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In Search of Lost Concreteness: A Discourse in Physics and Philosophy

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Abstract

We inquire into the compenetration of philosophical and physical approaches to nature through Unger's distinction between higher and first order of discourse. Inside such a scheme we analyse from a philosophical standpoint the current physical paradigm, in a way similarly encountered in Smolin's work. We then follow in depth an example of how similar viewpoints are achieved through considerations stemming from the internal technical discourse present in the literature, exploring the works of Isham and Sorkin.

We thus advocate the view that the different orders should complement each other in driving physical innovation, while also recontextualising how theoretical abstraction relates to our immediate perception.

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[0] Note: in the following the word "chapter" refers to the bold text in the table of contents, the word "section" refers to the the double digit text (e.g., Preliminaries: Quantisation on General Spaces) and the word "subsection" to the triple digit one (e.g., Space-Time Foam)

Chapter 1

The Higher-Order Discourse: The Mathematisation of Nature

1.1 Introduction

If you open a book about the history of western thought the first name you might come across is the one of Thales, closely followed by Anaximander and Anaximenes. They lived around the VI century B.C. in Miletus, on the Ionian coast, and because of this they are remembered as the Ionian or Milesian school of philosophy. However, there is another name they are known as: they were the Natural Philosophers. They have this name because they were the first people to ponder about the nature of reality not in mythological terms, but in rational ones.

To the question "What is the world made of?" they answered in terms of water, for water was that substance which could be solid, liquid, gaseous and was the nutriment of all things, or in terms of air, for it was boundless and most capable of change while also making possible the breath of life.

It is easy now to look upon these theories and smile, but this is what Science in its embryonic stage was like. The Greek genius was lucid and logical, they asked important questions and were fervent mathematicians. From there stemmed a long line of natural philosophers, the naïveté of the Milesian school giving space to ever more acute insights into the nature of things, up to as late as the beginning of the XXth century, with exponents such as Mach, Poincaré and Whitehead.

The attribute "natural philosopher" begs the question of just what natural philosophy is, indeed one might easily confuse it with philosophy of science, an active research discipline present in many universities worldwide. To the contrary, natural philosophy is a now almost dead genre of philosophy.
The differences lie in their object of study, in the methods they use and in the goals of each discipline: while philosophy of science focuses on the interpretation of known theories and on the study of the practice of science, natural philosophy shares its subject matter with science proper, i.e., the world itself.

Hence, when engaging in controversy with science it only does so as part of a larger argument about nature, its goal being to distinguish what scientists indeed discover about the world from the metaphysical baggage they come equipped with when interpreting what they discover. It does so not in an interpretive and analytical manner, the way philosophy of science proceeds, but in a distinct revisionist fashion, thus dictating an agenda for science itself.

This might sound rather peculiar to someone who thinks proper scientists should not and do not have any metaphysical baggage; while we might say the early natural philosophers viewed the world through glasses tinged by the writings of Hesiod and Homer, thereby articulating their cosmology as if it was essentially a work of tragic art, surely modern scientists are unencumbered by such lenses.

To understand how ideas devoid of any technical or mathematical content can claim to speak to the direction of a technical discipline like physics we report Unger's distinction between a first-order discourse and a higher-order discourse[US14].

A first-order discourse finds its origin within a particular discipline or subfield, that is, where that discipline finds itself at a certain moment in time. Hence, it uses the accepted methods, presuppositions and arguments characteristic of that science. It is important to note that through contrasting views and new discoveries the end point of the process can result in a fundamental change about the very same presuppositions, methods of analysis and argument it had as baggage.

A higher-order discourse in contrast addresses such presuppositions directly, judging them on the basis of considerations both in and out of the discipline's paradigm. Through this process a higher-order discourse typically suggests a change in those presuppositions. Such a move would then hope to give a new way to look at the known, in order for us to chart a new path towards the unknown. Using Whitehead's definition of philosophy as the critic and creator of abstractions we evince the importance of this discipline as the terrain for the development of such proposals, though this runs counter to how philosophy is seen in many contemporary philosophical departments.

Indeed, there is a modern tendency to deny the importance of such higher-order discourse, regarding it as mere speculation too often influenced by direct perception. In such a view authority lies in the specialised sciences and the only higher-order discourse needed is the uselessness of higher-order discourses in general.

We argue here that it is dogmatic to believe either discourse to be more important; higher-order discourse simply has for too long been disregarded, hence it is the one which needs defending. By this we mean that normally progress comes from the internal conflicts of first-order discourses, but as the controversies augment in magnitude they might unknowingly trespass into higher-order discourse. The mark of progress would then reside in a dynamic exchange between different orders of discourse, and a gradual disappearance of rigid divisions.

We have now only made a claim, i.e., that much science comes with in-built presuppositions at the outset, and supra-empirical (that is, not justified by experiments) interpretations at the closing. This essay finds its purpose in showing the reasons for such a claim in the first chapter, critically evaluating these biases and proposing alternatives.
The second chapter will consider a perspective from a first-order discourse standpoint, in particular we explore the question of the deep structure of space-time, something intimately connected to the search for a quantum theory of gravity. The first-order example explored here-in will focus mainly on a thematical thread inaugurated by Isham but stemming from considerations first found in Wheeler. We see how notions of concreteness in the sense explored in the first chapter follow in distinct ways in later work by Isham and Sorkin. It follows that the narrative of the second part will be historical and that though the final themes constitute promising research directions today we will offer only a conceptual introduction to such frameworks. However, such conceptual introductions will come from a standpoint not very present in the literature.

Both parts complement each other: without the latter, the first chapter might be accused of being idle speculation, void of any physical content; without the former, the second chapter is a neutral historical perspective without any non-technical insights to be gained from it. As previously stated, the goal of the first chapter forces us to dip our toes into philosophy. We thus embark on a brief speculative review of the history of science and more particularly of what we will proceed to call the Newtonian paradigm, which has been the playing field of all of physics since the XVII century, and see the presuppositions at play in it.

1.2 The Newtonian Paradigm

The lenses through which Nature was investigated changed a few times throughout the course of western history. We claimed that the first "scientific" ones had a theatrical flair to it: through them Nature was differentiated so that each thing could be provided with its appropriate end. The celestial spheres were for things eternal and the lower ones for things immanent, each had a role in the great drama of Nature. Why then bother with the "How?" when we have the "Why?"? Why bother with the patience for minute observation, when we have the grandiose scheme of things?

This trend was characteristic of almost all Greek philosophy: there was a focus on the transcendental as opposed to the immanent, on cosmology as opposed to everyday physics, an attitude reaching its apex in the works of Plato and his realm of forms, where concepts in their purest essence lie, not corrupted by their participation in an imperfect world.

Science cannot bloom in such a worldview, for it is that imperfect world that is the one investigated by Science, hence it was with the advent of the thought of Plato's greatest student that the scientific embryo could keep developing. We are here talking about Aristotle of course, the precursor of Science: in the famous painting by Raffaello "The School of Athens" we see him discussing with his mentor; while the latter points his finger towards the sky, Aristotle by a gesture of his hand asks him to pay attention to the world of the senses instead.

detail of "The School of Athens"

When the worries of the natural philosophers were encountered by the genius of Aristotle, the question of the ultimate substance of nature still had not been settled. The answers he gave, and more specifically the concepts employed by him, form the backbone upon which the scientific mindset stands.

Aristotle is the father both of classical logic and of the metaphysics that most influenced the course of European thought, indeed for 1500 years he was simply referred to as "The Philosopher". Something explicitly expressed in his theory of logic but which is ubiquitous in our way of speaking is the notion of a subject upon which we predicate things, it is the most basic type of affirmative proposition.

Accordingly, built both in his metaphysics and in the way we talk is the notion that the basic constituent of nature is something, call it matter, which has changing properties. Natural notions of space and time derive from this picture of the world, for when looking at something for a period of time, dividing space divides matter, but dividing time leaves matter intact. This entails that time is not a fundamental property of matter, but an accident. Hence the answer that Aristotle gave to the Ionians is that the world is made of successions of instantaneous configurations of matter.[Whi25]

This was part of a much more sophisticated system of thought than the one which inaugurated western philosophy three centuries prior; however, for all its merits, Aristotelian Science was a science based on classification. What the new Science needed was a way to test hypotheses, to make models, to peer into the future; it needed measure. This came with the rediscovery of mathematics and a newly found interest in nature (mainly under the form of alchemy and magic) occurring during the Renaissance.

Mathematics under this new vest came to assume the rôle of an oracle, it became not only the language through which the laws of nature were written but the one that alone was capable of divining them. They reflected the unwavering belief in an orderly cosmos, an idea born of Greek world but kept alive by the medieval creed of the rationality of God's creation. Indeed, it is no accident that the early modern philosophical tradition most closely connected with mathematics, rationalism, is also the one most closely connected with theology.

Nature through its timeless laws came to participate in the timelessness of mathematics, this gave to space a primacy over time which only increased with the passing of the centuries and the gradual advance of Science. Something like the law of Gravitation is determined solely by the configuration of the masses at a given instant, hence honoring the classical Aristotelian ontology, and the configurations determine their own changes through a mathematically expressed law.

Time in this sense is reduced to a parameter expressing movement along a trajectory, potentiality disappears in favor of strict determinism where law-governed phenomena are explored inside a configuration space bounded by initial conditions. This formed the beginning of a tradition which at times explicitly denied or in all cases devalued the reality of time, which culminated with the advent of special and general relativity. Through special relativity came the denial of a global time, this was enforced by the relativity of simultaneity; while with the popular block-universe view of general relativity time explicitly became just another space dimension.

These two elements, the devaluing of the reality of time and the view of mathematics as the prophet of the inner workings of the universe fit like a jigsaw puzzle with one another. In the following sections we will disentangle these two twin notions, arguing for a return to concreteness, both in our view of space-time and in our view of mathematics.

1.3 A Process View of the World

The very first sentence of this essay cited three names and credited to them the idea that the world was born of a certain element, be it water or air. An historian of philosophy might have complained about this claim, arguing that this really only describes the thought of two of the three Milesians. While Thales posited water, and Anaximenes air, as the first principle of the world, Anaximander was a much more cryptic philosopher. At the dawn of western thought he wrote:

> « Whence things have their origin, thence also their destruction happens, according to necessity; for they give to each other justice and recompense for their injustice, in conformity with the ordinance of time »

Though the fragmentary evidence we have of him does not allow us to easily interpret it, we glimpse here the embryo of an idea which, though common in the Eastern philosophical tradition, became uncommon in the Western one: one prioritising becoming over being.
In their book on the reality of time[US14] Lee Smolin and Roberto Unger put the accent on how the brief history of cosmology as a science requires a switch to this view of the world, thus prioritising again first-order arguments. While this is certainly true, in this essay we want to put to trial the very metaphysics upon which much of modern science is based, thus recuperating the aforementioned higher-order discourse.

We reiterate that the seasoned physicist may be weary of argumentation about nature based on philosophical arguments and direct experience, a view with representatives such as Hawking and Weinberg. We might argue that this is a reasonable position insofar research is done within the bounds of what Kuhn calls the normal phase of science[Kuh70]. In Kuhn's scheme, science has its paradigm shifts, and such paradigm shifts are often aided by a moderate dose of philosophy. However, rather than paradigm shifts this essay champions an exchange between different orders of discourse, which would also imply a blurring between normal and revolutionary science.

At any rate, we can only ask the reader to withhold judgement until the end of this first part, for when compounded with the view of mathematics and causation we propose, the picture which emerges amounts to a reinterpretation of what has been discovered up until now and not a cancellation thereof, or a return to a pre-scientific view of the world. It also allows us to reason about what is often seen with mysticism, like the "unreasonable" effectiveness of mathematics in physics, and to quite simply have a more scientific view of the world.

Let us start by critically considering what is most concrete in our perception of the real world and what is more abstract, i.e., what requires us to construct categories of thought. What is delivered to the senses is a whole, it is a mesh of what we came to habitually consider matter situated in space and time, we see what we may call event. Starting from the primitive notion of an event is something which sounds rather unnatural, but one can reason that recognition of objects situated in space-time, the very first stage of abstraction, allows us to recognise what endures and what changes in the world: likely a better way to survive than to passively be prey of the currents of undifferentiated process.

Accordingly, abstraction of the most varied kind can give us profound insight into the nature of things, but we have to be conscious of the fact we are abstracting, and in abstraction we are choosing to discard a part of the world to more precisely analyse another. In this sense Science in the Newtonian paradigm is more readily suitable to the study of what endures and what changes in nature, that is, the study of objects. However, in building a theory of space-time we have to be careful whether in such a move we are not discarding essential information. A century old argument like Zeno's arrow shows we ought to proceed very carefully, and the problem of time in quantum gravity might only be the most recent warning. If we start instead from the more concrete fact, that the most primitive feature of the world, as it is delivered to the senses, is the notion of an event, then we can reason about the plural and relational character of events, An event can "extend over" other events the same way a slab of wood contains molecules, an analogy which brings us to speculate about the possible existence of event particles, as opposed to infinitesimal space-time points.

These notions would then give us a way to derive notions of space and time in a non-axiomatic manner, for these are precisely the product of the relational character between said events. The relational character between two non-intersecting events may then only be termed causal, in the same sense it is used in relativity (that is, it cannot be space-like, otherwise there would be no relation). For the savvy physicist this might sound like a hidden manifesto for causal set theory, but we are in fact here closely following considerations first put forward in a context informed by relativity by A. N. Whitehead[1] some 70 years before the 1987 paper by Bombelli et al introducing the idea of a causal set[Whi20]. In fact, as we will see in the second part, causal set theory is historically almost exclusively the product of first-order discourses and not such higher-order ones.

[1] For these considerations are to be found in more general character for the first time in Nagarjuna, a buddhist philosopher from the II-III century A.D., and more generally in the Abhidharma buddhist tradition. I thank Paolo Bassani for bringing this to my attention.

This view gives time a rôle very different from the one in the Newtonian paradigm; as explained in the precedent section science has classically held what we may call a filmographic view of time, where the flow of time is the result of the juxtaposition of "photographs", with the numbering of the photographs being the parameter we were talking about before. This reflects an attitude which Bergson will term in his timeless essay "Sur les données immédiates de la conscience" the spatialisation of time. This helped bring forth, and was reinforced by, a tendency born with Galileo; that which the father of phenomenology Edmund Husserl will term the mathematisation of nature[Hus36]. This term stands for the confusion of the relation between the physical world and the idealised representation in mathematical language, the move which instead of considering mathematical notions as abstractions from experience, reverses the order and lets the models become prescriptive for how reality really is.

