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Solutions in projected massive gravity

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Abstract

This thesis reviews the background of massive gravity and its new extension theory, projected massive gravity (PMG). The first part of this thesis consists of addressing some outstanding problems and their solutions in the historical development of massive gravity, such as the van Dam–Veltman–Zakharov discontinuity, Boulware-Deser (BD) ghost, and Vainshtein screening. In particular, starting with massless and massive spin-1 and spin-2 fields, we show how gauge invariance can be restored through Stückelberg fields, and finally arrive at a ghost-free massive theory. In the second part, the new BD ghost-free theory PMG is introduced by abandoning a global translation invariance, which has 5 dynamical degrees of freedom. PMG has attracted attention because of it allows for stable cosmological solutions without infinite strong coupling. We thus provide complete details of the derivation of cosmological background equations. The last section comprises the original work. An investigation into black hole solutions in PMG is conducted and the static spherically symmetric solutions are obtained in a concrete model. We also discuss the time-dependent metrics in the end.

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Notation

Throughout this thesis, d represents the number of space-time dimensions and we will work in units where the reduced Planck constant $\hbar$ and the speed of light c are equal to 1. The gravitational Newton constant G_N is related to the reduced 4d Planck mass $M_{\rm Pl} = \frac{1}{\sqrt{8\pi G_N}}$. We also adopt the 'space-like convention' for the metric, i.e., mainly + convention $(- + \cdots +)$. Space indices are denoted by $i, j, \cdots = 1, \cdots, d-1$ while time-like direction represented by 0, $x^0 = t$. $\eta_{\mu\nu}$ represent the flat Minkowski metric.

We use the symmetric convention: $(a, b) = \frac{1}{2}(ab + ba)$ and $[a, b] = \frac{1}{2}(ab - ba)$. For the tensors: $T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)$, $T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)$. The squares of vectors and tensors often represent contractions, for instance, $A_\mu^{\ 2} = A_\mu A^\mu$, $F_{\mu\nu}^{\ 2} = F_{\mu\nu}F^{\mu\nu}$, etc$\cdots$. Moreover, square brackets of a tensor indicate the trace of tensor, for instance $[\mathbb{X}] = \mathbb{X}^\mu{}_\mu$, $\left[\mathbb{X}^2\right] = \mathbb{X}^\mu{}_\nu \mathbb{X}^\nu{}_\mu$, etc$\cdots$. We also use the notation $\Pi_{\mu\nu} = \partial_\mu\partial_\nu\pi$, and $\mathbb{I} = \delta^\mu{}_\nu$.

For the Riemann and Einstein curvature tensors, we will use the conventions:

$$
\begin{aligned}
R^\mu{}_{\nu\alpha\beta} &= \partial_\alpha \Gamma^\mu{}_{\nu\beta} - \partial_\beta \Gamma^\mu{}_{\nu\alpha} + \Gamma^\mu{}_{\sigma\alpha}\Gamma^\sigma{}_{\nu\beta} - \Gamma^\mu{}_{\sigma\beta}\Gamma^\sigma{}_{\nu\alpha}, \\
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,
\end{aligned}
\tag{1}
$$

where $\Gamma^\nu{}_{\mu\rho} = \frac{1}{2} g^{\nu\alpha}\left(\partial_\rho g_{\alpha\mu} + \partial_\mu g_{\alpha\rho} - \partial_\alpha g_{\mu\rho}\right)$ is the Christoffel symbol. We also obtain the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R, via contraction: $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ and $R = g_{\alpha\beta}R^{\alpha\beta} = R^\alpha{}_\alpha$

In 4d space-time we denote ∇_μ as covariant derivatives. The d'Alembertian will then be defined as $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. We will use the dot operators to refer to time derivatives $\dot{} \equiv \frac{d}{dt}$, and use the prime operators to refer to radial derivatives $' \equiv \frac{d}{dr}$.

Acronyms

BD	Boulware–Deser
BH	Black Hole
dofs	degrees of freedom
dRGT	de Rham–Gabadadze–Tolley
EH	Einstein-Hilbert
FLRW	Friedmann–Lemaître–Robertson–Walker
FP	Fierz–Pauli
GR	General Relativity
PMG	Projected Massive Gravity
vDVZ	van Dam–Veltman–Zakharov

Contents

1 Introduction

General relativity (GR) is widely considered to be the correct theory to describe the force of gravity at low energies or large distances developed by Albert Einstein [1]. The correctness of GR has been verified with high precision, ranging from laboratory scale to the solar system scale [2]–[7]. However, on galactic and cosmological scales, current experimental observations are not accurate enough to confirm or disprove GR. The correctness of GR thus remains hypothetical. In fact, within the realm of gravity and cosmology, several questions remain unsolved such as the old cosmological constant problem [8]. In addition, in 1998, astronomers were astonished to find that the expansion of the Universe is accelerating [9], [10]. This discovery of the late-time acceleration of the Universe has presented one of the most challenging problems in theoretical physics [11], leading physicists to explore alternatives to GR on large scales.

Modifications to gravity offer an intriguing approach to addressing the cosmological constant problem. One approach involves modifying the way gravity responds to the cosmological constant [12]. There is a braneworld theory that postulates that we are living in a 4d membrane embedded in a higher dimensional spacetime. Based on the braneworld theory, Georgi Dvali, Gregory Gabadadze, and Massimo Porrati proposed a beautiful self-accelerating solution to explain the accelerating expansion of the universe, known as the DGP model [13]. In the DGP model, a 3-brane is embedded in a 5d spacetime, where gravity in the 5d bulk is described by 5d GR, which induced the 4d gravity on the 3-brane. This solution wherein the Universe's expansion is determined by the ratio between the 4d and 5d Newton constants, without the need for a cosmological constant [14]. Unfortunately, there is no viable model that effectively embodies these novel concepts in a consistent way.

We are confronted with a significant question: why is it challenging to modify GR? David Lovelock proved that Einstein's equations are the only possible equations of motion derived from a 4d action which is a function of the 4d spacetime metric up to second order. It is also known as Lovelock's theorem [15], [16]. From a more modern perspective, Refs. [17]–[19] point out that GR is the unique theory of massless spin-2 particle (graviton) with Lorentz invariance build in. Therefore, if we want to modify GR, we need to introduce one or more of the following conditions [11]: extra degrees of freedom, higher derivatives, higher dimensional spacetime and non-locality. In this thesis, our focus will be on the concept that gravity is still mediated by a spin-2 graviton while respecting Lorentz invariance, but with the consideration that this graviton to be massive.

The construction of a graviton's mass can be traced back to the work of Markus Fierz and Wolfgang Pauli in 1939 [20]. The straightforward approach to constructing a theory of massive gravity is to simply add a mass term to the linearized Einstein-Hilbert (EH) action, known as the Fierz-Pauli (FP) action. This is the theoretical study of massive gravity from a field theory perspective. However, the massive spin-2 field, i.e. FP action, in the linear theory has 5 physical degrees of freedom (dofs) and does not recover to GR as the graviton mass m goes to zero. This discontinuity was recognized as an artificial outcome arising from the linear theory. Moreover, the real challenge arises when we are attempting to extend this theory to a massive gravity theory with interaction. As we know, GR is a fully non-linear and diffeomorphism invariant (general covariant) theory. For massive gravity, the diffeomorphism is broken by the mass term while the full non-linearity is still present. It is the non-linearity that makes the construction complicated. Nearly seven decades after the inception of the massive gravity theory, Claudia de Rham, Gregory Gabadadze and Andrew Tolley (dRGT) first proposed a theoretically consistent theory with a non-linear mass term [21], [22]. Furthermore, the dRGT theory has simultaneously avoided two elements that make the massive gravity theory problematic for many years.

The first element is van Dam–Veltman–Zakharov (vDVZ) discontinuity [23], [24]. As mentioned earlier, a massive spin-2 field propagates 5 dofs even in the limit as $m \to 0$, whereas GR only propagates 2 dofs as a theory of massless spin-2 fields. This subtlety violates the principle of continuity in physics. This vDVZ discontinuity was soon attributed to the fact that not all of the extra dofs introduced by the graviton mass decouple as $m \to 0$. In fact, in the massless limit, a massive graviton becomes a massless graviton coupled with a longitudinal graviton, rather than a massless graviton in GR. Consequently, some phenomenological predictions given by the massive gravity in the massless limit deviate from those of GR. Arkady Vainshtein provided a resolution to this puzzle in 1972 [25]. As the graviton mass decreases, non-linearities of the theory strengthen due to the scalar mode undergoing non-linear fluctuations. This leads to higher-order derivative terms surpassing the magnitude of the conventional kinetic term, a phenomenon known as Vainshtein screening.

However, the story of dofs is still not over. In 1972, David Boulware and Stanley Deser found that the fully non-linear massive gravity theories exhibit a ghost-like instability, resulting in the presence of the sixth dof in these theories [26]. This is the second element of concern in dealing with massive gravity theories, now known as the Boulware–Deser (BD) ghost. Fortunately, new theories born later, such as DGP model and dRGT theory mentioned earlier, have cleverly avoided BD ghost.

Moreover, the dRGT theory even admits an open Friedmann–Lemaître–Robertson–Walker (FLRW) solution, where its mass terms mimic the behavior of a cosmological constant [27]. Thus, the dRGT theory should be an important theory promising to solve the late-time acceleration problem. However, this theory was subsequently shown to potentially have strong coupling problem [28] and a non-linear ghost instability [29]. Hence, it is necessary to explore additional extensions of non-linear massive gravity in order to obtain a stable cosmological solution.

In this thesis, we shall focus on an extension of generalized dRGT theory, which abandons translation invariance. This extension results in a theory that is different from dRGT-type constructions by directly projecting out one dof, now known as projected massive gravity (PMG)[30]. PMG was proposed in 2020 and has been demonstrated to possess a self-accelerating solution without theoretical instabilities. In addition to the solutions in cosmology, black hole (BH) solutions also deserve further investigation, as they serve as a crucial phenomenological basis for testing the correctness of the gravity theory.

This thesis is organized as follows: In Chapter 2, we establish the formalism for massive and massless spin-1 and spin-2 fields, with a particular emphasis on the Stückelberg language for both the Proca and Fierz–Pauli fields. A Brief introduction to the vDVZ discontinuity and its resolution Vainshtein screening are shown in Section 2.3 and Section 2.4. The BD ghost and ghost-free theory are finally discussed at the end of this chapter. In Chapter 3, we start with the massive gravity with non-minimal coupling and generalize the dRGT mass terms by using disformal transformation acted on the reference metric. The necessary conditions to eliminate the would-be BD ghost are summarized in Section 3.1.3. In Section 3.3, we investigate the cosmology solutions based on the equations of motion of PMG and derive the corresponding background equations. Section 3.4 consists of original work, we investigate the BH solutions in PMG and discuss the possibility of the time-dependent metrics. We then devoted to summary and provide perspective on future developments. As this thesis is intended to review the background of massive gravity and its extension PMG, we provide in the main body and appendix numerous non-trivial computational details that may are not presented in the existing literature.

2

2
Massive gravity in a nutshell

This chapter mainly follows with Ref. [31] and Ref. [32], offering a brief introduction to massive gravity within the context of historical development, showing how some important results are inspired by the spin-1 field theory, and finally arriving at the ghost-free massive gravity that we really care about.

Contents

2.1 Spin-1 field

In order to get a more intuitive understanding of massive spin-2 fields, we could start with a review of the spin-1 field.

2.1.1 Maxwell kinetic term

Consider a vector field A_μ in 4d Minkowski manifold, indices are raised and lowered with respect to the flat Minkowski metric. From Lorentz invariance and locality, the kinetic term can be generally given by

$$\mathcal{L}_{\text{kin}}^{\text{spin-1}} = a_1 \partial_\mu A^\nu \partial^\mu A_\nu + a_2 \partial_\mu A^\mu \partial_\nu A^\nu + a_3 \partial_\mu A^\nu \partial_\nu A^\mu, \tag{2.1}$$

where a_1, a_2 and a_3 are undetermined constants. Up to a boundary term, we notice that the contribution of the first and third terms is equal. Thus we can set $a_3 = 0$ from now on.

Helicity-0 and Helicity-1 mode

Before going on to discuss how to decompose the vector field A_μ to continue simplifying the kinetic term, let's introduce an important concept. The Lorentz transformation rule for bosons can be characterized by an integer $h \geq 0$, i.e., helicity. When $h = 0$, these massless particles can be effectively described by a scalar field that possible to introduce any sort of interaction terms that maintain Lorentz invariance. Consequently, there exists a lot of potential self-consistent interacting theories concerning spin-0 particles. When $h = 1$, such massless particles can be carried by a vector field A_μ, which is fixed to be the Maxwell action. Moreover, for the case $h \geq 1$, if we consider the interactions that exhibit manifest Lorentz symmetry and locality, the field must carry a gauge symmetry. When $h = 2$, the required gauge symmetry is linearized diffeomorphism invariance. Refs. [18], [19], [33] show that asking for self-consistent interactions essentially leads to a unique outcome, namely GR with full general coordinate invariance. In the end, Ref. [34] shows that there are no self-interactions that can be written for the case $h \geq 3$.

Now, let us back to discussing the behavior of the different dofs in this theory. In principle, the Lorentz vector field A_μ can have up to 4 dofs in 4d spacetime. A priori, A_μ can be split as

$$A_\mu = A_\mu^\perp + \partial_\mu \chi. \tag{2.2}$$

where χ is a longitudinal (or helicity-0) mode which has 1 dof, $A_\mu^\perp$ is a transverse (or helicity-1) mode, i.e. $\partial^\mu A_\mu^\perp \equiv 0$, which has 3 dofs. Therefore, $\mathcal{L}_{\text{kin}}^{\text{spin-1}}$ becomes,

$$\begin{aligned}\mathcal{L}_{\text{kin}}^{\text{spin-1}} &= a_1\partial_\mu\left(A^{\perp\nu}+\partial^\nu\chi\right)\partial^\mu\left(A_\nu^\perp+\partial_\nu\chi\right)+a_2\partial^\mu\left(A_\mu^\perp+\partial_\mu\chi\right)\partial^\nu\left(A_\nu^\perp+\partial_\nu\chi\right)\\ &= a_1\partial_\mu\left(A^{\perp\nu}+\partial^\nu\chi\right)\partial^\mu\left(A_\nu^\perp+\partial_\nu\chi\right)+a_2\partial^\mu\partial_\mu\chi\partial^\nu\partial_\nu\chi\\ &= a_1\left(\partial_\mu A^{\perp\nu}\partial^\mu A_\nu^\perp+\partial_\mu\partial^\nu\chi\partial^\mu\partial_\nu\chi\right)+a_2\partial^\mu\partial_\mu\chi\partial^\nu\partial_\nu\chi.\end{aligned} \tag{2.3}$$

where we have removed the mixing terms $\partial_\mu A^{\perp\nu}\partial^\mu\partial_\nu\chi$ and $\partial_\mu\partial^\nu\chi\partial^\mu A_\nu^\perp$ by integrating by parts. Now the kinetic term is separated into the part of longitudinal (helicity-0) mode and the part of transverse (helicity-1) mode. For the longitudinal mode χ, we have

$$\begin{aligned}\mathcal{L}_{\text{kin}}^{\chi} &= a_1\partial_\mu\partial^\nu\chi\partial^\mu\partial_\nu\chi+a_2\partial^\mu\partial_\mu\chi\partial^\nu\partial_\nu\chi\\ &= a_1\partial^\mu\partial_\mu\chi\partial^\nu\partial_\nu\chi+a_2(\Box\chi)^2\\ &= (a_1+a_2)\,(\Box\chi)^2,\end{aligned} \tag{2.4}$$

here the d'Alembertian in flat Minkowski is $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and the first term of the second equality holds after integrations by parts. It is easy to see that this kinetic term for the field of longitudinal mode involves higher spacetime derivatives unless $a_1 = -a_2$. We now demonstrate the Ostrogradsky instability[1] by introducing Lagrange multiplier $\tilde{\chi}(x)$. Thus, the Lagrangian for the field χ is equivalent to

$$\mathcal{L}_{\text{kin}}^{\chi} = (a_1+a_2)\left(\tilde{\chi}\Box\chi-\frac{1}{4}\tilde{\chi}^2\right), \tag{2.5}$$

Here, we should notice about that the constraint equation for $\tilde{\chi}(x)$:

$$\tilde{\chi} = 2\Box\chi, \tag{2.6}$$

can be obtained by deriving the equation of motion with respect to $\tilde{\chi}(x)$. After changing the variables $\chi = \phi_1+\phi_2$ and $\tilde{\chi} = \phi_1-\phi_2$, the kinetic term of χ with two scalar fields ϕ_1 and ϕ_2 take the form

$$\mathcal{L}_{\text{kin}}^{\chi} = (a_1+a_2)\left(\phi_1\Box\phi_1-\phi_2\Box\phi_2-\frac{1}{4}\left(\phi_1-\phi_2\right)^2\right). \tag{2.7}$$

We directly see that two scalar fields ϕ_1 and ϕ_2 consistently associated with opposite kinetic terms, indicating that one of them is a ghost and we should choose $a_1 = -a_2$ to avoid this generic pathology at the classical level.

[1] Ref. [35] pointed out that there are 2 dofs hidden in the field χ with an opposing sign kinetic term, also known as Ostrogradsky's theorem.

We now proceed to discus the remaining helicity-1 mode $A^{\perp}_{\mu}$, the kinetic term takes the form

$$\mathcal{L}_{\text{kin}}^{\text{helicity-1}} = a_1 \left(\partial_\mu A^{\perp}_{\nu}\right)\left(\partial^\mu A^{\perp\nu}\right). \tag{2.8}$$

If we choose $a_1 = -1/2$, this local kinetic term of the spin-1 field is Maxwell field

$$\begin{aligned}\mathcal{L}_{\text{Maxwell}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\ &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\\ &= -\frac{1}{2}\partial_\mu A^\nu \partial^\mu A_\nu\\ &= \mathcal{L}_{\text{kin}}^{\text{helicity-1}},\end{aligned} \tag{2.9}$$

where we fixed the Lorenz gauge. Here, there exist $U(1)$ gauge symmetry in this massless free spin-1 Maxwell field,

$$A_\mu \to A_\mu + \partial_\mu \xi, \tag{2.10}$$

which can be used to fix the gauge of our choice.

We can perform a spacetime split so-called $(3+1)$-split for convenience. There are 3 dofs in A_i, and the Coulomb gauge $\partial_i A^i = 0$ can eliminate one of them. Thus, A_i contains no longitudinal mode,

$$A_i = A_i^{\text{transverse}} + \partial_i A^{\text{longitudinal}}, \tag{2.11}$$

where the the Coulomb gauge sets the longitudinal mode $A^{\text{longitudinal}} = 0$. The Maxwell action (2.9) under this splitting becomes,

$$\begin{aligned}\mathcal{L}_{\text{kin}}^{\text{spin-1}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}F_{\mu\nu}\partial^\mu A^\nu\\ &= -\frac{1}{2}(\partial_t A_i - \partial_i A_t)\partial^t A^i - \frac{1}{2}(\partial_i A_t - \partial_t A_i)\partial^i A^t - \frac{1}{2}(\partial_i A_j - \partial_j A_i)\partial^i A^j\\ &= -\frac{1}{2}(\partial_t A_i)^2 - \frac{1}{2}(\partial_i A_t)^2 - \frac{1}{2}(\partial_i A_j)^2\end{aligned} \tag{2.12}$$

where we have removed the terms $\partial_i A_t \partial^t A^i$, $\partial_t A_i \partial^i A^t$ and $\partial_j A_i \partial^i A^j$ by integrating by parts and by Coulomb gauge fixing. The time component A_t does not exhibit a kinetic term and appears instead as a Lagrange multiplier. Thus we can impose the constraint

$$\partial_i \partial^i A_t \equiv 0. \tag{2.13}$$

Then Maxwell action becomes,

$$\mathcal{L}_{\text{kin}}^{\text{spin-1}} = -\frac{1}{2}(\partial_t A_i)^2 - \frac{1}{2}(\partial_i A_j)^2 = -\frac{1}{2}\left(\partial_\mu A_i^{\text{transverse}}\right)^2, \tag{2.14}$$

which only propagating 2 dofs in $A_i^{\text{transverse}}$.

In conclusion, to prevent any ghosts producing along with the helicity-0 mode, the form of the Maxwell kinetic term of the vector field and the fact that the 4d massless vector field only propagates 2 dofs are not artificial choices.

2.1.2 Proca mass term

We now move on to discuss this theory in the massive vector field. We now add a covariant mass term to Maxwell action, i.e., Proca action

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu. \tag{2.15}$$

Notice that even with the addition of the Proca mass term, the kinetic term has been uniquely fixed in order to avoid the ghost. However, this mass term breaks the gauge symmetry (2.10), and then the Coulomb gauge can no longer be chosen. Similarly, we decompose the vector field A_μ as before

$$A_\mu = A_\mu^\perp + \partial_\mu \hat{\chi}, \tag{2.16}$$

but now we have $\chi = m\hat{\chi}$, so the Proca action becomes,

$$\begin{aligned}\mathcal{L}_{\text{Proca}} &= -\frac{1}{2}\left(\partial_\mu A_\nu^\perp\right)\left(\partial^\mu A^{\perp\nu}\right) - \frac{1}{2}m^2\left(A_\mu^\perp + \partial_\mu\hat{\chi}\right)^2 \\ &= -\frac{1}{2}\left(\partial_\mu A_\nu^\perp\right)^2 - \frac{1}{2}m^2\left(A_\mu^\perp\right)^2 - \frac{1}{2}\left(\partial_\mu \chi\right)^2,\end{aligned} \tag{2.17}$$

where we have dropped the term $m^2 A_\mu^\perp \partial_\mu \hat{\chi}$ by integrating by parts. We immediately see the massive vector field has 3 dofs. The transverse mode $A_\mu^\perp$ propagates 2 dofs and longitudinal mode χ propagates 1 dof. For the first time, we encounter discontinuity in the number of dofs between the case of massive and massless fields. Moreover, it is obvious that the Proca action (2.17) does not recover to Maxwell action (2.9) as $m \to 0$.

