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Testing Quantum Gravity: Foundations and Prospects

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Abstract

In this review we will examine the recent proposals for lab based tests of quantum gravity and examine their viability. The bulk of this dissertation will focus, not on the experimental feasibility but rather, whether or not they would serve as tests for quantum gravity; proving the existence of the graviton. With this goal in mind we will start off by reviewing the foundations of quantum mechanics, examining what features features in physics are genuinely classical or genuinely quantum and then how these may be identified experimentally. We will then examine the minimal theory of quantum gravity, the most promising table-top test for quantum gravity and how it fairs under scrutiny.

Acknowledgements

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Contents

Chapter 1

Introduction

1.1 Historical Background

Ever since the inception of quantum mechanics as a fundamental theory of microscopic phenomena, it has been the goal of physicists to describe all classically known phenomena as quantum phenomena. A quantum description of electromagnetic interactions was provided with the postulation[1, 2] and experimental[3] discovery of the photon, followed by the subsequent quantisation of the electromagnetic field[4]. Naturally, the next step has been to quantise the other fundamental interaction that has a classical description: gravity.

The quantisation of gravity has faced issues. The primary one being the non-renormalisability of the graviton as a quantised linearisation of Einstein's field equation within the framework of quantum field theory. In spite of this, arguments have been put forwards to justify the existence of a quantum theory of gravity [5, 6] with several recent arguments presenting themselves as no-go theorems for classicality [7, 8] although historically, issues have been found with some arguments[9, 10].

It has none the less been the general goal of the theoretical physics community to find such a theory for the better half of the last seventy years with numerous theories having been proposed; the most notable having been string theory and loop quantum gravity. All theories of quantum gravity thus far are plagued with issues and continuously make predictions that are either later falsified or impossible to test within any reasonable time frame.

Given this state of affairs one may ask if there is any need to produce a quantum theory of gravity? Unfortunately, even if there is no quantum theory of gravity, the current state of physics is tremendously incomplete. The current standard model of particle physics leaves much to be desired, lacking UV completion and calculations requiring regularisation schemes. Furthermore, at

energies where the Compton wavelength of a free particle approaches that particles Schwarzschild radius, one would expect a black hole to form. We not only lack a description of this process but according to semi-classical quantum gravity[11] the black hole should randomly radiate thermal energy before evaporating. The unitarity of quantum mechanics requires that information cannot be lost, contradicting the statement that the radiation should be random. At the opposite end of length scales, general relativity predicts the existence of a singularity inside the centre of a black hole and at the start of the universe where the notion of space and time break down. The philosophical consensus being that space-time should be defined everywhere, contrary to this prediction. Regardless of whether or not there is a theory of quantum gravity, the fact cannot be avoided that there are problems inherent to the current model of physics, which are centered around gravitational interactions over very small length scales and high energies.

In the light of the dichotomy that these theoretical arguments present, one may ask what experiments tell us about a quantum theory of gravity? So far there have been several tests for phenomena[12, 13, 14, 15] that are shared by all theories which predict that gravity should be in quantum nature. Most notably we see that single quantum particles are affected by the gravitational field of a classical body and that the equivalence principle is preserved for quantum particles[12]. These do not confirm that gravity is in fact quantum in nature for reasons that we will explore later on. More recently there have been a series of proposals for table-top experiments that would aim to test for quantum gravity by searching for evidence of gravitationally induced entanglement[16][17]. These proposals are proving to be seminal. They are looking to be the first practically feasible experimental proposals that would be able to falsify the existence of the graviton. But they are not without flaws.

1.2 Structure of report

The focus of this review will be to examine what criterion must be met for a phenomena to be quantum. In the first section we shall start by examining the fundamental concepts of what entanglement is, how we distinguish between quantum and classical phenomena, both in theory and in the lab looking at quantum witnesses and entanglement measures.

In the second section we shall move onto exploring these first proposed table-top tests for quantum gravity. We will start by giving a break down of the minimal covariant theory of quantum gravity, perturbativly quantised gravity; then how one could produce entanglement in the table-top regime using this formalism. Having given a break down of the benefits of this proposal we shall

then go over the critiques that have been made, exploring alternative sources of entanglement and breifly mention some other proposals.

Chapter 2

A Review of the Foundations

Before examine quantum gravity and the tests to propose we must first understand what it means for something to be quantum. With this in mind we shall first do an in-depth review of quantum states, entanglement and the operations one can perform with it, the difference between quantum and classical states, and how one characterises this phenomena. In this review we shall focus on bipartite states, that is states which are composed of two distinct subsystems. While one can extend many of the definitions that we shall present to N subsystems, using the definitions often becomes unreasonably difficult and the dynamics in many cases is not well understood[18]. We shall also reduce these to two level(qubit) bipartite systems to further simplify our scenario because it is the type of system that is most often encountered in a experimental setting.

2.1 A Reminder of Quantum Ensembles

2.1.1 Quantum states and measurement

Definition (Density matrix)**.** Given we have a set of states and weights, $\{|\psi_i\rangle\}_{i\in I}$ and $\{w\}_{i\in I}$ respectively, we define the density matrix by

$$\hat{\rho} = \sum_{i\in I} w_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \quad \mathrm{Tr}\left(\hat{\rho}\right) = 1.$$

By definition it is Hermitian and positive semi-definite. We often refer to the density matrix as the state of the system with the general form written above being designated a mixed state. The state density matrix composed of the single "pure" state $|\psi\rangle$, $\hat{\rho} = |\psi\rangle\langle\psi|$ is known as a pure state. We define the purity of a state $P\left(\hat{\rho}\right) = \mathrm{Tr}(\hat{\rho}^2) \leq 1$ where for pure states we trivially have $P(\hat{\rho}) = 1$.

A pure state quite clearly represents at least a single quantum particle and if one considers that a state may exist in a Fock space, representing multiple particles, the question arises: what does a mixed state represent? Consider a statistical ensemble where a certain fraction of our population of states are of type A, say in a spin singlet state, and the rest are of type B, say in the spin triplet state. If there is no interaction between these states then it would be incorrect to describe them by single state composed of their superposition, as these states do not even have the same energy, let alone interfere. The probability that we select one or the other is the same as we would find in classical mechanics. However, we still wish to encode this into the description of our state. Seeing that two pure states, if not equal, will share no similar terms we designate our weights w to represent the fractional population of each of our states in the statistical ensemble. It is for these reasons that we call the diagonal elements "populations" and the off diagonal elements "coherences".

Definition (Separable states)**.** Suppose $\hat{\rho}_{AB...}$ is a state of composite systems; each labeled by A, B, ... respectively. A state is said to be separable if and only if it may be written in the form $\hat{\rho}_{AB...} = \sum_i w_i \hat{\rho}_A^i \otimes \hat{\rho}_B^i \otimes ...$ such that $\sum_i w_i = 1,\ w_i \in [0,1]$ and where each $\hat{\rho}_I^i$ are the states of a unique subsystem of the system I. If a state is not separable it is said to be entangled.

Definition (Projection valued measures)**.** Let $\mathscr{H}$ be a Hilbert space and M a indexing set for each possible measurement outcome of a process. A von-Neumann or projection valued measure(PVM) measurement is defined as the set of operators $\{\hat{P}_i : i \in M\}$ on $\mathscr{H}$ such that the following properties are satisfied[19]:

- (Hermitian). $\hat{P}_i = \hat{P}_i^\dagger\ \forall i \in M$;
- (Positive semi-definite). $\hat{P}_i \geq 0\ \forall i \in M$;
- (Idempotent and orthogonal). $\hat{P}_i \hat{P}_j = \delta_{ij} \hat{P}_i\ \forall i \in M$;
- (Resolution of the identity). $\sum_{i \in M} \hat{P}_i = \hat{\mathbb{I}}$.

The probability of obtaining a outcome i for a given state $|\psi\rangle$ is specified by $p_i = \langle\psi| \hat{P}_i |\psi\rangle = \mathrm{Tr}(\hat{P}_i \hat{\rho})$ and the post measurement state is

$$|\psi\rangle_{\mathrm{post}}^{(i)} = \frac{\hat{P}_i |\psi\rangle}{\sqrt{\langle\psi| \hat{P}_i |\psi\rangle}} \quad \text{or} \quad \rho_{\mathrm{out}}^{(i)} = \frac{\hat{P}_i \hat{\rho} \hat{P}_i}{\mathrm{Tr}(\hat{P}_i \hat{\rho})}.$$

Practically speaking we usually express such measures acting on our system with projection operators, $\hat{P} = |\alpha\rangle \langle\alpha|$, which when acted on a state, will force it into a particular eigenstate.

While easy to use and likely what most physics student use when they want to apply a measurement to their system, they are not very realistic. This is because they do not take into account the uncertainty in the measurement due to the presence of the ancillary system acting on them e.g. the resolution of the experimental apparatus. This unrealistic element restricts the types of measurements we may perform and can produce incorrect models; particularly in arrival time related problems[20, 21]. A more general measure is provided by POVM's:

Definition (Positive operator valued measures)**.** Let $\mathscr{H}$ be a Hilbert space and M a indexing set for each possible measurement outcome of a process. A projection operator valued measure(POVM) measurement is defined as the set of operators $\{\hat{\Pi}_i : i \in M\}$ on $\mathscr{H}$ such that the following properties are satisfied:

- (Hermitian). $\hat{\Pi}_i = \hat{\Pi}_i^\dagger \ \forall i \in M$;
- (Positive semi-definite). $\hat{\Pi}_i \geq 0 \ \forall i \in M$;
- (Resolution of the identity). $\sum_{i \in M} \hat{\Pi}_i = \hat{\mathbb{I}}$.

The probability for obtaining outcome i for a given state, specified by the density matrix ρ is given by $p_i = \mathrm{Tr}(\hat{\Pi}\hat{\rho})$.

We have a problem however; the lack of idempotency and orthogonality being necessary conditions does not allow us to necessarily define a new post measurement state. It is for this reason that we must introduce another operator; the Kraus measurement operators:

Definition (Kraus measurement operators)**.** We define the Kraus measurement operators $\hat{K}_i$ for each $\hat{\Pi}_i \in \{\hat{\Pi}_i : i \in M\}$ by $\hat{\Pi}_i := \hat{K}_i^\dagger \hat{K}_i$. In terms of the Kraus measurement operators we may define post measurement state as

$$\hat{\rho}_{\mathrm{post}}^{(i)} = \frac{\hat{K}_i \hat{\rho} \hat{K}_i}{\mathrm{Tr}(\hat{K}_i^\dagger \hat{K}_i \hat{\rho})}.$$

Kraus measurement operators are a special type of Kraus operator[22], that are defined by the looser condition that $\sum_i \hat{K}_i^\dagger \hat{K}_i \leq 1$, and by their definition they are not unique as for any unitary operator $\mathscr{U}$ we may define $\hat{W} = \hat{\mathscr{U}}\hat{K}_i \ \Rightarrow \ \hat{\Pi}_i = \hat{W}_i^\dagger \hat{W}_i$. This may be interpreted as the statement that there exists an arbitrary number of different measurement apparatus that can give the same measurement outcome. Another consequence of our definition is that measurements are no longer necessarily repeatable.

While it is good to have a consistent definition, what is currently written does not help us to practically construct a POVM operator. Newmark's theorem[23] states that any POVM subspace $U \subseteq \mathscr{H}$ may be constructed by a set of PVM's acting on a larger subspace. Practically, we may

use the fact that $\hat{\Pi}$ is Hermitian to expand it in terms of a appropriate basis of pointer states for our problem and work from there.

Example: Scattered density matrix

Let us consider an initial state of free two electrons $|p_1,\, s_1;\, p_2,\, s_2\rangle$ that scatter via an electromagnetic interaction into two free electrons. The density matrix for the initial states is given by

$$\hat{\rho} = \sum_{s,\,r} \int d^4p_1 d^4p_2 \;\; p_{s,\,r}(p_1,\, p_2)\, |p_1,\, s;\, p_2,\, r\rangle\, \langle p_1,\, s;\, p_2,\, r|\,.$$

We will trace out the momentum to just examine the density matrix for the spins of our system for computational purposes. Also because it is the spin correlations that are generally measured in the entanglement experiments.

$$\hat{\rho} = \sum_{s,\,r} p_{sr}\, |p_1,\, s;\, p_2,\, r\rangle\, \langle p_1,\, s;\, p_2,\, r|$$

By the definition of the density matrix one can see that the time evolved density matrix is given by

$$\hat{\rho}(t) = \hat{\mathscr{U}}(t,\, t_0)\hat{\rho}(t_0)\hat{\mathscr{U}}^{\dagger}(t,\, t_0)$$

for the time evolution operator $\hat{\mathscr{U}}(t,\, t_0)$. The scattering operator is $\hat{S} := \lim_{t\to\infty} \lim_{t\to-\infty} \hat{\mathscr{U}}(t,\, t_0)$ so naturally we may find our post scattering density matrix of final states as

$$\hat{\rho}(\infty) = \hat{S}\hat{\rho}(-\infty)\hat{S}^{\dagger} = \sum_{s} p_s \hat{S}\, |p,\, s\rangle\, \langle p,\, s|\, \hat{S}^{\dagger}.$$

We must still evaluate the scattering operator, thus we apply a POVM Kraus measurement operator, $\hat{\Pi} = \sum_q |q,\, r\rangle\, \langle q,\, r|$ to the exited states, obtaining the density matrix for the measured outcome[24]:

$$\hat{\rho} = \frac{\hat{\Pi}\hat{\rho}(\infty)\hat{\Pi}}{\mathrm{Tr}\left(\hat{\Pi}\hat{\rho}(\infty)\hat{\Pi}\right)} = \frac{\sum_{s,\,r,\,r} p_s\, |q,\, r\rangle\, \langle q,\, r|\, \hat{S}\, |p,\, s\rangle\, \langle p,\, s|\, \hat{S}^{\dagger}\, |q,\, r'\rangle\, \langle q,\, r'|}{\mathrm{Tr}\left(\sum_{s,\,r} p_s\, \langle q,\, r|\, \hat{S}\, |p,\, s\rangle\, \langle p,\, s|\, \hat{S}^{\dagger}\, |q,\, r\rangle\right)}.$$

