Imperial College London

Department of Physics

Theoretical Physics Group

Quantum Field theory in Reaction Diffusion process

Author:

Shen Qianchen

CID:01185858

Supervisor:

Dr. Tim Evans

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Abstract

In this dissertation we introduce the Doi-Peliti formalism to stochastic models. The detailed derivation of such field theoretic method is illustrated in a standard non-equilibrium stochastic process - the reaction diffusion process. We then present under such formalism, the observable is approximated via the perturbation theory and Feynman diagram. Examples with branching random walk and an imaginary model is shown.

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Chapter 1

Introduction

In this dissertation we are interested in the parallels between quantum field theory and stochastic process. Stochastic process is a time dependent observable that depends on sequence of random events, This model is widely used in various area such as grown of bacteria population, molecule dynamics, chemistry reaction, information theory and finance.

Quantum field theory is a theoretical framework developed starting in 1920s ,combining classical field theory and quantum mechanics to study interaction of subatomic particles[1]. The path integral formulation in Quantum field theory was initially introduced by Dirac [2] and subsequently advanced and popularized by Feynman [3].

The parallel between Quantum field theory and stochastic process is first noticed and introduced by Doi[4],[5], later developed and formed a comprehensive formalism by study of Peliti [6],[7],[8], Grassberger and Cardy[9],[10].

Stochastic processes is primarily studied in two equations: the Fokker-Planck [11][12](and equally Langevin equation[13]) and the master equation [14].There are two main field theoretic approach in non-equilibrium statistical mechanics, the first one starts with Langevin equation proceeds through path integral. is detailed illustrated in Vasil[15] and Tauber[16].

The second one is the Doi-Peliti formalism which is our focus in this dissertation, it starts with master equation expressed in states and operators, it is well reviewed and summarised by Cardy[17],[18],[19] and Pruessner[20]. The Doi-Peliti formalism is now known as common technique in the study of non-equilibrium stochastic process, such exact method can be applied at and away from the critical point, and it enables us applying approximation schemes, like the renormalization group and diagrammatic perturbation theory to the process.

In this paper we will construct the Doi-Peliti formalism with a standard reaction-diffusion model, followed the detailed instruction by Pruessner's note in chapter 6 in [20] with my calculation in each step.This paper is organized as followed. In chapter 2, we will derived the master equation from reaction-diffusion process and introduced the second quantisation of master equation in creation and annihilation operator and occupation number state. In chapter 3, we introduce the method of transition of the master equation to field theory. In chapter 4, we show all the non-bilinear term will be treated operturbatively and enter the system as loop expansion in diagrammatic expression, we followed Pruessner's choice, take random branching as example to illustrate how to find the correction from Feynman diagram. In chapter 5, we consider a random choice of a imaginary model that might not have a clear physical interpretation, and perform the Doi-Peliti formalism.

Chapter 2

Second Quantisation

2.1 Diffusion and reaction process

Now we aim to establish the standard model of a simple reaction diffusion process, here we choose to consider the system within a finite d dimensional lattice $\mathbb{Z}^d$ and the particles are randomly hopping to the nearest neighbouring site with a hopping rate, denoted by H. This is a Poisson process, which means that for each infinitesimal time step Δt, each particle will only engage in at most one such event, and the probability of each event remains the same. Taking to the continuum limit, this process is recognized as diffusion.

For such model we will also consider the spontaneous creation and spontaneous extinction, which means for every sites, particles may undergo extinction with a Poisson rate ε and particles will be created with a probability density β. In subsequent discussions, we will extend our consideration to include additional reaction such as branching and annihilating random walk, for now our current focus remians on the simple standard model to illustrate how the field theoretic method apply to such model and we will see the method can be easily generalised to more complicated model.

For start we will write down the master equation for the model and then find the field theoretic representation for it. We will consider the master equation for diffusion and spontaneous creation and extinction separately.

On our d-dimensional lattice, $q = 2d$ unit vectors denoted by $\mathbf{e}$ exist. These vectors comprise d pairs of basis vectors, each pointing in opposite directions. We denote the site coordinates by $\mathbf{x}$, and particle hopping takes place between $\mathbf{x}$ and its neighboring site $\mathbf{x}+\mathbf{e}$.

Following the notation in Pruessner's work[20], let us consider the probability, denoted as $P(\{n_{\mathbf{x}}\};t)$, of finding the system with $n_{\mathbf{x}}$ particles located at site $\mathbf{x}$ at time t. Here, $n_{\mathbf{x}}$ represents the set of particle numbers present on each site $\mathbf{x}$ within the lattice.

Master equation describing the evolution of the probability for a process transitioning from state α to state β takes a general form

$$\frac{dP(\alpha;t)}{dt} = \sum_{\beta} R_{\beta\to\alpha}P(\beta;t) - \sum_{\beta} R_{\alpha\to\beta}P(\alpha;t)$$

with $R_{\beta\to\alpha}$ is the rate of transition from state β to α.

Consider a system with only diffusion, the evolution of probability of finding the system in state $\{n_{\mathbf{x}}\}$ over time comprises two components: first, where the state $\{n_{\mathbf{x}}\}$ originates from another state, and the second one is where the state $\{n_{\mathbf{x}}\}$ transitions into different states. To analyze the process leading to $\{n_{\mathbf{x}}\}$, we shall narrow our focus to a specific site denoted as $\mathbf{x}$. The state that could reach state $\{n_{\mathbf{x}}\}$ is which a neighbouring site of $\{n_{\mathbf{x}}\}$ having a extra particle and the site $\mathbf{x}$ lacks one particle. A particle then undergoes a single hop from this neighboring site to $n_{\mathbf{x}}$ with a hopping frequency of H/q. Consequently, the overall influx contributing to the probability $P(n_{\mathbf{x}};t)$ is

$$\frac{H}{q}\sum_{\mathbf{e}}\left(n_{\mathbf{x}+\mathbf{e}}+1\right)P\left(\{n_{\mathbf{x}}-1,n_{\mathbf{x}+\mathbf{e}}+1\};t\right)$$

where the notation $P\left(\{n_x-1,n_{x+e}+1\};t\right)$ represents the state where there are $n_{x+e}+1$ particle at site $\mathbf{x}+\mathbf{e}$—one more than the desired count, and n_x-1 particles at site $\mathbf{x}$—one less than the targeted state. The summation $\sum_{\mathbf{e}}$ runs over total q nearest neighbouring site. The factor $(n_{\mathbf{x}+\mathbf{e}}+1)$ arises from the particle count on site $\mathbf{x}+\mathbf{e}$ as each particle may participate in diffusion independently hence contribute to the total rate of transition.

For the outflow of the probability $P(n_{\mathbf{x}};t)$, consider at state $\{n_{\mathbf{x}}\}$, there are $n_{\mathbf{x}}$ particles at site $\mathbf{x}$ hopping out with the rate H, the outflow of the probability $P(n_{\mathbf{x}};t)$ is $Hn_xP\left(\{n_x\};t\right)$. Hence taking into account the process at every site in the lattice, the master equation for diffusion is

$$\begin{aligned}\frac{d}{dt}P\left(\{n_{\mathbf{x}}\};t\right)_d =& - H\sum_{\mathbf{x}} n_{\mathbf{x}}P\left(\{n_{\mathbf{x}}\};t\right) + \frac{H}{q}\sum_{\mathbf{x}}\sum_{\mathbf{e}}\left(n_{\mathbf{x+e}}+1\right)P\left(\{n_{\mathbf{x}}-1,n_{\mathbf{x+e}}+1\};t\right)\\ =&\frac{H}{q}\sum_{\mathbf{x}}\sum_{\mathbf{e}}\left(\left(n_{\mathbf{x+e}}+1\right)P\left(\{n_{\mathbf{x}}-1,n_{\mathbf{x+e}}+1\};t\right)-n_{\mathbf{x}}P\left(\{n_{\mathbf{x}}\};t\right)\right)\end{aligned} \tag{2.1}$$

with the notation $P(\{n_{\mathbf{x}}-1\},t)$, we have to impose the condition $P(\{n\},t)=0$ for any negative n, to avoid the occur of a negative particle number.

Next we will consider the case involving solely spontaneous extinction and creation processes.Following a similar approach as in the case with diffusion, the process of spontaneous creation disregards the current particle number on the site, conversely the probability changes due to spontaneous extinction with extinction rate β is proportional to the current particle number $n_{\mathbf{x}}$,hence

$$\begin{aligned}\frac{d}{dt}P(\{n_{\mathbf{x}}\};t)_{e+c} =& \sum_{\mathbf{x}}\varepsilon\left(\left(n_{\mathbf{x}}+1\right)P\left(\{n_{\mathbf{x}}+1\};t\right)-n_{\mathbf{x}}P\left(\{n_{\mathbf{x}}\};t\right)\right.\\ &+\sum_{\mathbf{x}}\beta\left(P\left(\{n_{\mathbf{x}}-1\};t\right)-P\left(\{n_{\mathbf{x}}\};t\right)\right)\end{aligned} \tag{2.2}$$

Given the independence of diffusion, extinction, and creation processes, we can superimpose the master equation. Therefore, the resulting equation describing the entire system is as follow:

$$\boxed{\begin{aligned}\frac{d}{dt}P(\{n_{\mathbf{x}}\};t) =&\frac{H}{q}\sum_{\mathbf{x}}\sum_{\mathbf{e}}\left(\left(n_{\mathbf{x+e}}+1\right)P\left(\{n_{\mathbf{x}}-1,n_{\mathbf{x+e}}+1\};t\right)-n_{\mathbf{x}}P\left(\{n_{\mathbf{x}}\};t\right)\right)\\ &+\sum_{\mathbf{x}}\varepsilon\left(\left(n_{\mathbf{x}}+1\right)P\left(\{n_{\mathbf{x}}+1\};t\right)-n_{\mathbf{x}}P\left(\{n_{\mathbf{x}}\};t\right)\right.\\ &+\sum_{\mathbf{x}}\beta\left(P\left(\{n_{\mathbf{x}}-1\};t\right)-P\left(\{n_{\mathbf{x}}\};t\right)\right)\end{aligned}} \tag{2.3}$$

For the rest of the discussion of constructing field theory approach, we will stick to the model

2.2 States and operators

To study the master equation 2.3 in field theory, we shall first convert it into second quantisation representation in operator and states in Fock space. We introduce the occupation number representation of set of normalised basis vectors $|\{n_{\mathbf{x}}\}\rangle$. These basis satisfies the orthogonality as

$$\langle\{n_{\mathbf{x}}\} \mid \{m_{\mathbf{x}}\}\rangle = \prod_{\mathbf{x}} \delta_{n_{\mathbf{x}},m_{\mathbf{x}}}$$

For example, a particular state with occupation number as $|\{0,1,2,3,4,5\}\rangle$, then its orthogonality with bra-basis shows $\langle\{0,1,2,3,4,5\}|\{0,1,2,3,4,5\}\rangle = 1$ and $\langle\{0,1,2,3,4,6\}|\{0,1,2,3,4,5\}\rangle = 0$

The vacuum in this occupation number representation is defined to be a empty space, which is a state $|0\rangle = |\{n_{\mathbf{y}}\}\rangle$ such that $\forall_y n_y = 0$.

From the basis we can construct the mixed state as

$$|\psi(t)\rangle = \sum_{\{n_{\mathbf{x}}\}} P\left(\{n_{\mathbf{x}}\};t\right)|\{n_{\mathbf{x}}\}\rangle$$

the summation of $\{n_{\mathbf{x}}\}$ is over all possible occupation number, and each basis is weighted by the probability of that state. In fact the mixed state acts like a probability generating function. The mixed state act on a particular state give the probability of that state straightforward,

$$\langle\{n_{\mathbf{x}}\} \mid \psi(t)\rangle = P\left(\{n_{\mathbf{x}}\};t\right) \tag{2.4}$$

Similar to the standard quantum mechanics setting, we define the creation and annihilation operator act on the occupation number basis as:

$$\begin{aligned} \hat{a}^{\dagger}(\mathbf{x})|n_{\mathbf{x}}\rangle &= |n_{\mathbf{x}}+1\rangle \\ \hat{a}(\mathbf{x})|n_{\mathbf{x}}\rangle &= n_{\mathbf{x}}|n_{\mathbf{x}}-1\rangle \end{aligned} \tag{2.5}$$

Here for simplicity we abuse the notation $|n_{\mathbf{x}}\rangle$ a little bit, this represents a state with $n_{\mathbf{x}}$ occupation number specifically at site $\mathbf{x}$.

Similarly, those operator act on bra vectors as,

$$\begin{aligned} \langle n_{\mathbf{x}}|\hat{a}^{\dagger}(\mathbf{x}) &= \langle n_{\mathbf{x}}-1| \\ \langle n_{\mathbf{x}}|\hat{a}(\mathbf{x}) &= (n_{\mathbf{x}}+1)\langle n_{\mathbf{x}}+1| \end{aligned} \tag{2.6}$$

Following these definition, we have some useful calculation and properties, first the commutation relation can be verified by the occupation basis easily,

$$\left\langle n_{\mathbf{x}}\left|\left[\hat{a}(\mathbf{x}),\hat{a}^{\dagger}(\mathbf{x})\right]\right|n_{\mathbf{x}}\right\rangle = \left\langle n_{\mathbf{x}}\left|\hat{a}(\mathbf{x})\hat{a}^{\dagger}(\mathbf{x})-\hat{a}^{\dagger}(\mathbf{x})\hat{a}(\mathbf{x})\right|n_{\mathbf{x}}\right\rangle = (n_{\mathbf{x}}+1)-n_{\mathbf{x}} = 1$$

Similar calculation gives the full commutation relation in different point in space as

$$\left[\hat{a}(\mathbf{x}), \hat{a}^{\dagger}\left(\mathbf{x}'\right)\right] = \delta_{\mathbf{x},\mathbf{x}'}$$

An useful operator is the particle number operator, which has the occupation number vector as its eigenvector and give the particle number as its eigenvalue:

$$\hat{a}^{\dagger}(\mathbf{x})\hat{a}(\mathbf{x})\left|n_{\mathbf{x}}\right\rangle = n_{\mathbf{x}}\left|n_{\mathbf{x}}\right\rangle \tag{2.7}$$

From the knowledge of statistical mechanics, the expectation of particle number evaluated at a position $\mathbf{y}$ at time t is

$$\langle n\rangle(\mathbf{y};t) = \sum_{\{n_{\mathbf{x}}\}} P\left(\{n_{\mathbf{x}}\};t\right) n_{\mathbf{y}} \tag{2.8}$$

Notice the observable is not measured in the familiar way we have in quantum mechanics like $\langle\psi|\hat{O}|\psi\rangle$.

To measure the observable, we need to introduce a special state called abyss, which is a vector that will project any state to unity. To fulfill such property, the abyss shall take the form

$$\langle\Phi| = \sum_{\{n_{x'}\}} \langle\{n_{x'}\}|$$

It is clear that applying abyss to arbitrary state $|\{n_{\mathbf{x}}\}\rangle$ will have $\langle\Phi \mid \{n_{\mathbf{x}}\}\rangle = 1$ due to the orthogonality relation of state.