In order to distinguish the physical difference between a massless vector field and a massive vector field but with tiny mass, we can consider coupling the field to external sources

$$\mathcal{L}_{\text{sources}} = A_\mu J^\mu. \tag{2.18}$$

Here, the source should be conserved $\partial_\mu J^\mu = 0$ since the $U(1)$ symmetry is preserved in massless case. The transverse modes of the massless vector field $A_\mu^\perp$ produced by the source should satisfied $\Box A_\mu^\perp = J_\mu$. Therefore, the exchange amplitude $\mathcal{A}$ can be expressed as follows:

$$\mathcal{A}_{JJ'}^{\text{massless}} = \int \mathrm{d}^4x A_\mu^\perp J'^\mu = \int \mathrm{d}^4x J'^\mu \frac{1}{\Box} J_\mu, \tag{2.19}$$

where J_μ and J'_μ are two conserved sources mediated by a massless vector field. Moving on to the massive case, the source produces a massive vector field that should satisfy

$$\left(\Box - m^2\right) A_\mu^\perp = J_\mu, \tag{2.20}$$

and

$$\Box\chi = 0. \tag{2.21}$$

The transverse mode $A_\mu^\perp$ and the longitudinal mode χ of the massive vector field should be considered in the exchange amplitude

$$\begin{aligned} \mathcal{A}_{JJ'}^{\text{massive}} &= \int \mathrm{d}^4x \left(A_\mu^\perp + \partial_\mu\chi\right) J'^\mu \\ &= \int \mathrm{d}^4x A_\mu^\perp J'^\mu \\ &= \int \mathrm{d}^4x J'^\mu \frac{1}{\Box - m^2} J_\mu. \end{aligned} \tag{2.22}$$

Remember $\Box\chi = 0$, the longitudinal mode χ should not be excited by a conserved source. Consequently, even though the massive vector field propagates 3 dofs, the massless one only propagates 2 dofs, we cannot distinguish $\mathcal{A}_{JJ'}^{\text{massive}}$ and $\mathcal{A}_{JJ'}^{\text{massless}}$ between two conserved sources as $m \to 0$.

2.1.3 Stückelberg trick for spin-1 field

We will see more explicitly that there is a discontinuity in linear massless gravity and the linear massive gravity in massless limit in Section 2.3 later. In fact, the correct massless limit of linear massive gravity should be a massless gravity plus extra dofs which are a massless vector and a massless scalar coupling to the trace of the energy-momentum tensor. This extra scalar coupling leads to the well-known vDVZ discontinuity. As mentioned in Section 2.1.2, we encounter discontinuity in the number of dofs between the case of massive and massless fields. Stückelberg's trick can be used to find the correct limit so that there are no dofs gained or lost. This trick works by introducing extra new fields and gauge symmetries into the massive theory [36].

To introduce the idea, we consider the theory of a massive spin-1 field coupled to an external source, i.e., the action we just discussed

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu, \tag{2.23}$$

where the mass term breaks the gauge invariance, $\delta A_\mu = \partial_\mu \xi$. As it stands, the massless limit of the (2.23) is not smooth due to the discontinuity in the number of dofs. To reconcile this, we introduce a Stückelberg field ϕ, by making the replacement

$$A_\mu \to A_\mu + \partial_\mu \phi, \tag{2.24}$$

which is not a gauge transformation and not a decomposition of A_μ. So we created a new action from the original one, by the addition of a Stückelberg field ϕ. But $F_{\mu\nu}$ is still invariant under this replacement and this replacement only changes the mass term and coupling term for the action. Thus, the action (2.23) becomes,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 \left(A_\mu + \partial_\mu \phi\right)^2 + A_\mu J^\mu - \phi \partial_\mu J^\mu, \tag{2.25}$$

where we have integrated the last term by parts and this new action now is invariant under the gauge transformation

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu \xi, \\ \phi &\to \phi - \xi. \end{aligned} \tag{2.26}$$

If we set $\phi = 0$, i.e., fixing the unitary gauge, the action (2.25) can reduce to the original one (2.23). This implies that (2.25) and (2.23) are equivalent theories but the former uses more fields and gauge symmetry.

On the other hand, the Stückelberg trick is an excellent example of the fact that gauge symmetry is not a true symmetry of our theory, rather than a redundancy of description. Actually, any theory can become a gauge theory by introducing redundant variables. Or, the gauge symmetry can be eliminated by removing the redundant dofs in any gauge theory. From this case, we know that eliminating the redundancy may lead to the loss of something important. For instance, if we remove the gauge redundancy in electromagnetism, the Lorentz invariance is not manifest.

Coming back to the discussion that how the Stückelberg trick can preserve the number of dofs in massless limit. We first rescale the Stückelberg field $\phi \to \frac{1}{m}\phi$ to normalize the ϕ kinetic term, then the action (2.25) becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu - mA_\mu\partial^\mu\phi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + A_\mu J^\mu - \frac{1}{m}\phi\partial_\mu J^\mu, \tag{2.27}$$

with the gauge symmetry

$$\delta A_\mu = \partial_\mu\xi, \quad \delta\phi = -m\xi. \tag{2.28}$$

Notice that, we should consider the conserved source now, since the last term of (2.27) diverges as $m \to 0$ [37]. So (2.27) with conserved source in massless limit becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + A_\mu J^\mu, \tag{2.29}$$

with the new gauge symmetry

$$\delta A_\mu = \partial_\mu\xi, \quad \delta\phi = 0. \tag{2.30}$$

Now we can clearly recognize that of the 3 dofs of massive field, two into the massless vector and one into the scalar. In the massless limit, we are left with a completely decoupled free scalar and a massless vector interacting with the source.

2.2 Spin-2 field

For a spin-2 field, the kinetic term is also uniquely fixed by the requirement that free of any ghost-like instability. This is the well-known Einstein–Hilbert (EH) action.

2.2.1 Einstein–Hilbert kinetic term

Similar to the case in spin-1 field, we first consider a symmetric Lorentz rank-2 field but now is tensor $h_{\mu\nu}$ (and its trace h) in 4d Minkowski manifold. From Lorentz invariance and locality, the kinetic term can be generally given by

$$\mathcal{L}_{\rm kin}^{\rm spin-2} = \frac{1}{2}\partial^\alpha h^{\mu\nu}\left(b_1\partial_\alpha h_{\mu\nu} + 2b_2\partial_{(\mu}h_{\nu)\alpha} + b_3\partial_\alpha h\eta_{\mu\nu} + 2b_4\partial_{(\mu}h\eta_{\nu)\alpha}\right), \tag{2.31}$$

where b_1, b_2, b_3, and b_4 are undetermined constants and would be determined by preventing the occurrence of higher derivative terms similar to the spin-1 field. This symmetric tensor field $h_{\mu\nu}$ has 10 components, which can be split as a transverse tensor $h^T_{\mu\nu}$ and a vector field χ_μ,

$$h_{\mu\nu} = h^T_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)}, \tag{2.32}$$

where $h^T_{\mu\nu}$ is also symmetric tensor which carries 6 components and χ_μ carries 4 components. As we analyzed in the spin-1 field, the terms that contain higher derivatives for longitudinal modes χ_μ would imply a ghost. Similarly, we can avoid the ghosts by tuning the undetermined coefficient $b_{1,2,3,4}$. The potentially dangerous parts of the kinetic term are

$$\begin{aligned}\mathcal{L}^{\text{spin}-2}_{\text{kin}} \supset & \,(b_1 + b_2)\,\chi^\mu \Box^2 \chi_\mu + (b_1 + 3b_2 + 2b_3 + 4b_4)\,\chi^\mu \Box \partial_\mu \partial_\nu \chi^\nu \\ & - 2h^{T\mu\nu}\left((b_2 + b_4)\,\partial_\mu\partial_\nu\partial_\alpha\chi^\alpha + (b_1 + b_2)\,\partial_\mu \Box \chi_\mu \right. \\ & \left. + (b_3 + b_4)\,\Box\partial_\alpha\chi^\alpha \eta_{\mu\nu}\right).\end{aligned} \tag{2.33}$$

All of these terms should disappear and we thus obtain the relationship between these 4 undetermined coefficients

$$b_4 = -b_3 = -b_2 = b_1. \tag{2.34}$$

In order to follow the standard conventions, we set $b_1 = -1/4$ from now on. Then, the only possible local and Lorentz invariant kinetic term for a spin-2 field is the Einstein–Hilbert one

$$\mathcal{L}^{\text{spin}-2}_{\text{kin}} = -\frac{1}{4} h^{\mu\nu} \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = -\frac{1}{4} h^{T\mu\nu} \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h^T_{\alpha\beta} \tag{2.35}$$

where we defined the Lichnerowicz operator $\hat{\mathcal{E}}$ acting on $h_{\mu\nu}$

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = -\frac{1}{2}\left(\Box h_{\mu\nu} - 2\partial_{(\mu}\partial_\alpha h^\alpha_{\nu)} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\left(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta}\right)\right). \tag{2.36}$$

Moreover, there also exists gauge symmetry in this massless tensor field and is invariant under

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}. \tag{2.37}$$

Notice that, whether the tensor field has mass or not does not affect the form and gauge invariance of the kinetic term, as long as the kinetic term is restricted to a local and Lorentz invariant. However, the mass term and/or any other self-interacting potential in the vector field do break the gauge invariance.

2.2.2 Fierz–Pauli mass term

Moving to the spin-2 tensor field with mass. A priori, there are two possible contributions of the mass terms. The more general mass term can be written as

$$\mathcal{L}_{\text{mass}} = -\frac{1}{8}m^2\left(h_{\mu\nu}^2 - Ah^2\right), \tag{2.38}$$

where A is a dimensionless parameter and we will show that the theory is only stable for $A = 1$, which is also called the Fierz-Pauli tuning.

In order to restore the diffeomorphism invariance that was broken by adding the mass term, we now again introduce the Stückelberg fields but have four components χ_μ to make the mass term invariant under linear diffeomorphisms. This is a trick similar to the Abelian-Higgs mechanism of electromagnetism. Then the mass term becomes

$$\mathcal{L}_{\text{mass}} = -\frac{1}{8}m^2\left(\left(h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)}\right)^2 - A\left(h + 2\partial_\alpha\chi^\alpha\right)^2\right), \tag{2.39}$$

which is invariant under the following gauge transformations

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}, \tag{2.40}$$

$$\chi_\mu \to \chi_\mu - \frac{1}{2}\xi_\mu. \tag{2.41}$$

The kinetic term for Stückelberg fields is

$$\mathcal{L}^{\chi}_{\text{kin}} = -\frac{1}{2}m^2\left(\left(\partial_\mu\chi_\nu\right)^2 - A\left(\partial_\alpha\chi^\alpha\right)^2\right). \tag{2.42}$$

The terms in parentheses in Eq. 2.42 precisely have the same form as the longitudinal mode χ kinetic term for spin-1 field (2.4) with $a_1 = 1$ and $a_2 = A$. As analyzed in the spin-1 field, it is necessary to choose $a_1 = a_2$ in order to avoid higher derivatives. In other words, the only combination for the longitudinal component of these Stückelberg fields is $A = 1$. As a result, the unique mass term is the well-known FP mass term which is free from an Ostrogradsky instability [20], [35]

$$\mathcal{L}_{\text{FP mass}} = -\frac{1}{8}m^2\left(\left(h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)}\right)^2 - \left(h + 2\partial_\alpha\chi^\alpha\right)^2\right). \tag{2.43}$$

Besides, if Stückelberg fields χ_μ are set to zero, i.e., choosing the unitary gauge, the FP mass term becomes

$$\mathcal{L}_{\text{FP mass}} = -\frac{1}{8}m^2\left(h_{\mu\nu}^2 - h^2\right). \tag{2.44}$$

Consequently, we obtain the linearized FP action

$$\mathcal{L}_{\text{FP}} = \mathcal{L}_{\text{kin}}^{\text{spin}-2} + \mathcal{L}_{\text{FP mass}} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}_{\mu\nu}^{\alpha\beta}h_{\alpha\beta} - \frac{1}{8}m^2\left(h_{\mu\nu}^2 - h^2\right). \tag{2.45}$$

Propagating degrees of freedom and helicity decomposition

The Stückelberg fields χ^a can be split further into a transverse mode A^a and a longitudinal mode π,

$$\chi^a = \frac{1}{m}A^a + \frac{1}{m^2}\eta^{ab}\partial_b\pi, \tag{2.46}$$

where m is the normalization factor. After substitution of χ^a in terms of A^a and π. The linearized FP action becomes

$$\begin{aligned}\mathcal{L}_{\text{FP}} = &-\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}_{\mu\nu}^{\alpha\beta}h_{\alpha\beta} - \frac{1}{2}h^{\mu\nu}\left(\Pi_{\mu\nu} - [\Pi]\eta_{\mu\nu}\right) - \frac{1}{8}F_{\mu\nu}^2 \\ &-\frac{1}{8}m^2\left(h_{\mu\nu}^2 - h^2\right) - \frac{1}{2}m\left(h^{\mu\nu} - h\eta^{\mu\nu}\right)\partial_{(\mu}A_{\nu)}\end{aligned} \tag{2.47}$$

where we defined $\Pi_{\mu\nu} = \partial_\mu\partial_\nu\pi$ for convenience and $[\Pi]$ is its trace with respect to the background Minkowski metric $\eta^{\mu\nu}$. We can see that the terms on the first line represent the kinetic terms of the A^a and π fields, respectively. The second line represents the mass terms and mixing term. However, the field π is mixing with the field $h_{\mu\nu}$. But we can diagonalize this mixing by shifting $h_{\mu\nu} = \tilde{h}_{\mu\nu} + \pi\eta_{\mu\nu}$ and the $\mathcal{L}_{\text{FP}}$ can arrive at

$$\begin{aligned}\mathcal{L}_{\text{FP}} = &-\frac{1}{4}\tilde{h}^{\mu\nu}\hat{\mathcal{E}}_{\mu\nu}^{\alpha\beta}\tilde{h}_{\alpha\beta} - \frac{3}{4}(\partial\pi)^2 - \frac{1}{8}F_{\mu\nu}^2 \\ &-\frac{1}{8}m^2\left(\tilde{h}_{\mu\nu}^2 - \tilde{h}^2\right) + \frac{3}{2}m^2\pi^2 + \frac{3}{2}m^2\pi\tilde{h} \\ &-\frac{1}{2}m\left(\tilde{h}^{\mu\nu} - \tilde{h}\eta^{\mu\nu}\right)\partial_{(\mu}A_{\nu)} + 3m\pi\partial_\alpha A^\alpha.\end{aligned} \tag{2.48}$$

We now can identify the different dofs and helicity decomposition present in FP action: $h_{\mu\nu}$ represents the helicity-2 mode and has 2 dofs, A_μ represents the helicity-1 mode and has 2 dofs, and π represents the helicity-0 mode and has 1 dof. To sum up, the massive spin-2 field propagates 5 dofs in 4d spacetime.

Moreover, for massive spin-2 field, the coupling to matter now occurs by

$$h_{\mu\nu}T^{\mu\nu} = \tilde{h}_{\mu\nu}T^{\mu\nu} + \pi T, \tag{2.49}$$

where $T^{\mu\nu}$ is external stress-energy tensor and T is its trace. Unlike in the case of massive spin-1 field, now the helicity-0 mode couples to conserved sources but the helicity-1 mode does not. So the generic sources will excite the two helicity-2 polarization of a third helicity-0 polarization, which could potentially lead to significant outcomes (the origin of the vDVZ discontinuity).

2.3 van Dam–Veltman–Zakharov discontinuity

Similar to the spin-1 field case, the massive spin-2 field also propagates more dofs than the massless one. In Section 2.1.2, we show that there are no observational signatures for the spin-1 field due to the exchange amplitude $\mathcal{A}_{JJ'}^{\text{massive}}$ is the same as $\mathcal{A}_{JJ'}^{\text{massless}}$ with $m \to 0$. This is because no external source excites the helicity-0 mode in massive spin-1 field. However, as discussed in Section 2.2.2, the external sources will excite both helicity-2 polarization and helicity-0 polarization. In order to see the dramatic consequences more explicitly, we will show the discrepancy between the massless limit of massive spin-2 field and massless spin-2 field by computing the gravitational exchange amplitude, which is also known as vDVZ discontinuity [23], [24].

2.3.1 Massless spin-2 field

Let us start with the massless spin-2 field, the theory in this case is diffeomorphism invariant. So in order to ensure that the symmetry is preserved when considering coupling to external sources (with the form $h_{\mu\nu}T^{\mu\nu}$). This requires the stress-energy tensor $T^{\mu\nu}$ should be conserved

$$\partial_\mu T^{\mu\nu} = 0. \tag{2.50}$$

Thus, the massless spin-2 field response to a conserved external source

$$\mathcal{L} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + \frac{1}{2M_{\text{Pl}}}h_{\mu\nu}T^{\mu\nu} \tag{2.51}$$

Then the linearized Einstein equation can be obtained by solving its equation of motion:

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = \frac{1}{M_{\rm Pl}} T_{\mu\nu}. \tag{2.52}$$

Since the tensor field $h_{\mu\nu}$ is invariant under the gauge transformation (2.40), we can impose a gauge fixing condition in linearized gravity. In this spin-2 field, the analog of the Lorenz gauge is called the de Donder (or harmonic) gauge

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0. \tag{2.53}$$

Under the de Donder gauge, the linearized Einstein equation (2.52) then reduces to

$$\Box h_{\mu\nu} - \frac{1}{2}\Box h \eta_{\mu\nu} = \frac{-2}{M_{\rm Pl}} T_{\mu\nu}, \tag{2.54}$$

and taking the trace of this equation, we have

$$\Box h = \frac{-2}{M_{\rm Pl}} T. \tag{2.55}$$

Substituting it back into the linearized Einstein equation (2.52),

$$\Box h_{\mu\nu} = -\frac{2}{M_{\rm Pl}} \left(T_{\mu\nu} - \frac{1}{2} T \eta_{\mu\nu} \right). \tag{2.56}$$

Thus, we can define the propagator for a massless spin-2 field

$$G^{\rm massless}_{\mu\nu\alpha\beta} = \frac{f^{\rm massless}_{\mu\nu\alpha\beta}}{\Box}, \tag{2.57}$$

where $f^{\rm massless}_{\mu\nu\alpha\beta}$ is the polarization tensor,

$$f^{\rm massless}_{\mu\nu\alpha\beta} = \eta_{\mu(\alpha}\eta_{\nu\beta)} - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}. \tag{2.58}$$

As a result, the gravitational exchange amplitude between two sources $T_{\mu\nu}$ and $T'_{\mu\nu}$ via a massless spin-2 field is given by

$$\mathcal{A}^{\rm massless}_{TT'} = \int {\rm d}^4 x h_{\mu\nu} T'^{\mu\nu} = -\frac{2}{M_{\rm Pl}} \int {\rm d}^4 x T'^{\mu\nu} \frac{1}{\Box} \left(T_{\mu\nu} - \frac{1}{2} T \eta_{\mu\nu} \right). \tag{2.59}$$

2.3.2 Massive spin-2 field

Moving on to the massive case, and consider the linearized FP action response to a conserved external source

$$\mathcal{L} = -\frac{1}{4} h^{\mu\nu} \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} - \frac{m^2}{8}\left(h^2_{\mu\nu} - h^2\right) + \frac{1}{2M_{\mathrm{Pl}}} h_{\mu\nu} T^{\mu\nu}, \tag{2.60}$$

then we can obtain the modified linearized Einstein equation

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} + \frac{1}{2} m^2 \left(h_{\mu\nu} - h\eta_{\mu\nu}\right) = \frac{1}{M_{\mathrm{Pl}}} T_{\mu\nu}. \tag{2.61}$$

Here, we need to solve this modified linearized Einstein equation for $h_{\mu\nu}$ and the calculations are similar to the massless case but more complicated. We consider the trace and the divergence of the Lichnerowicz operator acting on $h_{\mu\nu}$, we obtain

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} \eta^{\mu\nu} = \Box h - \partial_\alpha \partial_\beta h^{\alpha\beta}, \tag{2.62}$$

$$\partial^\mu \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = 0. \tag{2.63}$$

Then, taking double derivatives of the modified linearized Einstein equation (2.61) and combining with the Eq. 2.62 and Eq. 2.63 one can find that

$$\begin{aligned} &\partial^\mu \partial^\nu \left(\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} + \frac{1}{2} m^2 \left(h_{\mu\nu} - h\eta_{\mu\nu}\right)\right) = \frac{1}{M_{\mathrm{Pl}}} \partial^\mu \partial^\nu T_{\mu\nu} \\ \Longrightarrow\ &\Box h - \partial^\mu \partial^\nu h_{\mu\nu} = \frac{-2}{m^2 M_{\mathrm{Pl}}} \partial^\mu \partial^\nu T_{\mu\nu} = \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} \eta^{\mu\nu}. \end{aligned} \tag{2.64}$$

Then, we consider the trace of the modified linearized Einstein equation (2.61)

$$\begin{aligned} &\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} \eta^{\mu\nu} - \frac{3}{2} m^2 h = \frac{1}{M_{\mathrm{Pl}}} T \\ \Longrightarrow\ &\frac{-2}{M_{\mathrm{Pl}}} \partial^\mu \partial^\nu T_{\mu\nu} - \frac{3}{2} m^2 h = \frac{1}{M_{\mathrm{Pl}}} T \\ \Longrightarrow\ &h = \frac{-2}{3m^2 M_{\mathrm{Pl}}} \left(T + \frac{2}{m^2} \partial^\mu \partial^\nu T_{\mu\nu} \right). \end{aligned} \tag{2.65}$$

Taking the derivative of the Eq. 2.61 and combining the above results,

$$\begin{aligned} &\partial^\mu \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} + \partial^\mu \left(\frac{1}{2} m^2 \left(h_{\mu\nu} - h\eta_{\mu\nu}\right)\right) = \frac{1}{M_{\mathrm{Pl}}} \partial^\mu T_{\mu\nu} \\ \Longrightarrow\ &\frac{1}{2} m^2 \partial^\mu h_{\mu\nu} - \partial_\nu \left[\frac{-2}{3m^2 M_{\mathrm{Pl}}} \left(T + \frac{2}{m^2} \partial^\mu \partial^\nu T_{\mu\nu} \right)\right] = \frac{1}{M_{\mathrm{Pl}}} \partial^\mu T_{\mu\nu} \\ \Longrightarrow\ &\partial_\mu h^\mu{}_\nu = \frac{2}{m^2 M_{\mathrm{Pl}}} \left(\partial_\mu T^\mu_\nu + \frac{1}{3} \partial_\nu T + \frac{2}{3m^2} \partial_\nu \partial_\alpha \partial_\beta T^{\alpha\beta} \right). \end{aligned} \tag{2.66}$$

From Eq. 2.64 we can see that for a conserved source $\partial^\mu T_{\mu\nu}$, the linearized Ricci scalar vanishes $\partial_\mu\partial_\nu h^{\mu\nu} - \Box h = 0$, which can be considered to be the origin of vDVZ discontinuity [23], [24].