2.1.2 Identifying entanglement

By definition, entanglement is effectively a correlation found between subsystems whereby measuring one subsystem will inform you about the properties of the other subsystem. There is nothing

inherently special about such a correlation as plenty of them exist in the classical world. For example, if you have a red and blue ball each inside a box, you then close the boxes and swap them so you don't know which ball is in which box. Then take one of the boxes far away. you will know that if your box has a blue ball in it then the other box has a red ball in it and vice versa. What's special about entanglement is due to the fact that prior to measurement each subsystem is in a superposition of states. This superposition, by the properties of the wave function, is non-local in nature. Thus measurement results in a non-local action that changes the other subsystem. Historically this has been quite a contentious issue, with Einstein, Podolski and Rosen[25] arguing that this shows that quantum mechanics cannot be a complete theory as we don't see any non-local correlations in the classical world due to some non-local action. They instead advocated that quantum mechanics must posses local hidden variables. To address this problem John. S. Bell[25] developed a inequality that, in principle, could be used to practically test for such variables. Bell's original theorem had a lot of ideal constraints in place and made it practically unusable to an experimentalist. It is for this reason the the CHSH inequalities were developed:

Lemma (CHSH inequalities)**.** Suppose we have a experiment in which the pair of variables (A_1, A_2) are measured on one subsystem and (B_1, B_2) on another subsystem that is spatially separated. If we assume the existence of some local hidden variables then we have the experimental constraint

$$|\langle A_1 B_1\rangle + \langle A_1 B_2\rangle + \langle A_2 B_1\rangle - \langle A_2 B_2\rangle| \leq 1$$

On the theoretical end we may instead consider these to be operators, in which case we define the CHSH operator as

$$\hat{\mathcal{B}}_{\text{CHSH}} := \hat{A}_1 \otimes (\hat{B}_1 + \hat{B}_2) + \hat{A}_2 \otimes (\hat{A}_2 - \hat{B}_2), \quad \hat{A}_1 = \mathbf{a}_1 \cdot \hat{\boldsymbol{\sigma}}$$

for a unit vector $\mathbf{a}_1$ describing the measurements that are performed. The CHSH inequality then becomes

$$|\operatorname{Tr}(\hat{\mathcal{B}}_{\text{CHSH}}\hat{\rho})| \leq 2$$

is satisfied $\forall\hat{\rho}$ so long as local hidden variables are in place. More generally, there is e Cirel'son's inequality

$$\left|\left\langle \hat{\mathcal{B}}_{\text{CHSH}} \right\rangle\right| = |\operatorname{Tr}(\hat{\mathcal{B}}_{\text{CHSH}}\hat{\rho})| \leq 2\sqrt{2}$$

for a expectation value taken with respect to the state of our system $|\psi\rangle$.

As it is famously known, these inequalities were violated, thus quantum mechanics possesses no local hidden variables. This has proved remarkably profound with the development of techniques in

quantum information such as quantum teleportation and entanglement swapping[26]. Furthermore, the fact that quantum mechanics displays this feature of non-local interaction, it has been so significant that some people have proposed that we actually define entanglement by the presence of any non-classical features of our system[27]. Since Bell's proposal, there has been a plurality of other such inequalities that have been developed which are now collectively known as Bell theorems. A very nice feature of Bell theorems is that they serve as a test informing us whether or not we are dealing with an entangled state.

Definition (Entanglement Witness)**.** A entanglement witness W is a positive Hermitian operator action on our Hilbert space $\mathscr{H}$ this is not positive definite, but yields positive expectation values $\forall$ separable pure states $|\psi\rangle$. Since any separable state can be expressed as a convex sum of projectors onto pure separable states, we see that the entanglement witness will also be non-negative with respect to any separable mixed state.

Entanglement witnesses are good from an academic standpoint as they tell us whether our state is separable. They are practically useful as well, telling us whether we have useful entangled states. As useful as the CHSH inequalities are to an experimentalist, they are generally a bit impractical from a more academic standpoint if you only want to check for whether there is entanglement present in your system. For this reason another technique has been developed involving the partial transpose:

Definition (Partial transposition)**.** For a bipartite state $\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B = \sum_{i,j,k,\ell} p_{ij,\,k\ell} \, |i\rangle \langle j| \otimes |k\rangle \langle \ell|$, the partial transpose with respect to B is defined as

$$\hat{\rho}^{T_B} := \sum_{i,j,k,\ell} p_{ij,\,k\ell} \, |i\rangle \langle j| \otimes |\ell\rangle \langle k|$$

Lemma (Positive partial transpose criterion)**.** Given we have a separable density matrix $\hat{\rho}$, the partial transpose of this state $\hat{\rho}^{T_B}$ must have a non-negative spectrum.

If we are dealing with a pure state $\hat{\rho}$ then simply taking the partial trace over a subsystem $\hat{\rho}_B = \mathrm{Tr}_A(\hat{\rho})$ is sufficient. It turns out that if we have a pure state then $\mathrm{Tr}(\hat{\rho}_B^2) = 1$ then there is no entanglement present but if we have a inpure state $\mathrm{Tr}(\hat{\rho}_B^2) < 1$ then there is entanglement present.

2.1.3 Examples of Entangled systems

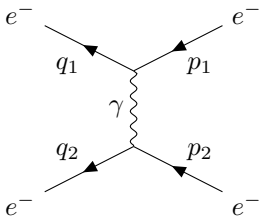
Example (Entanglement via Møller scattering)**.** We will now demonstrate that two particles can become entangled after any interaction, no matter how weak, as long as it happened and we account for some dynamical variable before and after the event. For simplicity let us assume that we are able prepare a pair of non-coupled, non-relativistic pair of electrons in the same state before we turn on some interaction, leaving us with the pure state

$$|\psi\rangle = \big(\alpha_1 |p_1, \uparrow\rangle + \beta_1 |p_1, \downarrow\rangle\big) \otimes \big(\alpha_2 |p_1, \uparrow\rangle + \beta_2 |p_1, \downarrow\rangle\big)$$

where $\alpha_1^2 + \beta_1^2 = 1 = \alpha_2^2 + \beta_2^2$. Working in QED, the tree-level contributions to the scattering matrix

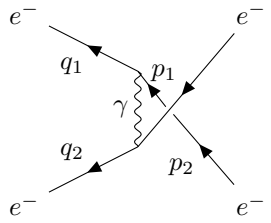
$$\langle \text{out}| \hat{S} |\text{in}\rangle = \langle \text{out}| \left(\hat{\mathbb{I}} + i\hat{T}\right) |\text{in}\rangle = 1 + (2\pi)^4 \delta^{(4)} \left(\sum_i p_i - q_i\right) i\mathcal{M}, \quad \text{are}$$

$i\mathcal{M}_t =$



$$= i\frac{\mathsf{e}^2}{\varepsilon_0}\overline{u}(q_1)\gamma^\mu u(p_1)\frac{\eta_{\mu\nu}}{t^2}\overline{u}(q_2)\gamma^\nu u(p_2)$$

$i\mathcal{M}_u =$



$$= -i\frac{\mathsf{e}^2}{\varepsilon_0}\overline{u}(q_2)\gamma^\mu u(p_1)\frac{\eta_{\mu\nu}}{u^2}\overline{u}(q_1)\gamma^\nu u(p_2).$$

We know from the Dirac equation that

$$u_s(p) = \begin{pmatrix} \sqrt{\boldsymbol{p}\cdot\boldsymbol{\sigma}}\xi_s \\ \sqrt{\boldsymbol{p}\cdot\overline{\boldsymbol{\sigma}}}\xi_s \end{pmatrix}$$

where ξ_s give the 2-component spin orientation. Thus in the non-relativistic limit we may write $\overline{u}(q,\, r)\gamma^i u(p,\, s) \approx 0$, $\overline{u}(q,\, r)\gamma^0 u(p,\, s) \approx 1$ which allows us to state

$$\mathcal{M} = \frac{\mathsf{e}^2}{\varepsilon_0 |\mathbf{p}_1 - \mathbf{q}_1|^2}$$

where the u-channel contribution becomes negligible. If we work in the non-relativistic limit we may write the transfer matrix as

$$\hat{T} = \sum_{n=0}^{\infty} \hat{V}(\hat{\Pi}_{LS}\hat{V})^n = \hat{V} + \hat{V}\hat{\Pi}_{LS}\hat{V} + \hat{V}\hat{\Pi}_{LS}\hat{V}\hat{\Pi}_{LS}\hat{V} + ...$$

where $\hat{\Pi}_{LS} := (E - \hat{\mathcal{H}}_0 + i\varepsilon)^{-1}$ is the Lippman-Schwinger kernal. Applying the 1st order Born approximation obtain

$$\begin{aligned}(2\pi)^4\tilde{V}(\mathbf{p}_1 - \mathbf{q}_1)\delta^{(4)}(p-q) \approx \langle q|\,\hat{T}\,|p\rangle &= (2\pi)^4\delta^{(4)}\left(\sum_i p_i - q_i\right)\mathcal{M} \\ \Rightarrow \quad V(\mathbf{r}) &= \frac{\mathsf{e}^2}{\varepsilon_0}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{k}|^2}, \quad \mathbf{k} = \mathbf{p}_1 - \mathbf{q}_1 \\ &= \frac{\mathsf{e}^2}{8i\pi^2\varepsilon_0 r}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dk \frac{e^{ikr} - e^{-ikr}}{k + i\epsilon} = \frac{\mathsf{e}^2}{4\pi\varepsilon_0 r}.\end{aligned}$$

Spin dependant positions

In the case that the position of the particles is dependant on their spin orientation and that the distance between them is held constant, due to some experiment involving a Stern-Gerlach apparatus, we may naively apply the time evolution operator to our states to obtain the time evolved state

$$\begin{aligned}|\psi(\tau)\rangle_{12} = \hat{\mathscr{U}}(\tau)\,|\psi(t=0)\rangle_{12} &= \exp\left(-i\frac{\hat{\mathcal{H}}_{free} + \hat{V}}{\hbar}\tau\right)|\psi(t=0)\rangle_{12} \\ &= \frac{e^{i\phi_0}}{2}\left(e^{i\phi_{\uparrow_1\uparrow_2}}\,|\uparrow_1\uparrow_2\rangle + e^{i\phi_{\downarrow_1\downarrow_2}}\,|\downarrow_1\downarrow_2\rangle + e^{i\phi_{\downarrow_1\uparrow_2}}\,|\downarrow_1\uparrow_2\rangle + e^{i\phi_{\uparrow_1\downarrow_2}}\,|\uparrow_1\downarrow_2\rangle\right)\end{aligned}$$

where $\phi_0 = \frac{-E_{free}\tau}{\hbar}$, $\phi_{\updownarrow_1\updownarrow_2} = -\frac{\mathsf{e}^2\tau}{\hbar\varepsilon_0 r_{\updownarrow_1\updownarrow_2}}$ which are clearly entangled.

Relativistic spin corrections

If we now simplify our initial state further to show that if we consider higher order relativistic correction, an external apparatus is not even needed to assist in creating entanglement. Using the exact formula[28]

$$\overline{u}(q,\,r)\gamma^i u(p,\,s) = (\xi^r)^\dagger\left(-\frac{i(\mathbf{p}+\mathbf{q})}{2m} + \frac{\boldsymbol{\sigma}\times(\mathbf{p}-\mathbf{q})}{2m}\right)\xi^s$$

and only considering the $\boldsymbol{\sigma}\times$ term we obtain by Fourier transforming our scattering matrix the dipole interaction matrix term

$$V = -\frac{\mathsf{e}\boldsymbol{\sigma}_1}{2m} \cdot \left(\nabla \times \left(\frac{\mathsf{e}\boldsymbol{\sigma}_2}{2m} \times \nabla \frac{1}{4\pi\varepsilon_0 r} \right) \right) = J\left(2\delta^{s_1 r_1}\delta^{s_2 r_1} - \delta^{s_1 r_1}\delta^{s_2 r_2}\right)$$

where the last equality applies if both our initial states are oppositely aligned along, say, the z-axis. $\boldsymbol{\sigma}_i$ is the vector of Pauli matrices multiplied by the direction of each incoming particle, and s_i and r_i indicate the spins of our incoming and outgoing particles respectively. Performing the time evolution as we did before we may obtain the final state after time τ

$$|\psi(\tau)\rangle_{12} = e^{i\phi_0}\big(\cos(2J\tau)\,|{\downarrow_1}{\uparrow_2}\rangle - i\sin(2J\tau)\,|{\uparrow}{\downarrow}\rangle\big)$$

where for the time $\tau = \pi/4J$ we obtain

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}\big(|{\downarrow_1}{\uparrow_2}\rangle - i\,|{\uparrow_1}{\downarrow_2}\rangle\big)$$

our maximally entangled Bell state.

It should be noted that while all of the examples presented here are with spins the exact same treatement could be done with a continuous variable such as momentum[29].

2.2 Quantum vs Classical Dynamics

2.2.1 Quantum decoherence and realism

Definition (Decoherence)**.** Suppose we have a density matrix $\hat{\rho}$ and a environment state $|\psi\rangle\langle\psi|_E$ so the collective state of our system is $\hat{\rho}\otimes|\psi\rangle\langle\psi|_E$. When we evolve the entire system coupling between the state of interest $\hat{\rho}$ and the environment state can cause all of the off-diagonal elements of the density matrix to vanish. We are then left with a classical ensemble of states. The process where a system loses its coherences is called decoherence.

Decoherence is effectively the same as removing all interference terms from your probability distribution. While decoherence is often claimed to solve the measurement problem this is not do so without additional interpretation or postulated dynamics for while decoherence does provide us with a classical ensemble of states, each state is still in a superposition until it is measured. To illustrate this most clearly let us consider the density matrix produced by the single particle

superposition state $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$

$$\hat{\rho} = \frac{1}{2}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1| \big) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

. If we evolve this matrix so that the off-diagonals disappear then we will still be left with a single particle that is still in a superposition of states until measured. A true definite state would correspond to only a single element of the density matrix being occupied. Decoherence does however provide a solution to the measurement basis problem[30], giving a mechanism for how a preferred basis is arrived at upon measurement and giving one a route for deriving Born's rule with having to postulate the axiom in the first place.

