A Holographic Approach to the Black Hole Information Paradox

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Abstract

This dissertation focuses on how recent developments in holography could be applied to the black hole evaporation process, to get closer to resolving the black hole information paradox. We begin by presenting some of the required background on Hawking radiation and the AdS/CFT correspondence, before focusing on the quantum extremal surfaces prescription for holographic entanglement entropy. We show how quantum extremal surfaces and islands recover the Page curve and show that this description of the system is consistent with unitarity, as well as how this description suggests information might be conserved.

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Chapter 1

Introduction

Black holes have long been of interest to theoretical physicists. Since Karl Schwarzschild first discovered black holes as features of spherically symmetric solutions to Einstein's field equations, there have been questions about what insights black holes might give in the context of the fundamental theories of physics. Hawking's landmark publication in 1973 showed using quantum field theory in a curved spacetime (an intermediate step towards a theory of quantum gravity) that black holes emit radiation as a perfect black body, and so have a defined temperature. In radiating black holes slowly "evaporate". This groundbreaking result was the first step towards considering black holes as quantum systems, but threw up many questions in itself. One central question concerned information. If one threw a book into a black hole which proceeded to evaporate into Hawking radiation, what happens to the information contained in the book? This idea forms a part of what is known as the black hole information paradox.

There are currently many competing ideas about how this paradox could be resolved. A small minority of theorists believe that information is indeed lost; the majority of notable ideas aim to show that the information can be recovered. Some theories, such as the fuzzball resolution, opt to modify common understanding about the geometry of the black hole itself in order to recover the missing entropy. However, most widely accepted ideas about the paradox believe that in order to fully understand where the information goes a theory of quantum gravity must be formulated. The most well-explored candidate theories of quantum gravity is string theory. The string theory community argue that the horizon is of great importance in understanding the black hole system. String theorists have been looking at the paradox from a number of different angles. One such angle has been through the holographic principle. Recent developments in the AdS/CFT, the most explored example of holography to date, have had a great influence on the string theory community's ideas about black holes. This dissertation aims to give an overview of how AdS/CFT and the holographic principle more generally have recently been used by theorists to understand the entropy of the black hole system, and how this approach makes the unitarity of the black hole system manifest. It aims to give a brief overview of the key literature, along with accompanying derivations.

We begin by outlining some key results from classical considerations of black holes, going over the development of uniqueness theorems as well as the laws of black hole mechanics and their relation to the laws of thermodynamics. We also review some basic notions of the entropy of quantum systems. We then consider quantum field theory on a curved spacetime, following Hawking, to arrive at his formulation of Hawking radiation. We are then in a position to precisely frame the black hole information paradox, and how we might go about viewing black holes as quantum systems. We then aim to provide a brief overview of AdS/CFT and holography, before reviewing recent literature on holographic solutions to the black hole information paradox.

Throughout the dissertation we use a mostly-plus Minkowski metric signature, $\eta_{\mu\nu} = diag(-,+,..,+)$. We take the physical constants $\hbar = c = k_B = G_N = 1$.

Chapter 2

Background

2.1 Classifying Black Holes

Throughout the 20th century, after Karl Schwarzschild's initial discovery of black holes, there were many efforts to expand on this and find other exact black hole solutions. One such solution, the Kerr-Newman metric, describes a charged, rotating black hole in a vacuum. The Kerr-Newman metric [33] in Boyer-Lindquist coordinates is

$$\begin{aligned} ds^2 = &- \frac{\Delta - a^2 sin^2\theta}{\Sigma}\, dt^2 \; - 2a\, sin^2\theta\, \frac{r^2 + a^2 - \Delta}{\Sigma}\, dt\, d\phi \\ &+ \frac{(r^2 + a^2)^2 - \Delta}{\Sigma}\, sin^2\theta\, d\phi^2 \; + \frac{\Sigma}{\Delta}\, dr^2 \; + \Sigma\, d\theta^2 \end{aligned}$$

where

$$\Sigma = r^2 + a^2 cos^2\theta$$

$$\Delta = r^2 - 2Mr + a^2 + e^2.$$

Upon inspection of the metric it can be seen that the three parameters that can describe individual black holes with this metric are M, a, and e. Here, a is the total angular momentum J per unit mass,

$$a = \frac{J}{M},$$

and e can be expressed in terms of electric and magnetic charges Q and P as

$$e = \sqrt{Q^2 + P^2}\,.$$

After the publication of this metric, many uniqueness theorems emerged. In 1967, Israel conjectured that for static, asymptotically flat, vacuum black hole spacetimes that are “suitably regular” on and outside the event horizon, then the spacetime must be isometric to the Schwarzschild solution [27]. Further theorems by Carter in 1971 [10], Robinson in 1975 [41], Bunting and Masood in 1987 [7], and Sudarsky and Wald in 1992 [6] developed this idea. Carter and Robinson conjectured the no-hair theorem, stating that all stationary electrovac black hole solutions can be described by the three parameters that appear in the Kerr-Newman metric: mass M, total angular momentum J, and electromagnetic charge e. The various uniqueness theorems have been put together to show that the Kerr-Newman metric is the most general stationary, asymptotically flat isolated black hole solution to Einstein's field equations.

2.2 The Laws of Black Hole Mechanics

In 1973, Bardeen, Carter, and Hawking published a paper [3] which outlined the four laws of black hole mechanics. Emerging only from classical general relativity, there are clear parallels that can be drawn between the laws of black hole mechanics and the laws of thermodynamics.

