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MSc in Quantum Fields and Fundamental Forces

# Massive gravity and Positivity bounds


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**Abstract**

With many theoretical obstacles now overcome, massive gravity has recently become a field of renewed vitality. In this thesis, we first review some outstanding problems of massive gravity in its historical development and the recent progress in obtaining solutions. In particular, we provide the proof of absence of BD ghost in dRGT theory. We then reproduce the construction of the positivity bounds, which could be a powerful tool to verify the existence of standard UV completion for EFTs. We also give some examples to show how it restricts the parameter space. Finally, we explore what unitarity combined with renormalizability may imply polynomial constraints for generic SM-like theories, and discuss whether this also feasible for EFTs.

# Contents

# 1 Introduction

With impeccable phenomenological consistency, general relativity (GR) is widely accepted as the correct theory describing the force of gravity, at least over the range of scales so far probed. On the other hand, the search for alternative theories of gravity is a challenge with a long history with significant recent interest. Rather than being a purely academic exercise, the development of alternative theories is paramount in order to provide an essential test for GR. Furthermore, there exist remaining puzzles over the role of gravity, for instance the old cosmological constant problem and the origin of the late-time acceleration of the Universe, have also motivating the search of alternatives.

In order to find an alternative gravity theory, one should first consider a modification of the underlying assumptions that go into the construction of GR. From a modern perspective GR describes an interacting theory of massless spin-2 particles with Lorentz invariance build in. Thus the most straightforward way to modify these properties might be to break Lorentz invariance and or include states with additional spins. This has been explored in many literatures such as [64]. Another well-known alternative theory is so-called higher spin gravity which maintains Lorentz invariance but describes gravity by including higher spin states. We refer the reader to [79, 4] for further details. Despite the progress made in the above two possibilities, in this review we will instead explore another alternative: based on the notion that gravity continues to be propagated by gravitons which respect Lorentz invariance and have spin-2, but where now the graviton is massive. From the viewpoint of particle physics, this consideration seems natural since particles that mediate electroweak forces also have acquired masses by the Higgs mechanism.

The study of massive spin-2 field theory can be traced back to 1939 when the well-known Fierz-Pauli action was found. The construction of a linear theory including a single massive spin-2 field is relatively simple. However, the real obstacle appears if one tries to extend it as an interacting theory. In GR, external matter coupling forces local Lorentz symmetry and the gauge invariance of a massless spin-2 field to become fully non-linear diffeomorphism invariance (covariance). This gauge symmetry is inherited in massive gravity theories and still plays a crucial role, even though the additional mass term will break it. As a result, constructing a theory of massive spin-2 fields become complicated and challenging.

Searching for a massive gravity theory alternative to GR, one should first concern ourselves with the change of degrees of freedom as it is the most elementary factor in physics. A massive spin-2 field in a Lorentz invariant theory should propagate five degrees of freedom while a massless one only propagates two as in the case of GR. These extra DoFs become the origin of an inconsistency with GR, which known as van Dam-Veltman-Zakharov (vDVZ) discontinuity [78, 84]. The resolution is given by the so-called Vainshtein mechanism which was first considered in the 1970s for the naive non-linear Fierz-Pauli action of massive gravity and has further been generalized to many different massive gravity models in recent decades.

Another related problem is that non-linear extensions of massive gravity usually admit higher derivative terms. In the naive non-linear Fierz-Pauli action, a sixth ghost degree of freedom arises from self-interactions with higher derivatives, namely Boulware-Deser (BD) ghost [7]. Fortunately, there are several theories evading the BD ghost have been successfully constructed. For instance, the Dvali-Gabadadze-Porrati (DGP) model [33, 34, 32] with a soft graviton (resonance) mass, and in particular, a ghost-free massive gravity known as de Rham-Gabadadze-Tolley (dRGT) theory [18].

Despite the fact that ghost-free massive gravity has made a great step forward in raising the energy scale of the dominant interactions to $\Lambda_3$, the theory is still strongly restricted to this low energy level which even much below the Planck scale. It should be interpreted as a low energy effective field theory (LEEFT) and will be break down at the cutoff, where the unknown new degrees of freedom may introduce. On the other hand, it is now widely known that not all effective field theories admit a local Lorentz invariant UV completion [76, 1]. This has stimulated interest in exploring the existence of possible UV completion for those modified gravity theories.

In the absence of an established framework for the standard UV approach, the explicit form of the finite UV completion remains unknown. Nevertheless, the IR physics could still teach people some lessons about the theory in high energy. It has long been recognized that the physical requirements of locality, unitarity and crossing symmetry could together provide non-trivial constraints on the scattering S-matrix for a Lorentz invariant theory [62, 35]. With the requirements of analyticity, some of these constraints may be derived by expressing the scattering amplitude in terms of dispersion relations with a finite number of subtractions. Assuming the existence of a possible local Lorentz invariant UV completion of the LEEFT, with some studies of positive properties for the scattering amplitude and its derivatives, it could allow us to construct so-called-positivity bounds [21, 22, 20, 23] for the linear combinations of free parameters in the Wilsonian effective action. For the past few decades, these bounds have been pushed away from the forward scattering limit to provide further constraints on the LEEFT and well generalized for the theories of particle with spin.

Although there is no existing evidence to prove that any UV complete theory must satisfy positivity bounds, it can at least help us to verify that whether an EFT admits a possible local Lorentz invariant UV completion. Furthermore, demanding that such a UV completion exists for gravitational EFTs, one can derive constraints for the parameter space of massive gravity theories, and this may sometimes even give inspirations for the UV approach.

This review is organized as follows: We start by linear Fierz-Pauli theory and the vDVZ discontinuity, which is the first significant phenomenological obstacle for massive gravity in Part 1. We then introduce the Stückelberg formalism for massive spin-1 field and spin-2 field, and explain the necessity of non-linearity for the gravitational theories with the appearance of the BD ghost. In section 4 we provide the ADM proof of the absence of ghost for the well-known ghost-free massive gravity (dRGT), explore the decoupling limit of the theory and resolve the vDVZ discontinuity for both $\Lambda_5$ and $\Lambda_3$ theories by corresponding Vainshtein mechanism. The other massive gravity theories, for instance DGP model, are briefly introduced in section 5. In Part 2, we first perform the construction of positivity bounds for scalar and give massive Galileon EFT as an example. In the use of transversity formalism, which has more explicit form of crossing symmetry relations, we generalize the positivity bounds for particle with spins in the following section. The Proca EFT is then studied as the simplest example for the spin-1 field. The search of UV completion possibility for massive gravity theory are concluded by its positivity bounds analysis in the section 8. Finally, in section 9 we explore another methodology in different of positivity bounds to show the unitarity imposes more constraints on generic renormalizable theories, and discuss the feasibility for EFTs.

# 2 General relativity and Fierz-Pauli action

## 2.1 Theory of massless spin 2 field

General relativity (GR) is widely considered to be the correct theory for describing gravity at low energies or large distances [47]. The discovery of GR undoubtedly led Einstein to construct a fully non-linear theory describing the dynamics of spacetime itself in geometric terms, starting from the equivalence principle and general coordinate invariance. The discovery of GR has undoubtedly led to a major step forward in the understanding of the physical world. However, in a parallel universe where Einstein did not exist, the equivalence theory of GR might also be found in a completely different but more mathematically logical path a few decades later.

The alternative way to approach GR is most likely in field theory language, by the study of particle symmetries. From a real physical world point of view, the basic degrees of freedom are particles which carry mass and spin. In the language of field theory, this means that the degrees of freedom are carried by fields. To describe long range macroscopic forces, one should consider only bosonic fields which carry integer spin by virtue of the spin statistics theorem, since fermion do not build classical coherent states. While by considering the source like solution $\sim \frac{1}{r}e^{-mr}$ of the Klein-Gordon equation $(\Box - m^2)\psi = 0$, it is natural to believe long range forces should be described by massless fields to evade exponential suppression.

The massless bosonic fields can be classified by their transformation rule under rotations. It is characterized by an integer called helicity $h \geqslant 0$. The path to building a theory for massless particles with certain helicity is straightforward. Assuming a priori special relativity, the principles of Lorentz invariance give the corresponding gauge symmetry for particles characterized by helicity. Finally, writing all possible self-interactions preserve gauge symmetry (consistent with Lorentz invariance) to describe the dynamics of the physics system. For helicity 2 particles, the linearized general coordinate invariance is the required gauge symmetry. The covariance of self-interactions leads to a theory of massless helicity-2 particle essentially uniquely to GR[47, 38, 58, 81, 31, 8, 36, 80]. On the other hand, the equivalence principles and general coordinate invariance do not uniquely suggest GR. The gauge symmetries are redundancies rather fundamental properties [47]. As we will see in the following section, it is possible to introduce redundant variables to restore the general coordinate diffeomorphisms for any Lagrangian. As long as people realize that GR is the theory of a non-trivially interacting massless helicity 2 particle rather the (full) theory of graviton, they will not be surprise that GR is not UV complete at the quantum level, and must be viewed as an effective field theory with a cutoff at Planck mass $M_P$.

## 2.2 Fierz-Pauli mass term

A straightforward way to modify the gravity of GR is by constructing a mass term for graviton. A theory of massive gravity is a theory propagating massive spin-2 particle(s). Naively, one may simply add a graviton mass term for Einstein-Hilbert action, so in the m=0 limit theory goes back to GR. A priori assumption is that the mass term for a spin-2 field $h_{\mu\nu}$ contain only 2 power of h and no derivatives, therefore the generic mass term can be written as

$$\mathcal{L}_{mass} = -\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - ah^2) \ , \tag{2.1}$$

where $a$ is a dimensionless parameter and the indices are raised and lowered with respect to the Minkowski metric. The mass term is manifestly breaking the diffeomorphism invariance. We will show how to restore it by introducing Stückelberg fields. On the other hand, there is no known symmetry to enforce the parameter a to a particular value.

However, we shall always take $a = 1$ called Fierz-Pauli tuning. Violating this tuning by take $a \neq 1$ would lead Fierz-Pauli action propagating a scalar ghost with mass $m_g^2 = m^2\frac{4a-1}{2(1-a)}$, in addition to the massive spin-2 particle. With Fierz-Pauli tuning approach, it goes to infinity and is thus non-dynamical. We will see this more explicitly in the next section with Stückelberg field introduced.

**Degree of freedom**

A massive gravity theory should propagate different degrees of freedom to massless theory. We shall first study if the Fierz-Pauli action propagates the correct number of DoFs for a massive spin 2 particle.

The full Fierz-Pauli action contains all possible contractions of two powers of $h$ and up to two derivatives is

$$\begin{aligned} S = \int dx &- \frac{1}{2}\partial_k h_{\mu\nu}\partial^k h^{\mu\nu} + \partial_\mu h_{\nu k}\partial^\nu h^{\mu k} - \partial_\mu h_{\mu\nu}\partial^\nu h + \frac{1}{2}\partial_k h \partial^k h \\ &- \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) \ . \end{aligned} \tag{2.2}$$

Indeed, in the $m = 0$ limit, it obtains exactly the Einstein-Hilbert action at linear level. By Legendre transformation, the dynamical spatial canonical momenta are

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}^{ij} - \dot{h}\delta^{ij} - 2\partial^{(i}h^{j)0} + \partial^k h^{0k}\delta^{ij} \ . \tag{2.3}$$

Conversely,

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{D-2}\pi_{kk}\delta_{ij} + 2\partial_{(i}h_{j)0} \ . \tag{2.4}$$

Our Hamiltonian is then

$$\begin{aligned}\mathcal{H} =&\frac{1}{2}\pi_{ij}^2 - \frac{1}{2}\frac{1}{D-2}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j h_{kk} \\ &- \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2(h_{ij}h_{ij} - h_{ii}^2) \ .\end{aligned} \tag{2.5}$$

Now the Fierz-Pauli action can be written in the form containing no time-like components $h_{0i}$ and $h_{00}$ with time derivatives

$$\begin{aligned}S = \int d^D x \pi_{ij}\dot{h}_{ij} - \mathcal{H} + 2h_{0i}(\partial_j \pi_{ij}) + m^2 h_{0i}^2 \\ + h_{00}(\nabla^2 h_{ij} - \partial_i\partial_j h_{ij} - m^2 h_{ii}) \ .\end{aligned} \tag{2.6}$$

Manifestly, the component $h_{00}$ is a Lagrange multiplier. Solving the equation of motion will give a constraint

$$C = \nabla^2 h_{ij} - \partial_i\partial_j h_{ij} - m^2 h_{ii} = 0 \ . \tag{2.7}$$

Integrating out the component $h_{0i}$ by

$$h_{0i} = -\frac{1}{m^2}\partial_j \pi_{ij} \ . \tag{2.8}$$

The Hamiltonian then becomes

$$\mathcal{H} \to \mathcal{H} + \frac{1}{m^2}(\partial_j \pi_{ij})^2 \ . \tag{2.9}$$

The Poisson bracket enforces another constraint

$$\{H, C\}_{PB} = -\frac{1}{D-2}m^2 \pi_{ii} - \partial_i\partial_j \pi_{ij} = 0 \ . \tag{2.10}$$

where $H = \int d^D \mathcal{H}$.

For $D = 4$, two symmetric 3x3 tensors $h_{ij}$ and $\pi_{ij}$ span a 12-dimensional phase space. The two secondary class constraints move 2 phase freedom for the space, so the massive graviton and its conjugate momenta carry a total of 10 degrees of freedom, i.e., a massive spin 2 particle has 5 degrees of freedom as required. We shall later see how these 5 DoFs assign to Stückelberg fields. In the $m = 0$ limit, there will have more constraints and thus leave only 2 degrees of freedom as the correct number for a massless spin 2 particle in GR.

It is worth mentioning that if Fierz-Pauli tuning is violated, the component $h_{00}$ no longer appears linearly and does not enforce a constraint. Consequently, the total 12 degrees of freedom excited and the extra DoFs propagate the mentioned scalar ghost.

## 2.3 vDVZ discontinuity

Since the massive gravity theory propagates more degrees of freedom, it is not surprising that there may appear some inconsistencies between massless limit of Fierz-Pauli action and the purely massless theory. One should keep the following statement in mind: a massless limit of massive gravity is not a theory propagating only single massless graviton [47]. A sufficient example is Van Dam–Veltman–Zakharov discontinuity, which could be shown by a study of the physical propagator of the theory.

### Massless propagator

We start by considering purely massless theory equivalent to GR. Setting m=0 in Fierz-Pauli action and couple gravity to a conserved (requiring by general diffeomorphism) external source $T^{\mu\nu}$

$$\mathcal{L} = -\frac{1}{4}h^{\mu\nu}\hat{\varepsilon}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + \frac{1}{2M_P}h_{\mu\nu}T^{\mu\nu} \ . \tag{2.11}$$

Where $\hat{\varepsilon}^{\alpha\beta}_{\mu\nu}$ called Lichnerowicz operator

$$\hat{\varepsilon}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} = -\frac{1}{2}(\Box h_{\mu\nu} - 2\partial_{(\mu}\partial_{\alpha}h^{\alpha}_{\nu)} + \partial_{\mu}\partial_{\nu}h - \eta_{\mu\nu}(\Box h - \partial_{\alpha}\partial_{\beta}h^{\alpha\beta})) \ . \tag{2.12}$$

Solving its equation of motion obtain linearized Einstein equation

$$\hat{\varepsilon}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} = \frac{1}{M_P}T_{\mu\nu} \ . \tag{2.13}$$

Since the diffeomorphism is a gauge symmetry, we can fix the gauge by choosing the de Donger gauge or called harmonic gauge

$$\partial^{\mu}h_{\mu\nu} - \frac{1}{2}\partial_{\nu}h = 0 \ . \tag{2.14}$$

The Einstein equation reduces to simply

$$\Box h_{\mu\nu} = -\frac{2}{M_P}(T_{\mu\nu} - \frac{1}{2}T\eta_{\mu\nu}) \ . \tag{2.15}$$

Plug it back to rewrite Lagrangian in the form of $\mathcal{L} = -1/4 h_{\mu\nu}\mathcal{O}^{\mu\nu\alpha\beta}h_{\alpha\beta}$ and use the relation

$$\mathcal{O}^{\mu\nu\alpha\beta}G_{\alpha\beta,\sigma\lambda} = \frac{i}{2}(\delta^{\mu}_{\sigma}\delta^{\nu}_{\lambda} + \delta^{\nu}_{\sigma}\delta^{\mu}_{\lambda}) \ . \tag{2.16}$$

We obtain the propagator for a massless spin-2 field

$$G_{\mu\nu\alpha\beta} = -\frac{i}{p^2}(\eta_{\mu(\alpha}\eta_{\nu\beta)} - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}) \ . \tag{2.17}$$

### Massive propagator

Now let us consider Fierz-Pauli action with non-vanishing mass. The modified Einstein equation is then

$$\hat{\varepsilon}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} + \frac{1}{2} m^2 (h_{\mu\nu} - h\eta_{\mu\nu}) = \frac{1}{M_P} T_{\mu\nu} \ . \tag{2.18}$$

The equation of motion is more complicated than the massless case and may not be easy to derive out the propagator immediately. We thus first consider the trace and the divergence separately [15],

$$h = -\frac{1}{3m^2 M_P}(T + \frac{2}{m^2}\partial_\alpha \partial_\beta T^{\alpha\beta}) \ , \tag{2.19}$$

$$\partial_\mu h^\mu_\nu = \frac{1}{m^2 M_P}(\partial_\mu T^\mu_\nu + \frac{1}{3}\partial_\nu T + \frac{2}{3m^2}\partial_\nu \partial_\alpha \partial_\beta T^{\alpha\beta}) \ . \tag{2.20}$$

We obtain the equation of motion in a more familiar form by plugging them back

$$\begin{aligned}(\Box - m^2) h_{\mu\nu} = &- \frac{1}{M_P}[T_{\mu\nu} - \frac{1}{3} T\eta_{\mu\nu} - \frac{2}{m^2}\partial_{(\mu}\partial_\alpha T^\alpha_{\nu)} + \frac{1}{3m^2}\partial_\mu \partial_\nu T \\ &+ \frac{1}{3m^2}\partial_\alpha \partial_\beta T^{\alpha\beta}\eta_{\mu\nu} + \frac{2}{3m^4}\partial_\mu \partial_\nu \partial_\alpha \partial_\beta T^{\alpha\beta}] \ .\end{aligned} \tag{2.21}$$

One can then derive the generic propagator by defining some new operator, but here we are only interested in the case of massless limit. Taking $m \to 0$ and recall diffeomorphism invariance enforce source to be conserved $\partial_\mu T^\mu_\nu \to 0$ in this limit. The equation then simply reduces to

$$\Box h_{\mu\nu} = -\frac{1}{M_P}[T_{\mu\nu} - \frac{1}{3} T\eta_{\mu\nu}] \ . \tag{2.22}$$

The propagator in massless limit is thus

$$G_{\mu\nu\alpha\beta} = -\frac{i}{p^2}(\eta_{\mu(\alpha}\eta_{\nu\beta)} - \frac{1}{3}\eta_{\mu\nu}\eta_{\alpha\beta}) \ . \tag{2.23}$$

Explicitly, we see there is a discontinuity of coefficient between purely massless propagator 1/2 and massless limit propagator 1/3. It implies the linear Fierz-Pauli should not be the full story of massive gravity. There might be some non-linear interactions that would affect the physical system as gravity mass goes small. This problem is resolved by the Vainshtein mechanism [77] in 1972 as we shall see in Section 4.3.

# 3 Stückelberg field

## 3.1 Massive Photon

We first consider a massive spin-1 particle coupled to a source as a simpler example.

$$S = \int d^D x - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu \ , \tag{3.1}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual field strength tensor. Since the mass term breaks the gauge symmetry $\delta A_\mu = \partial_\mu \Lambda$, this theory propagates 3 degrees of freedom different from 2 for a massless vector. However, the Lagrangian is not smooth in the $m \to 0$ limit: we lost 1 degree of freedom when the theory returned to Maxwell.

To keep track of the would-lost DoFs and restore the gauge symmetry, we introduce a new scalar field $\phi$ and replace $A_\mu$ by

$$A_\mu \to A_\mu + \partial_\mu \phi \ . \tag{3.2}$$

The action is then

$$S = \int d^D x - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 (A_\mu + \partial_\mu \phi)^2 + (A_\mu + \partial_\mu \phi) J^\mu \ . \tag{3.3}$$

It is manifestly invariant under the following simultaneous gauge transformation

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu \Lambda \ , \\ \phi &\to \phi - \Lambda \ . \end{aligned} \tag{3.4}$$

In this replacement, we are not decomposing our field $A_\mu$ into transverse and longitudinal parts as in QED. In QED, such a gauge residual gauge freedom is nothing but redundancy. In contrast, we are adding a new field $\phi$ as a physical longitudinal mode carries the extra 1 degree of freedom for the massive spin-1 particle. To show this statement more explicitly, let us rewrite the Lagrangian by firstly rescaling $\phi \to \frac{1}{m}\phi$ to normalize its kinetic term, and then integrating by parts

$$\begin{aligned} S = \int d^D x - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu - m A_\mu \partial^\mu \phi \\ - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + A_\mu J^\mu - \frac{1}{m} \phi \partial_\mu J^\mu \ . \end{aligned} \tag{3.5}$$

It suggests a massive spin-1 particle is described by 2 fields: helicity-1 vector $A_\mu$ and helicity-0 scalar $\phi$. This Lagrangian thus propagates correct degrees of freedom for the massive spin-1 particle and has the expected gauge symmetry

$$\delta A_\mu = \partial_\mu \Lambda, \delta\phi = -m\Lambda \ . \tag{3.6}$$

We note in the $m \to 0$ limit, the external source should be restricted to be conserved $\partial^\mu J_\mu = 0$ to evade infinity strongly coupling $\frac{1}{m}\phi\partial_\mu J^\mu$. It is consistent with conserved source preserving gauge invariance in purely massless theory. Taking m=0, the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + A_\mu J^\mu \ . \tag{3.7}$$

Clearly, the transverse and longitudinal mode both survive, and no degree of freedom is lost at the limit. The loss of DoF tends to be the origin of the appearance of physical difference between a massive theory with tiny mass and purely massless theory. Here, we show that for massive spin-1 particle, there is no vDVZ discontinuity.

For Maxwell theory, the propagator is simply

$$G_{\mu\nu} = \frac{-i\eta_{\mu\nu}}{p^2} \ . \tag{3.8}$$

For massive photon (3.5), we have the equation of motion

$$(\Box - m^2)A_\mu = -J_\mu, \Box\phi = \frac{1}{m}\partial_\mu J^\mu \ . \tag{3.9}$$

However, we have seen above the source should be conserved and the scalar fully decouple $\Box\phi = \frac{1}{m}\partial_\mu J^\mu \to \Box\phi = 0$. The propagator is therefore obtained by

$$G_{\mu\nu} = \frac{-i\eta_{\mu\nu}}{p^2 + m^2} \ , \tag{3.10}$$

which is smooth in the limit of $m \to 0$.

## 3.2 Stückelberg-ing Graviton

Now let us turn to massive spin-2 particle

$$\mathcal{L} = \mathcal{L}_{kin} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \lambda h_{\mu\nu}T^{\mu\nu} \ . \tag{3.11}$$

The mass term breaks the gauge symmetry $\delta h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}$ for the massless graviton. We introduce Stückelberg field $A_\mu$ to restore it by replacing

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu \ . \tag{3.12}$$

This looks like a gauge transformation, so the kinetic term remains invariant. Integrating by parts, the action becomes

$$
\begin{aligned}
S = & \mathcal{L}_{kin} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}m^2(\partial_\mu A_\nu + \partial_\nu A_\mu)(\partial^\mu A^\nu + \partial^\nu A^\mu) \\
& - m^2 h_{\mu\nu}(\partial^\mu A^\nu + \partial^\nu A^\mu) + 2m^2[(\partial_\mu A^\mu)^2 + h\partial_\mu A^\mu] \\
& + \lambda(h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu)T^{\mu\nu} \\
& = \mathcal{L}_{kin} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} - 2m^2(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu) \\
& + \lambda h_{\mu\nu}T^{\mu\nu} - 2\lambda A_\mu \partial_\nu T^{\mu\nu} \ ,
\end{aligned}
\tag{3.13}
$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ as usual. The Lagrangian is then invariant under the fowling gauge transformation

$$
\begin{aligned}
h_{\mu\nu} &\to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} \ , \\
A_\mu &\to A_\mu - \xi_\mu \ .
\end{aligned}
\tag{3.14}
$$

Normalizing the vector kinetic term by rescaling $A_\mu \to \frac{1}{m}A_\mu$, we see the Lagrangian now propagates a tensor field and a vector field both carry 2 degrees of freedom. However, recall that for a massive spin-2 particle we should have a total of 5 degrees of freedom, so we need secondary Stückelberging $A_\mu$ to evade unsmooth at $m \to 0$ limit

$$
A_\mu \to A_\mu + \partial_\mu \phi \ . \tag{3.15}
$$

The action is then

$$
\begin{aligned}
S = \int & d^4x \mathcal{L}_{kin} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} \\
& - 2m^2(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu) - 2m^2(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi) \\
& + \lambda h_{\mu\nu}T^{\mu\nu} - 2\lambda A_\mu\partial_\nu T^{\mu\nu} + 2\lambda\phi\partial_\mu\partial_\nu T^{\mu\nu} \ .
\end{aligned}
\tag{3.16}
$$

To see this Lagrangian now propagates correct number of degrees of freedom, we first rescale $A_\mu \to \frac{1}{m}A_\mu, \phi \to \frac{1}{m^2}\phi$ to normalize kinetic (or kinetic-like) terms

$$
\mathcal{L} = \mathcal{L}_{kin}[h] - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi) + \mathcal{L}_{mass}[h] + \mathcal{L}_{source}[\phi, A, T, h] \ . \tag{3.17}
$$

The vector kinetic term is explicit, but the scalar is still kinetically mixed with the tensor field. Consider field redefinition

$$
h_{\mu\nu} = h'_{\mu\nu} + \pi\eta_{\mu\nu} \ , \ \phi = \pi \ . \tag{3.18}
$$

The action rearranged as

$$
\begin{aligned}
S = \int & d^4x\mathcal{L}_{kin}[h'] - \frac{1}{2}m^2(h'_{\mu\nu}h'^{\mu\nu} - h'^2) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + 3\pi(\Box + 2m^2)\pi \\
& - 2m^2(h'_{\mu\nu}\partial^\mu A^\nu - h'\partial_\mu A^\mu) + 3(m^2 h'\pi + 2m\pi\partial_\mu A^\mu) \\
& + \lambda h'_{\mu\nu}T^{\mu\nu} + \lambda\pi T - \frac{2}{m}\lambda A_\mu\partial_\nu T^{\mu\nu} + \frac{2}{m^2}\lambda\pi\partial_\mu\partial_\nu T^{\mu\nu} \ .
\end{aligned}
\tag{3.19}
$$

2 gauge symmetries obtained by

$$
\delta h'_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)} + m\Lambda\eta_{\mu\nu} \ , \ \delta A_\mu = -m\xi_\mu \ , \delta A_\mu = \partial_\mu\Lambda \ , \ \delta\pi = -m\Lambda \ . \tag{3.20}
$$

We note in section 2, that only tensor field $h_{\mu\nu}$ couple with an external source. However, here we have an extra interaction $\lambda\pi T$ still survives even in the m=0 limit. This is the origin of the vDVZ discontinuity. Imposing the following gauge conditions [66, 53]

$$
\partial^\nu h'_{\mu\nu} - \frac{1}{2}\partial_\mu h' + mA_\mu = 0 \ , \tag{3.21}
$$

$$
\partial_\mu A^\mu + m(\frac{1}{2}h' + 3\pi) = 0 \ . \tag{3.22}
$$

and diagonalizing the action with the gauge-fixing term

$$
\begin{aligned}
S' + S_{gauge} = \int & d^4x\frac{1}{2}h'_{\mu\nu}(\Box - m^2)h'^{\mu\nu} - \frac{1}{4}h'(\Box - m^2)h' \\
& + A_\mu(\Box - m^2)A^\mu + 3\pi(\Box - m^2)\pi \\
& + \lambda h'_{\mu\nu}T^{\mu\nu} + \lambda\pi T - \frac{2}{m}\lambda A_\mu\partial_\nu T^{\mu\nu} + \frac{2}{m^2}\lambda\pi\partial_\mu\partial_\nu T^{\mu\nu} \ .
\end{aligned}
\tag{3.23}
$$

The propagator of $h'_{\mu\nu}$ is then

$$
G^{h'}_{\mu\nu\alpha\beta} = -\frac{i}{p^2 + m^2}(\eta_{\mu(\nu}\eta_{\alpha\beta)} - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}) \ . \tag{3.24}
$$

which is now smooth at the $m \to 0$ limit. Also, the Lagrangian propagates the correct number of degrees of freedom and preserves DoFs in the massless limit.

At this step so far, we can see how ghost appears if Fierz-Pauli tuning is violated. For the first Stückelberg decomposition $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}A_{\nu)}$, one can no longer obtain an exact product $-1/2m^2F_{\mu\nu}F^{\mu\nu}$ of field stress, but with some additional terms $\sim (\partial A)(\partial A)$. Consequently, Stückelberg scalar appears a kinetic term with four derivatives $\sim (\Box\phi)^2$ which carries two degrees of freedom with one of ghost-like [26, 27].