Consequently, when combined with the results provided by Eq. 2.65 and Eq. 2.66, the modified linearized Einstein equation becomes

$$\begin{aligned}\left(\Box - m^2\right) h_{\mu\nu} =& -\frac{2}{M_{\mathrm{Pl}}}\left[T_{\mu\nu} - \frac{1}{3}T\eta_{\mu\nu} - \frac{2}{m^2}\partial_{(\mu}\partial_\alpha T^\alpha_{\nu)} + \frac{1}{3m^2}\partial_\mu\partial_\nu T\right.\\ &\left. + \frac{1}{3m^2}\partial_\alpha\partial_\beta T^{\alpha\beta}\eta_{\mu\nu} + \frac{2}{3m^4}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta T^{\alpha\beta}\right]\\ =& -\frac{2}{M_{\mathrm{Pl}}}\left[\tilde{\eta}_{\mu(\alpha}\tilde{\eta}_{\nu\beta)} - \frac{1}{3}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta}\right]T^{\alpha\beta},\end{aligned} \tag{2.67}$$

where we have defined

$$\tilde{\eta}_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu. \tag{2.68}$$

Thus, we can define the propagator for a massive spin-2 field

$$G^{\mathrm{massive}}_{\mu\nu\alpha\beta} = \frac{f^{\mathrm{massive}}_{\mu\nu\alpha\beta}}{\Box - m^2}, \tag{2.69}$$

where $f^{\mathrm{massive}}_{\mu\nu\alpha\beta}$ is the polarization tensor,

$$f^{\mathrm{massive}}_{\mu\nu\alpha\beta} = \tilde{\eta}_{\mu(\alpha}\tilde{\eta}_{\nu\beta)} - \frac{1}{3}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta}. \tag{2.70}$$

As a result, the gravitational exchange amplitude between two sources $T_{\mu\nu}$ and $T'_{\mu\nu}$ via a massive spin-2 field is given by

$$\mathcal{A}^{\mathrm{massive}}_{TT'} = \int \mathrm{d}^4x h_{\mu\nu}T'^{\mu\nu} = -\frac{2}{M_{\mathrm{Pl}}}\int \mathrm{d}^4x T'^{\mu\nu}\frac{f^{\mathrm{massive}}_{\mu\nu\alpha\beta}}{\Box - m^2}T^{\alpha\beta}. \tag{2.71}$$

Notice that, there is no a priori reason to restrict ourselves to conserved sources in massive case. But in order to compare this result with the massless case, the two sources should be conserved as $m \to 0$. Therefore, the amplitude exchanged via a massive spin-2 field but in the massless limit is

$$\mathcal{A}^{m\to 0}_{TT'} = -\frac{2}{M_{\mathrm{Pl}}}\int \mathrm{d}^4x T'^{\mu\nu}\frac{1}{\Box}\left(T_{\mu\nu} - \frac{1}{3}T\eta_{\mu\nu}\right), \tag{2.72}$$

which is not consistent with the result (2.59) of the massless field.

The difference between the exchange amplitudes of massless graviton and massive graviton in massless limit is the well-known vDVZ discontinuity. Arkady Vainshtein gave the resolution to this discontinuity problem in 1972 [25]. He argued that there is no reliable non-linear behavior of massive gravity in the limit of small mass since massive gravity theory becomes strongly coupled with a low energy scale. We shall do a simple calculation to show how this happens in the next section (see Ref [32] for more details).

2.4 Vainshtein radius

We start looking at static spherical solutions for the FP massive gravity and follow the steps of the perturbation method in GR in the Appendix C.2.2. The spherically symmetric static metric (off-diagonal metrics would be more general but we limit ourselves to the diagonal ansatz) can be written as

$$\mathrm{d}s^2 = g_{\mu\nu}dx^\mu dx^\nu = -B(r)dt^2 + A(r)dr^2 + C(r)r^2 d\Omega^2. \tag{2.73}$$

The vacuum solution satisfies the equations $T_{\mu\nu} = 0$, combing this ansatz and the equations of motion from the full non-linear GR action, i.e.

$$\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right) + \sqrt{-g^{(0)}}\frac{m^2}{2}\left(g^{(0)\mu\alpha}g^{(0)\nu\beta}h_{\alpha\beta} - g^{(0)\alpha\beta}h_{\alpha\beta}g^{(0)\mu\nu}\right) = 0, \tag{2.74}$$

with $h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu}$, where $g^{(0)}_{\mu\nu}$ is the absolute metric which is flat Minkowski in spherical coordinates,

$$g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{2.75}$$

we can obtain tt, rr and $\theta\theta$ ($\phi\phi$ equation gives the same result) components equations respectively,

$$\begin{aligned}
&4BC^2m^2r^2A^3 + \left(2B(C-3)C^2m^2r^2 - 4\sqrt{A^2BC}\left(C - rC'\right)\right)A^2 \\
&+ 2\sqrt{A^2BC}\left(2C^2 - 2r\left(3A' + rA''\right)C + r^2A'C'\right)A + C\sqrt{A^2BC}r^2\left(A'\right)^2 = 0, \\
&\frac{4\left(B + rB'\right)A^2 + \left(2r^2A'B' - 4B\left(C - rA'\right)\right)A + Br^2\left(A'\right)^2}{A^2BC^2r^2} - \frac{2(2A + B - 3)m^2}{\sqrt{A^2BC}} = 0, \\
&-2B^2C^2m^2rA^4 - 2B^2C^2(B + C - 3)m^2rA^3 \\
&-\sqrt{A^2BC}\left(2C'B^2 + \left(rB'C' - 2C\left(B' + rB''\right)\right)B + Cr\left(B'\right)^2\right)A^2 \\
&+ B\sqrt{A^2BC}\left(CrA'B' + B\left(4CA' - rC'A' + 2CrA''\right)\right)A - B^2C\sqrt{A^2BC}r\left(A'\right)^2 = 0.
\end{aligned} \tag{2.76}$$

We demand the solution to be asymptotically flat (or rather expand the above equations around the flat space solution), which also leads to

$$B_0(r) = 1, \quad C_0(r) = 1, \quad A_0(r) = 1. \tag{2.77}$$

2

The expansion can be defined as

$$\begin{aligned} B(r) &= B_0(r) + \epsilon B_1(r) + \epsilon^2 B_2(r) + \cdots , \\ C(r) &= C_0(r) + \epsilon C_1(r) + \epsilon^2 C_2(r) + \cdots , \\ A(r) &= A_0(r) + \epsilon A_1(r) + \epsilon^2 A_2(r) + \cdots . \end{aligned} \tag{2.78}$$

where ϵ is a parameter that counts the order of non-linearity. Plugging the expansion expression of $A(r)$, $B(r)$ and $C(r)$, we can obtain the differential equations for the expansion of each power of ϵ. At $\mathcal{O}(0)$ order, we have $0 = 0$, i.e., $A_0 = B_0 = C_0 = 1$ are solutions to the full non-linear equations. At $\mathcal{O}(\epsilon)$ order we have

$$\begin{aligned} &2\left(m^2r^2 - 1\right) A_1 + \left(m^2r^2 + 2\right) C_1 + 2r\left(-3A_1' + C_1' - rA_1''\right) = 0, \\ &-\frac{1}{2}B_1 m^2 + \left(\frac{1}{r^2} - m^2\right) A_1 + \frac{r\left(A_1' + B_1'\right) - C_1}{r^2} = 0, \\ &rA_1m^2 + rB_1m^2 + rC_1m^2 - 2A_1' - B_1' + C_1' - rA_1'' - rB_1'' = 0, \end{aligned} \tag{2.79}$$

which could lead to

$$-3rB_1m^2 + 6B_1' + 3rB_1'' = 0. \tag{2.80}$$

Combing these equations with the boundary condition, we obtain the solution

$$\begin{aligned} B_1(r) &= -\frac{8GM}{3}\frac{e^{-mr}}{r}, \\ C_1(r) &= -\frac{8GM}{3}\frac{e^{-mr}}{r}\frac{1+mr}{m^2r^2}, \\ A_1(r) &= \frac{4GM}{3}\frac{e^{-mr}}{r}\frac{1+mr+m^2r^2}{m^2r^2}, \end{aligned} \tag{2.81}$$

where we have selected the integration constant that corresponds to other physical results. We can continue in this way to any order of ϵ, and obtain the expansion in $mr \ll 1$ limit,

$$\begin{aligned} B(r) &= 1 - \frac{8}{3}\frac{GM}{r}\left(1 - \frac{1}{6}\frac{GM}{m^4r^5} + \cdots\right), \\ C(r) &= 1 - \frac{8}{3}\frac{GM}{m^2r^3}\left(1 - 14\frac{GM}{m^4r^5} + \cdots\right), \\ A(r) &= 1 + \frac{4}{3}\frac{GM}{4\pi m^2r^3}\left(1 - 4\frac{GM}{m^4r^5} + \cdots\right). \end{aligned} \tag{2.82}$$

The dots represent higher order in the non-linearity expansion in the parameter ϵ. Moreover, the non-linear expansion is an expansion of parameters r_V/r, where

$$r_V \equiv \left(\frac{GM}{m^4}\right)^{1/5} \tag{2.83}$$

is the Vainshtein radius. We can see that r_V grows to infinity as $m \to 0$. Analogy to the GR, the Schwarzschild radius can be considered as the cut-off scale for the linear theory of linearised GR, the solution of linear perturbation theory cannot be trusted within Vainshtein radius. Thus we have reason to believe that the vDVZ discontinuity is the true non-linear solution shows a smooth limit [38], [39].

2.5 Non-linear Stückelberg decomposition

There are only two ways to non-linearly complete linear diffeomorphism in spin-2 field, one is linear diffeomorphism in the full theory and the other is full non-linear diffeomorphism. It is possible to write self-interactions which preserve linear diffeomorphism, but there are no interactions between matter and spin-2 field which preserve linear diffeomorphism. So any theory of gravity must exhibit full non-linear diffeomorphism which leads to GR.

2.5.1 Reference metric

Now we would like to extend the theory non-linearly, so we may need for extending the theory about different reference (or rather fiducial) metric $f_{\mu\nu}$. Interestingly, Ref. [26] also discussed whether it was possible to construct a massive gravity theory without using a reference metric at all. It was shown that the only consistent alternative is to consider a function of the metric determinant which is equivalent to the cosmological constant. Strictly speaking, the notion of spin is only meaningful when representing a Lorentz group, thus the theory of massive spin-2 field is only meaningful when Lorentz invariance is preserved, i.e., $f_{\mu\nu} = \eta_{\mu\nu}$.

2.5.2 Non-linear Stückelberg field

At the linearized level, the mass for gravity was not built by the full metric $g_{\mu\nu}$, but by the fluctuation of the reference metric $h_{\mu\nu}$. Notice that this reference metric does not transform as a tensor under general coordinate transformations. This result is already known at the linear level because the FP mass term (2.38) breaks linearized diffeomorphism invariance. Nevertheless, we have discussed previously that the gauge symmetry can always be restored by Stückelberg fields, which amounts to replacing the reference metric to

$$\eta_{\mu\nu} \longrightarrow \eta_{\mu\nu} - \frac{2}{M_{\rm Pl}}\partial_{(\mu}\chi_{\nu)}, \tag{2.84}$$

(where we worked with the flat Minkowski metric as the reference metric,) and transforming χ_μ under linearized diffeomorphism. Thus the combination $h_{\mu\nu} - 2\partial_{(\mu}\chi_{\nu)}$ remains invariant.

Now the Stückelberg trick should be promoted to a fully covariant realization and non-linearly since the symmetry is replaced by general covariance and non-linearly realized. We can 'formally' restore covariance by including four Stückelberg fields ϕ^a, and promoting the reference metric $f_{\mu\nu}$ to tensor $\tilde{f}_{\mu\nu}$ [40],

$$f_{\mu\nu} \longrightarrow \tilde{f}_{\mu\nu} = \partial_\mu\phi^a\partial_\nu\phi^b f_{ab}, \tag{2.85}$$

where the Stückelberg fields ϕ^a transform as scalars, thus $\tilde{f}_{\mu\nu}$ transforms as a tensor under coordinate transformations. Besides, the Stückelberg fields can reduce to $\phi^a = x^a$ and the reference metric can recover $\tilde{f}_{\mu\nu} = f_{\mu\nu}$ in unitary gauge. As a result, a theory of massive gravity can be constructed by a scalar Lagrangian of the tensors $\tilde{f}_{\mu\nu}$ and $g_{\mu\nu}$.

In general, it is useful to construct the following tensor quantity in massive gravity,

$$\mathbb{X}^\mu{}_\nu = g^{\mu\alpha}\tilde{f}_{\alpha\nu} = \partial^\mu\phi^a\partial_\nu\phi^b f_{ab}. \tag{2.86}$$

In unitary gauge, we have

$$\mathbb{X} = g^{-1}f = g^{\mu\alpha}f_{\alpha\nu}. \tag{2.87}$$

Ref. [41] also provided an alternative way to Stückelberize the reference metric $f_{\mu\nu}$. The new tensor quantity is

$$g^{ac}f_{cb} \to \mathbb{Y}^a{}_b = g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^c f_{cb}. \tag{2.88}$$

But both matrices $\mathbb{X}^\mu{}_\nu$ and $\mathbb{Y}^a{}_b$ have the same eigenvalues and there is no difference between them when they are used to define the massive gravity Lagrangian [42].

2.5.3 Non-linear Fierz–Pauli mass term

Ref. [40] provides a straightforward example of a non-linear extension of the FP mass term,

$$\mathcal{L}_{\rm FP}^{(\rm non-lin1)} = -m^2 M_{\rm Pl}^2\sqrt{-g}\left(\left[(\mathbb{I}-\mathbb{X})^2\right] - [\mathbb{I}-\mathbb{X}]^2\right), \tag{2.89}$$

which is invariant under non-linear coordinate transformations. Ref. [26] also gives another way to generalize the FP mass non-linearly as follows

$$\mathcal{L}_{\rm FP}^{(\rm non-lin2)} = -m^2 M_{\rm Pl}^2 \sqrt{-g}\sqrt{\det \mathbb{X}}\left(\left[\left(\mathbb{I}-\mathbb{X}^{-1}\right)^2\right] - \left[\mathbb{I}-\mathbb{X}^{-1}\right]^2\right). \tag{2.90}$$

Prior to this, the linear FP action for massive gravity can be extended non-linearly in various ways. However, most of these generalizations unavoidably introduce the BD ghost non-linearly. In fact, a theory of non-linear extension of the FP mass term which free of the BD ghost is unique (with up to two constant parameters) [31].

2.6 Boulware-Deser ghost

BD ghost is generally considered to be the sixth dof that appears at the non-linear level, which leads to instability in non-linear massive gravity [43]. In order to observe the emergence of the BD ghost at the non-linear level, we choose the easiest way: following the Stückelberg trick non-linearly [43], [44].

We will do the further helicity decomposition of the Stückelberg field first. Focusing on the flat (Minkowski) reference metric, $f_{\mu\nu} = \eta_{\mu\nu}$, the Stückelberg field can be further split in $\phi^a = x^a - \frac{1}{M_{\rm Pl}}\chi^a$ (where a is a Lorentz index). Combing with Eq. 2.46, we obtain the non-linear generalization of the Stückelberg trick

$$\begin{aligned}\tilde{f}_{\mu\nu} =& \ \partial_\mu\phi^a\partial_\nu\phi^b\eta_{ab} \\ =& \ \eta_{\mu\nu} - \frac{2}{M_{\rm Pl}}\partial_{(\mu}\chi_{\nu)} + \frac{1}{M_{\rm Pl}^2}\partial_\mu\chi^a\partial_\nu\chi^b\eta_{ab} \\ =& \ \eta_{\mu\nu} - \frac{2}{M_{\rm Pl}m}\partial_{(\mu}A_{\nu)} - \frac{2}{M_{\rm Pl}m^2}\Pi_{\mu\nu} \\ &+ \frac{1}{M_{\rm Pl}^2m^2}\partial_\mu A^\alpha\partial_\nu A_\alpha + \frac{2}{M_{\rm Pl}^2m^3}\partial_\mu A^\alpha\Pi_{\nu\alpha} + \frac{1}{M_{\rm Pl}^2m^4}\Pi^2_{\mu\nu}.\end{aligned} \tag{2.91}$$

We now only keep the helicity-0 mode π, the tensor $\mathbb{X}^\mu{}_\nu$ defined in (2.86) becomes

$$\mathbb{X}^\mu{}_\nu = \delta^\mu{}_\nu - \frac{2}{M_{\rm Pl}m^2}\Pi^\mu{}_\nu + \frac{1}{M_{\rm Pl}^2m^4}\Pi^\mu{}_\alpha\Pi^\alpha{}_\nu, \tag{2.92}$$

and plugging this into the non-linear extension of the FP (2.89), then the mass term reads,

$$\mathcal{L}_{\rm FP,\pi}^{(\rm non-lin1)} = -\frac{4}{m^2}\left([\Pi^2]-[\Pi]^2\right) + \frac{4}{M_{\rm Pl}m^4}\left([\Pi^3]-[\Pi]\left[\Pi^2\right]\right) + \frac{1}{M_{\rm Pl}^2m^6}\left([\Pi^4]-[\Pi^2]^2\right). \tag{2.93}$$

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According to Ostrogradsky's theorem, the higher-order operators $\left([\Pi^3]-[\Pi]\left[\Pi^2\right]\right)$ and $\left([\Pi^4]-[\Pi^2]^2\right)$ propagate an additional dof, which always enters as a ghost. These operators might be irrelevant at the linear level, but the ghost always can be manifest by finding an appropriate background configuration $\pi=\pi_0+\delta\pi$. Thus,

$$\mathcal{L}^{(\mathrm{non-lin1})}_{\mathrm{FP},\pi} \supset \frac{4}{M_{\mathrm{Pl}}m^4} Z^{\mu\nu\alpha\beta}\partial_\mu\partial_\nu\delta\pi\partial_\alpha\partial_\beta\delta\pi, \tag{2.94}$$

where $Z^{\mu\nu\alpha\beta}=3\partial^\mu\partial^\alpha\pi_0\eta^{\nu\beta}-\Box\pi_0\eta^{\mu\alpha}\eta^{\nu\beta}-2\partial^\mu\partial^\nu\pi_0\eta^{\alpha\beta}+\cdots$. This implies that around a non-trivial background, the FP mass term propagates an additional dof which is a ghost, as known as the BD ghost. Besides, The mass of the BD ghost depends on the background configuration π_0,

$$m^2_{\mathrm{ghost}} \sim \frac{M_{\mathrm{Pl}}m^4}{\partial^2\pi_0}. \tag{2.95}$$

According to the Vainshtein mechanism, the field takes a large vacuum expectation value $\partial^2\pi_0 \gg M_{\mathrm{Pl}}m^2$, thus leads to the ghost with a tiny mass, $m^2_{\mathrm{ghost}} \ll m^2$.

2.6.1 Function of the Fierz–Pauli mass term

The fluctuations about flat spacetime

$$h_{\mu\nu}=M_{\mathrm{Pl}}\left(g_{\mu\nu}-\eta_{\mu\nu}\right), \tag{2.96}$$

now can be promoted to the tensor $H_{\mu\nu}$

$$H_{\mu\nu}=M_{\mathrm{Pl}}\left(g_{\mu\nu}-\tilde{f}_{\mu\nu}\right). \tag{2.97}$$

Combing with Eq. 2.91, we obtain

$$\begin{aligned} H_{\mu\nu} = &\quad h_{\mu\nu}+2\partial_{(\mu}\chi_{\nu)}-\frac{1}{M_{\mathrm{Pl}}}\eta_{ab}\partial_\mu\chi^a\partial_\nu\chi^b \\ = &\quad h_{\mu\nu}+\frac{2}{m}\partial_{(\mu}A_{\nu)}+\frac{2}{m^2}\Pi_{\mu\nu} \\ &-\frac{1}{M_{\mathrm{Pl}}m^2}\partial_\mu A^\alpha\partial_\nu A_\alpha-\frac{2}{M_{\mathrm{Pl}}m^3}\partial_\mu A^\alpha\Pi_{\nu\alpha}-\frac{1}{M_{\mathrm{Pl}}m^4}\Pi^2_{\mu\nu}. \end{aligned} \tag{2.98}$$

Thus, we can write a more general function of FP mass term as an extension [26]

$$\mathcal{L}^{\mathrm{function}}_{\mathrm{FP}} = -m^2\sqrt{-g}F\left(g^{\mu\nu}g^{\alpha\beta}\left(H_{\mu\alpha}H_{\nu\beta}-H_{\mu\nu}H_{\alpha\beta}\right)\right). \tag{2.99}$$

However, if $F' \neq 0$, there is no analytic function of F can avoid the non-linear propagation of the BD ghost. If $F'(0) = 0$, we can prevent the cubic higher-derivative interactions in π, but remove the mass term at the same time. Moreover, if we choose $F(0) \neq 0$ and $F'(0) = 0$, the theory is massless with respect to a specific reference metric and infinitely strong coupling with other backgrounds.