Hence we have the expectation of particle number can be evaluated by applying the abyss and our mixed state as

$$\begin{aligned}
\langle n\rangle(\mathbf{y};t) &= \left\langle\Phi\left|\hat{a}^{\dagger}(\mathbf{y})\hat{a}(\mathbf{y})\right|\psi(t)\right\rangle \\
&= \left\langle\Phi\left|\sum_{n_{\mathbf{x}}} P\left(\{n_{\mathbf{x}}\};t\right)\hat{a}^{\dagger}(\mathbf{y})\hat{a}(\mathbf{y})\right|\{n_{\mathbf{x}}\}\right\rangle \\
&= \left\langle\Phi\left|\sum_{n_{\mathbf{x}}} P\left(\{n_{\mathbf{x}}\};t\right) n_{\mathbf{y}}\right|\{n_{\mathbf{x}}\}\right\rangle \\
&= \sum_{n_{\mathbf{x}}} P\left(n_{\mathbf{x}};t\right) n_{\mathbf{y}}
\end{aligned} \tag{2.9}$$

From Eq 2.6, it is easy to show that a bra vector can be written as power of annihilation operators act on vacuum, which is $\langle 0|\hat{a}^{n}(\mathbf{y}) = n!\langle n_{\mathbf{y}}|$, for now we drop the position $\mathbf{y}$ and only focus on the occupation number, and similarly we can write the abyss state in the same way, which is

$$\langle\Phi| = \sum_n \langle n| = \sum_{n=0}^{\infty} \frac{1}{n!}\langle 0| a^n = \langle 0| e^a$$

More importantly, the creation operation act on the abyss as

$$\langle\Phi| \hat{a}^\dagger = \sum_{n=0}^{\infty}\langle n|\hat{a}^\dagger = 0 + \sum_{n=1}^{\infty}\langle n-1| = \langle\Phi| \tag{2.10}$$

The abyss is invariant under the creation operator, from this it is natural to keep our operator normal ordered,which means always keep all the $\hat{a}^\dagger$ on the left hand side to ease the calculation when evaluating by projected to abyss, which is how the observable will be evaluated in the further discussion. From now on we will keep the operator in normal order, this can always be done the using the commutation relation $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$

2.3 Master equation

Now we start to transform the master equation 2.3 in the language of state and operators. When look at the LHS of Eq2.3, it is natural to consider taking the time derivative of the mixed state as

$$\frac{d}{dt}|\psi(t)\rangle = \frac{d}{dt}\sum_{\{n_\mathbf{x}\}} P\left(\{n_\mathbf{x}\}\,; t\right)|\{n_\mathbf{x}\}\rangle$$

We wish to transform the master equation into the time evolution equation of the state as:

$$\frac{d}{dt}|\psi(t)\rangle = \hat{\mathcal{A}}|\psi(t)\rangle$$

where $\hat{\mathcal{A}}$ act as time evolution operator. The operator $\hat{\mathcal{A}}$ can be found from the RHS of the master equation 2.3, remind that the master equation contains three components of the process, we will also translate the operator into three terms, diffusion with hopping rate H, spontaneous creation with rate β and spontaneous extinction with rate ε,

$$\hat{\mathcal{A}} = \hat{\mathcal{A}}_H + \hat{\mathcal{A}}_\beta + \hat{\mathcal{A}}_\varepsilon$$

Set behind the diffusion term for a second, let consider the creation term first, it is the easiest term and illustrate the procedure.Taking the creation term in master equation and act on the mixed state

$$\begin{aligned}\hat{\mathcal{A}}_\beta|\psi(t)\rangle &= \sum_{\mathbf{x}}\sum_{\{n_\mathbf{x}\}}\beta\left\{P\left(\{n_\mathbf{x}-1\};t\right)-P\left(\{n_\mathbf{x}\};t\right)\right\}|\{n_\mathbf{x}\}\rangle \\ &= \sum_{\mathbf{x}}\sum_{n=0}\beta\left(P(n-1)|n\rangle - P(n)|n\rangle\right) \\ &= \beta\sum_{\mathbf{x}}\sum_{n=0}\left(P(n-1)\hat{a}^\dagger|n-1\rangle - P(n)|n\rangle\right)\end{aligned}$$

Notice here we drop the notation for position and time for simplicity. Note that in first term there is a undefined state $|-1\rangle$, but with restrict $P(-1)=0$ help us avoid such term, so we can sum up from $n=1$ and rearrange the occupation number in first term will gives,

$$\begin{aligned}\hat{\mathcal{A}}_\beta|\psi(t)\rangle &= \beta\sum_{\mathbf{x}}\sum_{\{n_\mathbf{x}\}}\left(P(\{n_\mathbf{x}\};t)\hat{a}^\dagger|(\mathbf{x})\{n_\mathbf{x}\}\rangle - P(\{n_\mathbf{x}\};t)|\{n_\mathbf{x}\}\rangle\right) \\ &= \beta\sum_{\mathbf{x}}(\hat{a}^\dagger(\mathbf{x})-1)|\psi(t)\rangle\end{aligned} \tag{2.11}$$

The extinction part can be proceed in a similar way, from Eq 2.3,

$$\begin{aligned}\hat{\mathcal{A}}_\varepsilon|\psi(t)\rangle =&\varepsilon\sum_{\mathbf{x}}\sum_{\{n_\mathbf{x}\}}\left((n_\mathbf{x}+1)\,P\left(\{n_\mathbf{x}+1\};t\right) - n_\mathbf{x}P\left(\{n_\mathbf{x}\};t\right)\right)|\{n_\mathbf{x}\}\rangle \\ =&\varepsilon\sum_{\mathbf{x}}\sum_{n=0}\left((n+1)P(n+1)|n\rangle - nP(n)|n\rangle\right)\end{aligned}$$

Again we drop the position and time for a second and only focus on occupation number, and recall the property of $\hat{a}$, $\hat{a}^\dagger$ and $\hat{a}\hat{a}^\dagger$,

$$\begin{aligned}&\sum_{n=0}P(n+1)(n+1)|n\rangle = \sum_{n=0}P(n+1)\hat{a}|n+1\rangle = \sum_{n=0}P(n)\hat{a}|n\rangle \\ &\sum_{n=0}P(n)n|n\rangle = \sum_{n=0}P(n)\hat{a}^\dagger\hat{a}|n\rangle\end{aligned}$$

the first line holds since $\hat{a}|0\rangle = 0$. Similarly we have the operator for extinction,

$$\hat{\mathcal{A}}_\varepsilon|\psi(t)\rangle = \varepsilon\sum_{\mathbf{x}}\left(\hat{a}(\mathbf{x}) - \hat{a}^\dagger(\mathbf{x})\hat{a}(\mathbf{x})\right) \tag{2.12}$$

(some standard procedure to apply)

Finally we consider the diffusion terms, this one is trickier than the previous two since there is a term $P\left(\{n_\mathbf{x}-1, n_{\mathbf{x+e}}+1\};t\right)$ in it, and $(\mathbf{x}+\mathbf{e})$ here describe the feature of lattice structure, the diffusion term is

$$\hat{\mathcal{A}}_H|\psi(t)\rangle = \frac{H}{q}\sum_{\{n_{\mathbf{x}}\}}\sum_{\mathbf{x}}\sum_{\mathbf{e}}\left\{(n_{\mathbf{x+e}}+1)\,P\left(\{n_{\mathbf{x}}-1,n_{\mathbf{x+e}}+1\};t\right)-n_{\mathbf{x}}P\left(\{n_{\mathbf{x}}\};t\right)\right\}|\{n_{\mathbf{x}}\}\rangle$$

The second term works just like what we did for extinction term, it is just $-\frac{H}{q}\sum_{\mathbf{x}}\hat{a}^\dagger(\mathbf{x})\hat{a}(\mathbf{x})$, for the first term again we slightly abuse the notation as

$$\begin{aligned}
&\frac{H}{q}\sum_{\{n_{\mathbf{x}}\}}\sum_{\mathbf{x}}\sum_{\mathbf{e}}(n_{\mathbf{x+e}}+1)\,P\left(\{n_{\mathbf{x}}-1,n_{\mathbf{x+e}}+1\};t\right)|\{n_{\mathbf{x}},n_{\mathbf{x+e}}\}\rangle\\
=&\frac{H}{q}\sum_{\{n_{\mathbf{x}}\}}\sum_{\mathbf{x}}\sum_{\mathbf{e}}(n_{\mathbf{x+e}}+1)\,P\left(\{n_{\mathbf{x}}-1,n_{\mathbf{x+e}}+1\};t\right)\hat{a}^\dagger(\mathbf{x})|\{n_{\mathbf{x}}-1,n_{\mathbf{x+e}}\}\rangle\\
=&\frac{H}{q}\sum_{\{n_{\mathbf{x}}\}}\sum_{\mathbf{x}}\sum_{\mathbf{e}}P\left(\{n_{\mathbf{x}}-1,n_{\mathbf{x+e}}+1\};t\right)\hat{a}(\mathbf{x}+\mathbf{e})\hat{a}^\dagger(\mathbf{x})|\{n_{\mathbf{x}}-1,n_{\mathbf{x+e}}+1\}\rangle
\end{aligned}$$

We can now absorb the probability term and basis state into the mixed state, combining the term $-\frac{H}{q}\sum_{\mathbf{x}}\hat{a}^\dagger(\mathbf{x})\hat{a}(\mathbf{x})$, we have the expression for diffusion operator

$$\begin{aligned}
\hat{\mathcal{A}}_H &= \frac{H}{q}\sum_{\mathbf{x}}\sum_{\mathbf{e}}\left(\hat{a}^\dagger(\mathbf{x})\hat{a}(\mathbf{x}+\mathbf{e})-\hat{a}^\dagger(\mathbf{x})\hat{a}(\mathbf{x})\right)\\
&= \frac{H}{q}\sum_{\mathbf{x}}\sum_{\mathbf{e}}\hat{a}^\dagger(\mathbf{x})(\hat{a}(\mathbf{x}+\mathbf{e})-\hat{a}(\mathbf{x}))
\end{aligned}$$

It looks in a nice form, but still not what we want for later calculation, consider the following identity

$$\left(\hat{a}^\dagger(\mathbf{x}+\mathbf{e})-\hat{a}^\dagger(\mathbf{x})\right)\left(\hat{a}(\mathbf{x}+\mathbf{e})-\hat{a}(\mathbf{x})\right)) = -\hat{a}^\dagger(\mathbf{x})(\hat{a}(\mathbf{x}+\mathbf{e})-\hat{a}(\mathbf{x}))-\hat{a}^\dagger(\mathbf{x}+\mathbf{e})(\hat{a}(\mathbf{x})-\hat{a}(\mathbf{x}+\mathbf{e}))$$

The first term and second term is in a similar form, the first term represent the process particle hop into site $\mathbf{x}$ and hop out from $\mathbf{x}$, and the second term represent the exact same process at site $\mathbf{x}+\mathbf{e}$. When sum over $\sum_{\mathbf{x}}\sum_{\mathbf{e}}$, these two term become the double of the expression in $\hat{\mathcal{A}}$. Hence the diffusion operator can be written as

$$\hat{\mathcal{A}}_H = -\frac{H}{2q}\sum_{\mathbf{x}}\sum_{\mathbf{e}}\left(\hat{a}^\dagger(\mathbf{x}+\mathbf{e})-\hat{a}^\dagger(\mathbf{x})\right)\left(\hat{a}(\mathbf{x}+\mathbf{e})-\hat{a}(\mathbf{x})\right) \tag{2.13}$$

Combine terms Eq 2.11, 2.12 and 2.13,the full expression of $\hat{\mathcal{A}}$ is

$$\boxed{\begin{aligned} \hat{\mathcal{A}}_{\varepsilon} &= \varepsilon \sum_{\mathbf{y}} \left(\hat{a}(\mathbf{y}) - \hat{a}^{\dagger}(\mathbf{y})\hat{a}(\mathbf{y})\right) \\ \hat{\mathcal{A}}_{\beta} &= \beta \sum_{\mathbf{y}} \left(\hat{a}^{\dagger}(\mathbf{y}) - 1\right) \\ \hat{\mathcal{A}}_{H} &= -\frac{H}{2q} \sum_{\mathbf{y}} \sum_{\mathbf{e}} \left(\hat{a}^{\dagger}(\mathbf{y}+\mathbf{e}) - \hat{a}^{\dagger}(\mathbf{y})\right)\left(\hat{a}(\mathbf{y}+\mathbf{e}) - \hat{a}(\mathbf{y})\right) \end{aligned}} \tag{2.14}$$

From the method we translate the master equation into operators, we find the key is to first apply the $|\{n_{\mathbf{x}}\}\rangle$ to the probability terms, and using the properties of creation and annihilation operators to eliminate all the occupation number related factor and rearrange the occupation number to match the particle number in the probability, then the operators can be move out the summation and be absorbed into mixed state.

From this chapter we see the method of second quantisation for a reaction diffusion process, as long as we are given the master equation of the process. For now we do not see any season a general Poisson process that can not be translate to such operator form.

In next chapter we will show the method to translate the problem into quantum field theory paradigm.

Chapter 3

Field theory

In this chapter we will transfer the master equation to quantum field theory. This can be done canonically similar to quantum field theory allowing us to derive the corresponding propagator and the Wick theorem. However in this article, following the method by Pruessner [20], we choose a more algebraic and rigorous way to discuss the exact calculation in the methods, and it will terms out a simple substitute of $\hat{a}$, $\hat{a}^\dagger$ by ϕ and ϕ^* yields the desired outcome. Starting from the master equation and the time evolution operator.

The master equation in state and operator is:

$$\frac{d}{dt}|\psi(t)\rangle = \hat{\mathcal{A}}|\psi(t)\rangle \quad \text{, where} \quad \hat{\mathcal{A}} = \hat{\mathcal{A}}_H + \hat{\mathcal{A}}_\beta + \hat{\mathcal{A}}_\varepsilon$$

A straightforward solution to this equation is $|\psi(t)\rangle = e^{\hat{\mathcal{A}}t}|\psi(0)\rangle$, hence the expectation of a observable is

$$\begin{aligned}\langle \hat{\mathcal{O}} \rangle &= \left\langle \Phi \middle| \hat{\mathcal{O}} \middle| \psi(t) \right\rangle \\ &= \left\langle \Phi \middle| \hat{\mathcal{O}} e^{\hat{\mathcal{A}}t} \middle| \psi(0) \right\rangle\end{aligned}$$

Now we introduce a initialisation operator $\mathcal{J}$ to generate the initial state $|\psi(0)\rangle$ from vacuum $|0\rangle$, as

$$\mathcal{J}|0\rangle = |\psi(0)\rangle$$

The expectation of observable becomes act on empty state

$$\langle \hat{\mathcal{O}} \rangle = \left\langle \Phi \middle| \hat{\mathcal{O}} e^{\hat{\mathcal{A}}t} \mathcal{J} \middle| 0 \right\rangle \tag{3.1}$$

3.1 Conservation of probability

In this section we will investigate the consequence of conservation of probability, and it shows that $\mathcal{A}$ can be properly rewritten under a shift of operator, called Doi-Shift

First consider the expectation of identity operator using equation 3.1, obviously it should be a identity

$$\langle \Phi | \mathbb{1} | \psi(t) \rangle = \left\langle \Phi | \mathbb{1} e^{\hat{\mathcal{A}} t} | \psi(0) \right\rangle = 1 \tag{3.2}$$

If the conservation of probability holds, also remind the property of abyss state, at time $t = 0$ there is

$$\langle \Phi | \psi(0) \rangle = \sum_{\{n_{\mathbf{x}}\}} P\left(\{n_{\mathbf{x}}\}; 0\right) \langle \Phi | \{n_{\mathbf{x}}\} \rangle = \sum_{\{n_{\mathbf{x}}\}} P\left(\{n_{\mathbf{x}}\}; 0\right) = 1 \tag{3.3}$$

Combining Eq 3.2 and 3.3, we have $\left\langle \Phi \left| e^{\hat{\mathcal{A}} t} - 1 \right| \psi(0) \right\rangle = 0$, by assuming the convergence, expand the operator in a small time t near $t = 0$, we required

$$\langle \Phi | (\hat{\mathcal{A}} t) + (\hat{\mathcal{A}} t)^2 / 2 + (\hat{\mathcal{A}} t)^3 / 3 + \cdots | \psi(0) \rangle = 0$$

In fact we further required $\langle \Phi | \hat{\mathcal{A}}^n | \psi(0) \rangle = 0$ for all $n \geq 1$, we will examine the case $\langle \Phi | \hat{\mathcal{A}} | \psi(0) \rangle = 0$,in fact we can use the master equation and by conservation of probability

$$\langle \Phi | \frac{d}{dt} | \psi(0) \rangle = \sum_{\{n_{\mathbf{x}}\}} \dot{P}(\{n_{\mathbf{x}}\}, 0) \langle \Phi | \{n_{\mathbf{x}}\} \rangle = \langle \Phi | \hat{A} | \psi(0) \rangle = 0$$

Since we choose $|\psi(0)\rangle$ randomly, this shall hold for arbitrary $|\psi(0)\rangle$, hence we shall show that for every occupation state $|\{n_{\mathbf{x}}\}\rangle$

$$\langle \Phi | \hat{\mathcal{A}} | \{n_{\mathbf{x}}\} \rangle = 0$$

Remind that we keep all our observable in normal order, all the creation operator are on the left, we consider those creation operator act on the abyss from the left, as the abyss is invariant under the creation operator from the left, as Eq2.10. Then we have

$$\langle \Phi | \hat{\mathcal{A}} | \{n_{\mathbf{x}}\} \rangle = \langle \Phi | \hat{\mathcal{A}}' | \{n_{\mathbf{x}}\} \rangle$$

where $\hat{\mathcal{A}}'$ is obtained by setting all creation operator to unity in $\hat{\mathcal{A}}$, hence it can be written as a polynomial of annihilation operators as

$$\hat{\mathcal{A}}' = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} \hat{a}$$

here we alter the coefficient by $\frac{1}{n!}$ since it will give a clearer expression in later calculation. Hence drop the position and focus on the occupation number now, for n order term

$$\langle \Phi | \frac{\alpha_m}{m!} \hat{a} | n \rangle = \frac{\alpha_m}{m!} n(n-1) \cdots (n-m) \langle \Phi | n-m \rangle = C_m^n \alpha_m$$

notice that this is nonzero for $n \leq m$.