## 3.3 Non-linear Stückelberg decomposition

### Necessity of GR

So far in our linear Fierz-Pauli action, we are only a prior consider an external source with satisfactory properties. However, one must consider non-linearity to describe nature more completely. In linearized theory, the gravity-matter coupling can be written as

$$\mathcal{L}_{matter}^{linear} = \frac{1}{2M_P} h_{\mu\nu} T_0^{\mu\nu} \ , \tag{3.25}$$

$$T_0^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - \frac{1}{2} (\partial\varphi)^2 \eta^{\mu\nu} \ . \tag{3.26}$$

The free Klein-Gordon equation $\Box\varphi = 0$ guarantees conservation of source $\partial_\mu T_0^{\mu\nu} = 0$ and thus preserves linearized diffeomorphism invariance. However, as an interacting theory, the coupling of matter and gravity enforces the massless scalar should satisfy following modified K-G equation

$$\Box\varphi = \frac{1}{M_P} [\partial^\alpha (h_{\alpha\beta} \partial^\beta \varphi) - \frac{1}{2} \partial_\alpha (h \partial^\alpha \varphi)] \ , \tag{3.27}$$

and the stress-energy tensor hence is no longer conserved

$$\partial_\mu T_0^{\mu\nu} = \frac{1}{M_P} [\partial^\alpha (h_{\alpha\beta} \partial^\beta \varphi) - \frac{1}{2} \partial_\alpha (h \partial^\alpha \varphi)] \partial^\nu \varphi \ . \tag{3.28}$$

It is natural to resolve this by adding non-linear interaction between gravity and matter,

$$\mathcal{L}_{matter}^{full} = \frac{1}{2M_P} h_{\mu\nu} T_0^{\mu\nu} + \frac{1}{2M_P^2} h_{\mu\nu} h_{\alpha\beta} T_1^{\mu\nu\alpha\beta} + \ldots \ . \tag{3.29}$$

This promotes the gauge symmetry from linearized diffeomorphism invariance to the fully non-linear coordinate transformation invariance, i.e., the covariance. Also, the stress-energy tensor enforced to be covariantly conserved $\nabla_\mu T^{\mu\nu} = 0$. It then leads us to GR: the theory of a massless spin-2 particle with a unique (up to Lovelock invariants) fully non-linear kinetic term

$$\mathcal{L}_{kin}^{linear} = -\frac{1}{4} h^{\mu\nu} \hat{\varepsilon}_{\mu\nu}^{\alpha\beta} h_{\alpha\beta} \to \mathcal{L}_{kin}^{covariant} = \frac{M_P^2}{2} \sqrt{-g} R[g] \ . \tag{3.30}$$

Following the above path, we see the general diffeomorphism is not a fundamental property, but rather an essential conclusion for the covariant theory of massless spin-2 particle. The appearance of the coupling itself changes the equation of motion of the matter field, makes the source non-conserved and so breaks linear diffs. Consequently, there are no interactions between matter and gravity to preserve linear diffs. While we successfully wrote self-interactions $h_{\mu\nu}\hat{\epsilon}^{\mu\nu}_{\alpha\beta}h^{\alpha\beta}$ with linear diffs, we should abandon them and only consider fully non-linear kinetic contribution $\sqrt{-g}R[g]$ to make the story continue.

**Stükelberg-ing non-linear gravity**

For a theory of massive gravity, the mass term is built out of fluctuation $h_{\mu\nu}$ which is not transformed as a tensor under diffeomorphism. Extending our massive gravity to a fully non-linear level, the mass term so breaks covariance. Nevertheless, as we have seen in linearized theory, the gauge symmetry is a 'redundancy' and one can always restore it by introducing Stückelberg fields. There are many equivalent ways with different advantages to using Stückelberg trick. Here we use the one which is convenient to keep track of the $h_{\mu\nu}$.

Under general diffeomorphism, the full metric transforms as

$$g_{\mu\nu}(x) \to \frac{\partial f^a}{\partial x^\mu}\frac{\partial f^b}{\partial x^\nu} g_{ab}(f(x)) \ . \tag{3.31}$$

Recall that the kinetic term $\sqrt{-g}R[g]$ is always gauge invariant. To restore the full diffs of the potential mass term built out of the $h_{\mu\nu}$ and leave the full metric unchanged, we consider introducing Stückelberg fields by making simultaneous replacement

$$f_{\mu\nu}(x) \to \tilde{f}_{\mu\nu}(x) = \partial_\mu\phi^a\partial_\nu\phi^b f_{ab}(\phi(x)) \ , \tag{3.32}$$

$$h_{\mu\nu}(x) \to H_{\mu\nu}(x) = (g_{\mu\nu}(x) - \partial_\mu\phi^a\partial_\nu\phi^b f_{ab}(\phi(x)))M_P \ . \tag{3.33}$$

where $\phi^a(x)$ $(a = 0, 1, 2, 3)$ are four scalars transform as

$$\phi^a \to \phi^a(f(x)) \ , \tag{3.34}$$

or infinitesimally,

$$\delta\phi^a = \xi^\nu\partial_\nu\phi^a \ , \ f^\mu(x) = x^\mu + \xi^\mu(x) \tag{3.35}$$

under diffeomorphisms. Now $\tilde{f}$ transform as a tensor under diffs, and so contractions like $g_{\mu\nu}\tilde{f}_{\mu\nu}$ transform as scalars. One can therefore construct gauge invariant potential $U(g, H)$ to introduce a non-linear extension of the Fierz-Pauli mass term.

### Helicity decomposition

We shall show more explicitly how Stückelberg fields $\phi^a$ decompose to different helicity modes. Expanding

$$\phi^a = x^a - \frac{1}{M_P}\chi^a \tag{3.36}$$

and only consider flat reference metric $f_{\mu\nu} = \eta_{\mu\nu}$ for simply, we obtain

$$H_{\mu\nu} = h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)} - \frac{1}{M_P}\eta_{ab}\partial_\mu\chi^a\partial_\nu\chi^b \ . \tag{3.37}$$

Again, splitting $\chi^a$ in terms of the helicity-1 and helicity-0 modes by

$$\chi^a = \frac{1}{m}A^a + \frac{1}{m^2}\eta^{ab}\partial_b\pi \ . \tag{3.38}$$

Then

$$\begin{aligned} H_{\mu\nu} =& h_{\mu\nu} + \frac{2}{m}\partial_{(\mu}A_{\nu)} + \frac{2}{m^2}\Pi_{\mu\nu} \\ & - \frac{1}{M_P m^2}\partial_\mu A^a\partial_\nu A_a - \frac{2}{M_P m^3}\partial_\mu A^a\Pi_{\nu a} - \frac{1}{M_P m^4}\Pi_\mu{}^a\Pi_{\nu a} \ . \end{aligned} \tag{3.39}$$

where $\Pi_{\mu\nu} = \partial_\mu\partial_\nu\pi$ is defined for future convenience and all indices are raised and lowered with respect to reference metric $f_{\mu\nu} = \eta_{\mu\nu}$. As discussed in linearized decomposition, the graviton described by helicity-2 part $h_{\mu\nu}$, helicity-1 part $A_\mu$ and helicity-0 part $\pi$.

## 3.4 Boulware-Deser ghost

A straightforward extension of the non-linear gravity mass term is replacing $\eta \to g, h \to H$ (and considering the change of measure) in the linear Fierz-Pauli mass term

$$\mathcal{L}_{FP} = -m^2M_P^2\ \eta^{\mu\nu}\eta^{\alpha\beta}(h_{\mu\alpha}h_{\nu\beta} - h_{\mu\nu}h_{\alpha\beta}) \tag{3.40}$$

to obtain

$$\mathcal{L}_{FP}^{(nl)} = -m^2M_P^2\ \sqrt{-g}[g^{\mu\nu}g^{\alpha\beta}(H_{\mu\alpha}H_{\nu\beta} - H_{\mu\nu}H_{\alpha\beta})] \ . \tag{3.41}$$

Constructing the tensor quantity

$$\mathbb{X}^\mu{}_\nu = g^{\mu a}\tilde{f}_{a\nu} = \partial^\mu\phi^a\partial_\nu\phi^b f_{ab} \ . \tag{3.42}$$

We can then rewrite (3.41) in a form that is manifestly diffeomorphism invariant

$$\mathcal{L}_{FP}^{(nl)} = -m^2 M_P^2 \sqrt{-g}([(1-\mathbb{X})^2] - [1-\mathbb{X}]^2) \ . \tag{3.43}$$

Focusing on helicity-0 mode $\pi$, we note that

$$\mathbb{X}^\mu{}_\nu \supset -\frac{2}{M_P m^2}\Pi^\mu{}_\nu + \frac{1}{M_P^2 m^4}\Pi^\mu{}_a \Pi^a{}_\nu \ . \tag{3.44}$$

These give terms with higher order derivatives in our extended mass term

$$\begin{aligned} \mathcal{L}_{FP}^{(nl)} \supset & -\frac{4}{m^2}([\Pi^2] - [\Pi]^2) + \frac{4}{M_P m^4}([\Pi^3] - [\Pi][\Pi^2]) \\ & + \frac{1}{M_P^2 m^6}([\Pi^4] - [\Pi^2]^2) \ . \end{aligned} \tag{3.45}$$

While the quadratic term can be viewed as a total derivative after integrating by parts, we still have interactions with higher derivatives in Lagrangian. By Ostrogradski's theorem[71, 83], this implies that the non-linear Firez-Pauli mass term propagates an additional ghostly scalar degree of freedom which called BD ghost. Taking an appropriate background configuration $\pi = \pi_0 + \delta\pi$, one can write

$$\begin{aligned} \mathcal{L}_{FP}^{(nl)} &\supset \frac{4}{M_P m^4} Z^{\mu\nu\alpha\beta}\partial_\mu\partial_\nu\delta\pi\partial_\alpha\partial_\beta\delta\pi \\ Z^{\mu\nu\alpha\beta} &= 3\partial^\mu\partial^\alpha\pi_0\eta^{\nu\beta} - \Box\pi_0\eta^{\mu\alpha}\eta^{\nu\beta} - 2\partial^\mu\partial^\nu\pi_0\eta^{\alpha\beta} + \dots \ . \end{aligned} \tag{3.46}$$

The ghost mass is then depending on the background

$$m_{gh}^2 \sim \frac{M_P m^4}{\partial^2 \pi_0} \ . \tag{3.47}$$

Around a flat background, the mass goes to infinity so the ghost 'freezes'. For this reason one have not seen the BD ghost in linear level as higher order operators $([\Pi^3] - [\Pi][\Pi^2])$ and $([\Pi^4] - [\Pi^2]^2)$ be irrelevant.

One may also consider a functional mass term to evade the BD ghost

$$\mathcal{L}_{FP}^{(nl)} = -m^2 M_P^2 \sqrt{-g} F[g^{\mu\nu}g^{\alpha\beta}(H_{\mu\alpha}H_{\nu\beta} - H_{\mu\nu}H_{\alpha\beta})] \ . \tag{3.48}$$

However, it does not help: the only choice of the function to prevent higher-derivative term $[\Pi^3] - [\Pi][\Pi^2]$ is $F'[0] = 0$ , but this removes the mass term as the price to pay[15].

# 4 Massive Gravity in dRGT theory

We have seen the theories of massive gravity are plagued by BD ghost instability at the non-linear level. Moreover, it is argued in [67, 85] that this instability is inevitable when constructing a gravity mass term. However, in this section we shall see how the ghost could be circumvented in the most well-known ghost-free massive gravity theory found by de Rham, Gabadadze and Tolley[16, 18]. In brief, they take a square root of (3.42) and introduce a tensor

$$\mathcal{K}^{\mu}{}_{\nu}\ [g,f] = \delta^{\mu}_{\nu} - (\sqrt{g^{-1}f}\ )^{\mu}{}_{\nu} \tag{4.1}$$

as a replacement of $1-\mathbb{X}$. Then constructing potential terms as

$$U \sim \sqrt{-g}\sum_{n=0}^{4}\frac{\alpha_n}{n!}\mathcal{L}_n[\mathcal{K}]\ , \tag{4.2}$$

or alternatively,

$$U \sim \sqrt{-g}\sum_{n=0}^{4}\frac{\beta_n}{n!}\mathcal{L}_n[\sqrt{g^{-1}f}\ ]\ , \tag{4.3}$$

where the Lagrangian $\mathcal{L}_n$ defined to

$$\begin{aligned}
\mathcal{L}_0[Q] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha\beta}\ ,\\
\mathcal{L}_1[Q] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu\alpha\beta}\ Q^{\mu'}_{\mu}\ ,\\
\mathcal{L}_2[Q] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha\beta}\ Q^{\mu'}_{\mu}Q^{\nu'}_{\nu}\ ,\\
\mathcal{L}_3[Q] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha'\beta}\ Q^{\mu'}_{\mu}Q^{\nu'}_{\nu}Q^{\alpha'}_{\alpha}\ ,\\
\mathcal{L}_4[Q] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha'\beta'}\ Q^{\mu'}_{\mu}Q^{\nu'}_{\nu}Q^{\alpha'}_{\alpha}Q^{\beta'}_{\beta}\ .
\end{aligned} \tag{4.4}$$

The coefficient relation is

$$\begin{pmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_3\\ \beta_4\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 & 1 & 1\\ 0 & -1 & -2 & -3 & -4\\ 0 & 0 & 2 & 6 & 12\\ 0 & 0 & 0 & -6 & -24\\ 0 & 0 & 0 & 0 & 24\end{pmatrix}\begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}\ . \tag{4.5}$$

In what follows, we prove this theory is indeed absent of BD ghost under general reference metric by ADM formulation. The proof in the Stückelberg language and in the vierbein formulation are referred to [3, 45]and [28, 50], respectively.

## 4.1 Absence of BD ghost in ADM formulation

We start by ADM formulation of GR, the full metric is decomposed to

$$N \equiv (-g^{00})^{-\frac{1}{2}} \ , \ N_i \equiv g_{0i}, \ \gamma_{ij} = g_{ij} \ . \tag{4.6}$$

The $N$ and $N_i$ are the lapse and shift functions respectively. We also define such analog for reference metric for later convenience

$$M \equiv (-f^{00})^{-\frac{1}{2}} \ , \ M_i \equiv f_{0i}, \ {}^3f_{ij} = f_{ij} = f \ . \tag{4.7}$$

The inverse metric in ADM parameterization is

$$g^{\mu\nu} = \begin{pmatrix} -1 & N^j \\ N^i & N^2\gamma^{ij} - N^iN^j \end{pmatrix} \ . \tag{4.8}$$

The Einstein-Hilbert action then reads

$$S_{kin} = M_P^2 \int d^4x[\pi^{ij}\partial_t\gamma_{ij} + NR^0 + N_iR^i] \ , \tag{4.9}$$

$$R^0 = \sqrt{\det\{\gamma\}}[R(\gamma) + \frac{1}{\det\{\gamma\}}(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij})] \ , \tag{4.10}$$

$$R^i = 2\sqrt{\det\{\gamma\}}\nabla_j(\frac{\pi^{ij}}{\sqrt{\det\{\gamma\}}}) \ . \tag{4.11}$$

The proof is based on the statement as follows. In the linear Fierz-Pauli theory, the lapse is a Lagrangian multiple, so its equation of motion provides a modified Hamiltonian constraint while the shifts would be determined in terms of $(\gamma, \pi)$. Expecting this situation is the same in non-linear theory, we suppose there exist three (invertible) functions

$$n_r = n_r(N, N^i, \gamma) \ , \ N_i = (N, n_j, \gamma) \text{ linear in } N \ , \tag{4.12}$$

such that one could rewrite action as

$$S[N, N_i] = \tilde{S}[N, n_r = n_j(N, N_i, \gamma)] \ , \tag{4.13}$$

and varying respect to $n_i$ give N-independent equation of motion so that $n_i$ can be entirely determined only by $(\gamma, \pi)$. That is, $n_i$ can be viewed as 're-defined shifts' variables that take the role of $N_i$ in linear theory. Conversely, one can also determine the $N_i$ as functions of $n_i$. The original equations of motion lead to the equivalent equations

$$\begin{aligned} \frac{\delta S}{\delta N_i} \equiv \left.\frac{\delta \tilde{S}}{\delta n_j}\right|_N \frac{\delta n_j}{\delta N_i} = 0 \ , \ \frac{\delta S}{\delta N} \equiv \left.\frac{\delta \tilde{S}}{\delta n_j}\right|_N \frac{\delta n_j}{\delta N} + \left.\frac{\delta \tilde{S}}{\delta N}\right|_n = 0 \ , \\ \Rightarrow \left.\frac{\delta \tilde{S}}{\delta n_j}\right|_N = 0 \ , \ \left.\frac{\delta \tilde{S}}{\delta N}\right|_n = 0 \ . \end{aligned} \tag{4.14}$$

Accordingly, there are 3 criteria for us to investigate the existence of a primary Hamiltonian constraint [44] :

i) The $n_i$ equations of motion should depend on lapse and shifts only through the three functions $n_i$, i.e., $F[\gamma, \pi, n_i(N, N^i, \gamma)] = 0$.

ii) The $N$ equation of motion could only involve $n_i$ and must be $N$-independent, so substituting the $n_i$ solution it becomes a constraint on $(\gamma, \pi)$. The action $\tilde{S}$ is thus linear in $N$.

iii) Since the action $\tilde{S}$ contains the term $R^i N_i = (N, n_j, \gamma)$, the linearity of $N_i = (N, n_j, \gamma)$ in N implies the expression must also be linear in $N$.

The most generic ghost-free massive gravity action can be written as[41]

$$S = M_P^2 \int d^4x \sqrt{-g}\ [R + 2m^2 \sum_{n=0}^{3} \beta_n\ Q_n(\sqrt{g^{-1}f})]\ , \tag{4.15}$$

with more explicit operators in analogue of (4.4)

$$\begin{aligned}
&Q_0(\mathbb{X}) = 1\ ,\\
&Q_1(\mathbb{X}) = [\mathbb{X}]\ ,\\
&Q_2(\mathbb{X}) = \frac{1}{2}([\mathbb{X}]^2 - [\mathbb{X}^2])\ ,\\
&Q_3(\mathbb{X}) = \frac{1}{6}([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2]\ , +2[\mathbb{X}^{\not\models}])\\
&Q_4(\mathbb{X}) = \frac{1}{24}([\mathbb{X}]^4 - 6[\mathbb{X}]^2[\mathbb{X}^2] + 3[\mathbb{X}^2]^2 + 8[\mathbb{X}][\mathbb{X}^3] - 6[\mathbb{X}^4])\ ,\\
&Q_k(\mathbb{X}) = 0 \text{ for } k > 4\ .
\end{aligned} \tag{4.16}$$

The four coefficients can be determined by 2-free parameters

$$\beta_n = (-1)^n(\frac{1}{2}(4-n)(3-n) - (4-n)\alpha_3 + \alpha_4)\ . \tag{4.17}$$

For simplicity, we first consider minimal massive gravity model ($\beta_0 = 3, \beta_1 = -1, \beta_2 = \beta_3 = 0$) with zero cosmological constant

$$S_{min} = M_P^2 \int d^4x \sqrt{-g}\ [R - 2m^2(\mathrm{Tr}\sqrt{g^{-1}f} - 3)]\ , \tag{4.18}$$

or equivalently,

$$\mathcal{L}_{min} = \pi^{ij}\partial_t\gamma_{ij} + NR^0 + N_iR^i - 2m^2\sqrt{\det\{\gamma\}}N(\mathrm{Tr}\sqrt{g^{-1}f} - 3)\ . \tag{4.19}$$

In ADM parameterization we have

$$
\begin{aligned}
N^2 g^{-1} f &= N^2 N^{-2} \begin{pmatrix} -1 & N^j \\ N^i & N^2\gamma^{ij} - N^i N^j \end{pmatrix} \begin{pmatrix} f_{00} & f_{0j} \\ f_{i0} & f_{ij} \end{pmatrix} \\
&= \begin{pmatrix} -f_{00} + N^j f_{j0} & -f_{0j} + N^j f_{ji} \\ N^i f_{00} + (N^2\gamma^{ij} - N^i N^j) f_{i0} & N^i f_{0j} + (N^2\gamma^{ij} - N^i N^j) f_{ij} \end{pmatrix} \\
&= \begin{pmatrix} -f_{00} + N^l f_{l0} & -f_{0j} + N^l f_{lj} \\ N^2\gamma^{il} f_{l0} - N^i(-f_{00} + N^l f_{l0}) & N^2\gamma^{il} f_{lj} - N^i(-f_{0j} + N^l f_{lj}) \end{pmatrix} .
\end{aligned}
\tag{4.20}
$$

By criteria 3, we suppose the shifts could be expressed by

$$
N^i = c^i(n, \gamma) + N d^i(n, \gamma) , \tag{4.21}
$$

where the variables $c^i$ and $d^i$ are manifestly independent of $N$. On the other hand, criteria 2 implies that the matrix (4.21) should take a form at most including 2 power of $N$

$$
N^2 g^{-1} f = E_0 + N E_1 + N^2 E_2 . \tag{4.22}
$$

Substituting (4.21) to (4.20), one obtains

$$
\begin{aligned}
N^2 g^{-1} f = &\begin{pmatrix} -f_{00} + c^l f_{l0} & -f_{0j} + c^l f_{lj} \\ c^i f_{00} - c^i c^l f_{l0} & c^i f_{0j} - c^i c^l f_{lj} \end{pmatrix} \\
&+ \begin{pmatrix} d^l f_{l0} & d^l f_{lj} \\ d^i f_{00} - d^i c^l f_{l0} - d^l c^i f_{l0} & d^i f_{0j} - d^i c^l f_{lj} - d^l c^i f_{lj} \end{pmatrix} N \\
&+ \begin{pmatrix} 0 & 0 \\ \gamma^{il} f_{l0} - d^i d^l f_{l0} & \gamma^{il} f_{lj} - d^i d^l f_{lj} \end{pmatrix} N^2 .
\end{aligned}
\tag{4.23}
$$

Comparing this to (4.22), these give the results

$$
\begin{aligned}
&E_0 = \begin{pmatrix} a_0 & a_j \\ -a_0 c^i & -a_j c^i \end{pmatrix} , \ E_1 = \begin{pmatrix} d^l f_{l0} & d^l f_{lj} \\ -a_0 d^i - c^i d^l f_{l0} & -a_j d^i - c^i d^l f_{lj} \end{pmatrix} , \\
&E_2 = \begin{pmatrix} 0 & 0 \\ (\gamma^{il} - d^i d^l) f_{l0} & (\gamma^{il} - d^i d^l) f_{lj} \end{pmatrix} ,
\end{aligned}
\tag{4.24}
$$

where we have introduced $a_\mu = -f_{0\mu} + c^l f_{l\mu}$ for compactivity. Equivalently, criteria 2 directly restricted

$$
N\sqrt{g^{-1} f} = A + NB \Rightarrow N^2 g^{-1} f = A^2 + N(AB + BA) + N^2 B^2 . \tag{4.25}
$$

Relations between these matrices then reads

$$
A^2 = E_0 \ , \ AB + BA = E_1 \ , \ B^2 = E_2 . \tag{4.26}
$$

To determine matrix $A$, we note

$$E_0^2 = (a_0 - c^l a_l) \begin{pmatrix} a_0 & a_j \\ -a_0 c^i & -c^i a_j \end{pmatrix} = x E_0 \ , x \equiv a_0 - c^l a_l \ . \tag{4.27}$$

gives $A = E_0/\sqrt{x}$. The matrix $B$ is straightforward to evaluate by

$$B = \begin{pmatrix} 0 & 0 \\ \sqrt{(\gamma^{il} - d^i d^l) f_{lk}} (f^{-1})^{kl} f_{lo} & \sqrt{(\gamma^{il} - d^i d^l) f_{lj}} \end{pmatrix} \ . \tag{4.28}$$

For later convenience, we introduce an important matrix $D$ through

$$\sqrt{x} D^i{}_j \equiv \sqrt{(\gamma^{il} - d^i d^l) f_{lj}} \ , \tag{4.29}$$

so that $B$ can be written in a more compact form

$$B = \sqrt{x} \begin{pmatrix} 0 & 0 \\ D^i{}_k (f^{-1})^{kl} f_{lo} & D^i{}_j \end{pmatrix} \ . \tag{4.30}$$

Also, one can easily check that the matrix $D$ has the following useful symmetric property

$$f_{ik} D^k{}_j = f_{jk} D^k{}_i \ , \tag{4.31}$$

which will be applied in what follow. Plugging the above results in the remaining equation $AB + BA = E$, we obtain

$$\begin{aligned} AB + BA &= \begin{pmatrix} a_0 & a_j \\ -a_0 c^i & -c^i a_j \end{pmatrix} \begin{pmatrix} 0 & 0 \\ D^i{}_k (f^{-1})^{kl} f_{lo} & D^i{}_j \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ D^i{}_k (f^{-1})^{kl} f_{lo} & D^i{}_j \end{pmatrix} \begin{pmatrix} a_0 & a_j \\ -a_0 c^i & -c^i a_j \end{pmatrix} \\ &= \begin{pmatrix} a_i D^i{}_k (f^{-1})^{kl} f_{l0} & a_i D^i{}_j \\ -c^i a_j D^j{}_k (f^{-1})^{kl} f_{l0} & -c^i a_l D^l{}_j \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ a_0 D^i{}_k (f^{-1})^{kl} f_{l0} - a_0 c^i D^i{}_j & a_j D^j{}_k (f^{-1})^{kl} f_{l0} - a_l c^i D^l{}_j \end{pmatrix} \\ &= \begin{pmatrix} a_i D^i{}_k (f^{-1})^{kl} f_{l0} & a_i D^i{}_j \\ a_0 D^i{}_k (f^{-1})^{kl} f_{l0} - a_j c^i D^j{}_k (f^{-1})^{kl} f_{l0} & a_j D^j{}_k (f^{-1})^{kl} f_{l0} - 2 a_l c^i D^l{}_j \end{pmatrix} \\ &= E_1 = \begin{pmatrix} d^l f_{l0} & d^l f_{lj} \\ -a_0 d^i - c^i d^l f_{l0} & -a_j d^i - c^i d^l f_{lj} \end{pmatrix} \ . \end{aligned} \tag{4.32}$$

This gives

$$d^i = c^j D^i{}_j - D^i{}_k (f^{-1})^{kl} f_{l0} = D^i{}_k (c^k - (f^{-1})^{kl} f_{l0}) \ . \tag{4.33}$$

Inspired by the proof in flat space in [43], we take a prior choice of $n^i$ as follows

$$n^i = c^i - (f^{-1})^{il} f_{l0} \ . \tag{4.34}$$

(4.33) then reduced to

$$d^i = D^i{}_j n^j \ . \tag{4.35}$$

By the definition (4.29), it gives

$$\sqrt{x} D^i{}_j \equiv \sqrt{(\gamma^{il} - D^i{}_k n^k D^l{}_h n^h) f_{lj}} \ . \tag{4.36}$$

At this stage, we can now determine all variables $c^i, d^i, D^i{}_j$ in terms of $n^i$ and $\gamma_{if}$ with arbitrary reference metric $f_{\mu\nu}$. In the case of minimal massive gravity, (4.36) can be analytically solved [44]

$$D = (\sqrt{\gamma^{-1} f Q}) Q^{-1} \ , \ Q^{-1} = \frac{1}{M^2 - n^k f_{kl} n^l} (1 - M^2 n n^{\intercal} f) \ . \tag{4.37}$$

Here we have treated the $n^i$ as a column vector and $n^{\intercal}$ as its transpose. The (4.19) in full ADM parametrization is

$$N^i = n^i + M^i + N D^i{}_k n^k \ . \tag{4.38}$$

Substituting it in the Lagrangian gives

$$\begin{aligned} \mathcal{L}_{min} =& \pi^{ij} \partial_t \gamma_{ij} + N R^0 + R^i [n^i + M^i + N D^i{}_k n^k] \\ & - 2m^2 \sqrt{\det\{\gamma\}} [\sqrt{x} + N \sqrt{x} \operatorname{Tr} D - 3N] \ . \end{aligned} \tag{4.39}$$

Varying respect to the $n^k$ using

$$\frac{\partial}{\partial n^k} \sqrt{x} = -\frac{1}{\sqrt{x}} n^j f_{ji} \frac{\partial n^i}{\partial n_k} \ , \tag{4.40}$$

$$\frac{\partial}{\partial n^k} (\sqrt{x} \operatorname{Tr} D) = -\frac{1}{\sqrt{x}} n^j f_{ji} \frac{\partial (D^i{}_l n^l)}{\partial n_k} \ . \tag{4.41}$$

gives the equation of motion

$$(R_i + 2m^2 \sqrt{\det\{\gamma\}} \frac{n^j f_{ji}}{\sqrt{x}}) (\frac{\partial (n^i + N D^i{}_l n^l)}{\partial n_k}) = 0 \ . \tag{4.42}$$

Note the quantity in the square bracket is exactly the Jacobi matrix $J^i{}_k = \frac{\partial N^i}{\partial n^k}$ which is invertible. Neglecting it and recall $\frac{\partial \mathcal{L}_{min}}{\partial n^k} = \frac{\partial \mathcal{L}_{min}}{\partial N^i} \frac{\partial N^i}{\partial n^k}$, the e.o.m reduce to

$$\frac{\partial \mathcal{L}_{min}}{\partial N^i} = R_i + 2m^2 \sqrt{\det\{\gamma\}} \frac{n^j f_{ji}}{\sqrt{x}} = 0 \tag{4.43}$$

and is N-independent as required. Surprisingly, with our choice of $n^i$, the criteria i) is automatically satisfied by imposing criteria ii) and iii). Using the definition in (4.25), $n^i$ could be explicitly determined as

$$n^i = \frac{-M}{\sqrt{4m^4 \det\{\gamma\} + R_k (f^{-1})^{kl} R_l}} (f^{-1})^{ij} R_j \ . \tag{4.44}$$

Finally, varying action with respect to the $N$ to obtain the $N$ equation of motion

$$R^0 + R_i D^i{}_j n^j - 2m^2 \sqrt{\det\{\gamma\}} [\sqrt{x}\, \mathrm{Tr}\, D - 3] = 0 \ . \tag{4.45}$$

Using the expression of $x$ and $D$ and the solution (4.44), this becomes a primary Hamiltonian constraint on the $(\gamma, \pi)$ and reduces the total number of phase freedom from 12 to 11.