If we would like to construct a ghost-free massive gravity theory, all higher derivative operators involving the helicity-0 mode $\left(\partial^2\pi\right)^n$ should be in the form of total derivatives in mass (potential) term.

2.7 Ghost-free massive gravity

A theory of massive gravity with coefficients tuned to avoid the BD ghost by incorporating all higher derivative operators as total derivatives was put forward by Cluadia de Rhan, Gregory Gabadadze and Andrew Tolley (dRGT) in 2010 [21], [22]. It was subsequently proved that the BD ghost for all orders and beyond the decoupling limit was completely absent [45], [46].

The action for the theory of ghost-free dRGT massive gravity is given by

$$S_{\mathrm{dRGT}} = \int d^4x\sqrt{-g}\frac{M_{\mathrm{Pl}}^2}{2}\left[R - 2m^2\sum_{n=0}^{4}\beta_n e_n\left(\mathbb{X}\right)\right] + S_{\mathrm{m}}[g,\psi], \tag{2.100}$$

where β_n are constant parameters, S_{m} is the matter action. The dRGT mass potential terms are built out of the n-th elementary symmetric polynomials e_n, which is constructed carefully to avoid the BD ghost and unnecessary to compute the eigenvalues of $\mathbb{X}$ to obtain,

$$\begin{aligned}
e_0(\mathbb{X}) &= 1,\\
e_1(\mathbb{X}) &= [\mathbb{X}],\\
e_2(\mathbb{X}) &= \frac{1}{2!}\left([\mathbb{X}]^2 - \left[\mathbb{X}^2\right]\right),\\
e_3(\mathbb{X}) &= \frac{1}{3!}\left([\mathbb{X}]^3 - 3[\mathbb{X}]\left[\mathbb{X}^2\right] + 2\left[\mathbb{X}^3\right]\right)\Big),\\
e_4(\mathbb{X}) &= \frac{1}{4!}\Big([\mathbb{X}]^4 - 6[\mathbb{X}]^2\left[\mathbb{X}^2\right] + 3\left[\mathbb{X}^2\right]^2 + 8[\mathbb{X}]\left[\mathbb{X}^3\right] - 6\left[\mathbb{X}^4\right]\Big).
\end{aligned} \tag{2.101}$$

The specific anti-symmetric combination of terms in each e_n is designed to prevents the BD ghost from becoming dynamical. Besides, according to the definition of $\mathbb{X}$ (2.86), we have introduced a reference metric $f_{\mu\nu}$ to construct the interaction term since we cannot construct a nontrivial interaction only consider $g_{\mu\nu}$ [47]. The reference metric transforms as a metric tensor under

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diffeomorphism, thus we have

$$(\mathbb{X}^2)^{\mu}{}_{\nu} = \mathbb{X}^{\mu}{}_{\alpha}\mathbb{X}^{\alpha}{}_{\nu}. \tag{2.102}$$

The matrix $\mathbb{X}$ is a tensor function of the tensor $H_{\mu\nu}$ which represents the covariantized metric perturbation. As discussed in Section 2.6.1, $H_{\mu\nu}$ was promoted from $h_{\mu\nu}$, and can be reduced to $h_{\mu\nu}$ in unitary gauge. Thus, relating $\mathbb{X}$ to $H_{\mu\nu}$, we have

$$\mathbb{X}_{\mu\nu} = \eta_{\mu\nu} + 2\mathcal{K}_{\mu\nu} - \eta^{\alpha\beta}\mathcal{K}_{\mu\alpha}\mathcal{K}_{\beta\nu}, \tag{2.103}$$

where we defined the extrinsic curvature

$$\mathcal{K}_{\mu\nu} = \eta_{\mu\nu} - \left(\sqrt{\partial_\mu\phi^a\partial_\nu\phi^b\eta_{ab}}\right) = \eta_{\mu\nu} - \left(\sqrt{\mathbb{X}}\right)_{\mu\nu} = \eta_{\mu\nu} - \sqrt{\eta_{\mu\nu} - H_{\mu\nu}}. \tag{2.104}$$

Same as the Fierz-Pauli case we analyzed previously, the massive field of dRGT also can be split into helicity-2 $h_{\mu\nu}$, helicity-1 A_μ and helicity-0 π fields. However, the covariantization and decoupling limit can ensure that the symmetry of dRGT is reduced to the symmetry of linearized GR plus $U(1)$ massive theory. Ref. [48] shows that the dRGT gives the known linearized gravity in the decoupling limit if $\partial^\mu\partial_\nu\pi$. We shall take a quick look how this happens. After some operations, we can replaced the fields $h_{\mu\nu}$, A_μ and π as

$$\begin{aligned} \tilde{h}_{\mu\nu} &= M_{\text{P1}}h_{\mu\nu}, \quad \hat{h}_{\mu\nu} = \tilde{h}_{\mu\nu} - \eta_{\mu\nu}\tilde{\pi}, \\ \tilde{A}_\mu &= M_{\text{P1}}mA_\mu, \quad \tilde{\pi} = M_{\text{P1}}m^2\pi. \end{aligned} \tag{2.105}$$

In decoupling limit, i.e. when both $M_{\text{Pl}} \to \infty$, $m \to 0, m^2M_{\text{Pl}} = \text{constant}$. The action of dRGT is invariant under

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \tag{2.106}$$

$$\delta A_\mu = \partial_\mu\pi, \tag{2.107}$$

$$\delta\pi = 0. \tag{2.108}$$

Consequently, Eq. 2.106 gives the result that same as Linearized GR. Eq. 2.107 gives the result that same as Maxwell.

There is another convention that using

$$\mathbb{K} = \mathbb{I} - \mathbb{X}. \tag{2.109}$$

Then the action of the dRGT theory becomes,

$$S_{\text{dRGT}} = \int d^4x\sqrt{-g}\frac{M_p^2}{2}\left[R - 2m^2\sum_{n=0}^{4}\alpha_n e_n\left(\mathbb{K}\right)\right] + S_{\text{m}}[g,\psi], \tag{2.110}$$

where α_n are dimensionless coupling constants which satisfy the relationship [49],

$$\beta_n = (4-n)!\sum_{i=n}^{4}\frac{(-1)^{i+n}}{(4-i)!(i-n)!}\alpha_i. \tag{2.111}$$

2.8 Summary

In this chapter, we have reviewed the massive gravity theory within the context of historical development. Starting with the spin-1 field theory, we found that the Proca mass term breaks the gauge invariance, and causes the discontinuity in the number of dofs which could be reconciled by the Stückelberg trick. We then performed a similar analysis for the spin-2 field, the Stückelberg fields with 4 components were introduced to restore the diffeomorphism invariance which was broken by adding the FP mass term. Subsequently, the vDVZ discontinuity was shown by computing the gravitational exchange amplitude for the case of the massless limit of the massive spin-2 field and massless spin-2 field. We also gave a simple calculation of the Vainshtein radius to be a resolution to this discontinuity problem. We then extended the theory non-linearly, but most generalizations unavoidably introduce the BD ghost which is the sixth dof that appears at the non-linear level. A healthy theory should avoid such instability. Thus, a theory of massive gravity with coefficients tuned to avoid the BD ghost by incorporating all higher derivative operators as total derivative was discussed at the end.

3

Projected massive gravity

This chapter mainly follows with Refs. [30], [50]. Starting with new massive gravity theories with non-minimal coupling which have 5 dynamical dofs and break the global translation invariance. We will focus on such a theory with a field space metric that is the same as the projected tensor on the space perpendicular to the Stückelberg field, known as projected massive gravity. After obtaining the equations of motion of PMG, we shall further investigate the cosmology and black hole solutions. It's worth noting that the investigation of black hole solutions in PMG, presented in the final section, constitutes original research.

Contents

3.1 Massive gravity with non-minimal coupling

The dRGT massive gravity theory we introduced in Section 2.7 admits an open-FLRW solution in which the dRGT mass terms behave as the cosmological constant [27]. However, Ref. [28] pointed out that the scalar and vector kinetic terms of perturbations around the open-FLRW background will vanishes, which may lead to a strong coupling problem and a non-linear ghost instability [29]. Therefore, in order for the non-linear mass gravity to have a stable cosmological solution, it is necessary to extend the theory further. In fact, many theories have been proposed to extend the dRGT by introducing other fields, such as Hassan-Rosen bimetric gravity [51], quasi-dilaton theory [52], and mass-varying massive gravity [53]. But now we would like to focus on the more interesting case that the massive gravity can be further extended without invoking additional dof. Refs. [54]–[58] have given different methods to investigate this extension intensively, which are based on theories invariant under the Poincaré symmetry in the internal field space, but none of them have been successful. The internal field space consisted of the Stückelberg fields ϕ^a which are introduced to restore the general covariance as discussed in the last Chapter 2. Nevertheless, there exists a natural extension of the dRGT theory (i.e., the total number of dofs remains the same and free of BD ghost) that abandons translation invariance while retaining global Lorentz invariance [59]. This extension is also known as the generalized massive theory, the constant parameters of this theory in the graviton mass potential are now promoted to be arbitrary functions of four Stückelberg fields ϕ^a. More importantly, there is no instability for all perturbations around the open-FLRW background [60]. An important reason why we chose the massive gravity theory with 5 dofs is that such theories exist in the Hamiltonian constraint in unitary gauge and thus are guaranteed to avoid the BD ghost [47]. In this section, we present two distinct extensions of the ghost-free massive gravity theory that preserve global Lorentz symmetry, primarily following Ref. [30].

3.1.1 Non-minimal coupling

Let us start with the action of dRGT (2.100 and 2.101) and the reference metric of this theory we now define as

$$f_{\mu\nu} \equiv \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b. \tag{3.1}$$

The action (2.100) is manifestly invariant under the Poincaré transformation in the internal field space. We now define a scalar function

$$X = \eta_{ab}\phi^a\phi^b = \phi^a\phi_a, \tag{3.2}$$

which can promote the constant parameters β_n to be functions of X if we abandon the global translation invariance $\phi^a \to \phi^a + c$. Such an extended massive gravity theory preserves global Lorentz invariance [59] and we would like to seek further extensions to this generalized massive gravity. Let us consider the conformal transformation of physical metric $g_{\mu\nu}$ utilizing the scalar

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X,

$$\tilde{g}_{\mu\nu} = G(X) g_{\mu\nu}. \tag{3.3}$$

Then performing this transformation to the action (2.100) that becomes,

$$\begin{aligned}
S_{\mathrm{dRGT}}^{\mathrm{deform}} &= \int d^4x \sqrt{-\tilde{g}} \left[\frac{M_{\mathrm{Pl}}^2}{2} R[\tilde{g}] - M_{\mathrm{Pl}}^2 m^2 \sum_{n=0}^{4} \beta_n e_n \left(\sqrt{\tilde{g}^{-1} f} \right) \right] \\
&= \int d^4x \sqrt{-g} G^2 \left[\frac{M_{\mathrm{Pl}}^2}{2} G^{-1} g^{\mu\nu} \tilde{R}_{\mu\nu} - M_{\mathrm{Pl}}^2 m^2 \sum_{n=0}^{4} \beta_n G^{-\frac{n}{2}} e_n \left(\sqrt{g^{-1} f} \right) \right] \\
&= \int d^4x \sqrt{-g} \left[\frac{M_{\mathrm{Pl}}^2}{2} G \left(R + \frac{3}{2} \nabla^\mu \log G \nabla_\mu \log G \right) - M_{\mathrm{Pl}}^2 m^2 \sum_{n=0}^{4} \beta_n G(X)^{\frac{4-n}{2}} e_n \left(\sqrt{g^{-1} f} \right) \right] \\
&= \int d^4x \sqrt{-g} \left[\frac{M_{\mathrm{Pl}}^2}{2} G \left(R + \frac{6 G_X^2}{G^2} \phi_a \phi_b g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b \right) - M_{\mathrm{Pl}}^2 m^2 \sum_{n=0}^{4} \tilde{\beta}_n(X) e_n \left(\sqrt{g^{-1} f} \right) \right]
\end{aligned} \tag{3.4}$$

where we defined the rescaled parameters as

$$\tilde{\beta}_n(X) \equiv \beta_n G(X)^{\frac{4-n}{2}}. \tag{3.5}$$

Thus, once the translation invariance is broken, the Einstein-Hilbert term can be non-minimally coupled with Stückelberg fields ϕ^a and the parameters $\tilde{\beta}_n$ now are the arbitrary functions of X. Moreover, the conformal scaling in the graviton mass terms can be interpreted as a redefinition of the reference metric,

$$\tilde{g}^{-1} f \to g^{-1} \left(G^{-1} f \right), \tag{3.6}$$

which implies that the reference metric can be deformed by appropriately contracting the Lorentz indices by η_{ab} and Stückelberg fields ϕ^a and introducing arbitrary functions of X.

3.1.2 Disformal deformations of the reference metric

Moving to the most general deformation of the reference metric,

$$\tilde{f}_{\mu\nu,I} = C_I(X) \eta_{ab} + D_I(X) \phi_a \phi_b \partial_\mu \phi^a \partial_\nu \phi^b, \tag{3.7}$$

where I is a label that will be assigned to each mass term and all $C_I = 1$ and $D_I = 0$ can be reduced to the case of dRGT massive gravity. Thus we can also define the square-root matrix in mass terms

$$Q^\mu{}_{\nu,I} \equiv \left(\sqrt{g^{-1} \tilde{f}_I} \right)^\mu{}_\nu, \tag{3.8}$$

Then for the action with general mass terms,

$$S = \int d^4x\sqrt{-g}\frac{M_{\rm Pl}^2}{2}\left[R[g] - 2m^2\mathcal{L}_{\rm mass}\right], \tag{3.9}$$

with

$$\begin{aligned}\mathcal{L}_{\rm mass} =& \beta(X)\left[Q_\beta\right] + \gamma_1(X)\left[Q_{\gamma_1}\right]^2 + \gamma_2(X)\left[Q_{\gamma_2}^2\right] + \delta_1(X)\left[Q_{\delta_1}\right]^3 + \delta_2(X)\left[Q_{\delta_2}\right]\left[Q_{\delta_2}^2\right] + \delta_3(X)\left[Q_{\delta_3}^3\right] \\ &+ \sigma_1(X)\left[Q_{\sigma_1}\right]^4 + \sigma_2(X)\left[Q_{\sigma_2}\right]^2\left[Q_{\sigma_2}^2\right] + \sigma_3(X)\left[Q_{\sigma_3}^2\right]^2 + \sigma_4(X)\left[Q_{\sigma_4}\right]\left[Q_{\sigma_4}^3\right] + \sigma_5(X)\left[Q_{\sigma_5}^4\right].\end{aligned} \tag{3.10}$$

Generally speaking, the BD ghost would be brought by the Stückelberg fields but we can avoid this if the action can be arranged one of the components of the Stückelberg fields has no dynamics. Ref. [59] introduced a new approach to derive BD ghost-free conditions, which is based on the degeneracy of the kinetic matrix of the Stückelberg fields and then generating primary and subsequent constraints to eliminate the BD ghost. However, it is not easy to derive the degeneracy conditions exactly due to the square-root form of the building block tensor (3.8). The action of the mass term can finally arrive at (see Ref. [30] for more details),

$$\mathcal{L}_{\rm mass} = \beta_1(X)e_1(Q) + \beta_2(X)e_2(Q) + \beta_3(X)e_3(Q) + \beta_4(X)e_4(Q), \tag{3.11}$$

which is the dRGT tuning. Here, we defined

$$Q \equiv \eta_{ab} + D(X)\phi_a\phi_b, \tag{3.12}$$

and

$$\beta_1 \equiv C_\beta\beta, \quad \beta_2 \equiv 2C_{\gamma_1}\gamma_1, \quad \beta_3 \equiv 3!C_{\delta_1}^{3/2}\delta_1, \quad \beta_4 \equiv 4!C_{\sigma_1}^{3/2}\sigma_1, \quad D \equiv \frac{D_{\gamma_1}}{C_{\gamma_1}}. \tag{3.13}$$

So there can only be one field space metric which explicitly depends on the ϕ^a is ghost-free [61], disformally related to the original Minkowski metric. Therefore, we can obtain the generalized massive gravity from dRGT theory with constant mass parameters, in which consider deformations of the field space metric with different conformal coefficients at different orders.

On the other hand, we can fix the disformal term in the field space metric that leaves all of the functions of mass terms unconstrained,

$$\frac{D_{\gamma_1}}{C_{\gamma_1}} = -\frac{1}{X}, \tag{3.14}$$

which is equivalent to having a field space metric proportional to a projection tensor (operator)

$$P_{ab} \equiv \left(\eta_{ab} - \frac{\phi_a\phi_b}{X}\right). \tag{3.15}$$

This tensor projects onto surfaces in the field space defined by normal vector ϕ^a and the conformal factors can be absorbed in the individual functions of the mass term. This clever combination ensures that the derivative in one of the directions does not exist in $\tilde{f}_{\mu\nu}$. Similar to some features of other Lorentz violating massive gravity theories [62], [63], the projected mass terms we are discussing now naturally lack the BD mode. But the difference is that the time direction in this theory remains unspecified due to the explicit dependence on ϕ^a.

3.1.3 Evading BD ghost

We now promote the previous analysis to include a non-minimal coupling $G(X)R$, and see its kinetic structure. We should consider a new action that contains terms responsible for the degeneracy of the kinetic matrix with the non-minimal coupling,

$$S = \int \mathrm{d}^4x\sqrt{-g}\frac{M_{\mathrm{Pl}}^2}{2}\left(G(X)R[g] + F(X)[Y] + A(X)[Z]\right), \tag{3.16}$$

where we defined

$$\begin{aligned} W^\mu_{\ \nu} &\equiv \left(g^{-1}f\right)^\mu_{\ \nu} \equiv g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b\eta_{ab}, \\ Y^\mu_{\ \nu} &\equiv \left(g^{-1}\tilde{f}\right)^\mu_{\ \nu} \equiv g^{\mu\nu}\phi_a\phi_b\partial_\mu\phi^a\partial_\nu\phi^b. \end{aligned} \tag{3.17}$$

where we set $C_I = 0$ and $D_I = 1$ in this case. Following similar steps to finding degenerate higher-order scalar tensor theories [64]–[66], we shall use the $(3+1)$ decomposition to investigate the degeneracy between metric variables and scalar fields ϕ^a in this theory. The partial derivative of four scalar fields can be decomposed by using the normal vector n^μ and the induced metric $\gamma_{\mu\nu}$ (see Ref. [30] for more details),

$$\partial_\mu\phi^a = -n_\mu\dot{\phi}^a + D_\mu\phi^a, \tag{3.18}$$

with

$$\dot{\phi}^a \equiv n^\mu\partial_\mu\phi^a, \quad D_\mu\phi^a \equiv \gamma^\nu_\mu\partial_\nu\phi^a. \tag{3.19}$$

The kinetic term of the action (3.16) can be written as

$$\mathcal{L}_{\mathrm{kin}} = \mathcal{A}^{ab}\dot{\phi}_a\dot{\phi}_b + \mathcal{C}^{a\mu\nu}\dot{\phi}_a\mathcal{K}_{\mu\nu} + \mathcal{F}^{\mu\nu\rho\sigma}\mathcal{K}_{\mu\nu}\mathcal{K}_{\rho\sigma}, \tag{3.20}$$

where $\mathcal{K}_{\mu\nu} \equiv \gamma_\mu{}^\rho \gamma_\nu{}^\sigma \nabla_\rho n_\sigma$ is the extrinsic curvature and we defined

$$\begin{aligned} \mathcal{A}^{ab} &= -A\eta^{ab} - F\phi^a\phi^b, \\ \mathcal{C}^{a\mu\nu} &= -4\eta^{ab}G_X\phi_b\gamma^{\mu\nu}, \\ \mathcal{F}^{\mu\nu\rho\sigma} &= G\left(\gamma^{\mu(\rho}\gamma^{\sigma)\nu} - \gamma^{\mu\nu}\gamma^{\rho\sigma}\right). \end{aligned} \tag{3.21}$$

Combing the canonical momenta $\frac{\delta\mathcal{L}}{\delta\dot{\phi}_a}$ and $\frac{\delta\mathcal{L}}{\delta\mathcal{K}_{\mu\nu}}$ with the existence of a primary constraint, we can obtain the degeneracy condition,

$$G(X)A(X) + G(X)F(X)X - 6G_X^2X = 0. \tag{3.22}$$

Therefore, the BD ghost does not exist as long as this condition is satisfied. Next, we shall include the mass terms obtained in the Section. 3.1.2. We first start with the option

$$\gamma_1 = -\gamma_2, \quad \delta_1 = -\frac{\delta_2}{3} = \frac{\delta_3}{2}, \quad \sigma_1 = -\frac{\sigma_2}{6} = \frac{\sigma_3}{3} = \frac{\sigma_4}{8} = -\frac{\sigma_5}{6}. \tag{3.23}$$