From this exposition we can thus draw the conclusion that the true difference between classical and quantum mechanics is the property that the value of any classical quantity is well defined at a particular point in time.

Definition (Realism)**.** The properties of the measured properties of particles depend only on the properties they had prior to measurement and are independent of measurement. Furthermore, the properties are well defined at any particular point in time, taking on specific values[31].

This is known to not be true for quantum particles contrary to what we see in the classical world with the Bell test violations completely ruling out any hope of recovering some sort of microscopic realism. This is still contrary to the property that macroscopic objects carry properties independent of the measurement process is known as macrorealism per se[32]. While arguably a more true definition for classicallity is, it does not take away from the fact that many of the distributions we see can be explained by both classical and quantum phenomena; entanglement as a criterion for quantumness reigns supreme. This inquiry does prompt the question for how we test for strict classicality through. The solution to this problem is addressed by Leggett-Garg test[33], a type of Bell inequality which while able to rule out certain cases of macrorealism are still a way off for being definitive tests for whether truly classical phenomena exists.

2.2.2 Local operations and classical communications

So far we have have seen some of the phenomena that quantum states display when they interact with each other. We have also seen that when a quantum state interacts with its environment we obtain a classical probability distribution. But is the only thing separating microscopic classical

and quantum interactions the presence of non-local correlations? As has been touched upon, the quantum information community has answered this question with the statement that any interaction which can produce entanglement is a quantum one; otherwise it is classical. To address what interaction could create entanglement, the paradigm of local operations and classical communications was created.

Definition (LOCC)**.** Consider two quantum systems that we shall label as A and B The most general quantum operation Λ is a stochastic physical operation with a trace non-increasing completely positive map $\hat{\rho} \mapsto \Lambda\hat{\rho}/\operatorname{Tr}(\Lambda\hat{\rho})$ that gives the new state for $\Lambda(\hat{\rho}) = \sum_i \hat{\mathscr{O}}_i\hat{\rho}\hat{\mathscr{O}}_i^\dagger$ for Kraus operators $\hat{\mathscr{O}}_i$ where the map is said to be deterministic if $\operatorname{Tr}(\hat{\mathscr{O}}\hat{\rho}) = 1$. Local operations and classical communications(LOCC) is the class of trace preserving deterministic channel type maps that satisfy one of the following hierarchy of conditions, each of which implies the previous condition:

- (Local operations)**.** A operation acting on both A and B may be written in the form $\Lambda_{AB} = \Lambda_A \otimes \Lambda_B$;
- (1-way LOCC operations)**.** Assuming that the interaction is propagating from A we may write this map as $\Lambda_{\overrightarrow{AB}}(\rho) = \sum_i (\hat{\mathscr{O}}_{iA} \otimes \hat{\mathbb{I}}_B)\big([\hat{\mathbb{I}}_A \otimes \Lambda_{iB}](\hat{\rho})\big)(\hat{\mathscr{O}}_{iA}^\dagger \otimes \hat{\mathbb{I}}_B)$. If the order of operations was acting in the other direction we would then simply swap which operators are identities and actions;
- (2-way LOCC operations)**.** Assuming that both systems are sending classical information to each other these operators may be composed in the following way: $\Lambda_1(\rho_{\hat{A}B}) = \sum_i \big((\hat{\mathscr{O}}_{iA} \otimes \hat{\mathbb{I}})\hat{\rho}_{AB}(\hat{\mathscr{O}}_{iA}^\dagger \otimes \hat{\mathbb{I}})\big) \otimes |i\rangle\langle i|_B$, $\Lambda_2(\rho_{\hat{A}B}) = \sum_i \big((\hat{\mathbb{I}} \otimes \hat{\mathscr{O}}_{iB})\hat{\rho}_{AB}(\hat{\mathbb{I}} \otimes \hat{\mathscr{O}}_{iA}^\dagger)\big) \otimes |i\rangle\langle i|_A$
- (Separable operations)**.** These are operations that can be constructed with the product of Kraus operators $\Lambda_{AB}^{\text{sep}}(\rho) = \sum_i (\hat{\mathscr{O}}_{iA} \otimes \hat{\mathscr{O}}_{iB})\hat{\rho}(\hat{\mathscr{O}}_{iA} \otimes \hat{\mathscr{O}}_{iB})^\dagger$.
- (Positive partial transpose operations)**.** These are operations Λ^{PPT} such that $(\Lambda^{PPT}[(\cdot)^{T_B}])^{T_B}$ is completely positive.

Based off the arguments we have made thus far, all of these are criterion for a classical interaction in their own right. For the bulk of this dissertation we shall focus on, at most, 2-way LOCC operations but generally only consider 1-way LOCC operations for simplicity. The rational behind this property is that if we have a local operation then it should only act at a point. Hence, we should only measure changes at the points where it acted. Conversely, a non-local operation could act at multiple points simultaneously, therefore we may see a correlation in two disconnected subsystems due to the simultaneous action of the operator. While we do see non-local correlations,

they are known to not be due to such a simultaneous action, justifying this postulate. There are however, non-local theories that have not been ruled out and could produce such affects but LOCC was not established with these effects in mind.

Similarly, we call such a sequence of operations a classical communication because classically we would apply a sequence of operations to each separate particle or subsystem system. We would then not see any additional correlation other than the local one for which we could retrace the subsystems history back to the initial action e.g. a series of collisions between particles. The use of the word non-separable is quite important, for conversely, the action of an operator on an entire subsystem that is separable may lead to the non-separability of that subsystem as all of the respective subsystems that it is composed of become entangled.

Technically speaking, LOCC was created assuming that the classical channels are modeled by the Koopman von-Neumann formulation of classical mechanics; see appendix A. As we shall see later on, if one assumes different classical models then this paradigm doesn't necessarily hold. With this thought in mind one might ask 'does LOCC actually prevent the formation of entanglement'?

Can't classical channels create entanglement?

Working in the Koopman von-Neumann paradigm, let us assume that we may map some classical observable corresponding to a point in phase space $(x,\, k)$ to a set of commuting operators $\{\hat{C}\}$ in some Hilbert space $\mathscr{H}_C$. That is, $\hat{C}$ has only one observable. This is then a part of the larger Hilbert space $\mathscr{H}_Q \otimes \mathscr{H}_{Q'} \otimes \mathscr{H}_C \otimes$ for the distinct quantum systems Q and Q'. Consider the spectral decomposition $\hat{C} = \sum_c c\hat{\Pi}_c$ corresponding to the PVM's $\{\hat{\Pi}_C\}$. We may take the initial density operator to have the diagonal form[10]

$$\hat{\rho}_{QQ'C} = \bigoplus_C \big(p(c)\hat{\rho}_{QQ'}(c) \otimes \hat{\rho}_C(c)\big) \text{ with respect to } \hat{C}\ .$$

The final density operator describing Q and Q' after a interaction is given by

$$\hat{\rho}^{\text{out}}_{QQ'} = \text{Tr}\left(\hat{\mathscr{U}}_{QC}\hat{\mathscr{U}}_{Q'C}\hat{\rho}_{QQ'C}\hat{\mathscr{U}}^{\dagger}_{Q'C}\hat{\mathscr{U}}^{\dagger}_{QC}\right) = \sum_c p(c)\hat{\mathscr{U}}_Q(c) \otimes \hat{\mathscr{U}}_{Q'}(c)\hat{\rho}_{QQ'}(c)\hat{\mathscr{U}}^{\dagger}_Q(c) \otimes \hat{\mathscr{U}}^{\dagger}_{Q'}(c)$$

where $\hat{\mathscr{U}}_{QC} = \bigoplus_c \left(\hat{\mathscr{U}}_Q(c) \otimes \hat{\mathbb{I}}_{Q'} \otimes \hat{\mathscr{U}}_C(c)\right)$ and $\hat{\mathscr{U}}_{Q'C} = \bigoplus_c \left(\hat{\mathbb{I}}_Q \otimes \hat{\mathscr{U}}_{Q'}(c) \otimes \hat{\mathscr{U}}_C(c)\right)$. As we can see, since the new density matrix is just a convex combination of local unitary operations on $\mathscr{H}_Q \otimes \mathscr{H}_{Q'}$ the quantum-classical interactions cannot increase entanglement between Q and Q'.

Scattering in QFT

While one may understand that a classical channel can't create entanglement, this may still feel a little presumptuous as we have assumed that our evolution operator is factorisable and can be applied successively. For an operator such as the scattering operator it is not so obvious we can do this; 'can we'? For a heuristic discussion let us consider Møller scattering but with a classical electromagnetic field, so our coupling term is $ieA_\mu(x)\hat{\overline{\psi}}(x)\gamma^\mu\hat{\psi}(x)$. In this case the scattering quantum fields do not interact. This is a consequence of the fact that the classical field is given vaguely by the form (neglecting indicies)

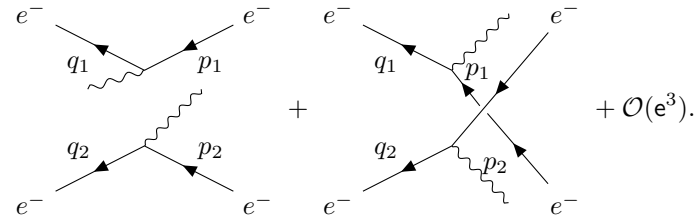
$$\phi(x) = H(x) + \int G(x,x')j(x')\,dx'$$

for a source $j(x)$ that depends on a fixed point in spacetime. The source for the different fields will not couple in any special way as we see in quantum mechanics because the field operators do not constructively interfere. To give a explicit example let us perform the Dyson series expansion of $\langle\Omega|\,\hat{\overline{\psi}}(q_1)\hat{\overline{\psi}}(q_2)\hat{\psi}(p_1)\hat{\psi}(p_2)\,|\Omega\rangle$ to the 2nd order. We then have an expression of the form

$$i\mathcal{M} =$$

$$(-ie^2)\int\frac{d^4p_1}{(2\pi)^4}\ldots e^{-ix\cdot p_i+\ldots}\int d^4w d^4z\,A_\mu(z)A_\nu(w)\Big(D_F^{ab}(x_1{-}z)\gamma^\mu_{bc}D_F^{cd}(z{-}y_1)D_F^{ef}(x_2{-}w)\gamma^\nu_{fg}D_F^{fh}(w{-}y_2){-}\ldots\Big)$$

where for simplicity and to save space we have left the out all the additional terms that are being integrated over in the Fourier transform with the assumption that the reader is familiar with what the fully expanded expression should look like after we have performed the Wick contractions. As can be seen, because the electromagnetic fields are reduced to functions of real numbers by the eigenstates we simply pull them out and they have the same action on path free interactions, the interaction being determined by the classical source j and gives us the follow diagrams assuming that the source is a $\delta-$function is not centered about zero

$i\mathcal{M} =$ 

$+\,\mathcal{O}(\mathsf{e}^3).$

There is thus no coupling between the states so when we evolve our system as we did in the QED example the output states will evolve according to a phase determined by our predetermined classical field. This phase will appear as a product of factors in front of each state, with each

factor corresponding to one of the free states which can be factored out with the normal state by absorbing the phase into the coefficients in front of each states; thus the system will not become entangled.

With these preliminary defintions out of the way we can move onto the next section: quantifying entanglement.

2.3 Characterising Entanglement

2.3.1 Building a entanglement quantifier

So far we have seen that we may characterise entanglement with Bell inequalities. We have also seen that there exists more and less entangled states composed of the same subsystems. Bell inequalities do not give us a way to compare such states. It is with this purpose in mind that we seek a more general set of procedures for characterising whether or not entanglement is present in a system.

Definition (Entanglement monotone)**.** A entanglement monotone $\mathcal{M}(\hat{\rho})$ is a quantity that does not increase under local operations and classical communications(LOCC); that is:

- (local operations) For a set of local operators $\{\hat{\mathcal{O}}_i\}$ that act on the jth subsystem of a system that is separable into n non-separable subsystems we see no increase in the monotone, $\mathcal{M}(\hat{\rho}) \geq \mathcal{M}\left(\sum_i \left(\hat{\mathbb{I}}^{\otimes(j-1)} \otimes \hat{\mathcal{O}}_i \otimes \hat{\mathbb{I}}^{\otimes(n+1-j)}\right)\hat{\rho}\left(\hat{\mathbb{I}}^{\otimes(j-1)} \otimes \hat{\mathcal{O}}_i \otimes \hat{\mathbb{I}}^{\otimes(n+1-j)}\right)^\dagger\right)$
- (classical communication) Given that we have a set of n sets of local operators $\left(\{\hat{\mathcal{O}}_i\}_k\right)_{k=1}^{n}$, each of which acts on the kth subsystem of our total system, the successive application of each transformation to each non-separable subsystem will leave the other subsystems unaffected, independent of the order that the transformations are applied, and we may write a similar inequality to the one above.

In the case of a pure bipartite system, $\hat{\rho} = \hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)}$ we can write the locality condition for a operation acting on the 1st of the two subsystems $\hat{\rho}^{(1)}$ as

$$\mathcal{M}(\hat{\rho}) \geq \mathcal{M}\left((\hat{\mathcal{O}}_1 \otimes \hat{\mathbb{I}})\hat{\rho}(\hat{\mathcal{O}}_1^\dagger \otimes \hat{\mathbb{I}})\right) = \mathcal{M}\left((\hat{\mathcal{O}}_1\hat{\rho}^{(1)}\hat{\mathcal{O}}_1^\dagger) \otimes \hat{\rho}^{(2)}\right)$$

and the classical communications condition is that our operations being local and classical may only act on one subsystem at a time and not increase the monotone e.g.

$$\hat{\rho} \mapsto \hat{\sigma} = \big(\hat{\mathscr{O}}_1 \otimes \hat{\mathbb{I}}\big)\hat{\rho}\big(\hat{\mathscr{O}}_1^\dagger \otimes \hat{\mathbb{I}}\big) \mapsto \hat{\tau} = \big(\hat{\mathbb{I}} \otimes \hat{\mathscr{O}}_2\big)\big(\hat{\mathscr{O}}_1 \otimes \hat{\mathbb{I}}\big)\hat{\rho}\big(\hat{\mathscr{O}}_1^\dagger \otimes \hat{\mathbb{I}}\big)\big(\hat{\mathbb{I}} \otimes \hat{\mathscr{O}}_2^\dagger\big)$$

$$\mathcal{M}(\hat{\rho}) \geq \mathcal{M}(\hat{\sigma}) \geq \mathcal{M}(\hat{\tau})$$

Postulate (Entanglement measure requirements)**.** As of the time of writing there is no consensus on the complete list of axioms that a entanglement quantifier must satisfy. However, it is generally agreed that the following axioms are necessary:

- A entanglement measure $\mathcal{M}$ should monotonically decrease under LOCC,
- Mixing two states $\hat{\rho}$, $\hat{\sigma}$ probabilistically can increase only classical correlations. This implies convexity of an entanglement measure; that is, $\mathcal{M}(p\hat{\rho} + (1-p)\hat{\sigma}) \leq p\mathcal{M}(\hat{\rho}) + (1-p)\mathcal{M}(\hat{\sigma})$,
- There should exist some maximally entangled state. That is, $\exists\sigma$ such that for all other $\hat{\rho}$ representing configurations of the constituents of the system, we may write $\mathcal{M}(\sigma) \geq \mathcal{M}(\hat{\rho})$.