However, arguing against the Galilean view has to reckon with the extraordinary success of mathematics in physics, what Wigner calls the unreasonable effectiveness of mathematics. We claim that such unreasonable effectiveness becomes very reasonable when one realises the character of mathematics as a whole: at the dawn of human thought mathematics arose as the perspective on Nature from a very specific standpoint, one abstracting away phenomenal particularity, but phenomenal particularity is always entrenched in time, and in the process of abstraction this latter dimension disappears as well. What remains is not the study of shapes and numbers but more generally the study of structure and abstract pattern. What Galileo and Newton brought us is then something of indeed immeasurable value, it is the discovery of a new way of constituting reality, for if the world is in some part structured then mathematics might be able to describe that structure, and as it turns out our cooled-down universe is indeed very structured.

Such a view superficially explains the success of mathematics in physics, but this argument must be laid down properly because, as it is now, lots of objections might be raised against it. Let us first take the objection raised by the expert in the domain: the mathematician. We will see how answering her queries sheds light on the limits of applicability of mathematics to the study of Nature.

The mathematician objects that mathematics has long since departed from the world of senses and every good mathematician has the feeling of discovering a timeless world outside the realm of sensory experience, recalling Raffaello's painting we might guess why such a view came to be defined as Platonism. Not only this but mathematics does not develop linearly, conjecture by conjecture; instead, every field comes laden with a certain tool kit and vast possibilities which relate to the rest of mathematics in ways that often take time to be unravelled and are unforeseen. Moreover, against what is often presupposed by non-specialists much of mathematics research is often neither deductive nor axiomatic: mathematics is infinitely richer than a formal system in the making. What the view proposed here seems to not not make sense of then is the unreasonable effectiveness of mathematics in mathematics.

To solve this conundrum we present a view that is suggested by a reading of another giant of modern philosophy: Hegel.
We claim that mathematics grows dialectically[2]: the abstractions born of enduring structure found in experience are plural and themselves structured wholes; as such, they share the possibility of a dialogue among each other. To make a more concrete example, geometrical relations describe a kind of structure that might be inherent in the formalisation of a structure of different nature, such as a group of transformations, and conversely one can analyse geometry through the viewpoint of groups of transformations.

While in Hegel's thought the dialogue between ideas replaces what was before as a synthesis, mathematics does not swap old ideas for a new one, instead it entertains the diverse syntheses and the old ideas on equal grounds, as part of the same network of ideas: mathematics itself. To understand the kind of dynamic which takes place think about a game like chess. The moment we stipulate the rules a host of theorems about that game become objectively true: they are not invented, for they have independent properties, nor discovered, for they did not exist before, they are instead evoked into existence. Now a game is born "in a vacuum", mathematical concepts are instead born of each other. A large part of mathematics is hence spent on the study of consequences of new modes of thought, consequences which are indeed genuinely discovered.

The mistake the Platonist does then is to think the product of the internal dialectic to be completely determined a priori. This of course cannot be the case in the strict sense of the word for mathematics cannot be reduced to an axiomatic system, thanks go to Gödel, but also in the larger sense it is a view which leaves things to be desired: the conceptualisation of a group by Galois or that of the diagonal argument by Cantor are best seen as feats of imaginative freedom which give us new ways of interpreting and relating the rest of mathematics; it does not make a lot of sense to talk about transfinite numbers outside of the conceptualisation of the diagonal argument, they are evoked into existence by it instead.

In this sense it does not serve us in any way to posit an abstract realm of all mathematics in the same sense that it does not serve us in any way to posit a realm of all possible ideas in human history that is being actualised. We can instead reason about why the "mathematical cosmos" feels so boundless: mathematics grows by unfathomably big steps, a new concept opens the door to a new, often large, set of relations which is indeed genuinely discovered; since mathematicians are relatively small in number, the class of potentially new mathematics in the adjacency of current mathematical knowledge dwarfs that which is indeed actualised.

[2] We here bring many examples from Smolin's and Unger's treatment; however, we believe Smolin's treatment to fall at times into mild conventionalist positions (thereby failing to make justice to the discipline's interconnectedness) and Unger's in mild empirical ones (thereby failing to make justice to the evoking pattern Smolin identifies). With the name dialectic we want to suggest instead an in-between similar in character to the process outlined by Hegel regarding the history of ideas, though an in-depth discussion is outside our scope.

What actualises certain dialogues and the ideas they're born of is a complex historical context, what actualises certain mathematical entities and charts the development of mathematics is instead the sense of importance mathematicians give to certain questions and methods over others, for at any point a large number of trivial open problems and valueless new definitions exist. In such a way mathematics is a self-fueling endeavour, "a divine madness of the human spirit" to quote Whitehead, a process which of its own accord abandons the certainty of being relevant to experience in exchange for the self-fulfilling promise of a never-ending advance.

In fact mathematics creative character as we described it is a particular instance of a much more general principle common to process thought, which is to believe in the radical reality of novelty. Indeed, for Whitehead novelty amounts to the most fundamental principle of the world: not only humans but Nature itself is capable of producing kinds of processes or forms which have no precedent. These take the form of novel phenomena described by novel laws: in biological evolution, cells, eukariotic cells, multicellullarity, species, all give constraints which make possible new variations and novel evokings. Therefore, we see this characterises emergence in a completely different way than does the Newtonian paradigm: in it emergence is always somehow approximate because theoretically we could descend to a more fundamental level described by the rearrangement of particles with timeless properties under timeless laws. But under the process viewpoint emergence has an irreducible meaning. The possibility of the coeval appearance of laws and the phenomena these laws regulate yields the greatest consequence for physics and the greatest limit of the Newtonian paradigm, a topic we will explore below.

Having answered the mathematician we can more precisely come to understand that in such a dialectical process arise modes of thought congenial to analyse a simulacrum of the world, one without phenomenal particularity or time. This also entails that mathematics is grossly underdetermined; for each equation we might add extra terms, consistent with symmetries, whose effects are merely too small to measure given the state-of-the-art technology. We should ignore these terms not because of a mystical belief in the simplicity of Nature, but because simpler expressions provide better tools.

Hence, if mathematics helps us in the study of Nature it is only by *modeling* patterns present in Nature, that is there is always a good reason for it:
Unger brings the example of the inverse square law[US14], in Newton's law of gravitation this holds because taking the Sun's gravitational field to be radial lines extending outward from its center, the force a planet at distance d experiences depends on the density of these lines. This density is determined by dividing the number of lines by the area of the surrounding surface, since the number of lines intersecting any spherical area around the Sun remains constant but the area increases as d^2 we have the inverse square law.

A more modern example which might be of interest to the young physicist, but which falls outside the scope of this essay, is the notion of a gauge symmetry, for which we recommend this insightful paper[Sch19].

This also means that mathematics needs reasons to be applicable to the study of Nature and we have to be critical of its use. Failing to do that results in what Whitehead aptly calls the fallacy of misplaced concreteness: an example is accepting string theory as a fundamental theory on the basis that it is mathematically complete. Alternatively, it is speaking of space-time as a manifold composed of infinitesimal points instead of critically reasoning that calculus and its extensions are only powerful tools[3] and the infinite and infinitesimals are abstractions not present concretely in Nature: consider the problem of singularities encountered in general relativity, it is more reasonable to believe this to be a product of mathematics and incomplete knowledge rather than physical truth. This is a leitmotiv which will be an important impulse for the first-order discourse analysed below.

Whitehead himself, in discussing the unphysicality of Euclidean geometry is led to try and build a theory of geometry[Whi20] (complete with notions such congruence, parallels, perpendiculars, etc) which dispenses with concepts such as points and lines, replacing them instead with the primitive concept of region and defining points as the ideal limit as we get smaller and smaller regions included in one another, instead of defining regions as collections of points[4]. More generally, the history of science should teach us to be careful when making ontological conclusions based on non-observable entities of a successful theory. In the context of relativity not only there are potential new theories which allow becoming, a subject we will explore below, but there even exist theories like shape dynamics which recover all the empirical hard core of GR while being consistent with a global absolute time[Bar11]. However, the greatest consequence of the mathematisation of Nature is a different one. The precious gift of mathematics to physics comes laced with the idea of timeless laws, expressible in the language of mathematics. Thus, the tragic vision of the Greeks dies hard, as it lives on in the scientific worldview under a new vest: the decrees of Fate, the three weaving goddesses stronger than even the gods of Olympus, being now the timeless order of nature.

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[3] Indeed it is not an issue in contexts using space-time as a background and at a coarse scale, thus not being too concerned with space-time itself.

[4] This brings him to build in 1920 a theory of regions based on the relation of inclusion called mereology (etimologically, the study of parts), he then complemented it in 1929 in his magnus opus "Process and Reality" with the relation of contact, giving rise to mereo-topology, also called connection theory. It is important to note that in his treatment this is a theory of space-time and not simply of space, hence points stand for idealised point-events. Moreover, what emerges is a largely algebraised theory of geometry, something which might ring a quantum bell in the physicist's mind. However, for all his acumen as a mathematician his theory would not be considered rigorous by present-day standards, though they have been formalised in the 70s they found their primary usage in formal ontology and computer science (especially computer vision) more than in mathematics proper. The similar concepts we will encounter in the first-order discourse, i.e., pointless topologies, were instead born of development intrinsic to mathematics; what one may call in this context first-order discourse with respect to mathematics, at the risk of confusion.

The process-relational characterisation of reality opens instead a different avenue for cosmology. This can be seen as the completion of an argument initiated by Ernst Mach[5] in the beginning of the XXth century, who understood that the fixed character of Newtonian space-time refers implicitly to degrees of freedom outside the system being described, such as a distant celestial object.
We do not feel the need here to argue for relationalism, as the process view without it boils down to the postulation of an external force that produces becoming in an otherwise passive universe, this easily leads us down a transcendental path that has no place in scientific discourse. Moreover, much of physics has actually been sensible to Mach's consequential criticisms to the Newtonian absolute view of space and time; indeed, these criticisms strongly influenced Einstein himself.

The reasons why we follow Unger in saying that Mach's thought is incomplete is that he held such relational views on space and time while holding very Newtonian views about causation. Under the Newtonian standpoint, causation only makes sense if it is under the influence of a law, and the law itself is timeless.

In such a view given the initial conditions everything is theoretically determined. This is ideally structured to inquire into small, replicable subsystems of the universe, as in these instances the configuration space aligns with the practical ability of the experimenter to choose any initial state for system preparation and hence test the general law. This also means that our discussion does not amount to a rejection of physics as it is practiced today. The problem is that we evidently cannot extend this paradigm to the whole universe.

This constrains physicists to accept as immune to scrutiny the laws of Nature or to recur to metaphysical ghosts, such as causally disconnected universes, thus putting an abrupt stop to a rational line of inquiry. These latter ones are worryingly viewed as scientifically valid when combined with the anthropic principle, but such a framework is incapable of yielding testable predictions.[6]

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[5] Again, we refer mainly to thinkers coeval with modern science, for relational views can be found in Leibniz and much of Eastern philosophy

[6] This can be broadly demonstrated by the following argument: the characteristics of a universe can be categorized into two groups. The first group comprises properties that contribute to creating conditions suitable for life to exist, such as the fine-structure constant's values and the difference in mass between protons and neutrons. The second group consists of properties that have minimal impact on the suitability of a universe for life, like the masses of second- and third-generation fermions, as long as they remain significantly heavier than the first generation. The first group of properties is essential and confirming them doesn't provide evidence for any specific cosmological scenario, as we already understand that the universe supports life. To explain this without assuming our existence, a non-circular argument is required. The second group of properties is assumed to be randomly distributed across the ensemble of universes, and since they aren't linked to life-friendliness, they will also be randomly distributed within the set of life-friendly universes. Consequently, no predictions can be made regarding them; whenever one is claimed on the basis of the anthropic principle, it can always be found not to be reliant on the anthropic principle after all.[US14] Smolin has wrote extensively on this matter.

The process view in positing causation and process as the most fundamental facets of reality instead is in line with the rational side of physics: for if physics is the inquiry into the regularities of Nature then, to quote one of the greatest American philosophers, C. S. Peirce, "Law is par excellence the thing that wants a reason".[Pei91] In such a view everything is under the dominion of time, laws as well; these are merely habits of nature, regularities that emerge from what would otherwise be lawless process, failing to work as recurrent connections among recurrent phenomena. Turning the Newtonian view of causation upon its head we can ponder the possible evolution or emergence of the laws we are familiar with and ask the questions "Why these laws?", "Why these constants?".

As we do not see evidence of a change in the laws or the parameters this must have happened in energy regimes yet unobserved. This accords with the intuition that our laws are only effective. There now appears to be a large amount of freedom in how we can approach this problem: was there a singularity or a bounce in our past? Did the evolution or emergence of laws happen all at once or incrementally? Is there a chain of ancestry? if so, is this ancestry linear or does it branch? Whatever the approach most proposed evolutionary scenarios share one common crucial characteristic: they yield testable predictions.

A branching-model example is offered by Smolin[US14], called cosmological natural selection. Here black holes are conjectured to give rise to new universes with different parametres and hence the parametres evolve to maximise the chance of black hole generation, hence yielding falsifiable predictions. Other examples can be found in the linear cyclic models of Turok and Penrose, both yielding falsifiable predictions, where one can imagine changes in effective laws happening as results of transitions among vacua "adjacent" to each other in some sense of the word. We say in "some sense" because we have to be mindful of the fact we may end up reproducing the Newtonian paradigm on a higher-level, with a configuration space of possible laws. This issue is discussed at length by Unger and Smolin together with a more in depth discussion of the first-order discourse regarding cosmology; for the scope of this essay it suffices to show that there do exist directions for future research.

We will instead focus on another example of discourse: the one most directly concerned with a quest for more concrete physics, in the Whiteheadian sense outlined above. What is noteworthy, and what complements the view presented in this chapter, is that these more concrete approaches stem historically not from views such as those we spoke of until now, but from first-order discourses: considerations about the quantum and about particular mathematics. This will result in a more concrete ontology in the work of Sorkin and in a more concrete epistemology in the work of Isham, both starting from work of J. A. Wheeler and mainly following the thread traced by Isham.