Plugging it into the degeneracy condition (3.22). Then kinetic term now becomes,

$$\mathcal{L}_{\rm kin} = \mathcal{F}^{\mu\nu\rho\sigma}\left(K_{\mu\nu} + \frac{G_X}{G}\phi_a\dot{\phi}^a\gamma_{\mu\nu}\right)\left(\mathcal{K}_{\rho\sigma} + \frac{G_X}{G}\phi_b\dot{\phi}^b\gamma_{\rho\sigma}\right) - A\dot{\phi}_a\dot{\phi}^a. \tag{3.24}$$

Notice that, under the condition $\gamma_1 = -\gamma_2$, $A = 0$ should be required to ensure the absence of the BD ghost. As a result, we obtain a new theory which is the extension of the generalized massive gravity. This ghost-free massive gravity requires a non-minimal coupling with curvature given by

$$S = \int d^4x\sqrt{-g}\frac{M_{\rm Pl}^2}{2}\left[G(X)R + \frac{6G_X^2}{G}[Y] - 2m^2\sum_{n=0}^{4}\beta_n(X)e_n\left(\sqrt{g^{-1}\tilde{f}}\right)\right] + S_{\rm m}\left[g_{\mu\nu},\psi\right]. \tag{3.25}$$

In this thesis, we are more interested in the option (3.14) which is nothing but the projection onto the ϕ^a direction. According to the projection operator (3.15), we can construct a new reference metric $\bar{f}_{\mu\nu}$ and the building block tensor $Z^\mu_{\ \nu}$

$$\bar{f}_{\mu\nu} = \left(\eta_{ab} - \frac{\phi_a\phi_b}{X}\right)\partial_\mu\phi^a\partial_\nu\phi^b = P_{ab}\partial_\mu\phi^a\partial_\nu\phi^b, \quad Z^\mu_{\ \nu} = \left(g^{-1}\bar{f}\right)^\mu_{\ \nu}. \tag{3.26}$$

Thus, under the degeneracy condition (3.22) the general non-minimal coupling action (3.16) can be transformed to

$$GR + F[Y] + A[W] = GR + \frac{6G_X^2}{G}[Y] + A[Z], \tag{3.27}$$

where we have used the relation $Z = W - Y/X$. We thus arrive at another new ghost-free massive gravity we call projected massive gravity (PMG). This theory can also have non-minimal coupling, which is given by

$$S = \int d^4x\sqrt{-g}\frac{M_{\rm Pl}^2}{2}\left[G(X)R + \frac{6G_X^2}{G}[Y] + m^2 U\left(X, [Z], \left[Z^2\right], \left[Z^3\right]\right)\right] + S_{\rm m}\left[g_{\mu\nu}, \psi\right], \tag{3.28}$$

where U is an arbitrary function of the mass potential terms that no longer of the form of the dRGT. The potential term $[Z]$ can be further generalized and higher order terms $[Z^n]$ denotes the trace of matrices $(Z^n)^\mu{}_\nu = Z^\mu{}_{\alpha_1} Z^{\alpha_1}_{\alpha_2} \cdots Z^{\alpha_{n-1}}_\nu$, which with $n \geq 4$ can be always reduced to lower order terms by Cayley-Hamilton theorem. Furthermore, the potential U including higher order remains consistent with the condition derived from the linear combination of the canonical momenta. Therefore, the projected massive gravity is the absence of the BD ghost. Ref. [30] show an explicit derivation of action (3.28) starting with the most general mass terms up to quadratic order composed of W and Y. Besides, the theory we presented was not the most general. Ref. [62] includes the term $G^{\mu\nu}\bar{f}_{\mu\nu}$ which also can be considered in our case without generating the BD ghost.

3.2 Equations of motion of PMG

If we want to do more phenomenological research on this new theory, it is very important to obtain the equation of motion of PMG. We start with summarizing this theory. The action given by PMG is

$$S_{\rm PMG} = \int d^4x\sqrt{-g}\frac{M_{\rm Pl}^2}{2}\left[G(X)R - \frac{6G_X^2}{G}[Y] + m^2 U\left(X, [Z], \left[Z^2\right], \left[Z^3\right]\right)\right] + S_{\rm m}\left[g_{\mu\nu}, \psi\right], \tag{3.29}$$

where $S_{\rm m}$ is the matter Lagrangian and

$$\begin{aligned}
&X \equiv \phi^a\phi_a = \eta_{ab}\phi^a\phi^b, \\
&Z^\mu_\nu \equiv \left(g^{-1}\bar{f}\right)^\mu{}_\nu, \quad Y^\mu_\nu \equiv \left(g^{-1}\tilde{f}\right)^\mu{}_\nu \\
&\bar{f}_{\mu\nu} \equiv P_{ab}\partial_\mu\phi^a\partial_\nu\phi^b = \left(\eta_{ab} - \frac{\phi_a\phi_b}{X}\right)\partial_\mu\phi^a\partial_\nu\phi^b, \\
&\tilde{f}_{\mu\nu} \equiv -\phi_a\phi_b\partial_\mu\phi^a\partial_\nu\phi^b,
\end{aligned} \tag{3.30}$$

here we adopt new convention (the definition of Y^μ_ν differs by a minus sign from the Section. 3.1.3) for convenience, which is consistent with Ref. [50].

In Appendix. A.2, we give a detailed derivation of how to obtain the modified Einstein equation from varying the action (3.29) with respect to $g_{\mu\nu}$. The equations of motion is,

$$M_{\rm Pl}^2\left[G(X)G_{\mu\nu}-\nabla_\mu\nabla_\nu G(X)+g_{\mu\nu}\nabla_\alpha\nabla^\alpha G(X)-\frac{6\left(G_X(X)\right)^2}{G(X)}\left(\tilde{f}_{\mu\nu}-\frac{1}{2}[Y]g_{\mu\nu}\right)\right]=T_{\mu\nu}^{(\rm mass)}+T_{\mu\nu}^{(\rm m)} \tag{3.31}$$

where $T_{\mu\nu}^{(\rm m)}$ is the energy-momentum tensor for matter content, and $T_{\mu\nu}^{(\rm mass)}$ is the effective energy-momentum tensor of the mass term defined as

$$T_{\mu\nu}^{(\rm mass)}=M_{\rm Pl}^2m^2\left(\frac{1}{2}g_{\mu\nu}U-U_{[Z]}\bar{f}_{\mu\nu}-2U_{[Z^2]}Z^\rho{}_{(\mu}\bar{f}_{\nu)\rho}-3U_{[Z^3]}Z^\rho{}_\sigma Z^\sigma{}_{(\mu}\bar{f}_{\nu)\rho}\right), \tag{3.32}$$

where we defined

$$U_{[Z^n]}\equiv\frac{\partial U}{\partial\left[Z^n\right]}. \tag{3.33}$$

We assume that the matter field obeys the standard conservation law,

$$\nabla^\mu T_{\mu\nu}^{(\rm m)}=0, \tag{3.34}$$

For matter content, we focus on the perfect fluid with no pressure ($p=0$), Eq. 3.34 can give the background matter equation (more on Appendix. B.1),

$$\dot{\rho}+3H\rho=0. \tag{3.35}$$

In massive gravity, we would have the Stückelberg equation come from the contracted Bianchi identity but is not independent with Eq. 3.31 and Eq. 3.34,

$$\nabla^\mu\left(\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\right)=\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\phi^a}\partial_\nu\phi^a. \tag{3.36}$$

3.3 Cosmology

In this section, we shall derive the background equations with an ansatz in the FLRW Universe. FLRW solution is one of the most crucial solutions in cosmology derived from GR, which assumes the universe satisfies the cosmological principle, i.e., homogeneous and isotropic.

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The line element for the physical metric $g_{\mu\nu}$ in FLRW Universe is given by

$$ds_g^2 = -dt^2 + a(t)^2 \left(\delta_{ij} - \frac{\kappa x^i x^j}{1+\kappa x^k x^k}\right) dx^i dx^j = -dt^2 + a(t)^2 \Omega_{ij} dx^i dx^j, \tag{3.37}$$

where κ is a curvature parameter that we adopt the convention $\kappa > 0$ represent the open universe, and Ω_{ij} is an induced metric on the constant time hypersurface defined as

$$\Omega_{ij} = \delta_{ij} - \frac{\kappa x^i x^j}{1+\kappa x^k x^k}. \tag{3.38}$$

In PMG, we should require FLRW symmetries for reference metrics as well as the homogeneity of the Lorentz-invariant scalar X. Thus, the requirement of homogeneity and isotropy constrains the configuration of the Stückelberg fields uniquely [27], which is given by

$$\phi^0 = f(t)\sqrt{1+\kappa\left(x^2+y^2+z^2\right)}, \quad \phi^i = f(t)\sqrt{\kappa}x^i, \tag{3.39}$$

where $f(t)$ is an arbitrary function of time. Therefore, the reference metrics corresponding to $\bar{f}$ and $\tilde{f}$ are

$$\begin{aligned} ds_{\bar{f}}^2 &= \bar{f}_{\mu\nu} dx^\mu dx^\nu = \kappa f^2 \Omega_{ij} dx^i dx^j, \\ ds_{\tilde{f}}^2 &= \tilde{f}_{\mu\nu} dx^\mu dx^\nu = -f^2 \dot{f}^2 dt^2. \end{aligned} \tag{3.40}$$

In this configuration, we can obtain the following results

$$X = -f^2(t), \quad [Y] = (f\dot{f})^2, \quad [Z] = g^{ij} f^2 \kappa \Omega_{ij}. \tag{3.41}$$

In the Appendix B.2, we follow with the derivation of Friedmann equations in GR (more on Appendix. B.1), show an explicit derivation of background equations in PMG starting with the modified Einstein equation (3.31) and combing the background matter equation (3.35) (or Stückelberg equation (3.36)). The results are summarized as follows:

$$3G\left[\left(H + \frac{\dot{G}}{2G}\right)^2 - \frac{\kappa}{a^2}\right] = \frac{\rho}{M_{\text{Pl}}^2} + \frac{\rho_g}{M_{\text{Pl}}^2}, \tag{3.42}$$

$$-2G\left[\partial_t\left(H + \frac{\dot{G}}{2G}\right) + \frac{\kappa}{a^2}\right] + \dot{G}\left(H + \frac{\dot{G}}{2G}\right) = \frac{\rho}{M_{\text{Pl}}^2} + \frac{\rho_g + p_g}{M_{\text{Pl}}^2}, \tag{3.43}$$

$$\dot{\rho}_g + 3H\left(\rho_g + p_g\right) - \frac{\dot{G}}{2G}\left(\rho_g - 3p_g + \rho\right) = 0, \tag{3.44}$$

where we defined the effective energy density ρ_g and effective pressure p_g with the graviton's mass

m

$$\rho_g = -\frac{1}{2}m^2 M_{\rm Pl}^2 U, \tag{3.45}$$

$$p_g = \frac{1}{2}m^2 M_{\rm Pl}^2 \left(U - 2\xi^2 U_{[Z]} - 4\xi^4 U_{[Z^2]} - 6\xi^6 U_{[Z^3]}\right), \tag{3.46}$$

with $\xi = \sqrt{\kappa} f/a$.

According to the studies in Ref. [30], the extended theory of PMG has a self-accelerating solution in open FLRW Universe and all perturbations are free of ghost and gradient instabilities, unlike the dRGT. In Ref. [50], the PMG was re-investigated in concrete models (minimal and non-minimal coupling model) without assuming the weak X-dependence, which is more meaningful due to the X-independence mass potential will lead to the strong coupling of the scalar perturbation [30]. Ref. [50] further puts observational constraints on the model parameter and cosmological parameters from the red-shift space distortion dataset and type Ia supernova, which shows that the model of PMG is consistent with these observations. The study of phenomenology makes PMG more potential. Moreover, these results demonstrate that the background evolution and the linear growth of structure at sub-horizon scales of PMG can be regarded as the dark energy model with the equation of state. One of the key motivations for exploring massive gravity is it provides a novel perspective on the cosmic acceleration issue.

3.4 Black hole

The existence and properties of black hole (BH) solutions are crucial for exploring the non-perturbative aspects of various theories of gravity. In addition, with the increasing understanding of astrophysical BHs and the development of more and more astronomical observation experiments, BH phenomenology has become increasingly important because it provides the possibility to verify the modified gravity. Massive gravity and its extensions like PMG in our case should certainly exhibit BH solutions. We would expect the solutions from massive gravity that closely resemble the Schwarzschild solution of GR if the Vainshtein mechanism is indeed correct. In fact, the BH solutions in massive gravity have been extensively investigated [67]–[74], but we should notice that the most physically relevant solutions are likely to be found in generic cases may not yield exact analytical solutions [31].

In this section, we shall first try to follow the classical method in Appendix C to find the BH solution in PMG. If we would like to find a Schwarzschild solution, which can assume the metric is

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static and spherically symmetric. Focusing on (t, r, θ, ϕ) coordinates, the metric can be written as

$$ds^2 = A(r)dt^2 + B(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{3.47}$$

The vacuum solution satisfies the equations $T^{(\mathrm{m})}_{\mu\nu} = 0$. The non-zero Einstein tensor are

$$\begin{aligned}
G_{tt} &= -\frac{A\left((-1+B)B + rB'\right)}{r^2 B^2}, \\
G_{rr} &= \frac{A - AB + rA'}{r^2 A}, \\
G_{\theta\theta} &= \frac{r\left(-rBA'^2 - 2A^2B' + A\left(-rA'B' + 2B\left(A' + rA''\right)\right)\right)}{4A^2B^2}, \\
G_{\phi\phi} &= \sin^2\theta G_{\theta\theta}.
\end{aligned} \tag{3.48}$$

We now consider a configuration of the Stückelberg fields similar to that of cosmology but time-independent, i.e. $\phi^0 = \sqrt{1+r^2}$, $\phi^r = r$ and $\phi^\theta = \phi^\phi = 0$. Under this ansatz, we can obtain the corresponding reference metrics

$$\begin{aligned}
ds^2_{\bar{f}} &= \bar{f}_{\mu\nu}dx^\mu dx^\nu = \frac{1}{1+r^2}dr^2, \\
ds^2_{\tilde{f}} &= \tilde{f}_{\mu\nu}dx^\mu dx^\nu = 0.
\end{aligned} \tag{3.49}$$

In this configuration, we can obtain the following results

$$[Y] = 0, \quad [Z] = \frac{g^{rr}}{1+r^2}. \tag{3.50}$$

The Lorentz-invariant scalar $X = -1$ implies that $G(X)$ is constant. For convenience, we consider the minimal coupling model, i.e. $G(X) = 1$. We now fix the configuration of the mass potential as

$$U\left(X, [Z], \left[Z^2\right], \left[Z^3\right]\right) = \lambda X + [Z]^2 - \left[Z^2\right], \tag{3.51}$$

where $\lambda \neq 0$ is a non-zero constant parameter since Refs. [30], [50] point out that the mass potential of PMG without X-dependence leads to the strong coupling problem. Consequently, the modified Einstein equation becomes,

$$M^2_{\mathrm{Pl}} G_{\mu\nu} = T^{(\mathrm{mass})}_{\mu\nu}, \tag{3.52}$$

with

$$T^{(\mathrm{mass})}_{\mu\nu} = M^2_{\mathrm{Pl}} m^2 \left[\frac{1}{2} g_{\mu\nu}\left(\lambda X + [Z]^2 - \left[Z^2\right]\right) - 2[Z]\bar{f}_{\mu\nu} + 2Z^\rho{}_{(\mu}\bar{f}_{\nu)\rho}\right]. \tag{3.53}$$

Plugging (3.49) and (3.50) in the modified Einstein equation, the combinations $\left([Z]^2 - \left[Z^2\right]\right)$ and

$(-2[Z]\bar{f}_{\mu\nu} + 2Z^{\rho}{}_{(\mu}\bar{f}_{\nu)\rho})$ will vanish, which leads to the following non-trivial equations

$$G_{tt} = -\frac{1}{2}g_{tt}m^2\lambda, \tag{3.54}$$

$$G_{rr} = -\frac{1}{2}g_{rr}m^2\lambda, \tag{3.55}$$

$$G_{\theta\theta} = -\frac{1}{2}g_{\theta\theta}m^2\lambda. \tag{3.56}$$

These differential equations can yield an exact analytical solution, i.e.,

$$A(r) = c_1 B^{-1}(r) = c_1\left(1 - \frac{c_2}{r} - \frac{\lambda m^2 r^2}{6}\right), \tag{3.57}$$

where c_1 and c_2 are undetermined constants. One can arrive at the metric

$$ds^2 = -\left(1 - \frac{2G_N M}{r} - \frac{\lambda m^2 r^2}{6}\right)dt^2 + \left(1 - \frac{2G_N M}{r} - \frac{\lambda m^2 r^2}{6}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{3.58}$$

When we compare it with the Schwarzschild-de Sitter metric provided in Appendix C.3.1, we can observe that

$$\frac{\Lambda r^2}{3} \sim \frac{\lambda m^2 r^2}{6}.$$

The new term we obtained from this modified Einstein equation can mimic the cosmological constant term in GR. Moreover, as presented in the Appendix C.3.2, we can easily extend this solution to the case of coupling charge q,

$$ds^2 = -\left(1 - \frac{2G_N M}{r} - \frac{\lambda m^2 r^2}{6} + \frac{q^2}{4r^2}\right)dt^2 + \left(1 - \frac{2G_N M}{r} - \frac{\lambda m^2 r^2}{6} + \frac{q^2}{4r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{3.59}$$

However, in the theories with two static, bi-diagonal metrics, Ref. [75] demonstrated that requiring both metrics to be simultaneously diagonal generically leads to coordinate-invariant singularities at the BH horizon. In specific solutions, introducing Stückelberg fields may allow us to render both metrics in diagonal form by having these fields absorb the off-diagonal terms. However, for generic solutions, it is expected that at least one metric will remain non-diagonal despite the presence of Stückelberg fields [31]. More particularly, in GR, we can use Schwarzschild coordinates which have a coordinate singularity at the horizon, we can always recognize that this is simply a coordinate artifact. But in massive gravity, if we choose Schwarzschild coordinate and make a choice for the Stückelberg fields which are regular at the horizon then we can able to compute some invariants (generally $I^{ab} = g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b$) which diverge at the horizon, leading to the emer-

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gence of a physical singularity in that region. In other words, we have already used up the gauge of freedoms, so we cannot fix the gauge again to remove the singularity. For a metric to serve as a valid description of a BH configuration, it must ensure the absence of physical singularities at the horizon. Hence, it is more promising to use a coordinate system that is manifestly regular at the horizon to ensure the resulting solution lacks any physical singularities at the horizon. Ref. [67] explains why choosing Schwarzschild-like coordinates is not a good idea and goes on to demonstrate the advantages of using alternative coordinate systems such as Kruskal-Szekeres, Eddington-Finkelstein, or Gullstrand-Painlevé coordinates.

3.4.1 Non-diagonal metrics

The more general (but not the most general) class of non-diagonal spherically-symmetric metrics in (t, r, θ, ϕ) coordinates could be written as follows

$$ds^2 = -A(r)dt^2 + 2B(r)dtdr + C(r)dr^2 + r^2 d\Omega^2, \tag{3.60}$$

where $A(r), B(r)$ and $C(r)$ are arbitrary functions. The non-zero Einstein tensor components are

$$\begin{aligned}
G_{tt} &= \frac{A\left(B^4 - rB^2A' + AB\left(B(-1+2C) + 2rB'\right) + A^2\left((-1+C)C + rC'\right)\right)}{r^2\left(B^2 + AC\right)^2},\\
G_{rr} &= \frac{-A^2(-1+C)C^2 + B^2\left(-B^2C + 2rCA' + 2rBB'\right) + A\left(rC^2A' + B^2\left(C - 2C^2 + rC'\right)\right)}{r^2\left(B^2 + AC\right)^2},\\
G_{tr} &= -\frac{B\left(B^4 - rB^2A' + AB\left(B(-1+2C) + 2rB'\right) + A^2\left((-1+C)C + rC'\right)\right)}{r^2\left(B^2 + AC\right)^2},\\
G_{\theta\theta} &= \frac{r\left(4B^2A' + CA'\left(2A - rA'\right) - 2B\left(2A + rA'\right)B' - A\left(2A + rA'\right)C' + 2r\left(B2^2 + AC^2\right)A''\right)}{4\left(B^2 + AA\right)^2},\\
G_{\phi\phi} &= \sin^2\theta G_{\theta\theta}.
\end{aligned} \tag{3.61}$$

The vacuum solution satisfies the equations $T^{(\mathrm{m})}_{\mu\nu} = 0$ and thus the modified Einstein equation gives,

$$G_{\mu\nu} = \frac{1}{2} g_{\mu\nu} m^2 \left(\lambda X + [Z]^2 - \left[Z^2\right]\right), \tag{3.62}$$

here X should be a constant. Notice that $(t$,$t)$ and $(t$,$r)$ components in the modified Einstein equation give the same result. If we continue to adopt the same ansatz for Stückelberg fields, i.e. $\phi^0 = \sqrt{1+r^2}$, $\phi^r = r$ and $\phi^\theta = \phi^\phi = 0$, we can still obtain the exact solution. From $(t$,$t)$ and $(r$,$r)$

components of the modified Einstein equations, we can obtain

$$
\begin{aligned}
A(r) &= \frac{\left(-12r + 2\lambda m^2 r^3 + 3c_1\right) c_2}{r}, \\
B(r) &= \pm\sqrt{\frac{c_2}{r}}\ \sqrt{-12r + 12rC(r) - 2\lambda m^2 r^3 C(r) - 3c_1 C(r)}.
\end{aligned}
\tag{3.63}
$$

3

where c_1 and c_2 are undetermined constants. One can arrive at the Eddington–Finkelstein-like metrics

$$
ds^2 = -\left(1 - \frac{2GM}{r} - \frac{\lambda m^2 r^2}{6}\right) dt^2 \pm 2dtdr + r^2 d\Omega^2, \tag{3.64}
$$

where $\pm$ represents ingoing and outgoing coordinates respectively.