The purpose of characterising the entanglement is ultimately to characterise the information that we may manipulate in a system. In classical mechanics this is naturally done by the Shannon entropy. Carrying the analogy over to quantum mechanics the von-Neumann entropy is given the following definition:

Definition (Entanglement entropy)**.** For any state $\hat{\rho}$ the von-Neumann entropy is defined as $S(\hat{\rho}) := -\operatorname{Tr}(\hat{\rho}\ln(\hat{\rho}))$. The entanglement entropy of a subsystem is then the von-Neumann entropy of the reduced density matrix representing that subsystem $\hat{\rho}_A :=$ When $S = 0$ there is said to be no entanglement and when $S = 1$ the state is said to be maximally entangled. One may find the entanglement entropy for subsystems by tracing out the other subsystems with the partial trace and then taking the entanglement entropy of the obtained reduced density matrix.

Some trivial points about the entangle entropy

- for a pure state $S(\hat{\rho}) = -\operatorname{Tr}(|\psi\rangle\langle\psi|\ln(|\psi\rangle\langle\psi|)) = \operatorname{Tr}(0) = 0$;
- The relation $(\lambda_1 - \lambda_2)\left(\frac{\partial S(\hat{\rho})}{\partial\lambda_1} - \frac{\partial S(\hat{\rho})}{\partial\lambda_2}\right) = (\lambda_1 - \lambda_2)\ln\left(\frac{\lambda_2}{\lambda_1}\right) \leq 0$ for eigenvalues of the density matrix λ_1, λ_2; thus it is an entanglement monotone and serves as a entanglement measure based off the current set requirements

The von-Neumann entropy measures the entropy of a quantum system in the same manner as the classical entropy does in classical mechanics. We have seen that the density matrix contains information about quantum and classical statistics and thus it measures both types of information for all states state. It is a well established phenomena that the entropy of a system will increase in classical mechanics, thus the entanglement entropy is, in general, not a entanglement monotone under LOCC. Therefore, despite the intuitive definition of the von-Neumann entropy as an entanglement measure it unfortunately is only valid for systems that may be represented by a pure state.

This motivates new definitions for characterising the entanglement of a system of which, there is a vast selection to choose from due to the complexity of the problem and the many approaches one can take. Here we will briefly go over some of the most commonly used measures and explain where they are derived from. The literature is vast, and increasingly niche problems have lead to a colossal number of different measures, which one may examine when they find a problem that requires specialisation.

2.3.2 Quantifying entanglement for mixed states

Entanglement of formation

Definition (Distillable entanglement)**.** Let $\hat{\rho}$ be a state. We define the distillable entanglement as

$$E_D := \sup\left\{r : \lim_{n\to\infty}\left(\inf_{\Psi} \mathrm{Tr}\left(\Psi\left(\hat{\rho}^{\otimes n}\right) - \Phi\left(2^{rn}\right)\right)\right)\right\}.$$

The distillable entanglement aims to characterise the amount of entanglement present by asking the question: suppose we have our state $\hat{\rho}$ and we perform LOCC operations such that in the asymptotic limit of performing infinitely many LOCC operations, our state $\hat{\rho}$ looks like a maximally entangled one, what is the ratio between the number of operations we need to perform on our set of maximally entangled states and $\hat{\rho}$ for them to look the same in this asymptotic limit? Complementary to the distillable entanglement there is the entanglement cost which instead measures the ratio of operations one needs to perform on the maximally entangled states against our state $\hat{\rho}$.

The distillable entanglement measure is arguably the truest measure of entanglement as it meets our desired requirements and provides what is likely the best measure at quantifying the total entanglement present. It suffers from the flaw that it is generally very difficult to calculate.

For this reason other entanglement measures have been proposed, derived from the distillable entanglement but easier to calculate, yet capturing different aspects of the entangled system.

Definition (Entanglement of formation)**.** For a mixed state $\hat{\rho}$ the entanglement of formation is defined as

$$E_F(\hat{\rho}) := \inf\left(\sum_i p_i E_{vN}(|\psi_i\rangle\langle\psi_i|)\right)$$

The entanglement of formation provides an upper bound for the distillable entanglement, acting as a convex roof. This is also a non-trivial calculation. It has been shown that for bipartite states we may reduce the computation required by first evaluating a value known as the concurrence.

Definition (Concurrence)**.** For the case of a bipartite two level system this is given by

$$\mathcal{C}(\hat{\rho}) := \max\{0,\, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

where λ_i are the square-roots of the eigenvalues of the matrix $\hat{\rho}\sigma_y \otimes \sigma_y \hat{\rho}^* \sigma_y \otimes \sigma_y$ and the entanglement of formation is

$$\mathcal{E}_F(\hat{\rho}) := S\left(\sqrt{2\,(1-\mathcal{C}(\hat{\rho})^2)}\right)$$

where $S(x) := -x\log(x) - (1-x)\log(1-x)$.

In the case of pure states, this all reduces to

$$\mathcal{C}(\hat{\rho}) = \sqrt{2\,(1-\mathrm{Tr}\,(\hat{\rho}^2))}$$

Logorithmic negativity

One might still argue that even the concurrence is a rather convoluted measure of entanglement. Looking back to our entanglement witness one may remember that we used the positivity of our eigenvalues of the partially traced state to check for entanglement. Inspired by this notion, an even simpler measure is proposed in the form of the negativity.

Definition (Logarithmic negativity)**.** For a bipartite state $\hat{\rho}$ we define the negativity of the state as

$$\mathcal{N} := \frac{\|\hat{\rho}^{T_B}\| - 1}{2} = \sum_i \frac{|\lambda_i| - \lambda_i}{2}, \quad \|\hat{\rho}\| = \sqrt{\hat{\rho}^\dagger \hat{\rho}}$$

for eigenvalues of $\hat{\rho}^{T_B}$, λ_i. This is not additive however so we define the logarithmic negativity,

$$E_{LN} := \log_2(\|\hat{\rho}^{T_B}\|) = \log_2(2\mathcal{N} + 1),$$

to remedy this issue. The logarithmic negativity acts as a upper bound to the distilable entanglement and is the easiest measure to calculate out of the ones presented thus far. Yet, logarithmic negativity doesn't necessarily detect all entangled states, fails to be a convex upper bound (unlike the negativity) and is not asymptotically continuous; failing to reduce to the entanglement entropy for pure states.

Chapter 3

Testing for Quantum Gravity

With the foundational preliminaries out of the way, we are now in a position to start discussing the properties of quantum gravity that we might look for in a laboratory setting. Before we discuss what we might look for we will first need to understand how we should expect gravity to behave as a quantum theory.

3.1 Gravity and quantum field theory

To consider gravity as a quantum theory involves quantising the gravitational interactions. To do this means to postulate that there exists a fundamental particle, the graviton, that would mediate the interactions between regions of non-zero energy density. While there are multiple proposals for theories of quantum gravity, they should in principle all reduce to a quantum field theory of gravity as an effective field theory. This effective field theory is non-renormalisable although we may still calculate the contributions from tree level interactions. Pioneered by Bryce deWitt[34, 35, 36], the covariant quantisation of gravity is known as perturbativly quantised gravity and may be derived as follows:

3.1.1 Perturbativly quantised gravity

The perturbativly quantised action

We know that we may describe gravitational waves by linearising the Einstein Hilbert action

$$S_{\mathrm{EH}}[g_{\mu\nu}] = \frac{\mathsf{c}^4}{16\pi\mathsf{G}} \int_{\mathcal{M}} d^4x \, \sqrt{-g} R$$

to obtain the Pauli-Feirz action. Considering that we generally obtain quantum particles by quantising the fields in a classical wave equation this could be a very easy starting point. However, we also want to ensure that we are preserving Lorentz covariance and unitarity. It would also be very convenient to have a generally scheme that we can build upon for future calculations. With this line of thought and for completeness, we would like to consider our gravitational waves on a arbitrary background. One may consider the graviton as a translation about a classical background field $\tilde{g}$. Let us consider our quantum field $g_{\mu\nu}$ and expand it about $\tilde{g}_{\mu\nu}$ so that $g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) + h_{\mu\nu}(x)$. To obtain our new Einstein-Hilbert action it seems reasonable to act $S_{\text{EH}}(\tilde{g}_{\mu\nu})$ with a translation operator[37]

$$S_{\text{EH}}(g_{\mu\nu}) = \exp\left[\int_{\tilde{\mathcal{M}}} d^4y\,\sqrt{-\tilde{g}}\Big(h_{\alpha\beta}(y)\frac{\delta}{\delta\tilde{g}_{\alpha\beta}(y)} + \partial_\gamma h_{\alpha\beta}(y)\frac{\delta}{\delta(\partial_\gamma\tilde{g}_{\alpha\beta}(y))} + \partial_\gamma\partial_\delta h_{\alpha\beta}(y)\frac{\delta}{\delta(\partial_\gamma\partial_\delta\tilde{g}_{\alpha\beta}(y))}\Big)\right] S_{\text{EH}}(\tilde{g}_{\mu\nu}).$$

The Einstein-Hilbert action up to quadratic order is then given by[38]

$$\begin{aligned} S_{\text{EH}}[g_{\mu\nu}] =& \big(S_{\text{EH}}[\tilde{g}_{\mu\nu}]\big) + \left(\frac{\mathsf{c}^4}{16\pi\mathsf{G}}\int_{\tilde{\mathcal{M}}} d^4x\,\sqrt{-\tilde{g}}h_{\mu\nu}\tilde{G}^{\mu\nu}\right) \\ &+ \left(\frac{\mathsf{c}^4}{16\pi\mathsf{G}}\int_{\tilde{\mathcal{M}}}\sqrt{-\tilde{g}}\Big(\frac{1}{2}\tilde{\nabla}_\alpha h_{\mu\nu}\tilde{\nabla}^\alpha h^{\mu\nu} - \frac{1}{2}\tilde{\nabla}_\alpha h\tilde{\nabla}^\alpha h + \tilde{\nabla}_\alpha h\tilde{\nabla}_\beta h^{\alpha\beta} - \tilde{\nabla}_\alpha h_{\mu\beta}\tilde{\nabla}^\beta h^{\mu\alpha} \right.\right. \\ &\left.\left. + \tilde{R}\left(\frac{1}{2}h_{\mu\nu}h^{\mu\nu} - \frac{h^2}{4}\right) + \tilde{R}^{\mu\nu}\big(hh_{\mu\nu} - 2h^\alpha_\mu h_{\nu\alpha}\big)\Big)\right) \\ &+ \mathcal{O}\left(S^{(3)}_{\text{EH}}[\tilde{g}_{\mu\nu}]\right) \end{aligned}$$

where the tilde over each term denotes that the term is a function of the background metric $\tilde{g}_{\mu\nu}$, $G^{\mu\nu}$ is the Einstein tensor and $h = \tilde{g}^{\mu\nu}h_{\mu\nu}$.

Path integral quantisation of gravity

To quantise gravity we then construct our path integral by starting with the partition function of the Einstein-Hilbert action S_{EH} and a source $J^{\mu\nu}$

$$\begin{aligned} Z[g_{\mu\nu},\,J^{\mu\nu}] &= \int \mathcal{D}h\,\exp\left(\frac{i}{\hbar}\left(S_{EH}[\tilde{g}_{\mu\nu},\,h_{\mu\nu}] + \int_{\tilde{\mathcal{M}}} d^4x\,\sqrt{-\tilde{g}}J^{\mu\nu}(x)h_{\mu\nu}(x)\right)\right) \\ &= \int \mathcal{D}h\,\exp\left(\frac{i}{\hbar}S[\tilde{g}_{\mu\nu},\,h_{\mu\nu},\,J_{\mu\nu}]\right). \end{aligned}$$

As with Yang-Mills theories we introduce quantum gauge invariance by noting

$$1 = \left[\prod_{\alpha=1}^{4}\int \mathcal{D}\epsilon_\alpha\right]\left(\det\left(\frac{\delta G_\nu(g'_{\mu\nu})}{\delta\epsilon_\nu}\right)\right)\delta^{(N)}\big(G_\mu[g_{\mu\nu}]\big)$$

for a gauge condition G and the gauge transformed field g'. Introducing the Faddeev-Poppv ghosts

$$\begin{aligned}\det\left(\frac{\delta G_\alpha(h'_{\mu\nu})}{\delta\epsilon_\beta}\right) &= \left[\prod_{\gamma=1}^{4}\prod_{\delta=1}^{4}\int\mathcal{D}c_\gamma\int\mathcal{D}\bar{c}_\delta\right]\exp\left(\frac{i\mathsf{c}^4}{16\pi\mathsf{G}\hbar}\int_{\mathcal{M}}d^4x\,\sqrt{-g}\bar{c}_\mu\frac{\delta G_\nu\left(g'_{\mu\nu}\right)}{\delta\epsilon_\mu}c^\nu\right)\\ &= \int\mathcal{D}c\int\mathcal{D}\bar{c}\,\exp\left(\frac{i}{\hbar}S_{\mathrm{FP}}[\tilde{g}_{\mu\nu},\,h_{\mu\nu},\,c_\alpha,\,\bar{c}_\alpha]\right)\end{aligned}$$