We are of course not arguing that the higher-order discourse is proved to be true because of that, rather each discourse would be weaker without the other, there is no hierarchy when it comes to importance and certainly no absolute truth. Equally, we are not claiming that their work is guaranteed to prove useful in the long run, indeed numerous technical shortcomings are present, but it is on the road to concreteness.

We now want to address a possible remark: indeed, one could claim we might as well have analysed the thought of philosophers supporting an eternalist view of time like Parmenides and McTaggart, accompanied by something like the fully constrained Wheeler-DeWitt equation as the corresponding first-order discourse. Rovelli is an example of someone who engages physically and philosophically in his work but threads this road. However, there are reasons the themes of concreteness and process cannot be swapped.

In first instance is the rebuttal of the *philosophical* arguments in favor of eternalism. We should note that a general remark of the type of "senses are not reliable", when it comes to such a central feature of experience as time, falls prey to the fact that we come to such conceptualisation from that same supposedly unreliable experience. This was the great criticism of phenomenology to the cartesian skeptical stance and the standpoint from which we start our inquiry. One can put a large part of eternalist argumentations in such a category: for example, McTaggart's argument can be solved with the distinction between definite and indefinite as opposed to the A and B conceptions of time [SV21b], again highlighting the most immediate data of consciousness. To delve deeper in such matters is outside our scope, but we should note that the strongest arguments in favor of eternalism are perceived as coming from science itself[7] which is exactly the topic we are disputing in this essay. This brings us to the other issue which is that the whole conception of philosophy as a counsellor to physics, as we argued herein, is inextricably linked with the process view of reality. Only from such a picture does the role of natural philosophy come into full force; to argue the converse is to place again mathematics as the ultimate charioteer of physics.

However, after all this one might still point out: what if the world is different from the picture we painted? What if there are multiverses and what if indeed the future does exist as much as the past does? The answer is that Science should not be concerned with what ifs, but with rational argumentation from empirical evidence. Thus, the road to concreteness should be the one most travelled before possibly being rejected; alas, it is too often ignored after the most cursory glance.

[7] Indeed, Rovelli himself in books like "the order of time", though engaging in philosophy, starts his inquiry from mostly physical considerations rather than purely philosophical ones

Chapter 2

The First-Order Discourse: the Deep Structure of Space-Time

2.1 Preliminaries: Quantisation on General Spaces

2.1.1 Space-Time Foam

Our discussion will begin with the physicist who most of all reasoned about the type of higher-order questions we investigated in the first part, John Archibald Wheeler. He proposed the concept of Law without Law[Whe83], and was a foremost advocate for pre-geometric constructions of space-time[Whe80], however the thread we will follow begins with a more historically fruitful facet of his work than are the above two, one regarding the nature of space-time at the Planck Scale.

It is ironic that while he is the first physicist to explicitly break free from the notion of space-time as a smooth continuum, he was for a time the foremost advocate for a reduction of all physics to geometry. In his 1962 paper on Geometrodynamics he says:

> "There is nothing in the world except empty curved space. Matter, charge, electromagnetism, and other fields are only manifestations of the bending of space. Physics is geometry."[Whe62]

Only two years later he disavowed this position as he questioned the validity of general relativity at the Planck scale, the same way one would question the validity of the theory of electrodynamics by considering the nature of quantum fluctuations. Boxed below is the theoretical derivation behind the concept of space-time foam[Whe64].

[0] ALL B&W IMAGES OF SECTION 2.1 COME FROM [Ish90][IKR91]

Wheeler starts by considering a harmonic oscillator of mass m and frequency ω. Quantum Mechanically the standard deviation from equilibrium as the energy decreases does not go to 0 but instead to $\hbar/2m\omega$. The probability amplitude $\psi(x)$ for a particle to be in one position or another is then given by $\psi(x) = e^{-m\omega x^2/2\hbar}$

In flat space-time an electromagnetic field can be described by fields oscillators, with each field oscillator having its own independent amplitude ξ_k. The probability amplitude for the ground state field to have a determinate configuration can then be expressed as $\psi(\{\xi_k\}) = e^{-\sum_k \xi_k^2}$.

For this configuration the amplitude of the electromagnetic field is given by a Fourier series with elementary standing waves $\mathbf{B}_k(x)$ written as $\mathbf{B}(x) = \sum_k \xi_k \mathbf{B}_k(x)$ and conversely given a distribution of the magnetic field $\mathbf{B}(x)$ one can invert the preceding Fourier series, find the coefficients ξ_k and recover the probability amplitude associated with $\mathbf{B}(x)$, that is:

$$\psi(\mathbf{B}(x)) = e^{-(1/16\pi^4\hbar c)\int \mathbf{B_1}(x)\mathbf{B_2}(x) r_{12} d^3x_1 d^3x_2}$$

A fluctuation in the field then is akin to asking for the probability that over a region of length L, the averaged magnetic field deviates from 0 by an amount $\delta\mathbf{B}$. This probability will be small by comparison with the probability for other field configurations unless the exponent in the expression above is of order of one or less: $(\delta\mathbf{B})^2 L^4/\hbar c \sim 1$ and in this sense field fluctuations in a domain of extension L are of order $(\delta\mathbf{B}) \sim \sqrt{\hbar c}/L^2$.

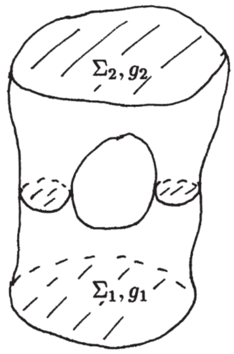
Similarly, when considering gravity, one infers thanks to dimensional arguments:

$$\delta\Gamma \sim \partial\delta g/\partial x \sim \sqrt{\hbar G/c^3}/L^2 \sim L^*/L^2$$

Where $L*$ is the Planck Length. The fluctuation in the metric itself is of order $\delta g = L^*/L$

Rather poetically Wheeler brings forth an analogy between the scale dependence of the fluctuations and the image of a pilot flying over the sea: to him it looks as if the body of water is perfectly flat, but if one were to zoom in we would see crests and breaking up into each other: this suggested the word space-time foam.

Here there is no ruling topology, rather there is a probability amplitude for each 3-geometry and the entire space-time is topologically non-trivial. Moreover, the fluctuations in the fields cause fluxes to thread throughout space: it is in this sense that the notion of virtual pair of particles being created and annihilated at every moment was born. This constituted the starting point for a variety of approaches considering a discretised space-time.



This result stands in contrast with the starting point of many canonical (both in the generalised and restricted sense of the term) approaches to quantum gravity[Ish90]. Take the Euclidean approach as an example: here the basic question that is asked is about the transition probability amplitude $K(\gamma_1, \Sigma_1; \gamma_2, \Sigma_2)$ to go from a metric γ_1 on a 3-Manifold Σ_1 to a metric γ_2 on a 3-manifold Σ_2 Without considering matter fields this is expressed as the functional integral:

$$K(\gamma_1, \Sigma_1; \gamma_2, \Sigma_2) = \sum_M \int e^{-S(g)} dg$$

Here $S(g)$ is the Einstein-Hilbert action and the sum is over every 4-manifold M whose boundary is the disjoint union of Σ_1 and Σ_2. The integral itself is over all possible Riemannian 4-metrics g on M which induce the 3-metrics γ_1 and γ_2.

It can be shown that this formalism relates back to the ADM formalism of canonical quantum gravity in the following way: if we fix γ_1 then it satisfies the Wheeler-DeWitt Equation in γ_2 appropriate for a Lorentzian space-time.[Ish90] This allows us to relate the canonical picture to an underlying space-time structure, and though it opens discussions such as whether to use Riemannian or Lorentzian metrics, what is of greatest interest for our discussion is that the sum over M suggests for quantum topology to be developed uniquely within the context of differential geometry. The failure of the functional equation above at the level of perturbative renormalizability aside, Wheeler's work as illustrated above suggests to be wary of assumptions about the smoothness of both the final spaces Σ_1 and Σ_2 and the interpolating 4-manifold M.

However, if the sum above is to include more singular topological spaces the question now becomes how to select the appropriate mathematics. Following this train of thought this discourse leads us to embark in a process of generalisation of the concept of space-time manifold, closely following in the next two subsections the discussion of Isham's seminar on quantum topology[Ish90].

2.1.2 Quantisation on Metric Spaces

In such a generalisation, let us first retain a notion of nearness of points as measured by a distance. Isham notes that if the 3-metric γ_{ab} is to become an operator in the canonical quantisation of general relativity so should the distance function induced by it. To see why that is the case recall that a space X with a distance function d is called a metric space, with the axioms for the distance function $d : X \times X \to \mathbb{R}_+$ being:

$$
\begin{aligned}
&d(x, y) \geq 0 \text{ and } d(x, y) = 0 \Leftrightarrow x = y \\
&d(x, y) = d(y, x) \\
&d(x, z) \leq d(x, y) + d(y, z)
\end{aligned}
$$

If the 1st axiom is relaxed simply to $d(x, y) \geq 0$ (i.e. there may be $x \neq y$ with $d(x, y) = 0$ then d is a pseudo-distance).

The distance d is also, confusingly, called a metric. From here on out we will reserve the name metric for the manifold metric and we will call d the distance. The relation between the two is the following; the distance induced by the metric γ on a 3-manifold Σ and the manifold itself constitute a metric space, specifically:

$$d(x, y) := \inf_{\lambda} \int (\gamma_{ab}(\lambda(t))\dot{\lambda}^a(t)\dot{\lambda}^b(t))^{1/2} dt \qquad (*)$$

where the infimum is over all piece-wise differentiable curves $t \to \lambda(t)$ in Σ which pass through x and y. It is now clear that if in the integral above γ_{ab} becomes an operator so does d.

Then an expectation value $\langle \psi, \hat{d}(x, y)\psi \rangle$ will satisfy the above conditions provided $\hat{\gamma}_{ab}\dot{\lambda}^a\dot{\lambda}^b$ is a well-defined positive operator (so that the square root may be taken).

However, implementing this idea does not get us very far. One problem is that the conventional canonical commutation relations

$$[\hat{\gamma}_{ab}(x), \hat{p}^{cd}] = i\hbar\delta^c_{(a}\delta^d_{b)}\delta(x, y)$$

are not compatible with the positivity condition outlined above: to see this consider an analogous construction of a quantum theory of a system with configuration space $\mathbb{R}_+$, we have:

$$[\hat{x}, \hat{p}] = i\hbar \Rightarrow e^{ia\hat{p}}\hat{x}e^{-ia\hat{p}} = \hat{x} + \hbar a$$

And hence the eigenvectors of $\hat{x}$ are not restricted to the positive real numbers. However, this can be resolved by the use of affine commutation relations[Ish84]: $[\hat{x}, \hat{p}] = i\hbar\hat{x}$

We quote the result that in a non-trivial irreducible representation of this algebra the $\hat{x}$-spectrum is either strictly positive or strictly negative, so we can make a choice consistent with the positivity condition.

What is the problem then? The thing is that in the expression for the distance $(*)$ we are smearing out the operator γ_{ab} over a tensor field concentrated along a curve, this will result in all likelihood in an operator that is still highly singular, so that taking the square root remains a problem. This difficulty puts a real stop at any further inquiry along these lines, however there is something else we may try: we note that the manifold's topology can be completely recovered from the distance function defined on it, i.e. the open sets are just union of open balls $B_\epsilon(y) = \{x \in X | d(x,y) < \epsilon\}$ and in particular it is independent from the choice of γ. This points to the fact that we can start with the distance function $\hat{d}$ as a canonical variable instead of the metric $\hat{\gamma}_{ab}$ and find the appropriate conjugate variable replacing $\hat{p}$. This does not lead us too far out of conventional physics, since $\hat{d}(x,y)$ can be regarded as a two-point quantum field. One thing we have to realise though is that since the distance function is non-negative we cannot recover the Lorentzian structure of space-time, this will be in contrast with approaches closer at heart to the ideas explored . However this discussion can be seen both as a generalisation of the Euclidean approach to quantum gravity and more simply as a description of the spatial part of physical space-time.

In the following [IKR91] we consider a finite topological space X for ease of treatment and consider $d(x,y)$ an observable whose configuration space is $\mathcal{M}(X)$ the space of distance functions on X. Each of these classical distance functions is the value of the observable for some state $|\Psi> \in \mathcal{H}$. This implies $\mathcal{M}(X) \subset spec(\hat{d}(x,y))$ and quantum transitions $|\Psi_1\rangle \to |\Psi_2\rangle$ correspond to a change in the distance structure, or even topology structure if the topology induced by the two distances is different.

	Quantum mechanics	Quantum metric topology
....	....	....
1.main object	particle in R^3	metric space X
2.configuration space	$R^3 = \{q_1, q_2, q_3\}$	$\mathcal{M}(X) = \{d(x,y)\}$
3.phase space	$\mathcal{F} = \{q_i, p_j\}$	?
4.quantization	$(q_i, p_j) \to (\hat{q}_i, \hat{p}_j)$	$d(x,y) \to \hat{d}(x,y)$
	$[\hat{q}_i, \hat{q}_j] = 0$	$[\hat{d}(x_1,x_2), \hat{d}(x_3,x_4)] = 0$
	$[\hat{p}_i, \hat{q}_j] = -i\delta_{ij}$	?
5.observables	$q_i, f(q)$	$d(x,y)$
	spec $\hat{q} = R^3$	$\mathcal{M}(x) \subset$ spec $\hat{d}$
6.Hilbert space of	$\|\Psi> \in \mathcal{H}$	$\|\Psi> \in \mathcal{H}$
quantum states	$f_\Psi(q) =< \Psi\|\hat{q}_i\|\Psi >$	$d_\Psi(x,y) =< \Psi\|\hat{d}(x,y)\|\Psi >$

It is clear that a typical state $|\Psi\rangle$ will not be a manifold, but we expect that for low-energy conditions the state is approximately semi-classical with X being a smooth manifold. In this case we can introduce local coordinates through an atlas and define the Riemannian metric to be given by:

$$\gamma_{ab}(x) = \frac{1}{2} \lim_{x \to y} \frac{\partial}{\partial x^a} \frac{\partial}{\partial y^b} [d(x,y)]^2$$

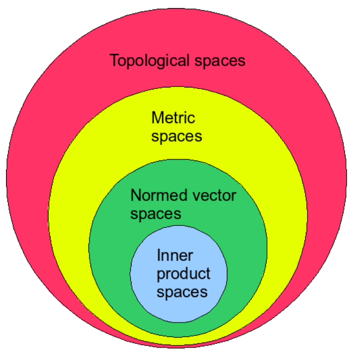
with x^a and y^b being the aforementioned local coordinates. What we now have to do is constructing a representation of $\mathcal{M}(X)$ so that we can later quantise it.

