

**INTEGRABLE SPIN CHAINS AND
SUPERCONDUCTIVITY**

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Abstract

This paper is organised into two sections, the first part is a review and the second a short exploitative inquiry into the integrability of a novel superconducting field theory. In the first part, a comprehensive review and derivation of the Coordinate Bethe Ansatz and the Algebraic Bethe Ansatz is conducted, focusing on the XXX Heisenberg Spin chain. Furthermore, there is a succinct overview of spin chains in the context superconductivity, with emphasis on the Hubbard model, $SU(2)$ Gross-Neveu Model and relativistic superconductivity. Then the short exploration, a simple, novel field theoretic model of superconductivity is considered, mainly to test whether or not it is integrable by synthesizing key attributes and principles from the Hubbard Model with the symmetries of the XXX spin chain. This model is found to be integrable.

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Review

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1.1 Introduction and Quantum Spin Chains

Spin chains are a rich field of study in condensed matter physics, both from a theoretical and experimental perspective, that have a myriad of applications in magnetisation and conduction. Spin chains are statistical models constructed, initially, of localised magnetic moments linearly arranged, interacting through spin-spin couplings. One of the original uses of a spin chain was by Werner Heisenberg, applying Pauli's exclusion principle to explain ferromagnetism and later anti-ferromagnetism by Hans Bethe in 1931.¹ Spin chains have proven to be instrumental even in modern condensed matter physics, for example, cited in the 2016 Nobel Prize in physics awarded to David Thouless, Duncan Haldane and Michael Kosterlitz "for theoretical discoveries of topological phase transitions and topological phases of matter" where specific chain parameters lead to the discovery of exotic phases of matter.^{2a,2b,3} Certain spin chains are integrable and can be solved exactly, which involves finding eigenvalues and eigenvectors of the Hamiltonian prescribed to that

particular spin chain. This was the feature of Bethe's influential 1931 paper in which he solved anti-ferromagnetically coupled spin chains, proving to be instrumental in various further integrable models.^{1,4}

The following paper is split into two parts. Part one will cover the Coordinate Bethe Ansatz and Algebraic Bethe Ansatz of the XXX spin chain, then address briefly the topic of spin chains in superconducting, specifically the Hubbard Model, $SU(2)$ Chiral Gross Neveu Model and a relativistic consideration of superconductivity. Part two will examine the integrability of a proposed relativistically consistent superconducting field theory that combines ideas and properties of the Hubbard Model and XXX Heisenberg spin chain and admits topological edge modes.

Integrability offers great insight into various elusive physical phenomena such as superconductivity, the ability for electrons to move through substances with zero resistance. The occurrence of superconductivity was attributed to the pairing of electrons with opposite spin below the Fermi level, called a Cooper Pair, in the original "pairing approximation" theorised by Bardeen, Cooper and Schrieffer, also known as BCS Theory, in 1957.⁵ This initially simplistic theory set the foundation for more sophisticated models of 1D superconducting in topological condensed matter.⁶ The Bethe ansatz is then extremely powerful in analysing the physics of both "classical" superconducting (BCS Theory etc.) and newer models of one-dimensional topological superconductors involving Majorana bound-states (MBS) along the edges of the material.^{7,8}

To study any kind of spin chain one needs to establish the concept of spin in a mathematically rigorous way. Here it will be set up using the representation of the $\mathfrak{su}(2)$ Lie algebra through the commutation relation (1.1.1) with the Levi-civita symbol, $\epsilon^{\alpha\beta\gamma}$ as the structure constants.

$$[S^\alpha, S^\beta] = i\epsilon^{\alpha\beta\gamma}S^\gamma, \quad \alpha, \beta, \gamma = 1, 2, 3 \quad (1.1.1)$$

In the fundamental representation, they are defined by the Pauli matrices as follows⁹:

$$S^1 = \frac{1}{2}\sigma^x, \quad S^2 = \frac{1}{2}\sigma^y, \quad S^3 = \frac{1}{2}\sigma^z \quad (1.1.2)$$

with the usual pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.1.3)$$

Then define the two spin raising and lowering operators, which will be greatly useful in analysing

the various Hamiltonians to come, as follows:

$$S^\pm = S^x \pm iS^y \quad (1.1.4)$$

which in the fundamental representation is:

$$S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (1.1.5)$$

The S^\pm and the S^z operators cumulatively describe the possible action on the spin states as such:

$$\begin{aligned} S^+ |\uparrow\rangle &= 0, & S^- |\uparrow\rangle &= |\downarrow\rangle, & S^z |\uparrow\rangle &= \frac{1}{2} |\uparrow\rangle \\ S^+ |\downarrow\rangle &= |\uparrow\rangle, & S^- |\downarrow\rangle &= 0, & S^z |\downarrow\rangle &= -\frac{1}{2} |\downarrow\rangle \end{aligned} \quad (1.1.6)$$

Using these operators one can construct various models of spin chains each with its own set of unique properties and symmetries that describe a plethora of natural phenomena to a remarkable degree. The spin states are defined in vector form as:

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One of the most powerful models is the Heisenberg Spin Chain. There are two possible states at each site ($|\uparrow\rangle, |\downarrow\rangle$), in a local Hilbert space $\mathcal{V} \in \mathbb{C}^2$. For a spin chain of length L , the total Hilbert space is given by the tensor product of the Hilbert spaces at each site, of dimension 2^L

$$\mathcal{V} = V_1 \otimes V_2 \otimes \dots \otimes V_L \quad (1.1.7)$$

where the Hilbert spaces are defined by $V_n = \mathbb{C}^2$ and characterised by the following Hamiltonian:

$$\hat{H} = \sum_{n=1}^L (J_x S_n^x S_{n+1}^x + J_y S_n^y S_{n+1}^y + J_z S_n^z S_{n+1}^z) \quad (1.1.8)$$

where J_x, J_y, J_z are parameters specifying the strength of spin interaction. The local spin operators S_n^α only interact with S_{n+1}^α , hence the is *nearest neighbour* only. *Medium* and *long*, range interactions are also of interest, though they are not nearly as well studied as the nearest neighbour models and tend to be more complicated in nature. Imposing conditions by equating some of the three parameters will also determine the symmetries of the spin chain, for example:

- $J_x = J_y = J_z$ is the isotropic XXX spin chain.
- $J_x = J_y \neq J_z$ is the anisotropic XXZ spin chain.
- $J_x \neq J_y \neq J_z$ is the fully anisotropic XYZ spin chain.

The models that are integrable can be solved by various versions of the Bethe Ansatz, starting with Hans Bethe's original proposal, now known as the Coordinate Bethe Ansatz ¹. This was originally done with the XXX spin chain but has since been expanded to incorporate various other models such as the XXZ and XYZ models mentioned above and more ^{10,11,12a,12b,12c}.

Henceforth the paper will focus on the $SU(2)$ -invariant isotropic XXX spin chain. The Hamiltonian of this model is *in involution* with the generators of the global $\mathfrak{su}(2)$ algebra, S^x, S^y, S^z , thus making it $SU(2)$ -invariant:

$$[\hat{H}, S^x] = [\hat{H}, S^y] = [\hat{H}, S^z] = 0, \quad (1.1.9)$$

with the translation-invariant sum of individual spins:

$$S^\alpha = \sum_n S_n^\alpha. \quad (1.1.10)$$

Consequently, \hat{H} and S^z can be diagonalised, as is the goal of the Bethe Ansatz.

1.2 Coordinate Bethe Ansatz of the XXX Spin Chain

The general idea of the Bethe Ansatz is to diagonalise the Hamiltonian. This process allows us to find the energy eigenvalues associated to the model, as well as the conserved charges prevalent in the physicality of the Hamiltonian. These quantities are necessary to define any integrable theory, and exploring them can be done elegantly through the imagery of a quasiparticle that defines the spin wave interactions in the theory called a Magnon.

1.2.1 Magnons

Let us start with the XXX spin chain Hamiltonian ⁹:

$$\hat{H}_{XXX} = -J \sum_{n=1}^L (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + S_n^z S_{n+1}^z) \quad (1.2.1)$$

Here the negative sign is chosen by convention to align with the ferromagnetic state when $J > 0$ (conversely one can also define the anti-ferromagnetic state $J < 0$), specific to the XXX Heisenberg Spin chain. Using the raising and lowering S_{\pm} operators described in (1.1.4) the Hamiltonian can also be written as :

$$\hat{H}_{XXX} = -\frac{J}{2} \sum_{n=1}^L (S_n^- S_{n+1}^+ + S_n^+ S_{n+1}^- + 2S_n^z S_{n+1}^z) \quad (1.2.2)$$

For simplification one can set $J = 1$ without loss of generality. From here the idea is to find the eigenvectors and eigenstates of \hat{H}_{XXX} . We start from the *pseudo-vacuum*, $|\Omega\rangle$, (ground) state with all spins up.

$$|\Omega\rangle = |\uparrow^L\rangle \quad (1.2.3)$$

One could also start with a ground state of all down spins without loss of accuracy: $|\Omega\rangle = |\downarrow^L\rangle$. $|\Omega\rangle$ can diagonalise the Hamiltonian and it follows (recalling that we set $J = 1$):

$$\hat{H}_{XXX} |\Omega\rangle = -\sum_{n=1}^L S_n^z S_{n+1}^z |\Omega\rangle = -\frac{L}{4} |\Omega\rangle = E_0 |\Omega\rangle. \quad (1.2.4)$$

Then the vacuum energy is

$$E_0 = -\frac{L}{4} \quad (1.2.5)$$

In this basis, the first excited state will have the n^{th} site as spin-down:

$$|n\rangle = |\uparrow\uparrow \dots \uparrow\downarrow \dots \uparrow\rangle. \quad (1.2.6)$$

More generally a state with N -excitations can be written using S_n^- acting on the vacuum state as:

$$|n_1, n_2, \dots, n_N\rangle = S_{n_1}^- S_{n_2}^- \dots S_{n_N}^- \cdot |\Omega\rangle \quad (1.2.7)$$

Now consider the linear combination of $|n\rangle$ states (noting the translation invariance of the Hamiltonian), this is then Bethe's original Ansatz ¹, with magnon momentum p :

$$|\Psi(p)\rangle = \sum_{n=1}^L e^{ipn} |n\rangle \quad (1.2.8)$$

A linear combination of the form in (1.2.8) is a magnon, a quasiparticle assigned to spin wave excitations. Before computing the action of \hat{H}_{XXX} on the wavefunction above, note the useful

relations:

$$\begin{aligned}\sum_{l=1}^L S_l^+ S_{l+1}^- |n\rangle &= |n+1\rangle, \\ \sum_{l=1}^L S_l^- S_{l+1}^+ |n\rangle &= |n-1\rangle, \\ \sum_{l=1}^L S_l^z S_{l+1}^z |n\rangle &= \frac{L-4}{4} |n\rangle.\end{aligned}\tag{1.2.9}$$

Then using the defined relations, the Hamiltonian acts on the wavefunction with the imposed periodic boundary condition on the spin chain of length L - $|n+L\rangle \equiv |n\rangle$ as follows:

$$\begin{aligned}\hat{H}_{XXX} |\Psi(p)\rangle &= \sum_{n=1}^L e^{ipn} \hat{H}_{XXX} |n\rangle \\ &= -\frac{1}{2} \sum_{n=1}^L e^{ipn} \left(|n+1\rangle + |n-1\rangle + \frac{L-4}{2} |n\rangle \right) \\ &= -\frac{1}{2} \left(e^{ip} + e^{-ip} + \frac{L-4}{2} \right) \sum_{n=1}^L e^{ipn} |n\rangle \\ &= -\left(\cos(p) + \frac{L-4}{4} \right) |\Psi(p)\rangle = E_1(p) |\Psi(p)\rangle\end{aligned}\tag{1.2.10}$$

Thus it is clear $|\Psi(p)\rangle$ is an eigenstate of \hat{H}_{XXX} with the energy eigenvalue $E_1(p)$. So, to define the energy of the magnon, $\epsilon(p)$, one only has to consider the difference between the ground state energy E_0 (zero magnon state) and the first excitation (one magnon state):

$$\epsilon(p) = E_1(p) - E_0 = 1 - \cos(p) \geq 0\tag{1.2.11}$$

This is nearly the full picture of the one magnon sector, however, there is no limit on p to a corresponding $|\Psi(p)\rangle$. To fix this, introduce the quantisation condition to limit the arbitrary freedom of p , following the momentum quantisation of a particle in finite space. Applying this to the wavefunction:

$$|\Psi(p)\rangle = \sum_{n=1}^L e^{ipn} |n\rangle = \sum_{n=1}^L e^{ipn} |n+L\rangle = \sum_{n=1}^L e^{ip(n-L)} |n\rangle\tag{1.2.12}$$

which requires

$$e^{ipL} = 1\tag{1.2.13}$$

therefore providing the quantised magnon momentum:

$$p = \frac{2\pi m}{L}, \quad m = 1, 2, \dots, L\tag{1.2.14}$$

This analysis can be extended to the two magnon sector and the N-magnon sector.