The secondary constraint is given by the time evaluation of the primary constraint. Integrating out shifts $N^i$, one can read off Hamiltonian

$$\mathcal{L}_{min} = \pi^{ij} \partial_t \gamma_{ij} - \mathcal{H}_0(\gamma_{ij}, \pi^{ij}, f) + NC(\gamma_{ij}, \pi^{ij}, f) \ , \tag{4.46}$$

$$H = \int d^3x (\mathcal{H}_0 - NC) \ . \tag{4.47}$$

The secondary constraint is then given by

$$\frac{d}{dt} C = \{C, H\} \ . \tag{4.48}$$

The detailed evaluation of the Poisson bracket is refereed to [42]. In brief, it shows that $\frac{dC}{dt}$ is independent of $N$ so become the secondary constraint on the $(\gamma, \pi)$. Furthermore, it also argued that there are no additional constraints are generated.

As a footnote, the Hamiltonian

$$\mathcal{H}_{min} = M \sqrt{4m^4 \det\{\gamma\} + R_k (f^{-1})^{kl} R_l} - R_i M^i \tag{4.49}$$

is manifestly positive for any reference metric $f_{\mu\nu}$ with $M^i = 0$ and $M > 0$, in particular, it is positive for flat space $f_{\mu\nu} = \eta_{\mu\nu}$ ($M = 1,\ M^i = 0$).

Now we generalize our conclusion to the generic 2-parameter theory of massive gravity

$$\mathcal{L} = \pi^{ij} \partial_t \gamma_{ij} + N R^0 + N_i R^i + 2m^2 \sqrt{\det\{\gamma\}} N (\sum_{n=0}^{3} \beta_n Q_n(\sqrt{g^{-1} f})) \ . \tag{4.50}$$

In the above discussion we have considered operators $Q_0$ and $Q_1$, so in what follows we study the remaining $Q_2$ and $Q_3$ to complete the proof.

We start by writing these in terms of $A$ and $B$ matrices we have determined

$$NQ_2(\sqrt{g^{-1}f}) = \operatorname{Tr} A \operatorname{Tr} B - \operatorname{Tr} AB + \frac{1}{2}N[(\operatorname{Tr} B)^2 - \operatorname{Tr} B^2] \ , \tag{4.51}$$

$$\begin{aligned} NQ_2(\sqrt{g^{-1}f}) =& \operatorname{Tr} AB^2 - \operatorname{Tr} AB \operatorname{Tr} B + \frac{1}{2} \operatorname{Tr} A[(\operatorname{Tr} B)^2 - \operatorname{Tr} B^2] \\ &+ \frac{1}{6}N[(\operatorname{Tr} B)^3 - 3 \operatorname{Tr} B \operatorname{Tr} B^2 + 2 \operatorname{Tr} B^3] \ . \end{aligned} \tag{4.52}$$

Varying the action with respect to $n^i$, we found that the $R^i N_i$ in the kinetic part gives a contribution

$$R_i \frac{\partial N^i}{\partial n^k} = R_i \frac{\partial}{\partial n^k}(n^i + N D^i{}_j n^j) \ , \tag{4.53}$$

which is proportional to the Jacobi matrix. To keep $N$ as a Lagrangian multiple, each of the potential terms should also give a contribution proportional to $J^i{}_k$ as well. We shall verify this in what follows. Using auxiliary variables we defined, express

$$\begin{aligned} &\operatorname{Tr} A = \sqrt{x}, \operatorname{Tr} B = \sqrt{x} \operatorname{Tr} D, \\ &\operatorname{Tr} B^2 = x \operatorname{Tr} D^2, \ \operatorname{Tr} B^3 = x^{\frac{3}{2}} \operatorname{Tr} D^3 \ , \\ &\operatorname{Tr} AB = -n^\intercal f D n \ , \ \operatorname{Tr} AB^2 = -\frac{1}{\sqrt{x}} n^\intercal f(\gamma^{-1} - D n n^\intercal D^\intercal) f n \ . \end{aligned} \tag{4.54}$$

Varying these gives

$$\begin{aligned} &\frac{\partial}{\partial n^k} \operatorname{Tr} A = -\frac{1}{\sqrt{x}} n^\intercal f \frac{\partial n}{\partial n^k} \ , \\ &\frac{\partial}{\partial n^k} \operatorname{Tr} B = -\frac{1}{\sqrt{x}} n^\intercal f \frac{\partial (Dn)}{\partial n^k} \ , \\ &\frac{\partial}{\partial n^k} \operatorname{Tr} B^2 = -2 n^\intercal f D \frac{\partial (Dn)}{\partial n^k} \ , \\ &\frac{\partial}{\partial n^k} \operatorname{Tr} B^3 = -3\sqrt{x} n^\intercal f D^2 \frac{\partial (Dn)}{\partial n^k} \ , \\ &\frac{\partial}{\partial n^k} \operatorname{Tr} AB = -n^\intercal f D \frac{\partial n}{\partial n^k} - n^\intercal f \frac{\partial (Dn)}{\partial n^k} \ , \\ &\frac{\partial}{\partial n^k} \operatorname{Tr} AB^2 = -\frac{1}{\sqrt{x}} (n^\intercal f D^2 n) n^\intercal f \frac{\partial n}{\partial n^k} - 2\sqrt{x} n^\intercal f D^2 \frac{\partial n}{\partial n^k} \\ &\qquad + \frac{2}{\sqrt{x}} (n^\intercal f D n) n^\intercal f \frac{\partial (Dn)}{\partial n^k} \ . \end{aligned} \tag{4.55}$$

Combine them to the (4.51) and (4.52), gets

$$\frac{\partial}{\partial n^k} NQ_2(\sqrt{g^{-1}f}) = n^\intercal f[D - \operatorname{Tr} D] \frac{\partial}{\partial n^k}(n + NDn) \ , \tag{4.56}$$

$$\begin{aligned} &\frac{\partial}{\partial n^k} NQ_3(\sqrt{g^{-1}f}) \\ &= -\sqrt{x} n^\intercal f[D^2 - D \operatorname{Tr} D + \frac{1}{2}((\operatorname{Tr} D)^2 - \operatorname{Tr} D^2)] \frac{\partial}{\partial n^k}(n + NDn) \ . \end{aligned} \tag{4.57}$$

Indeed, both satisfy the requirement as expected. So, varying the action, gives N-independent equations of motion

$$\begin{aligned} &R_i - 2m^2 \sqrt{\det\{\gamma\}} \frac{n^l f_{lj}}{\sqrt{x}} [\beta_1 \delta_i^{\ j} + \beta_2 \sqrt{x} (\delta_i^{\ j} D^m_{\ m} - D^j_{\ i}) \\ &+ \beta_3 x (\frac{1}{2} \delta_i^{\ j} (D^m_{\ m} D^n_{\ n} - D^m_{\ n} D^n_{\ m}) + D^j_{\ m} D^m_{\ i} - D^j_{\ i} D^m_{\ m})] = 0 \ , \end{aligned} \tag{4.58}$$

$$\begin{aligned} &R^0 + R_i D^i_j n^j + 2m^2 \sqrt{\det\{\gamma\}} [\beta_0 + \beta_1 \sqrt{x} D^j_{\ i} + \frac{1}{2} \beta_2 x (D^i_{\ i} D^j_{\ j} - D^i_{\ j} D^j_{\ i}) \\ &+ \frac{1}{6} \beta_3 x^{\frac{3}{2}} (D^i_{\ i} D^j_{\ j} D^k_{\ k} - 3 D^i_{\ i} D^j_{\ k} D^k_{\ j} + 2 D^i_{\ j} D^j_{\ k} D^k_{\ i})] = 0 \ . \end{aligned} \tag{4.59}$$

It may be difficult to solve them exactly as in the minimal case. However, it is enough to solve them perturbatively to show these indeed give a primary Hamiltonian constraint and reduced phase freedom. The proof of secondary constraint (4.48) is also given in [42]. Consequently, these two constraints eliminate the BD ghost in generic dRGT theory with an arbitrary reference metric.

## 4.2 Decoupling limit and $\Lambda_3$ scale

The generic interaction term included in the potential of massive gravity theories can be written in terms of helicity-0 $\pi$, helicity-1 $A_\mu$ and helicity-2 modes $h_{\mu\nu}$

$$\begin{aligned} \mathcal{L}_{j,k,l} &= m^2 M_P^2 (\frac{h}{M_P})^j (\frac{\partial A}{m M_P})^{2k} (\frac{\partial^2 \pi}{m^2 M_P})^l \\ &= \Lambda_{j,k,l}^{j+4k+3l-4} \ \ h^j (\partial A)^{2k} (\partial^2 \pi)^l \ , \end{aligned} \tag{4.60}$$

at the scale

$$\Lambda_{jkl} = (m^{2k+2l-2} \ M_P^{j+2k+l-2} \ \ )^{\frac{1}{j+4k+3l-4}} \ . \tag{4.61}$$

Manifestly, the first interaction involving dangerous operator $(\partial^2 \pi)^3$ excited at the scale $\Lambda_{j=0,k=0,l=3} \equiv \Lambda_5 = (M_P m^4)^{1/5}$. At and beyond this scale, such interactions will lead BD ghost instability. In what follows, we shall see the ghost-free massive gravity also rise the scale from $\Lambda_5$ to $\Lambda_{j=k=0,\ l\to\infty} = \Lambda_3 = (M_P m^2)^{\frac{1}{3}}$.

As in this case, we interested in interactions that only contain helicity-0 mode (j=k=0), so it is sufficient to consider

$$g^{\mu\nu}\Big|_{h=0} = \eta^{\mu\nu} \ , \tag{4.62}$$

$$\tilde{f}_{\mu\nu}\Big|_{h=A=0} = \eta_{\mu\nu} - \frac{2}{M_P m^2}\Pi_{\mu\nu} + \frac{1}{M_P^2 m^4}\eta^{\alpha\beta}\Pi_{\mu\alpha}\Pi_{\nu\beta} \ , \tag{4.63}$$

so the tensor introduced in (4.1) reads

$$\mathcal{K}^{\mu}{}_{\nu}\Big|_{h=A=0} = \frac{1}{M_P m^2}\Pi^{\mu}{}_{\nu} \ . \tag{4.64}$$

Setting $\alpha_0 = \alpha_1 = 0$ to obtain zero cosmological constant and tadpole, so the flat space is the vacuum solution. The potential $U[\mathcal{K}]$ up to the scale is then

$$\begin{aligned}\mathcal{L}_{mass} &= \frac{m^2 M_P^2}{4}\sqrt{-g}\Big|_{h=0}\sum_{n=2}^{4}\alpha_n\mathcal{L}_n[\mathcal{K}]\Big|_{h=A=0} \\ &= \frac{m^2 M_P^2}{4}\sum_{n=2}^{4}\alpha_n\mathcal{L}_n[\frac{\Pi_{\mu\nu}}{M_P m^2}] \\ &= \frac{1}{4}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha'\beta'}(\frac{\alpha_2}{m^2}\delta^{\mu'}_{\mu}\delta^{\nu'}_{\nu} + \frac{\alpha_3}{M_P m^4}\delta^{\mu'}_{\mu}\Pi^{\nu'}_{\nu} + \frac{\alpha_4}{M_P^2 m^6}\Pi^{\mu'}_{\mu}\Pi^{\nu'}_{\nu})\Pi^{\alpha'}_{\alpha}\Pi^{\beta'}_{\beta} \ .\end{aligned} \tag{4.65}$$

Since all these terms are total derivatives, interactions involving higher derivatives of the form $(\partial^2\pi)^l$ are thus harmless and would not lead to an Ostrogradsky instability. This proves the BD ghost is absent in dRGT theory in Stückelberg language, at least at a scale below to $\Lambda_3$. Actually, this is how ghost-free massive gravity was originally constructed in [16, 18]. At the scale $\Lambda_3$ or higher, more interactions with $(\partial^2\pi)^l$ excited and the BD ghost may still reappear. However, as we have proved in ADM parametrization in the last subsection, the 2-parameter dRGT theory is free of BD instability without any scaling restriction.

### Decoupling limit

Sending $M_P \to \infty$, $m \to 0$ and maintaining the scale $\Lambda_3$ fixed, we could investigate the theory at the scale $\Lambda_3$. The new interactions that arise are

$$j = 1 \ , \ k = 0 \ , \quad \forall\, l \geqslant 2 \tag{4.66}$$

and

$$j = 0 \ , \ k = 1 \ , \quad \forall \, l \geqslant 1 \ . \tag{4.67}$$

Here we are interested in helicity-2 and -0 coupling so focus on the case of (4.66) to see the interactions

$$\mathcal{L}^{DL}_{mass} = h_{\mu\nu} \bar{X}_{\mu\nu} \ , \tag{4.68}$$

where by [16, 18, 17],

$$\begin{aligned} \bar{X}_{\mu\nu} &= \frac{\delta}{\delta h_{\mu\nu}} \mathcal{L}_{mass} \Big|_{h=A=0} \\ &= \frac{M_P^2 m^2}{4} \frac{\delta}{\delta h_{\mu\nu}} (\sqrt{-g} \sum_{n=2}^{4} \alpha_n \mathcal{L}_n[\mathcal{K}]) \Big|_{h=A=0} \ . \end{aligned} \tag{4.69}$$

Using the relation

$$\frac{\delta \mathcal{K}^n}{\delta h_{\mu\nu}} \Big|_{h=A=0} = \frac{n}{2} (\Pi^{n-1}_{\mu\nu} - \Pi^n_{\mu\nu}) \ , \tag{4.70}$$

we obtain

$$\bar{X}_{\mu\nu} = \frac{\Lambda_3^3}{8} \sum_{n=2}^{4} \alpha_n ( \frac{4-n}{\Lambda_3^{3n}} X^{(n)}_{\mu\nu} \ [\Pi] + \frac{n}{\Lambda_3^{3(n-1)}} X^{(n-1)}_{\mu\nu} \ [\Pi]) \ , \tag{4.71}$$

with

$$\begin{aligned} & X^{(0)}_{\mu\nu}[Q] = 3! \eta_{\mu\nu} \ , \\ & X^{(1)}_{\mu\nu}[Q] = 2!([Q]\eta_{\mu\nu} - Q_{\mu\nu}) \ , \\ & X^{(2)}_{\mu\nu}[Q] = ([Q]^2 - [Q^2])\eta_{\mu\nu} - 2([Q]Q_{\mu\nu} - Q^2_{\mu\nu}) \ , \\ & X^{(3)}_{\mu\nu}[Q] = ([Q]^3 - 3[Q][Q^2] + 2[Q^3])\eta_{\mu\nu} \\ & \qquad\qquad - 3([Q]^2 Q_{\mu\nu} - 2[Q]Q^2_{\mu\nu} - [Q^2]Q_{\mu\nu} + 2Q^3_{\mu\nu}) \ , \\ & X^{(n \geqslant 4)}_{\mu\nu}[Q] = 0 \ . \end{aligned} \tag{4.72}$$

For more detail for the operators $X^{(n)}$ we refer the reader to appendix A in [47]. In the decoupling limit $M_P \to \infty$ the full metric $g_{\mu\nu} = \eta_{\mu\nu} + M_P^{-1} h_{\mu\nu}$ reduces to Minkowski metric and the standard Einstein-Hilbert kinetic term reduces to linearized version $h\varepsilon h$. Neglecting the helicity-1 modes, the full $\Lambda_3$-decoupling limit Lagrangian of ghost-free massive gravity (with flat reference metric) is

$$\mathcal{L}_{\Lambda_3} = \frac{1}{4} h^{\mu\nu} \hat{\varepsilon}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} + \frac{1}{8} h^{\mu\nu} (2\alpha_2 X^{(1)}_{\mu\nu} + \frac{2\alpha_2 + 3\alpha_3}{\Lambda_3^3} X^{(2)}_{\mu\nu} + \frac{\alpha_3 + 4\alpha_4}{\Lambda_3^6} X^{(3)}_{\mu\nu}) \ . \tag{4.73}$$

The tensors $X^{(n)}_{\mu\nu}$ are transverse and the equations of motion with respect to $h$ and $\pi$ involve derivatives no more than two. So, the theory is free of BD ghost instability in the decoupling limit and healthy even beyond this limit.

For completeness, we also give the full decoupling limit Lagrangian with vector modes [70]

$$\begin{aligned}
\mathcal{L}^{(0)}_{\Lambda_3} =& -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + \frac{1}{2}h^{\mu\nu}\sum_{n=1}^{3}\frac{a_n}{\Lambda_3^{3(n-1)}}X^{(n)}_{\mu\nu} \\
&+ \frac{3\beta_1}{8}\delta^{\alpha\beta\gamma\delta}_{abcd}\delta^a_\alpha\left(\delta^b_\beta F^c_\gamma\omega^d_\delta + 2\left[\omega^b{}_\beta\omega^c_\gamma + \frac{1}{2}\delta^b_\beta\omega^c{}_\mu\omega^\mu_\gamma\right](\delta+\Pi)^d_\delta\right) \\
&+ \frac{\beta_2}{8}\delta^{\alpha\beta\gamma\delta}_{abcd}(\delta+\Pi)^a_\alpha\left(2\delta^b_\beta F^c_\gamma\omega^d_\delta + \left[\omega^b{}_\beta\omega^c{}_\gamma + \delta^b_\beta\omega^c{}_\mu\omega^\mu_\gamma\right](\delta+\Pi)^d_\delta\right) \\
&+ \frac{\beta_3}{48}\delta^{\alpha\beta\gamma\delta}_{abcd}(\delta+\Pi)^a_\alpha(\delta+\Pi)^b_\beta\left(3F^c_\gamma\omega^d_\delta + \omega^c{}_\mu\omega^\mu{}_\gamma(\delta+\Pi)^d_\delta\right) ,
\end{aligned} \tag{4.74}$$

where the auxiliary Lorentz Stückelberg fields are introduced

$$\begin{aligned}
\omega_{ab} &= \int_0^\infty \mathrm{d}u e^{-2u}e^{-u\Pi^a_a{}'}F_{a'b'}e^{-u\Pi_b{}^{b\,\prime}} \\
&= \sum_{n,m}\frac{(n+m)!}{2^{1+n+m}\ n!m!}(-1)^{n+m}\left(\Pi^n\ F\ \Pi^m\right)_{ab} \ ,
\end{aligned} \tag{4.75}$$

with $F_{ab} = \partial_a A_b - \partial_b A_a$.

## 4.3 Vainshtein mechanism

### Vainshtein mechanism in $\Lambda_5$ theories

At the linear level, we have seen the helicity-0 mode couples to matter would give an extra fifth force and leads to vDVZ discontinuity. The resolution is well-known as the Vainshtein mechanism, which relies strongly on non-linearities. In theories with scale $\Lambda_5$, the BD ghost play a sufficient role in response to it.

In the decoupling limit

$$m \to 0\ ,\ M_P \to \infty\ ,\ T \to \infty\ , \qquad \Lambda_5\ ,\ \frac{T}{M_P}\ \text{ fixed}\ , \tag{4.76}$$

where T is the trace of matter source. The action contains only helicity-0 mode reads (Up to a total derivative)

$$S_\phi = \int d^4x - 3(\partial\pi)^2 + \frac{2}{\Lambda_5^5}[(\Box\pi)^3 - (\Box\pi)(\partial_\mu\partial_\nu\pi)^2] + \frac{1}{M_P}\pi T\ . \tag{4.77}$$

Around a source of mass M, the spherical solution at linear order goes like

$$\pi \sim \frac{M}{M_P}\frac{1}{r} \ . \tag{4.78}$$

The non-linearities become important until

$$r_V \sim (\frac{M}{M_P^2 m^4})^{\frac{1}{5}} \ , \tag{4.79}$$

which is called the Vainshtein radius. In section 3, we see the ghost has mass

$$m_{ghost}^2 \ (r) \sim \frac{M_P m^4}{\partial^2 \pi_0(r)} = \frac{\Lambda_5^5}{\partial^2 \pi_0(r)} \ . \tag{4.80}$$

We could not ignore the ghost in our EFT once its mass drops below scale $\Lambda_5$, so equivalently, it appears inside the radius

$$r_{ghost} \sim (\frac{M}{M_P})^{\frac{1}{3}}\frac{1}{\Lambda_5} \gg r_V \sim (\frac{M}{M_P})^{\frac{1}{5}}\frac{1}{\Lambda_5} \ . \tag{4.81}$$

Far outside the Vainshtein radius, the field remains the usual Coulombic form (4.74), while the non-linearities dominate inside the Vainshtein radius. We conclude

$$\begin{cases} \pi \sim \frac{M}{M_P}\frac{1}{r} & , \ r \gg r_V \\ \pi \sim (\frac{M}{M_P})^{\frac{1}{2}} \Lambda_5^{\frac{5}{2}} r^{\frac{3}{2}} & , \ r \ll r_V \end{cases} \ . \tag{4.82}$$

At distances much below the Vainshtein radius, the exciting ghost with a negative kinetic term mediates a long-range repulsive force as its mass goes small. This force exactly cancels the attractive force mediated by the healthy helicity-0 mode (a more explicit formulaic discussion referred to [29]).

As a result, we restore the general relativity inside the Vainshtein radius. In particular, in the massless limit $m \to 0$ the Vainshtein radius goes to infinity, which becomes the origin of the vDVZ discontinuity. Beyond the Vainshtein radius, the ghost 'frozen' so the helicity-0 mode generates a fifth force which is known as a screening mechanism for rendering a light scalar inactive at short distances through non-linearities [48, 49, 37].

This ghost-based mechanism is strongly relying on instability and may be viewed as only resolving a problem by introducing another. However, the radius of quantum correction is actually the same as the ghost radius. So, one may expect the unknown quantum effect would finally cure the ghost problem.

### Vainshtein mechanism in $\Lambda_3$ theories

As mentioned above, the resolution of vDVZ discontinuity relies on ghost instability in $\Lambda_5$ theories. However, we have seen theories that are free of ghosts at a scalar below $\Lambda_3$. Therefore, the Vainshtein mechanism in $\Lambda_3$ theories must be constructed by other methods and in fact depend on the source.

To see how this works, it is sufficient to consider the cubic Galileon term for $\pi$ self-interaction

$$\mathcal{L}^{(3)} = -\frac{1}{2}(\partial\pi)^2 - \frac{1}{\Lambda^3}(\partial\pi)^2\Box\pi + \frac{1}{M_P}\pi T \ . \tag{4.83}$$

Perturbatively, expanding

$$\pi = \pi_0 + \phi \ , \ T = T_0 + \delta T \ , \tag{4.84}$$

where $\pi_0$ is the background profile generated by the background source $T_0$ and the perturbation $\phi$ response to the fluctuation $\delta T$. In this background configuration

$$\begin{aligned} \mathcal{L} = &- \frac{1}{2}(\partial\phi)^2 + \frac{2}{\Lambda_3^3}(\partial_\mu\partial_\nu\pi_0 - \eta_{\mu\nu}\Box\pi_0)\partial^\mu\phi\partial^\nu\phi \\ &- \frac{1}{\Lambda_3^3}(\partial\phi)^2\Box\phi + \frac{1}{M_P}\phi\delta \ T \ . \end{aligned} \tag{4.85}$$

We introduce a new effective metric

$$Z^{\mu\nu} = \eta^{\mu\nu} + \frac{2}{\Lambda_3^3}X^{(1)\mu\nu}(\Pi_0) \ . \tag{4.86}$$

The Lagrangian then reduces to

$$\mathcal{L}^{(3)} = -\frac{1}{2}Z^{\mu\nu}(\Pi_0)\partial_\mu\phi\partial_\nu\phi + \frac{1}{M_P}\phi\delta T \ . \tag{4.87}$$

Symbolically, we see $Z \sim 1 + \partial^2\pi_0/\Lambda_3^3 \sim \partial^2\pi_0/\Lambda_3^3 \gg 1$ for large sources. By the nice argument in [68], one should also scale the space-like coordinates $x \to \hat{x}$ when canonically normalizing

$$\hat{\phi} = \sqrt{2}\phi \ . \tag{4.88}$$

However, here it is sufficient for us to see the essence of the Vainshtein mechanism by simply scaling $\phi \to \hat{\phi}$ to obtain

$$\mathcal{L}^{(3)} = -\frac{1}{2}(\partial\hat{\phi})^2 + \frac{1}{M_P\sqrt{2}}\hat{\phi}\delta T \ . \tag{4.89}$$

This manifestly shows the fluctuation-matter coupling $\hat{\phi}\delta T$ is considerable only at the effective scale

$$\Lambda_{eff} \equiv M_P\sqrt{2} \gg M_P \tag{4.90}$$

and thus, it is very suppressed by a large source. In the gravity massless limit $m \to 0$, our scale sent to zero and so $\Lambda_{eff} \to \infty$ as Z goes infinity. We therefore restored the continuity with GR.

Considering a point-like background source

$$T_0 = -\frac{M}{4\pi r^2}\delta(r) \ . \tag{4.91}$$

The Vainshtein radius is then

$$r_V \sim (\frac{M}{4\pi M_P})^{\frac{1}{3}}\frac{1}{\Lambda_3} \ . \tag{4.92}$$

We conclude the spherical solution of cubic Galileon

$$\pi(r) \sim \begin{cases} \frac{M}{M_P}\frac{1}{4\pi r^2} & , \ r \gg r_V \\ \frac{M}{M_P} r_v^{-\frac{3}{2}}\frac{1}{4\pi r^{1/2}} & , \ r \ll r_V \ . \end{cases} \tag{4.93}$$

At distances much larger than the Vainshtein radius, we recover a gravitational strength fifth force mediated by $\pi$. While at a short distance

$$\frac{F_\pi}{F_{Newton}} \sim (\frac{r}{r_V})^{\frac{3}{2}} \ll 1 \quad \text{for} \quad r \ll r_V \ . \tag{4.94}$$

This fifth force is extremely small compared to the standard gravity. As the end of this section, the two following diagram show more explicitly the regimes of different physical effects for $\Lambda_5$ and $\Lambda_3$ theory, respectively.

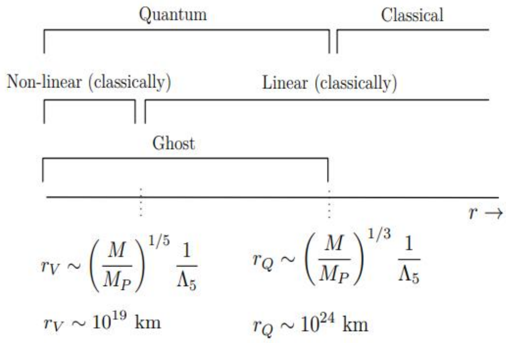


Figure 1: Regimes for $\Lambda_5$ massive gravity

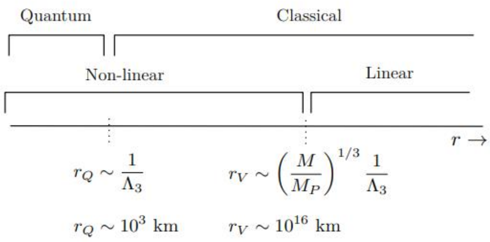


Figure 2: Regimes for $\Lambda_3$ massive gravity

# 5 Other theories of massive gravity

## 5.1 DGP Model

The Dvali-Gabadadze-Porrati Model [33, 34, 32] introduce gravity in a four-dimensional braneworld by construct an extra infinite-size dimension. The standard matter fields lying on the brane induce a curvature term. By integrating out the extra dimension, a momentum-dependent gravity mass term (or resonance) naturally arises in the effective 4d action.

Supposing a brane localized at y=0, the DGP 5d action is

$$S = \int d^4x\, dy \left( \frac{M_5^3}{4} \sqrt{-^{(5)}g}\ ^{(5)}R + \delta(y)[\frac{M_P^2}{2}\sqrt{-g}R[g] + \mathcal{L}_{matter}(g, \psi_i)] \right) , \tag{5.1}$$

where $M_5$ is the 5d plank mass and $\sqrt{-^{(5)}g}$, $^{(5)}R$ correspond to 5d metric $^{(5)}g$. The reason we choose the coefficient $M_5^3/4$ rather than $M_5^2/2$ is that we have been considering the whole extra dimension as a convention.