The equation for transformed $\hat{\mathcal{A}}'$ is then

$$\langle \Phi | \hat{\mathcal{A}}' | n \rangle = \sum_{m=0}^{n} C_m^n \alpha_m$$

We will prove this is zero by induction, clearly for n=0 we have $\alpha_0 = 0$, and $\alpha_1 = 0$ for n=1, and by induction all coefficient $\alpha_m = 0$.

Now we have prove that $\langle \Phi | \hat{\mathcal{A}} | n \rangle$ for all occupation number n as required. Another result is that when act $\hat{\mathcal{A}}$ to the abyss from the left vanishes

$$\langle \Phi | \hat{\mathcal{A}}' = \langle \Phi | \hat{\mathcal{A}} = 0$$

This result shows that abyss is invariant under left operation of time evolution as

$$\langle \Phi | \exp(\hat{\mathcal{A}} t) = \langle \Phi | (\mathbb{1} + \hat{\mathcal{A}} t + \cdots) = \langle \Phi |$$

More importantly, since all coefficient $\alpha_n = 0$, we have shown that setting all creation operation in $\hat{\mathcal{A}}$ to unity, the operation vanishes. This means we can "decompose" $\hat{\mathcal{A}}$ to some operator with a factor $(\hat{a}^\dagger - 1)$, so $\hat{\mathcal{A}}$ can be rewritten as

$$\begin{aligned} \hat{\mathcal{A}} &= \sum_{\mathbf{x}} (\hat{a}(\mathbf{x})^\dagger - 1) \bar{\mathcal{A}}(\mathbf{x}) \\ &= \sum_{\mathbf{x}} \tilde{a}(\mathbf{x}) \bar{\mathcal{A}}(\mathbf{x}) \end{aligned}$$

where we introduce a shift $\hat{a}^\dagger - 1 = \tilde{a}$ called Doi-shift, and $\bar{\mathcal{A}}(\mathbf{x})$ is some operator we obtained after making the shift. It is not only we extract a factor $\hat{a}^\dagger - 1$, we can even further have that all creation operator in such $\bar{\mathcal{A}}(\mathbf{x})$.

We do not make the transform now, later we will show the operators with all the creation operator $\hat{a}^\dagger$ Doi shifted to $\tilde{a}$.

3.2 Important integral

Instead of further investigating $\mathcal{A}$, in this section we will focus on a complex integral, which will prove highly useful in subsequent calculations

$$I = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-iz^* z} z^{*n} z^m \tag{3.4}$$

where the complex variable is $z = x + iy, z^* = x - iy$, and in polar coordinate $z = re^{i\theta}, z^* = re^{-i\theta}$, the exponential part becomes $e^{-z^* z} = e^{-r^2}$, the product becomes $z^{*n} z^m = r^{n+m} e^{i\theta(m-n)}$

Transform the integral to polar coordinate

$$\begin{aligned} I &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-r^2} r^{n+m} e^{i\theta(m-n)} \\ &= \int_{0}^{\infty} dr \int_{0}^{2\pi} d\theta r^{1+n+m} e^{-r^2} e^{i\theta(m-n)} \end{aligned}$$

Recall for integer n, m, the integral $\int_0^{2\pi} d\theta \exp(i\theta(m-n))$ is zero, unless $m = n$ then it gives 2π, and by substitution $u = r^2$, we have the integral

$$I = \delta_{mn} 2\pi \int_0^{\infty} r^{1+2n} e^{-r^2} dr = \delta_{mn} \pi \int_0^{\infty} u^n e^{-u} du = \delta_{mn} \pi n!$$

Introduce the wedge product, it is anticommute hence $v \wedge w = -w \wedge v$ and $v \wedge v = 0$, hence we rewrite $dxdy$ as

$$\begin{aligned} dz^* \wedge dz &= (dx - idy) \wedge (dx + idy) = dx \wedge dx + idx \wedge dy - idy \wedge dx + dy \wedge dy \\ &= 2idx \wedge dy. \end{aligned}$$

Absorb the pre-factor π and $2i$, the integral can be rewritten as

$$I = \int \frac{dz^* \wedge dz}{2\pi i} e^{-z^* z} z^{*n} z^m = \delta_{mn} m! \tag{3.5}$$

Note that in the integral z is a dummy variable, we can further substitute it to

$$\int \frac{dz^* \wedge dz}{2\pi i} e^{-(z+\xi)^*(z+\xi)} (z+\xi)^{*n} (z+\xi)^m = \delta_{mn} n!$$

In fact we would like to show that the equation still holds for z^* shift to $(z+\zeta)^*$ with $\xi \neq \zeta$, hence we will be treating the shift in z and z^* independently.

In the following we will show that

$$\mathrm{I_{mn}}(\zeta,\xi) = \int \frac{dz^* \wedge dz}{2\pi i} e^{-(z+\zeta)^*(z+\xi)}(z+\zeta)^{*n}(z+\xi)^m = \delta_{mn} n!$$

First we make the shift $z = z - \zeta$, and later substitute $u = \xi - \zeta$

$$\begin{aligned}
\mathrm{I_{mn}}(\zeta,\xi) &= \int \frac{dz^* \wedge dz}{2\pi i} \exp(-z^*(z+\xi-\zeta)) z^{*n}(z+\xi-\zeta)^m \\
&= \int \frac{dz^* \wedge dz}{2\pi i} e^{-z^* z} e^{-z^* u} z^{*n}(z+u)^m \\
&= \int \frac{dz^* \wedge dz}{2\pi i} e^{-z^* z} \left\{ z^{*n} \sum_{i=0}^{\infty} \frac{(-z^*)^i u^i}{i!} \sum_{j=0}^{m} z^j u^{m-j} C_j^m \right\} \\
&= \int \frac{dz^* \wedge dz}{2\pi i} e^{-z^* z} \sum_{i=0}^{\infty} \sum_{j=0}^{m} \frac{(-1)^i}{i!} C_j^m u^i u^{m-j} z^{*(n+i)} z^j
\end{aligned}$$

This returns to the integral we have obtained as Eq 3.5, hence the only terms in the summation that contribute is when $n+i=j$. As $j<m$ and $n \leq j$, so the integral is zero for $n>m$. We have make the shift $z = z-\zeta$, on the other case, we can also instead shift z to $z-\xi$, we can achieve the similar conclusion that integral is zero for $n<m$. We have now resume δ_{mn}, we will show the rest by assume $n<m$. By substitute $k=j-n$ and keep two powers of u for now, we will see that these two factor u reform the binomial expansion as

$$\begin{aligned}
\mathrm{I_{mn}}(\zeta,\xi) &= \delta_{n+i,j} \sum_{i=0}^{\infty} \sum_{j=0}^{m} \frac{(-1)^i}{i!} C_j^m u^i u^{m-j+} j! \\
&= \sum_{j=n}^{m} \frac{(-1)^{j-n}}{(j-n)!} C_j^m u^{j-n} u^{m-j} j! \\
&= \sum_{k=0}^{m-n} (-1)^k u^k u^{m-n-k} C_{k+n}^m \frac{(k+n)!}{k!} \\
&= \frac{m!}{(m-n)!} \sum_{k=0}^{m-n} (-1)^k u^k u^{m-n-k} C_k^{m-n} \\
&= \frac{m!}{(m-n)!} (u-u)^{m-n}
\end{aligned}$$

we will see the δ_{mn} is also shown in the result, and the integral is $m!$ for $m=n$, so we have

$$\mathrm{I_{mn}}(\zeta,\xi) = \int \frac{dz^* \wedge dz}{2\pi i} e^{-(z+\zeta)^*(z+\xi)}(z+\zeta)^{*n}(z+\xi)^m = \delta_{mn} n! \tag{3.6}$$

As a natural extension, consider a polynomial $f(z, z^*)$, the similar result can be obtained for a substitution of $(z+\zeta)^*$ and $(z+\xi)$

$$\int \frac{dz^* \wedge dz}{2\pi i} e^{-z^* z} f\left(z^*, z\right) = \int \frac{dz^* \wedge dz}{2\pi i} e^{-(z+\zeta)^*(z+\xi)} f\left(z^* + \zeta^*, z+\xi\right) \tag{3.7}$$

We wish to apply the integral to field theory $\phi\phi^*$ later, it will allow us to shift fields ϕ and ϕ^* independently.

3.3 Action of fields

In this section we aim to derive the the process in field theory, we will start with the expectation of observable and rewrite the time evolution $\exp(\hat{\mathcal{A}}t)$ with the special integral Eq3.6 we have in last section. As we are focus on the field theory at a point, to ease notation we can for now drop the dependence on position and work with occupation number.

Recall the expectation of observable is

$$\langle \hat{\mathcal{O}} \rangle = \left\langle \Phi \left| \hat{\mathcal{O}} e^{\hat{\mathcal{A}}t} \mathcal{J} \right| 0 \right\rangle$$

It is not very helpful to conduct the calculation with the operator $\exp(\hat{\mathcal{A}}t)$. Instead we choose to rewrite $\exp(\hat{\mathcal{A}}t)$ expand in a small time step, which is

$$\exp(\hat{\mathcal{A}}t) = \underbrace{(1+\Delta t\hat{\mathcal{A}})(1+\Delta t\hat{\mathcal{A}})(1+\Delta t\hat{\mathcal{A}})\cdots\cdots(1+\Delta t\hat{\mathcal{A}})}_{(t/\Delta t) \quad \text{terms}} \tag{3.8}$$

where Δt is a small and finite time step. This expression interpret the time evolution $\exp(\hat{\mathcal{A}}t)$ as the state is evolved in a small step with $(1+\Delta t\mathcal{A})$ each step, and there is total $t/\Delta t$ steps, all time steps add up to total time t. In addition, in each evolution step, we insert an identity, as this may first look weird, we will for now leave it as

$$(1+\Delta t\hat{\mathcal{A}}) = \mathbb{1}(1+\Delta t\hat{\mathcal{A}})\mathbb{1}$$

We wish to use the special integral we proved in Eq3.6 to generate a new representation of unity in fields and creation annihilation operator, and it will allows us to transform operator to fields without any further assumption. The unity can be written as

$$\begin{aligned}
&\int \frac{d\phi(t)^* \wedge d\phi(t)}{2\pi i} \exp(-\phi(t)^*\phi(t)) \exp\left(\phi(t)\hat{a}^\dagger\right) |0\rangle\langle 0| \exp\left(\phi(t)^*\hat{a}\right) \\
&= \int \frac{d\phi(t)^* \wedge d\phi(t)}{2\pi i} \exp\left(-\phi(t)^*\phi(t)\right) \sum_{n,m=0}^{\infty} \frac{\phi(t)^n \phi(t)^{*m}}{n!} |n\rangle\langle m| \\
&= \sum_n |n\rangle\langle n| = \mathbf{1}_t
\end{aligned} \tag{3.9}$$

here the expansion of $\exp(\phi(t)\hat{a}^\dagger)$ and $\exp(\phi(t)^*\hat{a})$ act on $|0\rangle\langle 0|$ from left and right, note that a $m!$ from creation cancel with the expansion $\frac{1}{m!}$

Then we insert the unity at each time step $\mathbb{1}_t$ between the time evolution of each time step, the time evolution is then calculated as

$$e^{\hat{A}_t} = \mathbb{1}_{t+\Delta t}(1+\Delta t\hat{A})\mathbb{1}_t(1+\Delta t\hat{A})\mathbb{1}_{t-\Delta t}\cdots \mathbb{1}_{\Delta t}(1+\Delta t\hat{A})\mathbb{1}_0$$

Focus on each bracket at a time step $\mathbb{1}_{t+\Delta t}(1+\Delta t\hat{A})\mathbb{1}_t$, we would like to sandwich the evolution operator from each side by identity, but we only takes out the part after and before the $|0\rangle\langle 0|$ part in Eq 3.9, it is in the from

$$\begin{aligned}
\Xi(t+\Delta t, t) &= \langle 0| \exp(\phi^*(t+\Delta t)\hat{a})(1+\Delta t\hat{\mathcal{A}}) \exp(-\phi^*(t)\phi(t)) \exp(\phi(t)\hat{a}^\dagger)|0\rangle \\
&= e^{-\phi^*(t)\phi(t)} \sum_{n,m=0}^{\infty} \frac{\phi^*(t+\Delta t)^n \phi(t)^m}{m!} \langle n|(1+\Delta t\hat{\mathcal{A}})|m\rangle
\end{aligned} \tag{3.10}$$

note the creation operator in expansion rise the state in summation, and $\frac{1}{n!}$ from expansion cancel with annihilation operator on bra vector.

Hence using the expression $\Xi(t+\Delta t, t)$ to represent the series $\mathbb{1}_{t+\Delta t}(1+\Delta t\hat{A})\mathbb{1}_t$, collect the factors left in identity, especially note the left half of $\mathbb{1}_{t+\Delta t}$ and right half of $\mathbb{1}_0$, the time evolution $\exp(\hat{\mathcal{A}}t)$ is

$$\begin{aligned}
e^{\hat{A}t} =& \mathbb{1}_{t+\Delta t}(1+\Delta t\hat{A})\mathbb{1}_t(1+\Delta t\hat{A})\mathbb{1}_{t-\Delta t}\cdots \mathbb{1}_{\Delta t}(1+\Delta t\hat{A})\mathbb{1}_0 \\
=& \int \frac{d\phi^*(t+\Delta t) \wedge d\phi(t+\Delta t)}{2\pi i} \int \frac{d\phi^*(t) \wedge d\phi(t)}{2\pi i} \cdots \int \frac{d\phi^*(0) \wedge d\phi(0)}{2\pi i} \\
&\cdot \exp(-\phi^*(t+\Delta t)\phi(t+\Delta t) \exp\left(\phi(t+\Delta t)\hat{a}^\dagger\right) |0\rangle \\
&\cdot \Xi(t+\Delta t, t)\Xi(t, t-\Delta t)\cdot\cdots\cdot\Xi(\Delta t, 0) \\
&\cdot \langle 0| \exp(\phi^*(0)\hat{a}
\end{aligned} \tag{3.11}$$

Then we try to calculate each part in the integral, in the following we will divide the integral into three part, the series of $\Xi(t+\Delta t, t)$, the part on the left of it and the part on the right in the integral.