In fact, it is a general observation that two diagonal metrics have a common horizon, which is valid for any theory with more than one metric regardless of the field equations [75]. Equivalently, this implies that if reference metrics are the diagonal metrics without horizons, then the metric for a BH must be non-diagonal when working in unitary gauge [31].

3.4.2 Time-dependent metrics

We now consider with the more common choice for the Stückelberg fields, i.e. the unitary gauge $\phi^a = x^\mu \delta^a_\mu$, meaning $\phi^0 = t$, $\phi^r = r$ and $\phi^\theta = \phi^\phi = 0$. The choice of unitary gauge is usually done so that the reference metric takes the standard Minkowski form where Lorentz invariance is manifest. Under this gauge fixed, we can obtain the corresponding reference metrics

$$
\begin{aligned}
ds^2_{\bar{f}} &= \bar{f}_{\mu\nu} dx^\mu dx^\nu = \frac{-r^2}{-t^2 + r^2} dt^2 + 2\frac{tr}{-t^2 + r^2} dtdr + \frac{-t^2}{-t^2 + r^2} dr^2, \\
ds^2_{\tilde{f}} &= \tilde{f}_{\mu\nu} dx^\mu dx^\nu = t^2 dt^2 - 2trdtdr + r^2 dr^2
\end{aligned}
\tag{3.65}
$$

In the configuration of the unitary gauge, we can obtain the following results

$$
\begin{aligned}
& X = -t^2 + r^2 \Rightarrow G(X) = G(t, r), \\
& [Y] = g^{tt}\tilde{f}_{tt} + 2g^{tr}\tilde{f}_{tr} + g^{rr}\tilde{f}_{rr} = g^{tt}t^2 - 2g^{tr}tr + g^{rr}r^2, \\
& [Z] = g^{tt}\bar{f}_{tt} + 2g^{tr}\bar{f}_{tr} + g^{rr}\bar{f}_{rr} = \frac{-g^{tt}r^2 + 2g^{tr}tr - g^{rr}t^2}{-t^2 + r^2}.
\end{aligned}
\tag{3.66}
$$

Again, we consider the minimal coupling model ($G(X) = 1$), and thus we no longer need to concern with $\tilde{f}_{\mu\nu}$ and $[Y]$ in modified Einstein equation. Interestingly, we found that the combinations $([Z]^2 - [Z^2])$ and $(-2[Z]\bar{f}_{\mu\nu} + 2Z^\rho{}_{(\mu}\bar{f}_{\nu)\rho})$ will also vanish under the unitary gauge. Moreover, we can observe that t and r will always appear symmetric in this case, which means that we cannot construct a mass potential U that contains either t or r alone. Indeed, we can construct a mass potential for $U = 0$ under some choice of Stückelberg fields, but this will lead to the trivial case. Since we have to consider the more meaningful X-dependent mass potential, which means that U generally depend explicitly on t and r, i.e. modified Einstein equation now is time-dependent,

$$M_{\rm Pl}^2 G_{\mu\nu} = T_{\mu\nu}^{(\rm mass)}(r,t). \tag{3.67}$$

The most general non-diagonal spherically-symmetric metrics should be rewritten as

$$ds^2 = -A(r,t)dt^2 + 2B(r,t)dtdr + C(r,t)dr^2 + D(r,t)r^2 d\Omega^2, \tag{3.68}$$

thus the non-zero components of the Einstein tensor become more complex, potentially necessitating the use of numerical or perturbation methods to solve the vacuum modified Einstein equations since we have not discovered any analytical solutions in time-dependent metrics.

Let us now take a particular case as an example and discuss the feasibility of the perturbation method in the case of time-dependent metrics. We continue consider the mass potential configuration (3.51) for convenience, the vacuum modified Einstein equations thus become,

$$G_{\mu\nu} = \frac{1}{2} g_{\mu\nu} m^2 \lambda (r^2 - t^2). \tag{3.69}$$

Obviously, we can think of the graviton mass term as a perturbation term and treat λ as a small quantity. Thus the expansion can be defined as

$$\begin{aligned}
A(r,t) &= A_0(r,t) + \lambda A_1(r,t) + \lambda^2 A_2(r,t) + \cdots , \\
B(r,t) &= B_0(r,t) + \lambda B_1(r,t) + \lambda^2 B_2(r,t) + \cdots , \\
C(r,t) &= C_0(r,t) + \lambda C_1(r,t) + \lambda^2 C_2(r,t) + \cdots , \\
D(r,t) &= D_0(r,t) + \lambda D_1(r,t) + \lambda^2 D_2(r,t) + \cdots .
\end{aligned} \tag{3.70}$$

We demand the solution should expand around the flat space, which leads to

$$A_0(r,t) = 1, \quad B_0(r,t) = 0, \quad C_0(r,t) = 1, \quad D_0(r,t) = 1. \tag{3.71}$$

Or rather, (3.71) is a solution at $\mathcal{O}(0)$ order. At $\mathcal{O}(\lambda)$ order, the non-trivial components of the

Einstein tensor are

$$\begin{aligned}
G_{tt}^{\mathcal{O}(\lambda)} &= \frac{-D_1 + C_1 + r\left(-3D_1' + C_1' - rD_1''\right)}{r^2}\lambda, \\
G_{rr}^{\mathcal{O}(\lambda)} &= \frac{D_1 - C_1 + r\left(2\dot{B}_1 - r\ddot{D}_1 + A_1' + D_1'\right)}{r^2}\lambda, \\
G_{tr}^{\mathcal{O}(\lambda)} &= -\frac{\dot{D}_1 - \dot{C}_1 + r\dot{D}_1'}{r}\lambda, \\
G_{\theta\theta}^{\mathcal{O}(\lambda)} &= \frac{1}{2}r\left(2\dot{C}_1 + A_1' - B_1' + 2D_1' + r\left(-\ddot{B}_1 - \ddot{D}_1 + 2\dot{C}_1' + A_1'' + D_1''\right)\right)\lambda, \\
G_{\phi\phi}^{\mathcal{O}(\lambda)} &= \sin^2\theta G_{\theta\theta}^{\mathcal{O}(\lambda)}.
\end{aligned} \tag{3.72}$$

Combing Eq. 3.69 and (3.71), the cross term for Einstein tensor $G_{tr}^{\mathcal{O}(\lambda)}$ should have

$$-\frac{\dot{D}_1 - \dot{C}_1 + r\dot{D}_1'}{r}\lambda = \frac{1}{2}B_0 m^2\lambda(r^2 - t^2) = 0, \tag{3.73}$$

which gives $C_1(r,t) = D_1(r,t) + rD_1'(r,t) + f_1(r)$. Plugging the expression for $C_1(r,t)$ in $G_{tt}^{\mathcal{O}(\lambda)}$, we have

$$G_{tt}^{\mathcal{O}(\lambda)} = \frac{f_1(r) + rf_1'(r)}{r^2}\lambda, \tag{3.74}$$

which is a function of r, i.e., time-independent. However, according to Eq. 3.69, we have

$$G_{tt}^{\mathcal{O}(\lambda)} = -\frac{1}{2}A_0 m^2\lambda(r^2 - t^2), \tag{3.75}$$

which is contradicted with Eq. 3.74. More generally, as long as the mass potential term is a first order small quantity and time-dependent, this perturbation method fails if we expand around the flat space (Eq. 3.71). From a more intuitive and rigorous point of view, we may expand the solution around the Schwarzschild spacetime, i.e.,

$$A_0 = 1 - \frac{2G_N M}{r}, \quad B_0 = 0, \quad C_0 = \left(1 - \frac{2G_N M}{r}\right)^{-1}, \quad D_0 = 1. \tag{3.76}$$

which is also a solution at $\mathcal{O}(0)$ order. However, at $\mathcal{O}(\lambda)$ order, we shall come to a similar conclusion that $G_{tt}^{\mathcal{O}(\lambda)}$ is still independent of time. Furthermore, we could follow the same steps to prove that $G_{tt}^{\mathcal{O}(\lambda)}$ is always just a function of r as long as the $\mathcal{O}(0)$ order solution is of the following form,

$$A_0(r,t) = A_0(r), \quad B_0(r,t) = 0, \quad C_0(r,t) = C_0(r), \quad D_0(r,t) = d, \tag{3.77}$$

where d is a constant. Hence, if we intend to use the perturbation method to study that the correction term is first order small quantities and time-dependent, it is advisable to begin with non-diagonal metrics.

Indeed, while we demonstrate the challenges of studying time-dependent metrics using perturbation methods, the exploration of time-dependent matrices remains a valuable endeavor in the context of massive gravity. Rachel Rosen investigated the possibility of black hole solutions in massive gravity that can accommodate both a non-singular horizon and Yukawa asymptotics by adopting a time-dependent ansatz Ref. [70]. This work has shown that time-dependent BH solutions in massive gravity may offer a way to circumvent the problem of coordinate-invariant singularities at the horizon and smoothly recover the BH solutions of GR in the massless limit. Therefore, this also serves as inspiration for us to explore the time-dependent BH solutions in PMG, which may have even greater potential physical significance.

Conclusion and outlook

Through this thesis, we have undertaken a review of the background of massive gravity. We began by establishing the formalism for both massive and massless spin-1 and spin-2 fields, with a particular emphasis on the Stückelberg language for both the Proca and Fierz–Pauli fields. After introducing vDVZ discontinuity, Vainshtein radius and BD ghost in the order of historical development, we ended the first part with the dRGT theory which serves as the foundation for the investigation of its extended theory in the second part.

We then studied a novel massive gravity theory with non-minimal coupling, which is a generalization of massive gravity with the broken translation invariance. Starting with arbitrary mass functions, we found that the theory can be constructed using the reference metric $\bar{f}_{\mu\nu} = P_{ab}\partial_\mu\phi^a\partial_\nu\phi^b$, where the projection tensor $P_{ab} = \eta_{ab} - \phi_a\phi_b/X$ manifestly eliminates one of the Stückelberg fields. This projected massive gravity has a different mass term from the dRGT theory, and we have proved the absence of the BD ghost. We then investigated the equations of motion for PMG and derived the cosmological background equations. Additionally, we provided comprehensive derivation details that were not present in the original references.

Finally, BH solutions within the framework of PMG were studied for the first time. We obtained the static spherically symmetric solutions in both Schwarzschild-de Sitter-like and Eddington–Finkelstein-like metrics, under the ansatz of $\phi^0 = \sqrt{1+r^2}$, $\phi^r = r$ and $\phi^\theta = \phi^\phi = 0$. We inevitably need to address time-dependent metrics when adhering to the unitary gauge in PMG. Unfortunately, assuming spherical symmetry does not yield successful results when attempting to use perturbation theory to study time-dependent matrices. It is worth further discussion of whether perturbation theory fails in this case or if there is simply no time-dependent spherically symmetric solution. Besides, as a translation-breaking theory, PMG exhibits time variation in coupling constants. Furthermore, this theory needs confirmation regarding the existence and/or necessity of a screening mechanism, as it is disconnected from the dRGT construction. The investigation of local gravity tests, including the need for a screening mechanism in the PMG model, would be necessary.

A

Einstein-Hilbert

A.1 Varying the Einstein-Hilbert action

The EH action in GR is the action that yields the Einstein field equations through the stationary-action principle. The general EH action can be considered as the gravitational part plus the matter content part

$$S_{EH} = S_g + S_m = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{M_{\mathrm{Pl}}^2}{2}R + \mathcal{L}_{\mathrm{matter}}\right), \tag{A.1}$$

where we defined $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor matrix and scalar volume element $\mathrm{d}^4x\sqrt{-g}$ as integration measure. A physical law (equations of motion) can be recovered by demanding that the variation of the action with respect to the inverse metric $g^{\mu\nu}$ be zero. By Leibniz rule, the variation yielding,

$$\begin{aligned} 0 &= \delta S_{EH} \\ &= \int \left[\frac{M_{\mathrm{Pl}}^2}{2}\frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} + \frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{matter}}\right)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\mathrm{d}^4x \\ &= \int \left[\frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right) + \frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{matter}}\right)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x. \end{aligned} \tag{A.2}$$

We shall calculate it term by term.

For the variation of the determinant $\delta\sqrt{-g}$, we start with

$$\delta g = \delta \det\left(g_{\mu\nu}\right) = g g^{\mu\nu} \delta g_{\mu\nu}, \tag{A.3}$$

where we used the corollary of Jacobi's formula: $\det e^{B} = e^{\mathrm{tr}(B)}$. Thus,

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\delta g_{\mu\nu}\right) = -\frac{1}{2}\sqrt{-g}\left(g_{\mu\nu}\delta g^{\mu\nu}\right), \tag{A.4}$$

where we used the fact that

$$g_{\mu\nu}\delta g^{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu}. \tag{A.5}$$

For the variation of the Ricci scalar δR, we have

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = R_{\mu\nu}\delta g^{\mu\nu} + \ g^{\mu\nu}\delta R_{\mu\nu}. \tag{A.6}$$

Therefore, varying the EH action (A.2) now becomes

$$\begin{aligned}\delta S_{EH} &= \int \left[\frac{M_{\mathrm{Pl}}^2}{2}\left(R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \frac{g^{\mu\nu}\delta R_{\mu\nu}}{\delta g^{\mu\nu}}\right) + \frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{matter}}\right)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x \\ &= \int \left[\frac{M_{\mathrm{Pl}}^2}{2}\left(R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{matter}}\right)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x + \text{boundary term},\end{aligned} \tag{A.7}$$

where for the boundary term we have

$$\begin{aligned}\text{boundary term} &\sim \int\ \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \\ &= \int\ \mathrm{d}^4x\sqrt{-g}\left[g^{\mu\nu}\left(\nabla_\alpha\delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu\delta\Gamma^\alpha_{\alpha\mu}\right)\right] \\ &= \int\ \mathrm{d}^4x\sqrt{-g}\left[\nabla_\alpha\left(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu}\right) - \nabla_\nu\left(g^{\mu\nu}\delta\Gamma^\alpha_{\alpha\mu}\right)\right] \\ &= \int\ \mathrm{d}^4x\sqrt{-g}\nabla_\alpha\left(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\beta_{\beta\mu}\right) \\ &= \int\ \mathrm{d}^4x\sqrt{-g}\nabla_\alpha V^\alpha.\end{aligned} \tag{A.8}$$

where we have used the Palatini identity $\delta R_{\sigma\nu} \equiv \delta R^\rho{}_{\sigma\rho\nu} = \nabla_\rho\left(\delta\Gamma^\rho_{\nu\sigma}\right) - \nabla_\nu\left(\delta\Gamma^\rho_{\rho\sigma}\right)$ in the second line and $\nabla_\alpha g^{\mu\nu} = 0$ in the third line. Hence, by virtue of Stokes' theorem

$$\int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\alpha V^\alpha = \int_{\Sigma=\partial\mathcal{M}} d\Sigma_\alpha V^\alpha, \tag{A.9}$$

which proves that the boundary term does not contribute to the equations of motion and can be dropped out. Notice that the boundary term is in general non-zero since it also depends on its partial derivatives $\partial_\lambda \delta g^{\mu\nu} \equiv \delta\partial_\lambda g^{\mu\nu}$, so-called Gibbons–Hawking–York boundary term.

Moving the discussion on varying the EH action. In order for $\delta S_{EH} = 0$ to lead to the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{M_{\rm Pl}^2} T_{\mu\nu}, \tag{A.10}$$

we therefore combing with Eq. A.7

$$\frac{M_{\rm PI}^2}{2}\left(R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\right) + \frac{1}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_{\rm matter}\right)}{\delta g^{\mu\nu}} = 0, \tag{A.11}$$

then define the stress-energy tensor for matter content,

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_{\rm matter}\right)}{\delta g^{\mu\nu}}. \tag{A.12}$$

A.2 Varying the action of PMG

The PMG action can take the form of,

$$S_{\rm PMG} = \int {\rm d}^4 x \sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}\left(G(X)R - \frac{6G_X^2}{G}[Y] + m^2 U\left(X, [Z], \left[Z^2\right], \left[Z^3\right]\right)\right) + \mathcal{L}_{\rm matter}\right]. \tag{A.13}$$

Similarly, the variation yielding

$$\begin{aligned}
0 &= \delta S_{\rm PMG} \\
&= \int \left[\frac{M_{\rm Pl}^2}{2}\left(\frac{G\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} - \frac{6G_X^2}{G}\frac{\delta(\sqrt{-g}[Y])}{\delta g^{\mu\nu}} + \frac{\delta\left(\sqrt{-g}m^2 U\right)}{\delta g^{\mu\nu}}\right) + \frac{\delta\left(\sqrt{-g}\mathcal{L}_{\rm matter}\right)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}{\rm d}^4 x \\
&= \int \left[\frac{M_{\rm Pl}^2}{2} G\left(R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \frac{g^{\mu\nu}\delta R_{\mu\nu}}{\delta g^{\mu\nu}}\right) - \frac{M_{\rm Pl}^2}{2}\frac{6G_X^2}{G}\frac{\delta(\sqrt{-g}[Y])}{\sqrt{-g}\delta g^{\mu\nu}} + \frac{M_{\rm Pl}^2}{2}\frac{\delta\left(\sqrt{-g}m^2 U\right)}{\sqrt{-g}\delta g^{\mu\nu}} - \frac{T_{\mu\nu}^{(\rm m)}}{2}\right]\delta g^{\mu\nu}\sqrt{-g}\ {\rm d}^4 x \\
&= \int \left[M_{\rm Pl}^2 G\left(G_{\mu\nu} + \frac{g^{\mu\nu}\delta R_{\mu\nu}}{\delta g^{\mu\nu}}\right) - M_{\rm Pl}^2 \frac{6G_X^2}{G}\frac{\delta(\sqrt{-g}[Y])}{\sqrt{-g}\delta g^{\mu\nu}} + M_{\rm Pl}^2\frac{\delta\left(\sqrt{-g}m^2 U\right)}{\sqrt{-g}\delta g^{\mu\nu}} - T_{\mu\nu}^{(\rm m)}\right]\delta g^{\mu\nu}\ {\rm d}^4 x \\
&= \int\ {\rm d}^4 x\left[M_{\rm Pl}^2 G(X) G_{\mu\nu} - M_{\rm Pl}^2\frac{6G_X^2}{G}\frac{\delta(\sqrt{-g}[Y])}{\sqrt{-g}\delta g^{\mu\nu}} + M_{\rm Pl}^2\frac{\delta\left(\sqrt{-g}m^2 U\right)}{\sqrt{-g}\delta g^{\mu\nu}} - T_{\mu\nu}^{(\rm m)}\right]\delta g^{\mu\nu} \\
&\quad + \int\ {\rm d}^4 x\ M_{\rm Pl}^2 G(X) g^{\mu\nu}\delta R_{\mu\nu}.
\end{aligned} \tag{A.14}$$

Unlike the boundary term in the EH case, we should be very careful with the treatment of the boundary term now since $G(X)$ is coordinate-dependent. Thus

$$
\begin{aligned}
&\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2 G(X) g^{\mu\nu}\delta R_{\mu\nu} \\
=&\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2 G(X)\nabla_\alpha\left(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\beta_{\beta\mu}\right) \\
=&\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2 G(X)\nabla_\alpha V^\alpha \\
=&\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2 \left(\nabla_\alpha\left(V^\alpha G\right) - V^\alpha\nabla_\alpha G\right) \\
=&-\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2 V^\alpha\nabla_\alpha G \\
=&-\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2 \left[g^{\mu\nu}\left(\frac{1}{2}g^{\alpha\lambda}\left(\nabla_\nu\delta g_{\mu\lambda} + \nabla_\mu\delta g_{\nu\lambda} - \nabla_\lambda\delta g_{\mu\nu}\right)\right) - g^{\mu\alpha}\left(\frac{1}{2}g^{\beta\lambda}\left(\nabla_\mu\delta g_{\beta\lambda} + \nabla_\beta\delta g_{\mu\lambda} - \nabla_\lambda\delta g_{\mu\beta}\right)\right)\right]\nabla_\alpha G \\
=&-\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2 \left[g_{\mu\nu}\left(\nabla^\alpha\delta g^{\mu\nu}\right)\nabla_\alpha G - \left(\nabla_\mu\delta g^{\mu\nu}\right)\nabla_\nu G\right] \\
=&\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2 \left(g_{\mu\nu}\nabla^\alpha\nabla_\alpha G - \nabla_\mu\nabla_\nu G\right)\delta g^{\mu\nu},
\end{aligned}
\tag{A.15}
$$

where we have used the integration by parts many times. Then, for the term,

$$
\begin{aligned}
&\int \mathrm{d}^4x\left[-M_{\mathrm{Pl}}^2\frac{6G_X^2}{G}\frac{\delta(\sqrt{-g}[Y])}{\sqrt{-g}\delta g^{\mu\nu}}\right]\delta g^{\mu\nu} \\
=&-\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2\frac{6G_X^2}{G}\left[\frac{-\frac{1}{2}\sqrt{-g}\left(g_{\mu\nu}\delta g^{\mu\nu}\right)[Y] + \sqrt{-g}\delta(g^{\mu\nu}\tilde{f}_{\mu\nu})}{\sqrt{-g}}\right] \\
=&-\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2\frac{6G_X^2}{G}\left(\tilde{f}_{\mu\nu} - \frac{1}{2}[Y]g_{\mu\nu}\right)\delta g^{\mu\nu}.
\end{aligned}
\tag{A.16}
$$

For the term

$$
\begin{aligned}
&\int \mathrm{d}^4x\left[M_{\mathrm{Pl}}^2\frac{\delta\left(\sqrt{-g}m^2U\right)}{\sqrt{-g}\delta g^{\mu\nu}}\right]\delta g^{\mu\nu} \\
=&\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2\left[\frac{-\frac{1}{2}\sqrt{-g}\left(g_{\mu\nu}\delta g^{\mu\nu}\right)m^2U + \sqrt{-g}m^2\delta(U)}{\sqrt{-g}}\right] \\
=&\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2 m^2\left[-\frac{1}{2}g_{\mu\nu}U\delta g^{\mu\nu} + \delta\left(U\left(X,[Z],\left[Z^2\right],\left[Z^3\right]\right)\right)\right] \\
=&\int \mathrm{d}^4x\ M_{\mathrm{Pl}}^2 m^2\left[-\frac{1}{2}g_{\mu\nu}U + U_{[Z]}\bar{f}_{\mu\nu} + 2U_{[Z^2]}Z^\rho{}_{(\mu}\bar{f}_{\nu)\rho} + 3U_{[Z^3]}Z^\rho{}_\sigma Z^\sigma{}_{(\mu}\bar{f}_{\nu)\rho}\right]\delta g^{\mu\nu} \\
=&-\int \mathrm{d}^4x\ T^{\mathrm{mass}}_{\mu\nu}\delta g^{\mu\nu},
\end{aligned}
\tag{A.17}
$$

where we defined $T^{\mathrm{mass}}_{\mu\nu}$ as the effective energy-momentum tensor for massive graviton

$$
T^{\mathrm{mass}}_{\mu\nu} = M_{\mathrm{Pl}}^2 m^2\left(\frac{1}{2}g_{\mu\nu}U - U_{[Z]}\bar{f}_{\mu\nu} - 2U_{[Z^2]}Z^\rho{}_{(\mu}\bar{f}_{\nu)\rho} - 3U_{[Z^3]}Z^\rho{}_\sigma Z^\sigma{}_{(\mu}\bar{f}_{\nu)\rho}\right),
\tag{A.18}
$$

A

with $U_{[Z^n]} \equiv \frac{\partial U}{\partial [Z^n]}$.

