$$

defines the *tangent bundle*, denoted $TM \xrightarrow{\pi} M$.

Since $F := F_p = \pi^{-1}(p) = T_pM \cong \mathbb{R}^m$, this is indeed a fibre bundle. The sections, being maps from a point to a vector at that point, are simply vector fields. Using coordinate systems x^μ and y^μ for overlapping patches U_i and U_j containing $p = \pi(u)$, the vector corresponding to u is written as $v = V^\mu \frac{\partial}{\partial x^\mu}|_p = \tilde{V}^\mu \frac{\partial}{\partial y^\mu}|_p$. For U_i, the local trivialisation is then determined by

$$\phi_i^{-1}(u) = (p, \{V^\mu\}). \tag{2.1.3}$$

In order to be well-defined, these vector components must be related via change-of-coordinate transformations:

$$V^\mu = \left.\frac{\partial x^\mu}{\partial y^\nu}\right|_p \tilde{V}^\nu, \tag{2.1.4}$$

which allows the structure group to be identified as $GL(d, \mathbb{R})$.

Example. In the above manifold M, the coordinates x^μ describing $p \in U_i$ provide a basis $\{\frac{\partial}{\partial x^\mu}|_p\}$ for T_pM. In this basis, a *frame* at p is written as $A = \{X_1, \ldots, X_d\}$, where

$$X_\alpha = X_\alpha^\mu \frac{\partial}{\partial x^\mu}|_p \qquad 1 \leq \alpha \leq d \tag{2.1.5}$$

and the linear independence of the $\{X_\alpha\}$ is ensured only if (X_α^μ) is an invertible matrix. From this it follows that each element of $GL(m, \mathbb{R})$ completely describes an entire frame at a point p. Writing the set of all such frames at p as L_pM, one can now consider (in analogy with the previous example) the space

$$LM := \sqcup_{p \in M} L_pM, \tag{2.1.6}$$

where each element $u = (p, A) \in LM$ is specified by a point $p \in M$ and a frame $A \in L_pM$ at that point. Thus, given a tangent bundle $TM \xrightarrow{\pi} M$, one can always define the associated *frame bundle* $LM \xrightarrow{\pi} M$, with local trivialisation given by

$$\phi_i^{-1}(u) = (p, (X_\alpha^\mu)). \tag{2.1.7}$$

This is in fact a principal bundle. To see this, it helps to consider again the same set of vectors $\{X_\alpha\}$ at p, but now in the above-mentioned overlapping coordinate chart U_j, such that they are instead determined by $X_\alpha = \tilde{X}_\alpha^\mu \frac{\partial}{\partial y^\mu}|_p$. Equating the expressions, it must hold that the two frame-specifying matrices are related via

$$X^{\mu}_{\alpha} = \left.\frac{\partial x^{\mu}}{\partial y^{\nu}}\right|_{p} \tilde{X}^{\nu}_{\alpha} \tag{2.1.8}$$

This means that the fibre elements at p are all connected via transformations $t_{ij}(p) \in GL(d, \mathbb{R})$ and the structure group G is identical to F, as required.

Definition. If a frame bundle's structure group can be reduced to subgroup $G \subset GL(d, \mathbb{R})$, it is said to have a *$G$-structure*. The manifold M admits such a principal sub-bundle (sometimes denoted P_G) of the frame bundle if and only if there exists a global G-invariant tensor or spinor.

To get a better sense of how these two definitions are consistent with each other, it is instructive to list the general steps involved in explicitly building a G-structure. Since frames $A \in GL(d, \mathbb{R})$ for a generic vector space V map elements of the usual $\mathbb{R}^d$ basis into vectors living in V, they can be thought of as isomorphisms between the so-called "standard model" $\mathbb{R}^d$ and V. Bearing this in mind, the procedure involves picking an invariant object or "structure" and then specifying the following [26]:

1. The consistent definition of isomorphism relating vector spaces embedded with the chosen structure.

2. The "standard model" corresponding to the object. This is the form which the structure takes in $\mathbb{R}^d$.

3. The subgroup G of $GL(d, \mathbb{R})$ obtained by restricting to the "standard model" isomorphism condition.

4. The subset of frames for V that are compatible with the structure in the sense that they are preserved under the above-defined isomorphism.

Example. This is made concrete with a detailed walk-through of the case in which the selected structure is the metric, a positive and symmetric bilinear map defining the inner product

$$g : V \times V \to \mathbb{R}. \tag{2.1.9}$$

Having chosen the invariant object, the construction proceeds as follows:

1. An isomorphism $A : V \to \tilde{V}$ between two metric spaces (V, g) and $(\tilde{V}, \tilde{g})$ should satisfy

$$\forall\, u, v \in V : \tilde{g}(A(u), A(v)) = g(u, v). \tag{2.1.10}$$

Using matrix notation to phrase $g(u, v)$ as $u^T g v$, this translates to the condition that

$$A^T \tilde{g} A = g. \tag{2.1.11}$$

2. In $\mathbb{R}^d$, the standard inner product g_{std} is just the Euclidean "dot" product

$$g_{std}(u, v) = u^T v. \tag{2.1.12}$$

3. The set formed by writing the isomorphism condition for the standard model is

$$\{A \in GL(d, \mathbb{R}) : \forall\, u, v \in \mathbb{R}^d : \tilde{g}_{std}(A(u), A(v)) = g_{std}(u, v)\} \tag{2.1.13}$$

$$= \{A \in GL(d, \mathbb{R}) : A^T A = \mathbb{1}\} = O(d), \tag{2.1.14}$$

where the second line is obtained by substituting $\tilde{g}_{std}(A(u), A(v)) = (Au)^T Av = u^T A^T Av$ and noticing that this can be equal to $u^T v$ only if $A^T A = \mathbb{1}$. Thus, the existence of a globally-defined metric (the characterising feature of a Riemannian manifold) is equivalent to an $O(d)$-structure. Note that, since every smooth manifold admits a metric, it is always possible to reduce the structure group to $O(d)$.

4. The orthonormality condition

$$\forall\, \alpha, \beta \in \{1, \ldots, d\} : g(v_\alpha, v_\beta) = \delta_{\alpha,\beta} \tag{2.1.15}$$

for a frame $\{v_\alpha\}$ is preserved by the isomorphism A since

$$g(A^\alpha_\mu v_\alpha, A^\beta_\nu v_\beta) = A^\alpha_\mu A^\beta_\nu g(v_\alpha, v_\beta) = A^\alpha_\mu A^\alpha_\nu = \delta_{\mu,\nu}. \tag{2.1.16}$$

A Riemannian manifold is therefore seen as the sub-bundle $P_{O(d)}$ of $LM \xrightarrow{\pi} M$ containing only orthonormal frames.

Given the previous calculation, it is straightforward to modify these steps to find the G-structures resulting from defining other global tensors. Some familiar examples of such tensors include: *volume forms*, a class of top forms whose component is strictly-positive across all coordinate charts; and *parallelisations*, a set of globally-defined vector fields that define bases for the tangent spaces at each $p \in M$. Two other global objects relevant to this work are: real non-degenerate 2-forms, called *almost symplectic structures*; and (1,1) tensors, known as *almost complex structures*, defined by the requirement that their square gives the negation of the identity map on TM. The results of applying the procedure to these structures are listed below.

Name	Globally-Defined Invariant Tensor	G-Structure
Metric	g	$O(d)$
Volume Form	vol	$SL(d,\mathbb{R})$
Metric Volume Form	vol_g	$SO(d,\mathbb{R})$
Parallelisation	$\{v_\alpha\}$	$\mathbb{1}$
Almost Symplectic Structure	ω (real) $\implies d$ even	$Sp(d,\mathbb{R})$
Almost Complex Structure	J ($J^2 = -\mathbb{1}$) $\implies d$ even	$GL(d/2,\mathbb{C})$
Almost Hermitian Manifold	$\left.\begin{array}{ll} \omega, J & J^T\omega J = \omega \\ g, J & J^T g J = g \\ \omega, g & \omega^T g^{-1}\omega = g \end{array}\right\}$	$U(d/2)$

Table 2.1: G-structures and the global invariant tensors inducing them, adapted from [23]. The equivalence of the three conditions for an almost Hermitian structure is a consequence of the fact that the intersection of any two out of $GL(d/2,\mathbb{C})$, $O(d)$, and $Sp(d,\mathbb{R})$ is $U(d/2)$.

It is worth pausing for a moment to appreciate a subtle aspect of G-structures that is best understood through a concrete example. The following discussion thus focuses on examining the vector space of almost complex structures, but a similar pattern emerges for all of the structures described in the table above. An immediate implication of $J^2 = -\mathbb{1}$ is that the eigenvalues of J are $+i$ and $-i$. Note, however, that J is a map that acts on the real space TM. In order for the corresponding complex eigenvectors to be allowed, one therefore must extend the tangent bundle via complexification to $TM \otimes \mathbb{C}$. Then, the $(\pm i)$-eigenspaces at each $p \in M$ define fibre sub-bundles $L, \bar{L} \subset TM \otimes \mathbb{C}$. Because J is smooth, in each patch U_i it is possible to define two separate vector bases - one spanning L and the other spanning $\bar{L}$. These bases are not generally defined globally since the structure group $GL(d/2,\mathbb{C})$ allows for the presence of transition functions that mix the vector fields. Despite this, because the action of $GL(d/2,\mathbb{C})$ elements preserves J, the t_{ij} must maintain the decomposition $TM \otimes \mathbb{C} = L + \bar{L}$. Subbundles of TM such as L and $\bar{L}$ that have these properties are known as *distributions* and the dimension of the corresponding fibre is denoted $\mathrm{rank}(L)$.

Definition. An almost complex structure J on a $(d = 2n)$-dimensional manifold M

determines a local coframe of one-forms $\{\phi^\alpha\}$ up to a $GL(n,\mathbb{C})$ transformation [23], out of which one can construct a non-degenerate holomorphic n-form

$$\Omega = \phi^1 \wedge \ldots \wedge \phi^n. \tag{2.1.17}$$

In the subset of cases where M does not restrict it from being globally-defined, Ω is called a *fundamental form* and immediately equips M with a volume form $\Omega \wedge \overline{\Omega}$. Then, the existence of this object further reduces the structure group $GL(n,\mathbb{C})$ to $SL(n,\mathbb{C})$. Given an almost Hermitian manifold (a $U(n)$-structure), the additional presence of a fundamental form then defines an $SU(n)$*-structure*. Here, compatibility with the almost symplectic structure is understood as the requirement that

$$\omega \wedge \Omega = 0 \quad \text{and} \quad \Omega \wedge \overline{\Omega} \propto \omega^n. \tag{2.1.18}$$

For spin manifolds, these criteria for an $SU(n)$-structure are equivalent to the specification of a global spinor η. This is shown explicitly in the next chapter. This provides a crucial link between the geometric language introduced up to this point and the question of preserving supersymmetry that will be further exploited later in this dissertation.

2.2 Integrability and Torsion

The structures described in the previous section have been characterised by a topological condition, namely the existence of some globally-defined invariant tensor. Curious readers may be wondering why the names of many of these structures contain the qualifier "almost". The answer is that they are still missing a desirable ingredient: integrability. This takes the form of a differential criterion.

Definition. The concept of integrability poses the question of how one can restrict an over-determined system such that it becomes possible to obtain a solvable and consistent set of differential equations on a manifold M [27, 28]. Specifically, the distribution L is *integrable* if through each $p \in M$ described by coordinate x_0, there exists a solution $x^i(\sigma^1, \ldots, \sigma^{\mathrm{rank}(L)}; x_0)$ to

$$\frac{\partial x^i}{\partial \sigma^a} = X^i_a \tag{2.2.1}$$

in a neighbourhood of p. In the above set of equations, $\{X_a\}$ are basis vector fields locally spanning L. Notice from the above that if L is integrable, then $\bar{L}$ must be too. It therefore makes sense to speak of the integrability of the G-structure itself.

A very closely related concept is that of involutivity:

Definition. A distribution L is described as being *involutive* if it holds true that

$$X, Y \in \Gamma(L) \implies [X, Y] \in \Gamma(L). \tag{2.2.2}$$

It is a fundamental result in differential geometry that a distribution is integrable if and only if it is involutive. This is the Frobenius theorem [24]. Because of this, the two terms are often used interchangeably. In particular, it is generally simpler to specify the integrability condition of a particular structure in terms of its involutivity by specialising (2.2.2) case by case. This requirement is typically formulated in terms of vectors or differential forms:

- An almost complex structure J is integrable if

  $$\forall\, v, w \in \Gamma(TM) : N_J(v, w) := [v, w] + J[Jv, w] + J[v, Jw] - [Jv, Jw] = 0. \tag{2.2.3}$$

  J is then known as a *complex structure*, with the associated M becoming a complex manifold. The name *Nijenhuis tensor* is often used in reference to this map.

- An almost symplectic structure ω is integrable if

  $$\mathrm{d}\omega = 0. \tag{2.2.4}$$

  ω is then refered to as a *symplectic structure* and the corresponding M is then termed a symplectic manifold.

- An almost Hermitian manifold $(M,\, g,\, J,\, \omega)$ with J integrable but ω non-integrable is called a *Hermitian manifold*. If, on the other hand, ω is integrable whilst J is not, M is an *almost Kähler manifold*. The title of *Kähler manifold* is reserved for when J and ω are, respectively, complex and symplectic structures.
- The integrability condition for a fundamental n-form Ω is

  $$\mathrm{d}\Omega = 0. \tag{2.2.5}$$

  An $SU(n)$-structure $(M,\, g,\, J,\, \omega,\, \Omega)$ with J, ω, and Ω each satisfying their respective integrability criteria is a called a *Calabi-Yau manifold*. This is a sub-category of Kähler manifolds characterised by a Ricci-flat metric [20]. In terms of a global spinor

η, the equivalent Calabi-Yau definition is the requirement that η is covariantly conserved, as is demonstrated in Chapter 3.

Since the equations determining whether a G-structure is integrable can take vastly different forms depending on the particular invariant object being investigated, this discussion can be greatly simplified by rephrasing it through the concept of torsion. Requiring the absence of torsion, in a sense explained below, will be demonstrated to be exactly equivalent to the previously-stated integrability criteria for J and ω.

Definition. Suppose that manifold M has a G-structure induced by G-invariant (q, r) tensor Φ. A *G-compatible connection* ∇ is defined as one satisfying $\nabla\Phi = 0$. In a coordinate chart, this action on Φ takes the general form

$$\begin{aligned}(\nabla_\mu \Phi)^{\alpha_1 \ldots \alpha_q}_{\beta_1 \ldots \beta_r} = &\frac{\partial}{\partial x^\mu} \Phi^{\alpha_1 \ldots \alpha_q}_{\beta_1 \ldots \beta_r} \\ &+ \Gamma^{\alpha_1}_{\mu\rho} \Phi^{\rho\alpha_2 \ldots \alpha_q}_{\beta_1 \ldots \beta_r} + \ldots + \Gamma^{\alpha_q}_{\mu\rho} \Phi^{\alpha_1 \ldots \alpha_{q-1}\rho}_{\beta_1 \ldots \beta_r} \\ &- \Gamma^{\rho}_{\mu\beta_1} \Phi^{\alpha_1 \ldots \alpha_q}_{\rho\beta_2 \ldots \beta_r} - \ldots - \Gamma^{\rho}_{\mu\beta_r} \Phi^{\alpha_1 \ldots \alpha_q}_{\beta_1 \ldots \beta_{r-1}\rho} = 0.\end{aligned} \tag{2.2.6}$$

It is always possible to find such a connection since, if $\nabla\Phi \neq 0$, one can simply subtract the right-hand-side to get zero and relabel left-hand-side as $\nabla\Phi$. Recall that a G-structure is just a principal sub-bundle P_G of the frame bundle and notice that the second and third lines of the above expression are simply the adjoint action of Γ on Φ, with an extra free downstairs index μ. Because the first line does not contain any components of ∇, it cancels exactly when taking the difference $\Sigma \coloneqq \tilde{\nabla} - \nabla$ of two compatible connections. This resulting tensor Σ is therefore recognised as a section of the bundle $T^*M \otimes \operatorname{ad} P_G$, which is given the label K_G. The *torsion* $T(\nabla)$ of connection ∇ is defined by the map

$$T(\nabla)(v, w) \coloneqq \mathcal{L}^\nabla_v w - \mathcal{L}_v w, \tag{2.2.7}$$

where $\mathcal{L}^\nabla_v w$ is the Lie derivative of $w \in \Gamma(TM)$ with respect to $v \in \Gamma(TM)$ calculated by replacing the partial derivatives ∂ with ∇. Writing this using a coordinate basis $\{e^\mu = \frac{\partial}{\partial x^\mu}\}$ yields the component expression

$$T(\nabla)(v, w) = (\Gamma^\mu_{\nu\lambda} - \Gamma^\mu_{\lambda\nu}) v^\nu w^\lambda e^\mu. \tag{2.2.8}$$

This is explicitly antisymmetric with respect to its two lower indices, so $T(\nabla)$ is just a section of the bundle $TM \otimes \Lambda^2 T^*M$ often called W. Naturally, the torsion will depend on the choice of connection. This ambiguity can be resolved by relating the two aforementioned product bundles via the map [29]

$$\begin{aligned} \tau : K_G &\to W \\ \Sigma &\to \tau(\Sigma) \coloneqq T(\tilde{\nabla}) - T(\nabla). \end{aligned} \tag{2.2.9}$$

If the image of τ is written as W_G, then the quotient space $W_{\text{int}} \coloneqq W/W_G$ is by construction independent of the particular compatible connection chosen. It thus makes sense to think of W_{int} as containing the part of the torsion that is fundamental to P_G, and it is only if this space is trivial that the G-structure can be said to be torsion-free. Regardless of the choice of G-compatible ∇, the projection of $T(\nabla)$ onto W_{int} necessarily produces the same object: the *intrinsic torsion* T_{int} of P_G.

Example. To illustrate that the presence of intrinsic torsion is precisely the restriction that prevents a G-structure from being integrable, one can simply recast the integrability conditions in terms of $T(\nabla)$ (denoted as T_∇ below for clarity):

- A connection ∇ is compatible with an almost complex structure if $\nabla J = 0$. An immediate consequence of this is that $\nabla_v Jw = J\nabla_v w$. This relation, along with the defining property $J^2 = -\mathbb{1}$, can be used to insert trivial pairs of terms in order to rewrite the tensor N_J from Equation (2.2.3) as

$$\begin{aligned} -N_J(v,w) &= -[v,w] - J[Jv,w] - J[v,Jw] + [Jv,Jw] \\ &\quad + \nabla_v w + J\nabla_v Jw + J\nabla_{Jv} w - \nabla_{Jv} Jw \\ &\quad - \nabla_w v - J\nabla_w Jv - J\nabla_{Jw} v + \nabla_{Jw} Jv \\ &= \nabla_v w - \nabla_w v - [v,w] \\ &\quad + J\nabla_v Jw - J\nabla_{Jw} v - J[v,Jw] \\ &\quad + J\nabla_{Jv} w - J\nabla_w Jv - J[Jv,w] \\ &\quad - \nabla_{Jv} Jw + \nabla_{Jw} Jv + [Jv,Jw] \end{aligned} \tag{2.2.10}$$

  and substituting (2.2.7) into each of the four lines finally gives

$$-N_J(v,w) = T_\nabla(v,w) + JT_\nabla(v,Jw) + JT_\nabla(Jv,w) - T_\nabla(Jv,Jw). \tag{2.2.11}$$

  Having shown that the Nijenhuis tensor is proportional to the torsion, the integrability condition $N_J(v,w) = 0$ is automatically met if T_∇ vanishes. Despite this expression containing a particular ∇, it has been demonstrated above that each term containing ∇ cancels so the expression is independent of the choice of compatible connection. This means that a torsionless $GL(d/2,\mathbb{C})$-structure is an integrable complex manifold.

- For an almost symplectic structure, the compatibility condition for a connection is $\nabla\omega = 0$. Following a completely analagous calculation performed for J, the differential criterion for ω to define a symplectic manifold is easily re-expressed as

$$\mathrm{d}\omega(u,v,w) = \omega(T_\nabla(u,v),w) + \omega(T_\nabla(v,w),u) + \omega(T_\nabla(w,u),v). \tag{2.2.12}$$

  This proves that the ability to obtain a torsion-free compatible connection is equivalent to the existence of an integrable ω. Once again, this statement relies on the specification of such a ∇, which immediately falls out of the equation after plugging in (2.2.7). The choice of which $Sp(d,\mathbb{R})$-compatible connection to use is thus entirely arbitary and, as expected, any obstruction of integrability is due to the intrinsic torsion alone.