1.2.2 Two, Three and N-Magnon Sectors

For the two magnon situation consider the linear combination of states with excitations at two different sites n_1 and n_2 , in a very long spin chain so the sites are far apart, however, still in succession (choosing to keep p_1 to the left of p_2). Define the two magnon eigenstate as the linear combination:

$$|\Psi(p_1, p_2)\rangle = \sum_{1 \leq n_1 < n_2 \leq L} \psi(n_1|n_2) |n_1, n_2\rangle \quad (1.2.15)$$

Taking into account the two momenta and the linear combination of the two sites, one should expect a term involving $ip_1n_1 + ip_2n_2$ in the exponential. Then following the proposition of the first ansatz, the two magnon wavefunction ansatz is proposed:

$$\psi(n_1|n_2) = A(p_1, p_2)e^{i(p_1n_1+p_2n_2)} + B(p_1, p_2)e^{i(p_2n_1+p_1n_2)} \quad (1.2.16)$$

With momentum-dependent coefficients $A(p_1, p_2)$ and $B(p_1, p_2)$ which are not useful on their own, however, are crucial in a ratio that defines the scattering matrix¹³. The first half of the right hand side with coefficient $A(p_1, p_2)$ is the incoming wave, while the latter half with $B(p_1, p_2)$ is the reflected wave. If both $A \neq 0 \neq B$ then only the ration of the coefficients is physical.

$$S(p_1, p_2) = \frac{B(p_1, p_2)}{A(p_1, p_2)}. \quad (1.2.17)$$

Now, imposing that this is an eigenstate with the energy eigenvalue, E_2 :

$$\hat{H}_{XXX} |\Psi(p_1, p_2)\rangle = E_2 |\Psi(p_1, p_2)\rangle. \quad (1.2.18)$$

In this regime, then, there are two quantities to decipher and specify- the energy eigenvalue, $E_2(p_1, p_2)$, and the scattering matrix, $S(p_1, p_2)$. To proceed, decompose the sum in (1.2.15) and consider the following cases:

- Case 1: the two $|\downarrow\rangle$'s are not neighbours, i.e. $|n_1 - n_2| > 1$.
- Case 2: the two $|\downarrow\rangle$'s are neighbours, i.e. $n_2 = n_1 + 1$.

Then, by extending the relations in (1.2.9) to the two magnon sector:

$$\begin{aligned}\sum_{l=1}^L S_l^+ S_{l+1}^- |n_1, n_2\rangle &= |n_1 + 1, n_2\rangle + |n_1, n_2 + 1\rangle, \\ \sum_{l=1}^L S_l^- S_{l+1}^+ |n_1, n_2\rangle &= |n_1 - 1, n_2\rangle + |n_1, n_2 - 1\rangle, \\ \sum_{l=1}^L S_l^z S_{l+1}^z |n_1, n_2\rangle &= \frac{L-8}{4} |n_1, n_2\rangle,\end{aligned}\tag{1.2.19}$$

The constraint here on the Schrödinger equation in (1.2.18) to be true, is the coefficients of $|n_1, n_2\rangle$ must be known. First consider Case 1- distant magnons, i.e. not neighbours. comparing either side of the above equation:

$$\begin{aligned}E_2\psi(\mathbf{p}|n_1, n_2) &= -\frac{1}{2}(\psi(\mathbf{p}|n_1 - 1, n_2) + \psi(\mathbf{p}|n_1, n_2 - 1)) \\ &\quad - \frac{1}{2}\left(\psi(\mathbf{p}|n_1 + 1, n_2) + \psi(\mathbf{p}|n_1, n_2 + 1) - \frac{L-8}{4}\psi(\mathbf{p}|n_1, n_2)\right)\end{aligned}\tag{1.2.20}$$

For fixed n_1 and n_2 , consider the action of \hat{H}_{XX} on $|\downarrow\rangle$.

$$\begin{aligned}\psi(n_1 + 1, n_2) + \psi(n_1, n_2 + 1) &= (e^{ip_1} + e^{ip_2})\psi(\mathbf{p}|n_1, n_2), \\ \psi(n_1 - 1, n_2) + \psi(n_1, n_2 - 1) &= (e^{-ip_1} + e^{-ip_2})\psi(\mathbf{p}|n_1, n_2),\end{aligned}\tag{1.2.21}$$

The energy eigenvalues are then:

$$E_2(p_1, p_2) - E_0 = 2 - \cos(p_1) - \cos(p_2) = \epsilon(p_1) + \epsilon(p_2)\tag{1.2.22}$$

Which is just the sum of the energy of two magnons for the case of distant magnons. While this was derived from the condition $|n_1 - n_2| > 1$, substituting $E_2 = \epsilon(p_1) + \epsilon(p_2) - \frac{L}{4}$ into (1.2.18) the relation is valid for any n_1, n_2 (even $n_2 = n_1 + 1$). Now consider case 2- Neighbouring Magnon case. The wavefunction can be defined in terms of the magnon momentum and successive lattice sites- n_1 and $n_1 + 1$ - $\psi(\mathbf{p}|n_1, n_1 + 1)$. Then define, again, the relations of the spin operators acting on the states:

$$\begin{aligned}\sum_i S_i^+ S_{i+1}^- |n_1, n_1 + 1\rangle &= |n_1, n_1 + 2\rangle, \\ \sum_i S_i^- S_{i+1}^+ |n_1, n_1 + 1\rangle &= |n_1 - 1, n_1 + 1\rangle \\ \sum_i S_i^z S_{i+1}^z |n_1, n_1 + 1\rangle &= \frac{L-4}{4} |n_1, n_1 + 1\rangle,\end{aligned}\tag{1.2.23}$$

and following the argument in case 1, using the explicit definitions of the wavefunction and the Schrödinger equation, solve for the energy eigenvalue:

$$E_2\psi(\mathbf{p}|n_1, n_1 + 1) = -\frac{1}{2}\left(\psi(\mathbf{p}|n_1 - 1, n_1 + 1) + \psi(\mathbf{p}|n_1, n_1 + 2) - \frac{L-4}{4}\psi(\mathbf{p}|n_1, n_1 + 1)\right)\tag{1.2.24}$$

As mentioned above, (1.2.20) is applicable for any n_1, n_2 , so applying the specific case $n_2 = n_1 + 1$ to (1.2.20) results in:

$$E_2\psi(\mathbf{p}|n_1, n_1 + 1) = -\frac{1}{2} [\psi(\mathbf{p}|n_1 - 1, n_1 + 1) + \psi(\mathbf{p}|n_1, n_1)] \\ - \frac{1}{2} [\psi(\mathbf{p}|n_1 + 1, n_1 + 1) + \psi(\mathbf{p}|n_1, n_1 + 2)] - \frac{L-8}{4}\psi(\mathbf{p}|n_1, n_1 + 1) \quad (1.2.25)$$

Then taking the difference between (1.2.24) and (1.2.25) one finds:

$$\frac{1}{2} (\psi(\mathbf{p}|n_1, n_1) + \psi(\mathbf{p}|n_1 + 1, n_1 + 1)) - \psi(\mathbf{p}|n_1, n_1 + 1) = 0 \quad (1.2.26)$$

So the scattering matrix can be found as:

$$S(p_1, p_2) = -\frac{1 - 2e^{ip_2} + e^{i(p_1+p_2)}}{1 - 2e^{ip_1} + e^{i(p_1+p_2)}}. \quad (1.2.27)$$

Which can be rewritten using the trigonometric identities of the exponential:

$$S(p_1, p_2) = \frac{\frac{1}{2} \cot(\frac{p_1}{2}) - \frac{1}{2} \cot(\frac{p_2}{2}) - i}{\frac{1}{2} \cot(\frac{p_1}{2}) - \frac{1}{2} \cot(\frac{p_2}{2}) + i} \quad (1.2.28)$$

and has the properties:

$$S(p_1, p_2)S(p_2, p_1) = 1 \quad |S(p_1, p_2)|_c = 1.$$

Employing the periodic boundary conditions from before- $|n_1, n_2\rangle = |n_2 - L, n_1\rangle$ the eigenstate in case 2 is then:

$$|\Psi(p_1, p_2)\rangle = \sum_{n_1 < n_2} \psi(n_1, n_2) |n_1, n_2\rangle = \sum_{n_1 < n_2} \psi(n_1, n_2) |n_2 - L, n_1\rangle \\ = \sum_{n'_1 < n'_2} \psi(n'_2, n'_1 + L) |n'_1, n'_2\rangle \quad (1.2.29) \\ = \sum_{n_1 < n_2} \psi(n_2, n_1 + L) |n_1, n_2\rangle.$$

Where some relabeling was done to clean up the expressions, namely in the penultimate line- $n'_1 = n_2 - L$ and $n'_2 = n_1$ and in the final line the dummy indices were relabeled- n'_1 to n_1 and n'_2 to n_2 . By equating the first and final line of (1.2.30) the following statement can also be made:

$$\psi(n_1, n_2) = \psi(n_2, n_1 + L) \quad (1.2.30)$$

Then using the explicit form of $\psi(n_1, n_2)$ one finds the following set of quantisation conditions known as the Bethe Ansatz Equations (BAE):

$$e^{ip_1 L} S(p_1, p_2) = 1, \quad e^{ip_2 L} S(p_2, p_1) = 1 \quad (1.2.31)$$

The generalisation from here to the N-magnon sector is not yet clear, so the three-magnon sector must also be considered, and as it turns out the extension from there to the N-magnon sector is straightforward. Defining the eigenstate:

$$|\Psi(p_1, p_2, p_3)\rangle = \sum_{1 \leq n_1 < n_2 < n_3 \leq L} \psi(p|n) |n_1, n_2, n_3\rangle \quad (1.2.32)$$

With three different magnon momenta, $p = \{p_1, p_2, p_3\}$, there are $3! = 6$ possible permutations, so anticipating the upcoming generalisation, it is convenient to introduce some notation for more permutations of momenta. Firstly, name the permutations, here $\{1, 2, 3\}$ as σ where the set of all permutations is S_3 and $\sigma \in S_3$:

$$S_3 = \{\{1, 2, 3\}, \{2, 1, 3\}, \{1, 3, 2\}, \{2, 1, 3\}, \{2, 3, 1\}, \{3, 1, 2\}, \{3, 2, 1\}\} \quad (1.2.33)$$

Then we define a momentum, $p_{\{\sigma\}}$, for each permutation, σ , for example:

$$\sigma = \{1, 2, 3\}, \quad p_{\{\sigma\}} = \{p_1, p_2, p_3\}, \quad A(p_\sigma) = A(p_1, p_2, p_3) \quad (1.2.34)$$

Now writing down the three magnon wavefunction ansatz:

$$\psi(n) = \sum_{\sigma \in S_3} A(p_\sigma) e^{i(p_{\sigma(1)} n_1 + p_{\sigma(2)} n_2 + p_{\sigma(3)} n_3)} \quad (1.2.35)$$

Following the same method as in the one and two magnon sectors, the next step is determining the energy eigenvalue $E_3(p)$ and the amplitudes $A(p_\sigma)$. As there are 6 total terms in the wavefunction, there will be 5 ratios to determine and an overall 6 unknown quantities. The least tedious way to solve this is to take advantage of the naturality of the Bethe Ansatz and conjecture a structure for these objects based on physical intuition. Hence, the three magnon energy should continue to be a sum of all the individual magnons, which indeed is true ¹⁰.

$$E_3(p) - E_0 = \epsilon(p_1) + \epsilon(p_2) + \epsilon(p_3) \quad (1.2.36)$$

The ratio $\frac{A(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(13)})}{A(p_1, p_2, p_3)}$ is a little more complicated than the physical S-matrix in the two magnon sector. Bethe finds, that the ratio $\frac{A(p_{\sigma})}{A(p)}$ factorises into a *sequence of two-body S-matrices*, i.e. any $A(p_{\sigma})$ can be brought to $A(p)$ simply by exchanging two particles. This would look like:

$$A(\dots, p_j, p_k, \dots) = S(p_k, p_j)A(\dots, p_k, p_j, \dots), \quad \text{if } j > k \quad (1.2.37)$$

Then applying recursively an example of this rule with the two body S-Matrix, $S(p_i, p_j)$:

$$\begin{aligned} A(p_3, p_2, p_1) &= S(p_1, p_2)A(p_3, p_1, p_2) \\ &= S(p_1, p_2)S(p_1, p_3)A(p_1, p_3, p_2) \\ &= S(p_1, p_2)S(p_1, p_3)S(p_2, p_3)A(p_1, p_2, p_3) \end{aligned} \quad (1.2.38)$$

Then the ratio for this example:

$$\frac{A(p_3, p_2, p_1)}{A(p_1, p_2, p_3)} = S(p_1, p_2)S(p_1, p_3)S(p_2, p_3) \quad (1.2.39)$$

The 4 other amplitude ratios can be found using the same method, and with the periodic boundary conditions the resultant three equations are found:

$$\begin{aligned} e^{ip_1 L} S(p_1, p_2)S(p_1, p_3) &= 1 \\ e^{ip_2 L} S(p_2, p_1)S(p_2, p_3) &= 1 \\ e^{ip_3 L} S(p_3, p_1)S(p_3, p_2) &= 1 \end{aligned} \quad (1.2.40)$$

Now the extension to the N-Magnon sector is very straight forward in light of this new notation, starting with the eigenstate^{14,15}:

$$|\Psi(p)\rangle = \sum_{1 \leq n_1 < \dots < n_N \leq L} \frac{A(p_{\sigma})}{A(p)} e^{i(p_{\sigma(1)}n_1 + p_{\sigma(2)}n_2 + \dots + p_{\sigma(N)}n_N)} |n_1, \dots, n_N\rangle \quad (1.2.41)$$

with the energy eigenvalue, E_N :

$$\begin{aligned} \hat{H}_{XXX} |\Psi(p)\rangle &= E_N(p) |\Psi(p)\rangle \\ \text{with } E_N(p) &= E_0 + \sum_{k=1}^N \epsilon(p_k) \end{aligned} \quad (1.2.42)$$

This nearly concludes the full picture of the coordinate Bethe Ansatz. Bethe also found that the momentum variables tend to be a little troublesome when explicitly solving a model, thus he

introduced a change of variables from the momenta, p_k , to the *rapidities*, u_k .

$$e^{ip_k} = \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}, \quad u_k = \frac{1}{2} \cot\left(\frac{p_k}{2}\right) \quad (1.2.43)$$

Then, the BAE and the magnon energy, $\epsilon(p_k)$, can be written very simply in terms of the rapidity¹⁵:

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j \neq k}^N \frac{u_j - u_k + i}{u_k - u_j - i}, \quad (1.2.44)$$

$$k = 1, 2, \dots, N$$

$$\epsilon(p_k) = 1 - \cos(p_k) = \frac{2}{4u_k^2 + 1} \quad (1.2.45)$$

This way the Hamiltonian is diagonalised and described by the magnon picture which has proven to be instrumental in solving numerous integrable models in condensed matter. It has gone further to inspire various new analytical methods and re-derivations of these BAE's. One such formalism is the Algebraic Bethe Ansatz, also known as the Quantum Inverse Scattering method, which is discussed in the next section.