As a result, Eq. A.14 is reduced to

$$\frac{\delta S_{\rm PMG}}{\delta g^{\mu\nu}} = \int M_{\rm Pl}^2 \left[G(X) G_{\mu\nu} - \left(\tilde{f}_{\mu\nu} - \frac{1}{2}[Y] g_{\mu\nu} \right) + g_{\mu\nu} \Box G - \nabla_\mu \nabla_\nu G - \frac{T_{\mu\nu}^{(\rm mass)} + T_{\mu\nu}^{(\rm m)}}{M_{\rm Pl}^2} \right] \delta g^{\mu\nu} \, {\rm d}^4 x. \tag{A.19}$$

Thus, $\delta S_{\rm PMG}/\delta g^{\mu\nu} = 0$ leads to the modified Einstein equations in PMG,

$$M_{\rm Pl}^2 \left[G(X) G_{\mu\nu} - \nabla_\mu \nabla_\nu G(X) + g_{\mu\nu} \Box G(X) - \frac{6\left(G_X(X)\right)^2}{G(X)} \left(\tilde{f}_{\mu\nu} - \frac{1}{2}[Y] g_{\mu\nu} \right) \right] = T_{\mu\nu}^{(\rm mass)} + T_{\mu\nu}^{(\rm m)}. \tag{A.20}$$

B

Cosmology background equations

B.1 The Friedmann equations

The Friedmann equations are a set of equations in physical cosmology that govern the expansion of space in homogeneous and isotropic models of the universe within the context of GR. We shall derive these from Einstein's field equations for the FLRW metric and a perfect fluid with a given mass density ρ and pressure p. The stress-energy tensor for the matter is defined as

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu}, \tag{B.1}$$

which obeys the standard conservation law $\nabla_{\mu}T^{\mu\nu} = 0$. The 4-velocity of cosmological observers is given by,

$$u^{\mu} = (1, 0, 0, 0), \tag{B.2}$$

satisfying $u_{\mu}u^{\mu} = -1$. Then the energy-momentum tensor can take the form

$$T^{\mu}_{\ \nu} = g_{\mu\alpha}T^{\alpha\nu} = \mathrm{diag}(-\rho, p, p, p). \tag{B.3}$$

The FLRW metric in polar coordinates (t, r, θ, ϕ) takes the form

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = -dt^2 + a^2(t)\left[\frac{dr^2}{1+\kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{B.4}$$

where κ is the curvature parameter that $\kappa > 0$ corresponding to an open universe. We now can calculate the non-zero Ricci tensor in this metric,

$$\begin{aligned}
R_{tt} &= -3\frac{\ddot{a}}{a} = 3g_{tt}\frac{\ddot{a}}{a}, \\
R_{rr} &= \frac{a^2}{1+\kappa r^2}\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\kappa}{a^2}\right) = g_{rr}\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\kappa}{a^2}\right), \\
R_{\theta\theta} &= a^2r^2\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\kappa}{a^2}\right) = g_{\theta\theta}\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\kappa}{a^2}\right), \\
R_{\phi\phi} &= a^2r^2\sin^2\theta\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\kappa}{a^2}\right) = g_{\phi\phi}\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\kappa}{a^2}\right).
\end{aligned} \tag{B.5}$$

B

Thus the Ricci scalar

$$R = g^{\mu\mu}R_{\mu\nu} = 3\frac{\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\kappa}{a^2}\right) = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 - \frac{\kappa}{a^2}\right). \tag{B.6}$$

From Einstein's field equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_{\rm Pl}^2}T_{\mu\nu}, \tag{B.7}$$

we can obtain

$$R_{tt} - \frac{1}{2}g_{tt}R = \frac{1}{M_{\rm Pl}^2}T_{tt}, \tag{B.8}$$

from time components, which leads to the first Friedmann equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{\rm Pl}^2}\rho + \frac{\kappa}{a^2}, \tag{B.9}$$

where we defined the Hubble parameter $H \equiv \dot{a}/a$. In a similar way, the spatial component of the Einstein equations leads to

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 - \frac{\kappa}{a^2} = -\frac{p}{M_{\rm Pl}^2}. \tag{B.10}$$

we can rewrite this equation using the Hubble parameter time derivative

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2. \tag{B.11}$$

Thus, Eq. B.10 becomes,

$$\dot{H} = -\frac{\rho + p}{2M_{\rm Pl}^2} - \frac{\kappa}{a^2}, \tag{B.12}$$

which so-called second Friedmann equation. Combing the first and second Friedmann equations, one can obtain the acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\text{Pl}}^2}(\rho + 3p), \tag{B.13}$$

since it contains the expansion rate of the universe. We now further combine the Friedmann equation in another way by taking the partial time derivative of Eq. B.10, which gives

$$2H\dot{H} - \frac{1}{3M_{\text{Pl}}^2}\dot{\rho} = -2\kappa\frac{\dot{a}}{a^3}, \tag{B.14}$$

B

and then plugging the κ from the Eq. B.12, i.e.,

$$-\kappa = a^2\dot{H} + \frac{a^2(\rho + p)}{2M_{\text{Pl}}^2}, \tag{B.15}$$

which gives the continuity equation,

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{B.16}$$

We can also derive this from the conservation of the energy-momentum tensor,

$$\nabla_\mu T^\mu_{\ \nu} = \partial_\mu T^\mu_{\ \nu} + \Gamma^\mu_{\mu\alpha}T^\alpha_{\ \nu} - \Gamma^\alpha_{\mu\nu}T^\mu_{\ \alpha} = 0. \tag{B.17}$$

Consider the $\nu = t$ component of this equation and remember everything depends only on t and not on the spatial coordinates:

$$\begin{aligned}\nabla_\mu T^\mu_{\ t} =& \partial_\mu T^\mu_{\ t} + \Gamma^\mu_{\mu\alpha}T^\alpha_{\ t} - \Gamma^\alpha_{\mu t}T^\mu_{\ \alpha} \\ =& \partial_t T^t_{\ t} + \Gamma^\mu_{\mu t}T^t_{\ t} - \Gamma^t_{tt}T^t_{\ t} - \Gamma^j_{it}T^i_{\ j} \\ =& -\left[\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p)\right] \\ =& 0.\end{aligned} \tag{B.18}$$

The remaining components,

$$\nabla_\mu T^\mu_{\ i} = \partial_\mu T^\mu_{\ i} + \Gamma^\mu_{\mu\alpha}T^\alpha_{\ i} - \Gamma^\alpha_{\mu i}T^\mu_{\ \alpha} = 0. \tag{B.19}$$

Hence, we again obtain the continuity equation.

B.2 Background equations in PMG

The time-like components $(0,0)$ of the modified Einstein equation gives

$$G(X)G_{00} - \nabla_0\nabla_0 G(X) + g_{00}\nabla_\alpha\nabla^\alpha G(X) - \frac{6\left(G_X(X)\right)^2}{G(X)}\left(\tilde{f}_{00} - \frac{1}{2}[Y]g_{00}\right) = \frac{T^{(\mathrm{mass})}_{00}}{M^2_{\mathrm{Pl}}} + \frac{T^{(\mathrm{m})}_{00}}{M^2_{\mathrm{Pl}}}. \tag{B.20}$$

For the first term of LHS

$$G(X)G_{00} = 3G(X)\left(H^2 - \frac{\kappa}{a^2}\right). \tag{B.21}$$

B

For the term

$$\begin{aligned} &-\nabla_0\nabla_0 G(X) + g_{00}\nabla_\alpha\nabla^\alpha G(X)\\ =&-\partial_0\partial_0 G + \Gamma^\alpha_{00}\partial_\alpha G - \partial_\alpha\partial^\alpha G - \Gamma^\alpha_{\beta\alpha}\partial^\beta G\\ =&-\partial_0\partial_0 G + \partial_0\partial_0 G - \Gamma^\alpha_{0\alpha}\partial^0 G = \Gamma^\alpha_{0\alpha}\dot{G} = 3H\dot{G}. \end{aligned} \tag{B.22}$$

For the term

$$\begin{aligned} &-\frac{6\left(G_X(X)\right)^2}{G(X)}\left(\tilde{f}_{00} - \frac{1}{2}[Y]g_{00}\right)\\ =&-\frac{6}{G}\left(\frac{\partial G}{\partial t}\frac{\partial t}{\partial X}\right)^2\left(-\frac{1}{2}(f\dot{f})^2\right)\\ =&-\frac{3}{2G}\left(\frac{\dot{G}}{f\dot{f}}\right)^2\left(-\frac{1}{2}(f\dot{f})^2\right) = \frac{3\dot{G}^2}{4G}. \end{aligned} \tag{B.23}$$

For the RHS of the Eq. B.20

$$\frac{T^{(\mathrm{mass})}_{00}}{M^2_{\mathrm{Pl}}} + \frac{T^{(\mathrm{m})}_{00}}{M^2_{\mathrm{Pl}}} = \frac{-\frac{1}{2}m^2 M^2_{\mathrm{Pl}} U}{M^2_{\mathrm{Pl}}} + \frac{\rho}{M^2_{\mathrm{Pl}}} = \frac{\rho_g}{M^2_{\mathrm{Pl}}} + \frac{\rho}{M^2_{\mathrm{Pl}}}, \tag{B.24}$$

where we defined the effective energy density $\rho_g = -\frac{1}{2}m^2 M^2_{\mathrm{Pl}} U$. Therefore Eq. B.20 can be simplified to

$$3G\left[\left(H + \frac{\dot{G}}{2G}\right)^2 - \frac{\kappa}{a^2}\right] = \frac{\rho}{M^2_{\mathrm{Pl}}} + \frac{\rho_g}{M^2_{\mathrm{Pl}}}. \tag{B.25}$$

Similarly, if we consider the space-like components (i,i) of the modified Einstein equation

$$G(X)G_{ii} - \nabla_i\nabla_i G(X) + g_{ii}\nabla_\alpha\nabla^\alpha G(X) - \frac{6\left(G_X(X)\right)^2}{G(X)}\left(\tilde{f}_{ii} - \frac{1}{2}[Y]g_{ii}\right) = \frac{T^{(\mathrm{mass})}_{ii}}{M^2_{\mathrm{Pl}}} + \frac{T^{(\mathrm{m})}_{ii}}{M^2_{\mathrm{Pl}}}. \tag{B.26}$$

For the first term

$$G(X)G_{ii} = g_{ii}G(X)\left(-2\frac{\ddot{a}}{a} - H^2 + \frac{\kappa}{a^2}\right). \tag{B.27}$$

For the term

$$\begin{aligned}
&-\nabla_i\nabla_i G(X)+g_{ii}\nabla_\alpha\nabla^\alpha G(X)\\
=&-\partial_i\partial_i G+\Gamma^\alpha_{ii}\partial_\alpha G+g_{ii}\left(\partial_\alpha\partial^\alpha G+\Gamma^\alpha_{\beta\alpha}\partial^\beta G\right)\\
=&\,\Gamma^0_{ii}\partial^0 G+g_{ii}\left(\partial_0\partial^0 G+\Gamma^\alpha_{0\alpha}\partial^0 G\right)\\
=&\,g_{ii}H\dot{G}+g_{ii}\left(-\ddot{G}-3H\dot{G}\right)
\end{aligned} \tag{B.28}$$

For the term

$$\begin{aligned}
&-\frac{6\left(G_X(X)\right)^2}{G(X)}\left(\tilde{f}_{ii}-\frac{1}{2}[Y]g_{ii}\right)\\
=&-\frac{6}{G}\left(\frac{\partial G}{\partial t}\frac{\partial t}{\partial X}\right)^2\left(-\frac{1}{2}(f\dot{f})^2 g_{ii}\right)\\
=&-\frac{3}{2G}\left(\frac{\dot{G}}{f\dot{f}}\right)^2\left(-\frac{1}{2}(f\dot{f})^2 g_{ii}\right)\\
=&\,g_{ii}\frac{3\dot{G}^2}{4G}.
\end{aligned} \tag{B.29}$$

For the RHS of the Eq. B.26

$$\begin{aligned}
&\frac{T^{(\mathrm{mass})}_{ii}}{M^2_{\mathrm{Pl}}}+\frac{T^{(\mathrm{m})}_{ii}}{M^2_{\mathrm{Pl}}}\\
=&\frac{1}{M^2_{\mathrm{Pl}}}\left[\frac{1}{2}M^2_{\mathrm{Pl}}m^2\left(g_{ii}U-2U_{[Z]}\bar{f}_{ii}-4U_{[Z^2]}Z^\rho{}_{(i}\bar{f}_{i)\rho}-6U_{[Z^3]}Z^\rho{}_\sigma Z^\sigma{}_{(i}\bar{f}_{i)\rho}\right)\right]\\
=&g_{ii}\frac{1}{M^2_{\mathrm{Pl}}}\left[\frac{1}{2}M^2_{\mathrm{Pl}}m^2\left(U-2U_{[Z]}\frac{f^2\kappa}{a^2}-4U_{[Z^2]}\left(\frac{f^2\kappa}{a^2}\right)^2-6U_{[Z^3]}\left(\frac{f^2\kappa}{a^2}\right)^3\right)\right]\\
=&g_{ii}\frac{1}{M^2_{\mathrm{Pl}}}\left[\frac{1}{2}m^2M^2_{\mathrm{Pl}}\left(U-2\xi^2U_{[Z]}-4\xi^4U_{[Z^2]}-6\xi^6U_{[Z^3]}\right)\right]
\end{aligned} \tag{B.30}$$

where we defined $\xi=\frac{\sqrt{\kappa}f}{a}$. Therefore Eq. B.26 can be simplified to

$$G(X)\left(-2\frac{\ddot{a}}{a}-H^2+\frac{\kappa}{a^2}\right)-2H\dot{G}-\ddot{G}+\frac{3\dot{G}^2}{4G}=\frac{p_g}{M^2_{\mathrm{Pl}}}, \tag{B.31}$$

where we also defined effective energy pressure p_g

$$p_g=\frac{1}{2}m^2M^2_{\mathrm{Pl}}\left(U-2\xi^2U_{[Z]}-4\xi^4U_{[Z^2]}-6\xi^6U_{[Z^3]}\right). \tag{B.32}$$

Eq. B.25 plus Eq. B.31, one can obtain

$$-2G\left[\partial_t\left(H+\frac{\dot{G}}{2G}\right)+\frac{\kappa}{a^2}\right]+\dot{G}\left(H+\frac{\dot{G}}{2G}\right)=\frac{\rho}{M^2_{\mathrm{Pl}}}+\frac{\rho_g+p_g}{M^2_{\mathrm{Pl}}}. \tag{B.33}$$

Following the classical way to obtain the continuity equation, we can obtain the expression for κ

directly from Eq. B.31

$$\kappa = -\frac{a^2}{G}\left(\frac{p_g}{M_{\mathrm{Pl}}^2} + 2G\dot{H} + 3GH^2 + 2H\dot{G} + \ddot{G} - \frac{3\dot{G}^2}{4G}\right). \tag{B.34}$$

Secondly, we can use Eq. B.25 and Eq. B.31 to eliminate the κ to obtain the expression for $\dot{H}$

$$\dot{H} = -\frac{1}{6G}\frac{3p_g + \rho + \rho_g}{M_{\mathrm{Pl}}^2} - \frac{H\dot{G} + \ddot{G}}{2G} + \frac{\dot{G}^2}{2G^2} - H^2. \tag{B.35}$$

Finally, we can take the time derivative of the Eq. B.25 and combine it with Eq. B.34 and background matter equation (3.35), one can obtain

$$\dot{\rho}_g + 3H\left(\rho_g + p_g\right) - \frac{\dot{G}}{2G}\left(\rho_g - 3p_g + \rho\right) = 0. \tag{B.36}$$

Alternatively, we can obtain this modified continuity equation from the Stückelberg equation 3.36. From the LHS of the Stückelberg equation, we have

$$\begin{aligned}
&\nabla^\mu\left(\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\right)\\
=&\nabla^\mu\left[M_{\mathrm{Pl}}^2\left[G(X)G_{\mu\nu} - \nabla_\mu\nabla_\nu G(X) + g_{\mu\nu}\nabla_\alpha\nabla^\alpha G(X) - \frac{6\left(G_X(X)\right)^2}{G(X)}\left(\tilde{f}_{\mu\nu} - \frac{1}{2}[Y]g_{\mu\nu}\right)\right] - T_{\mu\nu}^{(\mathrm{mass})} - T_{\mu\nu}^{(\mathrm{m})}\right]\\
=&M_{\mathrm{Pl}}^2\left[G_{\mu\nu}\nabla^\mu G(X) - R_{\mu\nu}\nabla^\mu G(X) - \nabla^\mu\left(\left(\frac{3}{2G}\frac{\dot{G}^2}{(f\dot{f})^2}\right)\left(\tilde{f}_{\mu\nu} - \frac{1}{2}(f\dot{f})^2 g_{\mu\nu}\right)\right)\right] - \nabla^\mu T_{\mu\nu}^{(\mathrm{mass})}\\
=&M_{\mathrm{Pl}}^2\left[G_{\mu\nu}\nabla^\mu G(X) - R_{\mu\nu}\nabla^\mu G(X) - \nabla^\mu\left(\frac{3}{2G}\frac{\dot{G}^2}{(f\dot{f})^2}\tilde{f}_{\mu\nu}\right) + \frac{3}{4}\nabla_\nu\left(\frac{\dot{G}^2}{G}\right)\right] - \nabla^\mu T_{\mu\nu}^{(\mathrm{mass})}\\
=&0.
\end{aligned} \tag{B.37}$$

Here, we only consider the time-like components, i.e. $\nu = 0$,

$$
\begin{aligned}
&M_{\rm Pl}^2\left[G_{00}\nabla^0 G(X) - R_{00}\nabla^0 G(X) - \nabla^\mu\left(\frac{3}{2G}\frac{\dot{G}^2}{(f\dot{f})^2}\tilde{f}_{\mu0}\right) + \frac{3}{4}\nabla_0\left(\frac{\dot{G}^2}{G}\right)\right] - \nabla^\mu T^{(\rm mass)}_{\mu0}\\
=&M_{\rm Pl}^2\left[-3\left(H^2-\frac{\kappa}{a}\right)\dot{G} - 3\frac{\ddot{a}}{a}\dot{G} - \frac{9H\dot{G}^2}{2G} - \frac{3}{4}\partial_0\left(\frac{\dot{G}^2}{G}\right)\right] + \partial_0 T^{(\rm mass)}_{00} - \Gamma^\mu_{\mu\alpha}T^{\alpha\ (\rm mass)}_{\ 0} + \Gamma^\alpha_{\mu0}T^{\mu\ (\rm mass)}_{\ \alpha}\\
=&M_{\rm Pl}^2\left[-3\left(\frac{\ddot{a}}{a}+H^2-\frac{\kappa}{a}\right)\dot{G} - \frac{3}{4}\left(\frac{2\dot{G}\ddot{G}}{G} - \frac{\dot{G}^3}{G^2}\right) - \frac{1}{2}m^2\dot{U} - \frac{9H\dot{G}^2}{2G}\right] - \Gamma^\mu_{\mu0}T^{0\ (\rm mass)}_{\ 0} + \Gamma^i_{j0}T^{j\ (\rm mass)}_{\ i}\\
=&M_{\rm Pl}^2\left[-3\left(\frac{\ddot{a}}{a}+H^2-\frac{\kappa}{a}\right)\dot{G} - \frac{3}{4}\left(\frac{2\dot{G}\ddot{G}}{G} - \frac{\dot{G}^3}{G^2}\right) - \frac{1}{2}m^2\dot{U} - \frac{9H\dot{G}^2}{2G}\right] - 3HT^{0\ (\rm mass)}_{\ 0} + HT^{i\ (\rm mass)}_{\ i}\\
=&M_{\rm Pl}^2\left[-3\left(\frac{\ddot{a}}{a}+H^2-\frac{\kappa}{a}\right)\dot{G} - \frac{3}{4}\left(\frac{2\dot{G}\ddot{G}}{G} - \frac{\dot{G}^3}{G^2}\right) - \frac{1}{2}m^2\dot{U} - \frac{9H\dot{G}^2}{2G}\right] + 3H(\rho_g + p_g)\\
=&M_{\rm Pl}^2\left[-3\left(\frac{\ddot{a}}{a}+H^2-\frac{\kappa}{a}\right)\dot{G} - \frac{3}{4}\left(\frac{2\dot{G}\ddot{G}}{G} - \frac{\dot{G}^3}{G^2}\right) - \frac{9H\dot{G}^2}{2G}\right] + \dot{\rho}_g + 3H(\rho_g + p_g)\\
=&\dot{\rho}_g + 3H\left(\rho_g + p_g\right) - \frac{\dot{G}}{2G}\left(\rho_g - 3p_g + \rho\right),
\end{aligned}
\tag{B.38}
$$

where we combined Eq. B.25 and Eq. B.31 in the last equality. Again, we obtain the continuity equation as presented in (B.36).