Supergravity Overview 3

It is now time to introduce *supergravity*, a model in theoretical physics which incorporates general relativity into a supersymmetric framework in the form of a gauge theory. As explained in the introduction, this arises in its various forms as the low-energy limit of different superstring theories and M-theory. The specific supergravity that will be discussed in the following is type II, since it is the version most relevant to the branch of generalised geometry focused on in this work. In the first section, the key constituent fields and equations defining this theory are presented. In particular, the gauge symmetry of the two-form potential B will explain the somewhat unusual construction of the generalised tangent bundle in the following chapter. In the second section, an explicit derivation is given of the famous result [16] that a Calabi-Yau internal manifold is always produced when compactifying supergravity with all fields except the metric turned off. This calculation will illustrate both the strengths and limitations of the G-structure toolset developed over Chapter 2 in describing compactifications. Crucially, the failure of ordinary geometry to capture the whole story when flux is switched on motivates the need for the more sophisticated description provided later in the dissertation. The global references for this chapter are [15], [23], and [21].

3.1 Type II Supergravity

Coming in the variations IIA and IIB, type II supergravity is a theory in $d = 10$ containing the fields

$$\{g_{\mu\nu}, B_{\mu\nu}, \phi, A^{(n)}_{\mu_1\ldots\mu_n}, \psi^{\pm}_{\mu}, \lambda^{\pm}\}, \tag{3.1.1}$$

where $\pm$ refers to particular chirality patterns that do not affect this discussion, and n is odd and even for IIA and IIB, repectively. These can be split into two groups: the bosonic (metric $g_{\mu\nu}$, two-form potential $B_{\mu\nu}$, scalar dilaton ϕ, and RR n-form potentials

$A^{(n)}_{\mu_1 \dots \mu_n}$), and the fermionic (chiral gravitini $\psi^{\pm}_{\mu}$ and chiral dilatini $\lambda^{\pm}$).

Continuing to separately categorise bosons and fermions, the fields' contributions to the theory are summarised firstly in the bosonic action:

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2} \int \sqrt{-g} \left[e^{-2\phi} \left(R + 4(\partial\phi)^2 - \tfrac{1}{12} H^2 \right) - \tfrac{1}{4} \sum_n \tfrac{1}{n!} (F^{(B)}_{(n)})^2 \right], \tag{3.1.2}$$

with field strengths H and $F^{(B)}_{(n)}$ being respective shorthands for $\mathrm{d}B$ and $e^B \wedge \mathrm{d}A_{(n-1)}$. The first term is known as the *NSNS sector* and is the same in both theories whilst the sum in the second term, constituting the *RR sector*, is over odd n for IIA and even n for IIB. Note that this is technically a "pseudo-action" because supergravity contains a $F^{(B)}_{(n)}$ self-duality relation

$$F^{(B)}_{(n)} = (-)^{[n/2]} * F^{(B)}_{(10-n)} \tag{3.1.3}$$

that cannot be obtained as an equation of motion of S_{B} via variational calculus.

Secondly, the fermionic action reads:

$$\begin{aligned} S_{\mathrm{F}} = -\frac{1}{2\kappa^2} \int \sqrt{-g} \Big[& e^{-2\phi} \Big(2\bar{\psi}^{+\mu} \gamma^{\nu} \nabla_{\nu} \psi^{+}_{\mu} - 4\bar{\psi}^{+\mu} \nabla_{\mu} \rho^{+} - 2\bar{\rho}^{+} \not{\nabla} \rho^{+} \\ & \quad - \tfrac{1}{2} \bar{\psi}^{+\mu} \not{H} \psi^{+}_{\mu} - \bar{\psi}^{+}_{\mu} H^{\mu\nu\lambda} \gamma_{\nu} \psi^{+}_{\lambda} - \tfrac{1}{2} \rho^{+} H^{\mu\nu\lambda} \gamma_{\mu\nu} \psi^{+}_{\lambda} + \tfrac{1}{2} \rho^{+} \not{H} \rho^{+} \Big) \\ & + e^{-2\phi} \Big(2\bar{\psi}^{-\mu} \gamma^{\nu} \nabla_{\nu} \psi^{-}_{\mu} - 4\bar{\psi}^{-\mu} \nabla_{\mu} \rho^{-} - 2\bar{\rho}^{-} \not{\nabla} \rho^{-} \\ & \quad + \tfrac{1}{2} \bar{\psi}^{-\mu} \not{H} \psi^{-}_{\mu} + \bar{\psi}^{-}_{\mu} H^{\mu\nu\lambda} \gamma_{\nu} \psi^{-}_{\lambda} + \tfrac{1}{2} \rho^{-} H^{\mu\nu\lambda} \gamma_{\mu\nu} \psi^{-}_{\lambda} - \tfrac{1}{2} \rho^{-} \not{H} \rho^{-} \Big) \\ & - \tfrac{1}{4} e^{-\phi} \Big(\bar{\psi}^{+}_{\mu} \gamma^{\nu} \not{F}^{(B)} \gamma^{\mu} \psi^{-}_{\nu} + \rho^{+} \not{F}^{(B)} \rho^{-} \Big) \Big], \end{aligned} \tag{3.1.4}$$

where only terms up to quadratic in the fermions have been included. Here, ∇ denotes the Levi-Civita connection and the combination $\rho^{\pm} := \gamma^{\mu} \psi^{\pm}_{\mu} - \lambda^{\pm}$ arises naturally in generalised geometry.

Varying each action with respect to its contributing fields and demanding that the resulting first-order expressions vanish produces two sets of equations. Setting the fermions equal to zero, the bosons' equations of motion are typically stated in the combination:

$$\begin{aligned}
\mathcal{R}_{\mu\nu} - \tfrac{1}{4} H_{\mu\lambda\rho} H_{\nu}{}^{\lambda\rho} + 2\nabla_\mu \nabla_\nu \phi - \tfrac{1}{4} e^{2\phi} \sum_n \tfrac{1}{(n-1)!} F^{(B)}_{\mu\lambda_1\ldots\lambda_{n-1}} F_{\nu}^{(B)\lambda_1\ldots\lambda_{n-1}} &= 0, \\
\nabla^\mu \left(e^{-2\phi} H_{\mu\nu\lambda} \right) - \tfrac{1}{2} \sum_n \tfrac{1}{(n-2)!} F^{(B)}_{\mu\nu\lambda_1\ldots\lambda_{n-2}} F^{(B)\lambda_1\ldots\lambda_{n-2}} &= 0, \\
\nabla^2\phi - (\nabla\phi)^2 + \tfrac{1}{4}\mathcal{R} - \tfrac{1}{48} H^2 &= 0, \\
\mathrm{d}F^{(B)} - H \wedge F^{(B)} &= 0,
\end{aligned} \tag{3.1.5}$$

matching the form of the vanishing beta functions of the non-linear sigma model.

Since S_F includes at most quadratic terms, it can only be used to obtain the fermion equations of motion up to linear order. Its variation then yields:

$$\begin{aligned}
&\gamma^\nu \left[\left(\nabla_\nu \mp \tfrac{1}{24} H_{\nu\lambda\rho}\gamma^{\lambda\rho} - \partial_\nu \phi \right) \psi^\pm_\mu \pm \tfrac{1}{2} H_{\nu\mu}{}^{\lambda} \psi^\pm_\lambda \right] - \left(\nabla_\mu \mp \tfrac{1}{8} H_{\mu\nu\lambda}\gamma^{\nu\lambda} \right) \rho^\pm \\
&\qquad = \tfrac{1}{16} e^\phi \sum_n (\pm)^{[(n+1)/2]} \gamma^\nu \not{F}^{(B)}_{(n)} \gamma_\mu \psi^\mp_\nu , \\
&\left(\nabla_\mu \mp \tfrac{1}{8} H_{\mu\nu\lambda}\gamma^{\nu\lambda} - 2\partial_\mu\phi \right) \psi^{\mu\pm} - \gamma^\mu \left(\nabla_\mu \mp \tfrac{1}{24} H_{\mu\nu\lambda}\gamma^{\nu\lambda} - \partial_\mu \phi \right) \rho^\pm \\
&\qquad = \tfrac{1}{16} e^\phi \sum_n (\pm)^{[(n+1)/2]} \not{F}^{(B)}_{(n)} \rho^\mp .
\end{aligned} \tag{3.1.6}$$

Another crucial aspect of any supersymmetric theory is its supersymmetry variations, schematically written using parameter ϵ in the form

$$\delta(\text{boson}) = \epsilon(\text{fermion}) , \qquad \delta(\text{fermion}) = \epsilon(\text{boson}) . \tag{3.1.7}$$

For the theory in question, type II supergravity, $\epsilon = \epsilon^+ + \epsilon^-$ are a pair of chiral spinors.

For the boson variations, the right-hand-side contains only fermions. As was the case for the equations of motion, these are given only up to linear order and read:

$$\begin{aligned}
\delta e^a_\mu &= \bar{\epsilon}^+ \gamma^a \psi^+_\mu + \bar{\epsilon}^- \gamma^a \psi^-_\mu , \\
\delta B_{\mu\nu} &= 2\bar{\epsilon}^+ \gamma_{[\mu} \psi^+_{\nu]} - 2\bar{\epsilon}^- \gamma_{[\mu} \psi^-_{\nu]} , \\
\delta\phi - \tfrac{1}{4}\delta \log(-g) &= -\tfrac{1}{2}\bar{\epsilon}^+ \rho^+ - \tfrac{1}{2}\bar{\epsilon}^- \rho^- , \\
\left(e^B \wedge \delta A \right)^{(n)}_{\mu_1\ldots\mu_n} &= \tfrac{1}{2} \left(e^{-\phi} \bar{\psi}^+_\nu \gamma_{\mu_1\ldots\mu_n} \gamma^\nu \epsilon^- - e^{-\phi} \bar{\epsilon}^+ \gamma_{\mu_1\ldots\mu_n} \rho^- \right) \\
&\quad \mp \tfrac{1}{2} \left(e^{-\phi} \bar{\epsilon}^+ \gamma^\nu \gamma_{\mu_1\ldots\mu_n} \psi^-_\nu + e^{-\phi} \bar{\rho}^+ \gamma_{\mu_1\ldots\mu_n} \epsilon^- \right) ,
\end{aligned} \tag{3.1.8}$$

with the differing signs relating to IIA and IIB, respectively. The vectors $\{e_\mu\}$ specify an orthonormal frame induced by the metric, as desribed in Chapter 2.

The fermion variations, on the other hand, are:

$$\begin{aligned}\delta\psi_\mu^\pm &= \left(\nabla_\mu \mp \tfrac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda}\right)\epsilon^\pm + \tfrac{1}{16}e^\phi \sum_n (\pm)^{[(n+1)/2]} \not{F}^{(B)}_{(n)}\gamma_\mu\epsilon^\mp,\\ \delta\rho^\pm &= \gamma^\mu\left(\nabla_\mu \mp \tfrac{1}{24}H_{\mu\nu\lambda}\gamma^{\nu\lambda} - \partial_\mu\phi\right)\epsilon^\pm.\end{aligned} \tag{3.1.9}$$

As will be shown in the next section, these enforce non-trivial restrictions on both the internal manifold and background spacetime in even the most simple compactifications.

Now that the reader has seen the makeup of the theory, it is worth taking a moment to consider the symmetries of the NSNS bosonic sector as these are "built-in" as a fundamental building block of the generalised geometry that will be developed in the next chapter.

The potential B is not defined on a global frame but instead locally, taking the form $B_{(i)}$ in coordinate chart $U_i \subset M$. These local specifications are patched together on intersecting charts $U_i \cap U_j$ as

$$B_{(i)} = B_{(j)} - \mathrm{d}\Lambda_{(ij)}. \tag{3.1.10}$$

For this definition to be internally consistent, it must be ensured that over a triple intersection $U_i \cap U_j \cap U_k$, following the cyclic route $i \to j \to k \to \ i$ leaves $B(i)$ unchanged overall. This necessitates that the contributions from $\mathrm{d}\Lambda_{ij}$, $\mathrm{d}\Lambda_{jk}$, and $\mathrm{d}\Lambda_{ki}$ always perfectly cancel, which can be achieved generically by requiring that the one-forms Λ_{ij} obey

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = \mathrm{d}\Lambda_{(ijk)} \tag{3.1.11}$$

so that nilpotency guarantees the vanishing of the right-hand-side upon taking the exterior derivative of the equation above.

On top of the standard diffeomorphism invariance that characterises all tensors in general relativity, in supergravity the two-form potential $B_{(i)}$ in U_i is also invariant under gauge transformations:

$$B'_{(i)} = B_{(i)} - \mathrm{d}\lambda_{(i)}. \tag{3.1.12}$$

Plugging this and the corresponding expression for U_j into Equation (3.1.10), the transformation only preserves the patching rules if $\mathrm{d}\lambda_{(i)} = \mathrm{d}\lambda_{(j)}$. Explicitly,

$$\begin{aligned} B_{(i)} &= B_{(j)} - \mathrm{d}\Lambda_{(ij)} \\ B'_{(i)} + \mathrm{d}\lambda_{(i)} &= B'_{(j)} + \mathrm{d}\lambda_{(j)} - \mathrm{d}\Lambda_{(ij)} \\ \therefore \quad B'_{(i)} &= B'_{(j)} - \mathrm{d}\Lambda_{(ij)} \implies \mathrm{d}\lambda_{(i)} = \mathrm{d}\lambda_{(j)}, \end{aligned} \tag{3.1.13}$$

where $B_{(i)} = B'_{(i)} + \mathrm{d}\lambda_{(i)}$ and $B_{(j)} = B'_{(j)} + \mathrm{d}\lambda_{(j)}$ have been used to go from the first line to the second. This result means that the gauge symmetry is given by a global closed two-form $\omega = \mathrm{d}\lambda$, or, in other words, is characterised entirely by local one-form λ_i. Diffeomorphisms, meanwhile, are specified via a vector v and thus the two symmetries can be simultaneously described by the general variation

$$\delta_{v+\lambda} B_{(i)} = \mathcal{L}_v B_{(i)} - \mathrm{d}\lambda_{(i)}. \tag{3.1.14}$$

It is important to explicitly demand that this is coordinate-independent in order to agree with the patching of B:

$$\begin{aligned} \delta_{v+\lambda} B_{(i)} \quad &= \quad \delta_{v+\lambda} B_{(j)} \\ \mathcal{L}_v B_{(i)} - \mathrm{d}\lambda_{(i)} \quad &= \quad \mathcal{L}_v B_{(j)} - \mathrm{d}\lambda_{(j)} \\ \implies \quad \mathrm{d}\lambda_{(i)} \quad &= \quad \mathrm{d}\lambda_{(j)} + \mathcal{L}_v (B_{(i)} - B_{(j)}) \\ &= \quad \mathrm{d}\lambda_{(j)} - \mathcal{L}_v \mathrm{d}\Lambda_{(ij)} \end{aligned} \tag{3.1.15}$$

This can be satisfied by choosing to patch $\lambda_{(i)}$ as

$$\lambda_{(i)} = \lambda_{(j)} - i_v \mathrm{d}\Lambda_{(ij)}, \tag{3.1.16}$$

since then Cartan's formula allows one to calculate

$$\mathrm{d}\lambda_{(i)} = \mathrm{d}\lambda_{(j)} - \mathrm{d}\big(i_v \mathrm{d}\Lambda_{(ij)}\big) = \mathrm{d}\lambda_{(j)} - \mathcal{L}_v \mathrm{d}\Lambda_{(ij)} + i_v \mathrm{d}^2\Lambda_{(ij)} = \mathrm{d}\lambda_{(j)} - \mathcal{L}_v \mathrm{d}\Lambda_{(ij)}, \tag{3.1.17}$$

which is indeed the desired result.

3.2 Searching for Supersymmetric Solutions

As discussed in Chapter 1, when studying string theory or supergravity in $d = 10$ one is typically interested in finding solutions of the form

$$\mathcal{M}_{10} = \chi_6 \times M_4, \tag{3.2.1}$$

where χ is a compact six-dimensional internal manifold, and M_4 is a maximally-symmetric external spacetime. In seeking models matching observation, an essential aspect is maintaining Poincaré invariance in M_4. This requires a bosonic background in which each fermion is switched off, meaning that all of the boson variations in Equation (3.1.8) are automatically zero. The metric for the ten-dimensional manifold is then then split into

$$\mathrm{d}s^2_{\mathcal{M}} = e^{2A}\mathrm{d}s^2_{\mathcal{X}} + \mathrm{d}s^2_{M}\,, \tag{3.2.2}$$

with the warp factor A depending on the coordinates of the external spacetime.

The preservation of supersymmetry (a highly desirable feature - see Chaper 1) then corresponds to a single non-trivial condition: invariance under fermion variations. From Equation (3.1.9), it is clear that this criterion equates to having a non-zero *Killing spinor* ϵ whose supersymmetry variations vanish. Because these variations contain all of the information of the equations of motion via the Bianchi identities, any such ϵ satisfying these so-called *Killing spinor equations* is guaranteed to be a solution to all of the supergravity equations of motion.

Consider first the case where all fields but g are zero. This simplifies the Killing spinor equations to

$$\delta\lambda^{\pm} = \gamma^{\mu}\partial_{\mu}\phi\epsilon^{\pm} = 0\,, \qquad \delta\psi^{\pm}_{\mu} = \nabla_{\mu}\epsilon^{\pm} = 0\ . \tag{3.2.3}$$

The above metric ansatz allows one to split the parameters $\epsilon^{\pm}$ into the tensor product of four- and six-dimensional chiral spinors $\zeta_{\pm}$ and $\eta_{\pm}$, respectively. Noting that $\eta_{-} = \eta_{+}^{*}$ and $\eta_{-} = \eta_{+}^{*}$ (since ϵ is in the Majorana basis) and plugging the decomposition into Equation (3.2.3) finally gives the constraint

$$\nabla_{\mu}\eta_{\pm} = 0. \tag{3.2.4}$$

This packages together a topological criterion that there must be a global spinor η over χ_6 with the differential criterion that η is covariantly conserved. An immediate consequence of this statement is that the quantity $\eta\eta^{\dagger}$ is constant, so that it is possible to choose a normalisation $\eta_{+}\eta_{+}^{\dagger} = \eta_{-}\eta_{-}^{\dagger} = 1$ and define the tensor

$$J_m{}^n := i\eta_{+}^{\dagger}\gamma_m{}^n\eta_{+} = -i\eta_{+}^{\dagger}\gamma_m{}^n\eta_{+} \tag{3.2.5}$$

where $m, n \in \{1, \ldots, 6\}$. Using the Fierz identity [30], it follows that

$$J_m{}^n J_n{}^p = -\delta_m{}^p. \tag{3.2.6}$$

J can therefore be recognised as an almost complex structure. In components, the Nijenhuis tensor simply reads

$$N^p{}_{mn} = J_m{}^q \partial_{[q} J_{n]}{}^p - J_n{}^q \partial_{[q} J_{m]}{}^p = 0, \tag{3.2.7}$$

meaning that (see the integrability condition from Chapter 2) χ_6 is a complex manifold. Since the supergravity theory has a metric g which, crucially, has not been turned off, one can always construct a symplectic structure. Explicitly:

$$\omega = J_m{}^k g_{kn} \mathrm{d}x^m \otimes \mathrm{d}x^n = J_{mn} \mathrm{d}x^m \wedge \mathrm{d}x^n = i g_{\mu\bar{\nu}} \mathrm{d}x^\mu \wedge \mathrm{d}\bar{x}^\nu. \tag{3.2.8}$$

Taking the exterior derivative results in the expression

$$\mathrm{d}\omega = \partial\omega + \bar{\partial}\omega = i\partial_\alpha g_{\mu\bar{\nu}} \mathrm{d}x^\alpha \wedge \mathrm{d}x^\mu \wedge \mathrm{d}\bar{x}^\nu + i\partial_{\bar{\alpha}} g_{\mu\bar{\nu}} \mathrm{d}\bar{x}^\alpha \wedge \mathrm{d}x^\mu \wedge \mathrm{d}\bar{x}^\nu. \tag{3.2.9}$$

Using the antisymmetry of the wedge product, the partial derivatives can be replaced by covariant derivatives. The condition $\mathrm{d}\omega = 0$ is then satisfied provided that one can define a metric connection $\nabla g = 0$, which is always possible for Riemannian manifolds. This shows that χ_6 is a Kähler manifold because both J and ω are integrable. Finally, consider the nowhere vanishing holomorphic ($n = 3$)-form

$$\Omega = \frac{1}{6}\Omega_{mnp} \mathrm{d}z^m \wedge \mathrm{d}x^n \wedge \mathrm{d}x^p, \tag{3.2.10}$$

where Ω_{mnp} is built out of spinor η as $\eta_-^T \gamma_{mnp} \eta_-$. Since η is covariantly conserved, by construction it holds that $\nabla_{\bar{d}} \Omega_{abc} = 0$. Writing this out in coordinates and realising that because χ_6 is Kähler all the Γ terms must give zero, the expression reduces to $\bar{\partial}\Omega = 0$. Given that for $d = 6$ there are only three "holomorphic" dimensions, it is also true that $\partial\Omega = 0$. Combining these two results via $\mathrm{d} = \partial + \bar{\partial}$, this implies the closure and thus integrability of Ω. It has thus been demonstrated that, for a fluxless background, demanding the preservation of supersymmetry constrains the internal manifold to be Calabi-Yau.