Varying with respect to the metric gives five-dimensional Einstein equation

$$M_5^3\ {}^{(5)}G_{AB} = 2\delta(y)(T_{\mu\nu} - M_P^2 G_{\mu\nu})\delta_A^\mu \delta_B^\nu \tag{5.2}$$

with the stress-energy tensor $T_{\mu\nu}$ generated by matter fields $\psi_i$ confined on the brane. In what follows, we shall see how this action gives rise to a graviton mass. We start by expending the 5d metric perturbatively,

$$ds_5^2 = (\eta_{AB} + h_{AB})dx^A dx^B \tag{5.3}$$

and fix gauge by choosing 5d de Donder gauge $\partial_A h^A{}_B = 1/2\partial_B h^A{}_A$, the 5d Einstein tensor then reduces to

$$G_{AB} = -\frac{1}{2}\Box_5(h_{AB} - \frac{1}{2}h^c{}_c\eta_{AB})\ , \quad \Box_5 \equiv \Box + \partial_y^2\ . \tag{5.4}$$

The Einstein equation (5.2) manifestly implies ${}^{(5)}G_{\mu y} ={}^{(5)} G_{yy} = 0$, so

$$\begin{aligned}
&\begin{cases} \Box_5 h_{\mu y} = 0 \\ \Box_5(h_{yy} - \frac{1}{2}h^c{}_c) = \Box_5(h_{yy} - \frac{1}{2}h_{yy} - \frac{1}{2}h^\mu{}_\mu) \end{cases} \\
\Rightarrow &\begin{cases} h_{\mu y} = 0 \\ h_{yy} = h^\mu{}_\mu \end{cases} .
\end{aligned} \tag{5.5}$$

Using them, we can then drive the 4-dimensional part of de Donder gauge

$$\partial_\mu h^\mu{}_\nu = \partial_\nu h^\mu{}_\mu\ . \tag{5.6}$$

Substituting these relations in the Einstein equation (5.2) gives

$$-\frac{1}{2}M_5^3(\Box + \partial_y^2)(h_{\mu\nu} - h\eta_{\mu\nu}) = \delta(y)[2T_{\mu\nu} + M_P^2(\Box h_{\mu\nu} - \partial_\mu\partial_\nu h)]\ . \tag{5.7}$$

Schematically, the solution of this equation would exponentially suppress along the extra dimension

$$h_{\mu\nu}(x,y) \sim e^{-|y|\sqrt{-\Box}}\ h_{\mu\nu}(x)\ . \tag{5.8}$$

Integrating (5.7) along dimension y form $-\epsilon$ to $+\epsilon$, the equation of motion on the braneworld then becomes

$$M_P^2[(\Box h_{\mu\nu} - \partial_\mu\partial_\nu h) - \frac{M_5^3}{M_P^2}\sqrt{-\Box}(h_{\mu\nu} - h\eta_{\mu\nu})] = -2T_{\mu\nu}\ . \tag{5.9}$$

We found that $h_{\mu\nu} - h\eta_{\mu\nu}$ term appearing in equation of motion is exactly Fierz-Pauli combination. In a view of momentum Fourier space, gravity can be regarded as having an effective mass

$$m^2(\Box) = \frac{M_5^3}{M_P^2}\sqrt{-\Box} \sim m^2(k) = \frac{M_5^3}{M_P^2}k\ , \tag{5.10}$$

which strongly depends on the scale. Following the similar procedure in section 2.3, one can express the fluctuation $h$ as

$$h_{\mu\nu} = -\frac{2}{M_P^2}\frac{1}{\Box - m_0\sqrt{-\Box}}(T_{\mu\nu} - \frac{1}{3}T\eta_{\mu\nu} + \frac{1}{3m\sqrt{-\Box}}\partial_\mu\partial_\nu T) \ . \tag{5.11}$$

It contains the term $1/3T\eta_{\mu\nu}$ as in the case in (2.22), while in GR one should obtain a factor $1/2T\eta_{\mu\nu}$. This again explicitly shows the vDVZ discontinuity in DGP, which would also be cured by the Vainshtein mechanism [30].

**Stückelberg decomposition**

In the ADM parametrization, the metric split to

$$\begin{pmatrix} N^2 + N^\mu N_\mu & N_\mu \\ N_\mu & g_{\mu\nu} \end{pmatrix} \tag{5.12}$$

with the lapse $N = 1 + 1/2h_{yy}$ and the shift $N_\mu = g_{\mu y}$. The 5d Einstein-Hilbert kinetic term becomes

$$\mathcal{L}^{(5)}_{E-H} = \frac{M_5^3}{4}\sqrt{-g}N(R[g] + [\mathcal{K}]^2 - [\mathcal{K}^2]) \ , \tag{5.13}$$

with extrinsic curvature

$$\mathcal{K}_{\mu\nu} = \frac{1}{2N}(\partial_\nu g_{\mu\nu} - \nabla_\mu N_\nu - \nabla_\nu N_\mu) \ , \tag{5.14}$$

where $\nabla_\mu$ is the covariant derivative with respect to metric $g_{\mu\nu}$. The gauge-fixing term for de Donder gauge then expressed as

$$\begin{aligned}\mathcal{L}^{(5)}_{GF} &= -\frac{M_5^3}{8}(\partial_A h^A{}_B - \frac{1}{2}\partial_B h^A{}_A)^2 \\ &= -\frac{M_5^3}{8}[(\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h + \partial_y N_\nu - \frac{1}{2}\partial_\nu h_{yy})^2 \\ &\quad + (\partial_\mu N^\mu - \frac{1}{2}\partial_y h - \frac{1}{2}\partial_y h_{yy})^2] \ .\end{aligned} \tag{5.15}$$

Following the discussion in [59], we fix the residual linearized gauge symmetry of the bulk on the brane

$$\begin{aligned}\delta h_{\mu\nu} &= 2\partial_{(\mu}\xi_{\nu)} \ , \\ \delta N_\nu &= -\sqrt{-\Box}\xi_\mu \ , \\ \delta h_{yy} &= 0 \ ,\end{aligned} \tag{5.16}$$

by introducing the residual gauge-fixing term

$$\mathcal{L}^{(4)}_{GF} = -\frac{M_P^2}{4}(\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h + \frac{M_5^3}{M_P^2}N_\nu)^2 \ . \tag{5.17}$$

Performing field redefinitions in the analogy of Stückelberg decomposition (and normalizing) in Fierz-Pauli massive gravity

$$\begin{aligned} h_{\mu\nu} &= \frac{1}{M_P}(h'_{\mu\nu} + \pi\eta_{\mu\nu}) \ , \\ N_\mu &= \frac{1}{M_P\sqrt{m_0}}N'_\mu + \frac{1}{M_P m_0}\partial_\mu\pi \ , \\ h_{yy} &= -\frac{2\sqrt{-\Box}}{m_0 M_P}\pi \end{aligned} \tag{5.18}$$

with $m_0 = M_5^3/M_P^2$. We then obtain the linearized action in terms of helicity-2 $h'$ , -1 $N'$ and -0 $\pi$ modes, omitting the mass terms [59]

$$S_{DGP}^{lin} = \frac{1}{4}\int d^4x[\frac{1}{2}h'^{\mu\nu}\Box(h'_{\mu\nu} - \frac{1}{2}h'\eta_{\mu\nu}) - N'^{\mu}\sqrt{-\Box}N'_\mu + 3\pi\Box\pi] \ . \tag{5.19}$$

**Decoupling limit of DGP**

The operators included in boundary terms have a generic form

$$\Lambda_{n,k,l}\ \partial(h'_{\mu\nu})^n(N'_\mu)^k(\partial\pi)^l \ , \tag{5.20}$$

$$\Lambda_{n,k,l} = (M_P^{n+k+l-2}\ \ m_0^{k/2+l-1}\ )^{\ 1/(n+3/2k+2l-3)} \ . \tag{5.21}$$

The first dangerous operator $(\partial\pi)^3$ involving higher power of derivatives arises at scale $\Lambda_{n=0,k=0,l=3} \equiv \Lambda_3 = (M_P m_0^2)^{1/3}$. Taking the decoupling limit

$$m_0 \to 0 \ , \ M_P \to \infty \ , \quad \Lambda_3 \text{ fixed} \ . \tag{5.22}$$

The relevant helicity-0 self-interaction comes from the combination $(\partial\pi)^3$

$$\mathcal{L}_{\Lambda_3}^{DL} = \frac{1}{2\Lambda_3^3}(\partial\pi)^2\Box\pi \ . \tag{5.23}$$

Combining to (5.19), we obtain the decoupling limit of DGP at scale $\Lambda_3$

$$\mathcal{L}_{DGP}^{DL} = \frac{1}{8}h'^{\mu\nu}\Box(h'_{\mu\nu} - \frac{1}{2}h'\eta_{\mu\nu}) - \frac{1}{4}N'^{\mu}\sqrt{-\Box}N'_\mu + \frac{3}{4}\pi\Box\pi + \frac{1}{2\Lambda_3^3}(\partial\pi)^2\Box\pi \ . \tag{5.24}$$

## 5.2 Multi-gravity and Bi-gravity

### Multi-gravity

In the construction of the DGP model, we have integrated over the whole extra auxiliary dimension. Nevertheless, one can also consider a discretization of the 5th dimension, which will lead us to multi-gravity[50].

We start by considering N=2M+1 sites $y_i$ discretized from the extra dimension and their corresponding metric $g_i$ dominant for each site. The simplest multi-gravity action can be written as

$$S_{N\ mGR} = \frac{M_4^2}{2}\sum_{i=1}^{N}\int d^4x\sqrt{-g_i}\left(R[g_i] + \frac{m_N^2}{2}\sum_{n=0}^{4}\alpha_n^{(i)}\mathcal{L}_n[\mathcal{K}^\mu{}_\nu[g_i, g_{i+1}]]\right) . \tag{5.25}$$

where $M_4^2 = M_5^3/m$, $\alpha_2^{(i)} = -1/2$ and $\alpha_0^{(i)} = \alpha_1^{(i)} = 0$ to remain zero cosmological constant and non-tadpole. In this special case, one can clearly see from the form of the tensor $\mathcal{K}^\mu{}_\nu[g_i, g_{i+1}]$ that we have restricted each metric $g_i$ only interacts with its closest neighbours $g_{i-1},\ g_{i+1}$. Then we reduce the number of free parameters to 2N with $\alpha_3^{(i)} = (r_i + s_i),\ \alpha_4^{(i)} = r_i s_i$.

At the linear level, this multi-gravity action reads in terms of the Fourier transformed field variables $\tilde{h}_n$ as following

$$\mathcal{L} = \sum_{n=-M}^{M}[(\partial\tilde{h}_n)(\partial\tilde{h}_{-n}) + m_n^2\ \tilde{h}_n\ \tilde{h}_{-n}] + \mathcal{L}_{int} . \tag{5.26}$$

The mass spectrum is then given by

$$m_n = m_N \sin\left(\frac{n}{N}\right) . \tag{5.27}$$

It is manifestly the theory contains 2M massive spin-2 fields and one massless spin-2 field. To count the degrees of freedom, we note each massive spin-2 field contributes 5 DoFs while a massless spin-2 field contributes 2 DoFs. In addition, as the zero mode of the lapse and the shift contribute 3 DoFs [73], so we have a total of 5N degrees of freedom for the theory in 4 space-time coordinates.

### Bi-gravity

Bi-gravity is the special case of multi-gravity with only 2-site. While it can also be precisely interpreted as the ghost-free massive gravity with the now dynamical reference metric and so obtain an additional Einstein-Hilbert term in the Lagrangian. The bi-gravity action is then simply

$$\begin{aligned} S_{bi} =& \frac{M_P^2}{2}\int d^4x\sqrt{-g}R[g] + \frac{M_f^2}{2}\int d^4x\sqrt{-f}R[f] \\ &+ \frac{M_P^2 m^2}{4}\int d^4x\sqrt{-g}\sum_{n=0}^{4}\alpha_n\mathcal{L}_n[\mathcal{K}[g,f]] , \end{aligned} \tag{5.28}$$

where $\alpha_0 = \alpha_1 = 0, \ \alpha_2 = 1/2$ for consistency. For most generic cases, one can artificially manipulate the ratio $M_P/M_f$ by considering a non-trivial discretization by changing the 'weight' of each site or performing a conformal rescaling of the metrics, etc. To obtain the mass spectrum, we first expand two metrics into fluctuations about flat spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_P}\delta g_{\mu\nu} \ , \tag{5.29}$$

$$f_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_f}\delta f_{\mu\nu} \ . \tag{5.30}$$

To quadratic order, the (pseudo) linear bi-gravity action reads

$$S_{bi}^{(2)} = \int d^4x[-\frac{1}{4}\delta g^{\mu\nu}\tilde{\varepsilon}_{\mu\nu}^{\alpha\beta}\delta g_{\alpha\beta} - \frac{1}{4}\delta f^{\mu\nu}\tilde{\varepsilon}_{\mu\nu}^{\alpha\beta}\delta f_{\alpha\beta} - \frac{1}{8}m_{eff}^2\ (h_{\mu\nu}^2 - h^2)] \ . \tag{5.31}$$

The combination $h_{\mu\nu}^2 - h^2$ is exactly the Fierz-Pauli mass term. The corresponding massive spin-2 field (eigenstates) is

$$h_{\mu\nu} = (M_P^{-2} + M_f^{-2})^{-\frac{1}{2}}(\frac{1}{M_P}\delta g_{\mu\nu} - \frac{1}{M_f}\delta f_{\mu\nu}) \tag{5.32}$$

with the effective mass

$$m_{eff}^2 \ = m^2(1 + \frac{M_P^2}{M_f^2}) \ . \tag{5.33}$$

Extracting the kinetic term $h\varepsilon h$ from the non-mass part and rewriting the linear action to

$$S_{bi}^{(2)} = \int d^4x[-\frac{1}{4}h^{\mu\nu}\tilde{\varepsilon}_{\mu\nu}^{\alpha\beta}h_{\alpha\beta} - \frac{1}{4}l^{\mu\nu}\tilde{\varepsilon}_{\mu\nu}^{\alpha\beta}l_{\alpha\beta} - \frac{1}{8}m_{eff}^2\ (h_{\mu\nu}^2 - h^2)] \ . \tag{5.34}$$

We obtain the remaining massless spin-2 field $l_{\mu\nu}$ represented by the other combination

$$l_{\mu\nu} = (M_P^{-2} + M_f^{-2})^{-\frac{1}{2}}(\frac{1}{M_f}\delta g_{\mu\nu} + \frac{1}{M_P}\delta f_{\mu\nu}) \ . \tag{5.35}$$

In the case that the ratio $M_P/M_f \ll 1$, the massless and massive eigenstates are dominated by the fluctuations $\delta f$ and $\delta g$ respectively. Therefore, in the decoupling limit

$$M_f \to \infty \ , \quad M_P \ \text{ fixed} \tag{5.36}$$

we recover the ghost-free massive gravity with a single interacting massive graviton and a fully decoupled massless graviton.

### 5.3 Mass-varying gravity

The mass-varying gravity interprets the graviton mass as an effective potential of external scalar fields. This idea could be performed to various massive gravity theories, for instance, dRGT theory and bi-gravity. More generically, it can further generalize the formulation to varying all parameters $\alpha_n \to \alpha_n(\phi)$ with multiple fields $\phi_A$ $(A = 1, 2, 3, ..., N)$ [52]

$$\begin{aligned}\mathcal{L}_{MV}^{Generalized} =& \frac{M_P^2}{2} \int \mathrm{d}^4x \sqrt{-g} \left[ \Omega\left(\phi_A\right) R + \frac{1}{2} \sum_{n=0}^{4} \alpha_n\left(\phi_A\right) \mathcal{L}_n[\mathcal{K}] \right. \\ & \left. -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi_A \partial_\nu \phi^A - W\left(\phi_A\right) \right] \ ,\end{aligned} \tag{5.37}$$

where the tensor $\mathcal{K}$ is constructed as usual. The absence of BD ghost is presented in ADM formulation by constraint analysis in analogy to the argument in section 4.1 [51, 52]. A flexible graviton mass could lead to many interesting features, such as avoiding the Higuchi bound [46]. It is natural to consider the graviton mass(es) would depend on some moduli if gravity is an effective description from higher dimensions.

# 6 Positivity bounds for scalar

The unitarity, analyticity and crossing symmetry could imply strong constraints for the UV completion of EFTs. In this section, we will show how to derive these constraints into an infinity number of positivity bounds by using the known properties of 2-2 scattering amplitudes. The exact amplitudes are in principle incalculable since the explicit information of UV physics remains unclear. Nevertheless, the positivity bounds may practically be evaluated on the tree level at the low energy scale.

### 6.1 Dispersion relation

It is convenient to study the 2-2 scattering amplitude by expressing it in terms of the Mandelstam variables [61]

$$\begin{aligned} & s = -(p_1 + p_2)^2 = -(p_3 + p_4)^2 \ , \\ & t = -(p_1 - p_3)^2 = -(p_4 - p_2)^2 \ , \\ & s = -(p_1 - p_4)^2 = -(p_3 - p_2)^2 \ , \\ & \cos\theta = 1 + \frac{2t}{s - 4m^2} \end{aligned} \tag{6.1}$$

with respect to the Minkowski metric $\eta(-1,1,1,1)$. The physical requirement of unitarity gives the optical theorem

$$Im\ A(s,0) = \sqrt{s(s-4m^2)}\sigma(s)\ , \tag{6.2}$$

where $\sigma(s)$ is the total cross-section. This implies the generalization of Schwarz reflection principle

$$A^{A+B\to C+D}\ \ (s)^* = A^{C+D\to A+B}\ \ (s^*)\ . \tag{6.3}$$

Assuming the scattering is time reversal invariant, the above condition gives

$$\begin{aligned} A(s+i\varepsilon) - A(s-i\varepsilon) &= Re\ A(s+i\varepsilon) + iIm\ A(s+i\varepsilon) \\ &\quad - Re\ A(s-i\varepsilon) - iIm\ A(s-i\varepsilon) \\ &= 2iIm\ A(s+i\varepsilon)\ . \end{aligned} \tag{6.4}$$

Defining the s-channel 'absorptive' part of arbitrary function $f$ as

$$Abs_s\ f(s) = \frac{1}{2i} Disc\ f(s) = \frac{1}{2i}\lim_{\varepsilon\to 0}\left[f(s+i\varepsilon) - f(s-i\varepsilon)\right]\ ,\ \text{ for } s \gg 4m^2\ . \tag{6.5}$$

Then the absorptive part of scattering amplitude is exactly the imaginary part

$$Abs_s\ A(s) = Im\ A(s)\ . \tag{6.6}$$

To drive the dispersion relation, we start by assuming that the scattering amplitude is an analytic function of s with modulo poles and branch cuts in the usual places. By Cauchy's integral formula we have

$$A(s,t) = \frac{1}{2\pi i}\oint_c ds' \frac{A(s',t)}{s'-s}\ . \tag{6.7}$$

One can deform the contour $C$ inside the region in which $A$ is analytic as in Figure 3, and make use of (6.6) to obtain

$$\begin{aligned} A(s,t) =& \frac{\lambda}{m^2-s} + \frac{\lambda}{m^2-u} + \int_{C^{\pm}_{\infty}} ds' \frac{A(s',t)}{s'-s} \\ &+ \int_{4m^2}^{\infty} \frac{d\mu}{\pi}\left(\frac{Im\ A(\mu,t)}{\mu-s} + \frac{Im\ A(\mu,t)}{\mu-u}\right)\ , \end{aligned} \tag{6.8}$$

with the pole residues $\lambda = Res_{u=m^2}\ A(s,t) = -Res_{s=m^2}\ A(s,t)$ guaranteed by crossing symmetry. It is important to note that the contour integrals along semicircle $C^{\infty}_{\pm}$ in the upper/lower half plane are not finite in the limit $s' \to \infty$.

This suggests we perform two subtractions. Picking an arbitrary subtraction point $\mu_p$ which could be chosen for convenience, using the following identity

$$\begin{aligned}\frac{Im\ A(\mu,t)}{\mu-s} =&\frac{(s-\mu_p)^2}{(\mu-\mu_p)^2}\frac{Im\ A(\mu,t)}{\mu-s}+2\frac{(s-\mu_p)}{(\mu-\mu_p)^2}Im\ A(\mu,t)\\ &+\frac{(\mu-s)}{(\mu-\mu_p)^2}Im\ A(\mu,t)\ .\end{aligned} \tag{6.9}$$

One may rewrite the amplitude as

$$\begin{aligned}A(s,t) =&a(t)+\frac{\lambda}{m^2-s}+\frac{\lambda}{m^2-u}\\ &+\int_{4m^2}^{\infty}\frac{d\mu}{\pi}\left[\frac{(s-\mu_p)^2 ImA(\mu,t)}{(\mu-\mu_p)^2(\mu-s)}+\frac{(\mu-\mu_p)^2 ImA(\mu,t)}{(\mu-\mu_p)^2(\mu-\mu)}\right]\ ,\end{aligned} \tag{6.10}$$

where $a(t)$ absorbs all the remaining integral contributions and can be determined up to a constant by $t \leftrightarrow s$ crossing symmetry $A(s,t) = A(t,s)$. To continue to construct the positivity bounds, we shall first prove the positivity of t derivatives at the forward limit $t = 0$ in the physical region

$$\partial_t^n Im\ A(s,0) \equiv \frac{\partial^n}{\partial t^n} Im\ A(s,t)\bigg|_{t=0} > 0\ ,\quad \forall n \geqslant 0 \text{and } s \geqslant 4m^2\ . \tag{6.11}$$

Firstly, from the partial wave expansion of scattering amplitude implied by the optical theorem

$$A(s,t) = 16\pi\sqrt{\frac{s}{s-4m^2}}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)a_l(s)\ , \tag{6.12}$$

we obtain

$$\partial_t^n Im\ A(s,0) = 16\pi\sqrt{\frac{s}{s-4m^2}}\frac{2^n}{(s-4m^2)^n}\sum_{l=n}^{\infty}(2l+1)P_l^n(1)Im(a_l(s))\ . \tag{6.13}$$

Using the property of the Legendre polynomials $P_l^n = \partial_t^n P_l(1+t)\Big|_{t=0} \geqslant 0$, together with $Im(a_l(s)) =\mid a_l(s)\mid^2 + ... \geqslant 0$, it is straightforward to see that[1]

$$\partial_t^n Im\ A(s,0) \geqslant 0\ ,\quad \forall n\ ,\ \text{ for } s \geqslant 4m^2\ . \tag{6.14}$$

Up to this stage, one still need to consider the possibility that $\partial_t^n Im\ A(s,0) = 0$ for some $n_*$. However, we can rule out this possibility by the assumption of

[1] Also for n=0, non-trivial theories imply that $Im\ A(0,s) > 0$

analyticity [20]. The situation can be achieved by imposing $0 = Im(a_l(s)) \geqslant \mid a_l \mid^2$ for $l \geqslant n_*$, i.e., $a_l(s) = 0$ for $l \geqslant n_*$ so that $\partial_t^n A(s,0) = 0$ for $n \geqslant n_*$. We now assume that there exist the smallest $n_*$ satisfies $\partial_t^n Im\ A(s,0) = 0$ and consider dispersion relation (6.10) with choice $\mu_p = 0$

$$\begin{aligned} A(s,t) =& a(t) + \frac{\lambda}{m^2 - s} + \frac{\lambda}{-3m^2 + t + s} + s^2 \int_{4m^2}^{\infty} \frac{d\mu}{\pi} \frac{Im\ A(\mu,t)}{\mu^2(\mu - s)} \\ &+ (4m^2 - t - s)^2 \int_{4m^2}^{\infty} \frac{d\mu}{\pi} \frac{Im\ A(\mu,t)}{\mu^2(\mu - 4m^2 + t + s)} . \end{aligned} \tag{6.15}$$

Differentiating twice by s, get

$$\begin{aligned} \partial_s^2 A(s,t) =& \frac{2\lambda}{(m^2 - s)^3} + \frac{2\lambda}{(-3m^2 + t + s)^3} \\ &+ 2 \int_{4m^2}^{\infty} \frac{d\mu}{\pi} \frac{Im\ A(\mu,t)}{(\mu - s)^3} + 2 \int_{4m^2}^{\infty} \frac{d\mu}{\pi} \frac{Im\ A(\mu,t)}{(\mu - 4m^2 + t + s)^3} , \end{aligned} \tag{6.16}$$

then differentiating by t to obtain

$$\begin{aligned} \partial_t^{n_*} \partial_s^2 A(s,t) =& \frac{(2+n_*)!}{2!} \frac{2(-1)^{n_*}\lambda}{(-3m^2 + t + s)^{3+n_*}} \\ & 2 \sum_{m=0}^{n_*-1} \int_{4m^2}^{\infty} \frac{d\mu}{\pi} \frac{(-1)^{n_*-m} \partial_t^m Im\ A(\mu,t)}{(\mu - 4m^2 + t + s)^{3+n_*-m}} \frac{n_*!(2+n_*-m)!}{2!m!(n_*-m)!} + \dots , \end{aligned} \tag{6.17}$$

where the omitted terms or their t derivatives would vanish at $t = 0$ as in what follows. Acting the operator $(\partial_t - \partial_s)$ to reduces series

$$(-1)^{n_*}(\partial_t - \partial_s)^{n_*-1} \partial_s^{n_*} \partial_s^2 A(s,t) = 2 \int_{4m^2}^{\infty} \frac{d\mu}{\pi} \frac{\partial_t^{n_*-1} Im\ A(\mu,t)}{(\mu - 4m^2 + t + s)^{3+n_*}} \frac{(2+n_*)!}{2!n_*!} + \dots . \tag{6.18}$$

We note since by our assumption $n_*$ is the lowest value for $\partial_t^n Im\ A(s,0) = 0$, thus there must exist some region of $M \geqslant 4m^2$ with $\partial_t^{n_*-1} Im\ A(\mu,t) > 0$. Taking the forward limit t=0, the right-hand side of (6.18) is positive however the left-hand side equal to 0 by assumption, thus giving the contradiction

$$0 > 0 . \tag{6.19}$$

We therefore proved the positivity (6.11) in the physical region. Furthermore, for a scalar theory, we expect the amplitude to have a simple t-channel pole at $T = M^2$ with a necessarily real residue so that $Im\ A(s,t)$ has no poles at $t = m^2$. Consequently, $Im\ A(s,t)$ is then analytic in the region $\mid t \mid < 4m^2$, and we can therefore extend our positivity beyond the forward limit

$$\partial_t^n Im\ A(s,t) > 0 \ , \ \forall n \ , \quad \text{for } s \geqslant 4m^2 \text{ and } 0 \leqslant t < 4m^2 . \tag{6.20}$$

Finally, we remove all three poles $s, t, u = m^2$ by define

$$B(s,t) = A(s,t) - \frac{\lambda}{m^2 - s} - \frac{\lambda}{m^2 - u} - \frac{\lambda}{m^2 - t} \ . \tag{6.21}$$

For future convenience, define notations and variables

$$\bar{x} := x - \frac{4}{3}m^2 \ , \tag{6.22}$$

$$\bar{v} = \bar{s} + \frac{\bar{t}}{2} \ , \tag{6.23}$$

and choosing $\bar{\mu_p} = -1/2\bar{t}$, we obtain the final form of the dispersion relation as a function of $v^2$

$$B(s,t) = \tilde{B}(v(s,t),t) = b(t) + \int_{4m^2}^{\infty} \frac{d\mu}{\pi(\bar{\mu} + \frac{\bar{t}}{2})} \frac{2v^2 Im\ A(\mu,t)}{(\bar{\mu} + \frac{\bar{t}}{2})^2 - v^2} \ , \tag{6.24}$$

with redefined subtraction function $b(t) = a(t) - \frac{\lambda}{m^2 - t}$. The $s \leftrightarrow u$ symmetry then goes to $v \leftrightarrow -v$ symmetry.