C

Black hole solutions

C.1 The Coulomb solution

Before discussing the spherically symmetric metric, we start with the Coulomb solution as a warm-up. In electrodynamics, the electromagnetic four-potential $A_\mu(x^\nu)$ only depends on $r^2 = x^i x_i$ and t. Note that other components A_i in polar coordinates would transform non-trivially under spatial rotations, so spherical symmetry $SO(3)$ requires A_ϕ and A_θ set to zero. Thus,

$$A_i = A_r(r,t)\hat{r}, \quad A_0 = A_0(r,t).$$

The components of the electromagnetic field $F_{\mu\nu}$ in spherically symmetric ansatz have the structure $A_i = \frac{x^i}{r} A_r = \partial_i r A_r$, so $\partial_{[i} A_{j]} = 0 \rightarrow F_{ij} = 0$. Thus, we have the following results

$$\begin{aligned} B_i &= \frac{1}{2}\epsilon_{ijk} F^{jk} = 0, \\ F^{0i} &= E^i, \\ E_r &= (\partial_r A_0 - \partial_0 A_r) = \left(A_0' - \dot{A}_r\right), \\ E_i &= E_r \hat{r}. \end{aligned}$$

Here we are restricting to a situation without a spherically symmetric magnetic field B_r. Substituting the spherically symmetric ansatz into the Maxwell action, we have

$$I_{\text{Maxwell}} = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int d^4x \left(\mathbf{E}^2 - \mathbf{B}^2\right), \tag{C.1}$$

the reduced action

$$\begin{aligned} I^{\text{red}}_{\text{Maxwell}} &= \frac{1}{2}\int sin\theta d\theta d\phi \int r^2 drdtE_r^2 \\ &= 2\pi \int \left(\dot{A}_r - A_0'\right)^2 r^2 drdt. \end{aligned} \tag{C.2}$$

where 2π comes from $\int_{S^2} d^2\Omega$. Now we can vary the reduced action $I^{\text{red}}_{\text{Maxwell}}$ to obtain a contentful field equation. Notice that here we choose to vary the A_r and A_0 before imposing the Coulomb gauge, in order to obtain the constancy in time of the charge.

For the Eq. C.2, we have

$$\mathcal{L} = 2\pi \left(\dot{A}_r - A_0'\right)^2 r^2$$

and from Euler-Lagrange equations, we have

$$\frac{\delta \mathcal{L}}{\delta A_r} = \frac{\partial \mathcal{L}}{\partial A_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_r\right)} = 0. \tag{C.3}$$

where $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_r)}$ can be considered as $\partial_t \frac{\partial \mathcal{L}}{\partial\left(\dot{A}_r\right)} + \partial_r \frac{\partial \mathcal{L}}{\partial A_r'}$. Since $\frac{\partial \mathcal{L}}{\partial A_r} = 0$ and $\frac{\partial \mathcal{L}}{\partial A_r'} = 0$. Eq. C.3 becomes

$$0 = \frac{\delta \mathcal{L}}{\delta A_r} = -\partial_t \frac{\partial \mathcal{L}}{\partial\left(\dot{A}_r\right)} = -4\pi r^2 \partial_t \left(\dot{A}_r - A_0'\right)$$

Therefore, δA_r variation yields

$$\frac{\delta I^{\text{red}}_{\text{Maxwell}}}{\delta A_r} = 4\pi r^2 \frac{\partial}{\partial t}\left(A_0' - \dot{A}_r\right) = 0. \tag{C.4}$$

Similarly, δA_0 variation yields

$$\frac{\delta I^{\text{red}}_{\text{Maxwell}}}{\delta A_0} = 4\pi \frac{\partial}{\partial r}\left[r^2\left(A_0' - \dot{A}_r\right)\right] = 0. \tag{C.5}$$

We impose Coulomb gauge $A_r = 0$. Eq. C.5 can be simplified as

$$\left.\frac{I^{\text{red}}_{\text{Maxwell}}}{\delta A_0}\right|_{A_r=0} = 4\pi \frac{\partial}{\partial r}\left(r^2 A_0'\right) = 0. \tag{C.6}$$

According to the boundary condition $A_0 \to 0$ as $r \to \infty$, we can obtain the solution of this equation

$$A_0 = -\frac{q(t)}{r}. \tag{C.7}$$

Now, Eq. C.4 becomes

$$\left.\frac{I^{\text{red}}_{\text{Maxwell}}}{\delta A_r}\right|_{A_r=0} = 4\pi r^2 \dot{A}_0' = 4\pi \dot{q} = 0, \tag{C.8}$$

i.e. $\dot{q} = 0$, which indicates that charge is time-independent.

C.2 Spherically symmetric metric ansatz

We move on to the analogous problem in GR. A spherically symmetric metric in Cartesian coordinates takes the form

$$ds^2 = A(r,t)dx^i dx^i + B(r,t)x^i x^j dx^i dx^j,$$

where $r^2 = x^k x^k$. Then transform to the polar coordinates using

$$dx^i dx^i = dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2) = dr^2 + r^2 d\Omega^2, \quad x^i dx^i = r dr,$$

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where $d\Omega^2$ is the usual $SO(3)$ unit 2-sphere invariant element. Including the time coordinate, the corresponding spherically symmetric spacetime metric is

$$ds^2 = -D(r,t)dt^2 + \tilde{A}(r,t)dr^2 + E(r,t)r^2 d\Omega^2 + 2C(r,t)r dr dt,$$

where $\tilde{A}(r,t) = A(r,t) + B(r,t)r^2$ and note that A and B only occur in combination. Moreover, the residual coordinate transformation can modify these metric functions without changing any physical content. This is a type of gauge change also known as "proper" gauge change. The $r \to \tilde{r}(r,t)$ coordinate change can be used to set

$$E = 1$$

and further redefining other metric functions as

$$D = ab^2, \quad \tilde{A} = A + Br^2 = a^{-1}, C = bf/r,$$

where a, b and f are undetermined functions of r and t. The spherically symmetric spacetime metric then becomes

$$ds^2 = -ab^2 dt^2 + a^{-1} dr^2 + r^2 d\Omega^2 + 2bf dr dt.$$

Similar to the analysis in the Coulomb solution case, we can not set $f(r,t) = 0$ before variation, which can lead to Birkhoff's theorem (see Ref. [76]) for more details). However, we will use the gauge freedom to set $f = 0$ and begin with the more instructive choice that dropping the off-diagonal elements of the metric. Substituting this ansatz into the Einstein-Hilbert (EH) action

$I_{EH} = \int d^4x\sqrt{-g}R$ (here we set $\frac{1}{2\kappa} = 1$ for convenience and so $\sqrt{-g} = br^2 \sin\theta$), we can obtain the reduced action

$$\begin{aligned} I^{\text{red}}_{\text{EH}} &= \int sin\theta d\theta d\phi \int br^2 drdtR \\ &= 4\pi \int drdt \left[-b\left(-2+2a+4ra'+r^2a''\right) + r\left(-\left(4a+3ra'\right)b' - 2rab''\right)\right] \\ &\quad + \frac{r^2\left(a\dot{a}\dot{b} + b\left(2\dot{a}^2 - a\ddot{a}\right)\right)}{a^3b^2} \\ &= 8\pi \int drdt\,(b + rab') \\ &= 8\pi \int drdt\; b(r-ar)', \end{aligned} \tag{C.9}$$

where we have already used integrals by parts and omitted the boundary term. The respective variations then yield

$$\begin{aligned} \frac{\delta I^{\text{red}}_{\text{EH}}}{\delta a} &= rb' = 0 \Rightarrow b = b_0; \\ \frac{\delta I^{\text{red}}_{\text{EH}}}{\delta b} &= (r-ar)' = 0 \Rightarrow a = 1 - \frac{2G_N M}{r}. \end{aligned} \tag{C.10}$$

In fact, b_0 and M could be time-dependence but we won't consider that here. $b_0(t)$ can be set to 1 by fixing the remaining $t \to t'(t)$ gauge freedom. Finally, we arrive at the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right)dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{C.11}$$

where M can be considered as the mass of Schwarzschild BH.

C.2.1 From Einstein to Schwarzschild

Now, we would like to present another approach to arrive at the Schwarzschild solution, which might be more straightforward. Let's continue with (t, r, θ, ϕ) coordinates. The Schwarzschild solution is assumed to be static and spherically symmetric, describing the vacuum spacetime. The term "static" denotes a state where all metric components are time-independent, i.e. $\frac{\partial}{\partial t} g_{\mu\nu} = 0$. Moreover, all metric components should be unchanged under a time-reversal, i.e. $g_{0\mu} = g_{\mu 0} = 0$ for $\mu \neq 0$. Spherically symmetric means that the geometry of the spacetime is unchanged under $(t, r, \theta, \phi) \to (t, r, \theta, -\phi)$ and $(t, r, \theta, \phi) \to (t, r, -\theta, \phi)$, i.e., $g_{2\mu} = g_{\mu 2} = 0$ for $\mu \neq 2$ and $g_{3\mu} = g_{\mu 3} = 0$ for $\mu \neq 3$. To sum up, the metric can be written as

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2. \tag{C.12}$$

Across the hypersurfaces defined by constant values of t and r, a prerequisite is established, required that the metric should be a 2-sphere. Thus,

$$g_{22} = r^2, \quad g_{33} = r^2 \sin^2\theta. \tag{C.13}$$

According to the spherical symmetry, g_{00} and g_{11} should only depend on r on each radial line. We can define

$$g_{00} = A(r), \quad g_{11} = B(r). \tag{C.14}$$

hence, the metric takes the form

$$ds^2 = A(r)dt^2 + B(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{C.15}$$

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In order to determine the functions $A(r)$ and $B(r)$, we need to use the last but the most important condition, 'vacuum'. We first look at the Einstein field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{C.16}$$

where $T_{\mu\nu}$ is the stress-energy tensor and $G_{\mu\nu}$ is the Einstein tensor, is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{C.17}$$

The cosmological constant Λ here is set to 0 for the Schwarzschild case. The vacuum solution satisfies the equations $T_{\mu\nu} = 0$, i.e.,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \tag{C.18}$$

Taking trace for the Eq. C.18, one can obtain

$$R_{\alpha\beta} = 0 \Rightarrow \partial_\rho \Gamma^\rho_{\beta\alpha} - \partial_\beta \Gamma^\rho_{\rho\alpha} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\beta\alpha} - \Gamma^\rho_{\beta\lambda}\Gamma^\lambda_{\rho\alpha} = 0. \tag{C.19}$$

The non-zero Ricci tensor gives the following equations

$$R_{tt} = \frac{rBA'^2 + A\left(rA'B' - 2B\left(2A' + rA''\right)\right)}{4rAB^2}, \tag{C.20}$$

$$R_{rr} = \frac{A\left(4A + rA'\right)B' + rB\left(A'^2 - 2AA''\right)}{4rA^2B}, \tag{C.21}$$

$$R_{\theta\theta} = \frac{1}{2}\left(2 - \frac{2 + \frac{rA'}{A}}{B} + \frac{rB'}{B^2}\right), \tag{C.22}$$

$$R_{\phi\phi} = \sin^2(\theta) R_{\theta\theta}. \tag{C.23}$$

Combining Eq. C.20 and Eq. C.21 eliminates A'' and obtains

$$AB' + BA' = 0. \tag{C.24}$$

Substituting this result in Eq. C.22, one can obtain the general solutions

$$A = c_2\left(1 + \frac{c_1}{r}\right), \quad B = \left(1 + \frac{c_1}{r}\right)^{-1}, \tag{C.25}$$

where c_1 and c_2 are yet undetermined constants. Now we would like to obtain c_1 and c_2 by using the Newton limit. Recalled the Newtonian frame in GR, in which the line element of spacetime takes the form

$$\begin{aligned} \mathrm{d}s^2 &= -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) + \dots \\ &= -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)\left(\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega_2^2\right) + \dots \\ &= -(1+2\Phi)dt^2 + (1-2\Phi)dr^2 + r^2 d\Omega_2^2 + \dots \\ &\approx -(1+2\Phi)dt^2 + (1+2\Phi)^{-1}dr^2 + r^2 d\Omega_2^2. \end{aligned} \tag{C.26}$$

where $\Phi = -\frac{G_N M}{r} \ll 1$ is the gravitational potential. Notice that, we have introduced a new radial coordinate defined by $\tilde{r} = r(1+\Phi) = r - G_N M$. So, $d\tilde{r} = dr$ and $r^2 \approx \tilde{r}^2(1-2\Phi)$. Compare C.15 and C.26, one can obtain

$$c_1 = -2G_N M, \quad c_2 = -1. \tag{C.27}$$

Therefore, we arrive at the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right)dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{C.28}$$

C.2.2 Perturbation method

This time, we would like to find the spherically symmetric static solutions by using an expansion in powers of non-linearity. Continue focusing on (t, r, θ, ϕ) coordinates and $\Lambda = 0$. the most general spherically symmetric static metric can be written as

$$\mathrm{d}s^2 = -B(r)dt^2 + A(r)dr^2 + C(r)r^2 d\Omega^2. \tag{C.29}$$

For convenience, we can set $A(r) = C(r)$ by gauge fixing, thus the metric becomes,

$$\mathrm{d}s^2 = -B(r)dt^2 + A(r)\left[dr^2 + r^2 d\Omega^2\right]. \tag{C.30}$$

The vacuum solution satisfies the equations $T_{\mu\nu} = 0$. From Eq. C.18, we can obtain

$$\begin{aligned} 3r\left(A'\right)^2 - 4A\left(2A' + rA''\right) = 0, \\ 4B'A^2 + 2\left(2B + rB'\right)A'A + Br\left(A'\right)^2 = 0, \end{aligned} \tag{C.31}$$

from the tt and rr components equations. Similarly to the previous case, $\theta\theta$ and $\phi\phi$ components equations turn out to be redundant. The linear expansion of Eq. C.31 around the flat space solution is

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$$B_0(r) = 1, \quad A_0(r) = 1. \tag{C.32}$$

(If we demand the BH solution to be asymptotically flat, which also leads to $A_0 = B_0 = 1$). We now use the method of linearizing a non-linear differential equation about a solution. The expansion can be defined as

$$\begin{aligned} B(r) = B_0(r) + \epsilon B_1(r) + \epsilon^2 B_2(r) + \cdots, \\ A(r) = A_0(r) + \epsilon A_1(r) + \epsilon^2 A_2(r) + \cdots, \end{aligned} \tag{C.33}$$

where ϵ is a parameter that counts the order of non-linearity. Plugging the expansion expression of $A(r)$ and $B(r)$, we can obtain the differential equations for the expansion of each power of ϵ. Thus, we can obtain the solutions of the undetermined function at each order of ϵ, which can be used to solve for the next differential equations for higher order in ϵ. In the present case, $A_0 = B_0 = 1$ is a solution at $\mathcal{O}(0)$ order, which gives $0 = 0$. At $\mathcal{O}(\epsilon)$ order we have

$$A_1'' + \frac{2A_1'}{r} = 0, \quad B_1' + A_1' = 0. \tag{C.34}$$

Combing the boundary condition of asymptotically flat, i.e., $B_1 = A_1 = 0$ as $r \to \infty$, the general solution of the Eq. C.34 is left with an unfixed constant. We can choose it to reproduce the Schwarzschild solution. Thus, we obtain

$$B_1 = -\frac{2G_N M}{r}, \quad A_1 = \frac{2G_N M}{r}. \tag{C.35}$$

Plugging this results in the differential equations given by $\mathcal{O}(\epsilon^2)$ order, we have

$$\begin{aligned} \frac{3G^2M^2}{r^4} - \frac{2A_2'}{r} - A_2'' = 0, \\ \frac{7G^2M^2}{r^3} + B_2' + A_2' = 0. \end{aligned} \tag{C.36}$$

The boundary condition again requires that $B_2 = A_2 = 0$ as $r \to \infty$. The solution of Eq. C.36 left with an arbitrary constant which appears as the coefficient of a $\frac{1}{r}$ term. So, we can absorb it into the first order (i.e., set to zero) if we set $\epsilon = 1$ in the end. Thus, we obtain

$$B_2 = \frac{2G_N^2M^2}{r^2}, \quad A_2 = \frac{3G_N^2M^2}{2r^2}. \tag{C.37}$$

We can continue in this way to any order of ϵ, and obtain the expansion

$$\begin{aligned} B(r) = 1 - \frac{2G_NM}{r}\left(1 - \frac{G_NM}{r} + \cdots\right), \\ A(r) = 1 + \frac{2G_NM}{r}\left(1 + \frac{3G_NM}{4r} + \cdots\right). \end{aligned} \tag{C.38}$$

The dots represent higher order in the non-linear expansion in the parameter ϵ. Moreover, the non-linearity expansion is an expansion of parameters r_S/r, where

$$r_S \equiv 2G_NM, \tag{C.39}$$

is the Schwarzschild radius.

C.3 Extend Schwarzschild solutions

To facilitate a more straightforward elucidation, we shall proceed with the subsequent extension within the confines of the Birkhoff Theorem. Then the spherically symmetric ansatz takes the form

$$ds^2 = -a(r)b(r)^2dt^2 + a(r)^{-1}dr^2 + r^2d\Omega^2, \tag{C.40}$$

where a and b are functions only depend on r.

C

C.3.1 Schwarzschild-de Sitter

The EH action including the cosmological constant term becomes

$$I_{\text{SdS}} = \int d^4x\sqrt{-g}(R-2\Lambda), \tag{C.41}$$

and the reduced action becomes

$$I_{\text{SdS}}^{\text{red}} = 8\pi \int dr \left(b + rab' - \Lambda b r^2\right) \tag{C.42}$$

The respective variations then yield

$$\begin{aligned} \frac{\delta I_{\text{SdS}}^{\text{red}}}{\delta a} &= rb' = 0 \Rightarrow b = \text{constant}; \\ \frac{\delta I_{\text{SdS}}^{\text{red}}}{\delta b} &= (r-ar)' - \Lambda r^2 = 0 \Rightarrow a = 1 - \frac{2G_N M}{r} - \frac{\Lambda r^2}{3}. \end{aligned} \tag{C.43}$$

Thus, the Schwarzschild-de Sitter metric is

$$ds^2 = -\left(1 - \frac{2G_N M}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2G_N M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{C.44}$$

C.3.2 Reissner-Nordström solution on de Sitter

Now extend the above discussion to include the gravitationally coupled Maxwell action, i.e. Einstein-Maxwell theory. This extend action becomes

$$I_{\text{RNdS}} = \int d^4x\sqrt{-g}(R-2\Lambda) - \frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}. \tag{C.45}$$

where $\sqrt{-g} = br^2\sin\theta$. For the term

$$\begin{aligned} &-\frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{2}\int d^4x\sqrt{-g}\,(A_0')^2\,\frac{1}{b^2} \\ &= 2\pi\int dr\,(A_0')^2\,\frac{r^2}{b} \end{aligned} \tag{C.46}$$

Notice that, we have imposed the same ansatz for the gauge field A_μ and adopted the Coulomb gauge, the reduced action becomes

$$I^{\text{red}}_{\text{RNdS}} = 8\pi \int dr \left[-b(ar-r)' - \Lambda b r^2 + \frac{1}{4b} r^2 \left(A_0'\right)^2 \right]. \tag{C.47}$$

Following the steps of the previous analysis in the Coulomb case, one can obtain

$$A_0 = -\frac{q}{r}. \tag{C.48}$$

Similarly, δa variation gives the result that b is constant and then δb variation yields

$$\frac{\delta I^{\text{red}}_{\text{RNdS}}}{\delta b} = (r-ar)' - \Lambda r^2 - \frac{q^2}{4b^2 r^2} = 0 \Rightarrow a = 1 - \frac{2G_N M}{r} - \frac{\Lambda r^2}{3} + \frac{q^2}{4b^2 r^2}. \tag{C.49}$$

C

Once again, b can be absorbed into dt by changing the coordinate $t \to t'(t)$ and set $b = 1$. Thus, the metric of Reissner-Nordström solution with cosmological constant is

$$ds^2 = -\left(1 - \frac{2G_N M}{r} - \frac{\Lambda r^2}{3} + \frac{q^2}{4r^2}\right) dt^2 + \left(1 - \frac{2G_N M}{r} - \frac{\Lambda r^2}{3} + \frac{q^2}{4r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{C.50}$$

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