For the above solution, it is in fact possible to derive the corresponding four-dimensional effective action [31]. Such a calculation reveals this background to be an $\mathcal{N} = 2$ supergravity containing moduli. As discussed in the introduction, non-zero fluxes are therefore useful because they can provide mass-generating potentials to rid models of these massless scalars. Allowing some of the fields to be non-trivial alters the above picture in a drastic way. The presence of RR fields in (3.1.9) relates ϵ^+ and ϵ^-. As a result, the four-dimensional spinors $\zeta_\pm$ are no longer decoupled: the effective theory must now be

$\mathcal{N} = 1$. From a phenomenological perspective, this is preferable because it produces four-dimensional spacetimes that are closer to the observed universe. Unfortunately, the price to be paid for this is that the previous calculation breaks down from the start: $\nabla\eta \neq 0$ so the integrability conditions are no longer satified and χ_6 is no longer Calabi-Yau. This makes sense intuitively because the back-reaction of the fluxes with the metric modifies the geometry such that it cannot be Ricci-flat anymore.

Given that the tools that proved so powerful for the purely metric situation seem to be of far less use in less trivial cases, what *can* be said about the geometries resulting from flux compactifications? To find out, it will first be necessary to develop a more sophisticated machinery - generalised geometry.

Introduction to Generalised Geometry

4

Having been presented with ordinary differential geometry through the framing of bundles and G-structures, the reader is now properly equipped to tackle generalised geometry - the main focus of this dissertation. This will be developed in a manner mirroring Chapter 2, beginning with the generalised tangent bundle and ending with generalised Kähler structures. Throughout, special attention will be given to any similarities to type II supergravity, with a particular emphasis on the content and symmetries of the bosonic NSNS sector. The relevant references for this chapter are [15], [21], [32], [33], and [34].

4.1 Generalised Tangent Bundle

The pedagogical starting point for all the mathematics developed in Chapter 2 was the tangent bundle, an object that proved to be a fundamental building block for all that followed. After witnessing the wealth of intricate structures that were generated out of this, one might wonder what would happen if a slightly different initial object were chosen instead. In particular, what are the consequences of replacing each T_pM with $T_pM \oplus T_p{}^*M$ in the construction of TM?

Definition The *generalised tangent bundle* for a d-dimensional manifold M is defined as

$$E := \sqcup_{p\in M} T_pM \oplus T_p{}^*M. \tag{4.1.1}$$

Suppose M is equipped with an atlas $\mathcal{A}$. For the part of the ordinary tangent bundle TM covered by $U_i \in \mathcal{A}$, consider a section $v_{(i)}$. To avoid confusion, a section of the cotangent bundle T^*M over the same coordinate chart is denoted using the Greek letter $\lambda_{(i)}$. It follows that within local chart U_i, $V_{(i)} := v_{(i)} + \lambda_{(i)}$ is a section of E. This is a *generalised vector field*. For brevity, these objects will be referred to simply as generalised vectors.

Note that these sections of E have been defined locally, so it must now be specified how they relate to one another over non-trivial intersections $U_i \cap U_j$. In an attempt to weave the supergravity symmetries into the fabric of generalised geometry, the patching rules are chosen to be

$$v_{(i)} + \lambda_{(i)} = v_{(j)} + \big(\lambda_{(j)} - i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)}\big). \tag{4.1.2}$$

This is really two separate rules. The typical tensorial relation $v_{(i)} = v_{(j)}$ means that the $v_{(i)}$ globally specify a vector field. On the other hand, $\lambda_{(i)} = \lambda_{(j)} - i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)}$ (which can be recognised from Equation (3.1.16)) does not define a global one-form field. As a result, E is not the same as $TM \oplus T^*M$. It will later be shown that these two spaces become isomorphic only with a specific choice of frame - there exists a *non-canonical isomorphism.*

Because each generalised vector contains both a vector and one-form component, the generalised tangent bundle comes immediately equipped with an inner product without even having to build any more structure:

Definition. The norm squared of $V = v + \lambda \in \Gamma(E)$ is set as the interior product of λ with respect to v

$$\langle V, V \rangle \coloneqq i_v \lambda. \tag{4.1.3}$$

This being the interior product of V with itself, it makes sense to define the interior product between two arbitrary generalised vectors V and $W = w + \zeta$ as

$$\langle V, W \rangle \coloneqq \frac{1}{2}(i_v \zeta + i_w \lambda) = \begin{pmatrix} v & \lambda \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} w \\ \zeta \end{pmatrix} = V^T \eta W \coloneqq \eta(V, W), \tag{4.1.4}$$

which is indeed consistent with above since $(1/2)(i_v \lambda + i_v \lambda) = i_v \lambda$.

The use of matrix notation to rearrange the above expression highlights the emergence of an object η known as the *natural metric*, which induces an $O(d,d)$-structure. To show this explicitly, one can apply the standard procedure introduced in Chapter 2, where it was seen that a Riemannian metric g gives an $O(d)$-structure. In that example, g was specified as a bilinear and positive map. Here, the bilinearity of η is inherited from the bilinearity of $i_{(\cdot)}(\cdot)$. A quick calculation reveals the eigenvalues of η to be $\pm 1/2$. η therefore cannot be positive, and the relevant property to check for is non-degeneracy, which can be seen from the fact that the combination $i_v \zeta + i_w \lambda$ will vanish for all possible values of w and ζ only if both v and λ are zero. In other words, $\langle V, W \rangle = 0$ implies $V = 0$.

This means that η is a bilinear and non-degenerate map defining the inner product

$$\eta : E \times E \to \mathbb{R}. \tag{4.1.5}$$

Because of the unusual patching of generalised vectors, before proceeding it is important to also verify that this interior product is independent of the particular chart being used. If this were not the case, then the scalar $\langle V, V \rangle$ would not characterise an intrinsic property of V. Taking the interior product of the one-form patching rule on $U_i \cap U_j$ gives

$$i_{v_{(i)}} \lambda_{(i)} = i_{v_{(j)}} (\lambda_{(j)} - i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)}). \tag{4.1.6}$$

Expanding the bracket, it should be clear that the last term vanishes because the two interior products are symmetric with respect to i and j whereas the two-form $\mathrm{d}\Lambda_{(ij)}$ is antisymmetric. This leaves $i_{v_{(i)}} \lambda_{(i)} = i_{v_{(j)}} \lambda_{(j)}$, which proves that the quantity $i_v \lambda$ is globally defined.

Having established that η is actually a suitable metric, one now goes through the usual steps for defining G-structure, proceeding as:

1. An isomorphism $M : E \to E'$ between the two spaces (V, η) and (V', η') should satisfy

$$\forall\, V, W \in \Gamma(E) : \eta'(M(V), M(W)) = \eta(V, W). \tag{4.1.7}$$

2. As derived earlier, the standard inner product is just

$$\eta(V, W) = V^T \eta W. \tag{4.1.8}$$

3. The set formed by writing the isomorphism condition for the standard model is

$$\begin{aligned} &\{M \in GL(2d, \mathbb{R}) : \forall\, V, W \in \mathbb{R}^{2d} : \eta(M(V), M(W)) = \eta(u, v)\} \\ &= \left\{ M \in GL(2d, \mathbb{R}) : M^T \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} M = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \right\} := G, \end{aligned} \tag{4.1.9}$$

where the second line is obtained by substituting $\eta(M(V), M(W)) = (MV)^T \eta M W = V^T M^T \eta A W$ and noticing that this can be equal to $V^T \eta W$ only if $M^T \eta M = \eta$. To help identify group G, it is useful diagonalise the above matrix. Substituting

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} = O \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} O^{-1} \tag{4.1.10}$$

into Equation (4.1.9) and then rearranging allows group G to be written as

$$G = \left\{ M \in GL(2d, \mathbb{R}) : O^{-1} M^T O \cdot \mathrm{diag}(\mathbb{1}, -\mathbb{1}) \cdot OMO^{-1} = \mathrm{diag}(\mathbb{1}, -\mathbb{1}) \right\}. \tag{4.1.11}$$

With G in this suggestive form, it is now clear how to use the similarity transformation to construct an isomorphism that proves the natural metric is an $O(d,d)$-structure.

$$\begin{aligned} G \cong \phi(G) &:= \left\{ N = \phi(M) := O^{-1} M O : M \in G \right\} \\ &= \left\{ N \in GL(2d, \mathbb{R}) : N^T \cdot \mathrm{diag}(\mathbb{1}, -\mathbb{1}) \cdot N = \mathrm{diag}(\mathbb{1}, -\mathbb{1}) \right\} = O(d,d). \end{aligned} \tag{4.1.12}$$

4. Finally, it follows again by the argument made for the $O(d)$-structure that the special invariant frames are those that are orthonormal in the sense that the condition

$$\forall\, A, B \in \{1, \dots, 2d\} : \langle E_A, E_B \rangle = \eta_{AB} \tag{4.1.13}$$

for a frame $\{E_A\}$ is preserved by $O(d,d)$ transformations. Just as in ordinary geometry, this means that the corresponding principal bundle

$$F := \{(p, E_A) : P \in M, \{E_A\} \text{ is an orthonormal frame of E at } p\}, \tag{4.1.14}$$

known as the *generalised frame bundle*, has structure group $O(d,d)$.

To help better understand the symmetries of this group, it is useful to consider the general form of an $O(d,d)$ generator. To derive this, recalling that Lie algebra elements capture the properties of a group close to the identity, one can write an arbitrary $M \in O(d,d)$ as

$$M = \mathbb{1} + \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \mathcal{O}(\epsilon^2). \tag{4.1.15}$$

Plugging this into the second line of Equation (4.1.9) and only keeping terms up to linear order in ϵ, the condition reads

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} + \epsilon \begin{pmatrix} c + c^T & d + a^T \\ a + d^T & b + b^T \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{4.1.16}$$

which is only satisfied if $b = -b^T$, $c = -c^T$, and $d = -a^T$. After relabelling $b \to \beta$ and

$c \to B$, a generic $O(d,d)$ group generator is written as

$$\begin{pmatrix} a & \beta \\ B & -a^T \end{pmatrix}, \tag{4.1.17}$$

with $a : TM \to TM$ an element of $End(TM) \cong GL(d, \mathbb{R})$, $B : TM \to T^*M$ a two-form, and $\beta : T^*M \to TM$ an antisymmetric $(2,0)$ tensor. Via the exponential map, one finds the following three $O(d,d)$ subgroups:

1. *A-transformations.*

$$e^a \cdot V := \exp\left[\begin{pmatrix} a & 0 \\ 0 & -a^T \end{pmatrix}\right]\begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} e^a & 0 \\ 0 & e^{-a^T} \end{pmatrix}\begin{pmatrix} v \\ \lambda \end{pmatrix} = e^a \cdot v + e^{-a^T} \cdot \lambda. \tag{4.1.18}$$

   Just like a, $A := e^a$ is an element of $GL(d, \mathbb{R})$, so this subgroup can be identified as a $GL(d, \mathbb{R})$ embedding within $O(d,d)$.

2. *B-transformations.*

$$e^B \cdot V := \exp\left[\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\right]\begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix}\begin{pmatrix} v \\ \lambda \end{pmatrix} = v + (\lambda - i_v B), \tag{4.1.19}$$

   where, following the usual convention, $B \cdot v$ has been defined as $-i_v B$. The collection of elements e^B form the subgroup denoted G_B. Note that the action of these elements only affects the one-form component of V. As will become clear shortly, in order to set up this mathematical framework to describe type II supergravity, it is useful to demand that the two-from B is defined locally via the same patching rule as Equation (3.1.10).

3. *β-transformations.*

$$e^\beta \cdot V := \exp\left[\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} v \\ \lambda \end{pmatrix} = (v + \beta \cdot \lambda) + \lambda. \tag{4.1.20}$$

   This subgroup will be denoted G_β. Notice that, in contrast to e^B the action of the elements e^β changes the vector part of $V \in \Gamma(\tilde{E})$.

Combining the three types of transformation, a general $O(d,d)$ element can be represented as

$$M = \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T \end{pmatrix}. \tag{4.1.21}$$

It is worth pausing to reflect on these findings. After enlarging the tangent bundle to also include one-forms, the resulting structure group contains, as one might expect, an extension of the $GL(d, \mathbb{R})$ structure group of TM. As well as this, however, it also contains new symmetries characterised by B and β.

With the end result of capturing the NSNS bosonic sector in mind, one can slightly modify E to include an extra degree of freedom that will allow generalised geometry to capture the scalar dilaton field:

Definition. Given a generalised tangent bundle E, the *weighted generalised bundle* is given by

$$\tilde{E} := \det T^*M \otimes E. \tag{4.1.22}$$

With this slight tweak to E, the natural metric η now induces an $O(d,d) \times \mathbb{R}^+$ structure. In the previous discussion, it was seen that the invariant frames under the action of $O(d,d)$ elements were orthonormal. In the present context, the relevant frames are *conformal* in the sense that the condition

$$\forall\, A, B \in \{1, \ldots, 2d\} : \langle \hat{E}_A, \hat{E}_B \rangle = \Phi^2 \eta_{AB} \tag{4.1.23}$$

for a frame $\{\hat{E}_A\}$ is preserved by $O(d,d) \times \mathbb{R}^+$ transformations of the general form

$$M = C \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T \end{pmatrix}, \tag{4.1.24}$$

where C is just a scalar and all other symbols are unchanged from Equation (4.1.21). Here, the conformal factor Φ is just a section of $\det T^*M$. The union of all such frames for every $p \in M$ is the *weighted generalised frame bundle* $\tilde{F}$.

Over the local chart U_i in which point p is described by coordinates x^μ, an obvious choice of conformal basis is one simply obtained without using anything other than the coordinate vector basis and its dual: $\{\hat{E}_A\} = \partial/\partial x^\mu \cup \mathrm{d}x^\mu$. This is orthonormal (and thus trivially conformal) since the interior product of $\mathrm{d}x^\mu$ with respect to ∂x^ν is by construction δ^μ_ν.

What will be more useful for the purpose of this dissertation is a type of conformal frame that, for an arbitrary vector basis $\hat{e}_a$ and dual basis $\hat{\theta}^a$, is built using one of the locally-specified two-forms B encountered earlier in the context of the group of transformations G_B:

Definition. A *split frame* is defined via

$$\hat{E}_A = \begin{cases} \hat{E}_a = \det \hat{e}\,(\hat{e}_a + i_{\hat{e}_a} B) & \text{for } A = a \\ \hat{E}^a = \det \hat{e}\,\hat{\theta}^a & \text{for } A = a + d. \end{cases} \tag{4.1.25}$$

As a quick check that this is indeed conformal, it is necessary to verify that only the "off-diagonal" combinations have non-trivial inner products, and that these products are proportional to the identity. Explicitly,

$$\begin{aligned} \langle \hat{E}_a, \hat{E}_b \rangle &= \det \hat{e}^2 i_{\hat{e}_a} i_{\hat{e}_b} B = 0 \\ \langle \hat{E}^a, \hat{E}^b \rangle &= 0 \\ \langle \hat{E}_a, \hat{E}^b \rangle = \det \hat{e}^2 \frac{1}{2} i_{\hat{e}_a} \hat{\theta}^b &= \det \hat{e}^2 \frac{1}{2} \delta^b_a = \langle \hat{E}^b, \hat{E}_a \rangle, \end{aligned} \tag{4.1.26}$$

showing that Equation (4.1.23) is satisfied, with the conformal factor in this case simply $\Phi = \det \hat{e}^2$.

In order illustrate the importance of this basis, consider the weighted generalised vector $V = v + \lambda$ in $\tilde{E}$. Expanding this in the split frame using $V^A \hat{E}_A$

$$V = v^a \hat{E}_a + \lambda_a \hat{E}^a = v^a \det \hat{e}\, \hat{e}_a + \lambda_a \det \hat{e}\, \hat{\theta}^a v^a \det \hat{e}\, i_{\hat{e}_a} B \coloneqq V^{(B)} + i_v B, \tag{4.1.27}$$

one can identify the bijective map

$$f : V \to V^{(B)} = V - i_v B = v + (\lambda - i_v B). \tag{4.1.28}$$

This is known as the *splitting* produced by frame $\{\hat{E}_A\}$ and can be recognised from Equation (4.1.19) as the action of B-transformation $e^B \in G_B$. V, being a weighted generalised vector, is just a section of the weighted generalised tangent bundle $\tilde{E}$. Which space does the object $V^{(B)}$ live in? The answer to this becomes apparent after substituting the patching rules over $U_i \cap U_j$ for a generalised vector (4.1.2) with those of two-form

B (3.1.10):

$$\begin{aligned} \lambda_{(i)} &= \lambda_{(j)} - i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)} \\ &= \lambda_{(j)} - i_{v_{(j)}} (B_{(j)} - B_{(i)}) \\ \implies \lambda_{(i)} - i_{v_{(i)}} B_{(i)} &= \lambda_{(j)} - i_{v_{(j)}} B_{(j)}, \end{aligned} \tag{4.1.29}$$

where in the final line $v_{(i)} = v_{(j)}$ has been used. This result shows that the combination $\lambda - i_v B$ does not vary from patch to patch so that, for $V^{(B)}$, both the vector and one-form components are globally defined. Therefore, $V^{(B)}$ is a section of $TM \oplus T^*M$, and the splitting defines an isomorphism $\tilde{E} \cong TM \oplus T^*M$ As previously claimed, this isomorphism is non-canonical since it depends on the specific B chosen when constructing the split frame.

This restriction to split frames is equivalent to a reduction of the structure group of $\tilde{F}$ from $O(d,d) \times \mathbb{R}^+$ to G_{split}. To find out the nature of this subgroup, it is necessary to determine the general form of the transformations which relate split frames to one another. Just like in ordinary geometry, a generic change-of-basis matrix M changes the components of a generalised vector V as

$$V^A \to V'^A = M^A{}_B V^B. \tag{4.1.30}$$

The condition $V = V^A \hat{E}_A = V'^A \hat{E}'_A$ then implies that the frame components correspondingly transform as

$$\hat{E}_A \to \hat{E}'_A = \hat{E}_B (M^{-1})^B{}_A. \tag{4.1.31}$$

One can now proceed by considering the action for each part of an arbitrary $O(d,d) \times \mathbb{R}^+$ element (4.1.24) and identifying the parts that preserve the general form of a split frame, which is required to become

$$\hat{E}'_A = \begin{cases} \hat{E}'_a = \det \hat{e}' \left(\hat{e}'_a + i_{\hat{e}'_a} B' \right) & \text{for } A = a \\ \hat{E}'^a = \det \hat{e}' \, \hat{\theta}'^a & \text{for } A = a + d. \end{cases} \tag{4.1.32}$$

As derived in Equation (4.1.18), the effect of the A-transformation is

$$\hat{E}_A \to \det \hat{e} \left(\hat{e}_b (A^{-1})^b{}_a + (A^{-1})^b{}_a i_{\hat{e}_b} B + A^a{}_b \hat{\theta}^b \right) = \det \hat{e} \left(\hat{e}'_a + i_{\hat{e}'_a} B + \hat{\theta}'^a \right). \tag{4.1.33}$$

Since A is in the structure group $GL(d, \mathbb{R})$ of the ordinary frame bundle that relates vector bases, this already performs the desired action on the relevant affected parts of $\hat{E}_A$. Meanwhile, the scalar degree of freedom in (4.1.24) acts to multiply $\det \hat{e}$ by factor C. Comparing this with the determinant of relation $\hat{e}'_a = \hat{e}_b (A^{-1})^b{}_a$, namely

$$\det \hat{e}' = (\det A)^{-1} \det \hat{e}, \tag{4.1.34}$$

it is clear that the split frame form is only preserved if $C = (\det A)^{-1}$. Turning attention to the B-transformations, which will here be characterised using the symbol ω to avoid confusion with the two-form B that defines the split frame, these change $\hat{E}_A$ in the manner seen in Equation (4.1.19). As remarked earlier, only the one-form part is altered, becoming

$$\det \hat{e} \left(\hat{\theta}^a + i_{\hat{e}_a} B \right) \to \det \hat{e} \left(\hat{\theta}^a + i_{\hat{e}_a} B - i_{\hat{e}_a} \omega \right) = \det \hat{e} \left(\hat{\theta}^a + i_{\hat{e}_a} (B - \omega) \right). \tag{4.1.35}$$

For this to agree with (4.1.32), the combination $B - \omega$ should be identified with B'. Such a gauge transformation of the field B would clearly need to preserve the patching rule (3.1.10) in order for the splitting/isomorphism argument above to hold for the new frame. By Equation (3.1.12) and the calculation provided below it, this can only be the case if ω is closed. Finally, since the β-transformations modify the vector part of $\hat{E}_A \in \Gamma(\tilde{E})$ by adding a term $\beta \cdot \hat{\theta}^a$ that is not proportional to $\hat{e}_a$, they must be excluded from the reduced structure group. Putting all this together, elements of G_{split} must therefore be of the form

$$M = (\det A)^{-1} \begin{pmatrix} \mathbb{1} & 0 \\ \omega & \mathbb{1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T \end{pmatrix} = (\det A)^{-1} \begin{pmatrix} A & 0 \\ \omega A & (A^{-1})^T \end{pmatrix}, \tag{4.1.36}$$

which reveals that $G_{\text{split}} = GL(d, \mathbb{R}) \ltimes \mathbb{R}^{d(d-1)/2} \subset O(d,d) \times \mathbb{R}^+$ since the closed two-form ω has two antisymmetric indices that run from 1 to d. The semi-direct product has been used to emphasise that the $GL(d, \mathbb{R})$ group is acting on both the vector and one-form parts of generalised vectors, whilst the ω-transformations only affect the one-forms. Notice that the original scalar degree of freedom added to help capture the dilaton has been lost by restricting to frames of the form (4.1.25). To avoid this, it is convenient to instead consider a rescaling by function ϕ:

$$\hat{E}_A = \begin{cases} \hat{E}_a = e^{-2\phi} (\det \hat{e}) \left(\hat{e}_a + i_{\hat{e}_a} B \right) & \text{for } A = a \\ \hat{E}^a = e^{-2\phi} (\det \hat{e})\, \hat{\theta}^a & \text{for } A = a + d, \end{cases} \tag{4.1.37}$$

giving a class of frames called *conformal split frames* which form a principal bundle with structure group $G_{\text{split}} \times \mathbb{R}^+$.