An important point here is that these ghosts are actually spin-1 spinor fields, unlike the ghosts of Yang-Mills theories, due to the fields being spacetime vectors. To finish writing this expression we will have to choose an appropriate gauge condition that preserves unitarity. It turns our this is done by the generalised de Donder gauge where

$$G_\mu = \tilde{\nabla}^\nu h_{\mu\nu}(x) - \sigma\tilde{\nabla}_\mu h(x)$$

for a gauge fixing parameter σ that reduces to the regular de Donder gauge for $\sigma = 1/2$. To differentiate the gauge with respect to the infinitesimal displacement ϵ^μ we must transform the expression by performing the infinitesimal translation $x^\mu \mapsto x^\mu + \epsilon^\mu(x) + \mathcal{O}(\epsilon^2)$, finding that

$$\begin{aligned}g_{\mu\nu}(x) &\mapsto (J^{-1})^\alpha_\mu(x)(J^{-1})^\beta_\nu(x)\tilde{g}_{\alpha\beta}(x) + (h_{\mu\nu}(x) + 2\tilde{\nabla}_{(\mu}\epsilon_{\nu)}(x)) + \mathcal{O}(\epsilon^2)\\ &= \tilde{g}_{\alpha\beta}(x) + \kappa(h_{\mu\nu}(x) + 2\tilde{\nabla}_{(\mu}\epsilon_{\nu)}(x)) + \mathcal{O}(\epsilon^2) \quad \Rightarrow \quad h_{\mu\nu}(x) \mapsto h_{\mu\nu}(x) - 2\tilde{\nabla}_{(\mu}\epsilon_{\nu)}(x)\end{aligned}$$

obtaining

$$\frac{\delta G_\mu}{\delta\epsilon_\nu} = -\delta^\nu_\mu\tilde{\nabla}^\rho\tilde{\nabla}_\rho - (1-2\sigma)\tilde{g}^{\nu\rho}\tilde{\nabla}_\rho\tilde{\nabla}_\mu + 2\sigma\tilde{g}^{\nu\rho}\tilde{R}_{\mu\rho}$$

We may then write the Faddeev-Popov ghost action as

$$S_{\mathrm{ghost}} = \frac{\mathsf{c}^4}{16\pi\mathsf{G}}\int_{\tilde{\mathcal{M}}}d^4x\,\sqrt{-\tilde{g}}\bar{c}^\mu\left(2\sigma\tilde{R}_{\mu\nu}c^\nu - (1-2\sigma)\tilde{\nabla}_\nu\tilde{\nabla}_\mu c^\nu - \tilde{\nabla}_\sigma\tilde{\nabla}^\sigma c_\mu\right).$$

Lastly, we must evaluate our δ-functional, which may written as a path integral using a Nakanishi-Lautrup field:

$$\delta^{(N)}\big(G_\mu[g_{\mu\nu}]\big) = \int\mathcal{D}B\exp\left(\frac{i\mathsf{c}^4}{16\pi\mathsf{G}\hbar}\int_{\tilde{\mathcal{M}}}d^4x\,\sqrt{-\tilde{g}}B^\mu(x)G_\mu[g_{\mu\nu}(x)]\right).$$

Performing an analysis of charges will lead one to the conclusion that we may rewrite this expression in the form[39, 37]

$$\begin{aligned}S_{\mathrm{gauge}} &= \frac{-\mathsf{c}^4}{32\pi\xi\mathsf{G}}\int_{\tilde{\mathcal{M}}}d^4x\,\sqrt{-\tilde{g}}G_\mu[g_{\mu\nu}]G^\mu[g_{\mu\nu}]\\ &= \frac{-\mathsf{c}^4}{32\pi\xi\mathsf{G}}\int_{\tilde{\mathcal{M}}}d^4x\,\sqrt{-\tilde{g}}\left(\tilde{\nabla}^\nu h_{\mu\nu} - \sigma\tilde{\nabla}_\mu h\right)\left(\tilde{\nabla}_\rho h^{\mu\rho} - \sigma\tilde{\nabla}^\mu h\right)\end{aligned}$$

which reduces to the de Donder gauge for $\xi = 1/2$. With this we may write our BRST generating function as

$$\begin{aligned} Z[g_{\mu\nu},\, J^{\mu\nu}] &= \frac{\left[\int \mathcal{D}\epsilon \int \mathcal{D}h\right] \left|\det\left[\frac{\delta G_\alpha[g_{\mu\nu}]}{\delta \epsilon_\alpha}\right]\right| \delta^{(N)}\big(G_\alpha[g_{\mu\nu}]\big)}{\int \mathcal{D}\epsilon} \\ &= \int \mathcal{D}h \int \mathcal{D}B \int \mathcal{D}c \int \mathcal{D}\bar{c}\, \exp\left(iS_{\mathrm{BSRT}}[\tilde{g}_{\mu\nu},\, h_{\mu\nu},\, c_\alpha,\, \bar{c}_\alpha, B_\alpha]\right) \\ &= \int \mathcal{D}h \int \mathcal{D}c \int \mathcal{D}\bar{c}\, \exp\left(\frac{i}{\hbar}\big(S_{\mathrm{EH}} + S_{\mathrm{ghost}} + S_{\mathrm{gauge}}\big)\right) \end{aligned}$$

If we also want to add other fields to our system then we will have to also translate these fields into the appropriate spacetime configuration. Without knowing the form of these fields we may write the first order expansion of the matter component as

$$S_{\mathrm{M}}[g_{\mu\nu}] = S_{\mathrm{M}}[\tilde{g}_{\mu\nu}] + \int_{\tilde{\mathcal{M}}} d^4x\, \sqrt{-\tilde{g}} h_{\mu\nu} \tilde{T}^{\mu\nu} + \mathcal{O}\left(S^{(2)}_{\mathrm{M}}[\tilde{g}_{\mu\nu}]\right), \quad \tilde{T}^{\mu\nu} = \frac{\partial \mathscr{L}_{\mathrm{M}}[\tilde{g}_{\mu\nu}]}{\partial \tilde{g}_{\mu\nu}} - \frac{\tilde{g}^{\mu\nu}}{2} \mathscr{L}_{\mathrm{M}}[\tilde{g}_{\mu\nu}]$$

for the Hilbert stress-energy tensor $\tilde{T}^{\mu\nu}$. While we have explored a very general scheme for performing low energy graviton calculations, it is not the most general. We have not taken into account models with torsion present or the possibility of higher order corrections to the Einstein-Hilbert action that one may find in other theories of quantum gravity. A very similar analysis may be performed instead, using the Einstein-Cartan formalism[37].

The higher order corrections to the Einstein Hilbert action are of little concern as we can use the exact same procedure as we have done here. If there is torsion necessarily present in these higher order correction or intrinsic to the field at a lower level, then one can perform the exact same analysis but with additional terms in our translation operator to take into account contortion terms in the connection[37]. In spite of how simple adding these terms would be we have still neglected them in our construction because this dissertation will primarily focus on low-energy precision tests to detect quantum gravity. This is further justified by the fact that torsion doesn't propagate[40, 37]. Even through matter with intrinsic spin requires that torsion be present when the matter is coupled to gravity[41, 42], we are unlikely to see it anywhere except for in high energy interactions or cosmology.

Propagators of gravity

To do any calculation in QFT we will want to find the propagator and interaction vertices. Given the complexity of our Lagrangian density we will work with the 1PI effective action

$$\Gamma[\tilde{\phi},\, J] = W[\tilde{\phi},\, J] - \int d^4x\, J(x)\tilde{\phi}(x), \quad W[J] = -i\ln\left[Z[\tilde{\phi},\, J]\right], \quad \tilde{\phi}(x) = \frac{\delta W}{\delta J}.$$

In doing so we will inevitably set our background fields to zero, which in our case means working in the flat limit of our spacetime. Working in momentum space, the effective action at $\mathcal{O}(\hbar^2)$ is

$$\Gamma_{\hbar^2} = \frac{i\mathsf{c}^4}{64\pi\mathsf{G}}\int \frac{d^4k}{(2\pi)^4} h_{\mu\nu}(-k)\Big(-\tilde{g}^{\mu(\alpha}\tilde{g}^{\beta)\nu}k^2 + 2\left(1-\frac{1}{2\xi}\right)\delta^{\mu}_{(\gamma}\delta^{\nu}_{\rho)}\tilde{g}^{\gamma(\alpha}\tilde{g}^{\beta)\sigma}k^{\rho}k_{\sigma} \\ -\left(1-\frac{\sigma}{\xi}\right)\left(\eta^{\mu\nu}k^{\alpha}k^{\beta}+\eta^{\mu\nu}\eta^{\alpha\beta}k^2\right)\Big)h_{\alpha\beta}(k).$$

Taking the functional derivative and solving for the Green's function in the de Donder gauge, $\sigma = \xi = 1/2$, we obtain the propagator

$$\mu\nu \overset{\tilde{g}}{\sim\sim\sim} \alpha\beta \;= \Pi^{\mu\nu\alpha\beta}_{\tilde{g}} = -\frac{16\pi i\mathsf{G}}{\mathsf{c}^4}\frac{2\eta^{\mu(\alpha}\eta^{\beta)\nu}+\eta^{\mu\nu}\eta^{\alpha\beta}}{k^2+i\varepsilon}.$$

We can similarly assign a propagator to the Faddeev-Popov ghost

$$\mu \overset{f}{\cdots\cdots} \nu \;= \Pi^{\mu\nu}_{f} = -\frac{32\pi i\mathsf{G}}{\mathsf{c}^4}\frac{\eta^{\mu\nu}}{k^2+i\varepsilon}.$$

Example: Gravitational scattering of scalar fields

By placing a scalar field in curved space we may obtain the minimally coupled Lagrangian density for scalars to be

$$\mathscr{L}_{\phi}(g) = -\frac{\sqrt{-g}\hbar}{2\mathsf{c}}\left(g^{\mu\nu}\nabla_{\mu}\phi(x)\nabla_{\nu}\phi(x) + \frac{m^2\mathsf{c}^2}{\hbar^2}\phi(x)^2\right).$$

The propagator for this field will be the same as usual but we still need to find the term given by a interaction vertex. To now perturbativly quantise the gravitational field, we translate the Lagrangian density to obtain

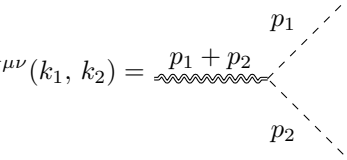
$$\mathscr{L}_{\phi}(g) = \mathscr{L}_{\phi}(\tilde{g}) - \int_{\tilde{\mathcal{M}}} d^4x\,\sqrt{-\tilde{g}}\frac{h_{\mu\nu}\hbar}{2\mathsf{c}}\left(\tilde{\nabla}^{\mu}\phi\tilde{\nabla}^{\nu}\phi - \frac{\tilde{g}^{\mu\nu}}{2}\left(\tilde{g}^{\mu\nu}\tilde{\nabla}_{\mu}\phi(x)\tilde{\nabla}_{\nu}\phi(x) + \frac{m^2\mathsf{c}^2}{\hbar^2}\phi(x)^2\right)\right) + \mathcal{O}(h^2).$$

We may then find the flat 1PI effective action to quadratic order in momentum space to be

$$\Gamma_{h\phi^2} = \frac{i\hbar}{2\mathsf{c}} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \phi(k_1) \left(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - \eta^{\mu\nu} \left(k_1 \cdot k_2 + \frac{m^2\mathsf{c}^2}{\hbar^2} \right) \right) \phi(k_2).$$

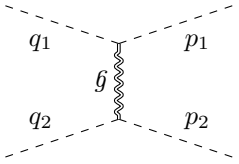
Functionally differentiating, we obtain the interaction vertex

$$V^{\mu\nu}(k_1,\, k_2) = \quad = \frac{i\hbar}{2\mathsf{c}} \left(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - \eta^{\mu\nu} \left(k_1 \cdot k_2 + \frac{m^2\mathsf{c}^2}{\hbar^2} \right) \right).$$



Performing scattering at the tree level we then obtain the t-channel diagram

$$i\mathcal{M}_t = \quad = \frac{16\pi i\mathsf{G}}{t\mathsf{c}^4} \Big((p_1 \cdot p_2)(q_1 \cdot q_2) + (p_1 \cdot q_2)(p_2 \cdot q_1) - (p_1 \cdot q_1)(p_2 \cdot q_2) + \frac{m^2\mathsf{c}^2}{\hbar^2} \left(p_1 \cdot k_1 + p_2 \cdot k_2 - 2\frac{m^2\mathsf{c}^2}{\hbar^2} \right) \Big).$$



The $s\&u$-channels may be obtained by merely permuting the momentum labels. Applying the same analysis as we did in the QED example, one may find that we will ultimately end up with Newton's law for gravitation

$$V(r) = -\frac{Gm^2}{r}.$$

3.2 Gravitational entanglement

3.2.1 Why previous results fail

As mentioned in the introduction, there have been a few phenomena witnessed where we have directly seen quantum objects interact with a gravitational field. These include the COW experiment[12] where the interference pattern produced by neutrons was observed, the Mashhoom effect[15] where the shift of energy levels in atoms was measured, and most recently the effect of anti-protons under a external gravitational field[43]; all caused by earths gravitational field. To the best of our understanding the planet Earth is a macroscopic, realistic object and there is no reason why its gravitational field shouldn't also be one. One may theoretically reproduce the exact same phenomena by implementing the interaction of a quantum particle with the gravitational field in the Koopman von-Neumann formalism. This may be done without worrying about how whether the

terms are produced by some non-realistic effects. It is apparent that if we want to observe quantum gravity that we will have to set up more careful experiments that involve at least two distinct quantum objects.

3.2.2 The BMV proposition

Since the first investigations into quantum gravity there have been several proposals for how one may witness true features of quantum gravity[16, 17, 44, 45, 46]. The most notable is the protocol in which two test masses with spin are held adjacently in a superposition of spatially localised states. Each particle must then travel down one of two paths via an interferometer, during which they would interact with each other gravitationally. These particles would be travelling slowly due to the desire to maximise gravitational interaction. Furthermore, due to the weakness of gravity we will have to place our quantum test masses in close proximity, but not too close so that electromagnetic Casimir effects could cause the entanglement instead. To tackle these tricky conditions as large a quantum mass as possible will be required to see the desired effects. In the absence of possessing quantum objects of the planetary mass range, this will call far the Newtonian limit of interactions.