1.3 Algebraic Bethe Ansatz of the XXX Spin Chain

The Algebraic Bethe Ansatz (ABA) originates from the Quantum Inverse Scattering method developed by Leningrad mathematicians Fadeev, Takhtajan, Reshetikhin, and Sklyanin, which was then expanded into its current formalism by physicists Korepin, Izergin and Slavnov to name a few¹⁶. The core mathematical operator of the ABA is the *Lax* operator, which acts as the spin chain generating object, explicitly defined as:

$$L_{n,a}(\lambda) = \lambda \mathbb{I}_n \otimes \mathbb{I}_a + i \sum_{\alpha \in \{x,y,z\}} S_n^\alpha \otimes \sigma_a^\alpha \quad (1.3.1)$$

Where σ_a^α are the Pauli matrices in the auxiliary Hilbert space, $H_a \in \mathbb{C}^2$ with a local physical Hilbert Space $V_n \in \mathbb{C}^2$ at each site n . The Lax operator depends on $\lambda \in \mathbb{C}$, the spectral parameter (eigenvalue of the Lax operator), and acts in the tensor product space - $\mathcal{H}_n \otimes \mathcal{H}_a$, of the quantum spin space, \mathcal{H}_n , the Hilbert space at site n , and complex auxiliary space, \mathbb{C}_a^2 . S_n^α is the spin operator¹⁷. It can also be written in matrix form as:

$$L_{n,a}(\lambda) = \begin{pmatrix} \lambda + iS_n^z & iS_n^- \\ iS_n^+ & \lambda - iS_n^z \end{pmatrix} \quad (1.3.2)$$

Then define the permutation operator- $\mathcal{P} \in \mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\mathcal{P} = \frac{1}{2} \left(\mathbb{I} \otimes \mathbb{I} + \sum_{\alpha} \sigma^{\alpha} \otimes \sigma^{\alpha} \right) \quad (1.3.3)$$

which acts by flipping the order as such:

$$\mathcal{P}(a \otimes b) = b \otimes a; \quad a \otimes b \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

One can also define the Lax operator in terms of the permutation operator in the auxiliary Hilbert spaces \mathbb{C}^2 :

$$L_{n,a}(\lambda) = \left(\lambda - \frac{i}{2} \right) \mathbb{I}_{n,a} + i\mathcal{P}_{n,a} \quad (1.3.4)$$

Now to understand if there is a simple relation between the products $L_{n,a_1}L_{n,a_2}$ and $L_{n,a_2}L_{n,a_1}$, it turns out that the equation (1.3.5) does this effectively provided one chooses the R-matrix acting on $H_{a_1} \otimes H_{a_2}$ as in (1.3.6) resulting in the fundamental commutation relation of the Lax operator, known as the RLL relation acting in the tensor triple product space $\mathfrak{h}_n \otimes V_1 \otimes V_2$:

$$R_{a_1,a_2}(\lambda - \mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu) = L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu) \quad (1.3.5)$$

with spectral parameters $\lambda \in V_1$ and $\mu \in V_2$ (to check that it works, see Appendix A). To solve the equation in (1.3.5), the R-matrix is defined with subscripts a_1 and a_2 in two auxiliary spaces $V_1, V_2 \in \mathbb{C}^2$:

$$R_{a_1,a_2}(\lambda) = \lambda \mathbb{I}_{a_1,a_2} + i\mathcal{P}_{a_1,a_2} \quad (1.3.6)$$

It is also interesting to note that comparing (1.3.1) and (1.36), the R-matrix and Lax Operator have the same form. The Lax operator can be interpreted as the object that generates an isotropic 1D spin chain in this setting¹⁵. The R-Matrix obeys the following properties^{16,17} with the simplified notation- $R_{a_1,a_2} \equiv R_{12}$ (proof found in Appendix B):

$$R_{12}(u, v)R_{13}(u, w)R_{23}(v, w) = R_{23}(v, w)R_{13}(u, w)R_{12}(u, v) \quad (1.3.7)$$

$$R_{12}(u, u) = \mathcal{P}_{12} \quad (1.3.8)$$

where \mathcal{P}_{12} is the permutation operator. (1.3.7) is the quantum Yang-Baxter equation and (1.3.8) is called the ‘‘regularity condition’’. The quantum Yang-Baxter equation plays a key role in integrable quantum models and arose in the works of Yang (1967)²⁰ and Baxter (1972)²¹.

Naturally, we can then consider the ordered product of the Lax operator for every spin space along the chain, then the Lax operator can be interpreted geometrically as the connection between each spin site, i.e. the transport from $n \rightarrow n + 1$. Thus defines the *monodromy* matrix:

$$T_a(\lambda) = L_{L,a}(\lambda)L_{L-1,a}(\lambda)\dots L_{1,a}(\lambda)$$

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad (1.3.9)$$

$T_a(\lambda)$ turns out to be a generating object for spin, Hamiltonian and raising & lowering operators. The monodromy matrix also fulfils a similar commutation relation as in (1.3.5), known as the RTT relation (proof in Appendix C):

$$R_{a_1,a_2}(\lambda - \mu)T_{a_1}(\lambda)T_{a_2}(\mu) = T_{a_2}(\mu)T_{a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu) \quad (1.3.10)$$

Having defined the monodromy matrix and the family of Yang-Baxter equations, consider the polynomial form of $T_{L,a}(\lambda)$ with the total spin, S^α ; $\alpha = x, y, z$, as the second highest order coefficient.

$$T_a(\lambda) = \lambda^L + i\lambda^{L-1} \sum_{\alpha} (S_{\alpha} \otimes \sigma^{\alpha}) + \dots \quad (1.3.11)$$

One defines the transfer matrix as the trace of the monodromy matrix, acting on $\mathbb{C}^2 \otimes L$:

$$F(\lambda) = \text{tr}T(\lambda) = A(\lambda) + D(\lambda)$$

$$[F(\lambda), F(\mu)] = 0 \quad (1.3.12)$$

It is straightforward to see the commutator in (1.3.12) vanishes:

$$[A(\lambda) + D(\lambda), A(\mu) + D(\mu)] = (A(\lambda) + D(\lambda))(A(\mu) + D(\mu)) - (A(\mu) + D(\mu))(A(\lambda) + D(\lambda))$$

So, if the operators $A(\lambda)A(\mu)$ and $D(\lambda)D(\mu)$ commute, the commutator in (1.3.12) is true. This illustrates another function of $F(\lambda)$, that generates a collection of commuting operators for different values of the spectral parameter. Now expanding in the non-trivial λ :

$$F(\lambda) = 2\lambda^L + \sum_{l=0}^{L-2} Q_l \lambda^l \quad (1.3.13)$$

which produces $L - 1$ commuting operators Q_l . For the Hamiltonian to be integrable, it must be a part of this family on commuting operators, in other words it must be *in involution* with the

conserved charges. Now we can proceed with the expansion around $\lambda = \frac{i}{2}$, and introducing an important observable- momentum, P :

$$e^{iP} = U, \quad (1.3.14)$$

which is a unique point in the auxiliary space (see Appendix D for the rationale of choosing the point $\lambda = \frac{i}{2}$ and the derivation of (1.3.14)). Then:

$$\left. \frac{d}{d\lambda} T_a(\lambda) \right|_{\lambda=\frac{i}{2}} = i^{L-1} \sum_n \mathcal{P}_{L,a} \dots \mathcal{P}_{n,a} \dots \mathcal{P}_{1,a} \quad (1.3.15)$$

Then repeating and taking the trace over the auxiliary space:

$$\left. \frac{d}{d\lambda} T_a(\lambda) \right|_{\lambda=\frac{i}{2}} = i^{L-1} \sum_n \mathcal{P}_{1,2} \dots \mathcal{P}_{n-1,n+1} \dots \mathcal{P}_{L-1,N} \quad (1.3.16)$$

Most permutations in (1.3.16) can be cancelled by multiplying the inverse of the unitary operator, U^{-1} , resulting in:

$$\left. \frac{d}{d\lambda} F_a(\lambda) F(\lambda)^{-1} \right|_{\lambda=\frac{i}{2}} = \left. \frac{d}{d\lambda} \ln F_a(\lambda) \right|_{\lambda=\frac{i}{2}} = \frac{i}{2} \sum_n \mathcal{P}_{n,n+1} \quad (1.3.17)$$

Recalling the Hamiltonian defined in (1.2.1) and the definition of \mathcal{P} in (1.3.3) the Hamiltonian can be rewritten in terms of the permutation operator:

$$\hat{H}_{XXX} = \frac{1}{2} \sum_n \mathcal{P}_{n,n+1} - \frac{N}{2} \quad (1.3.18)$$

Then comparing (1.3.18) and (1.3.17) the Hamiltonian is related to $F(\lambda)$ and thus belongs to the family of $L-1$ commuting operators that are generated by the trace of the monodromy matrix.

$$\hat{H}_{XXX} = \frac{i}{2} \left. \frac{d}{d\lambda} \ln F(\lambda) \right|_{\lambda=\frac{i}{2}} - \frac{N}{2} \quad (1.3.19)$$

Having set up the Lax and Permutation operators, the monodromy matrix and defining the set of commuting operators generated by the monodromy matrix, focus can be shifted on deriving the BAEs from this formalism. This entire buildup is to diagonalise the Hamiltonian as was done in the Coordinate Bethe Ansatz. The Algebraic Bethe Ansatz has also the extension to diagonalise the transfer matrix $F(\lambda) = A(\lambda) + D(\lambda)$. Now, use the explicit 4x4 matrix representation of the RTT relation, define the natural basis in the auxiliary space $\mathbb{C}^2 \otimes \mathbb{C}^2$ (Appendix E), so that the

R-matrix then looks like (where the zeros are dropped):

$$R(\lambda) = \begin{pmatrix} a(\lambda) & & & & \\ & b(\lambda) & c(\lambda) & & \\ & c(\lambda) & b(\lambda) & & \\ & & & & a(\lambda) \end{pmatrix} \quad (1.3.20)$$

where $a = \lambda + i$, $b = \lambda$ and $c = i$. Monodromy matrices $T_{a_1}(\lambda)$ and $T_{a_2}(\lambda)$ are of the form:

$$T_{a_1} = \begin{pmatrix} A(\lambda) & B(\lambda) & & \\ & A(\lambda) & B(\lambda) & \\ C(\lambda) & D(\lambda) & & \\ & C(\lambda) & D(\lambda) & \end{pmatrix} \quad (1.3.21)$$

$$T_{a_2} = \begin{pmatrix} A(\mu) & B(\mu) & & \\ C(\mu) & D(\mu) & & \\ & & A(\mu) & B(\mu) \\ & & C(\mu) & D(\mu) \end{pmatrix}$$

So the product of the two matrices:

$$T_{a_1}T_{a_2} = \begin{pmatrix} A(\lambda)A(\mu) & A(\lambda)B(\mu) & B(\lambda)A(\mu) & B(\lambda)B(\mu) \\ A(\lambda)C(\mu) & A(\lambda)D(\mu) & B(\lambda)C(\mu) & B(\lambda)D(\mu) \\ C(\lambda)A(\mu) & C(\lambda)B(\mu) & D(\lambda)A(\mu) & D(\lambda)B(\mu) \\ C(\lambda)C(\mu) & C(\lambda)D(\mu) & D(\lambda)C(\mu) & D(\lambda)D(\mu) \end{pmatrix}, \quad (1.3.22)$$

which can now be used in the RTT relation. From the explicit form of the RTT relation arises the set of commutator relations:

$$\begin{aligned} [B(\lambda), B(\mu)] &= 0; \\ A(\lambda)B(\mu) &= f(\lambda - \mu)B(\mu)A(\lambda) + g(\lambda - \mu)B(\lambda)A(\mu); \\ D(\lambda)B(\mu) &= h(\lambda - \mu)B(\mu)D(\lambda) + k(\lambda - \mu)B(\lambda)A(\mu); \end{aligned} \quad (1.3.23)$$

where the spectral parameter coefficients are:

$$\begin{aligned} f(\lambda) &= \frac{\lambda-i}{\lambda}; & g(\lambda) &= \frac{i}{\lambda} \\ h(\lambda) &= \frac{\lambda+i}{\lambda}; & k(\lambda) &= -\frac{i}{\lambda} \end{aligned} \quad (1.3.24)$$