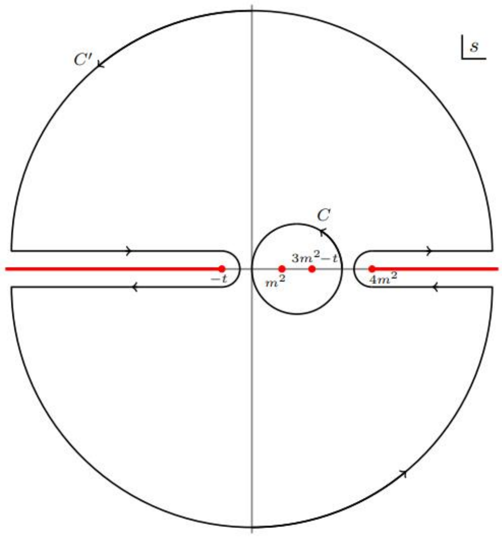


Figure 3: The scattering amplitude can be analytically continued to the entire complex s plane, with the poles at $s = m^2$ and $3m^2 - t$ and branch cuts along the real axis from $-t$ to $-\infty$ and from $4m^2$ to $\infty$

## 6.2 Positivity bounds

The left-hand side of (6.24) is the pole subtracted amplitude that can be evaluated at $s \sim m^2$ in the low energy effective field theory (LEEFT) while the right-hand side is fully depending on the possible UV completion and cannot be explicitly computed at this stage. However, given the positive condition (6.20) implied by unitarity and analyticity, we may expect there exist translated constraints on low energy amplitudes. With the form of the positivity (6.20), It is natural to consider that this information is included in derivatives of the amplitude, we therefore define

$$B^{(2N,M)}\ (t) = \frac{1}{M!}\partial_v^{2N}\partial_t^M \tilde{B}(v,t)\bigg|_v = 0\ ,\ \text{for}\ N \geqslant 1\ . \tag{6.25}$$

Explicitly,

$$B^{(2N,M)}\ (t) = \sum_{k=0}^{M} \frac{(-1)^k}{k!2^k} I^{(2N+k,M-k)}\ , \tag{6.26}$$

with manifestly positive integrals

$$I^{(q,p)}(t) = \frac{q!}{p!}\frac{2}{\pi}\int_{4m^2}^{\infty} \frac{d\mu\ \partial_t^p\ Im\ A(\mu,t)}{(\bar{\mu}+\frac{t}{2})^{q+1}} > 0\ . \tag{6.27}$$

Taking M=0 we see the simplest case with no t derivatives

$$B^{(2N,0)}\ (t) = I^{(2N,0)}\ (t)\ >\ 0\ . \tag{6.28}$$

The situation is more subtle for higher t derivatives: due to the sign structure $(-1)^k$ in series, one can not immediately construct a string of positivity constraints $B^{(2N,M)} \not> 0$. Alternatively, we need to construct a new series of quantities by recursion relation. Firstly, we note the integral inequality

$$I^{(q,p)} < \frac{1}{\mathcal{M}^2} I^{(q-1,p)} < \frac{q}{\mathcal{M}^2} I^{(q-1,p)}\ , \tag{6.29}$$

$$\mathcal{M}^2 = \text{Min}_{\mu \geqslant 4m^2}\ (\bar{\mu} + \frac{\bar{t}}{2}) = 2m^2 + \frac{1}{2}t\ . \tag{6.30}$$

Now consider single t derivative amplitude

$$B^{(2N,1)} = I^{(2N,1)} - \frac{1}{2} I^{(2N+1,1)} > I^{(2N,1)} - \frac{2N+1}{2\mathcal{M}^2} I^{(2N,0)}\ . \tag{6.31}$$

Using the property (6.28), we may view $Y^{(2N,0)} = B^{(2N,0)}$ as the first element of our list and define the second element as

$$\begin{aligned} Y^{(2N,1)} &= B^{(2N,1)} + \frac{2N+1}{2\mathcal{M}^2} I^{(2N,0)} \\ &= B^{(2N,1)} + \frac{2N+1}{2\mathcal{M}^2} Y^{(2N,0)} > I^{(2N,1)} > 0 \end{aligned} \tag{6.32}$$

and thus positive. Taking a step forward, we have, for a second t derivative

$$B^{(2N,2)} = I^{(2N,2)} - \frac{1}{2} I^{(2N+1,1)} + \frac{1}{8} I^{(2N+2,0)} \ . \tag{6.33}$$

It is then straightforward to perform addition and get a positive quantity. Since

$$\frac{2N+1}{2\mathcal{M}^2} Y^{(2N,1)} > \frac{2N+1}{2\mathcal{M}^2} I^{(2N,1)} > I^{(2N+1,1)} \ , \tag{6.34}$$

so

$$B^{(2N,2)} + \frac{2N+1}{2\mathcal{M}^2} Y^{(2N,1)} > I^{(2N,2)} + \frac{1}{8} I^{(2N+2,0)} > 0 \ . \tag{6.35}$$

This looks sufficient however we can take one more step forward to further restrict our bound by defining the manifestly positive quantity as follows

$$Y^{(2N,2)} = B^{(2N,2)} + \frac{2N+1}{2\mathcal{M}^2} Y^{(2N,1)} - \frac{1}{8} B^{(2(N+1),0)} \quad > I^{(2N,2)} > 0 \ . \tag{6.36}$$

From the above example, we show the positive quantities have the following schematical form

$$B^{(2N,M)} + B^{(2(N+1),M-2)} \quad + ... + B^{(2(N+1),0)} \quad + ... > I^{(2N,M)} > 0 \ , \tag{6.37}$$

where we have omitted the coefficients in front of $B$ and $I$. This motivated us to consider a generic linear combination splitting into odd and even parts

$$\begin{aligned}
\sum_{r=0}^{M/2} c_r B^{(2N+2r,M-2r)} &= \sum_{r=0}^{M/2} c_r \sum_{k=0}^{M-2r} \frac{(-1)^k}{k! 2^k} I^{(2N+2r+k,M-2r-k)} \\
&= \sum_{r=0}^{M/2} c_r \Bigg[ \sum_{k=0}^{M/2-2r} \frac{(-1)^{2\lambda}}{2\lambda! 2^{2\lambda}} I^{(2N+2r+\lambda,M-2(r+\lambda))} \\
&\qquad + \sum_{k=0}^{(M-1)/2-2r} \frac{(-1)^{2\lambda+1}}{(2\lambda+1)! 2^{2\lambda+1}} I^{(2N+2r+\lambda+1,M-2(r+\lambda)-1)} \Bigg] \\
&= \sum_{r=0}^{M/2} c_r \Bigg[ \sum_{r=0}^{k} \frac{2^{2(r-k)}}{(2k-2r)!} I^{(2N+2k,M-2k)} \\
&\qquad + \sum_{r=0}^{k} \frac{2^{2(r-k)-1}}{(2k-2r+1)!} I^{(2N+2k+1,M-2k-1)} \Bigg] \\
&= \sum_{k=0}^{M/2} \alpha_k I^{(2N+2k,M-2k)} \quad - \sum_{k=0}^{(M-1)/2} (-1)^k \beta_k I^{(2N+2k+1,M-2k-1)} \ ,
\end{aligned} \tag{6.38}$$

$$\alpha_k = \sum_{r=0}^{k} c_r \frac{2^{2(r-k)}}{(2k-2r)!}, \beta_k = (-1)^k \sum_{r=0}^{k} c_r \frac{2^{2(r-k)-1}}{(2k-2r+1)!} \ . \tag{6.39}$$

Now we precisely choose the coefficients so that the quantity has a similar structure to the right-hand side of (6.37), which requires

$$\alpha_0 = 1 \ , \ \alpha_k = 0 \qquad \text{for } k \neq 0 \ . \tag{6.40}$$

Simultaneously,

$$c_0 = \alpha_0 = 1 \ , \ \sum_{r=0}^{k-1} \frac{2^{2(r-k)}}{(2k-2r)!} c_r \ + \ c_k = \alpha_k = 0 \Rightarrow c_k = -\sum_{r=0}^{k-1} \frac{2^{2(r-k)}}{(2k-2r)!} c_r \ . \tag{6.41}$$

Since $\beta_k \geqslant 0$, we can perform addition to obtain

$$\begin{aligned} &\sum_{r=0}^{M/2} c_r B^{(2N+2r,M-2r)} \quad + \sum_{even\ k}^{(M-1)/2} \beta_k \ I^{(2N+2k+1,M-2k-1)} \\ &= I^{(2N,M)} + \sum_{k\ odd}^{(M-1)/2} \beta_k \ I^{(2N+2k+1,M-2k-1)} \quad > I^{(2N,M)} > 0 \ , \end{aligned} \tag{6.42}$$

which is manifestly positive. Using the inequalities (6.29) and (6.34), we finally construct positivity quantities

$$\begin{aligned} Y^{(2N,M)} =& \sum_{r=0}^{M/2} c_r B^{(2N+2r,M-2r)} \\ &+ \frac{1}{\mathcal{M}^2} \sum_{even\ k}^{(M-1)/2} (2(N+k)+1)\beta_k \ Y^{(2N+2k,M-2k-1)} \\ &\geqslant I^{(2N,M)} > 0 \ , \end{aligned} \tag{6.43}$$

which is also consistent with the previous example with respect to M=0, M=1 and M=2 case.

## 6.3 Massive Galileon EFT

The generalization of the non-linear Feriz-Pauli action acquires higher derivative interactions of helicity-0 mode $\pi$ which are in the Galileon form. It is therefore leading the study of the effective field theory for massive Galileons as the purely scalar part of the modified gravity theories [20, 69]. In what follows we shall consider the positivity bounds for the massive Galileon Lagrangian in 4d flat spacetime [69]

$$\mathcal{L}_{mGal} = -\frac{1}{2}(\partial\pi)^2 - \frac{1}{2}m^2\pi^2 + \frac{g_3}{3!\Lambda^3}\pi\left[[\pi]^2 - [\pi^2]\right] \\ + \frac{g_4}{4!\Lambda^6}\pi\left[[\pi^3] - 3[\pi][\pi^2] + 2[\pi^3]\right] + \ldots \; , \tag{6.44}$$

where we have used the usual notation $\Pi_{\mu\nu} = \partial_\mu\partial_\nu\pi$. As we will determine positivity bounds by 2-2 scattering amplitude at tree-level, only interactions up to quartic order contribute and higher terms are omitted in (6.44). In the centre of mass frame, the 2-2 scattering amplitude for the massive Galileon is then given by

$$A(s,t) = \frac{g_3^2}{16\Lambda^6}\left[\frac{s^2(s-4m^2)^2}{m^2-s} + \frac{t^2(t-4m^2)^2}{m^2-t} + \frac{u^2(u-4m^2)^2}{m^2-u}\right] + \frac{g_4}{4\Lambda^6}stu \; . \tag{6.45}$$

For a generic effective theory, the tree-level pole-subtracted amplitude $B(s,t)$ could express as an analytic function of the crossing symmetric variables

$$B(s,t) = \sum_{nm} \frac{a_{nm}}{\Lambda^{4n+6m}} x^n y^m \tag{6.46}$$

with

$$x = -(\bar{s}\bar{t} + \bar{t}\bar{u} + \bar{u}\bar{s}) \; , \; y = -\bar{s}\bar{t}\bar{u} \; , \tag{6.47}$$

where the bar-variables are defined in the way of (6.22). Here we then conclude the amplitude as

$$B(s,t) = a_{00} + a_{10}\, x + a_{01}\, y \tag{6.48}$$

with the coefficients

$$a_{00} = \frac{m^6}{\Lambda^6}\left[\frac{16g_4}{27} - \frac{295g_3^2}{144}\right] \; , \; a_{10} = \frac{m^2}{\Lambda^6}\left[-\frac{g_4}{3} + \frac{3g_3^2}{8}\right] \; , \; a_{01} = \frac{1}{\Lambda^6}\left[-\frac{g_4}{4} + \frac{3g_3^2}{16}\right] \; . \tag{6.49}$$

Applying the first two positivity bounds in (6.43) gives

$$Y^{(2,0)} : \quad a_{10} + a_{01}\bar{t} > 0 \; , \tag{6.50}$$

$$Y^{(2,1)} : \quad a_{01} + \frac{3}{2\Lambda_{th}^2}(a_{10} + a_{01}\bar{t}) > 0 \; , \tag{6.51}$$

where the scale $\Lambda_{th} \sim \mathcal{M}$ is the threshold that the analyticity and unitarity effectively hold. We first note (6.50) gives the essential requirement $a_{10} + 8/3m^2 a_{01} \geqslant 0$, i.e. $g_4/g_3^2 \leqslant 7/8$. On the other hand, it is clear if $a_{10} > 0$, $a_{01} \geqslant 0$ (so $g_4/g_3^2 \leqslant 3/4$), the two bounds are strongly satisfied. In summary, there are 3 scenarios distinguished:

i) For $g_4/g_3^2 > 7/8$, the positivity bounds are violated and there have no local, analytic, and Lorentz invariant UV completion for the massive Galileon.

ii) For $g_4/g_3^2 \leqslant 3/4$, the UV completion may exist and the Galileon mass can be taken to be arbitrarily small since bounds give no restriction on the threshold or the mass.

iii) For $3/4 < g_4/g_3^2 \leqslant 7/8$, the bound (6.51) at $\bar{t} \to 8/3m^2$ gives constraint on $\Lambda_{th}$

$$\Lambda_{th}^2 < 6m^2 \frac{7/8 - g_4/g_3^2}{g_4/g_3^2 - 3/4} \ . \tag{6.52}$$

This suggests, for a LEEFT, even the ratio $g_4/g_3^2$ can be larger than 3/4, it however should not get away from 3/4 because EFT requires a sufficient threshold $\Lambda_{th}^2 \gg m^2$.

**Analyticity**

At this stage, we have two different choices on the region of the ratio $g_4/g_3^2$. It is possible to further constraint the cutoff by also considering the analyticity together with perturbative unitarity. To see how this works, we first consider the case when any of the partial waves violate the optical theorem

$$32\pi\sqrt{\frac{s}{s-4m^2}}a_0(s) = (3g_3^2 - 4g_4)\frac{s^3}{24\Lambda^6} - (g_3^2 - 2g_4)\frac{2s^2m^2}{3\Lambda^6} + \mathcal{O}(\frac{sm^4}{\Lambda^6}) \ . \tag{6.53}$$

This implies that the generic strong coupling scale is

$$\Lambda_{sc} = \frac{\Lambda}{\mid g_4 - 3g_3^2/4 \mid^{1/6}} \ . \tag{6.54}$$

While in the massless limit $m \to 0$, if one performs the Galileon duality transformation [5, 24] to the Lagrangian, the combination $g_4 - 3g_3^3/4$ is exactly the new coefficient $g_4'$ of the quartic Galileon operator. If we artificially take $g_4 - 3g_3^2/4$ to be small to make the strong coupling scale large, we are also simultaneously switching off interactions. In contrast, it is more natural to take the tuning $\mid g_4 - 3g_3^2/4 \mid = 1$ and define the free parameter $\Lambda$ as the strong coupling scale. This tuning then implies

$$\Lambda_{th}^2 < \frac{1}{2}m^2 g_3^2 \tag{6.55}$$

in the region $3/4 < g_4/g_3^2 \leqslant 7/8$, which clearly violates the LEEFT requirement for the threshold. We, therefore, restrict the ratio to stay in only one region

$$g_4/g_3^2 \leqslant 3/4 \ . \tag{6.56}$$

### Vainshtein mechanism Compatibility

The phenomenological requirements enforce the Galileons to the Vainshtein screened region to suppress their fifth forces contributions. The Vainshtein mechanism thus requires the existence of a real regular solution to the Galileon equations in static and spherically symmetric configuration. In [69], it suggests Vainshtein works under such configuration only in the following case

$$g_3 > -\sqrt{g_4} \ , \ g_4 \geqslant 0 \ . \tag{6.57}$$

Combining it with (6.56), we conclude the requirements for the coefficients

$$g_4 > 0 \ , \ g_3 > \sqrt{\frac{4g_4}{3}} > 0 \ . \tag{6.58}$$

# 7 Positivity bounds for particle with spin

## 7.1 Helicity and Transversity formalism

It is common to use helicity formalism to calculate the scattering amplitude for particles with spin. However, this is inconvenient to construct positivity bounds as scalar case, since the crossing relations play an important role in construction but is highly non-trivial under helicity formalism. In contrast, in what follows we will introduce a so-called transversity formalism that can diagonalize the crossing relations.

For general 2-2 scattering amplitudes, one should consider particles with different masses and spins. Nevertheless, it is sufficient to see the essence of the simple case:

$$m_1 = m_2 = m_3 = m_4 = m \ , \ s_1 = s_3, \ s_2 = 2_4 \ . \tag{7.1}$$

We start by briefly reviewing the helicity formalism. As usual, we make use of Mandelstam variables and introduce auxiliary variables:

$$\mathcal{S} = s(s - 4m^2) \ , \ \mathcal{U} = u(u - 4m^2) \ . \tag{7.2}$$

### Helicity Formalism

The plane wave 2-particle states $|p\theta\phi\lambda_1\lambda_2\rangle$ and spherical wave 2-particle states $|pJM\lambda_1\lambda_2\rangle$ are related via [22]

$$|p\theta\phi\lambda_1\lambda_2\rangle = \sum_{J,M} \sqrt{\frac{2J+1}{4\pi}} D^J_{\mu\lambda}(\phi, \theta, 0) \, |pJM\lambda_1\lambda_2\rangle \ \ , \tag{7.3}$$

where J is angular momentum and Wigner D matrices [82] defined by

$$D^j_{m'm}(\alpha,\beta,\gamma) = e^{-i\alpha m'} d^j_{m'm}(\beta) e^{-i\gamma m} \ , \ d^j_{m'm}(\beta) = \langle jm' | \, e^{-i\beta J_y} \, | jm \rangle \ . \quad (7.4)$$

Consider the scattering between initial state $|i\rangle = |p_i 0 0 \lambda_1 \lambda_2\rangle$ and final state $|f\rangle = |p_f \theta\phi \lambda_3\lambda_4\rangle$ and splitting the S matrix as usual $\hat{S} = 1 + i\hat{T}$, the helicity amplitude $\mathcal{H}$ can be defined in the following way

$$\langle f | \, \bar{T} \, | i \rangle = (2\pi)^4 \delta^4(p_f - p_i) \mathcal{H}_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,\theta) \ , \quad (7.5)$$

$$\mathcal{H}_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,\theta) = 16\pi^2 \sqrt{\frac{s}{p_i p_f}} \langle p_f \theta\phi\lambda_3\lambda_4 | \, \hat{T} \, | p_i 0 0 \lambda_1\lambda_2 \rangle \ . \quad (7.6)$$

Inserting the complete spherical wave basis $\sum_{J,M} |p_f JM\lambda_3\lambda_4\rangle \langle p_f JM\lambda_3\lambda_4| = 1$, we obtain

$$\begin{aligned}\mathcal{H}_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,\theta) &= 16\pi^2 \sqrt{\frac{s}{p_i p_f}} \sum_{JM} \langle p_f \theta\phi\lambda_3\lambda_4 | p_f JM\lambda_3\lambda_4 \rangle \, T^J_{\lambda_1\lambda_2\lambda_3\lambda_4} \\ &\quad \langle p_i JM\lambda_1\lambda_2 | p_i 0 0 \lambda_1\lambda_2 \rangle \quad (7.7) \\ &= 4\pi \sqrt{\frac{s}{p_i p_f}} \sum_J (2J+1) e^{i\lambda\phi} d^J_{\lambda\mu}(\theta) T^J_{\lambda_1\lambda_2\lambda_3\lambda_4}(s) \ ,\end{aligned}$$

where $T^J_{\lambda_1\lambda_2\lambda_3\lambda_4}(s)$ called the partial wave helicity amplitude

$$T^J_{\lambda_1\lambda_2\lambda_3\lambda_4}(s) = \langle p_f JM\lambda_3\lambda_4 | \, \hat{T} \, | p_i JM\lambda_1\lambda_2 \rangle \quad (7.8)$$

is the scattering amplitude between two particles states of definite total angular momentum and definite individual helicities and $\lambda = \lambda_1 - \lambda_2, \ \mu = \lambda_3 - \lambda_4$ are defined for later convenience.

As angular momentum is conserved in the scattering, the S-matrix can be diagonalized to different partial wave blocks labeled by J

$$\hat{S} = 1 + i\hat{T} = \begin{pmatrix} \hat{S}^{J_1} & & & \\ & \hat{S}^{J_1} & & \\ & & \cdots & \\ & & & \square \end{pmatrix} = 1 + \begin{pmatrix} \hat{T}^{J_1} & & & \\ & \hat{T}^{J_1} & & \\ & & \cdots & \\ & & & \square \end{pmatrix} \ . \quad (7.9)$$

The partial unitarity then gives

$$\hat{S}^{J^\dagger} \hat{S}^J = (1 - i\hat{T}^{J^\dagger})(1 + i\hat{T}^{J^\dagger}) = 1 \Rightarrow i(\hat{T}^{J^\dagger} - \hat{T}^J) = \hat{T}^{J^\dagger} \hat{T}^J \ , \quad (7.10)$$

which implies $T^J_{\lambda_3\lambda_4\lambda_1\lambda_2}(s)^* = T^J_{\lambda_1\lambda_2\lambda_3\lambda_4}(s^*)$. Assuming the scattering is time reversal invariant, this then implies the absorptive part defined in (6.5) is just the imaginary part

$$Abs_s \ T^J_{\lambda_1\lambda_2\lambda_3\lambda_4} = Im \ T^J_{\lambda_1\lambda_2\lambda_3\lambda_4} \quad (7.11)$$

as in the scalar case.

We denote the s-channel $A + B \to C + D$ amplitude by $\mathcal{H}^s_{\lambda_1\lambda_2\lambda_3\lambda_4}$, and u-channel $A + \bar{D} \to C + \bar{B}$ amplitude by $\mathcal{H}^u_{\lambda_1\lambda_4\lambda_3\lambda_2}$, where the upper bar denotes the corresponding antiparticles. The $B \leftrightarrow D$ crossing relation is given by [22, 56, 39, 40]

$$\begin{aligned} \mathcal{H}^s_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,t,u) = (-1)^{2S_2} \sum_{\lambda'_i} e^{i\pi(\lambda'_1-\lambda'_3)}\ d^{S_1}_{\lambda'_1\lambda_1}(\chi_\mu)\ d^{S_2}_{\lambda'_2\lambda_2}(-\pi+\chi_\mu) \\ \times\ d^{S_1}_{\lambda'_3\lambda_3}(-\chi_\mu)\ d^{S_2}_{\lambda'_4\lambda_4}(\pi-\chi_\mu)\ \mathcal{H}^u_{\lambda'_1\lambda'_4\lambda'_3\lambda'_2}(u,t,s)\ , \end{aligned} \tag{7.12}$$

$$\cos\chi_\mu = \frac{-su}{\sqrt{\mathcal{SU}}}\ ,\ \sin\chi_\mu = \frac{-2m\sqrt{stu}}{\sqrt{\mathcal{SU}}}\ . \tag{7.13}$$

Manifestly, this is only trivial in the forward limit t=0 where $\chi_\mu = 0$

$$\mathcal{H}^s_{\lambda_1\lambda_2\lambda_3\lambda_4}\ (s,0,u) = \mathcal{H}^u_{\lambda_1\ -\lambda_4\ \lambda_3\ -\lambda_2}\ \ (u,0,s)\ . \tag{7.14}$$

The helicities flip sign since the momenta effectively reverse.

### Transversity formalism

The transversity eigenstates [22, 57, 56] are defined as a particular combination of the helicity eigenstates

$$|\overrightarrow{p}, S, \tau\rangle = \sum_\lambda u^S_{\lambda\tau}\,|\overrightarrow{p}, S, \lambda\rangle\ \ , \tag{7.15}$$

with the unitary matrix

$$u^S_{\lambda\tau} = D^S_{\lambda\tau}(\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2})\ , \tag{7.16}$$

which can diagonalize any of the Wigner $d^S$. It is then straightforward to drive the relation between the transversity amplitudes with helicity amplitudes

$$\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4} = \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} u^{S_1}_{\lambda_1\tau_1} u^{S_2}_{\lambda_2\tau_2} u^{S_1*}_{\lambda_3\tau_3} u^{S_2*}_{\lambda_4\tau_4} \mathcal{H}_{\lambda_1\lambda_2\lambda_3\lambda_4}\ . \tag{7.17}$$

The crossing relations are now simply

$$\mathcal{T}^s_{\tau_1\tau_2\tau_3\tau_4}(s,t,u) = (-1)^{2S_1+2S_2} e^{i\pi\sum_i \tau_i} e^{-i\chi_\mu\pi\sum_i\tau_i} \mathcal{T}^u_{-\tau_1\ -\tau_4\ -\tau_3\ \tau_2}(u,t,s)\ . \tag{7.18}$$

See detailed derivation in appendix B and D in [22]. If we only consider elastic transversities $\tau_1 = \tau_3,\ \tau_2 = \tau_4$, this reduces to

$$\mathcal{T}^s_{\tau_1\tau_2\tau_1\tau_2}(s,0,u) = e^{-i\chi_\mu\pi\sum_i\tau_i} \mathcal{T}^u_{-\tau_1\ -\tau_2\ -\tau_1\ \tau_2}(u,t,s)\ . \tag{7.19}$$

Further taking the forward limit t=0, the result is then simply as in helicity basis (7.14)

$$\mathcal{T}^{s}_{\tau_1\tau_2\tau_3\tau_4}(s,0,u) = \mathcal{T}^{u}_{-\tau_1\ -\tau_4\ -\tau_3\ -\tau_2}(u,0,s)\ . \tag{7.20}$$

The scattering amplitudes may have potential poles or branch cuts at $s = 0,\ s = 4m^2$ and $\sqrt{s+u} = 0$. For $s = 0$, the helicity amplitudes and transversity amplitudes are both regular [12] and so harmless. The factorizable singularities at $s = 4m^2$ can be removed by multiplying an appropriate factor. Finally, we can perform an appropriate combination for transversity amplitudes [63] to remove the branch cut at $s + u = 0$.

To summarize, we construct the regularized amplitudes

$$\mathcal{T}^{+}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) = (\sqrt{-su})^{\xi}\mathcal{S}^{S_1+S_2}(\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) + \mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,-\theta))\ , \tag{7.21}$$

$$\mathcal{T}^{-}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) = -i\sqrt{stu}(\sqrt{-su})^{\xi}\mathcal{S}^{S_1+S_2}(\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) - \mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,-\theta))\ , \tag{7.22}$$

where the parameter

$$\xi = \begin{cases} 1 & ,\ \text{if } S_1+S_2 \text{ is half integer} \\ 0 & ,\ \text{otherwise} \end{cases}\ . \tag{7.23}$$

These then have simply crossing relations and are free of kinematical singularities.

## 7.2 Positivity constraints

To study the positivity condition of the scattering amplitude, we shall look at its partial wave expansion. However, it is rather complicated under transversity formalism, since one cannot define a rotationally invariant notion of transversity in a state with only two particles [56, 72]. So we instead use the helicity partial wave expansion. As the system is symmetric with respect to rotations about the collision axis, we are free to set the interaction plane to lie along $\phi = 0$. Since the previous construction have removed all kinematic singularities of $\mathcal{T}^{+}_{\tau_1\tau_2\tau_3\tau_4}$, so the discontinuity comes from the physical part $T^{J}_{\lambda_1\lambda_2\lambda_3\lambda_4}$. We define the absorptive part

$$Abs_s\ \bar{T}^{J}_{\lambda_1\lambda_2\lambda_3\lambda_4}(s) = 4\pi(2J+1)\sqrt{\frac{s}{p_f p_f}}Abs_s\ T^{J}_{\lambda_1\lambda_2\lambda_3\lambda_4}\ . \tag{7.24}$$

Using (7.7), (7.17) and the symmetry properties

$$\mathcal{T}_{\tau_1\tau_2\tau_1\tau_2}(s,-\theta) = \mathcal{T}_{-\tau_1\ -\tau_2\ -\tau_1\ -\tau_2}(s,\theta)\ , \tag{7.25}$$

we find

$$Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2} = (\sqrt{-su})^{\xi}\mathcal{S}^{S_1+S_2} \sum_{J\lambda_1\lambda_2\lambda_3\lambda_4} u^{S_1}_{\lambda_1\tau_1} u^{S_2}_{\lambda_2\tau_2} u^{S_1*}_{\tau_1\lambda_3} u^{S_2*}_{\tau_2\lambda_4} (d^J_{\mu\lambda}(\theta) + d^J_{\mu\lambda}(-\theta)) \\ \times Abs_s\ \bar{T}^J_{\lambda_1\lambda_2\lambda_3\lambda_4}(s)\ . \tag{7.26}$$

Expanding the sum of Wigner matrix to the Fourier series

$$d^J_{\mu\lambda}(\theta) + d^J_{\mu\lambda}(-\theta) = 2e^{i\frac{\pi}{2}(\lambda-\mu)} \sum_{\nu=J}^{J} d^J_{\lambda\nu}(\frac{\pi}{2}) d^J_{\mu\nu}(\frac{\pi}{2}) \cos(\nu\theta) \tag{7.27}$$

and substituting it in (7.26), we obtain

$$Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2} = 2(\sqrt{-su})^{\xi}\mathcal{S}^{S_1+S_2} \sum_{J,\nu} \cos(\nu\theta) F^{J\nu}_{\tau_1\tau_2}(s)\ , \tag{7.28}$$

where

$$F^{J\nu}_{\tau_1\tau_2}(s) = \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} C^{\nu*}_{\lambda_1\lambda_2} Abs_s\ \bar{T}^J_{\lambda_1\lambda_2\lambda_3\lambda_4} C^{\nu}_{\lambda_3\lambda_4}\ , \tag{7.29}$$

$$\begin{aligned} C^{\nu}_{\lambda_3\lambda_4} &= u^{S_1*}_{\tau_1\lambda_3} u^{S_2*}_{\tau_2\lambda_4} e^{-i\frac{\pi}{2}\mu} d^J_{\mu\nu}(\frac{\pi}{2})\ , \\ C^{\nu*}_{\lambda_1\lambda_2} &= u^{S_1}_{\lambda_1\tau_1} u^{S_2*}_{\lambda_2\tau_2} e^{i\frac{\pi}{2}\lambda} d^J_{\lambda\nu}(\frac{\pi}{2})\ . \end{aligned} \tag{7.30}$$

We note the unitarity condition (7.10) implies

$$Abs_s\ T^J_{\lambda_1\lambda_2\lambda_3\lambda_4} = \frac{1}{2}\sum_N \langle p_f JM\lambda_3\lambda_4| \hat{T}^{J^\dagger} |N\rangle \langle N| \hat{T}^J |p_i JM\lambda_1\lambda_2\rangle\ , \tag{7.31}$$

where $\sum_N$ denotes the sum over all intermediate states. This suggests if one regards $\{\lambda_1\lambda_2\}, \{\lambda_3\lambda_4\}$ as matrix indices, then $Abs_s\ T^J_{\lambda_1\lambda_2\lambda_3\lambda_4}$ is a positive definite Hermitian matrix, and so as $Abs_s\ \bar{T}^J_{\lambda_1\lambda_2\lambda_3\lambda_4}$. consequently, we may conclude

$$\text{Unitarity} \Rightarrow F^{J\nu}_{\tau_1\tau_2}(s) \geqslant 0\ . \tag{7.32}$$

Now we have to first deal with the prefactor $(\sqrt{-su})^{\xi}$ to continue. In general, $\xi = 0$ corresponds to boson-boson or fermion-fermion scattering which has integer total spin, and $\xi = 1$ corresponds to boson-fermion scattering which has half-integer total spin. In what follows, we shall discuss these two situations separately.