4.2 Dorfman Derivative and Courant Bracket

Definition. Recall that the modification of E to $\tilde{E}$ gave rise to weighted generalised vectors. This construction can be extended to obtain *generalised tensors of weight p* as sections of

$$E_{(p)}^{\otimes n} = (\det T^*M)^p \otimes E \otimes \cdots \otimes E, \tag{4.2.1}$$

where it has been possible to build this space solely out of tensor products of E because the metric η can be used to explicitly define the isomorphism $E \cong E^*$ via

$$\begin{aligned} &\eta : E \to E^* \\ &V = V^A \hat{E}_A \to \eta(V, \cdot) = \eta_{AB} V^B \hat{E}^A. \end{aligned} \tag{4.2.2}$$

As a quick aside, note that spinor representations of $\mathrm{Spin}(d,d)$ can also be built into this formalism [12]. For the $O(d,d)$ group, the usual Clifford algebra takes the form $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$. This is expressed in terms of the Clifford action

$$V^A \Gamma_A \Psi_{(i)} = i_v \Psi_{(i)} + \lambda_{(i)} \wedge \Psi_{(i)} \tag{4.2.3}$$

defined for a spinor $\Psi_{(i)} \in \Gamma((\det\{T^*U_i\})^{1/2} \otimes \Lambda^\bullet T^*U_i)$ in a chart $U_i \in M$, which must obey patching rule $\Psi_{(i)} = e^{\mathrm{d}\Lambda_{(ij)}} \wedge \Psi_{(j)}$ in light of (4.1.2). Note that here $\Lambda^\bullet(M)$ denotes the set of polyforms on manifold M, where the term polyform refers to sums of forms of various ranks. As will be discussed further in Chapter 6, the mathematical properties of spinors can be recast in the context of polyforms. The $\mathrm{Spin}(d,d)$ bundles can then be identified as the two distinct spaces $S^\pm(E) \cong (\det\{T^*M\})^{-1/2} \otimes \Lambda^{\mathrm{even/odd}} T^*M$ containing spinors of opposite chirality. Mirroring the construction seen above for generalised tensors, weight-p spinors are understood as $\mathrm{Spin}(d,d) \times \mathbb{R}^+$ representations that live in $S^\pm_{(p)} = ({}^*M)^p \otimes S^\pm(E)$.

Having developed a generalised analogue of tensors, it is now possible to generalise the standard tensor maps found in ordinary geometry. One such map is the Lie derivative with respect to vector v, which acts on a generic (q,r) tensor ξ of weight p as [35]

$$\begin{aligned} (\mathcal{L}_v \xi)^{\mu_1 \ldots \mu_q}_{\nu_1 \ldots \nu_r} = v^\alpha \partial_\alpha \xi^{\mu_1 \ldots \mu_q}_{\nu_1 \ldots \nu_r} & \\ + (\partial_\alpha v^{\mu_1}) \, \xi^{\alpha \mu_2 \ldots \mu_q}_{\nu_1 \ldots \nu_r} + \cdots + (\partial_\alpha v^{\mu_q}) \, \xi^{\mu_1 \ldots \mu_{q-1} \alpha}_{\nu_1 \ldots \nu_r} & \\ - (\partial_{\nu_1} v^\alpha) \, \xi^{\mu_1 \ldots \mu_q}_{\alpha \nu_2 \ldots \nu_r} - \cdots - (\partial_{\nu_r} v^\alpha) \, \xi^{\mu_1 \ldots \mu_q}_{\nu_1 \ldots \nu_{r-1} \alpha} & \\ + p (\partial_\alpha v^\alpha) \xi^{\mu_1 \ldots \mu_q}_{\nu_1 \ldots \nu_r}. & \end{aligned} \tag{4.2.4}$$

In a similar manner to that seen for Equation (2.2.6), the combination $(\mathcal{L}_v\xi)^{\mu_1\ldots\mu_q}_{\nu_1\ldots\nu_r} - v^\alpha\partial_\alpha\xi^{\mu_1\ldots\mu_q}_{\nu_1\ldots\nu_r}$ is recognised as the adjoint action of $a^\mu{}_\nu = \partial_\nu v^\mu \in \mathfrak{gl}(d,\mathbb{R})$ on ξ. Focusing on the particular two cases of a weight-p vector w and weight-p one-form ζ, the above formula gives

$$\begin{aligned} (\mathcal{L}_v w)^\mu &= v^\alpha\partial_\alpha w^\mu - w^\alpha\partial_\alpha v^\mu + p(\partial_\alpha v^\alpha)w^\mu, \\ (\mathcal{L}_v\zeta)_\mu &= v^\alpha\partial_\alpha\zeta_\mu + (\partial_\mu v^\alpha)\zeta_\alpha + p(\partial_\alpha v^\alpha)\zeta_\mu. \end{aligned} \tag{4.2.5}$$

Setting the notation

$$\partial_M = \begin{cases} \partial_\mu & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d, \end{cases} \tag{4.2.6}$$

one can combine the two expressions in (4.2.5) to construct the generalised geometry counterpart to the Lie derivative:

Definition. The *Dorfman Derivative* of the weight-p generalised vector $W = w + \zeta \in \Gamma(E_{(p)})$ with respect to $V = v + \lambda \in \Gamma(E)$ is

$$(L_V W)^M := V^N\partial_N W^M + \left(\partial^M V^N - \partial^N V^M\right) W_N + p\left(\partial_N V^N\right) W^M, \tag{4.2.7}$$

with indices raised and lowered by using η as in (4.2.2). To understand this operator in terms of its action on the vector and one-form parts of W, these terms can be expanded as

$$\begin{aligned} &V^N\partial_N W^M = v^\alpha\partial_\alpha \begin{pmatrix} w^\mu \\ \zeta_\mu \end{pmatrix} = \begin{pmatrix} v^\alpha\partial_\alpha w^\mu \\ v^\alpha\partial_\alpha\zeta_\mu \end{pmatrix}, \\ &W_N\partial^M V^N = \eta_{NA}W^A\eta^{MB}\partial_B V^N = \frac{1}{2}\begin{pmatrix} \zeta_\alpha \\ w^\alpha \end{pmatrix} 2\begin{pmatrix} 0 \\ \partial_\mu \end{pmatrix}\begin{pmatrix} 0 \\ \partial_m u \end{pmatrix}\begin{pmatrix} v^\alpha \\ \lambda_\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \zeta_\alpha\partial_\mu v^\alpha + w^\alpha\partial_\mu\lambda_\alpha \end{pmatrix}, \\ &-W_N\partial^N V^M = -W^N\partial_N V^M = -w^\alpha\partial_\mu \begin{pmatrix} v^\mu \\ \lambda_\mu \end{pmatrix} = \begin{pmatrix} -w^\alpha\partial_\alpha v^\mu \\ -w^\alpha\partial_\alpha\lambda_\mu \end{pmatrix}, \\ &p\left(\partial_N V^N\right) W^M = p(\partial_\alpha v^\alpha)\begin{pmatrix} w^\mu \\ \zeta_\mu \end{pmatrix} = \begin{pmatrix} p(\partial_\alpha v^\alpha)w^\mu \\ p(\partial_\alpha v^\alpha)\zeta_\mu \end{pmatrix}. \end{aligned} \tag{4.2.8}$$

Using the relations in (4.2.5) and also writing $w^\alpha\partial_\mu\lambda_\alpha - w^\alpha\partial_\alpha\lambda_\mu$ in the form $-w^\alpha(\partial_\alpha\lambda_\mu - \partial_\mu\lambda_\alpha)$ so that it can be recognised as $-(i_w\mathrm{d}\lambda)_\mu$, the sum of the four terms above simplifies to

$$(L_V W)^M = \begin{pmatrix} (\mathcal{L}_v w)^\mu \\ (\mathcal{L}_v \zeta)_\mu - (i_w \mathrm{d}\lambda)_\mu \end{pmatrix}. \tag{4.2.9}$$

Analogously to how $(\mathcal{L}_v w)^\mu - v^\alpha \partial_\alpha w^\mu$ was seen to be the adjoint action in $\mathfrak{gl}(d, \mathbb{R})$, the combination $(L_V W)^M - V^N \partial_N W^M$ is identified as (the negative of) the $\mathfrak{g}_{\text{split}} \subset \mathfrak{o}(d,d) \oplus \mathbb{R}$ adjoint action on W. This can be shown by computing the lie algebra element m associated with a given $M \in G_{\text{split}}$. Note that for a generalised vector of weight p, the factor $(\det A)^{-1}$ in (4.1.36) generalises to $(\det A)^{-p}$. Then, one has

$$\begin{aligned} M &= (\det A)^{-p} \begin{pmatrix} A & 0 \\ \omega A & (A^{-1})^T \end{pmatrix} = (1 - p \operatorname{tr} a) \begin{pmatrix} \mathbb{1} + a & 0 \\ \omega(\mathbb{1} + a) & \mathbb{1} - a^T \end{pmatrix} \\ &\approx \mathbb{1} + \begin{pmatrix} a & 0 \\ \omega & -a^T \end{pmatrix} - p \operatorname{tr} a \coloneqq \mathbb{1} + m, \end{aligned} \tag{4.2.10}$$

where $a^\mu{}_\nu = \partial_\nu v^\mu \in \mathfrak{gl}(d, \mathbb{R})$ and $\omega_{\mu\nu}$, being a closed two-form, can always be locally written as $(\mathrm{d}\lambda)_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu$. Remembering that $\omega \cdot w$ was set as $-i_w \omega$ in (4.1.19), the adjoint action of this general element m of the Lie algebra $\mathfrak{g}_{\text{split}}$ is then

$$\begin{aligned} m^M{}_N W^N &= \begin{pmatrix} a^\mu{}_\alpha & 0 \\ \omega_{\mu\alpha} & -a^\alpha{}_\mu \end{pmatrix} \begin{pmatrix} w^\alpha \\ \zeta_\alpha \end{pmatrix} - p \operatorname{tr} a \begin{pmatrix} w^\mu \\ \zeta_\mu \end{pmatrix} \\ &= \begin{pmatrix} w^\alpha \partial_\alpha v^\mu - p(\partial_\alpha v^\alpha) w^\mu \\ w^\alpha(\partial_\alpha \lambda_\mu - \partial_\mu \lambda_\alpha) - (\partial_\mu v^\alpha)\zeta_\alpha - p(\partial_\alpha v^\alpha)\zeta_\mu \end{pmatrix} = - \begin{pmatrix} (\mathcal{L}_v w)^\mu - v^\alpha \partial_\alpha w^\mu \\ (\mathcal{L}_v \zeta)_\mu - (i_w \mathrm{d}\lambda)_\mu - v^\alpha \partial_\alpha \zeta_\mu \end{pmatrix} \\ &= -((L_V W)^M - V^N \partial_N W^M), \end{aligned} \tag{4.2.11}$$

thus proving the above claim and emphasising another parallel between the Dorfman and Lie derivatives.

Mirroring the usual procedure of ordinary geometry by setting $L_V f \coloneqq \mathcal{L}_v f = v^\alpha \partial_\alpha f$ and demanding that the Dorfman derivative commutes with contraction, the corresponding action on an arbitrary generalised tensor Ξ of weight p is found to be

$$\begin{aligned} (L_V \Xi)^{M_1 \ldots M_n} = {} & V^N \partial_N \Xi^{M_1 \ldots M_n} + \left(\partial^{M_1} V^N - \partial^N V^{M_1}\right) \Xi_N{}^{M_2 \ldots M_n} \\ & + \cdots + \left(\partial^{M_n} V^N - \partial^N V^{M_n}\right) \Xi^{M_1 \ldots M_{n-1}}{}_N + p \left(\partial_N V^N\right) \Xi^{M_1 \ldots M_n}, \end{aligned} \tag{4.2.12}$$

which matches the structure of (4.2.4) as expected.

The reader will be familiar with the well-known result in ordinary geometry that the Lie derivative of an unweighted vector $w \in \Gamma(TM)$ with respect to another unweighted vector $v \in \Gamma(TM)$ coincides with the Lie bracket $[v, w]$, where $[\cdot,\cdot]$ is a bilinear and antisymmetric map. This matching occurs in particular because of the form of the first expression in (4.2.5) once p is set to zero, which implies that $\mathcal{L}_v w + \mathcal{L}_w v = 0$. With a simple application of Cartan's formula, the corresponding combination in generalised geometry for weight-zero generalised vectors $V, W \in \Gamma(E)$ becomes

$$L_V W + L_W V = \begin{pmatrix} \mathcal{L}_v w + \mathcal{L}_w v \\ \mathcal{L}_v \zeta - i_w \mathrm{d}\lambda + \mathcal{L}_w \lambda - i_v \mathrm{d}\zeta \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{d} i_v \zeta + \mathrm{d} i_w \lambda \end{pmatrix} = 2\mathrm{d}\langle V, W\rangle, \quad (4.2.13)$$

which is necessarily finite since the inner product was shown earlier to be non-degenerate. This means that the Dorfman derivative - the generalisation of the Lie derivative - cannot coincide with an extension to the Lie bracket. Instead the counterpart to $[\cdot,\cdot]$ is constructed by antisymmetrising L:

Definition. The *Courant bracket* is the antisymmetric and bilinear map

$$\begin{aligned} &[\![\cdot,\cdot]\!] : \Gamma(E) \times \Gamma(E) \to \Gamma(E) \\ &\qquad (V, W) \to [\![V, W]\!] := \frac{1}{2}(L_V W - L_W V) \\ &\qquad\qquad = \frac{1}{2}\begin{pmatrix} \mathcal{L}_v w - \mathcal{L}_w v \\ \mathcal{L}_v \zeta - i_w \mathrm{d}\lambda - \mathcal{L}_w \lambda + i_v \mathrm{d}\zeta \end{pmatrix} \\ &\qquad\qquad = \frac{1}{2}\begin{pmatrix} [v, w] - [w, v] \\ \mathcal{L}_v \zeta - (\mathcal{L}_w \lambda - \mathrm{d} i_w \lambda) - \mathcal{L}_w \lambda + (\mathcal{L}_v \zeta - \mathrm{d} i_v \zeta) \end{pmatrix} \\ &\qquad\qquad = \begin{pmatrix} [v, w] \\ \mathcal{L}_v \zeta - \mathcal{L}_w \lambda - \frac{1}{2}\mathrm{d}\left(i_v \zeta - i_w \lambda\right), \end{pmatrix} \end{aligned} \quad (4.2.14)$$

where the antisymmetry property comes by construction and the bilinearity is inherited from that of $\mathcal{L}_{(\cdot)}(\cdot)$ and $i_{(\cdot)}(\cdot)$. Antisymmetrising Equation (4.2.7) with p set to zero immediately allows this to be written in an $O(d, d)$ covariant form as

$$[\![V, W]\!]^M = V^N \partial_N W^M - W^N \partial_N V^M - \frac{1}{2}\left(V_N \partial^M W^N - W_N \partial^M V^N\right). \quad (4.2.15)$$

The Lie bracket can be used to describe vector transformations in tangent space TM that result from diffeomorphisms on the manifold M. To better understand the corresponding role of the Courant bracket, one can consider how it is affected by the various types of $O(d,d)$ group action in Equation (4.1.21). If the A-transformations (4.1.18) are interpreted as representing a diffeomorphism $A := f : M \to M$, then the induced push-forward $f_* : TM \to TM$ and pull-back maps $f^* : T^*M \to T^*M$ must combine to act on some $V = v + \lambda \in Gamma(E)$ as

$$V \to F_*V := \begin{pmatrix} f_* & 0 \\ 0 & (f^{-1})^* \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} f_*v \\ (f^{-1})^*\lambda \end{pmatrix}. \tag{4.2.16}$$

Using the familiar results $f_*[v,w] = [f_*v, f_*w]$, $f_*\mathcal{L}_v\lambda = \mathcal{L}_{f_*v}(f^{-1})^*\lambda$, $f^*(\mathrm{d}\lambda) = \mathrm{d}(f^*\lambda)$, and $f_*i_v\lambda = i_{f_*v}(f^{-1})^*\lambda$ from ordinary differential geometry, application of F_* on $[\![V,W]\!]$ gives

$$\begin{aligned} F_*\,[\![V,W]\!] &= f_*([v,w] + \mathcal{L}_v\zeta - \mathcal{L}_w\lambda - \tfrac{1}{2}\mathrm{d}\,(i_v\zeta - i_w\lambda)) \\ &= [f_*v, f_*w] + \mathcal{L}_{f_*v}(f^{-1})^*\zeta - \mathcal{L}_{f_*w}(f^{-1})^*\lambda - \tfrac{1}{2}\,\mathrm{d}\big(i_{f_*v}(f^{-1})^*\zeta - i_{f_*w}(f^{-1})^*\lambda\big) \\ &= \big[\!\big[f_*v + (f^{-1})^*\lambda, f_*w + (f^{-1})^*\zeta\big]\!\big] = [\![F_*V, F_*W]\!]\,, \end{aligned} \tag{4.2.17}$$

meaning that the Courant bracket is invariant under A-transformations. Recalling the effect of B-transformations (4.1.19) on generalised vectors, one can check if they obey a similar property to the one just derived for diffeomorphisms. For e^B, the expression equivalent to the right-hand-side of (4.2.17) is

$$\begin{aligned} \big[\!\big[e^BV, e^BW\big]\!\big] &= [\![v + \lambda - i_vB, w + \zeta - i_wB]\!] \\ &= [\![v+\lambda, w+\zeta]\!] - \mathcal{L}_v i_w B + \mathcal{L}_w i_v B + \tfrac{1}{2}\,\mathrm{d}(i_v i_w B - i_w i_v B) \\ &= [\![v+\lambda, w+\zeta]\!] - \mathcal{L}_v i_w B + \mathcal{L}_w i_v + \mathrm{d}(i_v i_w B) \\ &= [\![v+\lambda, w+\zeta]\!] - \mathcal{L}_v i_w B + i_w\,\mathrm{d}(i_v B) \\ &= [\![v+\lambda, w+\zeta]\!] - i_{[v,w]}B - i_w\mathcal{L}B + i_w\,\mathrm{d}(i_v B) \\ &= [\![v+\lambda, w+\zeta]\!] - i_{[v,w]}B - i_w i_v \mathrm{d}B \\ &= e^B\,[\![V,W]\!] - i_w i_v \mathrm{d}B, \end{aligned} \tag{4.2.18}$$

which shows that e^B is a symmetry of the Courant bracket if B is closed. Repeating this procedure once more for the β-transformations yields $e^\beta\,[\![V,W]\!] \neq \big[\!\big[e^\beta V, e^\beta W\big]\!\big]$ in general, revealing that the symmetry group of the Courant bracket is nothing other than G_{split}. As discussed previously, G_{split} perfectly captures the NSNS bosonic sector diffeomorphism and gauge invariance. The implication of this result is that if the Dorfman derivative and Courant bracket are used to build geometric objects analogous to those of general relativity, these objects will automatically have the symmetries of type II supergravity encoded into their structure.

4.3 Generalised Metric and Other Structures

As was seen in Chapter 2, the introduction of a Riemannian metric reduces the $GL(d,\mathbb{R})$ structure group of the frame bundle to $O(d)$, which is its maximal compact subgroup. Attempting to mirror this, one can seek a globally-defined tensor G that reduces the $O(d,d)\times\mathbb{R}^+$ structure group of the weighted generalised frame bundle $\tilde{F}$ to its maximal compact subgroup $O(p,q)\times O(q,p)$, where $p+q=d$. Such a tensor can then appropriately be identified as the *generalised metric*. As will be shown shortly, restricting to the resulting induced principal sub-bundle $P\subset\tilde{F}$ equates to the specification of the entire NSNS sector, which consists of a Lorentzian metric g, a B-field, and a dilaton ϕ.