Now continuing with the form:

$$a(\lambda - \mu)B(\lambda)A(\mu) = c(\lambda - \mu)B(\mu)A(\lambda) + b(\lambda - \mu)A(\mu)B(\lambda) \quad (1.3.25)$$

and then interchanging the order $\lambda \leftrightarrow \mu$ one gets:

$$A(\lambda)B(\mu) = \frac{a(\mu - \lambda)}{b(\mu - \lambda)}B(\mu)A(\lambda) - \frac{c(\mu - \lambda)}{b(\mu - \lambda)}B(\lambda)A(\mu) \quad (1.3.26)$$

Using this process, the other pair-terms in (1.3.23) can be found explicitly. Now introducing $|\Omega\rangle$ which here will play as a reference state, similar to the pseudo-vacuum in the previous section:

$$C(\lambda) |\Omega\rangle = 0. \quad (1.3.27)$$

To decipher this state exactly, observe that in each \mathfrak{h}_n , there is a vector, ω_n that triangularises the Lax operator as follows:

$$L_m(\lambda)\omega_n = \begin{pmatrix} \lambda + \frac{i}{2} & * \\ 0 & \lambda - \frac{i}{2} \end{pmatrix} \omega_n \quad (1.3.28)$$

The * are expressions irrelevant to the diagonalisation process. So acting the monodromy matrix to this reference state results in the following:

$$T(\lambda) |\Omega\rangle = \begin{pmatrix} \alpha^L(\lambda) & * \\ 0 & \delta^L(\lambda) \end{pmatrix} |\Omega\rangle \quad (1.3.29)$$

where $\alpha(\lambda) = \lambda + \frac{i}{2}$ and $\delta(\lambda) = \lambda - \frac{i}{2}$. Then finally the terms from T_{a_1} and T_{a_1} are:

$$A(\lambda) |\Omega\rangle = \alpha^L(\lambda) |\Omega\rangle; \quad C(\lambda) |\Omega\rangle = 0; \quad D(\lambda) |\Omega\rangle = \delta^L(\lambda) |\Omega\rangle \quad (1.3.30)$$

so that $|\Omega\rangle$ is an eigenstate of $A(\lambda)$ and $D(\lambda)$ as well as $F(\lambda) = A(\lambda) + D(\lambda)$. The other eigenvectors, then, will be searched for in the form involving $B(\lambda_M)$:

$$|\Phi(\{\lambda\})\rangle = B(\lambda_1) \dots B(\lambda_M) |\Omega\rangle. \quad (1.3.31)$$

The crucial condition that $\Phi(\{\lambda\})$ must be an eigenvector of $F(\lambda)$ will result in an algebraic expression describing the relations between $\lambda_1 \dots \lambda_M$, i.e. the Algebraic Bethe Ansatz Equations.

The derivation starts with the application of (1.3.23), as above:

$$\begin{aligned} A(\lambda)B(\lambda_1)\dots B(\lambda_M)|\Omega\rangle &= \prod_{k=1}^l f(\lambda - \lambda_k)\alpha^L(\lambda)B(\lambda_1)|\Omega\rangle \\ &+ \sum_{k=1}^l M_k(\lambda, \{\lambda\})B(\lambda_1)\dots\hat{B}(\lambda_k)\dots B(\lambda_M)B(\lambda)|\Omega\rangle \end{aligned} \quad (1.3.32)$$

The coefficients M_k are very involved, however, a general expression can be derived quite straightforwardly from the first term, M_1 , by replacing $\lambda_1 \leftrightarrow \lambda_j$ due to the commutativity of $B(\lambda)$:

$$M_1(\lambda, \{\lambda\}) = g(\lambda - \lambda_1) \prod_{k=2}^M f(\lambda_1 - \lambda_k)\alpha^N(\lambda_1), \quad (1.3.33)$$

$$M_j(\lambda, \{\lambda\}) = g(\lambda - \lambda_j) \prod_{k \neq j}^l f(\lambda_j - \lambda_k)\alpha^N(\lambda_j). \quad (1.3.34)$$

Therefore the coefficients $a(\lambda), b(\lambda)$ and $c(\lambda)$ in the R-matrix satisfy sum rules, thus following the RTT relations. Consequently, for $D(\lambda)$:

$$\begin{aligned} D(\lambda)B(\lambda_1)\dots B(\lambda_M)|\Omega\rangle &= \prod_{k=1}^l h(\lambda - \lambda_k)\delta^N(\lambda)B(\lambda_1)|\Omega\rangle \\ &+ \sum_{k=1}^l N_k(\lambda, \{\lambda\})B(\lambda_1)\dots\hat{B}(\lambda_k)\dots B(\lambda_M)B(\lambda)|\Omega\rangle \end{aligned} \quad (1.3.35)$$

With N_k instead of M_k :

$$N_j(\lambda, \{\lambda\}) = k(\lambda - \lambda_j) \prod_{k \neq j}^l h(\lambda_j - \lambda_k)\delta^N(\lambda_j). \quad (1.3.36)$$

Then the observation can be made that $g(\lambda - \lambda_j) = -k(\lambda - \lambda_j)$, allowing for the cancellation of unwanted terms in (1.3.32) and (1.3.35) to apply $F(\lambda)\Phi(\{\lambda\}) = (A(\lambda) + D(\lambda))\Phi(\{\lambda\})$. This results in:

$$(A(\lambda) + D(\lambda))\Phi(\{\lambda\}) = \Lambda(\lambda, \{\lambda\})\Phi(\{\lambda\}) \quad (1.3.37)$$

where:

$$\Lambda(\lambda, \{\lambda\}) = \alpha^L(\lambda) \prod_{j=1}^l f(\lambda - \lambda_j) + \prod_{j=1}^l \delta^N(\lambda_j)h(\lambda - \lambda_j). \quad (1.3.38)$$

Under the conditions that $\{\lambda\}$ satisfies the conditions

$$\prod_{k \neq j}^l f(\lambda - \lambda_j)\alpha^L(\lambda_j) = \prod_{k \neq j}^l h(\lambda - \lambda_j)\delta^N(\lambda_j) \quad (1.3.39)$$

for $j = 1, \dots, M$ and applying the explicit expressions of $\alpha(\lambda)$, $\delta(\lambda)$, (1.3.24) and (1.3.39) the Algebraic Bethe Ansatz Equations are:

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}} \right)^L = \prod_{k \neq j}^l \frac{\lambda_j - \lambda_k + \frac{i}{2}}{\lambda_j - \lambda_k - \frac{i}{2}}. \quad (1.3.40)$$

This is equivalent to the BAE condition found in (1.2.46) using the magnon formalism. The algebraic method is another way to derive these powerful equations and has applications in more than just condensed matter theory, for example, it can be used to study various string theories. The only thing left to do now is apply these conditions to the shift operator to find the corresponding momentum and energy expressions. Continuing with the point $\lambda = \frac{i}{2}$, it is easy to notice the second term in $\Lambda(\lambda, \{\lambda\})$ vanishes, leaving a multiplicative eigenvalue:

$$U\Phi(\{\lambda\}) = i^L F \left(\frac{i}{2} \right) \Phi(\{\lambda\}) = \prod_j \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}. \quad (1.3.41)$$

Then taking the log, it turns out the momentum, P , eigenvalues are additive:

$$P\Phi(\{\lambda\}) = \sum_j p(\lambda) \Phi(\{\lambda\}) \quad (1.3.42)$$

with

$$p(\lambda) = \frac{1}{i} \ln \frac{\lambda + \frac{i}{2}}{\lambda - \frac{i}{2}}. \quad (1.3.43)$$

The Hamiltonian is also additive by extension, thus taking the differential: $\frac{d}{d\lambda} \ln \Lambda \Big|_{\lambda=\frac{i}{2}}$, the energy eigenvalues are:

$$\hat{H}_{XXX} \Phi(\{\lambda\}) = \sum_j \epsilon(\lambda_j) \Phi(\{\lambda\}) \quad (1.3.44)$$

where

$$\epsilon(\lambda) = -\frac{1}{2} \frac{1}{\lambda^2 + \frac{1}{4}} \quad (1.3.45)$$

Thus rearriving at the quasiparticle interpretation, where $\epsilon(\lambda)$ is the magnon energy and $p(\lambda)$ the magnon momentum. Consistently, λ , the spectral parameter, is the rapidity. The interesting extension here is that $B(\lambda)$ acts as the magnon creation operator. Finally, one can relate $\epsilon(\lambda)$ and $p(\lambda)$ in the following way:

$$\epsilon(\lambda) = \frac{1}{2} \frac{d}{d\lambda} p(\lambda) \quad (1.3.46)$$

with the dispersion relation:

$$\epsilon(p) = \cos p - 1. \quad (1.3.47)$$

1.4 Superconductivity and Spin Chains

Classical superconducting models rely heavily on various lattice spin chain models, the most successful thus far being the Hubbard Model of strongly correlated electrons^{18,19}. The Hubbard model is a fundamental model of electrons in the conduction band associated with a 4D Hilbert space. Where the Heisenberg spin chain deals with 1D spin interactions, the Hubbard model deals with a lattice of electrons involving kinetic energy, i.e. electron site hopping and coulomb interactions. Each lattice site can either have a single electron (with up or down spin) or a pair with opposing spins (due to Pauli exclusion). The standard Hubbard model Hamiltonian is¹⁹ :

$$H^{Hub} = -t \sum_i \sum_{\alpha=\uparrow,\downarrow} \left(c_{\alpha,i}^\dagger c_{\alpha,i} + c_{\alpha,i+1}^\dagger c_{\alpha,i+1} \right) + U \mathbf{n}_{\uparrow,i} \mathbf{n}_{\downarrow,i} \quad (1.4.1)$$

Where c, c^\dagger are the creation and annihilation operators (kinetic part), \mathbf{n} are the number densities (potential), t is the energy scale governing site hopping, found by the overlap of two wavefunctions of the pair of atoms in question, U is the energy scale at the site. The kinetic and potential parts of \mathbb{H} describe the hopping term which enables electrons to move between neighbouring sites and the number of electron pairs on each site respectively. The Hubbard model in 1D is integrable, thus there is a solution to the Yang-Baxter equation- the R-matrix that generates an infinite set of conserved charges *in involution* with the Hamiltonian. These mathematical objects codify the properties of the system in question and when explicitly solved, describe in detail the inner workings of the current model. In this lengthy process, there are numerous symmetries one can take advantage of, such as the “boost” symmetry of the Hamiltonian. Specific models will also demonstrate symmetric properties corresponding to Lie Algebras. A ubiquitous symmetry in successful models is the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ symmetry attributed to the kinetic part of \mathbb{H}^{Hub} , where the algebra is consistent with $\mathfrak{su}_C(2)$ charge symmetry and $\mathfrak{su}^\eta(2)$ spin symmetry which is expanded into a larger “centrally extended” $\mathfrak{su}(2|2)$ (an algebra also used extensively in AdS/CFT calculations)^{24,25,26}. This brand of Hubbard models can also be used to calculate the zero temperature susceptibility and magnetisation of the system to derive the “finite-temperature” Bethe Ansatz equations or “Thermodynamic” Bethe Ansatz equations, done initially by Yang in 1967²⁰, Lieb and Wu in 1968²² and extended to the finite-temperature setting by Takahashi and Shiba in 1972²³.