To start let us consider a metric space with just three distinct points. This entails that each metric is characterised by three non-negative numbers $t_1 = d(x_1, x_2), \quad t_2 = d(x_2, x_3), \quad t_3 = d(x_3, x_1)$ i.e. each metric is a point in $\mathbb{R}^3$. Taking the triangle inequality into consideration we have:

$$t_i \leq t_j + t_k, \text{for } i \neq j \neq k$$

which defines a subset of $\mathbb{R}^3$ whose boundary is delimited by three planes. Quantising a system with such a configuration space seems prohibitively hard, and this is just the case $|X| = 3$.

To advance we reformulate our problem in the context of normed vector spaces, which are metric spaces where objects can be said to have a length. It is then clear that if $||\cdot||$ is a norm, given two points $x, y \in V$ a distance function is induced by considering $||y - x||$. The image to the right shows the hierarchy of mathematical spaces we are considering and how determinate structures are induced by more specialised ones.



More precisely we will embed X in a vector space with a semi-norm N, which is a map $N : V \to \mathbb{R}_+$ satisfying:

$$N(\lambda v) = |\lambda| N(v) \quad \forall \lambda \in \mathbb{R} \text{ and } v \in V$$
$$N(u + v) \leq N(u) + N(v) \quad \forall u, v, \in V$$

If N also satisfies non-degeneracy, i.e. $N(v) = 0 \Rightarrow v = 0$ then N can be said to be a full norm.

We now have an embedding $i : X \to V$, which for finite X we can make canonical by taking $i(x_k) = e_k$ for $k < n$ and $i(x_n) = 0$, and as explained above we can construct a distance on X by considering

$$d(x, y) = N(i(x) - i(y)) \qquad (**)$$

It can be shown that given the embedding described above and this formula every metric can be recovered by considering some norm in V.

We can now quantise $\mathcal{M}(X)$ by varying the norm on V while leaving the embedding fixed (noting that we may have a gauge redundancy since different norms can give rise to the same distance). It does look like the problem is not much different from what we had previously trying to quantise $\mathcal{M}(X)$ directly, but the relations defining a norm are simpler in significant ways, which we shall now try and sketch.

The question is what is a suitable representation for $\mathcal{N}(V)$, the space of all norms on V? To be more precise we will deduce a dense subset of norms, such that any norm can be approximated arbitrarily well by a norm in this set, analogously to how any real number can be approximated arbitrarily well by a rational number. We now note that there exists a one to one correspondence between norms on V and subsets $C \subset V$ called barrels (absorbing [1], closed convex sets which include the origin of V and are symmetric about it, these can be interpreted as the unit balls with respect to the norm N).

The most basic barrels are so called polytopes, which on a finite space are just the convex hull of a finite number of points. It is these entities that are dense in the larger set of barrels. Polytopes are the barrels associated with seminorms characterised by sets of vectors $\Xi = \{\xi_i\}_{i=1}^{m}$:

$$N_{\infty,\Xi}(v) := \sup_{\xi_i \in \Xi} |\langle \xi_i, v \rangle|$$

We denote such a set of seminorms $\mathcal{N}_0(V)$ which is then dense in the set $\mathcal{N}(V)$. We will now proceed to construct hermitian operators $\hat{N}(v), v \in V$ which respect the operator analogues of the definition for a semi-norm given above and which commute which other:

$$[\hat{N}(u), \hat{N}(v)] = 0, \quad \forall u, v \in V$$

[1] A subset S of a vector space V is said to be absorbing if $\forall x \in V$ there is a number $c_x \geq 0$, such that, $\forall \lambda \in \mathbb{R}$ with $\lambda \leq c_x$, we have $\lambda x \in S$. Intuitively this means that one can scale S so as to include every point in the space V

Isham et Al. are unable to construct a a conjugate variable to $N(v)$ but their efforts lead them to postulate two more conditions:

1. There are canonical creation and annihilation operators $\hat{a}(\xi), \hat{a}^\dagger(\xi)$ satisfying standard bosonic commutation relations and all the operators of the systems are functions of them.

2. The norm operators are labeled by an integer label p and have the following commutation relations:

$$[\hat{N}_p(v)^p, \hat{a}(\xi)] = -|\langle \xi, v\rangle|^p \hat{a}(\xi)$$
$$[\hat{N}_p(v)^p, \hat{a}^\dagger(\xi)] = |\rangle \xi, v\langle|^p \hat{a}^\dagger(\xi)$$

These relations satisfy the operator requirements given by the seminorm definition, taking $\hat{N}_p(v)$ to be a positive function of $\hat{a}(\xi)$ and $\hat{a}^\dagger(\xi)$ given the positivity of norms.
A natural choice for $\hat{N}_p(v)$ is:

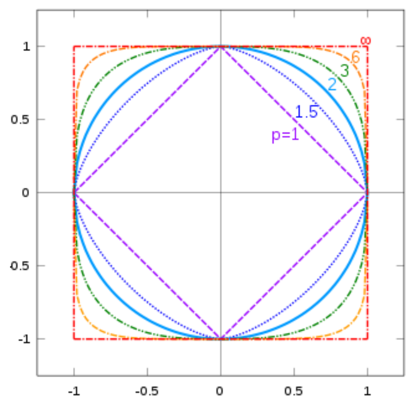
$$\hat{N}_p(v) = \left\{ \int |\langle \xi, v\rangle|^p \hat{a}^\dagger(\xi)\hat{a}(\xi) d\xi \right\}^{1/p}$$

And it becomes natural to use the Fock representation of the subalgebra given by the bosonic commutation relations, realising the Hilbert space $\mathcal{H}$ as the bosonic Fock space.
Now as usual the vacuum is annihilated by $\hat{a}(\xi)$ and $|\xi_1, \xi_2, ..., \xi_m\rangle = \hat{a}^\dagger(\xi_1)\hat{a}^\dagger(\xi_2)...\hat{a}^\dagger(\xi_m)|0\rangle$

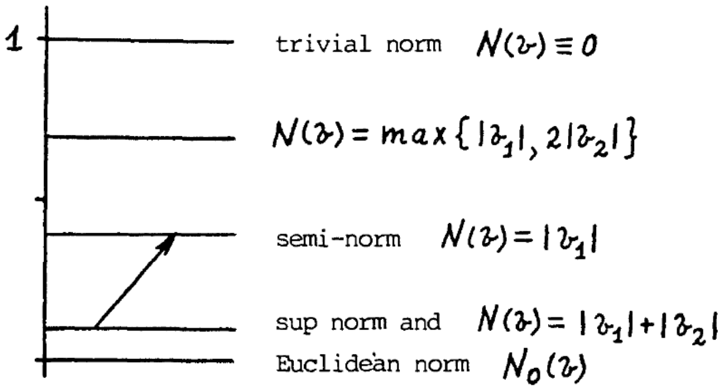
So one repeats the exercise we have done many times in QFT and gets an operator whose expectation value $\langle \xi_1, \xi_2, ..., \xi_m|\hat{N}_p|\xi_1, \xi_2, ..., \xi_m\rangle$ is $(\sum_{i=1}^m \langle \xi_i, v\rangle|^p)^{1/p}$ which when taking the formal limit $p \to \infty$ becomes:

$$\langle \xi_1, \xi_2, ..., \xi_m|\hat{N}_\infty|\xi_1, \xi_2, ..., \xi_m\rangle = \sup_{\xi_i \in \Xi} |\langle \xi_i, v\rangle|$$

Which are the norms given above! This is Quantum Norm Theory in a nutshell. To gain an intuition on why does the formal limit $(\langle \sum | \cdot \|^p \rangle)^{1/p}$ result in $\sup | \cdot |$ we show the image below which illustrates the nature of the p-norms in $\mathbb{R}^2$.



The last piece is constituted by the dynamics on $\mathcal{M}(X)$. We suppose there is a metric d_0 which represents the ground state of space-time. A natural choice for this would be the metric induced by the Euclidean norm ($p = 2$ in the image above) as this is what we usually associate with smoothness. Space-time foam is then given by excited states of our system, but to induce topology changes the required objects in $\mathcal{N}(V)$ are the semi-norms with very few spanning vectors ξ_i. Conversely, the full norms should correspond to low-excitation states.



Taking inspiration from the usual perturbative approach an hamiltonian which commutes with the norm operator is:

$$\hat{H}_0 := k \int |N_0(v) - \hat{N}(v)| \, dv$$

Picking k such that the "vacuum" state $|0\rangle$ corresponding to the trivial norm $N(v) = 0$ has energy one.

To see that it gives us the desired spectral properties consider again the simple case $|X| = 3$ implying V is two dimensional. If we rewrite dv in polar coordinates and scale out the length r from the vector v then the hamiltonian becomes:

$$\hat{H}_0 := \int r^2 dr \int |1 - \hat{N}(\theta)| \, d\theta, \qquad \text{Since } N(v) \geq 0$$

The integral over r is an infinite volume factor which can be renormalised. Applying this operator to the state $|0\rangle$ corresponding to the vacuum yields a result of one, due to the norm operator's action resulting in zero on the vacuum state. Conversely, when applied to the (generalized) state corresponding to the Euclidean norm, the operator produces a value of zero, as is expected. All remaining states yield energies ranging between these two extremes, where the seminorms exhibit higher-end energies on the scale, as we wanted.

To induce transitions between different levels we need a non-commuting interaction hamiltonian, again taking inspiration from usual QFT we pick:

$$\hat{H}_{int} := \int f(\xi)(\hat{a}^\dagger(\xi) + \hat{a}(\xi))\, d\xi$$

with $f(\xi)$ an arbitrary real function. Let us consider an example of a transition: take the same finite topological space X of dimensions 3, $V = \mathbb{R}^2$ and an initial state $|\psi_i\rangle = |e_1, e_2\rangle$ and a final one $|\psi_f\rangle = |e_1\rangle$.

These correspond to the sup norm $N_i(v) = max(|v_1|, |v_2|)$ and the seminorm $N_f(v) = |v_1|$ respectively. Using formula $(**)$ and the canonical embedding we obtain the distance $d_i(x_j, x_k) = 1 \quad \forall j \neq k$ and the pseudo-distance $d_f(x_1, x_2) = d_f(x_1, x_3) = 1,\ d_f(x_2, x_3) = 0$.

Note $\langle\psi_f|\hat{H}_{int}|\psi_i\rangle \neq 0$ and in the final state x_2 and x_3 can be identified so this represents a transition from a discrete topology on a set with three points to a discrete topology on a set with two points. this is what a topology transformation entails. We will define more carefully what we mean by discrete topology in the next page.

For this model to make contact with the usual smooth manifold picture we should take $|X|$ to be to the very least countable, and then uncountable. This has been pursued in the literature [Req06], and has possible consequences for QFT since what we are really doing by generalising the Fock construction to $|X|$ and hence $|V|$ to infinity is to build a QFT in which the arguments of the quantum fields belong to an infinite dimensional space.

However, we are interested in the dialectical dimension of these ideas, a dimension which gave rise to ideas close to the notions explored in the first part. Hence, instead of focusing on the infinite-dimensional case considered by Requardt we turn to the fact that not all topologies can be induced by a distance function, and that the notion of "nearness" can be described irrespective of any notion of distance. This leads us to a most ambitious aim, that of quantising the set of all topologies on X.

2.1.3 Quantisation on Topological Spaces

Recall that given a set X a topology τ is a collection of subsets of X, called the open sets, such that both X and the empty set $\varnothing$ are in τ, and such that τ is closed under arbitrary unions and under finite intersections of its open sets. Continuous functions are then defined as those trnasformations $f : X \to Y$ for which U open in Y implies $f^{-1}(U)$ open in X, note we do not need f to be bijective for us to be able to consider the pre-image of f.

We will now consider the set $\tau(X)$ of all topologies on a fixed set X. The main problem is the lack of any methodical way of quantising a system with a given configuration space Q: experience would suggest that the main hurdle to overcome is finding a suitable preferred algebra of observables on $Q = \tau(X)$. If Q were a manifold we could consider as q-variables the generators of the transformations along the p-direction on the phase space/cotangent bundle T^*Q, with conjugate p-variables being related to transformations of Q. Though $\tau(X)$ is certainly not a manifold we can still ask if there are any natural transformations of it which may represent a starting point.

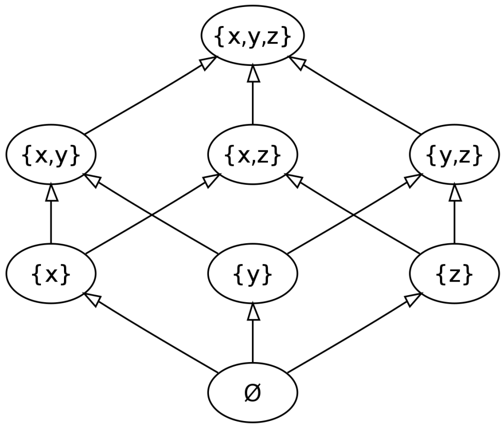
Looking at the definition of a topology one might think about varying the number of open sets a particular topology τ contains. However, appending a set A to a topology τ will generally not result in a topology, as the resulting set of subsets will generally not be closed under arbitrary unions or finite intersections. There exists however a minimal way to both add "missing sets" and substract "excess sets" so that the new set of subsets is indeed a topology, hence yielding a transformation on $\tau(X)$. What we will be working with is the lattice structure possessed by $\tau(X)$, to define what a lattice is and the machinery related to it we will be needing a brief mathematical excursus:

Posets The basic notion that will come back time and time again is that of a partially order set, or poset for short. This is a set X with a relation $\preceq$ on it which is:

1. Reflexive: $\forall x \in X,\ x \preceq x$
2. Antisymmetric: $\forall x, y \in X,\ x \preceq y$ and $y \preceq x$ implies $x = y$
3. Transitive: $\forall x, y, z \in X,\ x \preceq y$ and $y \preceq z$ implies $x \preceq z$

Note that there may be elements in X that are incomparable, but if it is true that either $x \preceq y$ or $y \preceq x$ then the relation $\preceq$ becomes a total order.
We shall use $\prec$ when $x \preceq y$ but $x \neq y$. We then say An element y in a poset X covers another element x if $x \prec y$ and there is no z such that $x \prec z \prec y$.