First we start with $\Xi(t+\Delta t, t)$, consider the term $\langle n|(1+\Delta t\hat{\mathcal{A}})|m\rangle$, where $\langle n|m\rangle = \delta_{nm}$, then we just need to focus on the part $\langle n|\Delta t\hat{\mathcal{A}}|m\rangle$. It is safe to assume that generally $\hat{\mathcal{A}}$ should be in the form of a polynomial of $\hat{a}$ and $\hat{a}^\dagger$, we can assume it is written as $\hat{\mathcal{A}} = \hat{a}^{\dagger\gamma}\hat{a}^\sigma$ with positive integer γ, σ, (normal ordered, as required), then by the definition of creation and annihilation operator

$$\left\langle n \left| \hat{a}^{\dagger\gamma}\hat{a}^\sigma \right| m \right\rangle = \frac{m!}{(m-\sigma)!}\delta_{n-\gamma,m-\sigma}$$

here the term only nonzero for $\gamma > n$ and $\sigma > m$.

Now use this result in the summation part in Eq 3.10

$$\begin{aligned}
&\sum_{n,m=0}^{\infty} \frac{\phi^*(t+\Delta t)^n\phi(t)^m}{m!}\left\langle n \left| \hat{a}^{+\gamma}\hat{a}^\sigma \right| m \right\rangle \\
&= \sum_{n=\gamma,m=\sigma}^{\infty} \frac{\phi^*(t+\Delta t)^n\phi(t)^m}{m!}\frac{m!}{(m-\sigma)!}\delta_{n-\gamma,m-\sigma} \\
&= \sum_{n=0,m=0}^{\infty} \frac{\phi^*(t+\Delta t)^{n+\gamma}\phi(t)^{m+\sigma}}{(m+\sigma)!}\frac{(m+\sigma)!}{m!}\delta_{n,m} \\
&= \phi^*(t+\Delta t)^\gamma\phi(t)^\sigma\sum_{m=0}^{\infty}\frac{\phi^{*m}(t+\Delta t)\phi^m(t)}{m!} \\
&= \phi^*(t+\Delta t)^\gamma\phi(t)^\sigma \exp\left\{\phi^*(t+\Delta t)\phi(t)\right\}
\end{aligned} \tag{3.12}$$

by rearranging the summation indices

And the other $\langle n|m\rangle$ term in Ξ is

$$\sum_{n,m=0}^{\infty} \frac{\phi^*(t+\Delta t)^n\phi(t)^m}{m!}\langle n|m\rangle = \sum_{m=0}^{\infty}\frac{\phi^*(t+\Delta t)^m\phi(t)^m}{m!} = \exp\left\{\phi^*(t+\Delta t)\phi(t)\right\} \tag{3.13}$$

Combine Eq 3.12 and 3.13 we have

$$\begin{aligned}
\Xi(t+\Delta t, t) &= \exp(-\phi^*(t)\phi(t))\left(\exp\left(\phi^*(t+\Delta t)\phi(t)\right)\left(1+\Delta t\phi^*(t+\Delta t)^\gamma\phi(t)^\sigma\right)\right) \\
&= \exp\left\{\left(\phi^*(t+\Delta t)-\phi^*(t)\right)\phi(t)\right\}\left(1+\Delta t\phi^*(t+\Delta t)^\gamma\phi(t)^\sigma\right)
\end{aligned} \tag{3.14}$$

Now $\Xi(t+\Delta t, t)$ in this from is a lot easier to dealt with.It is worth noting that now there is no more operator left in $\Xi(t+\Delta t, t)$, the nature of operator $\hat{\mathcal{A}} = \hat{a}^{\dagger\gamma}\hat{a}^\sigma$ is natural captured by $\phi^*(t+\Delta t)^\gamma \phi(t)^\sigma$, this is why we say this transform seems like easily substitute $\hat{a}^\dagger$ as ϕ^* and $\hat{a}$ as ϕ. We will then show the rest part of Eq 3.11 can also be rewritten as translate the operators to fields.

We have obtain the different representation of time evolution operator in 3.11, we will now consider the expectation of observable $\langle\hat{\mathcal{O}}\rangle = \left\langle \Phi \left| \hat{\mathcal{O}} e^{\hat{\mathcal{A}}t} \mathcal{J} \right| 0 \right\rangle$.We consider the part to the right of $\Xi(t+\Delta t, t)$ in this, which is the left part of $\Xi(t+\Delta t, t)$,(the last row in Eq 3.11), act on $\mathcal{J}|0\rangle$, we denoted as right cap C_r

$$C_r = \langle 0 \left| \exp\left(\phi^*(0)\hat{a}\right) \mathcal{J} \right| 0 \rangle \tag{3.15}$$

similar the left cap is the left part of $\Xi(t+\Delta t, t)$, (the second row of Eq 3.11), act on $\langle\Phi|\hat{\mathcal{O}}$ from the left, which is

$$C_l = \left\langle \Phi \left| \hat{O} \exp\left\{-\phi^*(t+\Delta t)\phi(t+\Delta t)\right\} \exp\left(\phi(t+\Delta t)\hat{a}^\dagger\right) \right| 0 \right\rangle \tag{3.16}$$

For the right cap, since the operator $\mathcal{J}$ act on vacuum as an initiator, it create the initial state from vacuum, we can assume that it is a power of creator, since it is normal ordered, any annihilator on the right and act on vacuum will kills the state. We assume that $\mathcal{J} = \hat{a}^{\dagger r}$

$$\begin{aligned} C_r &= \left\langle 0 \left| \sum_{n=0}^{\infty} \frac{\hat{a}^n}{n!} \phi^{*n}(0) \hat{a}^{\dagger r} \right| 0 \right\rangle \\ &= \sum_{n=0}^{\infty} \langle n \left| \phi^{*n}(0) \right| r \rangle = \phi^{*r}(0) \end{aligned} \tag{3.17}$$

For the right cap, we can assume the a random observable as $\hat{\mathcal{O}} = \hat{a}^{\dagger s}\hat{a}^l$, which is normal ordered. Remind that the abyss is invariant under any creator from the left as in Eq 2.10, the creator in $\hat{\mathcal{O}}$ will be absorbed by abyss, so we have $\hat{\mathcal{O}} = \hat{a}^l$, then the left cap is

$$\begin{aligned} C_l &= \exp\left(-\phi^*(t+\Delta t)\phi(t+\Delta t)\right) \left\langle \Phi \left| \hat{a}^l \sum_{n=0}^{\infty} \frac{\phi^n(t+\Delta t)}{n!} \hat{a}^{\dagger n} \right| 0 \right\rangle \\ &= \exp\left(-\phi^*(t+\Delta t)\phi(t+\Delta t)\right) \sum_{n=0}^{\infty} \left\langle \Phi \left| \hat{a}^l \frac{\phi^n(t+\Delta t)}{n!} \right| n \right\rangle \\ &= \exp\left(-\phi^*(t+\Delta t)\phi(t+\Delta t)\right) \sum_{n=l}^{\infty} \frac{\phi^n(t+\Delta t)}{n!} \frac{n!}{(n-l)!} \langle \Phi \mid n-l \rangle \\ &= \exp\left(-\phi^*(t+\Delta t)\phi(t+\Delta t)\right) \sum_{n=0}^{\infty} \frac{\phi^{n+l}(t+\Delta t)}{(n+l)!} \frac{(n+l)!}{n!} \langle \Phi \mid n \rangle \\ &= \exp\left(-\phi^*(t+\Delta t)\phi(t+\Delta t)\right) \phi^l(t+\Delta t) \exp(\phi(t+\Delta t)) \end{aligned} \tag{3.18}$$

Combining Eq 3.14, 3.17 and 3.18, we have the expectation of observable as

$$
\begin{aligned}
&\langle \hat{O} \rangle(t) = \left\langle \Phi \middle| \hat{O} \exp\left(\hat{A}t\right) J \middle| 0 \right\rangle \\
&= \int \left(\prod_{t'=0}^{t+\Delta t} \frac{d\phi^*(t') \wedge d\phi(t')}{2\pi i} \right) \cdot \exp\left(-\phi^*(t'+\Delta t)\phi(t'+\Delta t)\right) \left\langle \Phi \middle| \hat{O}\phi(t'+\Delta t)\hat{a}^\dagger \middle| 0 \right\rangle \\
&\cdot \left(\prod_{t'=0}^{t} \Xi\left(t'+\Delta t, t'\right) \right) \cdot \langle 0 \left| \exp\left(\phi^*(0)\hat{a}\right) \mathcal{J} \right| 0 \rangle \\
&= \int \mathcal{D}\phi \quad \underline{C_l} \underbrace{\left(\prod_{t'=0}^{t} \Xi\left(t'+\Delta t, t'\right) \right)} \cdot C_r \\
&= \int \mathcal{D}\phi \quad \underline{\phi^l(t'+\Delta t)} \exp\left\{ \underline{\phi(t'+\Delta t)\left(1-\phi^*(t'+\Delta t)\right)} + \underbrace{\sum_{t'=0}^{t} \left[\phi^*\left(t'+\Delta t\right) - \phi^*\left(t'\right)\right] \phi\left(t'\right)} \right\} \\
&\quad \cdot \left(\underbrace{\prod_{t'=0}^{t'+\delta t} \left(1 + \Delta t \phi^{*\gamma}\left(t'+\Delta t\right) \phi^\sigma\left(t'\right)\right)} \right) \phi^{*r}(0)
\end{aligned}
\tag{3.19}
$$

here the underline and underbracket is just to indicate where the term came from, Ξ or the caps. In the equation we use the notation similar to path integral

$$
\mathcal{D}\phi = \prod_{t'=0}^{t+\delta t} \frac{d\phi^*(t) \wedge d\phi(t')}{2\pi i}
$$

Consider the product over each time step Δt, when taking the step in a small limit, we can write each product as a expansion

$$
\prod_{t'=0}^{t} \left(1 + \Delta t \phi^*\left(t'+\Delta t\right)^\gamma \phi\left(t'\right)^\sigma\right) = \exp\left(\sum_{t=0}^{t} \Delta t \phi^*\left(t'+\Delta t\right)^\gamma \phi\left(t'\right)^\sigma \right)
$$

Hence is term can be add up into the exponent, the expectation will be in this more tidy from

$$
\langle \hat{O} \rangle(t) = \int \mathcal{D}\phi \quad \phi^l(t+\Delta t) \exp\left\{ \left(1-\phi^*(t+\Delta t)\right)\phi(t+\Delta t) + \sum_{t'=0}^{t} \mathcal{A}_{\Delta t}\left(t'\right) \right\} \phi^*(0)^r
\tag{3.20}
$$

where the action is

$$
\mathcal{A}_{\Delta t}\left(t'\right) = \left(\phi^*\left(t'+\Delta t\right) - \phi^*\left(t'\right)\right)\phi\left(t'\right) + \Delta t \phi^*\left(t'+\Delta t\right)^\gamma \phi\left(t'\right)^\sigma
$$

Now we have achieve the stage when there is no operator left in our expression, all the operators are captured by fields, but the action is still not very convenient to carry further calculation.

When the time step is in a small limit, the first term can be seem as a time derivative of ϕ^*

$$\mathcal{A}_{\Delta t}(t') = \Delta t \left(\dot{\phi}^*(t')\,\phi(t') + \phi^*(t')^{\gamma}\,\phi(t')^{\sigma} \right) + \Delta t^2 \cdots$$

The summation over time by a small time step of this action is just the Reimann sum to approximate the integral

$$\mathcal{A}' = \lim_{\Delta t \to 0} \sum_{t'=0}^{t} \mathcal{A}_{\Delta t}(t') = \int_0^t dt' \dot{\phi}^*(t')\,\phi(t') + \phi^*(t')^{\gamma}\,\phi(t')^{\sigma}$$

Note that there is a tricky term $(1 - \phi^*(t + \Delta t))\,\phi(t + \Delta t)$ in the exponent in Eq 3.20, we can now introduce a shift in ϕ^* to remove the non-linearity, it called the Doi-shift as

$$\widetilde{\phi}(t) = \phi^*(t) - 1 \tag{3.21}$$

This might look like a substitution, however we only shift the conjugate field independently, this is due to our result in Eq 3.6. We can also show the shift will not change the expectation by a simple example by consider

$$\begin{aligned}
\langle 0| \exp(a)\hat{a}^m \tilde{a}^n |0\rangle =& \left\langle 0 \left| \sum_{i=0}^{\infty} \frac{\hat{a}^i}{i!} \hat{a}^m (\hat{a}^\dagger - 1)^n \right| 0 \right\rangle \\
=& \left\langle 0 \left| \sum_{i=0}^{\infty} \frac{\hat{a}^i}{i!} \hat{a}^m \sum_{j=0}^{n} C_j^n \hat{a}^{\dagger j} (-1)^{n-j} \right| 0 \right\rangle \\
=& \sum_{i=0}^{\infty} \sum_{j=0}^{n} C_j^n \frac{(-1)^{n-j}}{i!} (i+m)! \delta_{i+m,j} \\
=& \sum_{j=m}^{n} C_{n-j}^n \frac{j!}{(j-m)!} (-1)^{n-j} \\
=& \frac{n!}{(n-m)!} \sum_{k=0}^{n-m} C_{n-m-k}^{n-m} (-1)^{n-m-k} \\
=& \frac{n!}{(n-m)!} (1-1)^{n-m} = n! \delta_{n,m}
\end{aligned}$$

going through the similar calculation as we did in the section 3.2 ,terms out it is exactly the same as before the shift.

Rewrite the action in terms of Doi-shifted field, using integration by part

$$\begin{aligned}\mathcal{A} &= \int_0^t dt' \quad \dot{\tilde{\phi}}(t')\,\phi(t') + \phi^\sigma(t')\left(\tilde{\phi}(t')+1\right)^\gamma \\ &= \tilde{\phi}(t)\phi(t) - \tilde{\phi}(0)\phi(0) + \int_0^t dt \left\{-\tilde{\phi}(t')\,\dot{\phi}(t') + \phi^\sigma(t')\left(\tilde{\phi}(t')+1\right)^\gamma\right\}\end{aligned}$$

now we shall look at the surface term, in Eq 3.20 there is a $(1-\phi^*(t+\Delta t))\phi(t+\Delta t) = -\tilde{\phi}(t+\Delta t)\phi(t+\Delta t)$ which cancel with the first surface term $\tilde{\phi}(t)\phi(t)$, the expectation of observable is now

$$\langle\hat{\mathcal{O}}\rangle(t) = \int \mathcal{D}\phi\,\phi^l(t)\exp(-\tilde{\phi}(0)\phi(0) + A)(\tilde{\phi}(t)+1)^r$$

The second surface term $-\tilde{\phi}(0)\phi(0)$ required more attention, as it terms out can be ignored in and did not mention too much in other discussion, in Pruessner's note[20] there is more detailed explanation, basically we note that there is no need to initiate the state from the vacuum and even more there is no difference to end the state with abyss or vacuum if the vacuum is absorbing, which means once the state hit vacuum, it stays empty. This is recognize when we look at initial time $t = 0$,

$$\int \frac{\mathrm{d}\tilde{\phi}^*(0)\wedge\mathrm{d}\phi(0)}{2\pi i}\exp(-\tilde{\phi}(0)\phi(0))(\tilde{\phi}(0)+1)^{\mathrm{r}} = \delta_{\mathrm{r},0}$$

remind that we set initiator $\mathcal{J} = \hat{a}^{\dagger r}$, the result $r = 0$ looks strange as we do not want to restrain the initial state in our assumption, but it is enforced in the field theory as the result we are probing the evolution of the system in a infinitesimal time step. Then we are able to take the system back to before the initial state, by a time t_0.

Consider a slightly adjustment to our expression of expectation of observable Eq 3.1

$$\langle\hat{\mathcal{O}}\rangle(\mathrm{t}) = \left\langle\Phi\left|\hat{\mathcal{O}}\exp(\hat{\mathcal{A}}\mathrm{t})\mathcal{J}\exp\left(\hat{\mathcal{A}}\left(-\mathrm{t}_0\right)\right)\right|0\right\rangle \tag{3.22}$$

Without going into detailed calculation, it turn out the system will be translate into fields in the same way as before. More importantly, this idea shows us the way to push back the system, even to infinity time in the past, $t_0 \to -\infty$ and also push forward to the future, $t \to \infty$. Hence the action is

$$\mathcal{A} = \int_{-\infty}^{\infty} dt' \left(-\tilde{\phi}(t')\,\dot{\phi}(t') + \phi^\sigma(t')\left(\tilde{\phi}(t')+1\right)^\gamma\right) \tag{3.23}$$

and the observable is

$$\langle\hat{O}\rangle(t) = \int \mathcal{D}\phi\,\phi^l(t)\exp(\mathcal{A})(\tilde{\phi}(t)+1)^r \tag{3.24}$$

Another advantage of pushing time backward and forward to infinity is that now we are free to perform the Fourier transform to work in momentum fields.