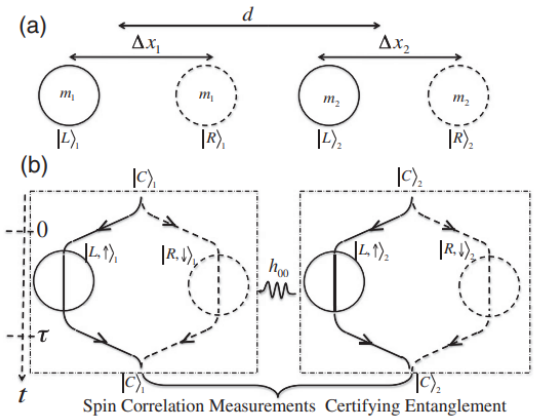


Figure 3.1: Here each particle is allowed to travel down one of two paths; $L_{1,2}$ or $R_{1\,2}$. Path L corresponding to a particle in the spin up state and R corresponding to a particle in the spin down state. The Casimir affect is a particularly difficult obstacle to overcome and modified apperatus have been proposed to combat it[47].

We could then measure the phases of the two particles and determine whether there is a correlation or not, establishing the existence of entanglement. With Newtons classical potential having been established as a mediator of quantum gravitational interactions we may consider the entanglement of the two masses; given in their initial states by

$$|\psi(t=0)\rangle_{12} = \frac{1}{2}\left(|L_1\rangle + |R_1\rangle\right) \otimes \left(|L_2\rangle + |R_2\rangle\right).$$

The time evolved state is thus given by

$$|\psi(\tau)\rangle_{12} = \mathscr{U}(\tau)\,|\psi(t=0)\rangle_{12} = \exp\left(-i\frac{\mathcal{H}_{free}+V_N}{\hbar}\tau\right)|\psi(t=0)\rangle_{12}$$
$$= \frac{e^{i\phi_0}}{2}\left(e^{i\phi_{L_1L_2}}\,|L_1L_2\rangle + e^{i\phi_{R_1R_2}}\,|R_1R_2\rangle + e^{i\phi_{R_1L_2}}\,|R_1L_2\rangle + e^{i\phi_{L_1R_2}}\,|L_1R_2\rangle\right)$$

in a similar manner to the example we gave in QED where $\phi_0 = \frac{-E_{free}\tau}{\hbar}$, $\phi_{L_1L_2} = -\frac{\mathsf{G}m^2\tau}{\hbar d_{H_iG_j}}$.

While the proposal was not as technically precise, a variation of this experiment was proposed at almost the exact same time; the authors referencing the more general use of some Stern-Gerlach based interferometer setup[16]. Both proponents of this protocol are given credit for its proposal in several papers[48, 10] so we shall thus refer to this protocal as the BMV (Bose, Marletto, Vledral) setup.

3.3 Criticisms of Non-Locality

Upon examining the methods that we have used to justify entanglement thus far one of the first criticisms one might make is how do we know that our generation of entanglement is local in the first place? Take for instance the time evolution operator that we have used to generate the entanglement in our states $\exp(-i\tau Gm^2/\hbar r)$. There is no obvious constraint on the how the action affects the particles due to time delays in our classical approximation nor for higher order corrections[28, 49].

3.3.1 Geometric implications of gravitational entanglement

Entanglement of geometries

One criticism of this protocol[50] is that the entanglement present here may actually be due to non-local gauge freedoms rather than being a display case for quantum gravity. In rebutal to this the following argument has been made[48]: Let us consider the BMV setup but now assume that we are dealing with a scale where we may consider more effects of general relativity. Let us assume we have the BMV scenario but with one of the paths significantly far away from the rest. Working in the weak field limit so that the distance metric is given by

$$ds^2 = \left(1 + \frac{2\phi(\boldsymbol{x})}{\mathsf{c}^2}\right)$$

where ϕ is the superposition of Newtonian potentials coming from each particle. For any particle for which we assume we may define some approximate radius R we have $\phi = -GM/r$ for a distance r from the centre of the particle if $r > R$. Otherwise we assume the potential is constant inside the particle and given by $\phi = -GM/R,\ r < R$. The metric inside each particle is thus given by

$$ds^2 = -\left(1 - \frac{2Gm}{R\mathsf{c}^2} - \frac{2Gm}{R_{ij}\mathsf{c}^2}\right)dt^2 + d\boldsymbol{x}^2$$

and the proper time measured by a clock inside each particle during a coordinate time lapse t is

$$\tau = \int_0^t \sqrt{1 - \frac{2Gm}{R\mathsf{c}^2} - \frac{2Gm}{d\mathsf{c}^2}} \approx t\left(1 - \frac{Gm}{R\mathsf{c}^2} - \frac{Gm}{R_{ij}\mathsf{c}^2}\right).$$

One may note here that the metrics described by our general metric are not diffeomorphic and hence not gauge invariant, thus we can rule out our future phenomena being due to gauge difference[48].

A tangent on quantum clock based tests

Let us now consider a different scenario. One that does not demonstrate any particular property that is pertinent to validating the BMV experiment, but is none the less very interesting, is the implication of quantum gravity at the tree level of interactions. Suppose we have a stationary observer at some sufficiently far distance away from the two clocks such that the observer cannot distinguish between any of the fields produced by the clocks. The observer then measures the local proper time duration τ on their ideal clock. One of these two clocks is constructed to be quantum in nature and the other is an ideal clock[46]. The quantum clock interaction term is given by the Hamiltonian $\hat{\mathcal{H}}_{\text{int}} = E_0 \left|0\right\rangle \left\langle 0\right| + E_1 \left|1\right\rangle \left\langle 1\right|$ and the initial state of the system, $\left|\psi\right\rangle = \frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}$.. In this state the optimum basis to measure the time is $\left|\pm\right\rangle = \frac{\left|0\right\rangle \pm \left|1\right\rangle}{\sqrt{2}}$. The passage of unit time is defined operationally by the orthogonalisation time; the time taken for a initial state to become orthogonal to itself. For a two level system this is given by $t_\perp \hbar\pi/\Delta E$.

Let us suppose that the second clock is localised at a coordinate distance $\mathbf{x}$. If we assume for simplicity that the energy state associated with the state $\left|0\right\rangle$ is zero then when our first clock is fixed in the $\left|0\right\rangle$ state it will have no effect on its surroundings. We assume that our second clock runs as it would in Minkowski space. When the 1st clock is in the state $\left|1\right\rangle$ however, we see that our clock would experience time dilation, running slower at a rate given by the second order approximation $\tau \mapsto \tau + \Delta\tau = \tau(1 + \mathsf{G}\Delta E/\mathsf{c}^4|\mathbf{x}|)$. If the quantum clock is in the initial superposition of states then the time elapsed will be in a superposition of states, having an intrinsic uncertainty relation depending on how close the clock is to a neighboring region of energy, described by $t_\perp \Delta t = \frac{\pi\hbar\mathsf{G}\tau}{c^4 x}$.

Consider now two gravitationally interacting clocks A and B, separated by a distance $\mathbf{x}$. We shall work with the Newtonian potential $V(x) = -\frac{\mathsf{G}m_A m_B}{|\mathbf{x}|}$ where m_A and m_B refer to the energies of the particles A and B respectively, divided by c^2, when observed in their rest frames. I shall refer to this as the bound mass. If we now assume that the mass of the free particles is small relative to the bound mass of the particle, we may write our two-clock Hamiltonian as

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_A + \hat{\mathcal{H}}_B - \frac{\mathsf{G}}{\mathsf{c}^4 x}\hat{\mathcal{H}}_A\hat{\mathcal{H}}_B.$$

Let us assume that the energies of both Hamiltonians are equal and that the initial state of the clocks are uncorrelated; $|\psi\rangle = (|0\rangle + |1\rangle)^{\otimes 2}/2$. Time evolving this state we obtain

$$|\psi(\tau)\rangle = \frac{1}{2}\left(|0\rangle|\varphi_0(\tau)\rangle + e^{-\frac{i\tau}{\hbar}\Delta E}|1\rangle|\varphi_1(\tau)\rangle\right)$$

$$|\varphi_0(\tau)\rangle = \left(|0\rangle + e^{-\frac{i\tau}{\hbar}\Delta E}|1\rangle\right), \quad |\varphi_1(\tau)\rangle = \left(|0\rangle + e^{-\frac{i\tau}{\hbar}\Delta E\left(1-\frac{\mathsf{G}\Delta E}{\mathsf{c}^4 x}\right)}|1\rangle\right)$$

This state is maximally entangled for the time $t_{\max} = \frac{\pi\hbar\mathsf{c}^4 x}{\mathsf{G}(\Delta E)^2} = \frac{\pi x}{\Delta E^2}$ in Plank units. Once in this maximally entangled state our clock ceases to act as a functional clock as it only outputs random times upon measurement.

While being a mildly interesting show case of the more exotic phenomena one might expect from quantum gravity it has been proposed that one could practically search for such effects in quantum clock. The reality is that, as indicated by the use of Plank units to simplify the expression, to see any actual effects in a quantum clock we would likely need to hold massive quantum clocks in superpositions for a very long time even begin to see differentiating effects, thus making this test likely not a practical one.

3.3.2 Obtaining local induced gravitational entanglement

The next and arguably more prevalent issue is lack of an obvious sign that our interaction was local. This is to say that there has been a delay in the evolution of one particle from what we would expect from a non-local interaction due to the causal separation. This is a problem of both theoretical importance, to determine that the generation of entanglement is a local phenomena, and of practical importance as one shall wish to test for the correct delay of evolution in a laboratory experiment to show that the interactions were local even when the correlation are not.

To address this we shall consider the same experiment but be systematic in how we find our time evolved state by working with the stationary phase approximation[51]. To start with, let us restate the assumptions of the BMV experiment but be specific. We consider two masses

$m_{a=1,\,2}$ each with some spin. At time t^i the particles are at initial positions x_a^i passing through an inhomogeneous magnetic field B_z, orientated orthogonally to the plane of motion of the particles that forces the particles into a path superposition due to spin coupling. We assume the spacetime curvature is locally flat so that we may use the linearised approximation and denote terms due to the gravitational perturbation by $\mathscr{F}$. We further assume that the coupling of B_z with $\mathscr{F}$ cause negligible backreaction and we shall thus follow the model in our derivation that $\mathscr{F}$ is the sole field consideration. We start by taking our partition function to be of the form

$$Z = \int \mathcal{D}\mathscr{F}'\mathcal{D}x' \exp\left(\frac{iS}{\hbar}\right)$$

and the joint evolution will be

$$\mathscr{U}_{i\to f} = \sum_\sigma |\sigma\rangle\langle\sigma| \otimes \mathscr{U}^\sigma_{i\to f}.$$

We assume we have an initial state that is separable at the time t^i. The boundary conditions will be the assumption that t^f is sufficiently far into the future that the field has time to relax in the vicinity of spin measurements and that these boundaries shall be the same for all spin configurations σ. Thus by the stationary phase approximation

$$\mathscr{U}^\sigma_{i\to f} \propto \int_i^f \mathcal{D}x' \exp\left(\frac{iS[x'_a,\,\mathscr{F}[x'_a]]}{\hbar}\right) |\psi^f\rangle\langle\psi^i|$$

between times t_i and t_f. For each spin configuration σ there is a magnetic field B_z. These paths can be taken as orthogonal states and the remaining integral approximated by a second stationary phase approximation, keeping only contributions from these paths

$$\mathscr{U}^\sigma_{i\to f} \propto \exp\left(\frac{iS^\sigma[x'_a,\,\mathscr{F}[x'_a]]}{\hbar}\right) |\psi^f\rangle\langle\psi^i|\,.$$

For the on-shell action we may write $S = S_0 + S_{\mathscr{F}}$ for S_0 which is independent of $\mathscr{F}$ thus making its contribution a global phase. We thus define $\phi_\sigma = S^\sigma[x'_a,\,\mathscr{F}[x'_a]]/\hbar$. To compute ϕ_σ we consider the Pauli-Feirz action and energy-momentum tensor

$$S_{\mathscr{F}} = \frac{\mathsf{c}^4}{32\pi\mathsf{G}}\int d^4x\,\left(-\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \frac{1}{2}\partial^\mu h\partial_\mu h\right) + \int d^4x\, h_{\mu\nu}T^{\mu\nu}.$$

Considering that we are effectively working under the assumption that our particle is following a classical trajectory, we can now solve this using the classical equations of motion $\Box h_{\mu\nu} = -16\pi\mathsf{G}T_{\mu\nu}/\mathsf{c}_4$, for $\partial^\nu h_{\mu\nu} = 0$ to obtain the desired form of ϕ_σ.