An important example is the partial order present on $P(X)$, the power set of a set X. To the right we give the example with $|X| = 3$. This implies a partial order on the set of topologies $\tau(X)$, where at the base of the diagram we have the indiscrete topology $\{\varnothing, X\}$ and at the top of the diagram the discrete topology $P(X)$ where every subset is open, along the tree we have $\tau_1 \preceq \tau_2$ when every τ_1-open set is a τ_2 open set: we then say that the topology τ_1 is coarser and the topology τ_2 is finer.



In any poset P a *join* or least upper bound of x and y is an element $x \vee y \in P$ such that $x \vee y$ is an upper bound of x and y, i.e. $x \preceq x \vee y$ and $y \preceq x \vee y$, and if there exists $z \in P$ such that $x \preceq z$ and $y \preceq z$ then $x \vee y \preceq z$. E.g., In the figure above $\{x, y\} = \{x\} \vee \{y\}$.
Similarly, a *meet* or greatest lower bound of x and y is an element $x \wedge y \in P$ such that $x \wedge y$ is a lower bound of x and y, i.e. $x \wedge y \preceq x$ and $x \wedge y \preceq y$, and if there exists $z \in P$ such that $z \preceq x$ and $z \preceq y$ then $z \preceq x \wedge y$. E.g., In the figure above $\{y\} = \{x, y\} \wedge \{y, z\}$.

Lattices A lattice is a poset $\mathcal{L}$ in which every pair of elements possess a join and meet. Hence, the image above is indeed a lattice, and so is the poset of topologies. Conversely a "tree" or an "upside tree" is not. Here are some definitions about lattices that will come in handy:

An Ideal is a subset $I \subset \mathcal{L}$ such that: $\quad x, y \in I \Rightarrow x \vee y \in I, \quad x \in I \text{ and } z \preceq x \Rightarrow z \in I$

A lattice $\mathcal{L}$ is said to be complete if a join and meet exist for any subset S of $\mathcal{L}$ (the definition only implies they exist for finite subsets).

A lattice $\mathcal{L}$ is said to be distributive if for all $x, y, z \in \mathcal{L}$ we have $x \wedge (y \vee z) = (x \wedge z) \vee (y \wedge z)$.

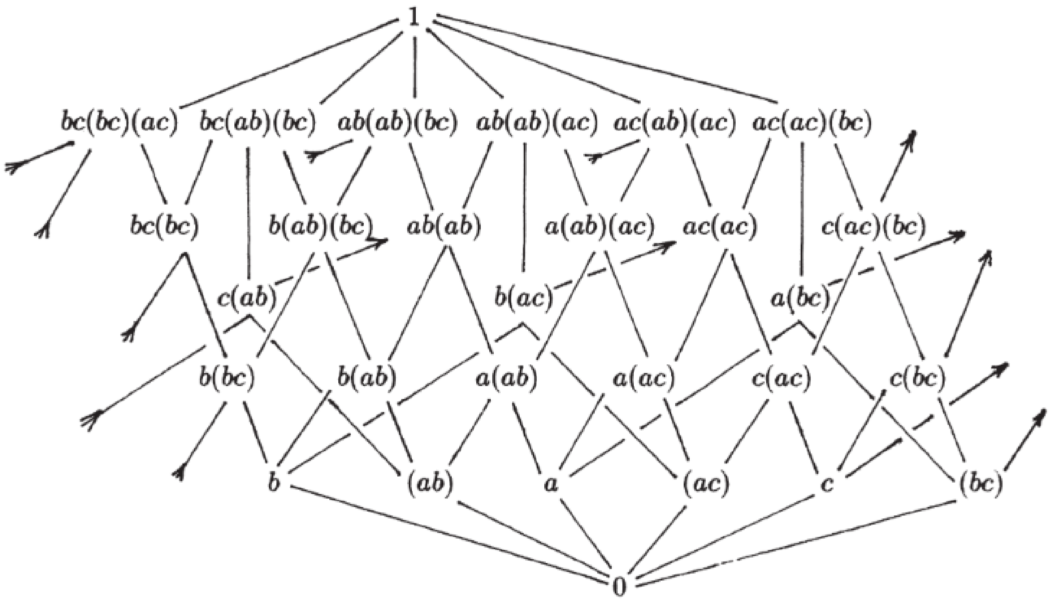
A lattice $\mathcal{L}$ is said to be complemented if it contains both a unit element (i.e. an element 1 such that $\forall x \in \mathcal{L},\ x \preceq 1$) and a null element (i.e., an element 0 such that $\forall x \in \mathcal{L},\ 0 \preceq x$) and if to each x in $\mathcal{L}$ there exists $x' \in \mathcal{L}$ such that $x \wedge x' = 0$ and $x \vee x' = 1$.

A Boolean algebra is a complemented, distributive lattice (the figure above is an example of this). Importantly, interpreting elements of the lattice as propositions, one obtains the logical structure of classical logic and standard mathematics: each proposition is complemented in the sense defined above and the distributive law applies! This will be important later on.

We can now turn back to the quantisation of $\tau(X)$[Ish89]. As a lattice it possesses the two operations:

$$\tau_1 \wedge \tau_2 := \tau_1 \cap \tau_2 = \{U \subset X | U \text{ is open in both } \tau_1 \text{ and } \tau_2\}$$
$$\tau_1 \vee \tau_2 := \text{ coarsest topology containing } \{U_1 \cap U_2 | U_1 \in \tau_1, U_2 \in \tau_2\}$$

To have a feeling about the lattice structure of $\tau(X)$ below is the lattice of topologies for our favorite $|X| = 3$ finite set.



The notation has been chosen for maximum typographical simplicity. For example, $ab(ab)(ac)$ means the topology whose open sets other than $\varnothing$ and X are the subsets $\{a\}, \{b\}, \{a, b\}$ and $\{a, c\}$

An important property is that the lattice of topologies is complete and atomic. That is, for each $A \subset X$, $\tau_A = \{\varnothing, X, A\}$ is an atom, i.e., it covers $0 = \{X, \varnothing\}$ (In the example above these are the first row starting from the bottom.). Furthermore, every topology τ is of the form $\tau = \bigvee\{\tau_A | \tau_A \preceq \tau\}$ Thus we have here a "minimal way" to add a subset A to a topology τ: by the join operation $\tau_A \vee \tau$.

Through this train of thought we identify a natural family of functions on $\tau(X)$ as:

$$R_\tau(\tau') := \begin{cases} 1, & \text{if } \tau \preceq \tau' \\ 0, & \text{otherwise} \end{cases}$$

Which for atoms entails that $R_{\tau_A}(\tau')$ is non-zero only if A is open in τ'.
We note these functions obey:

$$R_{\tau_1}(\tau')R_{\tau_2}(\tau') = 1 \Leftrightarrow \tau_1 \preceq \tau' \text{ and } \tau_2 \preceq \tau' \Leftrightarrow \tau_1 \vee \tau_2 \preceq \tau' \implies \quad R_{\tau_1}R_{\tau_2} = R_{\tau_1 \vee \tau_2} \qquad (***)$$

This also implies for all $\tau \in \tau(X)$: $(R_\tau)^2 = R_\tau$, so that when quantising $\{\hat{R}_\tau | \tau \in \tau(X)\}$ becomes a family of projection operators with above multiplication law.

Note that $(***)$ implies that $[\hat{R}_{\tau_1}, \hat{R}_{\tau_2}] = 0$ so that the algebra satisfied by the $\hat{R}$ operators is abelian. Moreover, the projector-like identity implies that the eigenvalues of $\hat{R}_\tau$ are 0 or 1. Hence, we should find a complete set of eigenvectors $|h\rangle$ such that:

$$\hat{R}_\tau |h\rangle = h(\tau)|h\rangle$$

With $h(\tau)$ being 0 or 1. We then have $h(\tau_1)h(\tau_2) = h(\tau_1 \vee \tau_2)$ We now define $I_h := \{\tau | h(\tau) = 1\}$.

Then

1. $\tau_1, \tau_2 \in I_h \Rightarrow \tau_1 \vee \tau_2 \in I_h$
2. if $\tau' \preceq \tau, h(\tau')h(\tau) = h(\tau)$. So If $\tau \in I_h, h(\tau') = 1$, hence $\tau' \in I_h$

These being exactly the properties for I_h to be an ideal! Conversely, if I is an ideal on $\tau(X)$ we define $h_I(\tau)$ to be the characteristic function of τ in I. In an ideal I, $\tau_1, \tau_2 \in I \Leftrightarrow \tau_1 \vee \tau_2 \in I$ hence $h(\tau_1)h(\tau_2) = h(\tau_1 \vee \tau_2)$. We thus conclude that there exists a bijection $h \leftrightarrow I_h$ and hence the domain space of quantum states is the set $I(\tau(X))$ of all ideals on $\tau(X)$.

This is the key to finding the nature of the domain of state vectors. Indeed, more naïvely we might have tried to produce representations of the $\hat{R}$-algebra by defining state vectors to be $\psi(\tau)$ with:

$$(\hat{R}_\tau \psi)(\tau') = \begin{cases} \psi(\tau'), & \text{if } \tau \preceq \tau' \\ 0, & \text{otherwise} \end{cases}$$

But this is only valid when $|\tau(X)|$ is finite, when the lattice is infinite it is possible for the domain of the state vectors to be a space of “distributional” topologies, We are thinking here in analogy with QFT:

In quantum field theory we usually cannot define an operator version of $(\phi(x))^2$, even if the configuration space is a vector space E the best we can do is consider polynomials in the smeared fields $\phi(f)$ where the test-function f belongs to E. It is then a result (find source) that the Hilbert space on which these fields are defined is isomorphic to one of the form: $L^2(E', d\mu)$ for some probability measure $d\mu$ on the dual space E' of E. The operator $\hat{\phi}(f)$ then acts on $\psi \in L^2(E', d\mu)$ as

$$(\hat{\phi}(f)\psi)(\chi) = \langle \chi, f \rangle \psi(\chi)$$

From the preceding analysis we find out that this space of "distributional topologies" is $I(\tau(X))$, the space of ideals on $\tau(X)$. Hence we can expect the Hilbert space to be of the form $L^2(I(\tau(X)), d\mu)$ for some measure μ on $I(\tau(X))$.

Moreover, in the same way the space of smooth functions is dense in the space of distributions (i.e. any distribution can be approximated arbitrarily closely by a smooth function) so $\tau(X)$ is dense in $I(\tau(X))$ (the embedding being $\tau \to \downarrow \tau := \{\tau' \in \tau(X) | \tau' \preceq \tau\}$, we will use this down arrow notation again below). In analogy with the above we can label $h_I(\tau)$ as $\langle I, \tau \rangle$ so that the eigenvalue equation derived earlier can be translated as the action:

$$(\hat{R}_\tau \psi)(I) = \langle I, \tau \rangle \psi(I)$$

on $\psi \in L^2(I(\tau(X)), d\mu)$. The last step in establishing this quantum theory on a solid footing would be to construct an inner product on the space of states. This entails studying the existence of suitable measures on $I(\tau(X))$. We refer to [Ish89] for a more thorough discussion of this fact.

We should consider now the construction of the complementary variables. We remark that the $\hat{R}_\tau$-operators constitute a representation of the $\vee$-operation. However, the lattice is not only atomic, it is also anti-atomic, i.e., there exist topologies τ_A which are covered by the discrete topology $1 = P(X)$ (In the example above it is the first row from the top) and such that every topology τ is of the form $\tau = \bigwedge \{\tau_A | \tau \preceq \tau_A\}$. Thus we have here a "minimal way" to substract a subset A from a topology τ: by the meet operation $\tau_A \wedge \tau$.

Hence it seems reasonable to complement the $\hat{R}_\tau$ operators with a representation for the $\wedge$-operation. The natural candidate function is $m_a : \tau(X) \to \tau(X)$ which sends τ to $a \wedge \tau$, with its extension to $I(\tau(X))$ being $m_a := \ (\downarrow a) \cap I$.
Thus on the Hilbert space $L^2(I(\tau(X)), d\mu)$ we define $\hat{M}_\tau \psi(I) := \psi(\ (\downarrow \tau) \cap I)$.
We then have the algebra:

$$\begin{aligned}
&\hat{R}_{\tau_1} \hat{R}_{\tau_2} = \hat{R}_{\tau_1 \vee \tau_2} \\
&\hat{M}_{\tau_1} \hat{M}_{\tau_2} = \hat{M}_{\tau_1 \wedge \tau_2} \\
&\hat{M}_{\tau_1} \hat{R}_{\tau_2} = \langle\ (\downarrow \tau_1), \tau_2 \rangle \hat{R}_{\tau_2} \hat{M}_{\tau_1}
\end{aligned}$$

We will thus regard this as being the basic algebra on the lattice $\tau(X)$. The associated hermitian operators are:

$$\hat{q}_\tau := (\hat{M}_\tau + \hat{M}_\tau^\dagger) \qquad\qquad \hat{\pi}_\tau := (\hat{M}_\tau - \hat{M}_\tau^\dagger)/i$$

The relation $(M_\tau)^2 = M_\tau$ then implies $(\hat{q}_\tau - 1)^2 - (\hat{\pi}_\tau)^2 = 1$, meaning that $\hat{q}_\tau$ and $\hat{\pi}_\tau$ are not independent variables.

It is important to note that such an operator $\hat{M}_\tau$ is not unitary but instead acts more as a creation or annihilation operator: as an example let X be a finite space so that quantum states are in $L^2(I(\tau(X)), d\mu) \sim \mathbb{C}^{|\tau(X)|}$ then if $|\tau'\rangle$ is an eigenstate of $\hat{R}_\tau$ then the adjoint operator $\hat{M}_\tau^\dagger$ acts to weaken the topology, while $\hat{M}_\tau$ strengthens it:

$$\hat{M}_\tau^\dagger|\tau'\rangle = (\mu(\tau')/\mu(\tau \wedge \tau')^{1/2}|\tau \wedge \tau'\rangle$$

$$\hat{M}_\tau|\tau'\rangle = \begin{cases} \frac{1}{\sqrt{\mu(\tau')}} \sum_{\{a|\tau\wedge a=\tau'} \sqrt{\mu(a)}|a\rangle, & \text{if } \tau' \preceq \tau \\ 0, & \text{otherwise} \end{cases}$$

A peculiar feature here is that the sum in the bottom expression will generally have more than one element, this has to do with the fact that the map $m_a : \tau(X) \to \tau(X)$ is not injective. Thus $\hat{M}_\tau$ is more than a simple creation operator, it also "broadens" the state in the sense that an eigenstate of the topology becomes a linear superposition of eigenstates.