Given that the generalised tangent bundle $\tilde{E}$ already came with a natural metric, a second metric G can be incorporated into the existing framework only if it is compatible with η. In order to satisfy the definition of a metric, $G=G_{MN}\hat{E}^M\otimes\hat{E}^N$ should be a symmetric and non-degenerate $(0,2)$ generalised tensor. As seen in the previous section, covariant $O(d,d)\times\mathbb{R}^+$ indices can be raised and lowered using η. It follows that the object $P=P^M{}_N\hat{E}_M\otimes\hat{E}^N$, with $P^M{}_N:=\eta^{MI}G_{IN}$, is a $(1,1)$ generalised tensor and so acts as an endomorphism $P:E\to E$ sending V to GV. Note that the symbol P has been introduced to avoid ambiguity when using index-free matrix notation. Having constructed this map P, one can now express the aforementioned compatibility criterion in a form matching Equation (2.1.10):

$$\forall V,W\in\Gamma(\tilde{E}):\eta(PV,PW)=\eta(V,W). \tag{4.3.1}$$

Using matrix notation to phrase $\eta(V,W)$ as $V^T\eta W$, this translates to the condition that $P^T\eta P=\eta$. Plugging in $P=\eta^{-1}G$ and using the symmetry of η, this is rearranged to give

$$\eta^{-1}G\eta^{-1}G=\mathbb{1}. \tag{4.3.2}$$

To see how the above expression constrains the form of G, one can write η^{-1} and G as

$$\eta^{-1}=2\begin{pmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{pmatrix},\qquad\qquad G=\begin{pmatrix}a&b\\c&d\end{pmatrix}, \tag{4.3.3}$$

where imposing $G^T=G$ yields $a^T=a$, $c^T=b$, and $d^T=d$. Inserting these 2×2 block matrices into (4.3.2) gives

$$4\begin{pmatrix} c^2+da & cd+db \\ ac+bc & ad+b^2 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{4.3.4}$$

Relabelling d as g^{-1} and the combination $d^{-1}c$ as B, it follows that:

- The vanishing of the components $ac+ba$ and $cd+db$ implies that $(ba)^T = a^T b^T = ac = -ba$ and also that $(cd)^T = d^T c^T = db = -cd$, so both ba and cd are antisymmetric.

- Because $(d^{-1})^T = (d^T)^{-1} = d^{-1}$ and $(d^{-1}c)^T = c^T d^{-1} = d^{-1}dc^T d^{-1} = -d^{-1}cdd^{-1} = -d^{-1}c$, fields g and B have the desired respective symmetry and antisymmetry.

- Solving for c and b in terms of the above fields, one has $c = dB = g^{-1}B$ and $b = (g^{-1}B)^T = -Bg^{-1}$.

- The diagonal component $4(c^2+da)$ must be equal to $\mathbb{1}$. Substituting the expressions for c and b and making a the subject gives $a = (1/4)g - Bg^{-1}B$.

Rescaling $g \to 2g$ to match the conventional form, G then becomes

$$G = \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix}. \tag{4.3.5}$$

To illustrate how this generalised metric is related to an $O(p,q) \times O(q,p)$ structure, it is instructive to consider a class of frames that explicitly split the $2d$-dimensional generalised tangent bundle into the direct sum of two sub-bundles C_+ and C_-, each of dimension d. Such frames can be built out of the union $\{\hat{E}^+_a\} \cup \{\hat{E}^-_{\bar{a}}\}$, where $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$ are arbitrary conformal frames for C_+ and C_-, respectively. In other words,

$$\begin{aligned} \langle \hat{E}^+_a, \hat{E}^+_b \rangle &= \Phi^2 \eta_{ab}, \\ \langle \hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}} \rangle &= -\Phi^2 \eta_{\bar{a}\bar{b}}, \\ \langle \hat{E}^+_a, \hat{E}^-_{\bar{a}} \rangle &= 0, \end{aligned} \tag{4.3.6}$$

which can be concisely expressed as

$$\langle \hat{E}'_A \hat{E}'_B \rangle = \Phi^2 \eta_{AB} = \Phi^2 \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix}, \tag{4.3.7}$$

where

$$\hat{E}'_A = \begin{cases} \hat{E}^+_a & \text{for } A = a \\ \hat{E}^-_{\bar{a}} & \text{for } A = \bar{a} + d. \end{cases} \tag{4.3.8}$$

Here, η_{ab} and $\eta_{\bar{a}\bar{b}}$ are both flat metrics with signature (p, q), so $-\eta_{\bar{a}\bar{b}}$ clearly has a (q, p) signature. Their appearance in the above combination inside the block matrix manifests the $O(p, q) \times O(q, p)$ symmetry since the prior factor acts only on $\hat{E}^+_a$ whilst the latter acts only on $\hat{E}^-_{\bar{a}}$. In fixing the section $\Phi \in \Gamma(\det T^*M)$ to be a non-vanishing and frame-independent quantity, $\{\hat{E}'_A\}$ defines an isomorphism between generalised tangent bundles $\tilde{E}$ and E. Notice the different form of η_{AB} compared to that in (4.1.4). This can be interpreted in terms of a generalised vielbein $\hat{E}_A{}^M$ which acts on the conformal split frame $\{\hat{E}_M\}$ from (4.1.37) to produce the above frame $\hat{E}'_A := \hat{E}_A{}^M \hat{E}_M$. In direct analogy with ordinary geometry, this vielbein is defined such that it diagonalises the natural metric so that it appears as in (4.3.7). Correspondingly, $\hat{E}_A{}^M$ also "flattens" the generalised metric G so that it takes the form

$$G(\hat{E}'_A, \hat{E}'_B) = G_{MN} \hat{E}^M{}_A \hat{E}^N{}_B = \Phi^2 G_{AB} = \Phi^2 \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix}, \tag{4.3.9}$$

which is equivalent to the expression

$$G = \Phi^{-2} \big(\eta^{ab} \hat{E}^+_a \otimes \hat{E}^+_b + \eta^{\bar{a}\bar{b}} \hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}} \big). \tag{4.3.10}$$

Going back to the distinct sub-bundles C_+ and C_- induced by $\{\hat{E}_M\}$ and noting that any inner product involving a generalised vector that is purely a vector or one-form always vanishes, Equation (4.3.6) can only hold if neither C_+ nor C_- contain such elements. This implies that any $V_+ \in \Gamma(C_+)$ can generally written as

$$V_+ = v + Mv = v + (g + B)v, \tag{4.3.11}$$

with $v \in \Gamma(TM)$ and $M : TM \to T^*M$ a generic $(0, 2)$ tensor that can be decomposed into the sum of symmetric and antisymmetric components g and B, respectively. By imposing that g is globally defined and that B is patched as (3.1.10), these are identified as the NSNS sector metric and B-field. By (4.3.6), the inner product of sections of C_+ with those of C_- must be equal to zero. This completely determines the general form of $V_- \in \Gamma(C_+)$ to be

$$V_- = v + (g - B)v, \tag{4.3.12}$$

since then

$$\langle V_+, V_- \rangle = \frac{1}{2}(i_x(g+B)v + i_v(B-g)v) = i_v i_v B = 0, \tag{4.3.13}$$

as required. The expressions (4.3.11) and (4.3.12) can then be used to explicitly write $\{\hat{E}'_A\}$ as

$$\begin{aligned} \hat{E}^+_a &= e^{-2\phi}\sqrt{-g}\left(\hat{e}^+_a + \hat{\theta}^+_a + i_{\hat{e}^+_a}B\right), \\ \hat{E}^-_{\bar{a}} &= e^{-2\phi}\sqrt{-g}\left(\hat{e}^-_{\bar{a}} - \hat{\theta}^-_{\bar{a}} + i_{\hat{e}^-_{\bar{a}}}B\right), \end{aligned} \tag{4.3.14}$$

in which Φ has been specified to be equal to $e^{-2\phi}\sqrt{-g}$ whilst $\{\hat{e}^+_a\}$ and $\{\hat{e}^-_{\bar{a}}\}$ are two independent d-dimensional vielbeins

$$g(\hat{e}^+_a, \hat{e}^+_b) = \eta_{ab}, \qquad g(\hat{e}^-_{\bar{a}}, \hat{e}^-_{\bar{b}}) = \eta_{\bar{a}\bar{b}}, \tag{4.3.15}$$

with respective duals $\hat{\theta}^{+a}$ and $\hat{\theta}^{-\bar{a}}$. As a consistency check, one can verify that (4.3.14) satisfies the conditions in (4.3.6) by calculating

$$\begin{aligned} \Phi^{-2}\langle \hat{E}^+_a, \hat{E}^+_b \rangle &= \tfrac{1}{2}(i_{\hat{e}^+_a}\hat{\theta}^+_b + i_{\hat{e}^+_b}\hat{\theta}^+_a) = \tfrac{1}{2}(i_{\hat{e}^+_a}(\eta_{bc}\hat{\theta}^{+c}) + i_{\hat{e}^+_b}(\eta_{ac}\hat{\theta}^{+c})) = \tfrac{1}{2}(\eta_{ba} + \eta_{ab}) = \eta_{ab} \\ \Phi^{-2}\langle \hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}} \rangle &= -\tfrac{1}{2}(i_{\hat{e}^-_{\bar{a}}}\hat{\theta}^-_{\bar{b}} + i_{\hat{e}^-_{\bar{b}}}\hat{\theta}^-_{\bar{a}}) = -\tfrac{1}{2}(i_{\hat{e}^-_{\bar{a}}}(\eta_{\bar{b}\bar{c}}\hat{\theta}^{-\bar{c}}) + i_{\hat{e}^-_{\bar{b}}}(\eta_{\bar{a}\bar{c}}\hat{\theta}^{-\bar{c}})) = -\tfrac{1}{2}(\eta_{\bar{b}\bar{a}} + \eta_{\bar{a}\bar{b}}) = \eta_{\bar{a}\bar{b}} \\ \Phi^{-2}\langle \hat{E}^-_{\bar{a}}, \hat{E}^+_b \rangle &= i_{\hat{e}^-_{\bar{a}}}(\hat{\theta}^+_b + i_{\hat{e}^+_b}B) + i_{\hat{e}^+_b}(-\hat{\theta}^-_{\bar{a}} + i_{\hat{e}^-_{\bar{a}}}B) = \eta_{bc}(\hat{e}^-_{\bar{a}})^d(i_{\hat{e}^+_d}\hat{\theta}^{+c}) - \eta_{\bar{a}\bar{c}}i_{\hat{e}^+_b}\hat{\theta}^{-\bar{c}} \\ &= \eta_{bc}(\hat{e}^-_{\bar{a}})^c - \eta_{bc}(\hat{e}^-_{\bar{a}})^c = 0. \end{aligned}$$

Substituting (4.3.14) into the definition of the generalised vielbein $\hat{E}_A{}^M$ seen earlier gives its explicit expression:

$$E = \frac{1}{\sqrt{2}}\begin{pmatrix} \hat{\theta}^+ - \hat{e}^{+T}B & \hat{e}^{+T} \\ -\hat{\theta}^- - \hat{e}^{-T}B & \hat{e}^{-T} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \hat{e}^{+T}(g-B) & \hat{e}^{+T} \\ -\hat{e}^{-T}(g+B) & \hat{e}^{-T} \end{pmatrix}. \tag{4.3.16}$$

Finally, plugging this into (4.3.9) gives the matrix formula for G in exactly the same form as (4.3.5). Recovering identical results from these two seemingly disparate routes demonstrates that specifying the general expression for frames which describe an $O(p,q)\times O(q,p)$ structure is in fact equivalent to defining an η-compatible generalised metric.

Having seen how to generalise the Riemannian structure from ordinary geometry, one can also consider the corresponding extension of almost complex structures:

Definition. A *generalised almost complex structure* is a map

$$\mathcal{J} : E \to E \tag{4.3.17}$$

that satisfies the property

$$\mathcal{J}^2 = -\mathbb{1}. \tag{4.3.18}$$

So far, this is a beat-for-beat adaptation of the construction of J in (2.1). However, the procedure is complicated by the fact the any generalised tangent bundle E comes automatically with a natural $O(d,d)$ metric, so one must demand that the two structures η and $\mathcal{J}$ are compatible with one another in the sense that

$$\mathcal{J}^T \eta \mathcal{J} = \eta. \tag{4.3.19}$$

This has the same form as the compatibility requirement (2.1) for g and J necessary to induce an almost Hermitian manifold.

One perhaps surprising aspect of generalised almost complex structures is that they incorporate not only ordinary almost complex structures J but also almost symplectic structures ω as

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^T \end{pmatrix}, \qquad \mathcal{J}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}. \tag{4.3.20}$$

Generalised geometry can thus be understood as unifying complex and symplectic geometry by placing the two sub-categories on equal footing as extreme cases of the object $\mathcal{J}$.

In complete analogy with what was described for the almost complex structure J in Chapter 2, $\mathcal{J}$ has the eigenvalues $+i$ and $-i$ which define associated sub-bundles $L_\mathcal{J}$ and $\overline{L}_\mathcal{J}$ of the complexified generalised tangent bundle $E \otimes \mathbb{C} = L_\mathcal{J} \oplus \overline{L}_\mathcal{J}$. Here, $L_\mathcal{J}$ and $\overline{L}_\mathcal{J}$, just like the C_+ and C_- sub-spaces seen earlier, are examples of *generalised distributions*. Adapting the concept of involutivity from ordinary geometry, a generalised almost complex structure is said to be integrable if

$$V, W \in \Gamma(L_\mathcal{J}) \implies [\![V, W]\!] \in \Gamma(L_\mathcal{J}). \tag{4.3.21}$$

Notice that this definition is almost identical to (2.2.2), with the Courant bracket now replacing the Lie bracket. Similarly to how it was used that $O(d) \cap GL(d/2, \mathbb{C}) = U(d/2)$ to identify the structure group of principal bundle associated with an almost Hermitian manifold, the intersection of $O(d,d)$ with $GL(d/2,\mathbb{C})$ implies that $\mathcal{J}$ induces a $U(d/2,d/2)$-structure.

Recall how it was shown explicitly in Chapter 2 that the integrability of an almost complex structure is equivalent to the vanishing of the intrinsic torsion of $GL(d/2,\mathbb{C})$. The following sections of this dissertation see the connection, torsion, and intrinsic torsion all generalised to fit within the mathematical framework that has been developed in this chapter. Crucially, these objects are all defined in step-by-step analogy with their ordinary geometry counterparts, with the only significant difference being that they live in the larger space of generalised tensors $E_{(p)}^{\otimes n}$. One of the most useful perks of this parallel approach is that many calculations made in ordinary geometry are still valid (with the appropriate replacements), so their results can be easily adapted. For example, rewriting (2.2.10) and (2.2.11) by substituting: $[\cdot,\cdot]$ with $[\![\cdot,\cdot]\!]$; J with $\mathcal{J}$; and ∇ with (see (5.1.1)) D, the existence of a generalised complex structure is then seen to be equivalent to the vanishing of the generalised intrinsic torsion of $U(d/2,d/2)$.

Definition. Two generalised complex structures J_1 and J_2 satisfying $J_1J_2 = J_2J_1$ define a *generalised Kähler structure* if the combination $G' := -J_1J_2$ forms a metric for the generalised tangent bundle E. To illustrate how this extends the ordinary notion of Kähler geometry, consider the Kähler manifold (M,g,J,ω) and use J and ω to build the generalised complex structures $\mathcal{J}_J$ and $\mathcal{J}_\omega$. A simple calculation then shows that

$$G' = -\mathcal{J}_J\mathcal{J}_\omega = \mathcal{J}_\omega\mathcal{J}_J = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}, \tag{4.3.22}$$

where the equivalence of the three given definitions of the $U(d/2)$ in (2.1) has been used to obtain a Hermitian metric through $g_{mn} := -\omega_{mp}J^p{}_n$. The above object G' is indeed a metric for E, inheriting non-degeneracy and bilinearity from g. In fact, G' is nothing other than a B-transformed generalised metric G, as can be seen explicitly by

$$e^B G' e^{-B} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} = \frac{1}{2}\eta^{-1}G. \tag{4.3.23}$$

The construction of a generalised Kähler manifold thus uniquely specifies a generalised metric G. As seen at the start of this section, G is equivalent to an $O(p,q)\times O(q,p)$ structure. Up to an isomorphism between abstract groups this is just $O(d)\times O(d)$, which implies that the structure group of the $(\mathcal{J}_1, \mathcal{J}_2)$-induced principal bundle is simply

$$U(d/2, d/2) \cap \Big(O(d) \times O(d)\Big) = U(d/2) \times U(d/2). \tag{4.3.24}$$

Geometrising Supergravity

5

In this chapter, the generalised geometry toolset that has been developed over the previous three sections will be expanded to include extensions of the connection, torsion, and curvature from ordinary geometry. In mirroring the typical procedure of general relativity by attempting to construct a generalisation of the Levi-Civita covariant derivative, one runs into a dilema: torsion-free and generalised metric compatible connections do exist but are not unique. The resolution of this ambiguity lies in the fact that the corresponding generalised Ricci tensors turn out to be independent of the connection used, meaning that the direct analogue of the Einstein-Hilbert action is in fact unique. This object will be recognised as the NSNS bosonic sector pseudo-action, strengthening the interpretation of generalised geometry as the natural language of supergravity. The main references for this part of the dissertation are [15], [12], [33], [32], [36], and [34].

5.1 Generalised Connection

Definition. A *generalised connection* for the generalised tangent bundle E is a map

$$\begin{aligned} D : \Gamma(E) \times \Gamma(E) &\to \Gamma(E) \\ (V, W) &\to D_v W, \end{aligned} \tag{5.1.1}$$

where, just like the usual affine connection on TM, $D_{(\cdot)}(\cdot)$ is bilinear and satisfies both the Leibniz and "directional derivative" rules

$$D_V(fW) = V[f]W + fD_V W, \qquad D_{fV} W = fD_V W, \tag{5.1.2}$$

for any function $f \in \mathcal{F}(M)$. Given a generic frame $\{\hat{E}_A\}$, the generalised connection components Ω are defined via

$$D_{\hat{E}_A} \hat{E}_B = D_A \hat{E}_B := \Omega_A{}^C{}_B \hat{E}_C \tag{5.1.3}$$

so that mirroring the standard differential geometry procedure and substituting $V = V^A \hat{E}_A$ gives

$$D_M V = D_M(V^A \hat{E}_A) = (\partial_M V^A + \Omega_M{}^A{}_B V^B)\hat{E}_A \coloneqq (D_M V)^A \hat{E}_A, \tag{5.1.4}$$

as expected. Since the generalised tangent bundle E comes equipped with the natural $O(d,d)$ metric $\eta = \eta_{AB}\hat{E}^A \otimes \hat{E}^B$, it is necessary to impose that this object is compatible with D. Expanding the condition $D\eta = 0$ in a generalised basis and using the fact that η_{AB} is a constant matrix, one has

$$\begin{aligned} D_M \eta &= D_M(\eta_{AB}\hat{E}^A \otimes \hat{E}^B) \\ &= (\partial_M \eta_{AB})\hat{E}^A \otimes \hat{E}^B + \eta_{AB}[(D_M \hat{E}^A) \otimes \hat{E}^B + \hat{E}^A(D_M \hat{E}^B)] \\ &= \eta_{AB}\Omega_M{}^A{}_C \hat{E}^C \otimes \hat{E}^B + \eta_{AB}\Omega_M{}^B{}_C \hat{E}^A \otimes \hat{E}^C \\ &= (\Omega_{MBA} + \Omega_{MAB})\hat{E}^A \otimes \hat{E}^B = 0, \end{aligned} \tag{5.1.5}$$

which means that

$$\Omega_M{}^{AB} = -\Omega_M{}^{BA}. \tag{5.1.6}$$

If, on top of this, compatibility of D with the $O(p,q) \times O(q,p)$ structure $P \subset \tilde{F}$ is demanded, then $DG = D\Phi = D\eta = 0$, where G is the generalised metric and Φ is the conformal factor. To inspect the implications of this on the Ω, one must use the appropriate $O(p,q) \times O(q,p)$ conformal frame. Φ, being a scalar section of $\det T^*M$, is compatible with D only if it is constant. Using this property and the diagonal $\{\hat{E}'_A\}$ basis from (4.3.7) and (4.3.9) to write η as $\Phi^{-2}(\eta^{ab}\hat{E}^+_a \otimes \hat{E}^+_b - \eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}})$, the η compatibility condition now reads

$$\begin{aligned} D_M \eta &= D_M(\Phi^{-2}\eta^{ab}\hat{E}^+_a \otimes \hat{E}^+_b - \Phi^{-2}\eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}}) \\ &= (\partial_M \Phi^{-2})\eta^{ab}\hat{E}^+_a \otimes \hat{E}^+_b + \Phi^{-2}(\partial_M \eta^{ab})\hat{E}^+_a \otimes \hat{E}^+_b + \Phi^{-2}\eta^{ab}[(D_M \hat{E}^+_a) \otimes \hat{E}^+_b + \hat{E}^+_a(D_M \hat{E}^+_b)] \\ &\quad - (\partial_M \Phi^{-2})\eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}} - \Phi^{-2}(\partial_M \eta^{\bar{a}\bar{b}})\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}} - \Phi^{-2}\eta^{\bar{a}\bar{b}}[(D_M \hat{E}^-_{\bar{a}}) \otimes \hat{E}^-_{\bar{b}} + \hat{E}^-_{\bar{a}}(D_M \hat{E}^-_{\bar{b}})] \\ &= \Phi^{-2}(\eta^{cb}\Omega_M{}^a{}_c + \eta^{ac}\Omega_M{}^b{}_c)\hat{E}^+_a \otimes \hat{E}^+_b - \Phi^{-2}(\eta^{\bar{c}\bar{b}}\Omega_M{}^{\bar{a}}{}_c + \eta^{\bar{a}\bar{c}}\Omega_M{}^{\bar{b}}{}_c)\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}} \\ &\quad + \Phi^{-2}(\eta^{cb}\Omega_M{}^{\bar{a}}{}_c - \eta^{\bar{a}\bar{c}}\Omega_M{}^b{}_{\bar{c}})\hat{E}^-_{\bar{a}} \otimes \hat{E}^+_b + \Phi^{-2}(\eta^{ac}\Omega_M{}^{\bar{b}}{}_c - \eta^{\bar{c}\bar{b}}\Omega_M{}^a{}_{\bar{c}})\hat{E}^+_a \otimes \hat{E}^-_{\bar{b}} \\ &= \Phi^{-2}(\Omega_M{}^{ab} + \Omega_M{}^{ba})\hat{E}^+_a \otimes \hat{E}^+_b + \Phi^{-2}(\Omega_M{}^{\bar{a}\bar{b}} + \Omega_M{}^{\bar{b}\bar{a}})\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}} \\ &\quad + \Phi^{-2}(\Omega_M{}^{\bar{a}b} - \Omega_M{}^{b\bar{a}})\hat{E}^-_{\bar{a}} \otimes \hat{E}^+_b + \Phi^{-2}(\Omega_M{}^{\bar{b}a} - \Omega_M{}^{a\bar{b}})\hat{E}^+_a \otimes \hat{E}^-_{\bar{b}} = 0. \end{aligned} \tag{5.1.7}$$