3.4 Fourier transform

The convention of Fourier transform we will use in this thesis is introduced

$$\phi(\mathbf{k},\omega) = \int dt\, d^d x\, \phi(\mathbf{x},t) \exp(i\omega t - i\mathbf{k}\cdot\mathbf{x}) \tag{3.25}$$

and

$$\phi(\mathbf{x},t) = \int \text{đ}\omega\, \text{đ}^d\mathbf{k}\, \phi(\mathbf{k},\omega) \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{x}) \tag{3.26}$$

where $\text{đ}\omega = d\omega/2\pi$

Notice that the fourier transform of $\phi^*(\mathbf{x},t)$ is not just the complex conjugate of $\phi(\mathbf{k},\omega)$, in face , we have $(\phi(\mathbf{k},\omega))^* = \phi^*(-\mathbf{k},-\omega)$. Focus on the transform of time for now to ease the notation, we will introduce the position later. In the case with the Doi-shifted field, it worth looking into

$$\tilde{\phi}(\omega) = \int dt\tilde{\phi}(t) \exp(i\omega t) = \int dt\, (\phi^*(t) - 1) \exp(i\omega t) = \phi^*(-\omega) - \not{\delta}(\omega) \tag{3.27}$$

where $\not{\delta}(\omega) = 2\pi\delta(\omega)$.

Recall our process have the master equation with evolution operator as in Eq2.14, the hopping and spontaneous extinction are all taking a bilinear part $\tilde{\phi}\phi$, which act as the dominant and effective part in the process as later in the discussion of perturbation theory. We will focus on the fourier transform of bilinear part of action, combine Eq 2.14 and Eq 3.23, we have

$$\mathcal{A}_0 = -\int_{-\infty}^{\infty} dt\, \tilde{\phi}\,(t')\, \dot{\phi}\,(t') + \varepsilon\tilde{\phi}\,(t')\,\phi\,(t') \tag{3.28}$$

Apply fourier transform to action $\mathcal{A}_0$

$$\begin{aligned}
&\mathcal{A}_0 = -\int_{-\infty}^{\infty} dt'\tilde{\phi}\,(t')\,\dot{\phi}\,(t') + \varepsilon\tilde{\phi}\,(t')\,\phi\,(t') \\
&= -\int_{-\infty}^{\infty} dt \int \text{đ}\omega' \int \text{đ}\omega''\, \tilde{\phi}\,(\omega'')\, \frac{\partial}{\partial t}\left(\phi(\omega)e^{-i\omega' t}\right) e^{-i\omega'' t'} + \varepsilon\tilde{\phi}\,(\omega'')\,\phi\,(\omega')\, e^{-i(\omega'+\omega'')t} \\
&= -\int_{-\infty}^{\infty} dt' \int \text{đ}\omega' \int \text{đ}\omega'' \left\{\tilde{\phi}\,(w'')\,(-i\omega')\,\phi\,(\omega') + \varepsilon\tilde{\phi}\,(\omega'')\,\phi\,(\omega')\right\} \exp\left(-i\left(\omega' + w''\right)t'\right) \\
&= -\int \text{đ}\omega' \int \text{đ}\omega''\widetilde{\phi}\,(w'')\,(-i\omega' + \varepsilon)\,\phi\,(w')\,\delta(w'' + \omega'') \\
&= -\int \text{đ}\omega\tilde{\phi}(-\omega)(-i\omega + \varepsilon)\phi(\omega)
\end{aligned} \tag{3.29}$$

hence the action is local except at $\omega = 0$ due to an extra term $\not{\delta}(\omega)$, the problem we need to think about is that in the calculation of expectation Eq 3.24 can the Gaussian integral still applied to our bilinear part

$$\frac{1}{\pi}\int dxdy \exp\left(-z^*\mathbf{A}z\right) = \frac{1}{\mathbf{A}} \tag{3.30}$$

Recall the result of Eq3.7 the integral invariant under independent shift in field, we have the equation in the form

$$\begin{aligned}\int \frac{d\phi^*\wedge d\phi}{2\pi i}\exp(-\tilde{\phi}(-\omega)\mathbf{A}\phi(\omega)) &= \int \frac{d\phi^*\wedge d\phi}{2\pi i}\exp(-(\phi^*(\omega)+\not{\delta}(\omega))\mathbf{A}\phi(\omega)) \\ &= \int \frac{d\phi^*\wedge d\phi}{2\pi i}\exp(-\phi^*(\omega)\mathbf{A}\phi(\omega))\end{aligned}$$

hence $\langle\hat{\mathcal{O}}\rangle(\omega)$ can be calculated using Gaussian result.

Now we proceed to fourier transform the space. Naturally the fourier transform will take a lattice form like

$$\mathcal{A}_0 = -\sum_{\mathbf{y}}\int \not{d}\omega\left(\phi^*\left(\mathbf{y},\omega\right)\left(-i\omega+\epsilon\right)\phi\left(\mathbf{y},\omega\right)\right)$$

However we can also take a continuum limit in space, consider the lattice with a lattice spacing a, absorbing all the renormalization and rescaling into the integral $\sum_{\mathbf{x}} a^d$ become $\int d^dx$, the current $\mathcal{A}_0$ is

$$\mathcal{A}_0 = -\int \not{d}^d\mathbf{k}\not{d}\omega\tilde{\phi}(-\mathbf{k},-\omega)(-i\omega+\varepsilon)\phi(\mathbf{k},\omega)$$

Look at the master equation for the process, the $\tilde{\phi}(-\mathbf{k},-\omega)(-i\omega+\varepsilon)\phi(\mathbf{k},\omega)$ interpret the part of time derivative and the spontaneous extinction part of master equation, there is also a hopping term that will gives contribution to effective action, the hopping term in field is

$$\mathcal{A}_H = -\frac{H}{2q}\int dt\sum_{\mathbf{x}}\sum_{\mathbf{e}}(\tilde{\phi}(\mathbf{x}+\mathbf{e},t)-\tilde{\phi}(\mathbf{x},t))(\phi(\mathbf{x}+\mathbf{e},t)-\phi(\mathbf{x},t)) \tag{3.31}$$

Similarly we take the continuum limit in space, to the first order is

$$\begin{pmatrix}\tilde{\phi}\left(\mathbf{x}+\mathbf{e}_x,t\right)-\tilde{\phi}(\mathbf{x})\\ \tilde{\phi}\left(\mathbf{x}+\mathbf{e}_y,t\right)-\tilde{\phi}(\mathbf{x})\\ \cdots\\ \tilde{\phi}\left(\mathbf{x}+\mathbf{e}_z,t\right)-\tilde{\phi}(\mathbf{x})\end{pmatrix}\cdot\begin{pmatrix}\phi\left(\mathbf{x}+\mathbf{e}_x,t\right)-\phi(\mathbf{x})\\ \phi\left(\mathbf{x}+\mathbf{e}_y,t\right)-\phi(\mathbf{x})\\ \cdots\\ \phi\left(\mathbf{y}+\mathbf{e}_z,t\right)-\phi(\mathbf{x})\end{pmatrix} = a\nabla\tilde{\phi}(\mathbf{x})\cdot a\nabla\phi(\mathbf{x})$$

with $\mathbf{e}_i$ denoted the basis of unit vectors, the summation over unit vectors shall runs over the opposite direction of these unit vectors, hence the hopping action is

$$\mathcal{A}_H = -\frac{H}{2q}\int dt \int d\mathbf{x}\ 2a^2\nabla\tilde{\phi}(\mathbf{x},t)\cdot\nabla\phi(\mathbf{x},t) \tag{3.32}$$

We do not want the term vanish when taking the continuum limit $a \to 0$, and when taking the continuum limit the hopping rate should stay the same, hence the hopping frequency $H \propto a^{-2}$ so that . We would like to we set $D = Ha^2/q$ to a constant, then the fourier transform of the hopping term is

$$\begin{aligned}
\mathcal{A}_H &= -D\int d^d\mathbf{x}\int đ^d\mathbf{k}'\int đ^d\mathbf{k}''\nabla\left(\tilde{\phi}\left(\mathbf{k}',-\omega\right)e^{i\mathbf{k}'\mathbf{x}}\right)\nabla\left(\phi\left(\mathbf{k}'',\omega\right)e^{i\mathbf{k}''\mathbf{x}}\right) \\
&= -D\int d^d\mathbf{x}\int đ^d\mathbf{k}'\int đ^d\mathbf{k}''\left(i\mathbf{k}'\right)\left(i\mathbf{k}''\right)\tilde{\phi}\left(\mathbf{k}',-\omega\right)\phi\left(\mathbf{k}'',\omega\right)e^{i(\mathbf{k}'+\mathbf{k}'')\mathbf{x}} \\
&= -D\int đ^d\mathbf{k}'\int đ^d\mathbf{k}''\left(i\mathbf{k}'\right)\left(i\mathbf{k}''\right)\delta\left(\mathbf{k}'+\mathbf{k}''\right)\widetilde{\phi}\left(\mathbf{k}',-\omega\right)\phi\left(\mathbf{k}'',\omega\right) \\
&= -D\int đ^d\mathbf{k}\quad \mathbf{k}^2\cdot\tilde{\phi}\left(-\mathbf{k},-\omega\right)\phi(\mathbf{k},\omega)
\end{aligned} \tag{3.33}$$

here we use $(i\mathbf{k}')(i\mathbf{k}'')\delta(\mathbf{k}'+\mathbf{k}'') = \mathbf{k}'\cdot\mathbf{k}''\delta(\mathbf{k}'+\mathbf{k}'')$

Then we are left with the spontaneous creation term, the field representation of this is

$$\mathcal{A}_\beta = \beta\int d^d\mathbf{x}dt\tilde{\phi}(\mathbf{x},t)$$

the fourier transform of this term is trivial,

$$\begin{aligned}
\mathcal{A}_\beta =&\beta\int d^d\mathbf{x}dt\int đ\omega đ^d\mathbf{k}\,\tilde{\phi}(\mathbf{k},\omega)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \\
&= \beta\int đ\omega đ^d\mathbf{k}\,\tilde{\phi}(\mathbf{k},\omega)\delta(\mathbf{k})\delta(\omega)
\end{aligned}$$

3.5 Field representation

Combining the hopping,spontaneous extinction and creation, we have finally arrive the field representation of our action

$$\boxed{\begin{aligned}
\mathcal{A} = -&\int đ^d\mathbf{k}đ\omega\,\tilde{\phi}\left(-\mathbf{k},-\omega\right)\left(-i\omega + D\mathbf{k}^2+\varepsilon\right)\phi(\mathbf{k},\omega) \\
+&\int đ^d\mathbf{k}đ\omega\,\beta\tilde{\phi}(\mathbf{k},\omega)\delta\left(\mathbf{k}'\right)\delta(\omega)
\end{aligned}} \tag{3.34}$$

The observable is measured as

$$\boxed{\begin{aligned}\langle\hat{\mathcal{O}}\rangle\left(x_1,t_1;x_2,t_2\right)&=\left\langle\phi^l\left(x_1,t_1\right)\phi^{*r}\left(x_2,t_2\right)\right\rangle\\&=\int\mathcal{D}\phi\,\phi^l\left(x_1,t_1\right)\exp(\mathcal{A})\left(1+\tilde{\phi}\left(x_2,t_2\right)\right)^r\end{aligned}} \tag{3.35}$$

In this chapter, we have introduce the general methods to perform fields theory to a reaction diffusion process.Form now on, for any reaction diffusion process with known master equation, we can easily arrived to such field representation. The standard procedure is, first rewrite the master equation in operator and state, then write down the time evolution operator with action $\hat{\mathcal{A}}$. Then we transform into the field representation by perform simple substitution of $\hat{a}\to\phi$ and $\hat{a}^\dagger\to\phi^*$. Next we introduce Doi-shift in field ϕ^* to $\tilde{\phi}$, then we have the fields in real space in the from Eq 3.23. Finally we perform the fourier transform to arrive the action in Eq 3.34.

What we still not quiet sure is how to analyze the process in field theory, in next chapter we will introduce the perturbation theory and how the Feynman Diagram simplify the analysis.

Chapter 4

Feynman Diagram

In this chapter we will start do some exact calculation about our process. We will apply the perturbation theory to the system, we will find the exact propagator in the bilinear term and dealt with other terms perturbatively and the correction fue to the can be shown with Feynman diagram, finally we will discuss the example with extra interacting terms.

4.1 Perturbation Theory

4.1.1 Bare propagator

First we focus on the bilinear part of our action, since we wish to apply the Gaussian result to the integral of observable in Eq 3.35, which is effective for our action except for the spontaneous creation term which is not in bilinear form. We will now set the $\beta = 0$ to obtain the bilinear result.

A important observable to calculate is the term $\tilde{\phi}\phi$ the correlation, which in this case is also known as bare propagator

$$
\begin{aligned}
&\left\langle \phi\left(\mathbf{k}_1, \omega_1\right) \tilde{\phi}\left(\mathbf{k}_2, \omega_2\right) \right\rangle \\
&= \int \mathcal{D}\phi \, \exp\left\{ - \int \not{d}^d\mathbf{k} \not{d}\omega' \, \tilde{\phi}(-\mathbf{k}, -\omega) \left(-i\omega + D\mathbf{k}^2 + \varepsilon \right) \phi(\mathbf{k}, \omega) \right\} \phi\left(\mathbf{k}_1, \omega_1\right) \tilde{\phi}\left(\mathbf{k}_2, \omega_2\right) \\
&= \frac{\delta\left(\mathbf{k}_1 + \mathbf{k}_2\right) \delta\left(\omega_1 + \omega_2\right)}{-i\omega + D\mathbf{k}^2 + \varepsilon}
\end{aligned}
\tag{4.1}
$$

this propagator is all called respond propagator,it measures how the system ”respond” to maybe a creation of particle somewhere in the process.

For clarity and get rid of the δ function in the discussion when not needed, now introduce the notation

$$\left\langle \phi(\mathbf{k},\omega)\tilde{\phi}\left(\mathbf{k}_0,\omega_0\right)\right\rangle = \delta\left(\mathbf{k}+\mathbf{k}_0\right)\delta\left(\omega+\omega_0\right)G_0(\mathbf{k},\omega) \tag{4.2}$$

with

$$G_0(\mathbf{k},\omega) = \frac{1}{-i\omega + D\mathbf{k}^2 + \varepsilon}$$

for simplicity sometime we just write $\langle \phi(\mathbf{k},\omega)\tilde{\phi}\left(\mathbf{k}_0,\omega_0\right)\rangle = G_0(\mathbf{k},\omega)$ as bare propagator. This is working as long as the space time invariance is valid, which is the reason why the δ function comes into the picture in the first place. However if the symmetry is broken, the solution can be adjust explicitly.

And it terms out, this is the only observable that we are going to need which has to go through such integration. The reason is that we could extract such propagator by Wick contraction from observable, hence we can apply the Wick Theorem, any observable in the power of $\tilde{\phi}$ and ϕ could be written as the sum of all the full contraction, in all possible pairing in ϕ and $\tilde{\phi}$, as example

$$\begin{aligned}
\left\langle \phi(\mathbf{k}_3,\omega_3)\phi(\mathbf{k}_2,\omega_2)\tilde{\phi}(\mathbf{k}_1,\omega_1)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle &= \left\langle \phi(\mathbf{k}_3,\omega_3)\tilde{\phi}(\mathbf{k}_1,\omega_1)\right\rangle \left\langle \phi(\mathbf{k}_2,\omega_2)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle \\
&+ \left\langle \phi(\mathbf{k}_3,\omega_3)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle \left\langle \phi(\mathbf{k}_2,\omega_2)\tilde{\phi}(\mathbf{k}_1,\omega_1)\right\rangle \\
&+ \left\langle \phi(\mathbf{k}_3,\omega_3)\phi(\mathbf{k}_2,\omega_2)\right\rangle \left\langle \tilde{\phi}(\mathbf{k}_1,\omega_1)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle
\end{aligned} \tag{4.3}$$

however the last term turns out to be zero, recall the integral identity in Eq 3.7, this term is case with $n=0$ and $m=2$ and the observable is zero by δ_{mn}.