Note that had we defined the minimal action to be the adding of a set A to τ (through $\tau \vee \tau_A$), then instead of starting with the $\hat{R}_\tau$ operators we would have started with the basic function:

$$S_\tau(\tau') := \begin{cases} 1, & \text{if } \tau \succeq \tau' \\ 0, & \text{otherwise} \end{cases}$$

satisfying $S_\tau S_{\tau'} = S_{\tau\wedge\tau'}$. This gives a complementary quantisation of the same lattice.

Having completed this discussion one last important property enjoyed by the lattice of topologies is its relation to Perm(X), which is the group of all permutations of X onto itself. Topologically speaking, if $\phi \in$ Perm(X) and τ is a topology on X then $\phi(\tau)$ is the topology whose open sets are $\{\phi(U)|U \in \tau\}$. The induced maps on $\tau(X)$ preserve lattice operations and thus are lattice automorphisms. Perm(X) can thus be regarded as the natural gauge group of the theory and $\tau(X)/$Perm(X) as the true "configuration space".

Given such a situation in most gauge theories we would fix a gauge on Perm(X) and quantise the remaining physical degrees of freedom but here specifying a gauge seems difficult. Other alternatives are quantising directly on $\tau(X)/$Perm(X) or quantising on $\tau(X)$ and enforcing Dirac-style constraints on physical observables/state vectors. The problem with the former alternative is that $\tau(X)/$Perm(X) does not inherit any lattice structure from $\tau(X)$. By this we mean that given two pairs of homeomorphic topologies on this space it is not the case that the join (or meet) of the first element of each pair is homeomorphic to the join (or meet) of of the second element of each pair. We will hence consider the latter option.

Since we have to enforce this on the Hilbert space $L^2(I(\tau(X)))$ we invoke a standard result from measure theory that guarantees, under quasi-invariance[2] of μ under the action of Perm(X), a unitary representation can be obtained by defining:

$$(\hat{U}(\Lambda)\psi)(I) = (d\mu_\Lambda(I)/d\mu)^{1/2}\psi(\Lambda^{-1}(I))$$

where the term $d\mu_\Lambda(I)/d\mu$ is the Jacobian (in measure theory called the Radon-Nikodym derivative) of μ_Λ with respect to μ.

Then the $\hat{R}_\tau$-variables transform covariantly with respect to Perm(X): $\hat{U}(\Lambda)\hat{R}_\tau\hat{U}(\Lambda)^{-1} = \hat{R}_{\Lambda(\tau)}$

The same is not true for the $\hat{M}_\tau$ since the Jacobian must satisfy an appropriate condition, we assume from now that such a measure has been found so that $\hat{U}(\Lambda)\hat{M}_\tau\hat{U}(\Lambda)^{-1} = \hat{M}_{\Lambda(\tau)}$. Again considering the case for finite X we then have that gauge invariant operators can be construed by defining:

$$\hat{R}_O := \sum_{\tau\in O}\hat{R}_\tau$$
$$\hat{M}_O := \sum_{\tau\in O}\hat{M}_\tau$$

Where O is an orbit of Perm(X) on $\tau(X)$. It is a result by Isham that closure holds for products of $\hat{R}_O$ only or $\hat{M}_O$ only[3], but the situation is trickier for mixed products: e.g., when $|X| = 3$

$$\hat{M}_{[b(a,c)]}\hat{R}_{[(a,b)]} = \hat{R}_{(a,b)}\hat{M}_{c(a,b)} + \hat{R}_{(b,c)}\hat{M}_{a(b,c)} + \hat{R}_{(c,a)}\hat{M}_{b(c,a)}$$

and the right hand-side is not a linear combination of terms $\hat{R}_O\hat{M}_{O'}$, though it is of the form $\sum\hat{R}_\tau\hat{M}_{\tau'}$ with (τ,τ') being in an orbit of Perm(X) on the Cartesian product $\tau(X)\times\tau(X)$. This indicates that gauge invariant operators are generally $\hat{T}_O = \sum_{(\tau,\tau')\in O}\hat{R}_\tau\hat{M}_{\tau'}$ and indeed one can prove the result $\hat{T}_O\hat{T}_{O'} = \sum n_i(O,O')T_{O_i}$ for integers n_i and where the sum is over orbits of Perm(X) in $\tau(X)\times\tau(X)$. We have hence found a set of gauge invariant operators, though only for the finite case.

[2] that is, the transformed measure μ_Λ and the original measure μ have the same sets of measure zero

[3] i.e., $\hat{R}_O\hat{R}_{O'} = \sum n_i(O,O')\hat{R}_{O_i}$ where the n_i are integers and the sum is over orbits O_i which pass through $\tau\vee\tau'$ for $\tau\in O$ and $\tau'\in O'$, a similar result holds for $\hat{M}_O$

At this point a natural question is what is the relationship between the metric topology approach of the previous section and the one based on the lattice structure of topologies. What is noteworthy is that the T_1 topologies, i.e., those for which any pair of points x, y are such that each one is contained in an open set that excludes the other, form a complete sublattice of $\tau(X)$, which shares with it the property that $\mathrm{Perm}(X)$ is its group of automorphisms; we can hence use them as an alternative space for quantisation. The upshot is that T_1 topologies are related to distance-functions in the sense that any such topology can be written as the lattice product of all finer metric topologies[Ish89].

Lastly, comes the question of the dynamics. Isham in his paper on the quantisation on the lattice of topologies provides, with reservations to be developed below, a canonical formulation similar to the one presented above. He notes that similarly to the simple harmonic oscillator the states $|\psi\rangle$ are eigenstates of $\hat{M}^{\dagger}_{\tau'} M_{\tau'}$. We can thus first build a family of "free" Hamiltonians by taking linear combinations of such operators and add other non-diagonal operators, yielding genuine quantum mechanical changes in topology[Ish89].

What are the reservations about then? The fact is that in trying to eschew concepts of differential geometry we are keeping a continuous label for time. Indeed, a more natural approach could be to render it discrete, Isham himself proposes as a starting point the replacing of the Schrödinger equation with the difference equation:

$$\psi(\tau, t) - \psi(\tau, t+1) = \sum_{\tau' \in N(\tau)} \psi(\tau', t) T(\tau', \tau)$$

With T being an evolution matrix and the sum is over topologies "near" τ: in the context of the lattice this might mean those topologies that cover τ or are covered by it. It remains to be seen whether this can recover a discretised type of dynamics. However, there is a much more powerful idea that lies in wait when one starts to consider the idea of a discrete structure of time.

To see how this all comes together we have to go back to why we are doing this discussion in the first place, i.e. the cobordism picture. We are considering those space-time configurations which should naturally appear in the integral but do not in the Euclidean approach to Quantum Gravity. This would involve the construction of an action principle, which raises the general question of how general relativity relates to this quantum scheme. This in turn requires us to ask the deeper question of what is the status of the set X and its points and how do smooth manifolds arise from this picture. It is in this light that the work of Sorkin appears to be a natural continuation of this picture initiated by Isham. Hence, we will be turning to his work now.

2.2 On the Road to Concreteness

2.2.1 Finitary Substitutes and Causal Sets

Sorkin inserts himself in this discussion by reviewing Isham's work and arguing that the assignment of the cardinality of the real numbers to space is indeed a non-justified assumption and that he sees reasons to pass to a discrete picture.

Here we do not mean to say that Sorkin *starts off* from Isham's aforementioned work. In fact, his discreteness tendencies resulted from studying Regge Calculus[4] as a graduate student[Sor95] for the reasons about discreteness given below; equally, we could have explored his work on topological geons which also starts from Wheeler's quantum foam. Given enough time it would have been interesting and enlightening to discuss all of the above but the scope of this second chapter lies primarily in giving an example of the kind of first-order considerations which come to trespass on higher-order ones in the context of concreteness explained in the first chapter. As such we are hereby presenting a somewhat simplified, and more importantly self-contained, narrative of one of these.[5] Indeed, his insertion in the discussion we are having at this point is a paradigmatic example of the kind of first-order discourse which naturally arises in physics. The reasons he lists for a switch to a discrete picture have mainly to do with singularities: he talks about the family of infinites arising in quantum field theory, traditionally dealth through normalisation, the singularities of general relativity we alluded to in the first chapter, and the infinite black hole entropy that arises when one tries to count the degrees of freedom of the horizon directly.
He hence answers Isham's query about the status of points by arguing that from a physical standpoint, points which cannot be distinguished by elements of a covering of the space should be identified. He then suggests that it is this set that should be regarded as the set of real "spatial" points. We briefly present his treatment below.

Given a topology assume we can pick a finite subtopology $\mathcal{U} \subset \mathcal{T}$ and define an equivalence relation on S in the following way:

$$x \sim y \Longleftrightarrow \forall U \in \mathcal{U}, x \in U \Leftrightarrow y \in U$$

Let $\mathbf{F}(\mathcal{U})$ be the quotient of S by this equivalence relation $\sim$, this is called the T_0 quotient of S [6], and $f(\mathcal{U}) : S \to \mathbf{F}(\mathcal{U})$ be the map taking each point $x \in S$ to the equivalence class to which it belongs. The open sets of $\mathbf{F}(\mathcal{U})$ are those whose inverse image under f belongs to $\mathcal{U}$.

---

[4] what engineers would call a finite element description of space-time, as Sorkin describes it

[5] The reality is often more complex and the reasons to pass from a research paradigm to another one are complex, as Sorkin explains in his insightful paper on the "intellectual history" of the causal set idea[Sor95]

[6] A space is T_0 is for every pair of distinct points there exists one open set containing one and not the other.

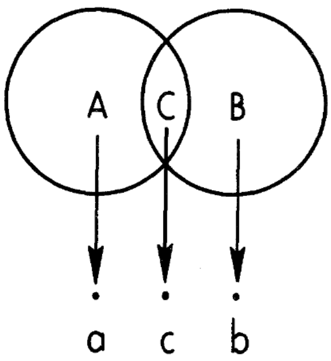


Fig. 1. A covering and its T_0-quotient.

We give as an example the situation to the left (taken from Sorkin's paper[Sor91a]), where S is the union of two disks, we imagine these to correspond to two coarse points i.e. the open cover $\mathcal{U} = \{A \cup C, B \cup C, C, S, \varnothing\}$
Then the associated quotient space has 3 points $\mathbf{F}(\mathcal{U}) = \{a, b, c\}$ with open sets $\{a, c\}, \{b, c\}, \{c\}, F, \varnothing$. It is clear that the finite approximation contains as much information about S as $\mathcal{U}$ provides.

As a consequence of the fact that $\mathbf{F}$ is finite we have that it is closed under arbitrary intersections, hence any $x \in F$ has a smallest neighbourhood as $\Delta(x) = \cap\{A|x \in A\}$.

We can thus convert the ordering relation that comes from subsets ($U \preceq V$if $U \subseteq V$) into a relation on elements: $x \preceq y$ if $\Delta(x) \subseteq \Delta(y)$ or equivalently if $x \in \Delta(y)$. It is easy to note that $\preceq$ is transitive and reflexive, i.e. it is a pre-order.

What is remarkable is that, conversely, given any relation with these properties one can obtain a topology on $\mathbf{F}$: $\Delta(x) = \{y|y \preceq x\}$, where the open sets are unions of such smallest neighbourhoods. Now we can ask what are the restrictions we have to enforce on $\mathbf{F}$ to obtain a T_0 space. We note that $\mathbf{F}$ fails to be T_0 when there exist distinct points for which any open set containing one contains the other as well, that is when $\Delta(x) = \Delta(y)$ or equivalently that there exist circular relations $x \preceq y \preceq x$. We conclude that $\mathbf{F}$ is T_0 only if $\preceq$ is a partial order, hence we will be assuming that to be the case from now on.

We can define the order-dual $V(x) = \{y|x \preceq y\}$ and say that a poset is finitary if and only if both $\Delta(x)$ and $V(x)$ are finite for all x. Considering finite (or locally finite) topological spaces allows us (by definition) to define homotopy and homology groups on them, which is not normally the case for lattice theories. It can be shown for example that the finite topology derived from an open covering of the circle is $\mathbf{Z}$ [Sor91a]. We hence proved that the finite space $\mathbf{F}(\mathcal{U})$ to a certain extent approximates S and that it can be described in terms of an order relation.
To properly conclude this discussion we would need to show that as $\mathcal{U}$ grows finer the approximation of S by $\mathbf{F}(\mathcal{U})$ grows exact. Without such a result the idea that posets could be seen as an equivalent picture to the manifold would be hard to entertain. Since there is no notion of a "topology on the space of topologies" with respect to which to take such a limit, Sorkin achieves the proof through the mathematically advanced notion of an inverse system. This proof is rather involved and does not strictly interest the kind of discussion we're carrying out, we hence redirect the curious reader to the original paper[Sor91a].

The one facet we should be commentating on though is that such a well-defined inverse system limit can also be obtained for the simplexes of Regge Calculus, but that as one adds finer and finer sets the simplexes of the complex do not all get small when regarded as subsets of this limiting space. Thus though a convergence exists mathematically, physically we would want successive approximations correspond to successively smaller scales of physical size. The reasons for this is that the nerve complex arising in Regge's calculus only keeps track of the mutual intersection of a finite open covering, hence it does not encode all the overlap information[Sor95]. This is yet another example of how mathematics does contain hints if one is critical about its use.

Much of this work on finitary substitutes stems from ideas announced in 1983 when Sorkin was working in such a strict topological and "canonical" framework, with the topological substratum underpinning space at a fixed time. Indeed, in reviewing Isham's work Sorkin highlights how it fits into this canonical picture of quantum gravity as a whole and Sorkin himself briefly had a go with a fully topological theory of space-time.