In the same basis, the generalised metric was seen in the last chapter to take the form $\Phi^{-2}(\eta^{ab}\hat{E}^+_a \otimes \hat{E}^+_b + \eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}})$. Realising that this is just the η expression with the signed flipped on the second term, the above calculation can immediately be adapted to write the G compatibility condition as

$$\begin{aligned} D_M G = \Phi^{-2}(\Omega_M{}^{ab} + \Omega_M{}^{ba})\hat{E}^+_a \otimes \hat{E}^+_b + \Phi^{-2}(\Omega_M{}^{\bar{a}\bar{b}} + \Omega_M{}^{\bar{b}\bar{a}})\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}} \\ + \Phi^{-2}(\Omega_M{}^{\bar{a}b} + \Omega_M{}^{b\bar{a}})\hat{E}^-_{\bar{a}} \otimes \hat{E}^+_b + \Phi^{-2}(\Omega_M{}^{\bar{b}a} + \Omega_M{}^{a\bar{b}})\hat{E}^+_a \otimes \hat{E}^-_{\bar{b}} = 0. \end{aligned} \tag{5.1.8}$$

These criteria can simultaneously hold only if

$$\Omega_{Mab} = -\Omega_{Mba}, \qquad \Omega_{M\bar{a}\bar{b}} = -\Omega_{M\bar{b}\bar{a}}, \qquad \Omega_{M\bar{a}b} = \Omega_{Mb\bar{a}} = 0. \tag{5.1.9}$$

Notice that the first equation reveals that $\Omega_M{}^a{}_b$ lives in the Lie algebra $\mathfrak{o}(p,q)$, the second reveals that $\Omega_M{}^{\bar{a}}{}_{\bar{b}}$ lives in $\mathfrak{o}(q,p)$, and the third reveals that D has an action on C_+ that is independent of C_- and vice versa. In other words, the generalised connection maps $V = V^A \hat{E}'_A = v^a_+ \hat{E}^+_a + v^{\bar{a}}_- \hat{E}^-_{\bar{a}}$ into

$$D_M v^A = \begin{cases} \partial_M v^a_+ + \Omega_M{}^a{}_b v^b_+ & \text{for } A = a \\ \partial_M v^{\bar{a}}_- + \Omega_M{}^{\bar{a}}{}_{\bar{b}} v^{\bar{b}}_- & \text{for } A = \bar{a}. \end{cases} \tag{5.1.10}$$

In a similar manner to that seen for the Lie (4.2.1) and Dorfman (4.2.7) derivatives, the generalised connection can be extended to an action on a weighted generalised vector. Using once again a generic frame $\{\hat{E}_A\}$, comparison with (4.2.7) suggests that Equation (5.1.4) must be modified for $V \in E_{(p)}$ by including a term proportional to V^A:

$$(D_M V)^A = \partial_M V^A + \Omega_M{}^A{}_B V^B \hat{E}_A - p\Lambda_M V^A, \tag{5.1.11}$$

where an arbitrary generalised (0,1) tensor Λ_M has been included in order to respect the index structure of the other terms. Specialising to the weight-one generalised tangent space $\tilde{E}$, which was earlier noted to have an $O(d,d) \times \mathbb{R}^+$ structure, this final term can be absorbed by a relabelling of the connection components as

$$\tilde{\Omega}_M{}^A{}_B \coloneqq \Omega_M{}^A{}_B - \Lambda_M \delta^A{}_B. \tag{5.1.12}$$

More generally, by considering appropriate combinations of generalised tensors and using that $Df = \partial f$ for $f \in \mathcal{F}(M)$, the action of D on a weight-p generalised tensor Ξ is found to be

$$\begin{aligned} D_M \Xi^{A_1 \dots A_n} = \partial_M \Xi^{A_1 \dots A_n} + \Omega_M{}^{A_1}{}_B \Xi^{B A_2 \dots A_n} \\ + \dots + \Omega_M{}^{A_n}{}_B \Xi^{A_1 \dots A_{n-1} B} - p\Lambda_M \Xi^{A_1 \dots A_n}. \end{aligned} \tag{5.1.13}$$

Focusing in instead on the spinors $\Psi \in \Gamma(S^{\pm}_{(p)})$ described in (4.2.3), the generalised connection becomes

$$D_M\Psi = \left(\partial_M + \tfrac{1}{4}\Omega_M{}^{AB}\Gamma_{AB} - p\Lambda_M\right)\Psi. \tag{5.1.14}$$

For the purposes of this dissertation, it will also prove useful consider how to lift an ordinary connection ∇ to obtain a generalised connection on $\tilde{E}$. This is achieve by using the conformal split frame (4.1.37) to write $V = w^a\hat{E}_a + \lambda_a E^a$, where $w = w^a(\det e)\hat{e}_a \in \Gamma((\det T^*M) \otimes TM)$ and $\zeta = \zeta_a(\det e)e^a \in \Gamma((\det T^*M) \otimes T^*M)$. The connection ∇ is then straightforwardly embedded into the generalised geometry framework as

$$(D^{\nabla}_M V^A)\hat{E}_A = \begin{cases} (\nabla_\mu v^a)\hat{E}_a + (\nabla_\mu\lambda_a)E^a & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d. \end{cases} \tag{5.1.15}$$

5.2 Generalised Torsion

Definition. The *generalised torsion* of a generalised connection D is defined via the map

$$\mathcal{T}(V) \cdot \Xi = L^D_V\Xi - L_V\Xi, \tag{5.2.1}$$

where Ξ is an arbitrary generalised tensor and $V \in \Gamma(E)$. Direct comparison reveals that this is essentially just to (2.2.7), with the Dorfman derivative L playing the role of $\mathcal{L}$. In order for $\mathcal{T}(V) \cdot \Xi$ to live in the same space as Ξ, the components of $\mathcal{T}(V)$ must have one upstairs index and one downstairs index and can thus be decomposed as $\mathcal{T}(V)^M{}_N = V^P\mathcal{T}^M{}_{PN}$. To obtain a general formula for $\mathcal{T}^M{}_{PN}$, it is sufficient to consider the action on a weighted generalised vector W rather than tensor Ξ. One can make use of the fact that a conformal frame $\{\hat{E}_A\}$ can be used to construct an orthonormal frame $\{\Phi^{-1}\hat{E}_A\}$, to write $V \in \Gamma(E)$ in L_V as $V = V^A\Phi^{-1}\hat{E}_A$ whilst writing $W = W^A\hat{E}_A$ as usual for $W \in \Gamma(\tilde{E})$. With $\mathcal{T}(V) \cdot W \in \Gamma(\tilde{E})$ denoted as $\mathcal{T}_{(V)}W$ to simplify notation, expanding in this basis and using (5.1.12) and (4.2.7) with $p = 1$ yields

$$\begin{aligned}
\mathcal{T}_{(V)}W &= (\mathcal{T}_{(V)}W)^A\hat{E}_A = (\mathcal{T}_{(V)}(W^C\hat{E}_C))^A\hat{E}_A = [(\mathcal{T}_{(V)}W^C)(\hat{E}_C)^A + W^C(\mathcal{T}_{(V)}\hat{E}_C)^A]\hat{E}_A \\
&= [(L^D_V W^C - L_V W^C)(\hat{E}_C)^A + W^C(L^D_V\hat{E}_B - L_V\hat{E}_C)]\hat{E}_A \\
&= [V^B(\tilde{\Omega}_B{}^A{}_C + \tilde{\Omega}^A{}_{CB} - \tilde{\Omega}_C{}^A{}_B)W^C + \tilde{\Omega}_D{}^D{}_B V^B(\delta^A_C W^C) - W^C V^B(L_{\Phi^{-1}\hat{E}_B}\hat{E}_C)^A]\hat{E}_A,
\end{aligned} \tag{5.2.2}$$

where the partial derivatives always cancel because of the form of (5.2.1). As discussed above, the component $(\mathcal{T}_{(V)}W)^A$ can be identified with $\mathcal{T}(V)^A{}_C W^C = V^B\mathcal{T}^A{}_{BC}W^C$, making it possible to simply read off the result that

$$\begin{aligned}\mathcal{T}_{ABC} &= \tilde{\Omega}_{BAC} + \tilde{\Omega}_{ACB} - \tilde{\Omega}_{CAB} + \tilde{\Omega}_D{}^D{}_B \eta_{AC} - (L_{\Phi^{-1}\hat{E}_B} \hat{E}_C)^D \eta_{DA} \\ &= -\tilde{\Omega}_{BCA} - \tilde{\Omega}_{ABC} - \tilde{\Omega}_{CAB} + \tilde{\Omega}_D{}^D{}_B \eta_{AC} - (L_{\Phi^{-1}\hat{E}_B} \hat{E}_C)^D \Phi^{-2} \langle \hat{E}_D, \hat{E}_A \rangle \\ &= -3\tilde{\Omega}_{[ABC]} + \tilde{\Omega}_D{}^D{}_B \eta_{AC} - \Phi^{-2} \langle \hat{E}_A, L_{\Phi^{-1}\hat{E}_B} \hat{E}_C \rangle.\end{aligned} \tag{5.2.3}$$

In complete analogy with the argument made below Equation (2.2.6), $\mathcal{T}$ must be section of the bundle $E \otimes \operatorname{ad} \tilde{F}$. To further highlight the parallels between ordinary and generalised geometry, the notation and terminology used in Chapter 2 can be recycled to give this space the label K_G. Then, just as in (2.2.9), for two arbitrary generalised connections D and $\tilde{D}$ one can define the map [29]

$$\begin{aligned}\tau: \quad & K_G \to W \\ & \Sigma \coloneqq \tilde{D} - D \to \tau(\Sigma) \coloneqq \mathcal{T}(\tilde{D}) - \mathcal{T}(D).\end{aligned} \tag{5.2.4}$$

As in the ordinary case, the image of τ is named W_G so that the quotient space $W_{\text{int}} \coloneqq W/W_G$ must by construction be independent of the specific choice of D and $\tilde{D}$. It therefore once again makes sense to interpret W_{int} as containing the part of the torsion that is core to the principal bundle $\tilde{F}$ - it is the space of the *generalised intrinsic torsion.* It is also useful to consider $U_G \coloneqq \ker \tau$. This is the space of compatible connections with a certain fixed torsion. It will in particular be relevant for the following discussion to help characterise scenarios in which it is possible to find several generalised metric-compatible, torsion-free generalised connections.

Remembering that the weighted generalised frame bundle is an $O(d,d) \times \mathbb{R}^+$ structure, $\operatorname{ad} \tilde{F}$ is clearly just the space $\mathfrak{o}(d,d) \oplus \mathbb{R}$, but how can this be expressed in terms of E? To make this more apparent, one can use the form of a general $\mathfrak{o}(d,d)$ element $m^I{}_J$ from (4.1.17) and notice that

$$m^{IJ} = m^I{}_K \eta^{KJ} = 2 \begin{pmatrix} a & \beta \\ B & a^T \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = 2 \begin{pmatrix} \beta & a \\ -a^T & C \end{pmatrix} = -2 \begin{pmatrix} \beta^T & a^T \\ -a & C^T \end{pmatrix} = -m^{JI}. \tag{5.2.5}$$

Therefore, $\mathfrak{o}(d,d)$ is isomorphic to the space of matrices with antisymmetric E indices. In other words, $\mathfrak{o}(d,d) \cong \Lambda^2 E$. Since this means that $\operatorname{ad} \tilde{F}$ is equivalent to $\Lambda^2 E \oplus \mathbb{R}$, one might guess that $\mathcal{T} \in \Gamma(E \otimes (\Lambda^2 E \oplus \mathbb{R})) \cong \Gamma((E \otimes \Lambda^2 E) \oplus E)$. However, rewriting Equation (5.2.3) in the coordinate frame described in Chapter 4 such that $L_{\hat{E}_B} \hat{E}_C$ vanishes and $\tilde{\Omega}_{[ABC]} = \Omega_{[ABC]} - \Lambda_{[A} \eta_{BC]} = \Omega_{[ABC]}$, the two surviving components of $\mathcal{T}$ are clearly just

$$\mathcal{T}^M{}_{PN} = (\mathcal{T}_1)^M{}_{PN} + (\mathcal{T}_2)_P\,\delta^M{}_N, \tag{5.2.6}$$

with

$$\begin{aligned}(\mathcal{T}_1)_{MNP} &= -3\tilde{\Omega}_{[MNP]} = -3\Omega_{[MNP]},\\ (\mathcal{T}_2)_M &= \tilde{\Omega}_Q{}^Q{}_M = +\Omega_Q{}^Q{}_M - \Lambda_M.\end{aligned} \tag{5.2.7}$$

The three antisymmetric indices on the first term and the single free E index on the second term reveal that the torsion must be a section of the space $\Lambda^3 E \oplus E$.

To make help get a better sense of how to calculate generalised torsion, one can adapt the general prescription followed above for the specific case of the generalised connection D^∇ from (5.1.15) which was defined using the conformal split frame (4.1.37). In particular, if ∇ is assumed to be torsion-free, it is interesting to investigate whether its resulting lift into the space E retains this property in the generalised $\mathcal{T} = 0$ sense. It is useful to realise that the first and second terms of the final line in (5.2.2) are precisely those entering the coordinate frame expressions for $\mathcal{T}_1$ and $\mathcal{T}_2$ (5.2.7), so can be recast as $V^B[(\mathcal{T}_1)^A{}_{BC} + (\mathcal{T}_2)_B\delta^A_C]W^C\hat{E}_A$. Bringing the remaining terms to the other side of the equation, one is left with the combination $\mathcal{T}(V)\cdot W + W^C V^B (L_{\Phi^{-1}\hat{E}_B}\hat{E}_C)^A\hat{E}_A$. As before, $L(V)\cdot W$ can be understood as $V^B T^A{}_{BC} W^C \hat{E}_A$, so dividing by $V^B W^C$ gives the relation

$$\mathcal{T}^A{}_{BC} + (L_{\Phi^{-1}\hat{E}_B}\hat{E}_C)^A = (T_1)^A{}_{BC} + (T_2)_B\delta^A_C \tag{5.2.8}$$

which is only valid if $\{\hat{E}_A\}$ is specified to be the coordinate frame. The objective of finding $\mathcal{T}_1$ and $\mathcal{T}_2$ for D^∇ is then reduced to an exercise in expanding $\mathcal{T}(V)\cdot W$ and $L_{\Phi^{-1}\hat{E}_B}\hat{E}_C$ in a generic conformal split frame and then imposing $\hat{e}_a = \partial_a$. The first of these two tasks is straightforward:

$$\begin{aligned}\mathcal{T}(V)\cdot W &= L^D_V W - L_V W = (\mathcal{L}^\nabla_v w + \mathcal{L}^\nabla_v \zeta - i_w \mathrm{d}^\nabla\lambda) - (\mathcal{L}_v w + \mathcal{L}_v\zeta - i_w\mathrm{d}\lambda)\\ &= (\mathcal{L}^\nabla_v w - \mathcal{L}_v w) + (\mathcal{L}^\nabla_v\zeta - \mathcal{L}_v\zeta) + i_w(\mathrm{d}\lambda - \mathrm{d}^\nabla\lambda)\\ &= i_w(\omega_c{}^b{}_a\lambda_b\hat{\theta}^c\wedge\hat{\theta}^a) = w^c\lambda_b(\omega_c{}^b{}_a - \omega_a{}^b{}_c)\hat{\theta}^a,\end{aligned} \tag{5.2.9}$$

where ω are the connection components in arbitrary TM frame $\{\hat{e}_a\}$ and the ordinary geometry torsion map (2.2.7) implies that the first two brackets must vanish for torsion-free ∇. Next, by inserting $\mathbb{1} = \Phi\Phi^{-1}$ and using the Liebniz rule, $L_{\Phi^{-1}\hat{E}_B}\hat{E}_C$ can be broken down into

$$L_{\Phi^{-1}\hat{E}_B}\hat{E}_C = (L_{\Phi^{-1}\hat{E}_B}\Phi)\Phi^{-1}\hat{E}_C + \Phi(L_{\Phi^{-1}\hat{E}_B}(\Phi^{-1}\hat{E}_C)). \tag{5.2.10}$$

Here, the first term is simple to calculate since Φ is a scalar so that

$$\begin{aligned} L_{\Phi^{-1}\hat{E}_B}\Phi &= \mathcal{L}_{\hat{e}_b}\Phi = \partial_b\Phi = \partial_b(e^{-2\phi}(\det e)) = e^{-2\phi}(\det e)\operatorname{tr}(e^{-1}\partial_b e) + \partial_b(e^{-2\phi})(\det e) \\ &= \Phi\hat{e}_a{}^\nu\partial_b(\hat{e}^a{}_\nu) - 2(\partial_b\phi)\Phi = \Phi\hat{e}_a{}^\nu\hat{e}_b{}^\mu\partial_\mu(\hat{e}^a{}_\nu) - 2\Phi i_{\hat{e}_b}\mathrm{d}\phi = -\Phi(i_{\hat{e}_b}i_{\hat{e}_a}\mathrm{d}e^a + 2i_{\hat{e}_b}\mathrm{d}\phi). \end{aligned}$$

A slightly more involved calculation reveals the $L_{\Phi^{-1}\hat{E}_B}\Phi^{-1}\hat{E}_C$ factor in the second term to be equal to [15]

$$L_{\Phi^{-1}\hat{E}_B}\Phi^{-1}\hat{E}_C = \begin{pmatrix} [\hat{e}_b,\hat{e}_c] + i_{[\hat{e}_b,\hat{e}_c]}B - i_{\hat{e}_b}i_{\hat{e}_c}dB & \mathcal{L}_{\hat{e}_b}\hat{\theta}^c \\ -\mathcal{L}_{\hat{e}_c}\hat{\theta}^b & 0 \end{pmatrix}_{AB\,.} \tag{5.2.11}$$

Switching to the coordinate frame so that $[\hat{e}_a,\hat{e}_b]$, $\mathcal{L}_{\hat{e}_b}\hat{\theta}^c$, and $i_{\hat{e}_b}i_{\hat{e}_a}\mathrm{d}e^a$ all vanish, the relation (5.2.8) is then

$$V^B[(\mathcal{T}_1)^A{}_{BC} + (\mathcal{T}_2)_B\delta^A_C]W^C\hat{E}_A = -2v^bW^C\delta^A_C(i_{\hat{e}_b}\mathrm{d}\phi)\hat{E}_A - v^bw^c(i_{\hat{e}_b}i_{\hat{e}_c}dB)_a\hat{E}^a. \tag{5.2.12}$$

Comparing coefficients of δ^A_C and accounting for the factor of two picked up when raising the index on $(\mathcal{T}_2)_B$ with η^{AB}, one can immediately read off that

$$\mathcal{T}_2 = (\mathcal{T}_2)^A\hat{E}_A = -4(i_{\hat{e}_b}\mathrm{d}\phi)\hat{\theta}^b = -4\mathrm{d}\phi. \tag{5.2.13}$$

Similarly, comparison of the remaining terms gives

$$\mathcal{T}_1 = (\mathcal{T}_1)^{ABC}\hat{E}_A\otimes\hat{E}_B\otimes\hat{E}_C = -4i_{\hat{e}_b}i_{\hat{e}_c}H)_a\hat{\theta}^b\otimes\hat{\theta}^c\otimes\hat{\theta}^a = -4H, \tag{5.2.14}$$

where $H \coloneqq \mathrm{d}B$ and the factor of four is included because both of the downstairs indices in $(T_1)^A{}_{BC}$ have been raised by η. This means in general that D^∇ is not torsion-free, with the obstruction coming from the quantities H and $\mathrm{d}\phi$. Note, however, that by subtracting two such objects so that their individual contributions to $\mathcal{T}_1$ and $\mathcal{T}_2$ perfectly cancel, it is indeed possible to obtain a torsionless generalised connection.