Notice that $\langle \phi(\mathbf{k},\omega)\phi(\mathbf{k}_0,\omega_0)\rangle$ is recognize as correlation function which measures the correlation of different particles, it is reasonable to have it zero in our bilinear model.

Now we will introduce the Feynman Diagram representation of the propagator, which is a straight line with arrow. We choose the convention that the arrow point from the right to the left, with the right end denoted the shifted creation field $\tilde{\phi}(\mathbf{k}_0,\omega_0)$, and the arrow points to left end with annihilation field $\phi(\mathbf{k},\omega)$, to indicate the direction of time (or say causality in Fourier transform).

$$
\begin{aligned}
\left\langle\phi(\mathbf{k},\omega)\tilde{\phi}(\mathbf{k}_0,\omega_0)\right\rangle &= \phi \longleftarrow \tilde{\phi} \\
&= \frac{\delta\left(\mathbf{k}_1+\mathbf{k}_0\right)\delta\left(\omega_1+\omega_0\right)}{-i\omega+D\mathbf{k}^2+\varepsilon}
\end{aligned} \tag{4.4}
$$

It is always important to keep the arrow in our propagator, to ensure the respond propagator connects ϕ and $\tilde{\phi}$. When proceed to perturbation later, the diagram representation will be proved very helpful.

It is convenient to work in Fourier transformed fields, one might also need to derive the propagator Eq 4.1 in real space time. Start with transforming the time, we have the standard result for G_0

$$
\begin{aligned}
G_0(\mathbf{k},t) &= \int \text{đ}\omega G(\mathbf{k},\omega)e^{-i\omega t} \\
&= \int \text{đ}\omega \frac{1}{-i\omega+D\mathbf{k}^2+\varepsilon}e^{-i\omega t} \\
&= \theta(t)\exp\left(-t\left(D\mathbf{k}^2+\varepsilon\right)\right)
\end{aligned}
$$

where $\theta(t)$ is a Heaviside Step function, which gives zero for negative argument, and 1 for positive. The full expression of bare propagator is

$$
\begin{aligned}
\left\langle\phi(\mathbf{k},t)\tilde{\phi}\left(\mathbf{k}_0,t_0\right)\right\rangle &= \int \text{đ}\omega\,\text{đ}\omega_0 e^{-i\omega t}e^{-i\omega_0 t_0}\left\langle\phi\left(\mathbf{k},\omega\right)\tilde{\phi}\left(\mathbf{k}_0,\omega_0\right)\right\rangle \\
&= \int \text{đ}\omega\,\text{đ}\omega_0 e^{-i\omega t}e^{-i\omega_0 t_0}\delta\left(\mathbf{k}+\mathbf{k}_0\right)\delta\left(\omega+\omega_0\right)G_0(\mathbf{k},\omega) \\
&= \delta\left(\mathbf{k}+\mathbf{k}_0\right)\int d\omega e^{-i\omega(t-t_0)}G_0(\mathbf{k},\omega) \\
&= \delta\left(\mathbf{k}+\mathbf{k}_0\right)\theta\left(t-t_0\right)\exp\left(-\left(t-t_0\right)\left(D\mathbf{k}^2+\varepsilon\right)\right)
\end{aligned} \tag{4.5}
$$

For Fourier Transform in space, we will use the Gaussian integral result

$$
\int \text{đ}^d\mathbf{k}\exp\left(-tD\mathbf{k}^2\right)e^{i\mathbf{k}\mathbf{x}} = \frac{1}{(4\pi Dt)^{d/2}}\exp\left(-\mathbf{x}^2/(4Dt)\right) \tag{4.6}
$$

Hence the Fourier transform in space of the bare propagator is

$$
\begin{aligned}
\left\langle\phi(\mathbf{x},t)\tilde{\phi}\left(\mathbf{x}_0,t_0\right)\right\rangle &= \int \text{đ}^d\mathbf{k}\int \text{đ}^d\mathbf{k}_0 e^{i\mathbf{k}\mathbf{x}}e^{i\mathbf{k}_0\mathbf{x}_0}\delta\left(\mathbf{k}+\mathbf{k}_0\right)\theta\left(t-t_0\right)\exp\left(-\left(t-t_0\right)\left(D\mathbf{k}^2+\varepsilon\right)\right) \\
&= \int \text{đ}^d\mathbf{k}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}_0)}\theta\left(t-t_0\right)\exp\left(-\left(t-t_0\right)\left(D\mathbf{k}^2+\varepsilon\right)\right) \\
&= \theta\left(t-t_0\right)\exp\left(-\varepsilon\left(t-t_0\right)\right)\int \text{đ}^d\mathbf{k}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}_0)}\exp\left(-\left(t-t_0\right)D\mathbf{k}^2\right) \\
&= \theta\left(t-t_0\right)\frac{\exp\left(-\varepsilon(t-t_0)\right)}{\left(4\pi D\left(t-t_0\right)\right)^{d/2}}\exp\left(-\frac{\left(\mathbf{x}-\mathbf{x}_0\right)^2}{4D\left(t-t_0\right)}\right)
\end{aligned} \tag{4.7}
$$

4.1.2 Perturbation

In this section we will deal with the rest of the action Eq 3.34, which is the spontaneous creation, a linear part $\beta\tilde{\phi}$. We can introduce a shift in field to absorb this part into bilinear part, however this will leads to several consequences to the system, one of that is it will leads to changes in initialisation, it might causes unstable in vacuum under perturbation. To apply a shift we might need to examine the extra term at initialisation, to avoid this, we will treat the spontaneous creation with perturbation theory.

We first start with the general theory, consider a action in the from $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, then the observable is measured as

$$\langle \mathcal{O} \rangle = \mathcal{N}^{-1} \int \mathcal{D}\phi \mathcal{O} \exp(\mathcal{A}) = \mathcal{N}^{-1} \int \mathcal{D}\phi \mathcal{O} \exp(\mathcal{A}_1 + \mathcal{A}_2))$$

with normalisation constant $\mathcal{N}$

Also consider a measure of observable under only $\mathcal{A}_1$ as basis,

$$\langle \mathcal{O} \rangle_1 = \mathcal{N}_1^{-1} \int \mathcal{D}\phi \mathcal{O} \exp\left(\mathcal{A}_1\right)$$

with notation $\langle \cdot \rangle_1$ denotes the basis $\mathcal{A}_1$. and the normalisation $\mathcal{N}_1$ is chosen so that $\langle \mathbb{1} \rangle_1 = \mathbb{1}$

Hence the full measure of observable can be expressed in term of the expectation in $\mathcal{A}_1$ basis as

$$\begin{aligned} \langle \mathcal{O} \rangle &= \mathcal{N}^{-1} \int \mathcal{D}\phi \mathcal{O} \exp(\mathcal{A}) \\ &= \mathcal{N}^{-1} \int \mathcal{D}\phi \hat{\mathcal{O}} \exp\left(\mathcal{A}_1\right) \exp\left(\mathcal{A}_2\right) \\ &= \frac{\mathcal{N}^{-1}}{\mathcal{N}_1^{-1}} \left\langle \mathcal{O} \exp\left(\mathcal{A}_2\right) \right\rangle_1 \end{aligned} \tag{4.8}$$

This is how the idea of perturbation theory works, we can then determine the observable with perturbation $\mathcal{A}_2$ by known $\langle \mathcal{O} \exp(\mathcal{A}_2) \rangle_1$.

So there are two parts in our action, the bilinear part with known solution and a linear part act as a source, then the action can be written as $\mathcal{A} = \mathcal{A}_{bil} + \mathcal{A}_{src}$,

$$\mathcal{A}_{bil} = -\int đ^d\mathbf{k} đ\omega \tilde{\phi}(-\mathbf{k}, -\omega) \left(-i\omega + D\mathbf{k}^2 - \varepsilon\right) \phi(\mathbf{k}, \omega)$$

and

$$\mathcal{A}_{src} = \int \not{d}^d\mathbf{k}d^d\omega\, \beta\tilde{\phi}(\mathbf{k},\omega)\delta(\mathbf{k})\delta(\omega)$$

In last section we have figure out the bare propagator which is the expectation in bilinear basis

$$\left\langle \phi\left(\mathbf{k},\omega\right)\tilde{\phi}\left(\mathbf{k}_0,\omega_0\right)\right\rangle_{bil} = \frac{\delta\left(\mathbf{k}+\mathbf{k}_0\right)\delta\left(\omega+\omega_0\right)}{-i\omega+D\mathbf{k}^2+\varepsilon}$$

To proceed calculation in perturbation theory, we state the Wick Theorem:

$$\left\langle \phi\left(\mathbf{k}_1,\omega_1\right)\phi\left(\mathbf{k}_2,\omega_2\right)\cdots\phi\left(\mathbf{k}_n,\omega_n\right)\tilde{\phi}\left(\mathbf{k}_1',\omega_1'\right)\cdots\tilde{\phi}\left(\mathbf{k}_m',\omega_m'\right)\right\rangle_{bil} = 0 \quad \text{for } n\neq m \tag{4.9}$$

and

$$\begin{aligned}\langle\phi\left(\mathbf{k}_1,\omega_1\right)\phi\left(\mathbf{k}_2,\omega_2\right)\cdots\phi\left(\mathbf{k}_n,\omega_n\right)\tilde{\phi}\left(\mathbf{k}_1',\omega_1'\right)\cdots\tilde{\phi}\left(\mathbf{k}_n',\omega_n'\right)\rangle_{bil}&\\ = \left\langle\phi_1\tilde{\phi}_1\right\rangle_{bil}\cdots\left\langle\phi_n\tilde{\phi}_n\right\rangle_{bil} + \text{ all possible paining in } \phi\tilde{\phi}.&\end{aligned} \tag{4.10}$$

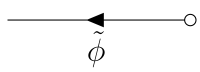
where in the right hand side we use $\phi(\mathbf{k}_n,\omega_n)=\phi_n$ and $\tilde{\phi}(\mathbf{k}_n,\omega_n)=\tilde{\phi}_n$ for simplicity, we can count that there is $n!$ ways to form $\phi\tilde{\phi}$ pairing.

Next we will consider the full expectation $\langle\cdot\rangle = \langle\cdot\exp(\mathcal{A}_{src})\rangle_1$, we will first look at the expansion of $\exp(\mathcal{A}_{acr})$ at $\mathbf{0}$

$$\exp\left(\hat{\mathcal{A}}_{src}\right) = \sum_{n=0}^{\infty}\frac{\beta^n}{n!}\phi^n(\mathbf{0},0) \tag{4.11}$$

Then the expectation $\langle\cdot\exp(\mathcal{A}_{src})\rangle_1$ will also be in the form of expansion correspond to the expansion of $\exp(\mathcal{A}_{src})$, we denote the field in expansion of $\exp(\mathcal{A}_{src})$ as internal fields and the fields from observable as external fields.

We also assign the Feynman diagram to the source, for a internal field $\tilde{\phi}(\mathbf{0},0)$ in source, we will follow the convention that Pruessner used, the contraction with another field ϕ will be expressed in diagram as



(4.12)

the bubble illustrate an "end" to any connecting annihilator field.

Since the terms in expansion of $\exp(\tilde{\phi})$ shall only match to annihilator fields by Wick theorem, we could its correction to a general observable that contain power of annihilator fields as

$$\begin{aligned}\langle \phi(\mathbf{k}_1,\omega_1)\cdots\phi(\mathbf{k}_n,\omega_n)\rangle &= \left\langle \phi(\mathbf{k}_1,\omega_1)\cdots\phi(\mathbf{k}_n,\omega_n)\exp(\hat{\mathcal{A}}_{src})\right\rangle_{bil} \\ &= \frac{\mathcal{N}_1}{\mathcal{N}}\prod_{i=1}^{n}\frac{\beta}{n!}\cdot n!\frac{\delta(\mathbf{k}_i)\,\delta(\omega_i)}{-i\omega_i + D\mathbf{k}_i^2+\varepsilon} \\ &= \frac{\mathcal{N}_1}{\mathcal{N}}\left(\frac{\beta}{\varepsilon}\right)^n\prod_{i=1}^{n}\delta(\mathbf{k}_i)\,\delta(\omega_i)\end{aligned} \tag{4.13}$$

Notice by Wick theorem, there is a factor $n!$ comes from the number of possible pairing, which cansels with the $\frac{1}{n!}$ from expansion.

We can examine the expectation of identity, which is just the $n=0$ case, using the above result

$$\mathbb{1} = \langle\mathbb{1}\rangle = \langle\mathbb{1}\exp(\mathcal{A}_{src})\rangle_{bil} = \frac{\mathcal{N}_1}{\mathcal{N}}$$

implies that $\mathcal{N}_1 = \mathcal{N}$. This tells that in perturbation theory, if the perturbative term gives 1 in its zeroth expansion, we will have $\langle\mathbf{1}\rangle = \langle\mathbf{1}\rangle_1$ and the normalisation constant $\mathcal{N} = \mathcal{N}_1$ which ensure the normalisation we have known in bilinear case, we can then keep "ignore" this in our calculation. However the higher order expansion tern in perturbation will enter the expectation of identity, we will not discuss in detail, but it turns out these "vacuum bubble" cancel in general case.

Then we consider when $n=1$, wecan examine this calculation using the result of bare propagator as Eq 4.1, we have

$$\begin{aligned}\langle\phi(\mathbf{k},\omega)\rangle &= \left\langle\phi(\mathbf{k},\omega)\exp\left(\hat{\mathcal{A}}_{src}\right)\right\rangle_{bil} \\ &= \langle\phi(\mathbf{k},\omega)\tilde{\phi}(\mathbf{0},0)\rangle_{bil} \\ &= \beta\frac{\delta(\mathbf{k})\delta(\omega)}{-i\omega+D\mathbf{k}^2+\varepsilon} \\ &= \frac{\beta}{\varepsilon}\delta(\mathbf{k})\delta(\omega)\end{aligned} \tag{4.14}$$

which is the global particle density, and its Fourier transform is

$$\begin{aligned}\langle\phi(\mathbf{x},t)\rangle &= \int \text{đ}\mathbf{k}\int \text{đ}\omega\;\frac{\beta}{\varepsilon}\delta(\mathbf{k})\delta(\omega)e^{-i\mathbf{k}\mathbf{x}}e^{-i\omega t} \\ &= \frac{\beta}{\varepsilon}\end{aligned} \tag{4.15}$$

Also consider the correction to bare propagator, using the results Eq 4.13, we can see that only the zeroth term could conttibute

$$\left\langle\phi(\mathbf{k},\omega)\tilde{\phi}\left(\mathbf{k}_0,\omega_0\right)\right\rangle=\left\langle\phi(\mathbf{k},\omega)\tilde{\phi}\left(\mathbf{k}_0,\omega_0\right)\mathbb{1}\right\rangle_{bil}=G_0(\mathbf{k},\omega)$$

The Fourier transform is

$$\begin{aligned}\left\langle\phi(\mathbf{k},t)\tilde{\phi}\left(\mathbf{k}_0,t_0\right)\right\rangle&=\left\langle\phi(\mathbf{k},t)\tilde{\phi}\left(\mathbf{k}_0,t_0\right)\right\rangle_{bil}\\&=\theta\left(t-t_0\right)\exp\left(-\left(t-t_0\right)\left(D\mathbf{k}^2+\varepsilon\right)\right)\end{aligned}\tag{4.16}$$

An interesting observable is the

$$\begin{aligned}\left\langle\phi(\mathbf{k},\omega)\phi^*\left(\mathbf{k}_0,\omega_0\right)\right\rangle&=\left\langle\phi(\mathbf{k},\omega)\left(1+\tilde{\phi}\left(\mathbf{k}_0,\omega_0\right)\right)\right\rangle\\&=\left\langle\phi(\mathbf{k},\omega)\tilde{\phi}\left(\mathbf{k}_0,\omega_0\right)\right\rangle+\langle\phi(\mathbf{k},\omega)\rangle\end{aligned}$$

which represent the particle density after placing a initial seed by creation ϕ^* . Using the result we obtained in Eq 4.13 and Eq 4.16 we have

$$\begin{aligned}\left\langle\phi(\mathbf{x},t)\phi^*\left(\mathbf{x}_0,t_0\right)\right\rangle&=\langle\phi(\mathbf{x},t)\rangle+\left\langle\phi(\mathbf{x},t)\tilde{\phi}\left(\mathbf{x}_0,t_0\right)\right\rangle\\&=\frac{\beta}{\varepsilon}+\theta\left(t-t_0\right)\frac{\exp\left(-\left(t-t_0\right)\varepsilon\right)}{\left(4\pi D\left(t-t_0\right)\right)^{2/2}}\exp\left(-\frac{\left(\mathbf{x}-\mathbf{x}_0\right)^2}{4D(t-t_0)}\right)\end{aligned}\tag{4.17}$$

4.2 Example: Branching

In this section we will consider an extra term in our process model, ”Branching”. We will followed the standard procedure we illustrate in previous chapters, from master equation in operators and state to its field representation, then branching will be dealt with perturbatively and the Feynman diagram of its contribution to the system will be drawn [21].