The $SU(2)$ spin chain models can be deformed and expanded upon to evaluate topological phases of matter often seen in topological superconductors allowing Majorana bound edge states. Topological protection of these states depends on the existence of an energy gap in the bulk of the material, however, provided there are enough symmetries, exponentially localised zero energy modes could also be found in a gapless system. One example is the 1D spin-triplet topological superconductor with fractional spin $\frac{1}{4}$ at either end of an open (exponentially localised spin) chain⁶. To solve this model Pasnoori et al. (2020) used a $U(1)$ -symmetric Thirring model with open boundary conditions on fermions- an anisotropic XXZ-type deformation of $SU(2)$ Chiral invariant Gross-Neveu Model with the Hamiltonian density:

$$H = -iv \left(\psi_{Ra}^\dagger \partial_x \psi_{Ra} - \psi_{La}^\dagger \partial_x \psi_{La} \right) + \psi_{Ra}^\dagger \psi_{Ra} [g_{\parallel} \sigma_{ab}^z \sigma_{cd}^z + g_{\perp} (\sigma_{ab}^x \sigma_{cd}^x + \sigma_{ab}^y \sigma_{cd}^y)] \psi_{Lc}^\dagger \psi_{Ld} \quad (1.4.2)$$

Where $\sigma^{x,y,z}$ are the Pauli matrices and $\psi_{L(R)}(x)$ are two component spinor fields corresponding to left (right) moving spin $\frac{1}{2}$ fermions with components $a = (\uparrow, \downarrow)$. \mathbb{H} can then be diagonalised using its exact eigenstates as it obeys the commutation $[\mathbb{H}, N] = 0$. Then to employ the Bethe Ansatz form, the set of momentum states, $k_j, j = 1 \dots N$, with energy eigenvalue $E = \sum_j k_j$ is:

$$|\{k_j\}\rangle = \sum_{Q, \vec{a}, \vec{\sigma}} \int \theta(x_Q) A_{\{a\}}^{\{\sigma\}}[Q] \prod_j^N e^{i\sigma_j k_j x_j} \psi_{a_j \sigma_j}^\dagger(x_j) |0\rangle \quad (1.4.3)$$

with spin configurations $\{a\} = \{a_1 \dots a_N\}$, chiral configurations $\{\sigma\} = \{\sigma_1 \dots \sigma_N\}$, various orderings of the N particles and with $\theta(x_Q)$ being the Heaviside function acting on a reference state, $|0\rangle$, known as the ‘‘Drained Fermi Sea’’. The amplitudes $A_a^\sigma[Q]$ refer to a chirality, σ , and spin, a , configuration for an electron in the system and are related by the particle-particle S matrix⁶:

$$S = \begin{pmatrix} 1 & & & \\ & \frac{\sinh(f)}{\sinh(f+\eta)} & \frac{\sinh(\eta)}{\sinh(f+\eta)} & \\ & \frac{\sinh(\eta)}{\sinh(f+\eta)} & \frac{\sinh(f)}{\sinh(f+\eta)} & \\ & & & 1 \end{pmatrix} \quad (1.4.4)$$

where the $\eta = -iu$ and f, u are related to g_{\parallel} and g_{\perp} by the following, where $(g_{\parallel}, g_{\perp})$ are the generic couplings from above.

$$\cos(u) = \frac{\cos(g_{\parallel})}{\cos(g_{\perp})}, \quad \frac{\sin(u)}{\tanh(f)} = \frac{\sin(g_{\parallel})}{\cos(g_{\perp})} \quad (1.4.5)$$

Additionally, W^{ij} is also defined as a scattering matrix that related amplitudes differing through

exchanging particles with identical chirality, given by $W^{ij} = P^{ij}$. Then both S and W matrices satisfy the YB and Reflection equations²⁷. These equations are then used to finally construct the Bethe equations in terms of “rapidities”, i.e. the Bethe roots, λ_β , with satisfy the Bethe equations:

$$\sum_{\sigma=\pm} N\Theta\left(\lambda_\alpha + \sigma\frac{f}{2u} - 2\Theta\left(\lambda_\alpha + \frac{i\tau\pi}{2u}, \frac{1}{2}\right)\right) = \sum_{\beta=1}^M \sum_{\sigma=\pm} \Theta(\lambda_\alpha + \sigma\lambda_\beta, 1) + 2i\pi I_\alpha \quad (1.4.6)$$

with the momenta, k_j , and defining $\Theta(x, y) = \log\left(\frac{\sinh(u(x+iy))}{\sinh(u(x-iy))}\right)$:

$$k_j = \frac{\pi n_j}{L} + \frac{i}{2L} \sum_{\beta=1}^M \sum_{\sigma=\pm} \Theta\left(\frac{f}{2u} + \sigma\lambda_\beta, \frac{1}{2}\right) \quad (1.4.7)$$

This set of non-linear equations are then used to analyse and evaluate various phases of the topological superconductor throughout the paper⁶.

These are just a couple of examples of the applications of the Bethe ansatz in superconducting models and how the evaluation of spin chains contributes to the future of condensed matter theory. Later in the research section, a Hamiltonian that combines the basic Hubbard Model and Heisenberg spin interactions will be considered to find the physicality of such a design in the relativistic regime. This section is here to illustrate the prevalence and importance of topological superconducting states and how powerful the Bethe Ansatz is to be applicable in so much more than the original anti-ferromagnetic setting. Not much is known about the effects of relativity in superconducting and an exploration of this type could prove to be useful in understanding the natural world. Before that is done, however, one needs to familiarise themselves with the existing theory on relativistic superconductivity.

1.4.1 Relativistic Superconducting Models

While “classical” models of superconductivity do a comprehensive job of describing numerous superconductive phenomenon through BCS theory, various papers have noted the prevalence of relativity in superconduction. Some examples are- Spin orbit coupling, which is relativistic to the second order of $\frac{v}{c}$, magnetic impurities in superconductors, spin susceptibility, Josephson currents and many other effects. There was also experimental proof of the theorised relativistically corrected Cooper pair masses, generally speaking, relativistic effects need to be taken into account while considering high temperature, heavy fermion superconductors. Another important, defining, characteristic of a material being superconductive is the Meissner effect, the rejection of an external

magnetic field due to self-consistent current screening which is also relativistic to the second order of $\frac{v}{c}$ ^{28,29}. While there isn't a comprehensive field theory of relativistic superconductivity, the phenomenological Ginzburg-Landau theory with Maxwell's equations has made some progress, though it doesn't capture the full picture of the underlying particle physics. The other main approach is to extend the BCS formalism to the relativistic regime and consider the relativistic generalisation of the so-called "Bogolubov-de Gennes" (BdG) equations of superconductivity, developed independently by Alexei Alexeyevich Abrikosov & Lev Gor'kov³⁰, Igor Yevgenyevich Dzyaloshinskii³¹, as well as by Nikolay Bogoliubov³², and David J. Thouless³³ in the 1950 and 60s. Capelle and Gross derive a relativistic Hamiltonian in their 1997 paper, "Relativistic Framework for Microscopic Theories of Superconductivity. I. the Dirac Equation for Superconductors"^{34a}, with a relativistic order parameter and covariant quantities:

$$\begin{aligned} \hat{H} = & \int d^3r \bar{\Psi}(r) [c\hat{\gamma} \cdot p + mc^2 + q\hat{\gamma}^\mu A_\mu] \Psi(r) \\ & - \frac{1}{2} \int d^3r d^3r' \{ \Psi^T(r) [\hat{\eta} \Delta^*(r, r') + \hat{\eta}^5 \Delta_P^*(r, r') \\ & + \hat{\eta}_V^\mu \Delta_{V,\mu}^*(r, r') + \hat{\eta}_A^\mu \Delta_{A,\mu}^*(r, r') + \hat{\eta}_T^{\mu\nu} \Delta_{T,\mu\nu}^*(r, r')] \Psi(r') + H.c. \} \end{aligned} \quad (1.4.8)$$

with the scalar constructed by $\hat{\eta}$ & Δ , the pseudoscalar constructed by $\hat{\eta}^5$ & Δ_P , the four-vector $\hat{\eta}_V^\mu$ & $\Delta_{V,\mu}$, the axial vector constructed by $\hat{\eta}_A^\mu$ & $\Delta_{A,\mu}$ and finally the tensor constructed using $\hat{\eta}_T^{\mu\nu}$ & $\Delta_{T,\mu\nu}$. The specifics of the construction are detailed in the paper and will not be the subject of this section, what is more relevant to the topic at hand is the general structure and analysis of the relativistic Bogolubov-de Gennes equations. Explicitly written in matrix form the Hamiltonian with the generalised BCS-type order parameter is:

$$\begin{aligned} \hat{H} = & \int d^3r \bar{\Psi}(r) [c\hat{\gamma} \cdot p + mc^2 + q\hat{\gamma}^\mu A_\mu] \Psi(r) \\ & - \frac{1}{2} \int d^3r d^3r' [P \Psi^T(r) \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \Psi(r') \Delta^*(r, r') + H.c.] \end{aligned} \quad (1.4.9)$$

with

$$\hat{\chi}(r, r') = \Psi^T(r, r') \hat{\eta} \Psi(r') \quad (1.4.10)$$

and

$$\hat{\eta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} i\hat{\sigma}_y & 0 \\ 0 & i\hat{\sigma}_y \end{pmatrix} \quad (1.4.11)$$

In the process of diagonalising the Hamiltonian the relativistic generalisation of what is called the Bogolubov-Valatin transformation is needed. Starting with the non-relativistic case:

$$\psi_{\uparrow}(r) = \sum_k (u_k(r)a_{k\uparrow} - v_k(r)^*a_{k\downarrow}^{\dagger}) \quad (1.4.12)$$

and

$$\psi_{\downarrow}(r) = \sum_k (u_k(r)a_{k\downarrow} - v_k(r)^*a_{k\uparrow}^{\dagger}) \quad (1.4.13)$$

where $u_k(r)$ and $v_k(r)$ can be found by imposing the requirement of diagonalising the Hamiltonian. To treat the more complicated phenomenon of relativistic models such as magnetic impurities, spin-orbit coupling, triple pairing and more, the transformation needs to be generalised to involve the spin degrees of freedom, using the subscripts σ and τ as spin-like quantum numbers.

$$\psi_{\tau}(r) = \sum_{\sigma k} (u_{\sigma\tau k}(r)a_{\sigma k} - v_{\sigma\tau k}(r)^*a_{\sigma k\uparrow}^{\dagger}) \quad (1.4.14)$$

Then replacing σ and τ with the spinor labels, the relativistic generalisation of (1.4.14) is:

$$\psi_i(r) = \sum_{ijk} [u_{ijk}(r)a_{jk} + v_{ijk}^*a_{jk}^{\dagger}]. \quad (1.4.15)$$

The transformation (1.4.15) has the requirement that it needs to be unitary and canonical to preserve the normalisation of quasiparticle wave functions and the anti-commutating relations of field operators respectively. Unitarity requires:

$$\int d^3r \sum_i [v_{ijk}(r)v_{ij'k'}^*(r) + u_{ijk}(r)u_{ij'k'}^*(r)] = \delta_{kk'}\delta_{jj'} \quad (1.4.16)$$

and

$$\int d^3r \sum_i [v_{ijk}(r)u_{ij'k'}(r) + u_{ijk}(r)v_{ij'k'}(r)] = 0, \quad (1.4.17)$$

while the canonical requirement define the conditions:

$$\sum_{kj} [u_{ijk}(r)v_{ij'k'}^*(r') + v_{ijk}^*(r)u_{ij'k'}(r')] = 0 \quad (1.4.18)$$

and

$$\sum_{kj} [u_{ijk}^*(r)u_{ij'k'}(r') + v_{ijk}(r)v_{ij'k'}^*(r')] = \delta_{ii'}\delta(r-r'). \quad (1.4.19)$$

The same results can be obtained by requiring the single-particle equations to be complete and orthonormal. The coefficients u_{ijk} & v_{ijk} have more conditions imposed on them for diagonalising the Hamiltonian, they are the particle and hole amplitudes i.e. four-component Dirac Spinors.

$$H = \sum_{jk} E_{jk} a_{jk}^\dagger a_{jk} + E_0, \quad (1.4.20)$$

with the ground state E_0 and the creation and annihilation operators for the quasiparticle called a Bogolon, with energy E_{ijk} . The conditions for u_{ijk} & v_{ijk} are conveniently laid out in the following set of integro-differential equations in matrix form:

$$\begin{pmatrix} \hat{h} & \mathcal{D} \\ -\mathcal{D}^* & -\hat{h}^* \end{pmatrix} \begin{pmatrix} u_{ijk}(r) \\ v_{ijk}(r) \end{pmatrix} = E_{jk} \begin{pmatrix} u_{ijk}(r) \\ v_{ijk}(r) \end{pmatrix} \quad (1.4.21)$$

where \hat{h} is the kernel of the Dirac Hamiltonian:

$$\hat{h} = \hat{\gamma}^0 [c\hat{\gamma} \cdot p + mc^2(1 - \hat{\gamma}^0) + q\hat{\gamma}^\mu A_\mu] \quad (1.4.22)$$

with \mathcal{D} being an integral operator containing the pair potential as a kernel:

$$\mathcal{D} = \int d^3r' \dots \Delta(r, r') \hat{\eta}. \quad (1.4.23)$$

Then the four-component spinors are:

$$u_{jk}(r) = \begin{pmatrix} u_{1jk} \\ u_{2jk} \\ u_{3jk} \\ u_{4jk} \end{pmatrix} \quad v_{jk}(r) = \begin{pmatrix} v_{1jk} \\ v_{2jk} \\ v_{3jk} \\ v_{4jk} \end{pmatrix} \quad (1.4.24)$$

(1.4.23) to (1.4.24) are the relativistic generalisations of the Bogolubov-de Gennes equations. This is a general summary of the BdG's for the Dirac Hamiltonian. The authors of the paper wrote

a subsequent article on the reduction of the Dirac BdG's to the vector Pauli BdG's, so that the 4x4 Dirac equation is reduced to the 2x2 Schrodinger type equation ^{34a,34b}. This then allows the expansion into first and second order of $\frac{v}{c}$ to explore the weakly relativistic effects in superconductivity.

Using the information in this section as precedent for a formalism that implicitly accounts for relativistic effects in BCS theory, the following section will endeavour to construct a theory in the Lagrangian notation and transform it into a relativistic Hamiltonian and further specifying the use of Majorana fermions as the spinor objects.

2

A New Model

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2.1 Constructing the Theory

This section describes an $SU(N)$, Gauge and Lorentz invariant Lagrangian to cover the interactions and particle spectrum of a field theory of “Classical”, relativistic and topological superconducting. The process includes adopting aspects of the $SU(3)$ -invariant non-abelian Yang-Mills Lagrangian of Quantum Chromodynamics (QCD) and combining it with a strongly coupled spin-spin interaction term and Hubbard Model, then simplifying to the $SU(2)$ -invariant abelian case attributed to Quantum Electrodynamics (QED).