If, in particular, ∇ is chosen to be the Levi-Civita connection for the ordinary metric g, one can use this result to construct a generalised connection that is both torsion-free and $O(p,q)\times O(q,p)$ compatible. Recalling the make-up of the relevant frame (4.3.14) and noting that the connection components $\omega^\pm$ obtained in each of the two orthonormal frames $\{\hat{e}^+_a\}$ and $\{\hat{e}^-_a\}$ must be gauge equivalent

$$\nabla_\mu v^\nu = \big(\partial_\mu v^a + \omega_\mu^{+a}{}_b v^b\big)(\hat{e}^+_a)^\nu = \big(\partial_\mu v^{\bar{a}} + \omega_\mu^{-\bar{a}}{}_{\bar{b}} v^{\bar{b}}\big)(\hat{e}^-_{\bar{a}})^\nu, \tag{5.2.15}$$

allows one to lift ∇ into E as in (5.1.15):

$$D^\nabla_M W^a = \begin{cases} \nabla_\mu w^a_+ & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d, \end{cases} \qquad D^\nabla_M W^{\bar{a}} = \begin{cases} \nabla_\mu w^{\bar{a}}_- & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d. \end{cases} \tag{5.2.16}$$

Because the Levi-Civita is compatible with g, one can use the familiar differential geometry result that ω is antisymmetric with respect to its second and third indices. In other words, $\omega^+_{\mu ab} = -\omega^+_{\mu ba}$ and $\omega^-_{\mu\bar{a}\bar{b}} = -\omega^-_{\mu\bar{b}\bar{a}}$. This implies that D^∇ is $O(p,q) \times O(q,p)$ compatible, as required.

As discussed in the previous chapter, the frames $\{\hat{e}^+_a\}$ and $\{\hat{e}^-_a\}$ are completely independent. Using this freedom to allign the two frames such that $e^+_a = e^-_a = e_a$, it follows that

$$\begin{aligned} W &= w^a_+ \hat{E}^+_a + w^{\bar{a}}_- \hat{E}^-_{\bar{a}} = w^a_+ \Phi(\hat{e}_a + \hat{\theta}_a + i_{\hat{e}_a} B) + w^a_- \Phi(\hat{e}_a - \hat{\theta}_a + i_{\hat{e}_a} B) \\ &= \left(w^a_+ + w^a_-\right) \hat{E}_a + (w_{+a} - w_{-a})\, \eta_{ab} \hat{\theta}^b = \left(w^a_+ + w^a_-\right) \hat{E}_a + (w_{+a} - w_{-a})\, \hat{E}^a. \end{aligned} \tag{5.2.17}$$

With this special choice of frame, the above expression shows that the construction of D^∇ in (5.1.15) coincides with that of (5.2.16). This allows the result $\mathcal{T}_1 = -4H$, $\mathcal{T}_2 = -4\mathrm{d}\phi$ to be borrowed from (5.2.13) and (5.2.14). As argued above, the introduction of a second set of components Σ can be used to construct a new generalised connection D:

$$D_M W^A = D^\nabla_M W^A + \Sigma_M{}^A{}_B W^B. \tag{5.2.18}$$

Given that D^∇ is $O(p,q) \times O(q,p)$ compatible so that Ω satisfies (5.1.9), the compatibility requirement for D implies (by exactly the same arguments) that the same symmetry conditions must also hold for the combination $\Omega + \Sigma$. Subtracting the two independent relations then gives

$$\Sigma_{Mab} = -\Sigma_{Mba}, \qquad \Sigma_{M\bar{a}\bar{b}} = -\Sigma_{M\bar{b}\bar{a}}, \qquad \Sigma_{M\bar{a}b} = \Sigma_{Mb\bar{a}} = 0. \tag{5.2.19}$$

Repeating the calculation in (5.2.13) and (5.2.14) with the extra Σ term in D must simply modify the torsion components to become

$$(T_1)_{ABC} = -4H_{ABC} - 3\Sigma_{[ABC]}, \qquad (T_2)_A = -4\mathrm{d}\phi_A - \Sigma_C{}^C{}_A. \tag{5.2.20}$$

In order to express the action of the generalised connection in terms of the TM indices and thus derive the general expression for a torsion-free D, it is necessary to first consider the embedding of the ordinary dual coordinate basis $\{dx^\mu\}$ in E:

$$\begin{aligned}\mathrm{d}x^\mu &= \hat{e}^\mu_a\eta^{ab}(e^a_\nu \mathrm{d}x^\nu) = \hat{e}^\mu_a\eta^{ab}\hat{e}^\nu_b i_{\partial_\mu}B = \hat{e}^\mu_a\eta^{ab}(\hat{e}^\nu_b\partial_\nu) + \tfrac{1}{2}\eta^{ab}(\hat{e}^{+\mu}_a\hat{e}^{+\nu}_b + \hat{e}^{-\mu}_a\hat{e}^{-\nu}_b\tfrac{\partial}{\partial x^\nu})\partial_\nu\\ &= \tfrac{1}{2}\hat{e}^{+\mu}_a(\eta^{ab}\hat{e}^+_b + \hat{\theta}^{+a} + \eta^{ab}i_{\hat{e}^+_b}B) + \tfrac{1}{2}\hat{e}^{-\mu}_{\bar{a}}(-\eta^{\bar{a}\bar{b}}\hat{e}^-_{\bar{b}} + \hat{\theta}^{-\bar{a}} - \eta^{\bar{a}\bar{b}}i_{\hat{e}^-_{\bar{b}}}B)\\ &= \tfrac{1}{2}(\hat{e}^{+\mu}_a\Phi^{-1}\hat{E}^{+a} - \hat{e}^{-\mu}_{\bar{a}}\Phi^{-1}\hat{E}^{-\bar{a}}).\end{aligned} \tag{5.2.21}$$

This immediately allows $\mathrm{d}\phi$ to be expressed in the frame $\{\Phi^{-1}\hat{E}_A\}$:

$$\mathrm{d}\phi = \partial_\mu\phi\mathrm{d}x^\mu = \tfrac{1}{2}\partial_a\phi\left(\Phi^{-1}\hat{E}^{+a}\right) - \tfrac{1}{2}\partial_{\bar{a}}\phi\left(\Phi^{-1}\hat{E}^{-\bar{a}}\right), \tag{5.2.22}$$

whilst H becomes

$$\begin{aligned}H &= H_{\mu\nu\lambda}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\lambda\\ &= \tfrac{1}{8}\Phi^{-3}H_{\mu\nu\lambda}(\hat{e}^{+\mu}_a\hat{E}^{+a} - \hat{e}^{-\mu}_{\bar{a}}\hat{E}^{-\bar{a}})\wedge(\hat{e}^{+\nu}_a\hat{E}^{+a} - \hat{e}^{-\nu}_{\bar{a}}\hat{E}^{-\bar{a}})\wedge(\hat{e}^{+\lambda}_a\hat{E}^{+a} - \hat{e}^{-\lambda}_{\bar{a}}\hat{E}^{-\bar{a}}).\end{aligned} \tag{5.2.23}$$

Expanding out these wedges and keeping track of the combinations of indices, these two calculations are summarised as

$$\mathrm{d}\phi_A = \begin{cases}\tfrac{1}{2}\partial_a\phi & A = a\\ \tfrac{1}{2}\partial_{\bar{a}}\phi & A = \bar{a}+d,\end{cases} \qquad H_{ABC} = \begin{cases}\tfrac{1}{8}H_{abc} & (A,B,C) = (a,b,c)\\ \tfrac{1}{8}H_{ab\bar{c}} & (A,B,C) = (a,b,\bar{c}+d)\\ \tfrac{1}{8}H_{a\bar{b}\bar{c}} & (A,B,C) = (a,\bar{b}+d,\bar{c}+d)\\ \tfrac{1}{8}H_{\bar{a}\bar{b}\bar{c}} & (A,B,C) = (\bar{a}+d,\bar{b}+d,\bar{c}+d).\end{cases}$$

Demanding that both $\mathcal{T}_1$ and $\mathcal{T}_2$ in (5.2.20) vanish is clearly equivalent to imposing the constraints

$$\Sigma_{[ABC]} = -\tfrac{4}{3}H_{ABC}, \qquad \Sigma_C{}^C{}_A = -4\mathrm{d}\phi_A. \tag{5.2.24}$$

Plugging the six $\mathrm{d}\phi$ and H components given above into these constraints, one obtains

$$\begin{aligned}&\Sigma_{[abc]} = -\tfrac{1}{6}H_{abc}, \quad \Sigma_{\bar{a}bc} = -\tfrac{1}{2}H_{\bar{a}bc}, \quad \Sigma_a{}^a{}_b = -2\partial_b\phi,\\ &\Sigma_{[\bar{a}\bar{b}\bar{c}]} = +\tfrac{1}{6}H_{\bar{a}\bar{b}\bar{c}}, \quad \Sigma_{a\bar{b}\bar{c}} = +\tfrac{1}{2}H_{a\bar{b}\bar{c}}, \quad \Sigma_{\bar{a}}{}^{\bar{a}}{}_{\bar{b}} = -2\partial_{\bar{b}}\phi.\end{aligned} \tag{5.2.25}$$

Crucially, the right-hand-side conditions are imposed on the symmetric degrees of freedom of Σ so can be satisfied independently of the remaining four conditions which, because $H = \mathrm{d}B$, are totally antisymmetric. This means that there is no obstruction to constructing a generalised $O(p,q) \times O(q,p)$ connection that has a vanishing generalised torsion. Rewriting $-2\partial_c\phi$ as

$$-2\partial_c\phi = -\tfrac{2}{d-1}(d-1)\partial_c\phi = -\tfrac{2}{d-1}(d\,\partial_c\phi - \partial_c\phi) = -\tfrac{2}{d-1}(\delta^a_a\partial_c\phi - \eta_{ac}\partial^a\phi) \tag{5.2.26}$$

and setting $d = 10$ with type II supergravity in mind, the most general form of a generalised connection that meets these criteria is then [15]

$$\begin{aligned}
D_a w^b_+ &= \nabla_a w^b_+ - \tfrac{1}{6}H_a{}^b{}_c w^c_+ - \tfrac{2}{9}\big(\delta_a{}^b\partial_c\phi - \eta_{ac}\partial^b\phi\big)w^c_+ + A^{+b}_a{}_c w^c_+,\\
D_{\bar a} w^b_+ &= \nabla_{\bar a} w^b_+ - \tfrac{1}{2}H_{\bar a}{}^b{}_c w^c_+,\\
D_a w^{\bar b}_- &= \nabla_a w^{\bar b}_- + \tfrac{1}{2}H_a{}^{\bar b}{}_{\bar c} w^{\bar c}_-,\\
D_{\bar a} w^{\bar b}_- &= \nabla_{\bar a} w^{\bar b}_- + \tfrac{1}{6}H_{\bar a}{}^{\bar b}{}_{\bar c} w^{\bar c}_- - \tfrac{2}{9}\big(\delta_{\bar a}{}^{\bar b}\partial_{\bar c}\phi - \eta_{\bar a\bar c}\partial^{\bar b}\phi\big)w^{\bar c}_- + A^{-\bar b}_{\bar a}{}_{\bar c} w^{\bar c}_-,
\end{aligned} \tag{5.2.27}$$

where the remaining degrees of freedom are expressed in terms of the tensors $A^{\pm}$. Once again, following the same arguments as in (5.1.9), the compatibility condition for these tensors implies that $A^+_{abc} = -A^+_{acb}$ and $A^-_{\bar a\bar b\bar c} = -A^-_{\bar a\bar c\bar b}$, whereas the torsion-free condition (5.2.24) can only be preserved if $A^+_{[abc]} = A^{+a}_a{}_b = A^-_{[\bar a\bar b\bar c]} = A^{-\bar a}_{\bar a}{}_{\bar b} = 0$.

The fact that there remain undetermined tensors $A^{\pm}$ in (5.2.27) leads to the conclusion that it is always possible to explicitly define a compatible and generalised-torsionless D for an $O(p,q) \times O(q,p)$ principal sub-bundle $P \subset \tilde F$ but that, unlike the Levi-Civita connection for Riemannian and pseudo-Riemannian manifolds, it is not unique. In the language of generalised intrinsic torsion, this means that $W_{\text{int}} = 0$ but U_G is non-trivial.

5.3 Generalised Curvature and Supergravity

In ordinary geometry, the curvature R of a given connection ∇ is given by the map

$$R(u,v)\cdot w = R(u,v,w) \coloneqq [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w. \tag{5.3.1}$$

Importantly, $R \in \Gamma(\Lambda^2 T^*M \otimes TM \otimes T^*M)$ has the tri-linear property $R(fu, gv, hw) = fgh\,R(u,v,w)\,\forall f,g,h \in \mathcal{F}(M)$, which means that it is a tensor. In attempting to define an appropriate extension of this object, it is natural to try to borrow the above expression and replace the Lie bracket with the Courant bracket:

Definition. The *generalised curvature* of a given generalised connection D is given by the map

$$\mathcal{R}\,(U,V)\cdot W = \mathcal{R}\,(U,V,W) \coloneqq [D_U, D_V]W - D_{[\![U,V]\!]}W. \tag{5.3.2}$$

A quick inspection of the corresponding property

$$\begin{aligned}\mathcal{R}(fU,gV,hW) &= D_{fU}D_{gV}hW - D_{gV}D_{fU}hW - D_{[\![fU,gV]\!]}hW \\ &= fgh([D_U,D_V]W - D_{[\![U,V]\!]}W) - \frac{1}{2}h\langle U,V\rangle D_{f\mathrm{d}g-g\mathrm{d}f}W\end{aligned} \tag{5.3.3}$$

reveals that $\mathcal{R}$ is not a generalised tensor. Note that Courant bracket relation [12] $[\![U,fV]\!] = f\,[\![U,V]\!] + \pi(U)[f]V - \langle U,V\rangle \mathrm{d}f$ has been used in the above calculation.

Despite this setback, it is still possible to construct other generalised tensors that can effectively measure generalised curvature. In particular, the focus will now turn to finding tensorial objects built out of torsion-free generalised connections D^∇ that not only extend the usual notion of curvature, but are also independent of the particular choice of D^∇. Recall from (5.2.27) that the ambiguity in this generalised connection is captured by the undetermined tensors $A^\pm$. The first objective is thus to build $O(p,q)\times O(q,p)$ covariant objects out of combinations of D^∇ which see the $A^\pm$ terms cancel.

Recalling that in the previous section it was argued the contractions $A^{+a}_{a}{}_{b}$ and $A^{-a}_{a}{}_{b}$ must vanish in order to not contribute to the torsion, it follows that

$$\begin{aligned}D_a w^a_+ &= \nabla_a w^a_+ - \tfrac{1}{2}\eta^{ac}H_{acb}w^b_+ - \tfrac{2}{9}(\delta^a_a\partial_b\phi - \eta_{ab}\partial^a\phi)w^b_+ - 2(\partial_a\phi)w^a_+ = \nabla_a w^a_+ - 2(\partial_a\phi)w^a_+,\\ D_{\bar a} w^{\bar a}_- &= \nabla_{\bar a} w^{\bar a}_- - \tfrac{1}{2}\eta^{\bar a\bar c}H_{\bar a\bar c\bar b}w^{\bar b}_- - \tfrac{2}{9}(\delta^{\bar a}_{\bar a}\partial_{\bar b}\phi - \eta_{\bar a barb}\partial^{\bar a}\phi)w^{\bar b}_- - 2(\partial_{\bar a}\phi)w^{\bar a}_- = \nabla_{\bar a} w^{\bar a}_- - 2(\partial_{\bar a}\phi)w^{\bar a}_-,\end{aligned} \tag{5.3.4}$$

where $\eta^{ac}H_{acb} = 0$ because η symmetrises two indices of the totally antisymmetric object H. Crucially, the $A^\pm$-dependence has dropped out of these two expressions. On top of this, one already has from (5.2.27) that $D_{\bar a}w^b_+$ and $D_a w^{\bar b}_-$ do not contain $A^\pm$. Thus, altogether these four objects are indeed unique.

One can equivalently consider spinor expressions, as is most appropriate when describing supergravity. Recalling the Chapter 4 discussion on going from $O(d,d)$ to $\mathrm{Spin}(d,d)$, assume now that the manifold has a $\mathrm{Spin}(p,q)\times\mathrm{Spin}(q,p)$ structure rather than $O(p,q)\times O(q,p)$. In the same way that an $O(p,q)\times O(q,p)$ structure led to the splitting of the generalised tangent bundle into $C_\pm$, the induced splitting is now into the sub-bundles

$S(C_\pm)$. Setting $p=0$ as required for $\epsilon^\pm \in \Gamma(S(C_\pm))$, (5.1.14) reduces to

$$\begin{aligned} D_M \epsilon^+ &= \partial_M \epsilon^+ + \tfrac{1}{4}\Omega_M{}^{ab}\gamma_{ab}\epsilon^+, \\ D_M \epsilon^- &= \partial_M \epsilon^- + \tfrac{1}{4}\Omega_M{}^{\bar{a}\bar{b}}\gamma_{\bar{a}\bar{b}}\epsilon^-, \end{aligned} \tag{5.3.5}$$

where γ^a and $\gamma^{\bar{a}}$ are the gamma matrices defined for the respective bundles $S(C_+)$ and $S(C_-)$. Then, reusing the same arguments made in the tensor case, the following four uniquely-specified operators are constructed from the above objects:

$$\begin{aligned} D_{\bar{a}}\epsilon^+ &= \left(\nabla_{\bar{a}} - \tfrac{1}{8}H_{\bar{a}bc}\gamma^{bc}\right)\epsilon^+, \\ D_a\epsilon^- &= \left(\nabla_a + \tfrac{1}{8}H_{a\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}\right)\epsilon^-, \\ \gamma^a D_a \epsilon^+ &= \left(\gamma^a\nabla_a - \tfrac{1}{24}H_{abc}\gamma^{abc} - \gamma^a\partial_a\phi\right)\epsilon^+, \\ \gamma^{\bar{a}} D_{\bar{a}} \epsilon^- &= \left(\gamma^{\bar{a}}\nabla_{\bar{a}} + \tfrac{1}{24}H_{\bar{a}\bar{b}\bar{c}}\gamma^{\bar{a}\bar{b}\bar{c}} - \gamma^{\bar{a}}\partial_{\bar{a}}\phi\right)\epsilon^-, \end{aligned} \tag{5.3.6}$$

where the first and second lines comes immediately from the middle two A-independent expressions in (5.2.27), whilst the third and forth lines use the remaining expressions as well as the identity $\gamma^a\gamma^{bc} = \gamma^{abc} + \eta^{ab}\gamma^c - \eta^{ac}\gamma^b$ to cancel any terms with $A^\pm$.