**BB or FF scattering**

We start by considering the forward limit $t = 0$, (7.28) is then simply gives the optical theorem

$$Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t=0,u)2\mathcal{S}^{S_1+S_2}\sum_{J=0}^{\infty}\sum_{\nu=J}^{J}F^{J\nu}_{\tau_1\tau_2}(s) > 0\ ,\ \forall s \geqslant 4m^2\ , \tag{7.33}$$

which is manifestly positive. Using the properties of the Chebyshev polynomials

$$N_{n,\nu} = \left.\frac{d^n\cos(\nu\theta)}{d\cos^n\theta}\right|_{\theta=0} = \prod_{k=0}^{n-1}\frac{\nu^2-k^2}{2k+1} \geqslant 0\ . \tag{7.34}$$

We find

$$\begin{aligned}
&\left.\frac{\partial^n}{\partial\cos^n\theta}Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,\theta)\right|_{\theta=0}\\
&= 2\mathcal{S}^{S_1+S_2}\sum_{J,\nu}\left.\frac{\partial^n}{\cos^n\theta}\cos(\nu\theta)F^{J\nu}_{\tau_1\tau_2}(s)\right|_{\theta=0}\\
&= 2\mathcal{S}^{S_1+S_2}\sum_{J,\nu}N_{n,\nu}F^{J\nu}_{\tau_1\tau_2}(s) > 0\\
&\Leftrightarrow \left.\frac{\partial^n}{\partial t^n}Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u)\right|_{t=0} > 0\ ,\ \forall s \geqslant 4m^2\ .
\end{aligned} \tag{7.35}$$

Which extends positivity conditions to arbitrary numbers of t derivatives. It further implies by analyticity of transversity amplitudes [60], the optical theorem can be analytically continued away from the forward limit to the finite positive t until the first pole at $t = m^2$ in generic case [2]

$$Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u) > 0\ ,\ \forall 0 \leqslant t < m^2 \text{ and } s \geqslant 4m^2\ . \tag{7.36}$$

**BF scattering**

In this case the prefactor $\sqrt{-su} = \sqrt{\mathcal{S}}\cos\left(\frac{\theta}{2}\right)$ is excited, by definition (7.28) we have

$$\begin{aligned}
Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,\theta) &= 2\mathcal{S}^{S_1+S_2+\frac{1}{2}}\sum_{J=\frac{1}{2}}^{\infty}\sum_{\nu=-J}^{J}\cos\left(\frac{\theta}{2}\right)\cos(\nu\theta)F^{J\nu}_{\tau_1\tau_2}(s)\\
&= \mathcal{S}^{S_1+S_2+\frac{1}{2}}\sum_{J,\nu}[\cos\left((\nu+\frac{1}{2})\theta\right)+\cos\left((\nu-\frac{1}{2})\theta\right)]F^{J\nu}_{\tau_1\tau_2}(s)\ .
\end{aligned} \tag{7.37}$$

[2]In special cases one can extend the range further, for instance, one can go to $t = 4m^2$ in purely scalar theory [21, 20]

It is also manifestly positive in the forward limit $Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}\Big|_{t=\theta=0} > 0$ differentiating it gives

$$\frac{\partial^n}{\partial t^n} Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,\theta)\Big|_{\theta=0} = \mathcal{S}^{S_1+S_2+\frac{1}{2}} \sum_{J,\nu} (N_{n,\nu+\frac{1}{2}} + N_{n,\nu-\frac{1}{2}}) F^{J\nu}_{\tau_1\tau_2}(s) > 0\ . \tag{7.38}$$

Again, by analytical continuation this implies

$$\frac{\partial^n}{\partial t^n} Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u) > 0\ ,\ \forall n \geqslant 0\ ,\ \text{for } \forall 0 \leqslant t < m^2 \text{ and } s \geqslant 4m^2\ . \tag{7.39}$$

**Left Hand Cut**

At present, we have only evaluated the positivity condition of the right-hand branch cut $s \geqslant 4m^2$ for scattering amplitudes. However, the positivity bounds would arise from dispersion relations which are strongly related to the properties of transversity amplitudes in the whole complex Mandelstam plane. It is important to consider the left-hand branch cut $u \geqslant 4m^2$ associated with the second pole $u = m^2$. On the other hand, one could easily follow the similar procedure for $\mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}$ to derive

$$\begin{aligned} Abs_s\ \mathcal{T}^{-}_{\tau_1\tau_2\tau_1\tau_2}(s,\theta) &= \frac{1}{\sqrt{s}} (\sqrt{-su})^{\xi} \mathcal{S}^{S_1+S_2+1} \sum_{J,\nu} \sin\theta \sin(\nu\theta) F^{J\nu}_{\tau_1\tau_2}(s) \\ &= -\frac{1}{2\sqrt{s}} (\sqrt{-su})^{\xi} \mathcal{S}^{S_1+S_2+1} \sum_{J,\nu} [\cos((\nu+1)\theta) + \cos((\nu-1)\theta)] F^{J\nu}_{\tau_1\tau_2}(s)\ . \end{aligned} \tag{7.40}$$

One cannot straightforwardly read off any positivity properties of either the discontinuity or its derivatives from it. Nevertheless, this expression could be useful to help us to determine the absorptive part along the left-hand cut. By $s \leftrightarrow u$ symmetry, we find

$$\begin{aligned} \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u) &= (\sqrt{-su})^{\xi} \mathcal{S}^{S_1+S_2} (\mathcal{T}^{s}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u) + \mathcal{T}^{s}_{-\tau_1\ -\tau_2\ -\tau_3\ -\tau_4}(s,t,u)) \\ &= (\sqrt{-su})^{\xi} \mathcal{S}^{S_1+S_2} \bigg( e^{+i\chi_\mu \sum_i \tau_i} \mathcal{T}^{u}_{\tau_1\tau_2\tau_1\tau_2}(u,t,s) \\ &\qquad + e^{-i\chi_\mu \sum_i \tau_i} \mathcal{T}^{u}_{-\tau_1\ -\tau_2\ -\tau_3\ -\tau_4}(u,t,s) \bigg)\ . \end{aligned} \tag{7.41}$$

Using expression in (7.13), define variables

$$C_+ = \frac{\mathcal{S}^{S-1+S_2}}{\mathcal{U}^{S_1=S_2}} \cos\left(\xi_\mu \sum_i \tau_i\right)\ ,\ C_- = -\frac{\mathcal{S}^{S-1+S_2}}{\mathcal{U}^{S_1=S_2}} \sin\left(\xi_\mu \sum_i \tau_i\right)\ . \tag{7.42}$$

Then (7.41) reduces to

$$\mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u) = C_+\mathcal{T}^{u+}_{\tau_1\tau_2\tau_1\tau_2}(u,t,s) + C_-\mathcal{T}^{u-}_{\tau_1\tau_2\tau_1\tau_2}(u,t,s)\ . \tag{7.43}$$

In the analogy of scattering angle $\theta$, we define the u-channel scattering angle

$$\cos\theta_u = 1 + \frac{2t}{u-4m^2}\ . \tag{7.44}$$

This copies the same analyticity from $\mathcal{T}^{s\pm}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u)$ to the $\mathcal{T}^{u\pm}_{\tau_1\tau_2\tau_1\tau_2}(u,t,s)$. The latter thus can also be expressed in terms of partial wave amplitudes, which contain all potential discontinuities. We thus define the u-channel absorptive part

$$\begin{aligned}
&Abs_u\ \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u)\\
&= \lim_{\varepsilon\to 0}\frac{1}{2i}[\mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u+i\varepsilon) - \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u-i\varepsilon)]\\
&= \lim_{\varepsilon\to 0}\frac{1}{2i}[\mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s-i\varepsilon,t,u) - \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s+i\varepsilon,t,u)]\\
&= C_+Abs_u\ \mathcal{T}^{u+}_{\tau_1\tau_2\tau_1\tau_2}(u,t,s) + C_-Abs_u\ \mathcal{T}^{u-}_{\tau_1\tau_2\tau_1\tau_2}(u,t,s)\ .
\end{aligned} \tag{7.45}$$

Here in the last line, we have substituted (7.43) in. Now, using expressions (7.28) and (7.40) with the replacement $\theta\to\theta_u,\ \mathcal{S}\to\mathcal{U},\ s\to u$, we can obtain the discontinuity of the s-channel amplitude across the LH cut in terms of cos and sin. Explicitly,

$$\begin{aligned}
&Abs_u\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t)\\
&= 2(\sqrt{-su})^{\xi}\mathcal{U}^{S_1+S_2}\sum_{J,\nu}[C_+\cos((\nu+1)\theta_u) - C_-\frac{\mathcal{U}}{2\sqrt{u}}\sin\theta_u\sin(\nu\theta_u)]F^{J\nu}_{\tau_1\tau_2}(s)\\
&= 2(\sqrt{-su})^{\xi}\mathcal{S}^{S_1+S_2}\sum_{J,\nu}\cos\left(\nu\theta_u - \chi_u\sum_i\tau_i\right)F^{u,J\nu}_{\tau_1\tau_2}(u)\ .
\end{aligned} \tag{7.46}$$

Again, we shall discuss it separately.

### BB or FF scattering

For $u>4m^2$ we first note

$$\sqrt{\mathcal{S}}e^{\pm i\chi_u} = \frac{1}{2}\sqrt{\mathcal{U}} + \frac{(\sqrt{u}\pm 2m)\sqrt{\mathcal{U}}}{4\sqrt{u}}e^{i\theta_u} + \frac{(\sqrt{u}\mp 2m)\sqrt{\mathcal{U}}}{4\sqrt{u}}e^{-i\theta_u} \tag{7.47}$$

and

$$\mathcal{S} = s(s-4m^2) = (u-4m^2)(1+\cos\theta_u)\frac{[u+4m^2+(u-4m^2)\cos\theta_u]}{4} \tag{7.48}$$

are both sum of positive quantities. They can be expressed as power series

$$\sqrt{\mathcal{S}}e^{\pm i\chi_u} = \sum_{p=-1}^{1} c_p^{\pm}(u)e^{ip\theta_u} \ , \ \text{with } c_p(u) > 0 \text{ for } u > 4m^2 \ . \tag{7.49}$$

On the other hand, since the amplitude is invariant under $\tau \to -\tau$, we can assume $\tau_1 + \tau_2 \geqslant 0$ without loss of generality. The prefactor of (7.46) then reduces to

$$\begin{aligned}
&2\mathcal{S}^{S_1+S_2} \sum_{J,\nu} \cos\left(\nu\theta_u - \chi_u \sum_i \tau_i\right) \\
&= \mathcal{S}^{S_1+S_2-\tau_1-\tau_2}\mathcal{S}^{\tau_1+\tau_2}(e^{i\nu\theta_u}e^{-i\chi_u\sum_i\tau_i} + e^{-i\nu\theta_u}e^{i\chi_u\sum_i\tau_i}) \\
&= \mathcal{S}^{S_1+S_2-\tau_1-\tau_2}[e^{i\nu\theta_u}(\sqrt{\mathcal{S}}e^{-i\chi_u})^{2(\tau_1+\tau_2)} + e^{-i\nu\theta_u}(\sqrt{\mathcal{S}}e^{i\chi_u})^{2(\tau_1+\tau_2)}] \\
&= \sum_{p=-2(\tau_1+\tau_2)-\nu}^{2(\tau_1+\tau_2)+\nu} c_{\nu,p}(u)e^{ip\theta_u} \ , \quad \text{with } c_{\nu,p}(u) > 0 \text{ for } u > 4m^2 \ .
\end{aligned} \tag{7.50}$$

Using the property $c_{\nu,-p} = c_{\nu,p}$ and recall $Abs_u \ \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}$ is real, we conclude

$$Abs_u \ \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u) = 2\sum_{J=0}^{\infty}\sum_{\nu=J}^{J}\sum_{p=0}^{2(\tau_1+\tau_2)+\nu} c_{\nu,p}(u)\cos(p\theta_u)F^{u,J\nu}_{\tau_1,\tau_2}(u) \ . \tag{7.51}$$

Combining with analyticity, it clearly implies the same set of positivity constraints with the s-channel absorptive part

$$Abs_u \ \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,0) > 0 \ , \ u \geqslant 4m^2 \ , \tag{7.52}$$

$$\frac{\partial^n}{\partial\cos^n\theta_u} Abs_u \ \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t)\bigg|_{\theta_u=0} > 0 \ , \ u \geqslant 4m^2 \ , \ \forall n \geqslant 0 \ , \tag{7.53}$$

$$\frac{\partial^n}{\partial t^n} Abs_u \ \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t)\bigg|_{t=0} > 0 \ , \ u \geqslant 4m^2 \ , \ \forall n \geqslant 0 \ , \tag{7.54}$$

$$\frac{\partial^n}{\partial t^n}\bigg|_u Abs_u \ \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u) \ , \ u > 4m^2 \ , \ \forall 0 \leqslant t < m^2 \ , \ n \geqslant 0 \ . \tag{7.55}$$

**BF scattering**

For $\xi = 1$, the additional factor $\sqrt{-su} = \sqrt{\mathcal{U}}\cos(\theta_u/2)$ excited, gives

$$\begin{aligned}
Abs_u \ \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u) =& \sqrt{\mathcal{U}}\mathcal{S}^{S_1+S_2}\sum_{J,\nu}[\cos\left((\nu+\frac{1}{2})\theta_u - 2\chi_u(\tau_1+\tau_2)\right) \\
&+ \cos\left((\nu-\frac{1}{2})\theta_u - 2\chi_u(\tau_1+\tau_2)\right)]F^{u,J\nu}_{\tau_1\tau_2}(u) \ .
\end{aligned} \tag{7.56}$$

As this does not change the situation too much difference, we can still follow the above arguments

$$Abs_u\ \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u) = \sum_{J,\nu}\sum_{p=0} D_{\nu,p}(u)\cos(p\theta_u)F^{u,J\nu}_{\tau_1\tau_2}(u)\ . \tag{7.57}$$

Once again implies

$$\left.\frac{\partial^n}{\partial t^n}\right|_u Abs_u\ \mathcal{T}^{s+}_{\tau_1\tau_2\tau_1\tau_2}(s,t,u)\ ,\ u > 4m^2\ ,\ \forall 0 \leqslant t < m^2\ ,\ n \geqslant 0\ . \tag{7.58}$$

## 7.3 Dispersion relation

In the scalar case, we define the pole-subtracted amplitude by removing all the 3 poles. However, this for the scattering of particles with spins is not convenient. Since the residue of the t-channel is a function of s, the subtraction of the pole will affect the behavior of the amplitude at large s and modifies the analyticity arguments as the residue may violate the Froissart bound at tree-level or finite loop [22]. Instead, we only remove s-channel pole at $s = m^2$ and u-channel pole at $u = m^2$, so consider

$$\begin{aligned}\tilde{\mathcal{T}}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t) =& \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t) - \frac{Res\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s=m^2,t)}{s-m^2} \\ &- \frac{Res\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s=3m^2-t,t)}{s+t-3m^2}\ .\end{aligned} \tag{7.59}$$

For any fixed point $-t < s < 4m^2$, by Cauchy's integral formula

$$\tilde{\mathcal{T}}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t) = \frac{1}{2\pi i}\oint_C ds' \frac{\tilde{\mathcal{T}}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s',t)}{(s'-s)}\ . \tag{7.60}$$

where the contour $C$ contains the poles $s' = m^2,\ 3m^2 - t$, and the point s, as shown in Figure 3. Following the same argument as in the scalar case, we deform the contour to $C'$ in Figure 3 and assume a Froissart bound applies [60]

$$\mid \tilde{\mathcal{T}}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t) \mid_{|s|\to\infty} < \mid s \mid^{N_S}\ , \tag{7.61}$$

where

$$N_S = 2 + 2(S_1 + S_2) + \xi\ . \tag{7.62}$$

Performing a sufficient number of subtractions to neglect the integral at the infinite arcs gives us the dispersion relation

$$\begin{aligned}\tilde{\mathcal{T}}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t) = \sum_{n=0}^{N_S-1} a_n(t)s^n &+ \frac{s^{N_S}}{\pi}\int_{4m^2}^{\infty} d\mu \frac{Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(\mu,t)}{\mu^{N_S}(\mu - s)} \\ &+ \frac{u^{N_S}}{\pi}\int_{4m^2}^{\infty} d\mu \frac{Abs_u\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(4m^2-t-\mu,t)}{\mu^{N_S}(\mu - u)}\ ,\end{aligned} \tag{7.63}$$

where the subtraction functions $a_n(t)$ are undetermined by analyticity.

## 7.4 Positivity bounds

The first positivity bound is simple to derive: taking the $N_S$ derivatives of dispersion relation, we eliminated the undetermined functions $a_n(t)$ to obtain

$$\begin{aligned} f_{\tau_1\tau_2}(s,t) &= \frac{1}{N_S!}\frac{d^{N_S}}{ds^{N_S}} Abs_s\ \tilde{\mathcal{T}}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t) \\ &= \frac{1}{2\pi i}\oint_C ds' \frac{Abs_s\ \tilde{\mathcal{T}}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s',t)}{(s'-s)^{N_S+1}} \\ &= \frac{1}{\pi}\int_{4m^2}^{\infty} d\mu \frac{Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(\mu,t)}{(\mu-s)^{N_S+1}} \\ &+ \frac{1}{\pi}\int_{4m^2}^{\infty} d\mu \frac{Abs_u\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(4m^2-t-\mu,t)}{(\mu-u)^{N_S+1}}\ . \end{aligned} \tag{7.64}$$

Using the positivity constraints on RH and LH branch cuts derived in section 7.2, we immediately infer that

$$f_{\tau_1\tau_2}(s,t) > 0\ ,\ \text{for}\ -t < s < 4m^2\ ,\ 0 \leqslant t < m^2\ . \tag{7.65}$$

To further proceed the construction of positivity bounds, we first define new variables in the way of the scalar case (6.22) and (6.23)

$$s = 2m^2 - \frac{t}{2} + v\ ,\ u = 2m^2 - \frac{t}{2} - v\ , \tag{7.66}$$

and then rewrite (7.62) to

$$\begin{aligned} f_{\tau_1\tau_2}(s,t) = \tilde{f}_{\tau_1\tau_2}(v,t) &= \frac{1}{\pi}\int_{4m^2}^{\infty} d\mu \frac{Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(\mu,t)}{(\mu-2m^2+t/2-v)^{N_S+1}} \\ &+ \frac{1}{\pi}\int_{4m^2}^{\infty} d\mu \frac{Abs_u\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(4m^2-t-\mu,t)}{(\mu-2m^2+t/2+v)^{N_S+1}}\ . \end{aligned} \tag{7.67}$$

Performing a single t derivative gives

$$\begin{aligned} \frac{\partial}{\partial t} f_{\tau_1\tau_2}(v,t) &= \frac{1}{\pi}\int_{4m^2}^{\infty} d\mu \frac{\partial_t Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(\mu,t)}{(\mu-2m^2+t/2-v)^{N_S+1}} \\ &+ \frac{1}{\pi}\int_{4m^2}^{\infty} d\mu \frac{\partial_t Abs_u\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(4m^2-t-\mu,t)}{(\mu-2m^2+t/2+v)^{N_S+1}} \\ &- \frac{(N_S+1)}{2\pi}\Big[\int_{4m^2}^{\infty} d\mu \frac{Abs_s\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(\mu,t)}{(\mu-2m^2+t/2-v)^{N_S+2}} \\ &+ \int_{4m^2}^{\infty} d\mu \frac{Abs_u\ \mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(4m^2-t-\mu,t)}{(\mu-2m^2+t/2+v)^{N_S+2}}\Big]\ . \end{aligned} \tag{7.68}$$

Using the integral inequalities satisfied for arbitrary positive function $\rho(\mu) > 0$ in an analogy of (6.29)

$$\frac{1}{\mathcal{M}^2}\int_{4m^2}^{\infty}\frac{d\mu\ \rho(\mu)}{(\mu-2m^2+t/2)^N} > \int_{4m^2}^{\infty}\frac{d\mu\ \rho(\mu)}{(\mu-2m^2+t/2)^{N+1}}\ . \tag{7.69}$$

We find

$$\begin{aligned}\frac{\partial}{\partial t}f_{\tau_1\tau_2}(0,t)+\frac{N_S+1}{2\mathcal{M}^2}f_{\tau_1\tau_2}(0,t) > \frac{1}{\pi}\int_{4m^2}^{\infty}d\mu\frac{\partial_t Abs_s\ \mathcal{T}^+_{\tau_1\tau_2\tau_1\tau_2}(\mu,t)}{(\mu-2m^2+t/2-v)^{N_S+1}}\\ +\frac{1}{\pi}\int_{4m^2}^{\infty}d\mu\frac{\partial_t Abs_u\ \mathcal{T}^+_{\tau_1\tau_2\tau_1\tau_2}(4m^2-t-\mu,t)}{(\mu-2m^2+t/2+v)^{N_S+1}} > 0\end{aligned} \tag{7.70}$$

and thus determine it as our second positivity bound

$$\frac{\partial}{\partial t}f_{\tau_1\tau_2}(0,t)+\frac{N_S+1}{2\mathcal{M}^2}f_{\tau_1\tau_2}(0,t) > 0\ ,\ \text{ for } 0\leqslant t < m^2\ . \tag{7.71}$$

Following the same argument in section 6, we can generalize our positivity bounds to any t derivatives and all even v derivatives of the amplitudes. To see this explicitly we shall use the similar notation for pole-subtracted amplitude as we are constructing positivity bounds for scalar

$$\tilde{B}_{\tau_1\tau_2}(v,t)=\tilde{\mathcal{T}}^+_{\tau_1\tau_2\tau_1\tau_2}(s=-2m^2-t/2+v,t)\ . \tag{7.72}$$

In the analogy of (6.25), we have

$$B^{(2N,M)}_{\tau_1\tau_2}(t)=\frac{1}{M!}\partial_v^{2N}\partial_t^{M}\tilde{B}_{\tau_1\tau_2}(v,t)\bigg|_{v=0}\ . \tag{7.73}$$

It can be further expressed in terms of $I^{(q,p)}_{\tau_1\tau_2}$ to

$$B^{(2N,M)}_{\tau_1\tau_2}(t)=\sum_{k=0}^{M}\frac{(-1)^k}{k!2^k}I^{(2N+k,M-k)}_{\tau_1\tau_2}\ ,\quad \forall N\geqslant\frac{N_S}{2}\ ,\ M>0\ , \tag{7.74}$$

where

$$I^{(q,p)}_{\tau_1\tau_2}=\frac{q!}{p!}\frac{1}{\pi}\int_{4m^2}^{\infty}d\mu\frac{[\partial_t^p Abs_s\ \mathcal{T}^+_{\tau_1\tau_2\tau_1\tau_2}(\mu,t)+\partial_t^p Abs_u\ \mathcal{T}^+_{\tau_1\tau_2\tau_1\tau_2}(4m^2-t-\mu,t)]}{(\mu+t/2-2m^2)^{q+1}} > 0\ . \tag{7.75}$$

It is then straightforward to write the positive quantities $Y^{(2N,M)}$ for particles with spin

$$\begin{aligned}Y^{(2N,M)}_{\tau_1\tau_2}(t)=&\sum_{r=0}^{M/2}c_r B^{(2N+2r,M-2r)}_{\tau_1\tau_2}(t)\\ &+\frac{1}{\mathcal{M}^2}\sum_{even\ \ k=0}^{(M-1)/2}(2(N+k)+1)\beta_k\ Y^{(2N+2k,M-2k-1)}_{\tau_1\tau_2}(t) > 0\ ,\end{aligned} \tag{7.76}$$

where the coefficients $c_r,\ \beta_k$ are same defined as in (6.39) and (6.41)

$$c_0 = 1\ ,\ c_k = -\sum_{r=0}^{k-1} \frac{2^{2(r-k)}}{(2k-2r)!} c_r\ ,\ \forall k \geqslant 1\ , \beta_k = (-1)^k \sum_{r=0}^{k} \frac{2^{2(r-k)-1}}{(2k-2r+1)!} c_r\ . \tag{7.77}$$

In summary, the general positivity bounds are

$$Y^{(2N,M)}_{\tau_1\tau_2}(t) > 0\ ,\ \forall N \geqslant \frac{N_S}{2}\ ,\ M > 0\ ,\ 0 \leqslant t < m^2\ . \tag{7.78}$$

## 7.5 Proca EFT

In this section, we discuss positivity bounds for theory with a single massive spin-1 field $A_\mu$ with cutoff $\Lambda_A \gg m$. This cut off is chosen to be sufficient to suppress the non-renormalizable operators $\partial/\Lambda_A \ll 1$ and so perturbative unitarity applies. The mass of the spin-1 field is introduced by the symmetry breaking scheme occurring at scale $\Lambda_\phi \gg m$ which is independent of $\Lambda_A$. This leading to an additional sector built out of the 'covariant' term of the helicity-0 mode $\phi$

$$D_\mu\phi = \partial_\mu\phi + mA_\mu\ . \tag{7.79}$$

The Lagrangian thus has a generic form

$$\mathcal{L} = \frac{\Lambda_A^4}{g_*^2}\mathcal{F}_1[\frac{\partial}{\Lambda_A}, \frac{F_{\mu\nu}}{\Lambda_A^2}] + \mathcal{F}_2[\frac{\partial}{\Lambda_\phi}, \frac{F_{D_\mu\phi}}{\Lambda_\phi^2}]\ , \tag{7.80}$$

where $\mathcal{F}_1$ and$\mathcal{F}_2$ denote dimensionless Lorentz scalars, and the overall coupling is chosen to be $g_* \ll 1$ to evade the possible strongly coupling near $\Lambda_A$, so we can apply the positivity bounds at tree level. For positivity bounds of 2-2 scattering, it is convenient to take unitary gauge $\phi = 0$ and sufficient to focus on the following contribution [23]

$$\begin{aligned} g_*^2\mathcal{L}^{\text{unitary}}_{\text{Proca}} \supset & -\frac{1}{4}F^\nu_\mu F^\mu_\nu - \frac{1}{2}m^2 A_\mu A^\mu + \frac{m^4 a_0}{\Lambda_\phi^4}\left(A_\mu A^\mu\right)^2 \\ & + \frac{m^4}{\Lambda_\phi^6}\left(a_3 A_\mu A_\nu \partial^\mu A_\rho \partial^\nu A^\rho + a_4 A_\mu A_\nu \partial_\rho A^\mu \partial^\rho A^\nu + a_5 A_\mu A^\mu \partial_\alpha A_\beta \partial^\beta A^\alpha\right) \\ & + \frac{1}{\Lambda_A^4}\left(c_1 F^\mu_\nu F^\nu_\rho F^\rho_\sigma F^\sigma_\mu + c_2\left(F^2_{\mu\nu}\right)^2\right) \\ & + \frac{m^4}{\Lambda_\phi^6}\left(C_1 A_\mu A^\nu F^{\alpha\mu}F_{\alpha\nu} + C_2 F^2_{\mu\nu}A_\alpha A^\alpha\right)\ . \end{aligned} \tag{7.81}$$

### Polarization vector

To compute the amplitudes, we first need to determine the polarization vectors in the transversity basis. Given the helicity spinors and anti-spinors that satisfied the Dirac equation $[-i\gamma \cdot \partial + m]\tilde{u}_\lambda e^{ip\cdot x} = 0$

$$
\begin{aligned}
\tilde{u}_+ &= -\frac{1}{\sqrt{2m(m+E)}} \begin{pmatrix} (E+m)\cos(\theta/2) \\ (E+m)\sin(\theta/2) \\ p\cos(\theta/2) \\ p\sin(\theta/2) \end{pmatrix} , \\
\tilde{u}_- &= -\frac{1}{\sqrt{2m(m+E)}} \begin{pmatrix} -(E+m)\sin(\theta/2) \\ (E+m)\cos(\theta/2) \\ p\sin(\theta/2) \\ -p\cos(\theta/2) \end{pmatrix} ,
\end{aligned} \tag{7.82}
$$

$$
\bar{\tilde{v}}_\lambda = \tilde{u}_\lambda^\intercal C \ , \ \bar{\tilde{u}}_\lambda = \tilde{v}_\lambda^\intercal C \ , \tag{7.83}
$$

where $C = -i\gamma^0\gamma^2$ is the charge conjugation matrix with the standard Dirac convention for the $\gamma$ matrices. The transversity spinors are then a particular combination of the helicity spinors

$$
u_\tau = \sum_\lambda u_{\tau\lambda}^{1/2} \tilde{u}_\lambda \tag{7.84}
$$

with Wigner matrix

$$
u_{\tau\lambda}^{1/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} . \tag{7.85}
$$

Explicitly,

$$
u_\tau(\theta) = \frac{e^{i\pi/4}}{\sqrt{4m(m+E)}} \begin{pmatrix} (E+m)e^{-i\tau(\theta+\pi/2)} \\ (E+m)e^{-i\tau(\theta-\pi/2)} \\ pe^{i\tau(\theta-\pi/2)} \\ -pe^{i\tau(\theta+\pi/2)} \end{pmatrix} . \tag{7.86}
$$

We can then construct the vector polarization for transversity states by the following relations [22]

$$
\epsilon^\mu_{\tau=\pm 1} = -\frac{1}{\sqrt{2}} \bar{v}_{\tau/2}\gamma^\mu u_{\tau/2}, \epsilon^\mu_0 = -\frac{1}{2}(\bar{v}_{1/2}\gamma^\mu u_{-1/2} + \bar{v}_{-1/2}\gamma^\mu u_{1/2}) \ . \tag{7.87}
$$

Explicitly,

$$
\begin{aligned}
\epsilon^\mu_{\tau=\pm 1}(\theta) &= \frac{i}{\sqrt{2}m}(p, E\sin\theta \pm im\cos\theta, 0, E\cos\theta \mp im\sin\theta) \ , \\
\epsilon^\mu_{\tau=0}(\theta) &= (0,0,1,0) \ ,
\end{aligned} \tag{7.88}
$$

which manifestly satisfy the conditions of orthogonality and completeness.