The goal is to construct a generalised $2D$ integrable Hamiltonian that admits relativistic properties and allows for topological phases such as Majorana Bound Edge states. The general basis of the Yang-Mills lagrangian was chosen to allow for cases of colour superconductivity and other general fermion superconductive cases, such as those observed in Neutron stars, without limiting the model to electrons alone. However, these phenomena will not be the focal point of this investigation. Instead, the Bethe Ansatz approach outlined in the review section will be used on the $2D$ Hamiltonian to be derived below, to prove integrability and compatibility with the greatly

successful Hubbard model analyses in an effort to consider known analytical methods through a new phenomenological perspective.

2

2.1.1 The Lagrangian and Hamiltonian

To begin, consider the non-abelian Yang-Mills Lagrangian:

$$\mathcal{L} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} \quad (2.1.1)$$

where $F_{\mu\nu}F^{\mu\nu}$ is a gauge invariant object and $F_{\mu\nu}$ is the field strength defined in terms of the vector potential, A_μ , and g is the coupling constant:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu^i, A_\nu^i] \equiv \frac{i}{g} [\mathcal{D}_\mu, \mathcal{D}_\nu], \quad (2.1.2)$$

Then consider the QCD lagrangian which includes the Yang-Mills term and the QCD interaction and mass term:

$$\mathcal{L}_{QCD} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_\alpha(i(\not{\partial}_\mu + ig\gamma^\mu T_{\alpha\beta}^a A_\mu^a))\psi_\beta - m^2\bar{\psi}_\alpha\psi_\alpha \quad (2.1.3)$$

where $\bar{\psi}_\alpha, \psi_\alpha$ are 4-component Dirac Spinors with $\alpha \in \{1, 2, 3, 4\}$ and the Dirac adjoint is defined $\bar{\psi} = \psi^\dagger\gamma^0$ and γ^μ and the standard clifford algebra gamma matrices with $\mu = 0, 1, 2, 3$ and the fifth gamma matrix defined $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Next to include the spin-spin interactions, recall the standard Hubbard Hamiltonian from (1.4.1). It can be extended to include the spin operators from Chapter One by simply adding the Heisenberg spin term- $\sum_{n=1}^L (J_x S_n^x S_{n+1}^x + J_y S_n^y S_{n+1}^y + J_z S_n^z S_{n+1}^z)$. In this model, however, the spin interactions are strongly coupled to off site interaction energy with the coefficient VJ . Then by making a change by replacing the creation and annihilation operators, $c_\alpha^\dagger, c_\alpha$, with the spinors, $\bar{\Psi}_\alpha, \Psi_\alpha$ and introducing the strongly coupled spin-spin interaction term $VJ^\mu \sum_{\mu=1}^3 (\bar{\Psi}_\alpha \gamma_{\alpha\beta}^\mu \Psi_\beta)^2$, the combined $SU(3)$ -invariant Lagrangian for this field theory of superconducting is:

$$\begin{aligned} \mathcal{L}_3 = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}_\alpha(i(\not{\partial}_\mu + ig\gamma_{\alpha\beta}^\mu T_{\alpha\beta}^a A_\mu^a))\Psi_\beta - m^2\bar{\Psi}_\alpha\Psi_\alpha - U(\bar{\Psi}_\alpha\Psi_\alpha)^2 - VJ^\mu \sum_{\mu=1}^3 (\bar{\Psi}_\alpha \gamma_{\alpha\beta}^\mu \Psi_\beta)^2 \\ & + \tau \sum_j [(\bar{\Psi}_\alpha(\vec{r}, t)\Psi_\alpha(\vec{r} + \vec{e}_j, t) + \Psi_\alpha(\vec{r}, t)\Psi_\alpha(\vec{r} - \vec{e}_j, t)) + c.c.] \end{aligned} \quad (2.1.4)$$

and then converting this to the Hamiltonian formulation:

$$\begin{aligned}
H_3 = & \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\Psi}_\alpha i g \gamma_{\alpha\beta}^\mu T^a A_\mu^a \Psi_\beta + (m^2 + U \bar{\Psi}_\alpha \Psi_\alpha) (\bar{\Psi}_\alpha \Psi_\alpha) \\
& + V J^\mu \sum_{\mu=1}^3 \left(\bar{\Psi}_\alpha \gamma_{\alpha\beta}^\mu \Psi_\beta \right)^2 - \tau \sum_j [(\bar{\Psi}_\alpha(\vec{r}, t) \Psi_\alpha(\vec{r} + \vec{e}_j, t) + \Psi_\alpha(\vec{r}, t) \bar{\Psi}_\alpha(\vec{r} - \vec{e}_j, t)) + c.c.]
\end{aligned} \tag{2.1.5}$$

Here the “ t ” hopping coefficient was replaced with τ so as not to confuse it with the time variable, $\mu, \nu \in \{0, 1, 2, 3\}$, α, β are spinors indices where each index is in two components and $T^a = \frac{\lambda^a}{2}$; $a \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ are the generators of the $\mathfrak{su}(3)$ Lie algebra defined by 3×3 traceless Gell-Mann matrices, λ^a . V is the kinetic coefficient paired with the spin parameter J . The spinor wavefunctions are defined in $3 + 1D$ with \hat{r} as the position coordinates and t as time. In the “lattice hopping” term \hat{e}_j describes a particle that is a given distance \hat{e}_j away, where the term “lattice” is used loosely to easily generalise down to the $2D$ lattice case. Finally, $\bar{\Psi}_\alpha, \Psi_\alpha$ are the quark fields, $F_{\mu\nu}$ is the gluon field strength tensor and A_μ^a are gluon fields. This defines the Non-Abelian Lagrangian of this Gauge field theory to allow for more general descriptions in higher dimensions.

The next step is to simplify to the $SU(2)$ case in $3 + 1D$ down to the $2D$ square lattice. To do this, replace the $SU(3)$ -QCD coupling constant, g , with the $SU(2)$ -QED coupling, e and replace the quark spinors $\bar{\Psi}_\alpha, \Psi_\alpha$ with fermion bispinors $\bar{\psi}_\alpha, \psi_\alpha$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is now the electromagnetic field strength tensor and A_μ is the covariant four-potential electromagnetic gauge field. For generality one can include an external field B_μ in the covariant derivative term- $\mathcal{D}_\mu = \partial_\mu + ieA_\mu + ieB_\mu$, however, $B_\mu = 0$ in this theory to account for the Meissner effect expelling an external magnetic field. This Lagrangian is:

$$\begin{aligned}
\mathcal{L}_2 = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}_\alpha (i(\not{\partial}_\mu + ie\gamma_{\alpha\beta}^\mu A_\mu)) \psi_\beta - m^2 \bar{\psi}_\alpha \psi_\alpha \\
& - U (\bar{\psi}_\alpha \psi_\alpha)^2 - V J^\mu \sum_{\mu=1}^3 \left(\bar{\psi}_\alpha \gamma_{\alpha\beta}^\mu \psi_\beta \right)^2 \\
& + \tau \sum_j [(\bar{\psi}_\alpha(\vec{r}, t) \psi_\alpha(\vec{r} + \vec{e}_j, t) + \bar{\psi}_\alpha(\vec{r}, t) \psi_\alpha(\vec{r} - \vec{e}_j, t)) + c.c.]
\end{aligned} \tag{2.1.6}$$

with the Hamiltonian:

$$\begin{aligned}
H_2 = & \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi}_\alpha i e \gamma_{\alpha\beta}^\mu A_\mu \psi_\beta + (m^2 + U \bar{\psi}_\alpha \psi_\alpha) (\bar{\psi}_\alpha \psi_\alpha) \\
& + V J^\mu \sum_{\mu=1}^3 \left(\bar{\psi}_\alpha \gamma_{\alpha\beta}^\mu \psi_\beta \right)^2 - \tau \sum_j [(\bar{\psi}_\alpha(\vec{r}, t) \psi_\alpha(\vec{r} + \vec{e}_j, t) + \psi_\alpha(\vec{r}, t) \bar{\psi}_\alpha(\vec{r} - \vec{e}_j, t)) + c.c.]
\end{aligned} \tag{2.1.7}$$

Finally, simplifying to the 2D lattice Hamiltonian:

$$\begin{aligned}
H_{lat} = & \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \bar{\psi}_\alpha i \gamma_{\alpha\beta}^u A_\mu \psi_\beta + \sum_i (m^2 + U \bar{\psi}_{\alpha,i} \psi_{\alpha,i}) (\bar{\psi}_{\alpha,j} \psi_{\alpha,j}) \\
& + V J^u \sum_{u=1}^3 \sum_{\langle i,j \rangle, \alpha} (\bar{\psi}_{\alpha,i} \gamma_{\alpha\beta}^u \psi_{\beta,j}) (\bar{\psi}_{\alpha,j} \gamma_{\alpha\beta}^u \psi_{\beta,i}) - \tau \sum_{\langle i,j \rangle} [(\bar{\psi}_{\alpha,i} \psi_{\alpha,j} + \psi_{\alpha,j} \bar{\psi}_{\alpha,i})]
\end{aligned} \tag{2.1.8}$$

Here, the vector displacements \hat{e}_j and position vector \hat{r} are replaced by lattice positions (i, j) where the $\langle i, j \rangle$ confine the model to nearest-neighbour interactions, though long-range interactions are not disallowed. A further specification is made to define Majorana fermions (fermions that are their own anti-particle), $\hat{\psi}_\alpha = \gamma^0 C \psi_\alpha^*$, where $\hat{\psi}$ is known as the Lorentz covariant conjugate, C is the charge conjugation operator and the subscript α denoting either an up spin or a down spin with each fermion admitting chirality L or R . The benefit of this choice is in defining the mass and kinetic interaction terms. The Dirac mass term can be written as $\bar{\psi}\psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$, and since ψ is a Majorana term, $\psi = \hat{\psi}$, which is the $\bar{\psi}\hat{\psi} = \bar{\psi}_L \hat{\psi}_R + \bar{\psi}_R \hat{\psi}_L$, which is Lorentz invariant. Taking a closer look at this Hamiltonian it is clear the interaction is a Hubbard-type interaction that has been extensively studied and has proven to be an integrable theory describing numerous physical superconducting phenomena. One final specification would be to choose the anti-ferromagnetic scheme in an isotropic XXX spin chain, recalling from before that $J = -1$, again, periodic boundary conditions are employed on the spin chain of length N : $|n + N\rangle \equiv |n\rangle$.

$$\begin{aligned}
H_{lat} = & \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \bar{\psi}_\alpha i \gamma_{\alpha\beta}^u A_\mu \psi_\beta + \sum_i (m^2 + U \bar{\psi}_{\alpha,i} \psi_{\alpha,i}) (\bar{\psi}_{\alpha,j} \psi_{\alpha,j}) \\
& - V \sum_{u=1}^3 \sum_{\langle i,j \rangle, \alpha} (\bar{\psi}_{\alpha,i} \gamma_{\alpha\beta}^u \psi_{\beta,j}) (\bar{\psi}_{\alpha,j} \gamma_{\alpha\beta}^u \psi_{\beta,i}) - \tau \sum_{\langle i,j \rangle} [(\bar{\psi}_{\alpha,i} \psi_{\alpha,j} + \psi_{\alpha,j} \bar{\psi}_{\alpha,i})]
\end{aligned} \tag{2.1.9}$$

It is important to note that while the Greek indices run from $\{0, 1, 2, 3\}$, the interactions are only occurring between sites on a square lattice, hence the bidimensionality of the model.