Definition. Having derived the expressions (5.3.4) for the A-independent quantities $D_a w_+^a$ and $D_{\bar{a}} w_-^{\bar{a}}$ as well as those for $D_{\bar{a}} w_+^b$ and $D_a w_-^{\bar{b}}$ (5.2.27), these can be combined into a unique tensor $\mathcal{R}_{\bar{a}b}$ defined equivalently via

$$\mathcal{R}_{a\bar{b}}\, w_+^a = [D_a, D_{\bar{b}}]\, w_+^a, \tag{5.3.7}$$

or

$$\mathcal{R}_{\bar{a}b}\, w_-^{\bar{a}} = [D_{\bar{a}}, D_b]\, w_-^{\bar{a}}. \tag{5.3.8}$$

This object is identified as the *generalised Ricci tensor* [15]. Meanwhile the ambiguity-free spinor expressions (5.3.6) can be contracted to form the finite scalar $\mathcal{R}$ from either

$$-\tfrac{1}{4}\mathcal{R}\epsilon^+ = \left(\gamma^a D_a \gamma^b D_b - D^{\bar{a}} D_{\bar{a}}\right)\epsilon^+, \tag{5.3.9}$$

or equivalently

$$-\tfrac{1}{4}\mathcal{R}\epsilon^- = \left(\gamma^{\bar{a}} D_{\bar{a}} \gamma^{\bar{b}} D_{\bar{b}} - D^a D_a\right)\epsilon^-. \tag{5.3.10}$$

As before, it is most convenient to align the two orthonormal TM frames by setting

$e_a^+ = e_a^-$. Expanding out the above defined tensor and scalar gives

$$\mathcal{R}_{ab} = R_{ab} - \tfrac{1}{4}H_{acd}H_b{}^{cd} + 2\nabla_a\nabla_b\phi + \tfrac{1}{2}e^{2\phi}\nabla^c(e^{-2\phi}H_{cab}), \tag{5.3.11}$$

where R_{ab} are the components of the ordinary Ricci scalar (5.3.1), and

$$\mathcal{R} = R + 4\nabla^2\phi - 4(\partial\phi)^2 - \tfrac{1}{12}H^2, \tag{5.3.12}$$

where $R = R^a{}_a$. Recalling the bosonic pseudo-action (3.1.2) for type II supergravity from Chapter 3, the NSNS sector is entirely captured by

$$S_{\text{NSNS}} = \frac{1}{2\kappa^2}\int \sqrt{-g}\, e^{-2\phi}\left(R + 4(\partial\phi)^2 - \tfrac{1}{12}H^2\right). \tag{5.3.13}$$

Comparison of the S_{NSNS} integrand with the expression for the generalised Ricci scalar in (5.3.12) reveals that they are equal after an integration by parts of the non-contributing $\nabla^2\phi$. In other words, if, mirroring the form of the usual Einstein-Hilbert action, one constructs a *generalised Einstein-Hilbert action* as

$$S_{\text{GEH}} = \frac{1}{2\kappa^2}\int \Phi\mathcal{R}, \tag{5.3.14}$$

the resulting expression is completely equivalent to S_{NSNS}. Indeed, varying S_{GEH} with respect to the generalised metric G gives $\mathcal{R}_{a\bar{b}} = 0$, an analogue of Einstein's vacuum equations which contains the equations of motion (3.1.5) for the metric g and two-form field B. On the other hand, from the variation of Φ one immediately obtains the dilaton equation repackaged as the vanishing of the generalised Ricci scalar $\mathcal{R} = 0$. In other words, by constructing unique operators out of combinations of expressions involving a generalised analogue of the Levi-Civita connection, it has been possible to effectively "geometrise" the NSNS sector of type II supergravity.

Flux Compactifications

6

With the mathematical toolset of generalised geometry properly developed, it is now time to return to the discussion at the end of Chapter 3. Recall that the ordinary geometry framework of G-structures was shown to be perfectly suited to describing fluxless compactifications of type II supergravity. It revealed that the preservation of supersymmetry is equivalent to the condition that the internal manifold is Calabi-Yau, with the resulting effective theory being $\mathcal{N} = 2$. This relied on the fact that setting all fields except the metric g to zero means that the fermion variations must vanish, and thus specifies the existence of a covariantly conserved global spinor (3.2.4). The inability of this set up to capture the full picture in the case where flux was switched then motivated the pursuit of a more sophisticated approach, particularly necessary to achieve the more desirable $\mathcal{N} = 1$ scenarios. In this Chapter, the internal manifolds in both cases will be understood as $SU(3) \times SU(3)$ structures, with the presence of RR fluxes acting as the obstruction to the specification of a generalised Calabi-Yau manifold. The global references here are [17], [18], [37], [24], [23], and [21].

6.1 Generalised Calabi-Yau Manifolds

In Chapter 5, it was shown that the Courant bracket was only invariant under B-transformations for closed B since (4.2.18)

$$\llbracket e^B V, e^B W \rrbracket = e^B \llbracket V, W \rrbracket - i_w i_v \mathrm{d}B, \tag{6.1.1}$$

for $V = v + \lambda \in \Gamma(E)$ and $W = w + \zeta \in \Gamma(E)$. The presence of the $i_w i_v \mathrm{d}B$ term in the above expression means that it is sometimes convenient relabel the Courant bracket through a shift to incorporate this term:

Definition. The *twisted Courant bracket* is defined as

$$[\![V,W]\!]_H = [\![V,W]\!] + i_w i_v H, \tag{6.1.2}$$

where three-from $H = \mathrm{d}B$. Thanks to this shift, it is then clear from the above that all B-transformations are symmetries of this new Courant bracket, with (6.1.1) becoming $\left[\!\left[e^B V, e^B W\right]\!\right]_{H-\mathrm{d}B} = e^B\, [\![V,W]\!]_H$. In order to use $[\![\cdot,\cdot]\!]_H$, it also makes sense for consistency to redefine other operators. In particular, the corresponding *twisted exterior derivative* is

$$\mathrm{d}_H = \mathrm{d} + H\wedge, \tag{6.1.3}$$

with three-form H again parameterising the "twisting" of the original map. Crucially, since $\mathrm{d}B$ is exact and thus globally closed, the nilpotency of d guarantees that of d_H.

The introduction of the twisted Courant bracket modifies the concepts described so far in a natural way. For example, earlier it was seen that a generalised almost complex structure $\mathcal{J}$ defines associated $\pm i$-eigenvalue sub-bundles $L_{\mathcal{J}}$ and $\bar{L}_{\mathcal{J}}$ of the complexified generalised tangent bundle via $E \otimes \mathbb{C} = L_{\mathcal{J}} \oplus \overline{L}_{\mathcal{J}}$. The criterion for $\mathcal{J}$ to be integrable was then defined in complete analogy with ordinary geometry (2.2.2) as

$$V, W \in \Gamma(L_{\mathcal{J}}) \implies [\![V,W]\!] \in \Gamma(L_{\mathcal{J}}). \tag{6.1.4}$$

Mirroring this, the *H-integrability* condition for $\mathcal{J}$ is just

$$V, W \in \Gamma(L_{\mathcal{J}}) \implies [\![V,W]\!]_H \in \Gamma(L_{\mathcal{J}}). \tag{6.1.5}$$

In the previous two chapters, spinors were described in the generalised geometry framework in terms of polyforms $\psi \in \Lambda^\bullet(M)$. To help understand why this identification is consistent, one can consider again the action (5.1.14) of a generalised vector $V = v+\lambda \in \Gamma(E)$ on a generic polyform, summarised by

$$V \cdot \psi = i_v\psi + \lambda \wedge \psi. \tag{6.1.6}$$

During the discussion on the $\mathrm{Spin}(d,d)$ group, this was termed the Clifford action. There are several ways to see that this name makes sense. Firstly, one way to define the Clifford algebra for the generalised tangent bundle E is via the condition [12] $V^2 = \langle V, V\rangle$. Applying above map twice, one has

$$V^2 \cdot \psi = i_v(i_v\psi + \lambda\wedge\psi) + \lambda \wedge (i_v\psi + \lambda\wedge\psi) = (i_v\lambda)\psi = \langle V,V\rangle\psi, \tag{6.1.7}$$

showing that the space of polyforms is indeed a module for the Clifford algebra. The second and more familiar way to express this algebra is $\{V, W\} = 2\eta(V, W)$ for $W = w + \zeta \in \Gamma(E)$. Indeed, this also holds in the present context:

$$\begin{aligned}\{V, W\} \cdot \psi &= V \cdot (W \cdot \psi) + W \cdot (V \cdot \psi) \\ &= i_v(i_w\psi + \zeta \wedge \psi) + \lambda \wedge (i_w\psi + \zeta \wedge \psi) + i_w(i_v\psi + \lambda \wedge \psi) + \zeta \wedge (i_v\psi + \lambda \wedge \psi) \\ &= (i_v\zeta \wedge \psi + i_w\lambda \wedge \psi) = 2\eta(V, W)\psi,\end{aligned} \tag{6.1.8}$$

as required. Since this is the standard relation for gamma matrices, polyforms are in fact appropriate objects to represent Weyl or Majorana spinors, confirming the approach taken at the end of the last section.

When describing spinors in this setting with the aim in mind of characterising and geometrising supersymmetry conditions, two other useful notions are isotropy and null spaces:

Definition. A generic sub-bundle L of the generalised tangent bundle E is said to be *isotropic* if it holds that

$$V, W \in \Gamma(L) \implies \langle V, W\rangle = 0. \tag{6.1.9}$$

Any spinor ψ can define a sub-bundle L_ψ of E via

$$L_\psi = \{V \in E : V \cdot \psi = 0\}. \tag{6.1.10}$$

This set of all generalised vectors that annihilate the parameterising spinor is called a *null space.* Such sets are examples of sub-bundles that satisfy the isotropy condition. To see this, note that if $V, W \in L_\psi$ then

$$\langle V, W\rangle\psi = \eta(V, W)\psi = \tfrac{1}{2}\{V, W\} \cdot \psi = V \cdot (W \cdot \psi) + W \cdot (V \cdot \psi) = 0. \tag{6.1.11}$$

This means that all null spaces are isotropic. If, in particular, L_ψ has rank d, it is referred to as a *maximally isotropic* space and the corresponding ψ is a *pure spinor.* In $d = 6$, which is of interest since it is the dimension of the internal manifold χ_6 (3.2.1) in type II supergravity compactifications, any Weyl spinor is automatically pure [38]. From a null space $\bar{L}_\psi$, one can construct the space

$$U_k = \Lambda^{d/2-k}\bar{L}_\psi \cdot \psi, \tag{6.1.12}$$

called the *filtration* defined by ψ. This provides a useful way to decompose the total space of polyforms via

$$\Lambda^\bullet(M)^{\mathbb{C}} = \bigoplus_{-d/2\leq k\leq d/2} U_k. \qquad (6.1.13)$$

It is always possible to define a complex-valued pure spinor such that its associated null space is a sub-bundle of $E \otimes \mathcal{C}$. Recognising this as the $(+i)$-eigenvalue distribution $L_{\mathcal{J}}$, it follows that there is a correspondence between a generalised almost complex structure $\mathcal{J}$ and a pure spinor, denoted $\psi_{\mathcal{J}}$, whose null space $L_{\psi_{\mathcal{J}}}$ is the space $L_{\mathcal{J}}$. Since this identification does not uniquely specify the spinor, determining $\psi_{\mathcal{J}}$ modulo an overall factor, it is referred to as a line bundle.

Given this link between generalised almost complex structures and spinors, one can restate geometric conditions for $\mathcal{J}$ in terms of the properties of $\psi_{\mathcal{J}}$ they translate to:

Example. Recalling that $\langle V, W\rangle$ for $V, W \in L_\psi$, the twisted Courant bracket must act on pure spinor ψ as

$$[\![V, W]\!]_H \cdot \psi = [\![\{V, \mathrm{d}_H\}, W]\!] \cdot \psi - \mathrm{d}\langle V, W\rangle \wedge \psi = V \cdot (W \cdot \mathrm{d}_H\psi). \qquad (6.1.14)$$

This relation allows the H-integrability conditon of the generalised almost complex structure J to be expressed as the vanishing of the combination $V \cdot (W \cdot \mathrm{d}_H\psi)$. From the above discussed property, this implies that $\mathrm{d}_H\psi = Z \cdot \psi$ for some $Z \in \Gamma(E)$.

Example. A sub-bundle of the generalised frame bundle F with a $U(d/2) \times U(d/2)$ structure group exists if and only if one can define on M pure spinor line bundles ψ_1 and ψ_2 that meet the following two criteria: ψ_2 is a section of U_0, with U_i being the filtration of ψ_1; and the metric uniquely defined by the $\mathcal{J}$ that $\psi_{1,2}$ correspond to is positive. Recall that in Chapter 4 it was shown that the reduction of the structure group to $U(d/2)\times U(d/2)$ is equivalent to the specification of an almost Kähler structure. If the above-described ambiguity contained in the undetermined overall factor for the spinors is eliminated by demanding a definite normalisation, the spinors are then defined globally and the structure group of F becomes $SU(d/2) \times SU(d/2)$. We can then normalize the pure spinors. Specifically, the normalization condition is

$$\langle\psi_1, \bar{\psi}_1\rangle = \langle\psi_2, \bar{\psi}_2\rangle \neq 0, \qquad (6.1.15)$$

Having collected all the necessary ingredients, it is finally possible to introduce the notion of generalised Calabi-Yau manifolds:

Definition. A given complex pure spinor ψ defines a *(twisted) weak generalised Calabi-Yau manifold* if both

$$\mathrm{d}_H\psi = 0, \qquad \text{and} \qquad \langle \psi, \bar{\psi} \rangle. \tag{6.1.16}$$

Such a manifold is generalized complex since $L_\psi = L_{\mathcal{J}_\psi}$. Importantly, $\mathrm{d}_H\psi = 0$ means that L_ψ (and therefore $\mathcal{J}$) is H-integrable. Note that the prefix "generalised" is slightly misleading here since this category of manifolds does not automatically encompass all ordinary Calabi-Yau structures.

Definition. A *(twisted) generalized Calabi-Yau manifold* is defined by pure ψ_1 and ψ_2 if they induce an $SU(d/2) \times SU(d/2)$ structure and also satisfy

$$\mathrm{d}_H\psi_1 = \mathrm{d}_H\psi_2 = 0. \tag{6.1.17}$$

Notice that the above is the condition that both $L_{\psi 1}$ and $L_{\psi 2}$ are integrable. As discussed earlier, this is equivalent to the specification of two generalised complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$. In light of (5.2.4) and the discussion at the end of Chapter 4, this is equivalent to the statement that the $SU(d/2) \times SU(d/2)$ structure defined in (6.1.15) has vanishing generalised intrinsic torsion such that it is possible to find a torsionless compatible connection. Comparison with the definition at the end of Chapter 4 reveals that this is a sub-category of the generalised Kähler manifold. Crucially (as will be seen in the next section), the ordinary Calabi-Yau manifolds introduced in Chapter 2 are now incorporated as a special case of this generalisation.

6.2 Geometrising the Supersymmetry Conditions

It is now time to return to the case of type II supergravity flux compactifications, which, as seen in Chapter 3, result in an $\mathcal{N} = 1$ four-dimensional effective theory. Notice that, from the two supersymmetry parameters η_1 and η_2, it is possible to construct a pair of globally-defined bispinors as

$$\begin{aligned} \Psi^+ &= e^{-\phi} e^{-B} \left(\eta_1^+ \otimes \overline{\eta}_2^+ \right) \in \Gamma(\Lambda^{\text{even}} T^* M), \\ \Psi_- &= e^{-\phi} e^{-B} \left(\eta_1^+ \otimes \overline{\eta}_2^- \right) \in \Gamma(\Lambda^{\text{odd}} T^* M), \end{aligned} \tag{6.2.1}$$

where the upper case symbol Ψ has been used to avoid confusion with the gravitini. Recall how the Clifford map was used to define the lifting of the generalised tangent bundle's $O(d,d)$-structure to $\mathrm{Spin}(d,d)$. From the discussion in the previous section, $\Psi_\pm$ are then recognised as polyform elements of the two opposite-helicity $\mathrm{Spin}(6,6)$ sub-bundles of E. It was also demonstrated earlier that each pure spinor corresponds to a generalised almost complex structure. Therefore, since both Ψ_+ and Ψ_- have a definite normalisation, they each define a separate $SU(3,3)$-structure. Compatibility then implies that an $SU(3) \times SU(3)$ structure is associated with any $\mathcal{N} = 1$ background.

For the type IIA theory, let Ψ_1 and Ψ_2 respectively represent Ψ_+ and Ψ_-, with the correspondence switched for IIB. Then, using the "twisted" framework developed piece by piece at the start of this Chapter, the fermion supersymmetry variations (3.1.9) can be re-expressed in terms of geometric constraints on the spinors as [17, 37]

$$\mathrm{d}_H(e^{3A}\Psi_1) = 0, \tag{6.2.2}$$

$$\mathrm{d}_H(e^{2A}\,\mathrm{Im}\,\Psi_2) = 0, \tag{6.2.3}$$

$$\mathrm{d}_H(e^{4A}\,\mathrm{Re}\,\Psi_2) = e^{4A} \star \lambda(F). \tag{6.2.4}$$

From these expressions, it is now clear how to interpret the differential conditions arising from preserving supersymmetry in flux backgrounds. The pure spinor $e^{3A}\Psi_1$ is closed with respect to the twisted exterior derivative so, in light of (6.1.16), the resulting internal space χ_6 must be a (twisted) weak generalised Calabi-Yau manifold (6.1.16). The RR sector n-from potentials $A^{(n)}_{\mu_1 \cdots \mu_n}$, which contribute to (6.2.2) via the field strength F, act as an obstruction that means that χ_6 is only a (twisted) generalised Calabi-Yau manifold (6.1.17) if they are set to zero as in the fluxless case.

As a final remark, the standard Calabi-Yau manifolds defined in Chapter 2 are recovered by choosing $\Psi_\pm$ to be

$$\Psi_+ = e^{-\phi}e^{-B}e^{i\omega}, \qquad \Psi_- = ie^{-\phi}e^{-B}\Omega, \tag{6.2.5}$$

where the two-form field B is specified to satisfy $\mathrm{d}B = 0$, whilst ϕ is fixed as a constant. Separately containing $e^{i\omega}$ and Ω, Ψ_+ and Ψ_- can from this expression be interpreted as respectively generalising symplectic and complex manifolds.

Conclusion

7

This dissertation presented a self-contained introduction to the topic of generalised geometry with a focus on applications to supergravity. After a brief overview of type II supergravity and a review of the relevant topics in complex "ordinary" differential geometry, which included discussions on fibre bundles, G-structures, and integrability, the $O(d,d) \times \mathbb{R}^+$ generalised geometry was built from the ground up. This pedagogical approach started with an extension of the tangent bundle that replaced each T_pM with $T_pM \oplus T_p{}^*M$, and then explored how various standard geometrical notions are affected by this modification. What resulted from this procedure was an elegant framework in which "generalised" versions of objects such as the Lie bracket, metric, and connection can be used to build further maps in a consistent and meaningful way. For example, the definition of the generalised torsion was seen to be completely analogous to the usual definition of torsion, with the Dorfman derivative playing the role of the Lie derivative. The categorisation of various geometries in terms of G-structures was found to translate seamlessly when going from ordinary to generalised geometry, and a notable pattern emerged whereby the globally-defined non-degenerate generalised tensors induced product group structures.

Two important applications of this sub-branch of mathematics were considered: the "geometrisation" of the NSNS sector of type II supergravity by Waldram, Coimbra, and Strickland-Constable [15]; and the description of flux compactifications in terms of generalised Calabi-Yau structures by Graña et al. [17, 18]. The former work relied on generalised geometry's ability to incorporate both diffeomorphism invariance and gauge invariance into a single geometric object. This dissertation showed how to exploit this feature, arguing that if the Dorfman derivative and Courant bracket are used to build geometric objects analogous to those of general relativity, those objects will automatically have the symmetries of type II supergravity encoded into their structure. The conclusion of this process was the construction of a generalised Einstein-Hilbert action, which was recognised as nothing other than the NSNS part of the bosonic pseudo-action. The latter paper showed how generalised geometry is able to succeed precisely where ordinary geometry failed, revealing the geometrical implications on the internal manifold that result from switching on the flux fields. In this dissertation, all the necessary ingredients to properly understand this result were collected and motivated physically, in the hope of rendering the field more accessible to newcomers.

Now equipped with a solid foundational understanding of generalised geometry and some of its applications, the reader is in a strong position to explore some of the interesting and powerful extensions to the topics discussed in this dissertation. For example, RR fields can be incorporated into the geometrisation of supergravity by instead considering exceptional generalised geometry [39, 40, 41], in which the generalised tangent bundle is constructed in such a way that it has an $E_{d(d)} \times \mathbb{R}^+$ structure group. The RR n-form potentials are seen to modify the generalised Einstein vacuum equation that was discussed in this dissertation, appearing as a source term. Exceptional generalised geometry is particularly significant because it of its versatility: it can be used to describe the full supergravities associated not only with type IIA and IIB string theories, but also with M-theory. Another exciting area that may be of interest to the reader is the identification of consistent truncations - in which the solutions of the reduced action are the same as those obtained by directly truncating the equations of motion of the original higher-dimensional theory. Early studies found that local group manifolds produce consistent truncations [42], but examples discovered on non-coset spaces such as S^5 [43] are difficult to understand using ordinary geometry. The answers, provided by Waldram and collaborators [44], lie in a conjecture that generalised Leibniz parallelisable manifolds give consistent truncations in the form of generalised Scherk-Schwarz reductions. In the work cited, this extended notion of parallelisability was shown to include all spheres S^d. More recently, a systematic approach for constructing truncations using singlet generalised intrinsic torsion was developed in [29, 45, 46].

Generalised geometry thus continues to prove to be a very active topic of research, attracting attention from both mathematicians and theoretical physicists. There is little sign of this activity slowing down anytime soon, and it is hoped that it will keep shedding light on the mysterious world of supergravity, string theory and M-theory.

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