**Scattering amplitudes**

There are only four independent elastic amplitudes. To the leading order of tree-level scattering, the pole-subtracted transversity amplitudes omitting overall factor $1/g_*^2$ are given by

$$\mathcal{T}^+_{0000} = 2s^2\tilde{s}^2\left(24\frac{m^4}{\Lambda_\phi^4}a_0 - 8\frac{m^6}{\Lambda_\phi^6}\left(a_4+C_1+2C_2\right) + 8\frac{m^2\left(6m^4+x\right)}{\Lambda_\phi^6}\left(\tilde{c}_1+2\tilde{c}_2\right)\right) \ , \tag{7.89}$$

$$\begin{aligned}\mathcal{T}^+_{-11-11} =& 2s^2\tilde{s}^2\left[\frac{x-4m^2\left(t-4m^2\right)}{\Lambda_\phi^4}\left(a_0-\frac{1}{2}\frac{m^2}{\Lambda_\phi^2}\left(a_4-4\left(\tilde{c}_1+2\tilde{c}_2\right)+C_1\right)\right)\right.\\ &\left.+\frac{3}{8}\frac{y}{\Lambda_\phi^6}\left(a_3+a_4-2a_5\right)-\frac{m^2su}{\Lambda_\phi^6}\left(\frac{3}{2}a_3-a_4+a_5+\frac{3}{2}C_1+2C_2\right)\right] \ ,\end{aligned} \tag{7.90}$$

$$\begin{aligned}\mathcal{T}^+_{0101} =& \frac{m^2s^2\tilde{s}\left(st-4m^2u\right)}{\Lambda_\phi^4}\left[4a_0-\frac{1}{2}\frac{u}{\Lambda_\phi^2}\left(a_3+C_1\right)+2\frac{s-t}{\Lambda_\phi^2}\tilde{c}_1\right.\\ &\left.+\frac{t}{\Lambda_\phi^2}\left(-a_4+a_5\right)-4\frac{2t-4m^2}{\Lambda_\phi^2}\tilde{c}_2+2\frac{t-4m^2}{\Lambda_\phi^2}C_2\right]\\ &+\frac{m^2s^2\tilde{s}^3(s-u)}{2\Lambda_\phi^6}\left(a_3+4\tilde{c}_1+C_1\right) \ ,\end{aligned} \tag{7.91}$$

$$\begin{aligned}\mathcal{T}^+_{1111} =& \frac{2s^2}{\Lambda_\phi^4}\left[\tilde{s}^2\left(t^2+t\tilde{s}+\tilde{s}^2\right)+4m^2s\left(8t^2+8t\tilde{s}+\tilde{s}^2\right)\right]\left(a_0+\frac{2m^2}{\Lambda_\phi^2}\left(\tilde{c}_1+2\tilde{c}_2-C_2\right)\right)\\ &+\frac{s^2\tilde{s}}{4\Lambda_\phi^6}\left[\tilde{s}^2\left(4m^2s-3tu\right)+16m^2t(t+\tilde{s})\left(3s-4m^2\right)\right]\left(a_3+a_4-2a_5\right) \ .\end{aligned} \tag{7.92}$$

$\tilde{s} = s - 4m^2$ and $\tilde{c}_{1,2} = c_{1,2}\Lambda_\phi^6/(m^2\Lambda_A^4)$ are defined for compactness and the Lorentz crossing-symmetric invariants are denoted as $x = -(st+su+ut)$ and $y = -stu$. By the definition of (7.62), taking $N_S = 2+4 = 6$ for Proca EFT here, we obtain the following definite positivity quantities

$$f_{00}(v,t) = 16\frac{m^2}{\Lambda_\phi^6}(\tilde{c}_1+2\tilde{c}_2) \ , \tag{7.93}$$

$$\begin{aligned}f_{-11}(v,t) =& \frac{2}{\Lambda_\phi^4}[a_0-\frac{m^2}{2\Lambda_\phi^2}(a_4-4(\tilde{c}_1+2\tilde{c}_2)+C_1)]\\ &+\frac{3t}{4\Lambda_\phi^6}(a_3+a_4-2a_5)+\frac{2m^2}{\Lambda_\phi^6}(\frac{3}{2}a_3-a_4+a_5+\frac{3}{2}C_1+2C_2) \ ,\end{aligned} \tag{7.94}$$

$$f_{01}(v,t) = \frac{m^2}{\Lambda_\phi^6}(a_3 + 4\tilde{c}_1 + C_1) \ , \tag{7.95}$$

$$f_{11}(v,t) = \frac{2}{\Lambda_\phi^4}[a_0 + \frac{2m^2}{\Lambda_\phi^2}(\tilde{c}_1 + 2\tilde{c}_2 - C_2)] + \frac{4m^2+3t}{4\Lambda_\phi^6}(a_3 + a_4 - 2a_5) \ . \tag{7.96}$$

**Indefinite Positivity Bounds**

Instead of positivity bounds for definite states $f_{\tau_1\tau_2}$, one can also construct a bound [23] for designed mixed transversity state $(\alpha, \beta)$

$$f_{\alpha\beta}(0,t) = \sum_{\tau_1\tau_2\tau_2\tau_4} \alpha_{\tau_1}\beta_{\tau_2}\alpha^*_{\tau_3}\beta^*_{\tau_4} f_{\tau_1\tau_2\tau_3\tau_4}(0,t) \ , \tag{7.97}$$

where $f_{\tau_1\tau_2\tau_3\tau_4}$ is defined by the same arguments of $f_{\tau_1\tau_2}$ without restriction of elastic scattering and $\mid \alpha \mid^2 = \mid \beta \mid^2 = 1$ by normalization. Now for the Proca EFT, this gives the forward limit bound

$$\begin{aligned}
f_{\alpha\beta}\Big|_{t=0} =& \frac{8}{\Lambda_\phi^4}\left(a_0 - \frac{1}{2}\frac{m^2}{\Lambda_\phi^2}(a_4 + C_1)\right)|\alpha_+|^2|\beta_+|^2 \\
&+ \frac{4m^2}{\Lambda_\phi^6}(a_3 - 2a_4 + 2a_5 + C_1 + 4C_2)\left(\mathrm{Re}\,[\alpha_0^*\alpha_+]\,\mathrm{Re}\,[\beta_0^*\beta_+] - \mathrm{Re}\left[\alpha_-^*\alpha_+\right]\mathrm{Re}\left[\beta_-^*\beta_+\right]\right) \\
&+ \frac{2m^2}{\Lambda_\phi^6}(a_3 + C_1)\left(|\alpha_+|^2|\beta|^2 + |\alpha|^2|\beta_+|^2\right) \\
&+ \frac{8m^2}{\Lambda_\phi^6}\tilde{c}_1\left(|\alpha_0|^2 + |\alpha_-|^2\right)\left(|\beta_0|^2 + |\beta_-|^2\right) \\
&+ \frac{8m^2}{\Lambda_\phi^6}(\tilde{c}_1 + 4\tilde{c}_2)\left(|\alpha_0\beta_0 - \alpha_-\beta_-|^2 - 2\,\mathrm{Im}\,[\alpha_0^*\alpha_-]\,\mathrm{Im}\,[\beta_0^*\beta_-]\right) \ ,
\end{aligned} \tag{7.98}$$

where for compactness we have defined variables

$$\alpha_\pm \equiv \frac{1}{\sqrt{2}}(\alpha_{-1} \pm \alpha_{+1}), \quad \beta_\pm \equiv \frac{1}{\sqrt{2}}(\beta_{-1} \pm \beta_{+1}) \ . \tag{7.99}$$

We first note that for the sum of the first and second line, the term proportional to $a_0$ dominates because the remaining terms are suppressed by the factor $m^2/\Lambda_\phi^2 \ll 1$. Therefore, we have our first requirement $a_0 > 0$.

Taking the choice $\beta_+ = 0$, the last three lines give

$$\begin{aligned}
&(a_3 + C_1)|\alpha_+|^2 + 4\tilde{c}_1(1 - |\alpha_+|^2) \\
&+ 4(\tilde{c}_1 + \tilde{c}_2)\bigg(|\alpha_0\beta_0 - \alpha_-\beta_-|^2 - 2\,\mathrm{Im}[\alpha_0^*\alpha_-]\,\mathrm{Im}[\beta_0^*\beta_-]\bigg) > 0 \ .
\end{aligned} \tag{7.100}$$

Now we shall discuss two cases $\Lambda_A^2 \gg \Lambda_\phi^3/m$ and $\Lambda_A^2 \sim \Lambda_\phi^3/m$ separately.

i) If $\Lambda_A^2 \gg \Lambda_\phi^3/m$ (i.e. $\tilde{c}_{1,2} \ll 1$), this simply implies $a_3 + C_1 > 0$, so we have

$$a_0 > 0 \ , \ a_3 + C_1 > 0 \text{ for } \Lambda_A^2 \gg \Lambda_\phi^3/m \ . \tag{7.101}$$

ii) If $\Lambda_A^2 \sim \Lambda_\phi^3/m$, up to a finite factor, we may assume $\Lambda_A^2 = \Lambda_\phi^3/m$. Taking $\alpha_0 = \beta_- = 0$ to eliminate the second line, we infer that $a_3 + C_1 > 0$ and $c_1 > 0$. On the other hand, choosing $\alpha_+ = \beta_- = \alpha_- = 0$ gives condition $c_1 + c_2 > 0$, so we conclude

$$a_0 > 0, \ c_1 > 0, \ c_1 + 2c_2 > 0 \text{ and } a_3 + C_1 > 0 \text{ for } \Lambda_A^2 = \Lambda_\phi^3/m \ . \tag{7.102}$$

It is important to note that taking $\alpha_{\pm 1} = \pm 1/\sqrt{2}$, $\alpha_0 = 0$ and $\beta_{\pm 1} = 1/\sqrt{2}$, $\beta_0 = 0$ leads to $f_{\alpha\beta}$ which violate the positivity bounds. This suggests one should further consider higher order operators $\mathcal{O}(m^4/\Lambda_\phi^8)$ for Proca EFT.

**First t derivative**

From the form of (7.93)-(7.96), it is easy to see that is only significant to take the first t derivative of them since the higher t derivatives only give 0. Recall the general form of the second positivity bound is

$$\frac{\partial}{\partial t} f_{\tau_1\tau_2}(0,t) + \frac{N_S + 1}{2\mathcal{M}^2} f_{\tau_1\tau_2}(0,t) > 0 \ , \text{ for } 0 \leqslant t < m^2 \ . \tag{7.103}$$

Given the leading contribution to the first t derivative of the indefinite bound

$$\left.\frac{\partial}{\partial t} f_{\alpha\beta}\right|_{t=0} = \frac{3}{4}\frac{a_3 + a_4 + 2a_5}{\Lambda_\phi^6} |\alpha_+|^2 |\beta_+|^2 + \ldots \ , \tag{7.104}$$

where here $N_2 = 2 + 4 = 6$ for massive spin-1 field and for tree-level bounds $\mathcal{M}^2 \sim \Lambda_{th}^2 \sim \Lambda_\phi^2$. For a weakly coupled UV completion, we may define $\Lambda_{th} = \Lambda_\phi$ so the interactions of the Goldstone arise from integrating out the massive modes. Substituting the results that have been evaluated in (7.103), we obtain

$$\begin{aligned} \frac{3}{4}\frac{(a_3 + a_4 - 2a_5)}{\Lambda_\phi^6} + \frac{7}{2\Lambda_\phi^2}\frac{8}{\Lambda_\phi^4}\left(a_0 - \frac{m^2}{2\Lambda_\phi^2}(a_4 + C_1)\right) \gtrapprox 0 \ , \\ \Rightarrow 3(a_3 + a_4 - 2a_5) + 112 a_0 \gtrapprox 0 \ . \end{aligned} \tag{7.105}$$

This clearly distinguished from the previous constraints (7.101) or (7.102) and thus gives new information on the parameter space.

# 8 Positivity bounds for massive spin-2 fields

## 8.1 Generic $\Lambda_5$ massive gravity

In the previous section, we have seen the first dangerous interaction for generic massive gravity theories arises at the scale $\Lambda_5 = (M_P m^4)^{1/5}$. If one expects a $\Lambda_5$ theory can be interpreted as a Wilsonian EFT with possible UV completion, to resolve the perturbative unitarity at the scale $\Lambda_5$, it will generate an infinite number of operators in the following form

$$\Delta\mathcal{L} = \Lambda_5^4 L_0(\frac{\Lambda_5^2}{m^2}\mathcal{K}^\mu{}_\nu, \frac{\nabla_\mu}{\Lambda_5}, \frac{R^\mu{}_{\nu\rho\sigma}}{\Lambda_5^2}) \ , \tag{8.1}$$

where all notations are consistent with arguments in previous sections and $L_0$ denotes scalar operators. More explicitly, the tensor $\mathcal{K}^\mu{}_\nu$ is defined by

$$\mathcal{K}^\mu{}_\nu = 1 - \sqrt{g^{-1}f} = \delta^\mu{}_\nu - \sqrt{g^{\mu\rho}\partial_\rho\phi^a\partial_\nu\phi^b\eta_{ab}} \tag{8.2}$$

with Stückelberg decomposition

$$\phi^a = x^a - \frac{V^a}{mM_P} - \frac{\partial^a\pi}{m^2 M_P} \ . \tag{8.3}$$

One can further generalize this to a 'single scale-single coupling' theory [19] by introducing a weak coupling parameter $g_*$ and redefining

$$\Lambda_5 = (m^4 M_P g_*)^{\frac{1}{5}} \ , \tag{8.4}$$

which then becomes the cutoff of the EFT. For a weakly coupled UV completion, the amplitude contributions from higher derivative EFT corrections are suppressed by the factor $1/\Lambda_5^2$. Taking the threshold $\mathcal{M}^2 \sim \Lambda_5^2$, one may truncate the amplitudes to the leading order contribution and simplify the positivity bounds to [23]

$$f_{\tau_1\tau_2}(v,t) > 0 \quad , \ |v| \ll \Lambda_5 \ , \tag{8.5}$$

$$\frac{\partial}{\partial t} f_{\tau_1\tau_2}(v,t) > 0 \quad , \ |v| \ll \Lambda_5 \ , \tag{8.6}$$

$$\frac{\partial^{2N}}{\partial v^N}\frac{\partial^M}{\partial t^M} f_{\tau_1\tau_2}\bigg|_{t=v=0} > 0 \quad , \ \forall M \geqslant 1 \ , \ N \geqslant 0 \tag{8.7}$$

and for indefinite states

$$\frac{\partial^{2N}}{\partial v^N} f_{\alpha\beta}\bigg|_{t=v=0} > 0 \quad , \ \forall N \geqslant 0 \ . \tag{8.8}$$

It is therefore sufficient to focus on the finite number of terms from the mass potential up to quartic order in the manner

$$
\begin{aligned}
V(g,h) \supset & [h^2] - [h]^2 + (c_1 - 2)[h^3] \\
& + (-\frac{3}{2}c_1 + \frac{11}{4} + \Delta c)[h^2]h + (d_1 + 3 - 3c_1)[h^4] \\
& + (-\frac{1}{2}d_1 + \frac{3}{2}c_1 - \frac{45}{32} + \Delta d - \Delta c)[h^2]^2 \ .
\end{aligned}
\tag{8.9}
$$

$\Delta c = \Delta d = 0$ is precisely the tuning results in ghost-free massive gravity which raises the scale to the $\Lambda_3$. Remarkably, once the leading interactions contribute zero to the bounds, one can then apply the following EFT corrections to it and may obtain constraints on parameter space to any order in the EFT expansion by repeating the process.

### Polarization

For momenta $k^\mu = (\omega, 0, 0, k)$, the corresponding polarizations given by

$$
\begin{aligned}
\epsilon_{\mu\nu}^{(\tau=\pm 2)} &= \frac{1}{2m^2}\begin{pmatrix} k^2 & \pm ikm & 0 & kw \\ \pm ikm & -m^2 & 0 & \pm imw \\ 0 & 0 & 0 & 0 \\ kw & \pm imw & 0 & w^2 \end{pmatrix}, \\
\epsilon_{\mu\nu}^{(\tau=\pm 1)} &= \frac{1}{2m}\begin{pmatrix} 0 & 0 & ik & 0 \\ 0 & 0 & \mp m & 0 \\ ik \mp m & 0 & iw & \\ 0 & 0 & iw & 0 \end{pmatrix}, \\
\epsilon_{\mu\nu}^{(\tau=0)} &= \frac{1}{\sqrt{6}m^2}\begin{pmatrix} k^2 & 0 & 0 & kw \\ 0 & m^2 & 0 & 0 \\ 0 & 0 & -2m^2 & 0 \\ kw & 0 & 0 & w^2 \end{pmatrix}.
\end{aligned}
\tag{8.10}
$$

A general spin state then can be expressed as

$$
\epsilon_{\mu\nu}^{(\alpha)} = \sum_\tau \alpha_\tau \epsilon_{\mu\nu}^{(\tau)} \ . \tag{8.11}
$$

These polarizations are related to the standard SVT decomposition by

$$
\begin{pmatrix} \alpha_{T_1} \\ \alpha_{T_2} \\ \alpha_{V_1} \\ \alpha_{V_2} \\ \alpha_S \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} -1 & 0 & \sqrt{6} & 0 & -1 \\ 0 & 2 & 0 & -2 & 0 \\ -2 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ \sqrt{3} & 0 & \sqrt{2} & 0 & \sqrt{3} \end{pmatrix}\begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_{+1} \\ \alpha_{+2} \end{pmatrix}. \tag{8.12}
$$

It is more convenient to express the bounds in terms of vector $\alpha$.

#### Forward limit

First, consider the indefinite transversity bound $\frac{\partial^2}{\partial v^2} f_{\alpha\beta} > 0$ in the forward limit

$$2M_{\rm Pl}^2 m^6 \frac{\partial^2}{\partial v^2} f_{\alpha\beta}\bigg|_{t=0} = \frac{352}{9} \left|\alpha_S \beta_S\right|^2 \left(\Delta c\left(-6+9c_1-4\Delta c\right)-6\Delta d\right) + \frac{176}{3}\alpha_S^* \beta_S^* \left(\alpha_{V_1}\beta_{V_1}-\alpha_{V_2}\beta_{V_2}\right)\Delta c\left(3-3c_1+4\Delta c\right) \ . \tag{8.13}$$

As complex $\alpha$ and $\beta$ would not give stronger bounds, we shall consider them as real without loss of generality. Taking the choice of polarizations

$$\alpha_S = \beta_S = \epsilon \tag{8.14}$$

to be sufficiently small, (8.13) then reduces to

$$2M_{\rm P1}^2 m^6 \frac{\partial^2}{\partial v^2} f_{\alpha\alpha}\bigg|_{t=0} = \frac{176}{3}\left(\alpha_{V_1}\alpha_{V_1}-\alpha_{V_2}\alpha_{V_2}\right)\Delta c\left(3-3c_1+4\Delta c\right)\left(\epsilon^2+\mathcal{O}\left(\epsilon^4\right)\right) \ . \tag{8.15}$$

Since this should satisfy by an arbitrary value of $\alpha_{V_1}$ and$\alpha_{V_2}$, it gives the requirement

$$\Delta c = 0 \tag{8.16}$$

and together implies $\Delta d \leqslant 0$.

For another forward limit bound $f_{\alpha\beta}\Big|_{t=0}$, it is more complicated to analytically find the parameter constraints. A numerical minimization approximation in [23] found that analyticity prefers a large negative $\Delta d$ when people only consider the leading order contribution. However, applying the t derivative bounds completely reverses the situation.

#### First t derivatives

The first t derivative definite bound

$$\frac{\partial}{\partial t}\frac{\partial^2}{\partial v^2} f_{\tau_1\tau_2}(v,t) > 0 \tag{8.17}$$

only gives the same requirement $\Delta c = 0$ as before. While recalling for an EFT we have the underlying assumption $m^2 \ll \Lambda_5^2$. Considering the case $m^2 \ll |v| \ll \Lambda_5$, the first t derivative bound gives

$$\frac{\partial}{\partial t} f_{\tau_1\tau_2}(v,t) \propto \frac{m^2 v}{\Lambda_5^{10}}\Delta d + \mathcal{O}(\frac{m^4}{\Lambda_5^{10}}) > 0 \ . \tag{8.18}$$

Since the sign of v is not restricted, we obtain the second condition

$$\Delta d = 0 \ . \tag{8.19}$$

Surprisingly, unitarity and analyticity enforce the generic $\Lambda_5$ massive gravity to the special theory with $\Lambda_3$ massive gravity tuning. This further motivated that $\Lambda_3$ massive gravity is more possible to admit a Wilsonian UV completion, even though taking such tuning is not a prior natural thing to do.

## 8.2 $\Lambda_3$ Massive gravity

In the proof of the absence of ghost in section 4, we have seen the full 2-parameter dRGT theory in a simple form. It is however more explicit to see how the cutoff be raised to $\Lambda_3$ in ghost-free massive gravity by performing the tuning

$$c_1 = 2c_3 + \frac{1}{2}, \quad c_2 = -3c_3 - \frac{1}{2}, \quad d_1 = -6d_5 + \frac{3}{2}c_2 + \frac{5}{16}, \quad d_3 = 3d_5 - \frac{3}{4}c_3 - \frac{1}{16} \ . \tag{8.20}$$

**Forward limit**

The analytic results for the forward limit are rather difficult to obtain, while the strict approximation can be achieved by some numerical method. Consequently,

$$-0.0582 \approx \frac{9-2\sqrt{39}}{60} < c_3 < \frac{51+\sqrt{14613}}{546} \approx 0.315 \ , \tag{8.21}$$

$$\frac{1}{24}\left(-\frac{23}{8} + 9c_3 - 18c_3^2\right) < d_5 \lesssim 0.149 + 0.588c_3 + 3.21 \ , c_3^2 \tag{8.22}$$

$$d_5 > -0.122 - 0.461c_3 + 1.84c_3^2 \qquad \text{if} \quad -0.0582 \lesssim c_3 \lesssim 0 \ . \tag{8.23}$$

**First t derivative**

The first t derivative bound $\partial f_{SS}/\partial t$ gives a constraint in addition to $\Delta d = 0$ even if we have applied the $\Lambda_3$ tunning

$$25 + 4c_3(-37 + 63c_3) + 64d_5 > 0 \ . \tag{8.24}$$

**Higher t derivatives**

The residue of the t channel pole dominates when taking higher t derivatives

$$\mathrm{Res}_{t=m^2}\left[f_{\alpha\beta}\right] = \frac{1}{4} Y_\alpha Y_\beta + 12\left(c_1 - 1\right)^2 \alpha_S \alpha_{V_1} \beta_S \beta_{V_1} \ , \tag{8.25}$$

where the combinations

$$Y_\alpha = -2\alpha_T^2 + \left(3c_1 - 4\right)\alpha_V^2 + \left(6c_1 - 7\right)\alpha_S^2 \ . \tag{8.26}$$

With this must be positive for any definite states, the strictest constraint given by the scattering $\mathcal{T}_{0\pm 2}$. It gives

$$c_3 < \frac{2}{5} \quad \text{or} \quad c_3 > \frac{5}{6} \ , \tag{8.27}$$

which is automatically satisfied since the forward limit bounds already suggest $c_3 < 2/5$. As mentioned, if the EFT satisfies the leading forward and first t derivative bounds, then it will satisfy all higher t derivative bounds.

# 9 Unitarity implies more constraints?

The positivity bounds can be regarded as one sufficient example of how perturbative unitarity imposes important constraints on the physical Model. It is natural to think if there exist other manifestations in distinguishing positivity bounds that provide more constraints for generic theories. One possible consideration might be requiring the cancellation of unbounded growth of scattering amplitudes at the high energy level. This may lead to a set of specific relations among the coupling constants. We expect the feasibility of it on general grounds since the equations implied by perturbative unitarity uniquely reflect the spontaneously broken gauge structure [74, 13, 14] and thus might be derived by means of Slavnov-Taylor identities (STIs) as well [9]. Usually, constraints are appearing as polynomials which imply the relations between the physical coupling constants. These relations could help us to perform the renormalization when we are modeling the new physics as an extension of the standard model. For example, it may describe or restrict the general form of the dark matter interactions in realistic theory. Furthermore, if the above statement is also correct for effective field theories, it may provide additional constraints on their parameter space and more powerful tools to check the existence of their UV completion, together with positivity bounds.

## 9.1 The BRST invariance and Slavnov-Taylor identities

We start by considering a generic Lagrangian, which is an extension of the Standard Model (SM). It contains an arbitrary number of heavy scalars, fermion, and vector fields. As a starting point, we are mainly interested in bosonic interaction. Therefore, we will only study the vector and scalar fields in the following content. The interaction terms involving massless SM vector fields, photons and gluons, are fixed by $QED$ and $QCD$ gauge invariance. In particular, the massive-massless interaction terms are given in terms of the covariant derivative

$$(D_\mu)_{ij} = (\partial_\mu - ieQ_F A_\mu)\delta_{ij} - ig_s G^a_\mu T^a_{F,ij} \tag{9.1}$$

by the usual kinetic terms of the massive fields F. Here $T^a_{F,ij}$ and $G^a_\mu$ are representation of the respective gauge group $SU(3)$ and $U(1)$.

$$\begin{aligned}
\mathcal{L} = &\frac{i}{6} \sum_{v_1 v_2 v_3} g^{abc}_{v_1 v_2 v_3} (V^a_{v_1,\mu} V^b_{v_2,\nu} \partial^{[\mu} V^{c,\nu]}_{v_3} + V^c_{v_3,\mu} V^a_{v_1,\nu} \partial^{[\mu} V^{b,\nu]}_{v_2} + V^b_{v_2,\mu} V^c_{v_3,\nu} \partial^{[\mu} V^{a,\nu]}_{v_1}) \\
&+ \frac{1}{2} \sum_{v_1 v_2 s_1} g^{abc}_{v_1 v_2 s_1} V^a_{v_1,\mu} V^{b,\mu}_{v_2} h^c_{s_1} - \frac{i}{2} \sum_{v_1 s_1 s_2} g^{abc}_{v_1 s_1 s_2} V^{a,\mu}_{v_1} (h^b_{s_1} \partial_\mu h^c_{s_2} - (\partial_\mu h^b_{s_1}) h^c_{s_2}) \\
&+ \frac{1}{6} \sum_{s_1 s_2 s_3} g^{abc}_{s_1 s_2 s_2} (h^a_{s_1} h^b_{s_2} h^c_{s_3}) + \frac{1}{24} \sum_{s_1 s_2 s_3 s_4} g^{abcd}_{s_1 s_2 s_3 s_4} (h^a_{s_1} h^b_{s_2} h^c_{s_3} h^d_{s_4}) \\
&+ \frac{1}{8} \sum_{v_1 v_2 v_3 v_4} g^{abcd}_{v_1 v_2 v_3 v_4} (V^a_{v_1,\mu} V^{b,\mu}_{v_2} V^c_{v_3,\nu} V^{d,\nu}_{v_4}) + \frac{1}{4} \sum_{v_1 v_2 s_1 s_2} g^{abcd}_{v_1 v_2 s_1 s_2} (V^a_{v_1,\mu} V^{b,\mu}_{v_2} h^c_{s_1} h^d_{s_2}) \; .
\end{aligned} \tag{9.2}$$

It involves real vector fields $V_{v_i}$ and real physical scalar fields $h_{s_i}$ with non-zero masses $M_{v_i}$ and $M_{s_i}$, respectively. The indices $abc$ are color indices corresponding to $SU(3)$ gauge symmetry and do not strongly relate to the problem we study. In the following content, we will simply write $V_{v_i}$ as $V_i$ and $h_{s_i}$ as $s_i$.