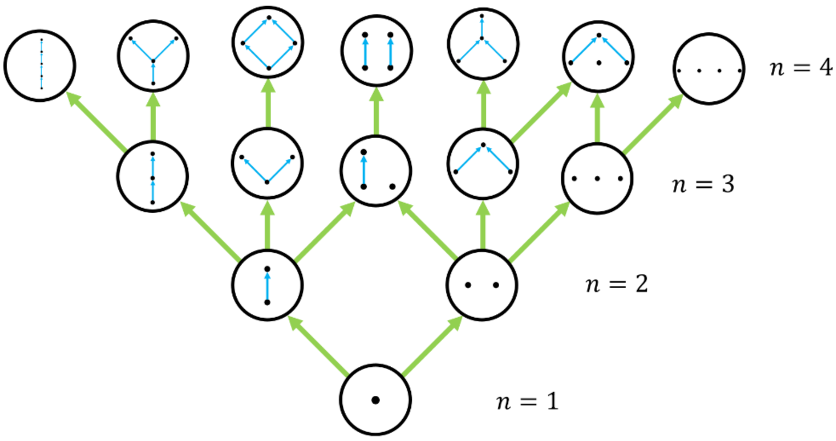
However, during this time period a semantics switch occurred, after Sorkin encountered the work by David Malament highlighting how much information about the manifold topological and differential structure is present in its causal structure[Sor95]. Accordingly, he exchanged the notion of partial order as topological inclusion for the notion of partial order as causal relation. These ideas were first put forward in a landmark 1987 paper by Bombelli et al[Bom+87].
More precisely, the causal set C is defined to be locally finite, i.e., $\forall x, z \in C, \{y \in C | x \preceq y \preceq z\}$ is finite, and the causal order in the continuum case only recovers the conformal metric $g_{ab}/|\det g|^{1/n}$, that is the volume is missing. However, in this context having a discrete substratum means we can take a fundamental volume, the most natural being the Planck volume, and obtain the missing information by simply counting.
It is for this reason that the CST slogan became: *order + number = geometry*

The problem is that causal sets should yield a Lorentzian manifold in some suitable limit: this belief is called the "Hauptvermutung", after a conjecture that was thought to be of great importance in the theory of manifolds, but ironically turned out to be false. Sorkin defines the criterion in the following way: the manifold (M, g) emerges from a causal set C iff C could have come from sprinkling points into M at unit density and endowing the sprinkled point with the order they inherit from the light-cone structure of g. Of course this is true in an appropriately coarse-grained regime, for causal sets are allowed to be chaotic at very small scales where only few space-time atoms are involved. The kind of result one would hope for is that given a causal set C and faithful embeddings $f : C \to M$ and $f' : C \to M'$ of C into M, M' there exists an approximate isometry (i.e., an isometry valid at volume scales much bigger than the fundamental volume) $h : M \to M'$ such that $f' = h \circ f$.

In the same paper proposing the german word he also proceeds to give a description of emergent metrical quantities, like dimension and distances, using only causet information[Sor91b].

However, because of the difficulty caused by the Hauptvermutung we see that already at the kinematics stage CST does encounter problems. That aside, the question that is most relevant to our discussion is does such a causal set indeed constitute a counterexample to the common belief that relativity implies a block-universe?
The first technical feature of GR often invoked by eternalists is Lorentz invariance. In this case CST boasts a rigorous result. As suggested in the discussion of the Hauptvermutung, given a Minkowski space-time $\mathcal{M}$ one can obtain a locally finite, uniformly random set of points by the Poisson process, where the probability of finding n points in a volume V is $P(n) = \frac{(\rho V)^n e^{-\rho V}}{n!}$ with ρ some fundamental density. Such a process is called sprinkling and endowing such a sprinkling with the order structure given by the light-cones we obtain a causal set. Then $C \approx \mathcal{M}$ iff C arises "with high probability" from $\mathcal{M}$. With such a setup, as opposed to say a regular lattice, it can be rigorously shown that with respect to the Poisson probability-measure it is impossible to deduce distinct timelike directions from the embedding[BHS09]. Indeed spatial hypersurfaces are a rather meaningless construct in CST, being anti-chains. The other technical feature would be general covariance, which we will discuss below after introducing dynamics.

The main prototype of dynamics for classical causal sets is given by classical sequential growth (CSG) models. In such a model elements of the growing causal set are added stochastically one by one, like a tree growing at its tips. Transitive percolation growth models can be developed but CSG is more naturally defined as a Markov process. To this end we define a Poscau $\mathcal{P}$, which is the configuration space of causal sets equipped with a natural partial order relation, we note new elements cannot be born to the past of any existing elements. This is the analogue of superspace in canonical quantum gravity.



In such a scheme the assignment of probability to a certain causet transition (typical in transitive percolation models) is replaced by a generic map between the set of Poscau links and the interval $[0, 1]$. Physically we would also want for the birth of a new element e to only be affected by the elements to its past, that is, the ratio of transition probabilities between a causet C to new causets C_1, C_2 with elements e_1, e_2 should be equivalent to the ratio with the same elements being born without the spectator set $\{\, y \in C \mid y \not\preceq e_1, e_2 \,\}$., a condition called Bell causality.

With such a setup we can try to reason about general covariance in a discrete setting[Sor07]. At the kinematical level the individual space-time elements carry no identifiers so what has physical meaning is only the isomorphism equivalence class of the poset. This entails invariance under relabeling, akin to invariance under coordinate change in GR. Then, on the space of labeled causets $\tilde{\Omega}$ one can define a probability measure $\tilde{\mu}$, and obtain the induced measure μ on the quotient set Ω whose members are the isomorphism classes. Thus, we obtain general covariance at the dynamical level in the passage from $\tilde{\mu}$ to μ. However, GR being a deterministic theory, such invariance implies invariance at the level of the equations of motion as well. CST, being a stochastic theory, does not share this feature; instead to reproduce the Einstein equations in the classical limit $\tilde{\mu}$ should itself be relabeling invariant. Such an invariance condition is in fact baked into the CSG model: for any finite causet C of cardinality n the probability to arrive at C after n steps is independent of the birth's order of C's elements. It follows that both Lorentz invariance and discrete covariance are consistent with a growing causal set. What is crucial here is that such a scheme gives lieu to a localised version of becoming, and that Sorkin was guided by this principle in formulating CSG[Sor07]. This is what valuing of higher-order discourse can look like, a fact we shall further comment on in the conclusion.

Such discussions aside, it is clear CSG can only be a stepping stone and real causal sets are of quantum mechanical nature. A physically well-motivated model remains to be found, with the greatest obstacle being a quantum analogue of Bell causality. In the mean-time an intermediate approach is of sum-over-histories type, where one sums quantum amplitudes over the space of causets, yielding a partition function with a discretised Einstein-Hilbert action[Ben13].
Even lacking an explicit form; the need for the quantum dynamics to be of path-integral form, the major contender being Sorkin's quantum measure theory[7] in its path-integral formulation[Sor17], imply that from a purely heuristic standpoint something remarkable can be said.

[7] this approach starts from the remark that while there is a double slit interference, there is no triple slit interference. Accordingly, for disjoint A, B, C instead of satisfying $\mu(A \cup B) = \mu(A) + \mu(B)$ as for classical probability measures, the quantum measure is taken to satisfy $\mu(A \cup B \cup C) = \mu(A \cup B) + \mu(B \cup C) + \mu(A \cup C) - \mu(A) - \mu(B) - \mu(C)$. Under this view quantum mechanics is a generalisation of classical stochastic processes.

Indeed, if one takes from the CSG models the feature that the growing cardinality N of the causet behaves as some kind of time parameter (more precisely that with respect to which probabilities are normalised) then just as one does not sum over time in the path integral formulation of quantum mechanics, we would not expect to sum over causets of different cardinality.
However, since N directly reflects the space-time volume $\mathcal{V}$ this would entail to sum over fixed $\mathcal{V}$ in the gravitational path integral. Such a restricted path integral is exactly the one arising in unimodular quantum gravity, and within such a formulation Λ is conjugate to $\mathcal{V}$. What is key here is realising that because of Lorentz invariance the correspondence between number and volume cannot be exact but instead subject to Poisson fluctuations of order $\sqrt{N}$. Hence, in keeping N fixed we only fix $\mathcal{V}$ up to fluctuations of order $\sqrt{\mathcal{V}}$, which correspond to fluctuations in Λ.
Accordingly:

$$\Delta\Lambda \sim 1/\Delta\mathcal{V} \sim \frac{1}{\sqrt{\mathcal{V}}} \sim \frac{1}{\sqrt{N}}$$

Taking the fundamental volume to be the Planck volume and the 4-volume to be the observable universe from the big bang up until now yields $N \sim 10^{240}$ and hence we obtain $\Delta\Lambda \sim 10^{-120}$ in natural units, that is, the measured magnitude of Λ.[Ahm+04] Of course, this does not answer the question of why the "target value" would be precisely 0[8]; still, Sorkin made this estimate before the empirical confirmation of Λ, hence constituting the sole genuine prediction of a theory of Quantum Gravity.

Afterwards, we shall see another approach to quantum CST dynamics in what is again a remarkable generalisation of the discussion of the previous section by Isham; the key here is to realise that a poset is an example of a category. However, let us orderly follow our much hailed discursive narrative.

[8] Sorkin conjectures that only $\Lambda = 0$ might be stable against the destructive interference induced by non-manifold fluctuations of the causal set

2.2.2 Pointless Spaces and Non-Continuum Physics

Isham learnt about Sorkin's finitary substitutes in 1983 and developed the idea in a different direction, which though non-processual is still relevant to our thread of going towards concreteness. Indeed, he argues that what is really being affirmed as a result of Sorkin's discussion is that the most important feature of space-time is not points, rather it is the open sets and the relations between them[Ish90]. We here get something very close to Whitehead's mereotopology. However, as mentioned in the first chapter, in the century or so that has passed since "Process and Reality", Mathematics managed to ground these pointless spaces upon a more rigorous basis.

What should be noted in the discussion about lattices is that a topology on a space X constitutes a sublattice of $P(X)$. We may then wonder to what extent can we reconstruct the space X from the lattice alone. When considering this possibility the first step should be in trying to define what a point is in such a construction. What is the one question we can ask about a point? Whether it belongs to a certain set or not, we hence obtain a collection of mappings $h_x : \tau \to \{0, 1\}$ defined on open sets O by:

$$h_x(O) = \begin{cases} 1, & \text{if } x \in O \\ 0, & \text{otherwise} \end{cases}$$

So that we have a homomorphism from the lattice τ onto the lattice $\{0, 1\}$ of two points.

What would a mathematician do in such a case? Why generalise, of course. Let us then define a generalised point as any homomorphism from the lattice onto $\{0, 1\}$ and call the set of all such homomorphisms $\mathrm{pt}(\tau)$. We can build a topology on such a set by defining the open sets to be the subsets $\{h \in \mathrm{pt}(\tau) | h(O) = 1\}$ with O being any τ-open subset of X and remark that the map $h : X \to \mathrm{pt}(\tau)$ such that $x \to h_x$ is continuous with respect to this topology.

However, we are certainly not required to consider a lattice of open sets! To understand which lattices are allowed consider that the lattice of open sets of a topology is closed under arbitrary unions ("join" in the language we're using) and if we define the meet ot be the interior of the intersection of subsets then such a lattice becomes a complete sublattice of $P(X)$. Furthermore, it is infinitely distributive, that is for any collection of open sets Z, $A \wedge \bigvee Z = \bigvee \{A \wedge B | B \in Z\}$.

Let us then generalise once again, and define a complete infinitely distributive lattice to be the purely algebraic definition of a topology-like structure. Such an object is called a *locale*.
Recalling that a continuous function $f : (X, \tau_1) \to (X, \tau_2)$ induces a homomorphism from the lattice τ_2 to τ_1 we take homomorphisms between locales to be the continuous maps. In such a pointless topology the only thing that matters is relations of inclusion, nextness, overlap and such.

Such relations are then subject to quantum fluctuations which could be expressed as a generalisation of the above. The classical limit of such a theory would yield a collection of regions behaving like some covering of open sets of a differentiable manifold. We could even consider the application of non-commutative topologies in a manner analogous to Connes' non-commutative geometry. Such constructions are called quantales, explored around the turn of the century by Mulvey [Mul01]. There is a lot of ground potentially to be covered, and in some way this might be perceived as the coronation of Whitehead's ideas about point-free geometry: of a theory of points without points; but at this point Isham instead of repeating a very similar discussion to the above wonders whether our enterprise to do away with the continuum is not inconsistent after all.

Indeed, he argues that the continuum is an ingredient not only in our conception of space-time but also in our use of real and complex numbers in quantum theory, and there is a danger of bringing a prioris from the very start. To understand this point of contention it is important to understand the origin and role of the continuum in standard quantum theory, we will thus follow a quick Nietzschean style genealogical inquiry of such a concept.

Let us begin by considering the paradigmatic example of a point particle moving in one dimension $\psi_t(x)$.[Ish02] the x and t dimensions have been explored at length in the preceding section of this paper, i.e., the relation with measurement and the continuum of space and time; it might be considered a category mistake, to use a term dear to Ryle, to apply the continuum to something which, by all indications, might not be continuous at all. We note that such an a priori does not disappear with quantisation: e.g., if quantum states are defined to be cross-sections of some vector bundle over the classical configuration space then the domain of these state functions is still continuous. This carries over to the value of the wave-function when operators with continuous spectra are considered: e.g. $(\hat{x}\psi)(x) := x\psi(x)$

However, more fundamentally real numbers are used in the expression of probabilities, via their relative frequency interpretation. This entails in principle for a measurement to be repeated a great number of time N and for a probability associated to an outcome k_i to be the ratio N_i/N with N_i being the number of times the outcome k_i arose. This bounds probabilities between 0 and 1, and if the limit $N \to \infty$ were to be taken we would get the ensemble $[0, 1]$. This is a natural setup in an instrumentalist theory, however in a realist theory "probability" must be understood differently. In this sense we can make a connection to the discussion of the first chapter, by interpreting probability as potentiality we can make a concrete step towards seeing novelty as fundamental. Examples of such approaches are discussed by Verde & Smolin[SV21b] and Zafiris & and Epperson[EZ13].

Returning to our discussion, it follows that under a different interpretation there is no reason to consider as the probability valued space a subset of the real numbers, all we need is an ordered set, so that outcomes can be expressed as more or less likely than others. However, we do not need for this set to be totally ordered. What we get through this considerations is instead a notion of "generalised" truth values for propositions.