Branching is a process of reproduction, which is widely used in several subject such as population of bacteria[22], avalanche [23],[24] and network [25]. We define each particle can branch to two particles with Poisson rate σ, in the language of reaction, this is $A\rightarrow 2A$ reaction.

4.2.1 Action

So the master equation 2.3 will have an extra term correspond to branching, which is

$$\dot{P}\left(\left\{n_{\mathbf{x}}\right\};t\right)=\text{ previous term }+\sigma\sum_{\mathbf{x}}\left(n_{\mathbf{x}}-1\right)P\left(\left\{n_{\mathbf{x}}-1\right\};t\right)-n_{\mathbf{x}}P\left(\left\{n_{\mathbf{x}}\right\},t\right)\tag{4.18}$$

As we have done before, this term willl be translate to operator and state by act on occupation number basis

$$\sigma \sum_{\mathbf{x}} \sum_{n_{\mathbf{x}}} \left[(n_{\mathbf{x}} - 1) P\left(\{n_{\mathbf{x}} - 1\}; t\right) |n_{\mathbf{x}}\rangle - n_{\mathbf{x}} P\left(\{n_{\mathbf{x}}\}; t\right) |n_{\mathbf{x}}\rangle \right] \tag{4.19}$$

The first term

$$(n-1)P(n-1)|n\rangle = \hat{a}^\dagger (n-1)P(n-1)|n-1\rangle = \hat{a}^\dagger \hat{a}^\dagger a P(n-1)|n-1\rangle$$

The second term

$$nP(n)|n\rangle = \hat{a}^\dagger a P(n)|n\rangle$$

Hence the master equation in operator and state are

$$\frac{d}{dt}|\psi\rangle = \mathcal{A}|\psi\rangle$$

with the evolution operator $\mathcal{A}$, and we perform the Doi shift $\tilde{a} = \hat{a}^\dagger - 1$

$$\hat{\mathcal{A}}_\sigma = \sigma \left(\hat{a}^{\dagger 2} \hat{a} - \hat{a}^\dagger \hat{a} \right) = \sigma \tilde{a} \hat{a}^\dagger \hat{a} = \sigma \tilde{a} \hat{a} + \sigma \tilde{a}^2 \hat{a} \tag{4.20}$$

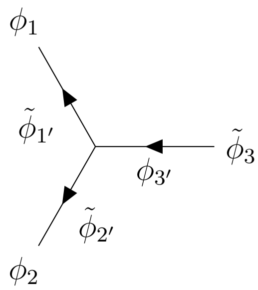
Notice the first term is bilinear in $\hat{a}\tilde{a}$, in field representation it will enter the bilinear part and gives contribution to the mass, this part indicate the increase in particle number due to branching process. The second term is a vertex, which represent the correlation generated due to branching.

We can easily write down the field representation of action.

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_{bil} + \mathcal{A}_\beta + \mathcal{A}_\sigma \\
&= -\int đ^d\mathbf{k} đ\omega\, \tilde{\phi}(-\mathbf{k}, -\omega)(-i\omega + D\mathbf{k}^2 + \varepsilon - \sigma)\phi(\mathbf{k}, \omega) \\
&+ \int đ^d\mathbf{k} d\omega\;\; \beta\tilde{\phi}(\mathbf{k}, \omega)\delta(\mathbf{k})\delta(\omega) \\
&+ \int đ^d\mathbf{k}_1 đ\omega_1 đ^d\mathbf{k}_2 đ\omega_2\;\; \sigma\tilde{\phi}\left(\mathbf{k}_1, \omega_1\right) \tilde{\phi}\left(\mathbf{k}_2, \omega_2\right) \phi\left(-\left(\mathbf{k}_1 + \mathbf{k}_2\right), -\left(\omega_1 + \omega_2\right)\right)
\end{aligned} \tag{4.21}$$

We shall look into what kind of correction the σ vertex will generate. The vertex provides three internal fields ϕ, $\tilde{\phi}$, $\tilde{\phi}$, then by Wick theorem it shall connect to other fields with ϕ, ϕ, $\tilde{\phi}$, then apparently an observable with correction from vertex is $\langle \phi(\mathbf{k}_1, \omega_1)\phi(\mathbf{k}_2, \omega_2)\tilde{\phi}(\mathbf{k}_3, \omega_3)\rangle$. The diagrammatic representation of this term is a vertex with 1 in-leg and 2 out-leg, connected to internal fields,

$$\left\langle \phi_1\phi_2\tilde{\phi}_3 \right\rangle \quad = \tag{4.22}$$



Here we simply denotes $\phi_n = \phi(\mathbf{k}_n, \omega_n)$.

Notice the arrow direction should always keep consistent, it it very important that contraction only between ϕ and $\tilde{\phi}$.

It is obvious that the only term from $\exp(\mathcal{A}_\sigma)$ gives contribution to $\langle \mathbb{1} \rangle$ is the zeroth order in expansion, again we have $\mathcal{N} = \mathcal{A}$, we do not worry about the normalisation, though $\mathcal{N}_1$ has changed since there are extra branching σ term in bilinear action.

From Eq 4.1, we can write down the bare propagator,

$$\left\langle \phi(\mathbf{k}, \omega)\tilde{\phi}\left(\mathbf{k}_0, \omega_0\right) \right\rangle = \frac{\delta\left(\mathbf{k} + \mathbf{k}_0\right)\delta\left(\omega + \omega_0\right)}{-i\omega + D\mathbf{k}^2 + \varepsilon - \sigma} \tag{4.23}$$

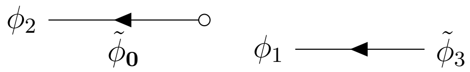
notice the term $\varepsilon - \sigma$, the bare propagator denoted as $G_0(\mathbf{k}, \omega; \varepsilon - \sigma)$

4.2.2 Diagrammatic result

Now we will start calculate the observable $\langle \phi(\mathbf{k}_1, \omega_1)\phi(\mathbf{k}_2, \omega_2)\tilde{\phi}(\mathbf{k}_3, \omega_3) \rangle$. We do not proceed this by expand the $\exp(\mathcal{A}_{pert})$ to avoid difficult calculation, we can now find out the contribution by Feynman diagram .

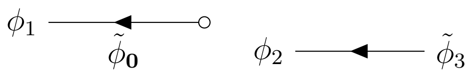
First notice there are terms not generated by vertex, but from the uniform source with $\mathcal{A}_\beta$, they are

$$\left\langle \phi_1\tilde{\phi}_3 \right\rangle_{bil} \beta \left\langle \phi_2\tilde{\phi}(\mathbf{0}, 0) \right\rangle_{bil} = \tag{4.24}$$



and

$$\left\langle \phi_2\tilde{\phi}_3 \right\rangle_{bil} \beta \left\langle \phi_2\tilde{\phi}(\mathbf{0}, 0) \right\rangle_{bil} = \tag{4.25}$$



Using the result Eq 4.23 and Eq 4.16, we could read off the propgator

$$\begin{aligned}\left\langle \phi_1\tilde{\phi}_3\right\rangle_{bil}\beta\left\langle \phi_2\tilde{\phi}(\mathbf{0},0)\right\rangle_{bil} =& \frac{\delta\left(\mathbf{k}_1+\mathbf{k}_3\right)\delta\left(\omega_1+\omega_3\right)}{-i\omega_1+D\mathbf{k}_1^2+\epsilon-\sigma}\beta\frac{\delta\left(\mathbf{k}_2\right)\delta\left(\omega_2\right)}{-i\omega_2+D\mathbf{k}_2^2+\epsilon-\sigma}\\ =& G_0(\mathbf{k}_1,\omega_1;\varepsilon-\sigma)\frac{\beta}{\varepsilon-\sigma}\delta\left(\mathbf{k}_1+\mathbf{k}_3\right)\delta\left(\omega_1+\omega_3\right)\delta\left(\mathbf{k}_2\right)\delta\left(\omega_2\right)\end{aligned} \tag{4.26}$$

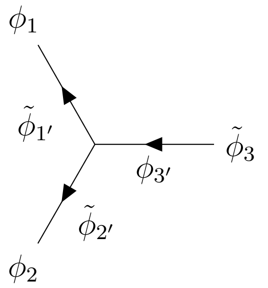
Fourier transform the result, we have

$$\left\langle \phi_1\tilde{\phi}_3\right\rangle_{bil}\beta\left\langle \phi_2\tilde{\phi}(\mathbf{0},0)\right\rangle_{bil} = \frac{\beta}{\varepsilon-\sigma}G_0(\mathbf{x}_1-\mathbf{x}_3,\omega_1-\omega_3;\varepsilon-\sigma) \tag{4.27}$$

and

$$\left\langle \phi_2\tilde{\phi}_3\right\rangle_{bil}\beta\left\langle \phi_1\tilde{\phi}(\mathbf{0},0)\right\rangle_{bil} = \frac{\beta}{\varepsilon-\sigma}G_0(\mathbf{x}_2-\mathbf{x}_3,\omega_2-\omega_3;\varepsilon-\sigma) \tag{4.28}$$

Secondly we will calculate the contribution from the branching vertex, reading the Feynman diagram we have



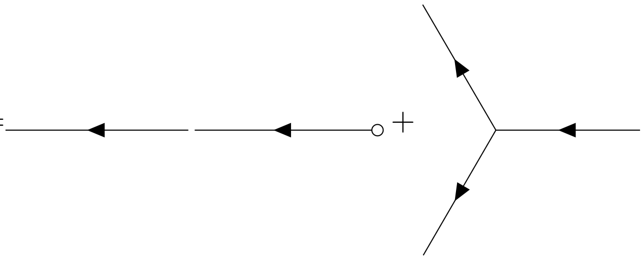
$$\begin{aligned}=& \int \not{d}^d\mathbf{k}_1'd\omega_1'\not{d}^d\mathbf{k}_2'd\omega_2'\not{d}^d\mathbf{k}_3'd\omega_3'\left\langle \phi_1\tilde{\phi}_{1'}\right\rangle_{bil}\left\langle \phi_2\tilde{\phi}_{2'}\right\rangle_{bil}\left\langle \phi_{3'}\tilde{\phi}_3\right\rangle_{bil}\\ &\times\sigma\delta\left(\mathbf{k}_1'+\mathbf{k}_2'+\mathbf{k}_3'\right)\delta\left(\omega_1'+\omega_2'+\omega_3'\right)\\ =&\sigma\frac{\delta\left(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3\right)\delta\left(\omega_1+\omega_2+\omega_3\right)}{\left(-i\omega_1+D\mathbf{k}_1^2+\epsilon-\sigma\right)\left(-i\omega_2+D\mathbf{k}_2^2+\epsilon-\sigma\right)\left(i\omega_0+D\mathbf{k}_3^2+\epsilon-\sigma\right)}\end{aligned} \tag{4.29}$$

Fourier transform it we have

$$=\sigma\int dt'd^d\mathbf{x}'G_0\left(t_1-t',\mathbf{x}_1-\mathbf{x}'\right)G_0\left(t_2-t',\mathbf{x}_2-\mathbf{x}'\right)G_0\left(t'-t_3,\mathbf{x}'-\mathbf{x}_3\right) \tag{4.30}$$

Combining the result of source and vertex, we have

$$\left\langle \phi_1 \phi_2 \tilde{\phi}_3 \right\rangle =$$



$$
\begin{aligned}
&= \frac{\beta}{\varepsilon - \sigma} \left\{ G_0(\mathbf{x}_1 - \mathbf{x}_3, \omega_1 - \omega_3) + G_0(\mathbf{x}_2 - \mathbf{x}_3, \omega_2 - \omega_3) \right\} \\
&\quad + \sigma \int dt' d^d\mathbf{x}' G_0\left(t_1 - t', \mathbf{x}_1 - \mathbf{x}'\right) G_0\left(t_2 - t', \mathbf{x}_2 - \mathbf{x}'\right) G_0\left(t' - t_3, \mathbf{x}' - \mathbf{x}_3\right)
\end{aligned}
\tag{4.31}
$$

Chapter 5

Example: Imaginary Model

5.1 Action

In this chapter we would like to see how the constructed theory work on a more complicated model, instead of investigate in a realist model, we could try on an imaginary model. We would like to see what the other diagram we could have and how they work, maybe a process gives a vertex $\phi\phi\tilde{\phi}$ similar to the branching example, without considering its physical interpretation we randomly write down a extra term to master equation that represent some imaginary stochastic process.

$$\begin{aligned} & \sum \left(g_1 \hat{a}^\dagger \hat{a}^\dagger \hat{a} + g_2 \hat{a}^\dagger \hat{a}\hat{a}\right) \\ = & \sum g_1(\tilde{a}+1)(\tilde{a}+1)\hat{a} + g_2(\tilde{a}+1)\hat{a}\hat{a} \\ = & \sum g_1 \tilde{a}\tilde{a}\hat{a} + 2g_1\tilde{a}\hat{a} + g_1\hat{a} + g_2\tilde{a}\hat{a}\hat{a} + g_2\hat{a}\hat{a} \end{aligned} \tag{5.1}$$

Notice that although the first row looks like a branching and $A + A \to A$ merging, it is not the correct master equation of that. The annihilation is discussed in [21], merging is in [8].