For stress energy tensor for N point masses is given by

$$T^{\mu\nu}(t,\,\mathbf{x}) = \sum_{n=1}^{N} m_n \delta^{(3)}(\mathbf{x}-\mathbf{x}_n(t)) V_n^{\mu\nu}(t), \quad \text{for} \quad V_n^{\mu\nu}(t) = \gamma_n(t) v_n^{\mu}(t) v_n^{\nu}(t)$$

with $v^{\mu}(t)_n$ being the 4-velocity for the nth mass. The retarded wave equation for all times is then

$$\begin{aligned} h_{\mu\nu}(t,\,\mathbf{x}) &= \frac{4\mathsf{G}}{\mathsf{c}^4}\int d^3\mathbf{x}'\frac{T_{\mu\nu}(\mathbf{x}',\,t_r)}{|\mathbf{x}-\mathbf{x}'|} = \frac{4\mathsf{G}}{\mathsf{c}^4}\sum_{n=1}^{N} m_n \int d^3\mathbf{x}'\frac{\delta^{(3)}(\mathbf{x}'-\mathbf{x}_n(t_r))\overline{V}_n^{\mu\nu}(t_r)}{|\mathbf{x}-\mathbf{x}'|} \\ &= \frac{4\mathsf{G}}{\mathsf{c}^4}\sum_{n=1}^{N} m_n \int dt' \int d^3\mathbf{x}'\frac{\delta^{(3)}(\mathbf{x}'-\mathbf{x}_n(t_r)\delta(t'-t_r))\overline{V}_n^{\mu\nu}(t_r)}{|\mathbf{x}-\mathbf{x}|} \\ &= \frac{4\mathsf{G}}{\mathsf{c}^4}\sum_{n=1}^{N} m_n \int dt' \frac{\delta(t'-t_r))\overline{V}_n^{\mu\nu}(t')}{|\mathbf{x}-\mathbf{x}_n(t')|} = \frac{4G}{c^4}\sum_{n=1}^{N}\frac{m_n \mathsf{c}\overline{V}_n^{\mu\nu}(t_n)}{|\mathbf{d}_n|\mathsf{c}-\mathbf{d}_n\cdot\mathbf{v}_n(t_n)} \end{aligned}$$

where $\mathsf{c}t_r = \mathsf{c}t - |\mathbf{x}-\mathbf{x}_n(t)|$ is the retarded time, $\overline{V}_n^{\mu\nu} = V_n^{\mu\nu} - \eta^{\mu\nu}\eta_{\alpha\beta}V_n^{\alpha\beta}$, $\mathbf{d}_{ab}^{\sigma} = \mathbf{x}_b^{\sigma} - \mathbf{x}_a^{\sigma}$ and $t_{ab}^{\sigma}(t) = t - |\mathbf{d}_{ab}^{\sigma}(t)|$. Plugging our solution back in we may now write our on shell action as

$$S_{\mathscr{F}} = \frac{\mathsf{G}}{\mathsf{c}^4}\sum_{a,\,b}\int dt\,\frac{m_a m_b \mathsf{c}\overline{V}_a^{\mu\nu}(t_{ab})\overline{V}_{b\mu\nu}(t)}{|\mathbf{d}_{ab}|\mathsf{c}-\mathbf{d}_{ab}\cdot\mathbf{v}_a(t_{ab})} \approx \frac{\mathsf{G}}{2}\sum_{a,\,b}^{a\neq b}\int dt\,\frac{m_a m_b}{|\mathbf{d}_{ab}^{\sigma}(t)|}$$

Observable retardation

Now for our 2-particle case let us take the particles to be at rest at a distance d for all times $t < t_1$, $t > t_2$. Between t_1 and t_2 is when the particles undergoes spin dependant motion. The setup is assumed to be such that $\mathsf{c}(t_2 - t_1) < d$, so that the non-stationary parts of the worldlines are spacelike separated. From time $t_3 = t_2 + d/\mathsf{c}$ the retarded position of each particle with respect to the other is again constant, thus generating no entanglement. Let $x_a^{s_a}$ now be the displacement of particle a from its initial position due to a external magnetic field B_z. ϕ_σ is a sum of integrals which we can break up as

$$\int_{t_i}^{t_f}\frac{dt}{d_{21}^{\sigma}(t)} = \int_{t_i}^{t_1}\frac{dt}{d} + \int_{t_2}^{t_3}\frac{dt}{d - x_1^{s_1}(t)} + \int_{t_2}^{t_3}\frac{dt}{d + x_2^{s_2}(t_{12}(t))} + \int_{t_3}^{t_f r}\frac{dt}{d}.$$

The phase will now be of the form $\phi_\sigma = C + \phi_{s_a} + \phi_{s_b} + C'$. Practically, to achieve a measurable spacelike separation the experimentalist could vary the initial velocity of our particles. If we consider the scenario where we have maximal entanglement and are thus likely to see a correlation we will require $\Delta\phi \approx (m/m_p)^2(\Delta x/d)^2(\mathsf{c}\tau/(d-\Delta x))$ with $m \gg m_p$ for the plank mass m_p. The current proposals above would operate at $m \approx 10^{-10}m_p$ thus without a colossal experimental

apparatus that can sustain coherence times, a table top experiment to verify this property looks unlikely in the foreseeable future.

3.3.3 Non-local string-gravity

In the case of a torsion free theory of gravity in the absence of other fields it has been noted that the most general theory may be written down as [52]

$$S = \frac{1}{16\pi \mathsf{G}} \int d^4x\sqrt{-g}\left(R + \beta\left(R\mathcal{F}_1\left(\frac{\Box}{M_s^2}\right)R + R_{\mu\nu}\mathcal{F}_2\left(\frac{\Box_s}{M_s^2}\right)R^{\mu\nu} + R_{\mu\nu\rho\sigma}\mathcal{F}_3\left(\frac{\Box}{M_s^2}\right)R^{\mu\nu\rho\sigma}\right)\right)$$

where M_s is the fundamental length scale of non-locality which in string theory corresponds to the string length scale.

If we set $\mathcal{F}_3 = 0$ up to quadratic order in the metric perturbation of the flat background we can keep the graviton as the only dynamical degree of freedom by imposing the condition $2\mathcal{F}_1 = -\mathcal{F}_2$. Expanding the metric about the Minkowski background $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, we obtain

$$S = \frac{1}{4}\int d^4x h_{\mu\nu}\left(1 - \mathcal{F}_1\left(\frac{\Box}{M_s^2}\right)\frac{\Box}{M_s^2}\right)\Pi^{\mu\nu\rho\sigma}h_{\rho\sigma}\Pi(h^3)$$

One may ultimately obtain the T^{00} components of the gravitational potential to be

$$-\frac{\mathsf{G}m}{r}\int_0^{\frac{M_s r}{2}} e^{-x^2}\,dx.$$

We can thus differentiate between certain non-local theories of quantum gravity however, as is the case with many of these more technical phenomena we are unlikely to have the technology to reach the levels of sensitivity to rule out such phenomena for many decades.

3.4 Other models of classical-quantum interactions

Although the BMV protocols do describe a method for witnessing gravitationally induced entanglement, given that there aren't any non-local interactions, it is not quite as clear as some authors proclaim[17, 49] that we do actually have a demonstration of quantum gravity. This is because they have based there arguments entirely on the authority of LOCC. As we have touched upon throughout this dissertation there are in fact multiple ways of describing classical systems within a quantum framework and just as many arguments for what is meant by 'classical'. What approach

one takes can affect validity of these experiments with the Koopman von-Neumann representation being just one approach. In the original proposals for the BMV type experiments, arguments were made[16, 53] as for why we cannot have a classical mediator of entanglement using quantum information type arguments analogous to what we provided in our section on LOCC. We will expand on this to other classical formulations.

3.4.1 Mean field theories

In a mean field theory, such as semi-classical quantum gravity[11] the classical system is represented by a phase space coordinate $(x,\, p)$, the quantum system by the state $|\psi\rangle$ and the dynamics are specified by the Hamiltonian operator $\hat{\mathcal{H}}(\mathbf{x},\, p)$, which is parameterised by the classical phase space coordinates. The equations of motion for the classical and quantum evolution of our state are thus given by Schrödinger's equation and Hamilton's equations of motion

$$i\hbar\frac{\partial}{\partial t}\,|\psi\rangle = \hat{\mathcal{H}}(x,\, p)\,|\psi\rangle\,, \quad \frac{dx^i}{dt} = \frac{\partial}{\partial p^i}\langle\hat{\mathcal{H}}\rangle, \quad \frac{dp^i}{dt} = -\frac{\partial}{\partial x^i}\langle\hat{\mathcal{H}}\rangle$$

for $\langle\hat{\mathcal{H}}\rangle = \langle\psi|\,\hat{\mathcal{H}}(x,\, p)\,|\psi\rangle$; or in the relativic case[54]

$$\frac{dx^\mu}{d\tau} = \frac{\partial}{\partial p^\nu}\big(\langle\hat{\mathcal{H}}\rangle g^{\mu\nu}\big), \quad \frac{dp^\mu}{d\tau} = -\frac{\partial}{\partial x^\nu}\big(\langle\hat{\mathcal{H}}\rangle g^{\mu\nu}\big),$$

for proper time τ. The quantum and classical dynamics are coupled in a non-linear manner, thus making mean field models incompatible with an embedding of the classical system in a quantum system. We shall still show that entanglement cannot be created in such a system for completeness: for two quantum systems Q and Q', consider a coupling to a classical system C via the mean field Hamiltonian

$$\hat{\mathcal{H}}(x,\, p) = \hat{\mathcal{H}}(x,\, p)\otimes\hat{\mathbb{I}}_{Q'} + \hat{\mathbb{I}}_{Q}\otimes\hat{\mathcal{H}}(x,\, p).$$

Initially we have $|\psi\rangle_0 = |\psi_Q\rangle\otimes|\psi_{Q'}\rangle$, thus the time evolved state will be given by

$$\begin{aligned}|\psi\rangle_t &= \left(\mathcal{T}\left\{\exp\left(-\frac{i}{\hbar}\int_0^t \hat{\mathcal{H}}_Q\big(x(s),\, p(s)\big)\,ds\right)\right\}|\psi_Q\rangle\right)\otimes\left(\mathcal{T}\left\{\exp\left(-\frac{i}{\hbar}\int_0^t \hat{\mathcal{H}}_{Q'}\big(x(s),\, p(s)\big)\,ds\right)\right\}|\psi_{Q'}\rangle\right)\\ &=: |\psi_Q\rangle_t\otimes|\psi_{Q'}\rangle_t\end{aligned}$$

and the classical phase space trajectories are given by

$$\frac{dx^i}{dt} = \frac{\partial}{\partial p^i}\Big({}_t\langle\psi_Q|\, \hat{\mathcal{H}}_Q\big(x(t),\, p(t)\big)\, |\psi_Q\rangle_t + {}_t\langle\psi_{Q'}|\, \hat{\mathcal{H}}_{Q'}\big(x(t),\, p(t)\big)\, |\psi_{Q'}\rangle_t \Big),$$
$$\frac{dp^i}{dt} = -\frac{\partial}{\partial x^i}\Big({}_t\langle\psi_Q|\, \hat{\mathcal{H}}_Q\big(x(t),\, p(t)\big)\, |\psi_Q\rangle_t + {}_t\langle\psi_{Q'}|\, \hat{\mathcal{H}}_{Q'}\big(x(t),\, p(t)\big)\, |\psi_{Q'}\rangle_t \Big).$$

As we can see, both before and after the evolution our states are separable; implying the absence of the formation of any entanglement.

3.4.2 Classical ensemble models

Let Q and Q' be quantum particles with positions q and q' and consider the ensemble Hamiltonian

$$\mathcal{H}[P,\, S] = \mathcal{H}_{QC}(P,\, S) + \mathcal{H}_{Q'C}(P,\, S) = \int g_1 P(\partial_q S) + g_2 P q'(\partial_x S)\, dq dq' dx$$

where g_1 and g_2 are coupling constants and we have used a one-dimensional system for illustrative convenience. Here $\mathcal{H}[P,\, S]$ is just a interaction Hamiltonian where we have neglected any kinetic terms by making g_i sufficiently large and considering interactions over a short time interval. Observables of Q are invariant under the action of $\mathcal{H}_{Q'C}$ alone and similarly for Q' and $\mathcal{H}_{QC}$. The equations of motion governing this system are

$$\frac{\partial P}{\partial t} = -g_1 x \frac{\partial P}{\partial q} - g_2 q' \frac{\partial P}{\partial x}, \quad \frac{\partial S}{\partial t} = -g_1 x \frac{\partial S}{\partial q} - g_2 q' \frac{\partial S}{\partial x}$$

which are solved by

$$P_t(q,\, q',\, x) = P_0\left(q - g_1 t x + \frac{1}{2} g_1 g_2 t^2 q',\, q',\, x - g_2 t q'\right)$$
$$S_t(q,\, q',\, x) = S_0\left(q - g_1 t x + \frac{1}{2} g_1 g_2 t^2 q',\, q',\, x - g_2 t q'\right).$$

The hybrid wave function thus evolves linearly according to

$$\psi_t(q,\, q',\, x) = \exp\left(-\frac{it}{\hbar}(g_1 \hat{p}\hat{x} + g_2 \hat{q}'\hat{k})\right) \psi_0(q,\, q',\, x)$$

where $\hat{p}$, $\hat{k}$ denote the momentum operators $\left(\frac{\hbar}{i}\right)\partial_q$, $\left(\frac{\hbar}{i}\right)\partial_x$ conjugate to the position operators $\hat{q}$, $\hat{x}$ respectively. Thus, this conjugation-ensemble dynamics corresponds to a formal embedding of the classical particle C into a quantum system evolving under the Hamiltonian operator $\hat{\mathcal{H}} = g_1 \hat{p}\hat{x} + g_2 \hat{q}'\hat{k}$.

To show that this model may crate entanglement suppose that our ensembles are initially

independent:

$$P_0(q,\ q',\ x) = P_Q(q)P_{Q'}(q')P_C(x), \quad S_0 = S_Q(q) + S_{Q'}(q') + S_C(x)$$

corresponding to the factorisable function $\psi_0(q,\, q',\, t) = \psi_Q(q)\psi_{Q'}(q')\psi_C(x)$. It follows that if a measurement is made on x after the interaction with $x = a$, the statistics of Q and Q' is described by the wave function

$$\psi_{t|a}(q,\, q') = K_a\psi_t(q,\, q',\, a) = K_a\psi_Q\left(q - g_1ta + \frac{1}{2}g_1g_2tq'\right)\psi_{Q'}(q')\psi_C(a - g_2tq')$$

for normalisation K_a. This wave function clearly does not factorise, thus we have a entangled state.

While this proposal is incredibly intriguing and arguably controversial, breaking our standard understanding of what is classical and quantum it has been noted[8] that this setup is not GPT complete. That is to say that there are redundancies in the system that does not make the theory predictive for all measurements. It is still unclear whether this will have a significant effect on the theory or if other pathologies will be found.

3.4.3 A interpretation of classical-quantum coupling

Suppose that we have a particle interacting with a classical potential $V(\mathbf{x})$ that depends on the particles position $\mathbf{x}$. If the particle is quantum in nature then its position will be described by a operator which will appear to be some real number when the particle is measured. It stands to reason that the potential will couple to the distribution of where we may find the particle and thus the classical potential becomes a operator $V(\hat{r})$. Note that this has nothing to do with the quantum nature of the potential and is merely a reflection of the fact that it is coupled to a quantum system; a property that is used when solving for the hydrogen atom and one could arguably use to entangle the particles in BMV experiment.