How can such a formalism be applied as a realist framework for physical theories with arbitrary space-times? Isham makes the example of topological spaces of course but, more fittingly given our discussion, he also makes the example of causal sets[Ish02]. In fact, as seen before, from the point of view of the poset relation, a CST history dynamics is mathematically analogous to a canonical dynamics for topological spaces.
He argues, in line with Sorkin's quantum measure theory, that only a history formalism can be a proper candidate for CST dynamics. Indeed, he starts from a scheme very close to Sorkin's: Hartle generalised quantum mechanics in their path integral formulation. Originally such an approach was born of the consistent histories interpretation of QM, which starts from the observation that in conventional quantum mechanics if one starts from an initial state ρ_0 at t_0 the joint probability of finding a set of properties $\alpha_1, \alpha_2, ..., \alpha_n$ at times $t_1, t_2, ..., t_n$ is

$$p(\alpha|\rho_0) = tr_{\mathcal{H}}(C_\alpha \rho_0 C_\alpha^\dagger)$$

with the class operator $C_\alpha^\dagger = U(t_0, t_1)\alpha_1 U(t_1, t_2)\alpha_2 ... U(t_{n-1}, t_n)\alpha_n U(t_n, t_0)$ written in term of projection operators α_i. From this, consistent histories posits that, under precise conditions, this probability assignment is meaningful even for a closed system, with no state-vector reduction induced by a measurement. The satisfaction of such conditions is determined by the behaviour of a complex-valued function of pairs of histories, called the decoherence functional, measuring their interference and defined as:

$$d_{(H,\rho)}(\alpha, \beta) = tr(C_\alpha \rho C_\beta^\dagger)$$

A complete set of histories (i.e.,one such that the realisation of one of them excludes all the others and one of them must occur) $\alpha, \beta, \gamma, ...$ is d-consistent if the decoherence function of any two distinct histories is null, in that case the probability of a history α is $d(\alpha, \alpha)$.

The suggestion by Hartle and Gell-Mann is for the history to be regarded as the primitive entity, not just as a time-ordered sequence of projection operators, marking the passage to a path-integral formulation. Isham then axiomatised this theory in the language of quantum logic, taking on one side the set of propositions about histories $\mathcal{UP}$ and on the other the set of decoherence functions $\mathcal{D}$[Ish97].

It follows from the previous discussion that the quantum history formalism should adapt to the background causal set as to both the choice of the space $\mathcal{UP}$ of propositions about the "universe" and the choice of space where decoherent functions take their values. As mentioned at the end of the last subsection, in yet another remarkable generalisation of the discussion of the previous section he explored the first steps of such an approach by quantising directly on the poset structure[Ish05], which is an example of a category; more below. Though the example he explores in his paper is rudimentary, he manages to define the operators from which the decoherence functional would be construed.

The central problem of such an approach is the large number of incompatible d-consistent sets. Classically this has been dealt by positing a unique consistent set of histories to be realised, possibly yielding a sort of many-worlds interpretation. Isham instead explores a rigorous way to accept the plethora of d-consistent sets as different "world-views" or "classical snapshots" on a system, handling them all at the same time. Indeed, the mathematical structure of such a collection suggests a novel logic in which probabilistic propositions are, accordingly, contextual, and not binary-valued (i.e., not just "true" or "false").

This, yields an almost "classical theory" of quantum mechanics in a sense Isham explains more in depth in[Ish10]. In short, realism is baked into the mathematical formulation of classical physics through the language of set theory:

- To each system S we have the configuration space $\mathcal{S}$ of states and at each moment the system is in one and only one of these states.
- Any physical quantity A is represented by a function $\breve{A} : \mathcal{S} \to \mathbb{R}$ from the set of states to the real numbers. Hence we talk of the value of A in a certain state.
- Propositions are of the form $A \in \Delta$, for Δ a Borel subset of $\mathbb{R}$. These can be viewed as properties of the system, represented as $\breve{A}^{-1}(\Delta) \subseteq \mathcal{S}$, i.e., those states s such that $\breve{A}(s) \in \Delta$.

Such a representation, in keep with our intuition, crucially implies that the logical structure of classical physics is that of a Boolean algebra [9].
Thus we now have a more formal understanding of what has been alluded to in the first chapter: Newtonian physics interpreting things as the bearers of properties.

Crucially, in quantum theory such a view is foiled by the Kochen-Specker theorem, which prohibits hidden-variable theories positing it's possible to consistently represent all aspects of physical reality simultaneously disregarding the context (in our discussion a complete set of histories) of the analytical viewpoint under considerations.

[9] we did say to keep that definition handy

Isham wants to point out that set theory with its Boolean structure is naturally geared towards classical physics; therefore, he tries to generalise the Newtonian scheme to categories other than that of sets.

We recall here that a category is a collection of objects together with a collection of morphisms. For each object a there exists the identity arrow $i_a : a \to a$ and to each morphism f are associated a pair of objects, i.e., $f : a \to b$.
Furthermore if there is a morphism $g : b \to c$ then there exists the arrow $g \circ f : a \to c$ satisfying the associative law. $\mathrm{Hom}(a, b)$ represents the set of morphisms from a to b.

We hence see that the category of sets is a category whose objects are sets and whose morphisms are set maps. Similarly many structured sets are categories where morphisms are maps preserving that structure (e.g., the category of groups with homomorphisms or the category of topological space with homeomorphisms). However categories need not have structured sets as their objects; as noted above a poset $\mathcal{P}$ is also a category whose objects are elements p, q of $\mathcal{P}$ with a morphism from p to q existing iff $p \preceq q$. However, in analogy with subsets or subgroups there exists the notion of sub-objects, formally defined as injective homomorphisms.

A representation of a physical system in a general category would hence have the following form:

- There are two special objects: the state object Σ and the quantity value object $\mathcal{R}$
- A physical quantity A is represented by a morphism $\breve{A} : \Sigma \to \mathcal{R}$
- propositions are represented by sub-objects of Σ.

The problem is that we would want for our physical system to have a logical structure, and sub-objects of a category generally do not have one. There exists however a category where that is the case: a topos. Such a category has many possible definitions, but the one more in line with our line of inquiry is that it can be interpreted as a generalisation of the notion of a set. More precisely, they are categories with:

- "initial" and "terminal" objects $\mathbf{0}$ and $\mathbf{1}$, which are objects such as that for every other object a there exists exactly one morphism $\mathbf{0} \to a$ and $a \to \mathbf{1}$ respectively. These are the equivalent of the empty set and singleton sets in set theory.
- products and co-products of objects, which in the case of sets are the cartesian product and disjoint union respectively. Additionally, given two objects a and b there exists the exponentiation object $b^a = \mathrm{Hom}(a, b)$
- a sub-object classifier Ω which is the analogue of the set $\{0, 1\}$ when interpreted as the codomain of the characteristic function $\chi_A : A \to \{0, 1\}$ for a subset A. Thus to each sub-object a in a topos one has a characteristic arrow $\chi_a : a \to \Omega$. As the power set of A is in bijection with the set of characteristic functions $\{0, 1\}^A$ one has the power object of a, Ω^a.

Crucially, the logic that comes with it is not necessarily classical logic but it is an intuitionistic logic (i.e., non-binary valued) and instead of a Boolean algebra one gets a Heyting algebra, which is a bounded distributive lattice, not necessarily complemented; so that the law of the excluded middle (either p or not p) does not apply. In the words of Isham: "The propositions in such a theory admit a 'neo'-realist interpretation in the sense that there is a 'way things are', but this is specified by generalised truth values that may not be just true (1) or false (0)"[Ish10]

The internal logic can then be made contextual by choosing an appropriate topos[10]. This is the beginning of the Topos Physics of Döring and Isham[DI10], yielding a theory of quantum mechanics without the continuum; we refer to their seminal work for further discussion. Rolling the thread back, the history formalism in such topos-theoretic language in principle yields a recipe to obtain a theory of quantum causal sets in an alternative way to Sorkin's quantum measure theory.

Given our general focus, we should note that Topos Physics heavily relies on the Newtonian scheme of a state space[11], though as mentioned in the first chapter to go beyond that paradigm is an issue for the problem of laws more than quantum theory; it follows that the more restricted category of quantum cosmologists should take the considerations at the end of section 1.3 into account.

Given our discussion, more fitting is instead the remark that topoi, with their intrinsic logical structure, can be used as alternative foundations to set theory for mathematics. Hence each and every topos is a whole mathematical universe with its own set of possibilities. As an example, since the proof in analysis that there exist no infinitesimal numbers relies on the law of the excluded middle, which as we said is generally missing in intuitionistic logics, one has topoi where infinitesimals truly exist: these give rise to the field of synthetic differential geometry. With the due limitations of mathematics outlined by the discussion in the first chapter, such topoi open a Pandora's box of possibilities regarding (possibly more faithful) representations of the physical world in mathematical language.

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[10] for the adepts, to represent quantum mechanics they choose the topos of presheaves over the category of contexts $\mathcal{V}(\mathcal{H})$, the objects of $\mathcal{V}(\mathcal{H})$ being the commutative subalgebras of bounded operators on $\mathcal{H}$, i.e. sets of commuting operators

[11] In fact, in an ironic twist given our discussion, Isham's and Döring's seminal paper on topos physics is titled "What is a thing?", after a book by Heidegger, Husserl's greatest student. Here they draw the parallel between the quest for a realist theory of QM and the question by the philosopher. From this they quote Heidegger's answer to his own question:

> "A thing is always something that has such and such properties, always something that is constituted in such and such a way. This something is the bearer of the properties; the something, as it were, that underlies the qualities."

We can put this answer in the context of the first chapter by finding out that among the greatest influences on Heidegger was Aristotle, and indeed this is a very Aristotelian answer. Whitehead gives a different answer in his book "The Concept of Nature" where he uses the term "object":

> "..., the theory of objects is the theory of the comparison of events. Events are only comparable because they body forth permanences. We are comparing objects in events whenever we can say, 'There it is again.' Objects are the elements in nature which can 'be again.'"

Chapter 3

Conclusion

In this essay we have highlighted the difference between a first-order and higher-order discourse, and explored how general considerations of philosophical nature can point to research directions in a technical field like physics. We then explored in depth how ideas similar to the ones in the first chapter emerged from an example of internal physical discourse.

What is of note is that besides being the product of "raw" first-order discourse, in both the somewhat unrelated physical approaches of section 3.2 there appear instances of "higher-order" motifs, exemplifying at a smaller scale the kind of mixed-order thinking which we argued should become common in physical theorising. In Sorkin's case we have a telling example of how a process-picture of reality informed the markovian CSG dynamics, in Isham's Topos Physics we saw an example of a theory being critical of the a prioris inherent in determinate mathematical representations.
More generally, theories do not live in a vacuum, but inside the minds of the theoreticians. Therefore, the mental picture researchers have of such theories and their potential extensions is a very important facet of theorising.

At the greater scale, beyond singular research programs, the crucial take-away point is that both discourses should complement each other. In such a union it is clear that more than forming a singular whole the parts will have heterogeneous roles. The physical approach is the one which *aims* at working incrementally[1], trying to integrate as a synthesis the various empirical and theoretical findings in seeming contrast with each other. This approach has center-stage since it is the one with respect to which progress is measured: the conceptualisation of of space-time topology built upon Wheeler's space-time foam, together with the relations to causal set theory and quantum logic we have explored herein, represent important milestones in our attempt at understanding the deep fabric of space-time, independently of any higher-order stance.

[1] indeed, some philosophers of science like Feyerabend argue that progress in science is most aptly described as a succession of different techniques and values, similarly to art[Fey84], not necessarily comparable with each other.

The other approach, which is usually ignored, is the philosophic one. This disparagement comes both from those who misunderstand its role and from those who recognise its ambitions.

The former stance is due to the misconception that philosophy is either estranged from science or it is only a commentator of physical concepts and views. This is a consequence of the analytical turn in philosophy, which came with a great loss in philosophy's importance for scientific progress. However, even more foolish in the eyes of many scientists would be philosophy's meddling in the exact sciences: i.e., the role we are advocating here for it. In such a discussion we transformed it from mere commentator to critic and even creator of physical concepts. Vested this way, it becomes the starter rope to the lawn-mower of physical discovery, when this latter one halts due to conceptual deficiency. It is clear that such role is dynamical and is constantly informed by scientific discovery itself.

The fact that this is a starter rope, and not a reset button, is very important: we have indeed criticized the ingrained metaphysics upon which physics is based, but the goal of this essay is not to blame those metaphysics, as they have been incredibly useful. Rather it is to draw attention to the fact that they are there, and it is worthwhile questioning and re-contextualising them when their domain of limitation approaches.

This brought us to the second thread of this essay, which lies in the thematic of concreteness. Surveying once again the arguments of section 1.3 about process, mathematics and causality we can appreciate the incredible work that physics has yet to do, having extended its domain to territories once seemingly forbidden. It is in such a light that the arguments of this essay should also be seen as victories for the discipline itself. Indeed, we are nowadays in a climate reminding the one before the quantum and relativistic revolutions, when it was thought that physics had discovered all the salient features of reality. Similarly, today it is often felt that physics is almost completed and the last piece of the puzzle is a grand unified theory accounting for gravity. The greatest lesson that process philosophy can give to science is instead to recognise the never-ending quest that is the study of nature.

With such a discussion being closed, we wish to explore one last facet of concreteness as we described it. Indeed, returning to concreteness in our discipline more generally means valuing how new physics is actually born: in the corridors of universities, in conferences around the world and in the bedrooms of physicists before they rest. In short, it is realising that physics is a human practice, similarly to the arts, and that any attempt at concealing that humanity inevitably does violence to the discipline itself. That is why pedagogy in physics would do well to proceed the same way humanities do: historically and possibly through the study of the thought of individuals or groups of individuals, as related to the empirical discoveries made in the meantime.

This should not entail a corresponding sacrifice in technicality. Instead, it should be akin to what, surely in very modest fashion, we have been trying to accomplish in the second chapter of this essay. A pedagogy of this sort would allow young physicists to understand the current form of the discipline and appreciate its achievements even more, priming insight into the process of theoretical innovation.

Highlighting the dimension of discovery one would also be reminded about the very nature of Nature; that we are not the product of an abstract web of physical concepts, but the sons and daughters of a world of flesh; a world towards which a physicist's eye should keep gazing with utmost curiosity. Thus, stealing Whitehead's phrasing about philosophy:

> Physics should begin in wonder.
> And, at the end,
> when physical thought has done its best,
> the wonder should remain.

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