Follow the derivation of field theory we have discussed, the simple translation rule to field , $\hat{a} \to \phi$ and $\tilde{a} \to \tilde{\phi}$, We can see the action consist of different terms

$$\mathcal{A} = \mathcal{A}_{bil} + \mathcal{A}_{src} + \mathcal{A}_{end} + \mathcal{A}_{branch} + \mathcal{A}_{merge} + \mathcal{A}_{\phi\phi} \tag{5.2}$$

Notice that there is a bilinear term in our new process, hence it enters the bilinear term and also the bare propagator, it is the obly term in our model gives contribution to mass.

$$\mathcal{A}_{bil} = -\int \not{d}^d\mathbf{k}\not{d}\omega\tilde{\phi}(-k,-\omega)\left(-i\omega + D\mathbf{k}^2 + \varepsilon - 2g_1\right)\phi(\mathbf{k},\omega)$$

The source term is the same as previous case,

$$\mathcal{A}_{src} = \int \not{d}^d\mathbf{k}\not{d}\omega\, \beta\tilde{\phi}(\mathbf{k},\omega)\delta(\mathbf{k})\delta(\omega)$$

there is a term similar to the source, which is linear in ϕ, it is a end term

$$\mathcal{A}_{end} = \int \not{d}^d\mathbf{k}\not{d}\omega\, g_1\phi(\mathbf{k},\omega)\delta(\mathbf{k})\delta(\omega)$$

Again we have the branching vertex

$$\mathcal{A}_{branch} = \int \not{d}^d\mathbf{k}_1 d\omega_1 \not{d}^d k_2 \not{d}\omega_2\, g_1\tilde{\phi}\left(\mathbf{k}_1,\omega_2\right)\tilde{\phi}\left(\mathbf{k}_2,\omega_2\right)\phi\left(-\left(\mathbf{k}_1+\mathbf{k}_2\right),-\omega_1\right)$$

We have another vertex oppose to branching, the vertex has internal fields ϕ, ϕ and $\tilde{\phi}$, connect to $\tilde{\phi}$, $\tilde{\phi}$ and ϕ, here we call it merge vertex

$$\mathcal{A}_{merge} = \int \not{d}^d\mathbf{k}_1 d\omega_1 \not{d}^d\mathbf{k}_2 d\omega_2\, g_2\tilde{\phi}\left(\mathbf{k}_1+\mathbf{k}_2,\omega_1+\omega_2\right)\phi\left(-\mathbf{k}_2,-\omega_1\right)\phi\left(-\mathbf{k}_2,\omega_2\right)$$

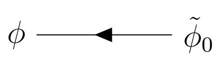
At last we have a term quadratic in ϕ, it is not very common in most realist model, still we are interested in at this point how to proceed this with our theory

$$\mathcal{A}_{\phi\phi} = \int \not{d}^d\mathbf{k}\not{d}\omega\, g_2\phi(\mathbf{k},\omega)\phi(-\mathbf{k},-\omega)$$

5.2 Propagator and Vertex

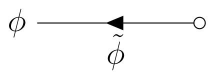
Applying the Diagram theory we derived from the last two chapter, now it is straight forward to draw the Feynman diagram for these propagator and vertices and write down the corresponding solution without too many calculations.

First is the bare propagator, notice the there is a g_1 term enters the mass similar to Eq 4.23.



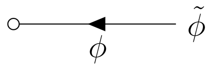
$$= \frac{\delta(\mathbf{k}+\mathbf{k}_0)\delta(\omega+\omega_0)}{-i\omega + D\mathbf{k}^2 + \varepsilon - g_1} \tag{5.3}$$

similarly we have the source term



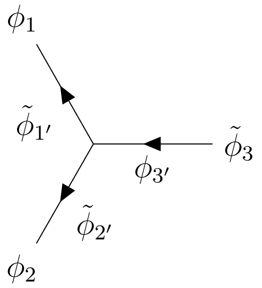
$$= \frac{\beta\delta(\mathbf{k})\delta(\omega)}{-i\omega + D\mathbf{k}^2 + \varepsilon - g_1} \tag{5.4}$$

Similar to the calculation of source, when go through the calculation of $\langle\tilde{\phi}\rangle$, we have the end term which is linear in ϕ



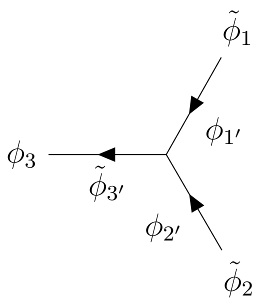
$$= \frac{g_1\delta(\mathbf{k})\delta(\omega)}{i\omega + D\mathbf{k}^2 + \varepsilon - g_1} \tag{5.5}$$

Similar to the branching vertex in Eq 4.29, we have



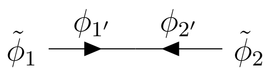
$$= \frac{g_1\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta(\omega_1+\omega_2+\omega_3)}{\left(-i\omega_1 + D\mathbf{k}_1^2 + \varepsilon - g_1\right)\left(-i\omega_2 + D\mathbf{k}_2^2 + \varepsilon - g_1\right)\left(i\omega_3 + D\mathbf{k}_3^2 + \varepsilon - g_1\right)} \tag{5.6}$$

For the merge vertex, we have 2 internal ϕ and 1 internal $\tilde{\phi}$ connect to 2 $\tilde{\phi}$ and 1 ϕ, we have



$$\begin{aligned} &= \int đ^d\mathbf{k}_1' đ\omega_1' đ^d\mathbf{k}_2' đ\omega_2' đ^d\mathbf{k}_3' đ\omega_3' \left\langle \phi_{1'}\tilde{\phi}_1\right\rangle_{bil} \left\langle \phi_{2'}\tilde{\phi}_2\right\rangle_{bil} \left\langle \phi_3\tilde{\phi}_{3'}\right\rangle_{bil} \\ &\quad \times \sigma\delta\left(\mathbf{k}_1'+\mathbf{k}_2'+\mathbf{k}_3'\right)\delta\left(\omega_1'+\omega_2'+\omega_3'\right) \\ &= \frac{g_2\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta(\omega_1+\omega_2+\omega_3)}{\left(i\omega_1 + D\mathbf{k}_1^2 + \varepsilon - g_1\right)\left(i\omega_2 + D\mathbf{k}_2^2 + \varepsilon - g_1\right)\left(-i\omega_3 + D\mathbf{k}_3^2 + \varepsilon - g_1\right)} \end{aligned} \tag{5.7}$$

At last we have the term quadratic in ϕ, it has 2 internal field ϕ each connect to a field $\tilde{\phi}$, the diagram of it is a vertex with two propagator pointing into it, later we will see how it will contribute in Feynman diagram, now we calculate it as



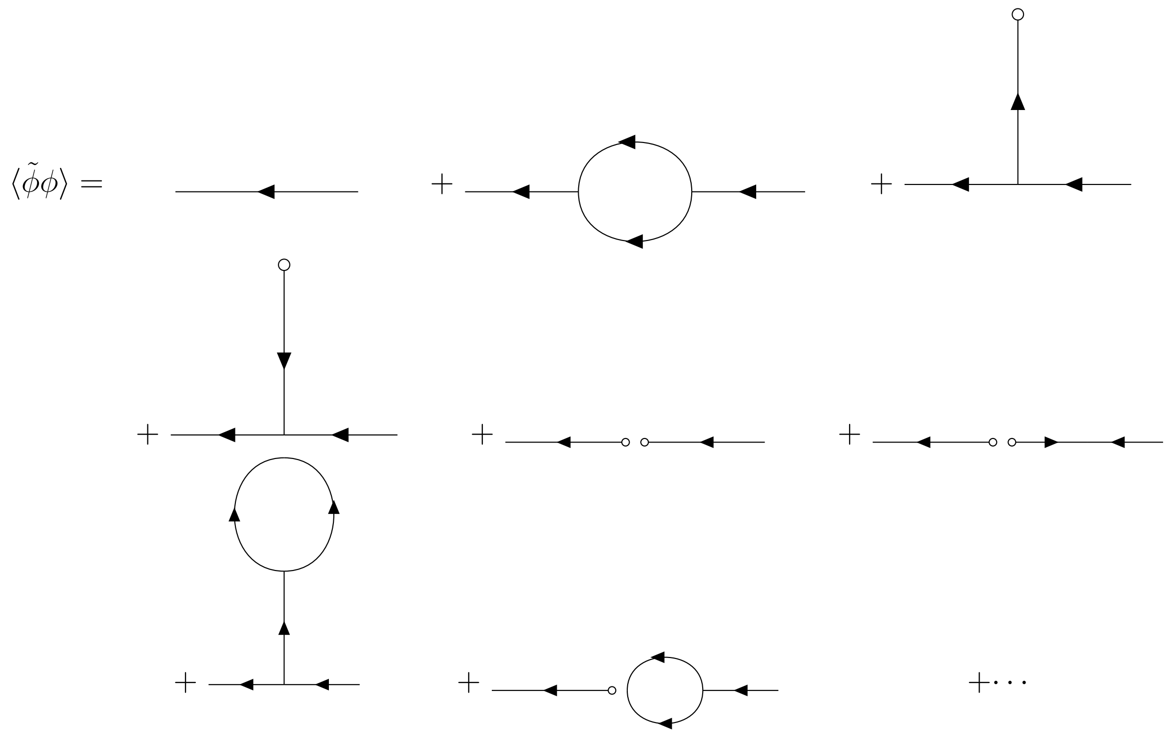
$$\begin{aligned} &= \int đ^d\mathbf{k}_1' đ\omega_1' đ^d\mathbf{k}_2' đ\omega_2' \langle\phi_{1'}\tilde{\phi}_1\rangle_{bil}\langle\phi_{2'}\tilde{\phi}_2\rangle_{bil} \cdot g_2\delta(\mathbf{k}_{1'}+\mathbf{k}_{2'})\delta(\omega_{1'}+\omega_{2'}) \\ &= \frac{g_2\delta(\mathbf{k}_1+\mathbf{k}_2)\delta(\omega_1+\omega_2)}{\left(i\omega_1 + D\mathbf{k}_1^2 + \varepsilon - g_1\right)\left(i\omega_2 + D\mathbf{k}_2^2 + \varepsilon - g_1\right)} \end{aligned} \tag{5.8}$$

The method of calculate the diagram is rather simple, the intuitive idea is count the propagator and enforce the conservation of momentum of internal fields, and keeping track of the direction of arrow is crucial.

5.3 Feynman Diagram

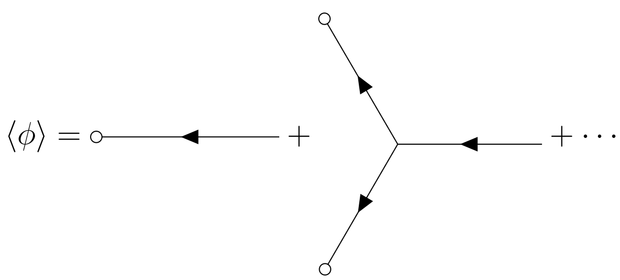
Without going further, we would like to see with our current propagator and vertices, what kind of correction will they form.

First to notice, with our vertices structure, the normalisation is apparently not conserved $\mathcal{N} \neq \mathcal{N}_1$, also the in this case $\langle \phi\tilde{\phi}\rangle \neq \langle \phi\tilde{\phi}\rangle_{bil}$, we can see there are loops contribute to the respond



They do not seems to cancel, these are also apparently contributions to expectation of identity with some vary in external legs, hence there is non-vanished vacuum bubbles. If we took this theory further, we will have to worried about such normalisation problem.

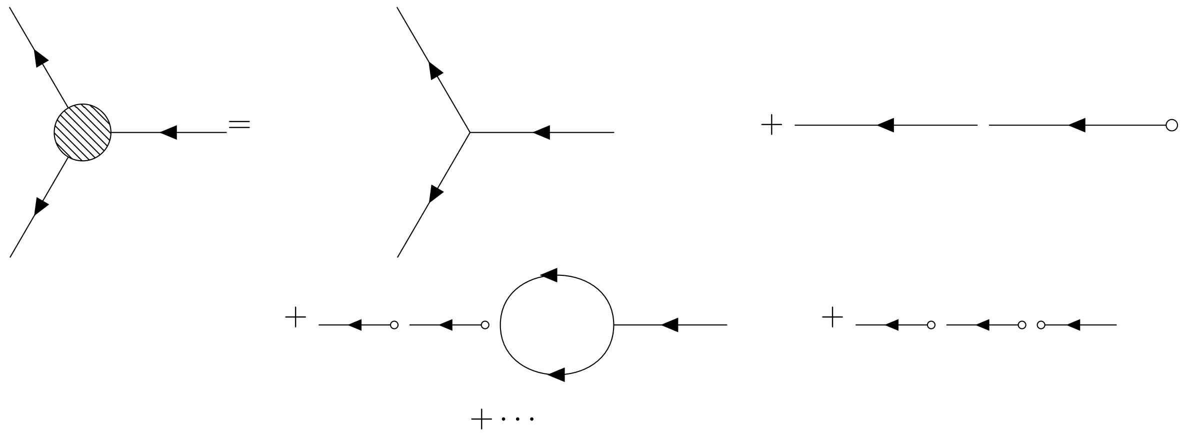
Another observable we shall pay attention to is the $\langle\phi\rangle$ as



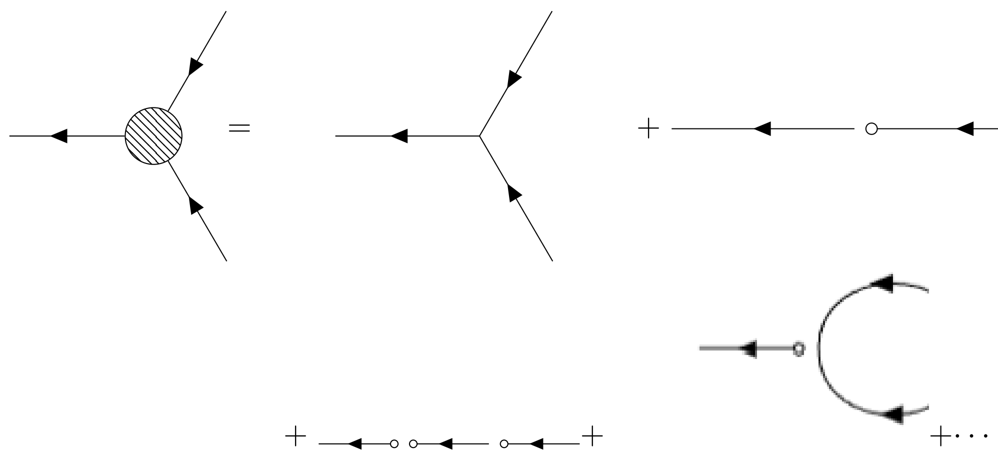
(5.9)

Such correction to the measure of field may suggest we are not expanding at the right place. A method to try is that we could diagonalize the action, we could make a shift to ϕ and $\tilde{\phi}$, maybe a rotation or other transformation

Also we would like to see the effect to the case of vertices, we will consider $\langle \phi\phi\tilde{\phi} \rangle$ and $\langle \phi\tilde{\phi}\tilde{\phi} \rangle$, using the diagrammatic representation as below, notice the shaded area represent the different loops.



and



In these diagram we avoid the loop on external legs like those we discussed in $\langle \phi\tilde{\phi} \rangle$. It consists of the disconnected diagrams and the branching and merging vertices as expected, similar to the branching example discussed in last chapter. The different terms in this example is due to the strange $\phi\phi$ vertex we have in this model. After all we are considering an imaginary model, such term is not common in general study of non-equilibrium stochastic process, the realistic physical interpretation is not clear. We are hoping with proper renormalisation [8], transformation of field and further methods applied, such diagram will become more sensible.

Chapter 6

Conclusion

6.1 Summary

In this paper, we introduce the Doi-Peliti Formalism on a standard non-equilibrium stochastic model. The idea is to apply the method in Quantum Field Theory in reaction diffusion process, then by treating the non-linearity perturbatively, some complicated result can be approximated by Feynman Diagram.

After the detailed calculation, we have a simple procedure that can be generally applied, which is:

- Obtain the master equation of the process.
- Rewrite the system as a mixed state in occupation number basis, translate the master equation in terms of creation and annihilation operator.
- Derive the action as time evolution operator.
- Apply Doi-shift $\tilde{a} = \hat{a}^\dagger - 1$.
- A simple "substitution" of $\hat{a} \to \phi$, and $\tilde{a} \to \tilde{\phi}$
- Take the Fourier transform to momentum space and have the expression of action.
- Divide the action into bilinear part and perturbation part.
- Read off the bare propagator from the bilinear action.

- Identify the vertex type of the perturbation by identify the internal field.
- Using the Wick theorem and Feynman Diagram to calculate the desired observable.

6.2 Further discussion

In chapter 5 we applied the above procedure in an imaginary model, the procedure proceed well until the problem occur in working perturbation theory. To make further progress, we have to consider other difficulty, such as renormalisation, transformation in fields, and symmetry breaking etc.

The calculation of observable in this article is not going very deep, our main focus in these calculation is the particle number $\phi\phi^*$. However, other statistical and dynamical observable can also be calculated under such formalism. For example, probability distribution, moment generating function and more ovservable in branching is determined in [26].

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