Is this a reasonably argument? The classical potential which was quantised in the case of Bose's protocol[17] is actually not a local interaction as it assumes a Euclidean spacetime. We may still justify the use of this potential for our evolution because it is what our quantum interactions reduced to but if we want to show that the same is true for a local classical potential we will need to work within a quantum field theory framework.

If the position operator of a particle, say a photon, is in fact the position operator of the particle that it is coupled to, say a electron, then this will mean that the creation and annihilation

operators for the photon are the creation and annihilation operators for the electron e.g. in the case of scalar fields this looks like

$$\hat{\mathbf{x}} = \int \mathbf{x}\hat{a}\hat{a}^{\dagger}\, d^3x.$$

Not only does this mean that the field would now have to be composed of all the appropriate quantum fields to allow for interactions but that our field would take on the statistics of the different fields. The photon would be a Fermion when it comes from interactions with electrons and we know that this isn't true. Not only is this not true but the formulation is inconsistent so we do in fact require that interacting fields exist independently of a source. Alternatively, we may say that the field is purely classical but that it should just couple to the position of the quantum particle during the scattering process. In this scenario however, we return to one of the other classical formulations, thus unless someone is able to formulate such an exotic theory it is likely safe to assume that it doesn't exist.

The fact that this protocol will not vindicate quantum gravity does not invalidate the protocol. A experiment following this protocol would be the first of its kind to demonstrate any gravitational interaction between two objects that are quantum in nature. If no correlations were observed, it would demonstrate that gravity is not quantum in nature, therefore falsifying most of the contended theories we have today, changing the course of modern research in theoretical physics. On the other hand, if entanglement is demonstrated it may provide an opportunity to test for other properties of gravity, which may have remained aloof so far.

Chapter 4

Conclusion

In this dissertation we have explored a variety of phenomena in quantum information and quantum gravity. In the realm of quantum information we have explored the notion of what it means for two particles to be entangled, the basic phenomena that this leads to, how to characterise entanglement, what one can mean by the terms quantum and classical, what the physics community considers quantum and classical in the form of LOCC and how this affects our perception of physics. We have also seen how this effects the dynamics of gravity as a quantum and a classical theory with the most notable phenomena in quantum gravity being that the entanglement of quantum clocks can cause them to loose their ability to reliably keep time.

While we have seen that it should be possible to entangle two masses via a gravitational interaction it does not necessarily imply that gravity is in fact a quantum theory with at least one alternative classical theory presenting itself as a candidate. This candidate is not without faults so more investigation will need to be done to see if there are any pathologies present in this theory. Furthermore, the experimental realisation of any gravitational entanglement is still at least several years away given the current technology available. In spite of this, the validation of gravitational entanglement would still be a massive leap forwards for the theoretical physics community with proposals being presented to distinguish between different mechanisms for entanglement. The value in such a experiment would not just be that it would be the first step towards verifying the existence of the graviton but the experiment would act as a probe, potentially allowing us to determine whether spacetime is discrete[55] and if there are non-local actions present.

As it stands other experiments are being proposed to remove loopholes and make achieving the experiments a more obtainable goal. The experiment would either steer us in a new direction of how we model gravity, be the first step in validating all the work done by the community over the past 70 years or, in the case of a negative result, falsify almost all of this work and take theoretical physics into a new paradigm of uncharted territory. It is undeniable that the work involved is of

high importance and with the current rate of progress we should be able to see the first fruits of the labour done by the entire theoretical physics community within the next decade or two.

Appendix A

Quantum Models of Classical Mechanics

A.1 Koopman von-Neumann mechanics

A.1.1 A brief summary of KvN mechanics

Consider a classical ensemble of particles, each of which we may allocate a phase space coordinate $\varphi = (x,\, k)$ and governed by a Hamiltonian $\mathcal{H}$[56]. If we take the continuum limit on this ensemble and decide to pick a state $(x,\, k)$ at random we may allocate a probability P to this state based of what percentage off our statistical admixture this state occupies. According to classical mechanics the probability density ρ of the ensemble has to evolve according to the Liouville equation

$$i\frac{\partial \rho}{\partial t} = \hat{L}\rho, \quad \hat{L} = -i\frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial}{\partial q_i} + i\frac{\partial \mathcal{H}}{\partial q_i}\frac{\partial}{\partial p_i}.$$

Koopman and von-Neumann postulated that we may apply the same evolution to a state on a Hilbert space $|\psi\rangle \in \mathscr{H}$ so that

$$i\frac{\partial}{\partial t}\psi(\varphi) = \hat{L}\psi(\varphi).$$

To have a Hilbert space, we must have a scalar product on our states. We thus define the bilinear product

$$\langle \tau|\psi\rangle = \int d\varphi\, \tau^*(\varphi)\psi(\varphi) = \|\psi\|^2 = \int d\varphi\, \rho(\varphi)$$

$$\Rightarrow \quad \left\langle \tau|\hat{L}\psi \right\rangle = \left\langle \hat{L}\tau|\psi \right\rangle$$

$\hat{L}$ is Hermitian and our new probability density ρ obeys the Liouville equation. An interesting consequence of this is that unlike in normal quantum mechanics the evolution of the phase of our state is independent of the evolution of our probability distribution. In normal quantum mechanics we can show this if we consider the Schrödinger equation and the probability density $\rho(x,\, t) = \|\psi(x,\, t)\|^2$ which satisfies $\partial_t \rho = -\nabla \cdot \mathbf{j}$ for probability current

$$\mathbf{j} = -\frac{i\hbar}{2m}\left(\psi^* \nabla\psi - \psi \nabla\psi^*\right).$$

If we write $\psi = \sqrt{\rho} e^{\frac{iS}{\hbar}}$ where we define S by $\mathbf{j} = \rho \nabla S/m$, demonstrating that the evolution does couple to the phase. These differences arise primarily from the fact that the Liouvillian contains only first order derivatives while the Schödinger Hamiltonian operator contains second order derivatives.

An important consequence that one may infer from this and by studying further examples is that there is no interference present in the wave functions of KvN mechanics. This implies that the position and momentum of every particle commutes with each other. Furthermore, unlike in quantum mechanics we can know the state of a particle with arbitrary precision at any time with the average momentum of the classical particle.

A.1.2 The abstract KvN Hilbert space

Building the Hilbert space

With the observation that $\hat{p}$ and $\hat{q}$ can be considered a complete set of commuting observables that form a continuous spectrum on $\mathbb{R}^*$, the extended real number line, the orthonormality condition allows us to use the complete set of eigenvalues as a basis for our Hilbert space. These relations are given by

$$\langle q',\, p' | q'',\, p'' \rangle = \delta(q' - q'')\delta(p' - p'') \qquad \int dqdp\, |q,\, p\rangle \langle q,\, p| = 1.$$

In this basis $\hat{q}$ and $\hat{p}$ are diagonal while $-i\partial_q$ $(-i\partial_p)$ are defined by

$$\langle q',\, p'| - i\frac{\partial}{\partial q}\left(-i\frac{\partial}{\partial p}\right)|\psi\rangle = i\frac{\partial}{\partial q'}\left(-i\frac{\partial}{\partial p'}\right)\langle q',\, p'|\psi\rangle$$

and are Hermitian. One may show that $[\hat{q},\, -i\partial_q] = i = [\hat{p},\, -i\partial_p]$ and thus we define the operators $\hat{\lambda}_q = -i\partial_q$and $\hat{\lambda}_p = -i\partial_p$ respectivly. One may further show that the Liouville operator is a particular representation of the equation $i\partial_t |\psi(t)\rangle = \hat{\lambda}_a \omega^{ab} \partial_b \mathcal{H} |\psi(t)\rangle$ that one may define over the Hilbert space. This reduces to the Liouville equation when we take the inner product with $\langle q,\, p|$.

Superselection rules

In classical mechanics we identify observables with functions of phase space variables φ. Now using our operational formalism, this is equivalent to the postulate that all observables are only functions of $\hat{\varphi} = (\hat{q}, \hat{p})$. Thus our algebra is Abelian and $\hat{\varphi}$ are thus superselection operators. A consequence of this follows if we consider two states corresponding to different eigenvalues of the superselection operators, say $|\varphi_1\rangle$ and $|\varphi_2\rangle$. Then there does not exist an observable $\mathcal{O}(\hat{\varphi})$ connecting them since $\langle\varphi_1| \mathcal{O}(\hat{\varphi}) |\varphi_2\rangle$. A further consequence of this is that an arbitrary state cannot be prepared by diagonalising a complete set of observables. The state $|\psi_1\rangle = \alpha_1 |\varphi_1\rangle + \alpha_2 |\varphi_2\rangle$, $\alpha_1 \neq \alpha_2$ is thus a mixed state and the expectation value of the operator $\mathcal{O}$ is

$$\langle\mathcal{O}\rangle = \frac{\langle\psi| \mathcal{O}(\hat{\varphi}) |\varphi\rangle}{\langle\psi|\psi\rangle} = \frac{|\alpha_1|^2 \langle\varphi_1| \mathcal{O}(\hat{\varphi}) |\varphi_1\rangle + |\alpha_2|^2 \langle\varphi_2| \mathcal{O}(\hat{\varphi}) |\varphi_2\rangle}{|\alpha_1|^2 \langle\varphi_1|\varphi_1\rangle + |\alpha_2|^2 \langle\varphi_2|\varphi_2\rangle}.$$

The expectation values calculated are therefore the same as those calculated using a decoherent mixed density matrix

$$\hat{\rho} = |\alpha_1|^2 |\varphi_1\rangle \langle\varphi_1| + |\alpha_2|^2 |\varphi_2\rangle \langle\varphi_2|$$

which is inferred from the formula $\langle\mathcal{O}\rangle = \mathrm{Tr}(\hat{\rho}\mathcal{O}(\varphi))/\,\mathrm{Tr}(\hat{\rho})$. Coherent superpositions of pure states are thus impossible and one automatically gets mixed states.

This can be rephrased by saying that the relative phase between $|\varphi_1\rangle$ and $|\varphi\rangle_2$ cannot be measured using only observables of the form $\mathcal{O}(\hat{\varphi})$. These considerations can be further extended to the continuous superpositions of states $|\varphi\rangle$ but we run into an error: the states $|\psi_1\rangle = \int d\varphi\, \psi(\varphi) |\varphi\rangle$ and $|\psi_2\rangle = \int d\varphi\, \psi(\varphi) e^{iA(\varphi)} |\varphi\rangle$ are now indistinguishable despite being two different vectors in the same Hilbert space. To avoid this redundancy we have to forbid any superpositions of eigenstates of the superselction operators and consider only statistical admixtures. This means that in the continuous case we write our density matrix as

$$\hat{\rho} = \int d\varphi \rho(\varphi) |\varphi\rangle \langle\varphi|$$

and we must generally construct the Hilbert space as a direct sum of eigensubspaces

$$\mathscr{H} = \bigoplus_{\{\varphi_i\}} \mathscr{H}(\{\varphi_i\})$$

for different eigenvalues φ_i corresponding to the superselection observables $\hat{\varphi}$. Hence, we can see that all operators acting over a collection of Hilbert spaces must also be constructed as a direct sum over those Hilbert spaces.

A.2 Mechanics of ensembles

A.2.1 Ensembles in classical mechanics

An ensemble in configuration space Z is described at a time t by a probability density $P_t(Z)$ on Z[10]. We reqiure that the dynamics be described by an action principle, implying that we must have some canonically conjugate variable to the probability $S_t(Z)$ on the configuration space and an ensemble Hamiltonian functional[10] $\mathcal{H}[P,\, S]$ with equations of motion

$$\frac{\partial P}{\partial t} = \frac{\delta \mathcal{H}}{\delta S}, \quad \frac{\partial S}{\partial t} = -\frac{\delta \mathcal{H}}{\delta P}.$$

Observables are represented by functionals of P and S, the Poisson brackets of observables A and B are defined as

$$\{A,\, B\} = \int dz \left(\frac{\delta A}{\delta P}\frac{\delta B}{\delta S} - \frac{\delta B}{\delta P}\frac{\delta A}{\delta S} \right)$$

with integration replaced by summation in discrete regions of configuration space. From this we have $\{P,\, \mathcal{H}\} = \partial_t P$, $\{S,\, \mathcal{H}\} = \partial_t S$ and thus for all observables A we may write $\{A,\, \mathcal{H}\} = \partial_t A$. Observables will generally need to satisfy certain conditions to ensure that the probabilities remain positive and normalised. In the case of the Hamiltonian, these conditions are that the addition of a global constant to S will leave the Hamiltonian invariant by the conservation of probability and that $\partial_t P$ vanishes when P does by the positivity of probability[57].

To show that this construction is equivalent to classical phase space as a classical model, let us consider a classical configuration space labelled by positions x and the observable C_f corresponding to the phase space configuration function $f(x,\, k)$ which is defined by

$$C_f[P,\, S] := \int dx\, P f(x,\, \nabla_x S).$$

This is numerically equivalent to the ensemble average of f, provided one associates the momentum $k = \nabla_x S$ with x. Evaluating the Poisson bracket for the arbitrary observables C_f and C_g we may write $\{C_f,\, C_g\} = C_{\{f,\, g\}}$ where $\{f,\, g\}$ is the canonically defined phase space Poisson bracket. Thus the Poisson bracket for classical ensembles is isomorphic to the classical phase space bracket.

A.2.2 Ensembles in quantum mechanics

Now moving onto developing the quantum version of configuration ensembles, we consider a configuration space labelled by the possible outcomes q of some complete basis set $\{|q\rangle\} \subseteq \mathscr{H}$. The observable $Q_{\hat{M}}$ corresponding to $M = M^\dagger$ is defined by the functional

$$Q_{\hat{M}}[P,\, S] := \langle\psi|\, \hat{M}\, |\psi\rangle$$

where $|\psi\rangle$ is the wave function defined via $\langle q|\psi\rangle = \sqrt{P(q)}e^{iS(q)/\hbar}$ and thus we may write $\{Q_{\hat{M}},\, Q_{\hat{N}}\} = Q_{[\hat{M},\, \hat{N}]/(i\hbar)}$ for the usual commutator. In summary, we shown that the Poisson bracket for quantum ensembles is isomorphic to the quantum commutator, we can therefore construct the usual Schödinger equation.

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