2.2 Integrability

To begin, define a Lax Operator, L , acting in the tensor product space $\mathcal{H}_n \otimes \mathcal{H}_a$ with the Hilbert space at site n and spectral parameter $\lambda \in \mathbb{C}^2$, where they are placed consistently with the Hubbard

Model Lax operator³⁵:

$$L_n(\lambda) = \begin{pmatrix} f(\lambda) + A & B & C & D & 1 & & T_2 \\ B^* & f(\lambda) & -B & 0 & 0 & & 0 \\ -C^* & B^* & f(\lambda) & B & C^* & & 0 \\ E & 0 & B^* & f(\lambda) & -B & & -D \\ T_1 & 0 & C & B^* & f(\lambda) & & (-T_1 + BB^* - CC^*) \\ 1 & 0 & 0 & E & 1 & f(\lambda) + (-T_2 + DE)^{\frac{1}{2}} - BB^* + CC^* & \end{pmatrix} \quad (2.2.1)$$

where the components are defined:

$$\begin{aligned} A &= \left(\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\right), & B &= i^2 \sqrt{\bar{\psi}_{\alpha\beta}^\mu A_\mu \psi_\beta}, & C &= \sqrt{V} \bar{\psi}_{\alpha,n} \gamma_{\alpha\beta}^\mu \psi_{\beta,n}, \\ D &= m^2 + U(\bar{\psi}_{\alpha,n} \psi_{\alpha,n}), & E &= (\bar{\psi}_{\alpha,n} \psi_{\alpha,n}), \\ T_1 &= -\tau(\bar{\psi}_{\alpha,n} \psi_{\alpha,n}), & T_2 &= -\tau(\psi_{\alpha,n} \bar{\psi}_{\alpha,n}), \end{aligned} \quad (2.2.2)$$

with the function $f(\lambda) = \lambda - \frac{i}{2}$. Now, similar to the case in the review, choosing for now the point $\lambda = \frac{i}{2}$ to demonstrate some properties of this model, the full matrix with exact components are:

$$\begin{pmatrix} \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) & i^2 \sqrt{\bar{\psi}_{\alpha\beta}^\mu A_\mu \psi_\beta} & \sqrt{V} \bar{\psi}_{\alpha,n} \gamma_{\alpha\beta}^\mu \psi_{\beta,n} & m^2 + U(\bar{\psi}_{\alpha,n} \psi_{\alpha,n}) & 1 & & -\tau(\bar{\psi}_{\alpha,n} \psi_{\alpha,n}) \\ -i^2 \sqrt{\bar{\psi}_{\alpha\beta}^\mu A_\mu \psi_\beta} & 0 & i^2 \sqrt{\bar{\psi}_{\alpha\beta}^\mu A_\mu \psi_\beta} & 0 & 0 & & 0 \\ -\sqrt{V} \bar{\psi}_{\alpha,n} \gamma_{\alpha\beta}^\mu \psi_{\beta,n} & -i^2 \sqrt{\bar{\psi}_{\alpha\beta}^\mu A_\mu \psi_\beta} & 0 & i^2 \sqrt{\bar{\psi}_{\alpha\beta}^\mu A_\mu \psi_\beta} & \sqrt{V} \bar{\psi}_{\alpha,n} \gamma_{\alpha\beta}^\mu \psi_{\beta,n} & & 0 \\ (\bar{\psi}_{\alpha,n} \psi_{\alpha,n}) & 0 & -i^2 \sqrt{\bar{\psi}_{\alpha\beta}^\mu A_\mu \psi_\beta} & 0 & -i^2 \sqrt{\bar{\psi}_{\alpha\beta}^\mu A_\mu \psi_\beta} & & -(m^2 + U(\bar{\psi}_{\alpha,n} \psi_{\alpha,n})) \\ -\tau(\bar{\psi}_{\alpha,n} \psi_{\alpha,n}) & 0 & \sqrt{V} \bar{\psi}_{\alpha,n} \gamma_{\alpha\beta}^\mu \psi_{\beta,n} & -i^2 \sqrt{\bar{\psi}_{\alpha\beta}^\mu A_\mu \psi_\beta} & 0 & & \tau(\bar{\psi}_{\alpha,n} \psi_{\alpha,n}) + i \bar{\psi}_{\alpha\beta}^\mu A_\mu \psi_\beta - V(\bar{\psi}_{\alpha,n} \gamma_{\alpha\beta}^\mu \psi_{\beta,n})^2 \\ 1 & 0 & 0 & (\bar{\psi}_{\alpha,n} \psi_{\alpha,n}) & 1 & \tau(\bar{\psi}_{\alpha,n} \psi_{\alpha,n}) - (m^2 + U(\bar{\psi}_{\alpha,n} \psi_{\alpha,n}))(\bar{\psi}_{\alpha,n} \psi_{\alpha,n}) - \bar{\psi}_{\alpha,i} \gamma_{\alpha\beta}^\mu A_\mu \psi_\beta - V(\bar{\psi}_{\alpha,i} \gamma_{\alpha\beta}^\mu \psi_{\beta,j})(\bar{\psi}_{\alpha,j} \gamma_{\alpha\beta}^\mu \psi_{\beta,i}) & \end{pmatrix} \quad (2.2.3)$$

Then define the conserved quantities:

$$I_k = Tr[L^k]. \quad (2.2.4)$$

To see that these quantities are conserved, consider:

$$\begin{aligned} I_1 &= Tr[L] \\ &= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \tau(\bar{\psi}_{\alpha,n} \psi_{\alpha,n}) - (m^2 + U(\bar{\psi}_{\alpha,n} \psi_{\alpha,n}))(\bar{\psi}_{\alpha,n} \psi_{\alpha,n}) \\ &\quad - \bar{\psi}_{\alpha,i} \gamma_{\alpha\beta}^\mu A_\mu \psi_\beta - V(\bar{\psi}_{\alpha,i} \gamma_{\alpha\beta}^\mu \psi_{\beta,j})(\bar{\psi}_{\alpha,j} \gamma_{\alpha\beta}^\mu \psi_{\beta,i}) \end{aligned} \quad (2.2.5)$$

and

$$\begin{aligned}
I_2 &= Tr[L^2] \\
&= \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \bar{\psi}_\alpha i \gamma_{\alpha\beta}^u A_\mu \psi_\beta + \sum_i (m^2 + U \bar{\psi}_{\alpha,n} \psi_{\alpha,n}) (\bar{\psi}_{\alpha,n'} \psi_{\alpha,n'}) \\
&\quad - V \sum_{u=1}^3 \sum_{\langle n,n' \rangle, \alpha} (\bar{\psi}_{\alpha,n} \gamma_{\alpha\beta}^u \psi_{\beta,n'}) (\bar{\psi}_{\alpha,n'} \gamma_{\alpha\beta}^u \psi_{\beta,i}) - \tau \sum_{\langle n,n' \rangle} [(\bar{\psi}_{\alpha,n} \psi_{\alpha,n'} + \psi_{\alpha,n'} \bar{\psi}_{\alpha,n})]
\end{aligned} \tag{2.2.6}$$

is nothing but the lattice Hamiltonian listed previously, H_{lat} , which is also a conserved quantity and n, n' are successive lattice sites which can also be labelled n_1 and n_2 respectively. From here on, H_{lat} is referred to as H for simplicity of notation. The importance of the quantity in (2.25) is not immediately obvious and so it still needs to be checked whether or not it is actually conserved. This can be done by evaluating the commutator $[I_1, H]$, as a conserved quantity should commute with the Hamiltonian; which turns out to be true, i.e. $[I_1, H] = 0 = [I_1, I_2]$ as shown in Appendix F. Using (1.3.9) the monodromy matrix of this model can generally be constructed using the form of the monodromy matrix in the Hubbard model³⁶:

$$\mathcal{T}_a = \begin{pmatrix} B(\lambda) & \mathbf{B}(\lambda) & F(\lambda) \\ \mathbf{C}(\lambda) & \hat{A}(\lambda) & \mathbf{B}^*(\lambda) \\ C(\lambda) & \mathbf{C}^*(\lambda) & D(\lambda) \end{pmatrix} \tag{2.2.7}$$

where $\mathbf{B}(\lambda), \mathbf{C}^*(\lambda)$ have the form 4×1 , $\mathbf{B}^*(\lambda), \mathbf{C}(\lambda)$ have the form 1×4 , with associated scalar quantities $B(\lambda), C(\lambda), D(\lambda), F(\lambda)$ and $\hat{A}(\lambda)$ is a 4×4 matrix. Then the transfer matrix, $T = Tr(\mathcal{T})$, is defined with the matrix components $A_{11}(\lambda), A_{22}(\lambda), A_{33}(\lambda)$ and $A_{44}(\lambda)$:

$$T = B(\lambda) + A_{11}(\lambda) + A_{22}(\lambda) + A_{33}(\lambda) + A_{44}(\lambda) + D(\lambda) \tag{2.2.8}$$

Following the method of the Algebraic Bethe Ansatz, it's easy to see the transfer matrix expands non-trivially with the form of (1.3.13) producing $N - 1$ commuting operators:

$$T = (6f(\lambda))^N + \sum_{n=0}^{N-2} Q_n f(\lambda)^n, \tag{2.2.9}$$

in this case, however, by definition of the transfer matrix and the construction of L , it is evident that the Hamiltonian is already a part of this family, and thus integrable. To find the BAE's from here, the RTT relation must be followed. Using the short hand $T(\lambda) = T_\lambda$, the product of two

transfer matrices of different spectral parameters λ, θ :

$$T_\lambda T_\theta = \begin{pmatrix} B_B(\lambda) & \mathbf{B}(\lambda) & F(\lambda) \\ \mathbf{C}(\lambda) & \hat{A}(\lambda) & \mathbf{B}^*(\lambda) \\ C(\lambda) & \mathbf{C}^*(\lambda) & D(\lambda) \end{pmatrix} \begin{pmatrix} B_B(\theta) & \mathbf{B}(\theta) & F(\theta) \\ \mathbf{C}(\theta) & \hat{A}(\theta) & \mathbf{B}^*(\theta) \\ C(\theta) & \mathbf{C}^*(\theta) & D(\theta) \end{pmatrix} \quad (2.2.10)$$

then explicitly the product of the two matrices:

$$T_\lambda T_\theta = \begin{pmatrix} B_\lambda B_\theta + \mathbf{B}_\lambda \mathbf{C}_\theta + F_\lambda C_\theta & B_\lambda \mathbf{B}_\theta + \mathbf{B}_\lambda \hat{A}_\theta + F_\lambda \mathbf{C}_\theta^* & B_\lambda F_\theta + \mathbf{B}_\lambda \mathbf{B}_\theta^* + F_\lambda D_\theta \\ \mathbf{C}_\lambda B_\theta + \hat{A}_\lambda \mathbf{C}_\theta + \mathbf{B}_\lambda^* C_\theta & \mathbf{C}_\lambda \mathbf{B}_\theta + \hat{A}_\lambda \hat{A}_\theta + \mathbf{B}_\lambda^* \mathbf{B}_\theta^* & \mathbf{C}_\lambda F_\theta + \hat{A}_\lambda \mathbf{B}_\theta^* + \mathbf{B}_\lambda^* D_\theta \\ C_\lambda B_\theta + \mathbf{C}_\lambda^* \mathbf{C}_\theta + D_\lambda C_\theta & C_\lambda \mathbf{B}_\theta + \mathbf{C}_\lambda^* \hat{A}_\theta + D_\lambda \mathbf{C}_\theta^* & C_\lambda F_\theta + \mathbf{C}_\lambda^* \mathbf{B}_\theta^* + D_\lambda D_\theta \end{pmatrix}. \quad (2.2.11)$$

Next, employing the pseudo-vacuum to use as a reference state:

$$\begin{aligned} \mathbf{C}(\lambda, \theta) |\Omega\rangle &= 0, \\ \mathbf{C}^*(\lambda, \theta) |\Omega\rangle &= 0, \\ C(\lambda, \theta) |\Omega\rangle &= 0. \end{aligned} \quad (2.2.12)$$

as well as the bottom triangle of $\hat{A}(\lambda)$:

$$\hat{A}_{st}(\lambda) |\Omega\rangle = 0 \quad \text{if } s < t, s \neq t. \quad (2.2.13)$$

in order to make $|\Omega\rangle$ an eigenstate of the diagonal terms of the transfer matrix T and also an eigenstate of (2.2.8). This then suggests that operators $\mathbf{B}(\lambda), \mathbf{B}^*(\lambda)$ and $F(\lambda)$ are an analogue to the creation operators. Using this approach one can construct a set of eigenvectors in the same manner as Section 1.3:

$$|\Phi(\lambda)\rangle = \mathbf{B}(\lambda_1) F(\lambda) = B_a(\lambda) F^a(\lambda) |\Omega\rangle, \quad (2.2.14)$$

then conceptually one would expect to construct other higher states using a product of $\mathbf{B}(\lambda)$ and $F(\lambda)$, due to the commutation between two fields of type $\mathbf{B}(\lambda)$ which is evident from the RTT relation³⁶:

$$\mathbf{B}(\lambda) \mathbf{B}(\theta) = c_1(\mathbf{B}(\lambda) \mathbf{B}(\theta)) R(\lambda, \theta) - c_2 F(\lambda), \quad (2.2.15)$$

where $c_1, c_2 \equiv c_1(\lambda, \theta), c_2(\lambda, \theta)$ are constants that depend on the spectral parameters. Now that these parameters are defined, the diagonalisation can be completed. Starting with:

$$\left[B(\lambda) + \sum_{s=t=1}^4 \hat{A}(\lambda) + D(\lambda) \right] |\Phi_n(\lambda)_n\rangle = \Lambda(\lambda, \{\lambda_i\}) |\Phi_n(\lambda)\rangle, \quad (2.2.16)$$

then requiring the commutator relations between all the various fields in the transfer matrix. This then brings the model to what is essentially a Hubbard model solution, which has widely been proven to be integrable, i.e. the model is integrable.

Conclusions

The integrability of this new model is positive for three main reasons. Firstly, it is integrable. Manifestly, it is exactly solvable using a modification of the Hubbard model Bethe Ansatz method that is already quite well known thus allowing for deeper, scrutinous analysis with fewer gaps in the mathematical framework. Second, the model admits topological states, in particular, Majorana Bound edge states which can qualitatively be seen by limiting the lax connection to the edge of the square lattice. Consider an $N \times N$ square lattice grid M_{ab} with sites at (a, b) . The lax connection can be limited to:

$$\begin{aligned} a &= (1, 2, \dots, N) & \text{if } b &= 1 \text{ and } N \\ a &= 1 \text{ and } N & \text{if } b &= (1, 2, \dots, N) \end{aligned}$$

as a rudimentary set-up. Furthermore, due to the Hamiltonian of the model the interactions around the edge of the lattice are by definition superconducting, so here the topologically protected edge modes can be viewed as a Heisenberg style spin chain. Finally, the model is relativistic. Again, due to its Lorentz and Gauge invariant set-up, phenomena such as the Meissner effect, i.e. the exclusion of an external field, can be see as an intrinsically as a relativistic field theoretic effect by gauge fixing the external field $B_\mu = 0$ as applied in (2.1.6). Despite these favourable outcomes, there is still a lot to be done for any truly valuable result.