We first start with a brief introduction to the Goldstone-boson equivalence theorem. It is well known that the Lagrangian of Standard model is invariant under the Becchi-Rouet-Stora (BRS) transformations. For the gauge-bosons and fermion fields, the BRS transformation is a gauge transformation with $\delta\theta^a(x) = g\delta\lambda u^a(x)$,

$$\begin{aligned}
&\delta_{BRS} V^a_\mu(x) = \delta\lambda D^{ab}_\mu u^b(x) = \delta\lambda(\partial_\mu \delta^{ab} - g f^{abc} V^c_\mu(x)) u^b(x) \equiv \delta\lambda s V^a_\mu(x)) \; , \\
&\delta_{BRS} \psi_i(x) = \delta\lambda u^a(x) ig(\Gamma^a \psi(x))_i \equiv \delta\lambda s \psi_i(x) \; , \\
&\delta_{BRS} \bar{\psi}_i(x) = -\delta\lambda u^a(x) ig(\bar{\psi}(x)\Gamma^a)_i \equiv \delta\lambda s \bar{\psi}_i(x) \; .
\end{aligned} \tag{9.3}$$

where $\delta\lambda$ is an infinitesimal constant which anti-commutes with the ghost fields $u^a(x)$ and $\bar{u}^a(x)$ . The $s$ is the BRS operator defined as the left derivative with respect to $\delta\lambda$ of the BRS transformed fields with the product rule,

$$s(FG) = (sF)G \pm FsG \ . \tag{9.4}$$

The transformations of the ghost fields are chosen to keep $\mathcal{L}_{fix} + \mathcal{L}_{ghost}$ BRS-invariant.

$$\begin{aligned} &\delta_{BRS} u^a(x) = -\delta\lambda \frac{1}{2} g f^{abc} u^b(x) \equiv \delta\lambda s u^a(x) \ , \\ &\delta_{BRS} \bar{u}^a(x) = -\delta\lambda \frac{1}{\xi} C^a\{A; x\} \equiv \delta\lambda s \bar{u}^a(x) \ , \\ &\delta(\mathcal{L}_{fix} + \mathcal{L}_{ghost}) = 0 \ . \end{aligned} \tag{9.5}$$

The Slavnov-Taylor identities are the Ward identities under the BRS symmetry, which give rise to relations between Green's functions. The generic form of these identities could be written as

$$0 = \frac{\delta_{BRS}}{\delta\lambda} \langle T \prod_l \Psi_{I_l} \rangle \ , \tag{9.6}$$

where $\prod_l \Psi_{I_l}$ is a combination of arbitrary fields. Consider the Green function involving one anti-ghost field and other on-shell physical fields $\prod_l \Psi^{phy}_{I_l}$

$$0 = \frac{\delta_{BRS}}{\delta\lambda} \langle T \bar{u}^a \prod_l \Psi_{I_l} \rangle \ . \tag{9.7}$$

Using $\delta_{BRS}\bar{u}^a = -\delta\lambda \frac{1}{\xi_a} C^a \equiv \delta\lambda s \bar{u}^a$ and $s\Psi_{I_l} = 0$,

$$\begin{aligned} 0 &= \frac{\delta_{BRS}}{\delta\lambda} \langle T \bar{u}^a \prod_l \Psi_{I_l} \rangle = s \langle T \bar{u}^a \prod_l \Psi_{I_l} \rangle \\ &= -\frac{1}{\xi_a} \langle T C^a \prod_l \Psi^{phy}_{I_l} \rangle + \sum_k \sigma_k \langle T \bar{u}^a (\prod_{l<k} \Psi_{I_l})(s\Psi_{I_k}) \prod_{m>k} \Psi_{l_m} \rangle \\ &\Rightarrow 0 = \langle T C^a \prod_l \Psi^{phy}_{I_l} \rangle \ . \end{aligned} \tag{9.8}$$

Now putting our gauge-fixing term $C$ in our STIs

$$0 = \langle T\{\partial^\mu V^\mu_{\bar{v}} + \sigma_{\bar{v}} M_{\bar{v}} \xi_{\bar{v}} \phi_{\bar{v}}\} \prod_l \Psi^{phy}_{I_l} \rangle \ . \tag{9.9}$$

In momentum space, it reads

$$0 = \langle T\{k^\mu V^\mu_{\bar{v}} - i\sigma_{\bar{v}} M_{\bar{v}} \xi_{\bar{v}} \phi_{\bar{v}}\} \prod_l \Psi^{phy}_{I_l} \rangle \ . \tag{9.10}$$

In order to obtain a relation for S-matrix elements, we need to truncate the external legs. We first introduce the propagator matrix

$$G^{\bar{v}}_{(\mu\nu)} = \begin{pmatrix} G^{\bar{v}v}_{\mu\nu} & G^{\bar{v}\phi}_{\mu} \\ G^{\bar{\phi}v}_{\nu} & G^{\bar{\phi}\phi} \end{pmatrix} = \begin{pmatrix} g^T_{\mu\nu} G^{vv}_T + g^L_{\mu\nu} G^{vv}_L & k_\mu G^{v\phi}_L \\ k_\nu G^{v\phi}_L & G^{\phi\phi} \end{pmatrix} \ , \tag{9.11}$$

where $g^T_{\mu\nu} = (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})$ and $g^L_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}$. We rewrite our STIs (2.8) as

$$\begin{aligned}
0 &= \langle T(k^\mu V^\mu_{\bar{v}} - i\sigma_{\bar{v}} M_{\bar{v}} \xi_{\bar{v}} \phi_{\bar{v}}) \prod_l \Psi^{phy}_{I_l} \rangle \\
&= \begin{pmatrix} k^\mu & i\sigma_{\bar{v}} M_{\bar{v}} \xi_{\bar{v}} \end{pmatrix} \begin{pmatrix} \langle T V^\mu_{\bar{v}} X \rangle \\ \langle T \phi_{\bar{v}} X \rangle \end{pmatrix} \\
&= \begin{pmatrix} k^\mu & i\sigma_{\bar{v}} M_{\bar{v}} \xi_{\bar{v}} \end{pmatrix} \begin{pmatrix} 2\langle T V^\mu_{\bar{v}} X \rangle \\ 2\langle T \phi_{\bar{v}} X \rangle \end{pmatrix} \\
&= \begin{pmatrix} k^\mu & i\sigma_{\bar{v}} M_{\bar{v}} \xi_{\bar{v}} \end{pmatrix} \begin{pmatrix} G^{\bar{v}v}_{\mu\nu} \langle T \underline{V^\mu_v} X \rangle + G^{\bar{v}\phi}_{\mu} \langle T \underline{\phi_v} X \rangle \\ G^{\bar{\phi}v}_{\nu} \langle T \underline{V^\mu_v} X \rangle + G^{\bar{\phi}\phi} \langle T \underline{\phi_v} X \rangle \end{pmatrix} \\
&= \begin{pmatrix} k^\mu & i\sigma_{\bar{v}} M_{\bar{v}} \xi_{\bar{v}} \end{pmatrix} \begin{pmatrix} G^{\bar{v}v}_{\mu\nu} & G^{\bar{v}\phi}_{\mu} \\ G^{\bar{\phi}v}_{\nu} & G^{\bar{\phi}\phi} \end{pmatrix} \begin{pmatrix} \langle T \underline{V^\mu_v} X \rangle \\ \langle T \underline{\phi_v} X \rangle \end{pmatrix} \\
&= \begin{pmatrix} k^\mu & i\sigma_{\bar{v}} M_{\bar{v}} \xi_{\bar{v}} \end{pmatrix} \begin{pmatrix} g^T_{\mu\nu} G^{vv}_T + g^L_{\mu\nu} G^{vv}_L & k_\mu G^{v\phi}_L \\ k_\nu G^{v\phi}_L & G^{\phi\phi} \end{pmatrix} \begin{pmatrix} \langle T \underline{V^\mu_v} X \rangle \\ \langle T \underline{\phi_v} X \rangle \end{pmatrix} \\
&= \begin{pmatrix} k^\nu (G^{vv}_L + i\sigma_{\bar{v}} M_{\bar{v}} \xi_{\bar{v}} G^{v\phi}_L) & k^2 G^{v\phi}_L + i\sigma_{\bar{v}} M_{\bar{v}} \xi_{\bar{v}} G^{\phi\phi} \end{pmatrix} \begin{pmatrix} \langle T \underline{V^\mu_v} X \rangle \\ \langle T \underline{\phi_v} X \rangle \end{pmatrix} \ ,
\end{aligned} \tag{9.12}$$

where $X = \prod_l \Psi^{phy}_{I_l} = (...)_{ph}$. This yields

$$\begin{aligned}
& k^\nu \langle T \underline{V^\nu_v} (...)_{ph} \rangle = i\sigma_{\bar{v}} M_v A_v(k^2) \langle T \underline{\phi_v} (...)_{ph} \rangle \\
& \Rightarrow \langle T\{k^\mu \underline{V^\mu_v} - i\sigma_{\bar{v}} M_v A_v(k^2) \underline{\phi_v} (...)_{ph} \rangle = 0, \\
& A_v(k^2) = \frac{\Gamma^{V_v V_{\bar{v}}}_L + \frac{k^2}{\xi_v}}{M_v (M_v - i\sigma_v \Gamma^{V_v \phi_{\bar{v}}}_L)} \ .
\end{aligned} \tag{9.13}$$

Here we have defined the vertex matrix as the inverse of the propagator matrix

$$\begin{aligned}
& \Gamma^v_{(\mu\nu)}(k,-k) = \begin{pmatrix} g^T_{\mu\nu} \Gamma^{v\bar{v}}_T(k^2) + g^L_{\mu\nu} \Gamma^{v\bar{v}}_L(k^2) & k_\mu \Gamma^{v\bar{\phi}}_L(k^2) \\ k_\nu \Gamma^{\phi\bar{v}}_L(k^2) & \Gamma^{\phi\bar{\phi}(k^2)} \end{pmatrix} \ , \\
& G^{\bar{v}}_{(\mu\lambda)} \Gamma^{v(\lambda\nu)} = i \begin{pmatrix} \delta^\nu_\mu & 0 \\ 0 & 1 \end{pmatrix} \ .
\end{aligned} \tag{9.14}$$

In the one-loop approximation, the $A_v(k^2)$ could be written as

$$A_v(k^2) = 1 - \frac{\Sigma_L^{vv}(k^2)}{M_v^2} - \frac{\Sigma^{v\phi}(k^2)}{M_v} \ , \tag{9.15}$$

where $\tilde{\Gamma}_L^{vv} = M_v^2 - \Sigma_L^{vv}$ and $\tilde{\Gamma}_L^{v\phi} = M_v + \Sigma^{v\phi}$. The second and third terms are corresponding to the loop correction. Working on the tree level, the STIs have the simplest form

$$0 = \langle T\{k^\mu \underline{V_v^\mu} - i\sigma_v M_v \xi_v \underline{\phi_v}\}(...)_{ph}\rangle \ . \tag{9.16}$$

**Extract unphysical couplings**

The Goldstone-boson equivalence theorem is an important consequence of the Slavnov-Taylor identities (STIs) in spontaneously broken gauge theories. It states that the amplitudes for reactions involving high-energetic, longitudinal vector bosons are asymptotically proportional to the amplitudes where these are replaced by their associated would-be Goldstone bosons [6]. One important application is to study the relation between the unphysical would-be Goldstone-boson and the physical ones. The physical scattering then can be used to determine the unphysical couplings in terms of physical couplings, and this could be done specifically in the method of spontaneously symmetry breaking. For example, the coupling of a fermion to a Goldstone-boson is in a typical model of electronic symmetry breaking and related to the couplings of fermion and massive vector boson. These identities follow directly from the above Slavonv-Taylor identities. For instance, the bosonic 3-point couplings for the Lagrangian (9.2) can be derived as follows

$$\begin{aligned} &\langle T\{k_\rho \underline{V_{3,\rho}} - i\sigma_{\bar{v}_3} M_{v_3} \underline{\phi_3}\}(V_{1,\mu}V_{2,\nu})\rangle = 0 \\ &\Rightarrow g_{v_1 v_2 \phi_3} = i\sigma_{v_3} \frac{M_{v_2}^2 - M_{v_1}^2}{M_{v_3}} g_{v_1 v_2 v_3} \ , \end{aligned} \tag{9.17}$$

$$\begin{aligned} &\langle T\{k^\mu \underline{V_1^\mu} - i\sigma_{\bar{v}_1} M_{v_1} \underline{\phi_1}\}(s_1 s_2)\rangle = 0 \\ &\Rightarrow g_{\phi_1 s_1 s_2} = i\sigma_{v_1} \frac{M_{s_1}^2 - M_{s_2}^2}{M_{v_1}^2} g_{v_1 s_1 s_2} \ , \end{aligned} \tag{9.18}$$

$$\begin{aligned} &\langle T\{k^\nu \underline{V_2^\nu} - i\sigma_{\bar{v}_2} M_{v_2} \underline{\phi_2}\}(s_1 V_{1,\mu})\rangle = 0 \\ &\Rightarrow g_{v_1 \phi_2 s_1} = -i\sigma_{v_2} \frac{1}{2M_{v_2}} g_{v_1 v_2 s_1} \ , \end{aligned} \tag{9.19}$$

$$\begin{aligned} &\langle T\{k^\nu \underline{V_2^\nu} - i\sigma_{\bar{v}_2} M_{v_2} \underline{\phi_2}\}\{k^\rho \underline{V_3^\rho} - i\sigma_{\bar{v}_3} M_{v_3} \underline{\phi_3}\}(V_{1,\mu})\rangle = 0 \\ &\Rightarrow g_{v_1 \phi_2 \phi_3} \sigma_{v_2} \sigma_{v_3} \frac{M_{v_2}^2 + M_{v_3}^2 - M_{v_1}^2}{2M_{v_2} M_{v_3}} g_{v_1 v_2 v_3} \ , \end{aligned} \tag{9.20}$$

$$
\begin{aligned}
&\langle T\{k^\mu \underline{V_1^\mu} - i\sigma_{\bar{v}_1} M_{v_1} \underline{\phi_1}\}\{k^\nu \underline{V_2^\nu} - i\sigma_{\bar{v}_2} M_{v_2} \underline{\phi_2}\}(s_1)\rangle = 0 \\
&\Rightarrow g_{\phi_1\phi_2 s_1} = -\sigma_{v_1}\sigma_{v_2} \frac{M_{s_1}^2}{2M_{v_1}M_{v_2}} g_{v_1 v_2 s_1} \ ,
\end{aligned}
\tag{9.21}
$$

$$
g_{\phi_1\phi_2\phi_3} = 0 \ . \tag{9.22}
$$

**Jacobi identity**

As well as 3-point functions, the STIs allow us to extract 4-point couplings related to unphysical would-be Goldstone-boson. However, since to keep the interaction $\phi VVV$ Lorentz invariant, it would at least contains one partial derivative operator and then has a dimension greater than 4. This breaks the renormalizability of our theory, and thus $g_{\phi vvv} = 0$. This implies the corresponding 4-point function would yield an identity that includes only 3-point couplings and 4-point physical couplings. After substituting the expression of 3-point unphysical couplings in terms of physical ones, we may derive a polynomial only including physical couplings. Considering

$$
\langle T\{k^\mu \underline{V_{v_1}^\mu} - i\sigma_{\bar{v}_1} M_{v_1} \underline{\phi_{v_1}}\}(V_{2,\nu} V_{3,\rho} V_{4,\sigma})\rangle = 0 \ . \tag{9.23}
$$

The identity is manifestly containing both physical and unphysical couplings. we can however substitute the results in (9.17)-(9.22) to replace all unphysical couplings with expressions in terms of the physical ones. Schematically, the identity would ultimately come in a form

$$
\sum_{v_5} (gg + ...) + \sum_{s_5} (gg + ...) = 0 \ . \tag{9.24}
$$

Explicitly, we obtain

$$
\begin{aligned}
0 = &[\epsilon^*(k_3)\cdot\epsilon^*(k_4)][\epsilon(k_2)\cdot k_3]\{-2g_{v_1v_2v_3v_4} + \sum_{v_5}(g_{v_1v_2v_5}g_{v_3v_4\bar{v_5}} + 2g_{v_1v_4v_5}g_{v_2v_3\bar{v_5}})\} \\
&+ [\epsilon^*(k_3)\cdot\epsilon^*(k_4)][\epsilon(k_2)\cdot k_4]\{-2g_{v_1v_2v_3v_4} + \sum_{v_5}(-g_{v_1v_2v_5}g_{v_3v_4\bar{v_5}} + 2g_{v_1v_3v_5}g_{v_2v_4\bar{v_5}})\} \\
&+ [\epsilon(k_2)\cdot\epsilon^*(k_3)][\epsilon^*(k_4)\cdot k_2]\{g_{v_1v_2v_3v_4} + \sum_{v_5}(g_{v_1v_4v_5}g_{v_2v_3\bar{v_5}} - 2g_{v_1v_3v_5}g_{v_2v_4\bar{v_5}})\} \\
&+ [\epsilon(k_2)\cdot\epsilon^*(k_3)][\epsilon^*(k_4)\cdot k_3]\{g_{v_1v_2v_3v_4} + \sum_{v_5}(-g_{v_1v_4v_5}g_{v_2v_3\bar{v_5}} - 2g_{v_1v_2v_5}g_{v_3v_4\bar{v_5}})\} \\
&+ [\epsilon(k_2)\cdot\epsilon^*(k_4)][\epsilon^*(k_3)\cdot k_2]\{g_{v_1v_2v_3v_4} + \sum_{v_5}(g_{v_1v_3v_5}g_{v_2v_4\bar{v_5}} - 2g_{v_1v_4v_5}g_{v_2v_3\bar{v_5}})\} \\
&+ [\epsilon(k_2)\cdot\epsilon^*(k_4)][\epsilon^*(k_3)\cdot k_4]\{g_{v_1v_2v_3v_4} + \sum_{v_5}(-g_{v_1v_3v_5}g_{v_2v_4\bar{v_5}} + 2g_{v_1v_2v_5}g_{v_3v_4\bar{v_5}})\}\ ,
\end{aligned}
\tag{9.25}
$$

which is an identity with three independent coefficients. By taking a specific set of polarization vectors and momentums, it can lead to the Jacobi identity

$$
\sum_{v_5}(g_{v_1v_2v_5}g_{v_3v_4\bar{v_5}} + g_{v_1v_4v_5}g_{v_2v_3\bar{v_5}} + g_{v_3v_1v_5}g_{v_2v_4\bar{v_5}}) = 0\ . \tag{9.26}
$$

Furthermore, identity (9.25) also includes relations between physical 4-point couplings and 3-point couplings as non-trivial constraints. For example,

$$
g_{v_1v_2v_3v_4} = \sum_{v_5}(g_{v_1v_4v_5}g_{v_2v_3\bar{v_5}} + g_{v_1v_3v_5}g_{v_2v_4\bar{v_5}})\ . \tag{9.27}
$$

## 9.2 Feasibility for EFTs

Although it appears that unitarity may give polynomial constraints in distinguishing positivity bounds, it is rather strongly rely on the Standard Model-like behavior. If one considers the feasibility of these constraints for EFTs, the following requirements should be satisfied and an assumption has to be made: the standard S-matrix properties (unitarity, analyticity, etc.), the spontaneous symmetry breaking structure (SBGT's), the BRST invariance, and the existence of a weakly coupled UV completion with renormalizability[3]. Combining these would extremely restrict the form of an effective field

[3]By definition of Standard UV completion, it should be renormalizable. Nevertheless, from a physics point of view, the UV complete theory is only needed to be finite in the high energy level and the content of renormalization might be absent

theory. One may consider some relaxations of some requirements. However, the SBGT's and BRST symmetry is the precondition for applying the STIs, while violating the renormalizability would directly give trivial results. We shall see this by an example as follows.

### Proca EFT

Considering the Proca EFT in section 7.5 but in non-unitary gauge, so the Stükelberg/ Goldstone mode appears and the SBGT occurred with the heavy Higgs boson has integrated out

$$
\begin{aligned}
g_*^2\mathcal{L}_{\text{Proca}} \supset & -\frac{1}{4}F_\mu^\nu F_\nu^\mu - \frac{1}{2}\phi_\mu^2 + \frac{a_0}{\Lambda_\phi^4}\phi_\mu^4 + \frac{a_1}{\Lambda_\phi^3}\partial_\mu\phi_\nu\phi^\mu\phi^\nu \\
& + \frac{1}{\Lambda_\phi^6}\left(a_3\left(\phi_\mu\partial^\mu\phi_\nu\right)^2 + a_4\left(\partial_\mu\phi_\nu\phi^\nu\right)^2 + a_5\phi_\mu^2\partial_\alpha\phi_\beta\partial^\beta\phi^\alpha\right) \\
& + \frac{1}{\Lambda_A^4}\left(c_1 F_\nu^\mu F_\rho^\nu F_\sigma^\rho F_\mu^\sigma + c_2\left(F_\mu^\nu F_\nu^\mu\right)^2\right) \\
& + \frac{m^4}{\Lambda_\phi^6}\left(C_1\phi_\mu\phi^\nu F^{\alpha\mu}F_{\alpha\nu} + C_2\phi_\mu^2 F_{\alpha\beta}^2\right) \ ,
\end{aligned}
\tag{9.28}
$$

$$
\phi_\mu = D_\mu = \partial_\mu\phi + mA_\mu \ . \tag{9.29}
$$

To start, we take the gauge choice $C = \partial_\mu A^\mu - m\phi = 0$ and add the corresponding gauge-fixing term to eliminate the cross term $\phi\partial A$. To apply STIs, one should be careful of the physical part which should be inserted in (9.8). Although the field $A_\mu$ and $\phi$ carry degrees of freedom, they are not BRST invariant. Instead, one should only consider the field strength $F_{\mu\nu}$, which is manifestly BRST invariant. For the 4-point function, we have the STI

$$
< T\{\partial_\mu A^\mu - m\phi\}(F_{\nu\rho}F^{\rho\beta}F_\beta^\nu) >= 0 \ . \tag{9.30}
$$

Since the Goldstone mode has already explicitly appeared (so as the couplings) in the above Lagrangian, the interaction terms in the first, second, and fourth lines should manifestly only give 0 contributions for the STI (9.30). These terms should not give any new information about the parameter space. The only concern is whether terms in the third line would give a non-trivial contribution and ultimately lead to a constraint, which has a linear form

$$
ac_1 + bc_2 = 0 \ . \tag{9.31}
$$

To verify this possibility, we shall first write the third line in a more explicit form

$$
\begin{aligned}
& c_1 F^{\mu}_{\nu} F^{\nu}_{\rho} F^{\rho}_{\sigma} F^{\sigma}_{\mu} + c_2 \left(F^{\nu}_{\mu} F^{\mu}_{\nu}\right)^2 \\
& = c_1 (2\partial_\mu A_\nu \partial^\nu A^\rho \partial^\mu A_\sigma \partial^\sigma A_\rho - 4\partial_\mu A_\nu \partial^\nu A^\rho \partial^\mu A_\sigma \partial_\rho A^\sigma \\
& - 2\partial_\mu A_\nu \partial^\nu A^\rho \partial_\sigma A^\mu \partial^\sigma A_\rho + 2\partial_\mu A_\nu \partial^\nu A^\rho \partial_\sigma A^\mu \partial_\rho A^\sigma \\
& + 2\partial_\mu A_\nu \partial^\rho A^\nu \partial^\mu A_\sigma \partial_\rho A^\sigma + 2\partial_\mu A_\nu \partial^\rho A^\nu \partial_\sigma A^\mu \partial^\sigma A_\rho \\
& - 2\partial_\mu A_\nu \partial^\rho A^\nu \partial_\sigma A^\mu \partial_\rho A^\sigma) \\
& + 4c_2 (\partial_\mu A_\nu \partial^\mu A^\nu \partial^\rho A^\sigma \partial_\rho A_\sigma - 2\partial_\mu A_\nu \partial^\mu A^\nu \partial^\sigma A^\rho \partial_\rho A_\sigma \\
& + \partial_\mu A_\nu \partial^\nu A^\mu \partial^\sigma A^\rho \partial_\rho A_\sigma) \ .
\end{aligned}
\tag{9.32}
$$

According to the above argument, the STI can be reduced to

$$
< T\partial_\mu A^\mu (F_{\nu\rho} F^{\rho\beta} F^{\nu}_{\beta}) >= 0 \tag{9.33}
$$

with all vertices only coming from (9.32). This identity gives a trivial result only if contributions from terms which proportional to $c_1$ and $c_2$ respectively vanish by themselves. For simplicity, we evaluate the $c_2$ part of the right hand of the identity (9.33) to show the actual situation. Explicitly,

$$
\begin{aligned}
4c_2 \{ & [(p_1 \cdot p_2)(p_3 \cdot p_4)(p_1 \cdot p_3)(p_2 \cdot p_4) - 2(p_1 \cdot p_2)(p_1 \cdot p_3)(p_2 \cdot p_3)p_4^2 \\
& + (p_1 \cdot p_2)(p_1 \cdot p_3)(p_2 \cdot p_3)p_4^2] \\
& - [2(p_1 \cdot p_2)(p_3 \cdot p_4)(p_1 \cdot p_3)(p_2 \cdot p_4) - 2(p_1 \cdot p_2)(p_1 \cdot p_3)(p_2 \cdot p_4)(p_3 \cdot p_4) \\
& + (p_1 \cdot p_2)(p_1 \cdot p_3)(p_2 \cdot p_4)(p_3 \cdot p_4)] \\
& - [(p_1 \cdot p_2)(p_3 \cdot p_4)(p_1 \cdot p_2)(p_3 \cdot p_4) - 2(p_1 \cdot p_2)(p_1 \cdot p_4)(p_2 \cdot p_3)(p_3 \cdot p_4) \\
& + (p_1 \cdot p_2)(p_1 \cdot p_4)(p_2 \cdot p_3)(p_3 \cdot p_4)] \\
& + [(p_1 \cdot p_2)(p_3 \cdot p_4)(p_1 \cdot p_3)(p_2 \cdot p_4) - 2(p_1 \cdot p_2)(p_1 \cdot p_4)(p_2 \cdot p_4)p_3^2 \\
& + (p_1 \cdot p_2)(p_1 \cdot p_4)(p_2 \cdot p_4)p_3^2] \\
& - [(p_1 \cdot p_2)(p_3 \cdot p_4)(p_1 \cdot p_4)(p_2 \cdot p_3) - 2(p_1 \cdot p_2)(p_1 \cdot p_3)(p_2 \cdot p_3)p_4^2 \\
& + (p_1 \cdot p_2)(p_1 \cdot p_3)(p_2 \cdot p_3)p_4^2] \\
& + [2(p_1 \cdot p_2)(p_3 \cdot p_4)(p_1 \cdot p_4)(p_2 \cdot p_3) - 2(p_1 \cdot p_2)(p_1 \cdot p_4)(p_2 \cdot p_3)(p_3 \cdot p_4) \\
& + (p_1 \cdot p_2)(p_1 \cdot p_4)(p_2 \cdot p_3)(p_3 \cdot p_4)] \\
& + [(p_1 \cdot p_2)(p_3 \cdot p_4)(p_1 \cdot p_2)(p_3 \cdot p_4) - 2(p_1 \cdot p_2)(p_1 \cdot p_3)(p_2 \cdot p_4)(p_3 \cdot p_4) \\
& + (p_1 \cdot p_2)(p_1 \cdot p_3)(p_2 \cdot p_4)(p_3 \cdot p_4)] \\
& - [(p_1 \cdot p_2)(p_3 \cdot p_4)(p_1 \cdot p_4)(p_2 \cdot p_3) - 2(p_1 \cdot p_2)(p_1 \cdot p_4)(p_2 \cdot p_4)p_3^2 \\
& + (p_1 \cdot p_2)(p_1 \cdot p_4)(p_2 \cdot p_4)p_3^2]\} \ .
\end{aligned}
\tag{9.34}
$$

It is tedious but straightforward to check this quantity finally yields 0. We can thus conclude that the STIs trivially hold, and there is no polynomial constraint in analogy to (9.25)-(9.27) can be read off by unitarity and renormalizability, at least for non-interacting Proca EFT. It may be expected that coupling the theory to matter or considering multiple massive spin-1 fields could change this situation differently. However, even if these polynomial constraints can be true for the interacting Proca theory or some other LEEFTs, we are difficult to consider they have equivalent universality with positivity bounds. With identities in parameter space being more powerful constraints than positivity, the assumptions we made are extremely limiting in our search for feasible theories. Especially, the Higgs-like mechanism for massive gravity theories is still absent at present. This leads to no evidence that massive spin-2 theories should enjoy a spontaneous symmetry breaking structure. Furthermore, to the best of the author's knowledge, there has no BRST-like symmetry for gravitational theory in 4D or higher been yet explored. The renormalizability also does not tend to be the necessary property of all UV complete theories.

# 10 Discussion and Outlook

The origin of the concept of massive gravity can be traced back to a century ago, the consistency with GR was however only resolved in the past few decades. It is even more recently that the possibility of evading the ghost has been proven that continues the recent interest in massive gravity theories. The appearance of superluminalities with Vainshtein mechanism should be understood in more depth. The difficulty in finding fully-fledged cosmological and black holes solutions together with the absence of non-trivial spatially flat FRW solutions imply that the phenomenology of massive gravity theories is still waiting to be explored. On the other hand, there are also many unexplored questions about the non-perturbative quantum properties of massive theories, for instance the potential modification of black hole thermodynamics, Hawking radiation or holography by a massive graviton.

One of the major concerns of massive gravity is the existence of a possible UV completion. The ghost-free massive gravity is still in a low cutoff $\Lambda_3$, while the corresponding coefficient tuning is even non-natural (only technically natural). It would be a great step forward if a partial UV completion at least raises the cutoff to the Planck scale $M_P$. Such a UV completion may be sufficient to explain the naturalness problem. Nevertheless, the UV approach processes for these new gravitational EFTs which have a strong coupling scale associated with Vainshtein mechanism are yet difficult to establish. The information above the scale $\Lambda_3$ such as how would new degrees of freedom entry is still less known. And it is extremely important to re-consider the ghost problem beyond the $\Lambda_3$. The massive gravity is relatively well understood in AdS space. In recent years, there have been worked on holography of massive gravitons in AdS/CFT [2, 54, 55, 65] which proves the existence of possible UV completion of massive gravity in AdS space. It is also remarkable that massive gravity in 2d spacetime is UV complete since it would be equivalent to string theory through $T\bar{T}$ deformation [75].

Although there are some theories explicitly break the Lorentz invariance have been well-developed, it is more inclined to believe massive gravity admits (if that is the case) a standard Lorentz invariant UV completion analogous to Higgs mechanism which provides masses for vectors. Generically, without any explicit form of UV physics, unitarity, locality and analyticity still provide powerful constraints for EFTs. Assuming the potential UV completion is weakly coupled, positivity bounds can be applied on the tree-level scattering amplitudes and strictly restrict the parameter space of the Wilsonian effective action. This can on the other hand be turned into a tool to check the existence of UV completion. Violating any of these infinity bounds would directly imply the absence of any possible local, unitary and Lorentz invariant UV completion for the LEEFTs. In this review we have shown this from another perspective provides the necessity and (from some point of view) naturalness of $\Lambda_3$ massive gravity tuning.

The polynomial constraints might be powerful to provide strong constraints on the generic extension of renormalizable theories, it is however may not applicable for most of EFTs such as massive gravity. While there are many other useful tools for testing the existence of its UV completion in development, for instance, the causality constraints [11, 25, 10]. With many theoretical obstacles overcame but bring new sets of challenges, massive gravity remains an extremely young and active field that on its own path to becoming the alternative to GR.

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