Considering again with this model's strong association to the Hubbard. While it was shown that this model can be solved exactly, the full solutions have not yet been found- only a model of what these equations would look like. This was mainly due to a lack of time, however, it is promising to see an explicit lax operator that can be used to produce a family of commuting conserved quantities. Explicit derivation of the spectral parameter constants would be a favourable starting point. More so, an explicit illustration of the R-matrix with its Boltzmann weights would also be greatly beneficial, resulting in a comprehensive characterisation of the Bethe roots and by extension the energy eigenvalues, which could be measured. Then, to *stress test* the rudimentary edge state conception above, a thorough analysis of the Majorana modes and the associated symmetries would provide with a constructive research paper. Along this line, a perturbative expansion to clearly define a relativistic superconducting order parameter from the Lagrangian can and should be done to provide a better image of the phenomenology of this model. Finally, while the 2D case is

integrable, greater steps need to be taken to make any definitive statements about generalising this study to higher dimensional symmetries, for example an application of the nested Bethe Ansatz to the QCD Lagrangian listed in Section 2, \mathcal{L}_3 .

Overall, having started with the $SU(3)$ -invariant Lagrangian and simplified down to a two-dimensional lattice Hamiltonian with Hubbard, Spin-spin and electron-photon interactions has proven to be integrable and a potentially viable field theory. While there is some more work to make this a rigorous theory, the further generalisation of this model to the $SU(3)$ case could prove to be promising and would be of great interest to the development of the theory of superconducting.

A

Appendix A

To check the RLL relation, use the form in (1.3.6):

$$R_{a_1, a_2}(\lambda) = \lambda \mathbb{I}_{a_1, a_2} + i \mathcal{P}_{a_1, a_2} \quad (\text{A.0.1})$$

and the form in (1.3.4):

$$L_{n, a}(\lambda) = \left(\lambda - \frac{i}{2} \right) I_{n, a} + i \mathcal{P}_{n, a} \quad (\text{A.0.2})$$

Then we get on the left-hand side:

$$\begin{aligned} & (\lambda \mathbb{I}_{a_1, a_2} + i \mathcal{P}_{a_1, a_2}) \left(\left(\lambda - \frac{i}{2} \right) I_{n, a_1} + i \mathcal{P}_{n, a_1} \right) \left(\left(\lambda - \frac{i}{2} \right) I_{n, a_2} + i \mathcal{P}_{n, a_2} \right) \\ = & (\lambda \mathbb{I}_{a_1, a_2} + i \mathcal{P}_{a_1, a_2}) \left(\left(\lambda - \frac{i}{2} \right)^2 I_{n, a_1} I_{n, a_2} + \left(\lambda - \frac{i}{2} \right) I_{n, a_2} i \mathcal{P}_{n, a_1} \right) + \left(\lambda - \frac{i}{2} \right) I_{n, a_1} i \mathcal{P}_{n, a_2} - \mathcal{P}_{n, a_1} \mathcal{P}_{n, a_2} \end{aligned} \quad (\text{A.0.3})$$

Then using the general properties of the permutation operator:

$$\mathcal{P}_{n, a_1} \mathcal{P}_{n, a_2} = \mathcal{P}_{a_1, a_2} \mathcal{P}_{n, a_1} = \mathcal{P}_{n, a_1} \mathcal{P}_{a_2, a_1} \quad (\text{A.0.4})$$

along with the symmetry of \mathcal{P} :

$$\mathcal{P}_{a_2, a_1} = \mathcal{P}_{a_1, a_2}, \quad (\text{A.0.5})$$

we then get:

$$\left(\left(\lambda - \frac{i}{2} \right) I_{n, a_1} + i \mathcal{P}_{n, a_2} \right) \left(\left(\lambda - \frac{i}{2} \right) I_{n, a_1} + i \mathcal{P}_{n, a_1} \right) (\lambda \mathbb{I}_{a_2, a_1} + i \mathcal{P}_{a_2, a_1}) \quad (\text{A.0.6})$$

Which is the right-hand side of (A.0.3) and thus no different to the original statement of the RLL relation:

$$R_{a_1, a_2}(\lambda - \mu)L_{n, a_1}(\lambda)L_{n, a_2}(\mu) = L_{n, a_2}(\mu)L_{n, a_1}(\lambda)R_{a_1, a_2}(\lambda - \mu) \quad (\text{A.0.7})$$

B

B

Appendix B

We start with :

$$R_{12}(u, v)R_{13}(u, w)R_{23}(v, w) = R_{23}(v, w)R_{13}(u, w)R_{12}(u, v), \quad (\text{B.0.1})$$

then using:

$$R_{a_1, a_2}(\lambda) = \lambda \mathbb{I}_{a_1, a_2} + i \mathcal{P}_{a_1, a_2} \quad (\text{B.0.2})$$

we find of the left hand side:

$$\begin{aligned} & (\lambda \mathbb{I}_{a_1, a_2} + i \mathcal{P}_{a_1, a_2}) (\lambda \mathbb{I}_{a_1, a_3} + i \mathcal{P}_{a_1, a_3}) (\lambda \mathbb{I}_{a_2, a_3} + i \mathcal{P}_{a_2, a_3}) \\ &= (\lambda^2 \mathbb{I}_{a_1, a_2} \mathbb{I}_{a_1, a_3} + \dots - \mathcal{P}_{a_1, a_2} \mathcal{P}_{a_1, a_3}) (\lambda \mathbb{I}_{a_2, a_3} + i \mathcal{P}_{a_2, a_3}) \\ & \quad (\lambda^3 \mathbb{I}_{a_1, a_2} \mathbb{I}_{a_1, a_3} \mathbb{I}_{a_2, a_3} + \dots - i \mathcal{P}_{a_1, a_2} \mathcal{P}_{a_1, a_3} \mathcal{P}_{a_2, a_3}) \end{aligned} \quad (\text{B.0.3})$$

Now, using the general properties of the permutation operator:

$$\mathcal{P}_{n, a_1} \mathcal{P}_{n, a_2} = \mathcal{P}_{a_1, a_2} \mathcal{P}_{n, a_1} = \mathcal{P}_{n, a_1} \mathcal{P}_{a_2, a_1} \quad (\text{B.0.4})$$

and simplifying, we find:

$$R_{23}(v, w)R_{13}(u, w)R_{12}(u, v), \quad (\text{B.0.5})$$

which is the right-hand of (B.0.1).

The following proof is from¹⁵:

Using the following shorthand notation:

- $R_{a_1, a_2}(\lambda - \mu) = R_{12}$,
- $L_{n, a_1}(\lambda) = L_1$,
- $L_{n+1, a_1}(\lambda) = L'_1$,
- $L_{n, a_2}(\mu) = L_2$,
- $L_{n+1, a_2}(\mu) = L'_{a_2}$

$$\begin{aligned}
 & R_{12}L'_1L_1L'_2L_2 && \text{(commutativity of } L_1, L'_2) \\
 & = R_{12}L'_1L'_2L_1L_2 && \text{(due to RLL relation for } L_1, L_2 \text{ and } L'_1, L'_2) \\
 & = L'_2L'_1L_2L_1R_{12} && \text{(commutativity of } L'_1, L_2) \\
 & = L'_2L_2L_2L'_1L_1R_{12}
 \end{aligned}$$

which is the same as:

$$R_{a_1, a_2}(\lambda - \mu)T_{a_1}(\lambda)T_{a_2}(\mu) = T_{a_2}(\mu)T_{a_1}(\lambda)R_{a_1, a_2}(\lambda - \mu) \quad (\text{C.0.1})$$

D

D

Appendix D

$\lambda = \frac{i}{2}$ is quite unique for the reason:

$$L_{n,a}(\lambda = \frac{i}{2}) = i\mathcal{P}_{n,a} \quad (\text{D.0.1})$$

and naturally for any λ :

$$\frac{d}{d\lambda} L_{n,a}(\lambda) = \mathbb{I}_{n,a} \quad (\text{D.0.2})$$

Then the expansion of $F(\lambda)$ at $\lambda = \frac{i}{2}$, is:

$$T_a\left(\frac{i}{2}\right) = i^L \mathcal{P}_{L,a} \mathcal{P}_{L-1,a} \dots \mathcal{P}_{1,a} \quad (\text{D.0.3})$$

By permutating the string in (1.3.16) employing the property of \mathcal{P} stated earlier and noting the trace over the auxiliary space- $tr_a \mathcal{P}_{L,a} = \mathbb{I}_L$, the shift operator U in the hamiltonian can be defined:

$$U = i^{-L} tr_a T_L\left(\frac{i}{2}\right) = P_{1,2} P_{2,3} \dots P_{L,L-1} \quad (\text{D.0.4})$$

The function of \mathcal{P} can be rewritten as:

$$\mathcal{P}_{n_1, n_1} X_{n_2} \mathcal{P}_{n_1, n_2} = X_{n_1} \quad (\text{D.0.5})$$

Therefore, when applied to the operator U :

$$\begin{aligned}
 X_n U &= \mathcal{P}_{12} \dots X_n \mathcal{P}_{n-1,n} \mathcal{P}_{n,n+1} \dots \mathcal{P}_{L-1,L} \\
 &= \mathcal{P}_{12} \dots \mathcal{P}_{n-1,n} X_{n-1} \mathcal{P}_{n,n+1} \dots \mathcal{P}_{L-1,L} \\
 &= U X_{n-1}
 \end{aligned} \tag{D.0.6}$$

Then, using the fact that U is a unitary operator- $U^*U = UU^* = 1$ and the properties of \mathcal{P} - $\mathcal{P}^* = \mathcal{P}$; $\mathcal{P}^2 + \mathbb{I}$ one can write:

$$U^{-1} X_n U = X_{n-1} \tag{D.0.7}$$

Then using this unitary operator we can introduce an important observable- momentum, P , which by definition is an infinitesimal shift along the lattice ¹⁵:

$$e^{iP} = U. \tag{D.0.8}$$

E

Appendix E

Define the

$$e_1 = e_+ \otimes e_+, e_2 = e_+ \otimes e_-, e_3 = e_- \otimes e_+, e_4 = e_- \otimes e_- \quad (\text{E.0.1})$$

where:

$$e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (\text{E.0.2})$$

Then the matrix form of the permutation operator is:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{E.0.3})$$

Note also that $\omega_n = e_+$, i.e. that Ω exists in the Hilbert space as:

$$|\Omega\rangle = \prod_n \otimes w_n. \quad (\text{E.0.4})$$

F

Appendix F

To prove:

$$[I_1, H] = [L^1 = H] = 0, \quad (\text{F.0.1})$$

recall individual components:

$$\begin{aligned} A &= \left(\frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu)\right), & B &= i^2 \sqrt{\bar{\psi}_{\alpha\beta}^\mu A_\mu \psi_\beta}, & C &= \sqrt{V} \bar{\psi}_{\alpha,n} \gamma_{\alpha\beta}^\mu \psi_{\beta,n}, \\ D &= m^2 + U (\bar{\psi}_{\alpha,n} \psi_{\alpha,n}), & E &= (\bar{\psi}_{\alpha,n} \psi_{\alpha,n}), \\ T_1 &= -\tau (\bar{\psi}_{\alpha,n} \psi_{\alpha,n}), & T_2 &= -\tau (\bar{\psi}_{\alpha,n} \psi_{\alpha,n}). \end{aligned} \quad (\text{F.0.2})$$

substituted into the expression $[L^1, H]$:

$$\begin{aligned} & [A + (-T_2 + DE)^{\frac{1}{2}} - BB^* + CC^*, A^2 + BB^* - CC^* + DE + T_1 + T_2] = \\ & [A, A^2] + [A, BB^*] + [A, DE] - [A, CC^*] + [A, T_1] + [A, T_2] \\ & + [(-T_2 + DE)^{\frac{1}{2}}, A^2] + [(-T_2 + DE)^{\frac{1}{2}}, BB^*] + [(-T_2 + DE)^{\frac{1}{2}}, DE] \\ & - [(-T_2 + DE)^{\frac{1}{2}}, CC^*] + [(-T_2 + DE)^{\frac{1}{2}}, T_1] + [(-T_2 + DE)^{\frac{1}{2}}, T_2] + [-BB^*, A^2] + [-BB^*, BB^*] + [-BB^*, DE] \\ & - [-BB^*, CC^*] + [-BB^*, T_1] + [-BB^*, T_2] + [CC^*, A^2] + [CC^*, BB^*] + [CC^*, DE] \\ & - [CC^*, CC^*] + [CC^* T_1] + [CC^*, T_2]. \end{aligned} \quad (\text{F.0.3})$$

While this may look long and tedious, a number of these terms immediately turn out to be zero. Firstly, $T_1 = -\tau (\bar{\psi}_{\alpha,n} \psi_{\alpha,n})$ and $T_2 = -\tau (\bar{\psi}_{\alpha,n} \psi_{\alpha,n})$ will commute with all the operators as the combination of the two spinors are a Lorentz Scalar (also note the charge parity invariance of Majorana terms). Secondly, as this is the QED case in the Abelian theory, all the components will

commute with themselves. Then, similarly DE and CC^* are comprised of Lorentz scalars, thus leaving:

$$[A + (-T_2 + DE)^{\frac{1}{2}} - BB^* + CC^*, A^2 + BB^* - CC^* + DE + T_1 + T_2] = 0. \quad (\text{F.0.4})$$

Thus the commutator in (F.0.1) is true.

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