The 0th law of black hole mechanics states that for a stationary black hole spacetime with $T_{\mu\nu}$ obeying the dominant energy condition[1], the surface gravity κ on the future event horizon $\mathcal{H}^+$ is constant. The surface gravity on $\mathcal{H}^+$ has parallels with thermodynamic systems at equilibrium, which have constant temperature as stated by the 0th law of thermodynamics.

The 1st law of black hole mechanics is an expression of the conservation of energy in black holes. The three classical fundamental quantities of black holes are mass M, charge Q, and angular momentum J. The 1st law states that for stationary black hole with these fundamental quantities perturbed to a new state with $M + dM$, $Q + dQ$, and $J + dJ$, these quantities will change with respect to change in black hole area dA as:

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ,$$

where κ is the surface gravity on $\mathcal{H}^+$, Ω_H is the angular velocity, and Φ_H is the electric surface potential. We see the analogy with classical thermodynamics by re-

[1]Let a^μ be any causal 4-vector i.e. $a_\mu a^\mu \leq 0$ and $a^0 > 0$. Then the dominant energy condition requires the stress-energy tensor $T^{\mu\nu}$ to satisfy the the condition that $b^\mu = -T^{\mu\nu} a_\nu$ is causal.

lating entropy with the area of $\mathcal{H}^+$, and by relating the quantity $\frac{\kappa}{8\pi}$ to the temperature.

The 2nd law, Hawking's area theorem, states that for a spacetime where the weak energy condition and cosmic censorship holds, then the area of $\mathcal{H}^+$ is non-decreasing with time, or

$$dA \geq 0.$$

The 3rd law of black hole mechanics is the statement that it is impossible to reduce κ to 0 by any finite number of operations. This is analogous to the third law of thermodynamics, which states that absolute zero temperatures cannot be obtained by a finite number of operations.

From these laws of black hole mechanics, a notion of the thermodynamic entropy of a black hole, known as the Bekenstein-Hawking entropy emerges. This is given as a function of area of the future event horizon, A,

$$S_{BH}(A) = \frac{A}{4G_N}, \tag{2.1}$$

where Newton's gravitational constant G_N is explicitly included. Alternatively, by reintroducing Planck's constant and the speed of light to make the right hand side dimensionless, we may replace G_N in the above with l_p^2, the Planck length squared. Although we have derived this law classically, in inserting the Planck length we can begin to see how we might start thinking about the quantum nature of black hole entropy. Bekenstein also showed that entropy is proportional to the area of the future event horizon in 1973 using arguments from classical information theory [4], but Hawking precisely identified the constant of proportionality as $1/4G_N$.

2.3 The Generalised Second Law

Following the derivation of the Bekenstein-Hawking entropy, Bekenstein reconsidered the second law of black hole mechanics in more depth [5]. Bekenstein was concerned with reconciling previous notions of thermodynamic entropy with the Bekenstein-Hawking entropy. Bekenstein considered the effect of a highly entropic system entering a black hole; the entropy of spacetime outside the black hole, $S_{external}$ would decrease. In order to avoid violating the second and third laws of thermodynamics the black hole's entropy, S_{BH}, would have to increase enough to compensate for the loss in $S_{external}$. This led to the formulation of the generalised second law: for any black hole spacetime, the total entropy of the spacetime, which is known as the generalised entropy, cannot decrease. More formally,

$$dS_{gen} = dS_{BH} + dS_{external} \geq 0. \tag{2.2}$$

It is worth reiterating that this law has been obtained by restricting ourselves to classical general relativity. However, the generalised second law has important implications for our consideration of black holes as quantum systems, as we will see in later chapters.

2.4 Quantum Descriptions of Entropy

We now turn to quantum formulations of entropy.

Recall that all quantum systems can be described by state vector $|\psi\rangle$, which resides on the Hilbert space of all states $\mathcal{H}$. We can then define a density operator ρ, which acts on $\mathcal{H}$. Concretely, ρ is generally a linear combination of projections of state vectors, given by

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| , \tag{2.3}$$

where p_i is the probability of the system being in state $|\psi_i\rangle$. A pure state has only one such possibility, so can be expressed as

$$\rho = p |\psi\rangle \langle\psi| . \tag{2.4}$$

If it cannot be put into this form, the state is said to be mixed. Density operators are extremely useful in that they encode all information about the quantum system. From here we can find the expectation value of some general observable $\hat{A}$ as

$$\langle\hat{A}\rangle = tr(A\rho). \tag{2.5}$$

2.4.1 Fine-grained Entropy

We now arrive at our first formulation of entropy, the fine-grained entropy. This is the von Neumann entropy, which is given by

$$S_{vN} = -tr(\rho \, log \, \rho) = -\sum_i p_i \, log \, p_i, \tag{2.6}$$

where p_i are eigenvalues of ρ. For pure states, this quantity vanishes. We may think of this formulation of entropy as describing how little information we have about the precise quantum system [34].

We may use quantum statistical mechanics to note that thermal states are necessarily mixed states at the quantum level. This fact will be useful in our formulation of the black hole information paradox in the next chapter.

One important property of von Neumann entropy is that it is invariant under unitary time evolution, i.e. $S(\rho) = S(U\rho U^{-1})$. Thus we see that von Neumann entropy is constant under some time evolution of the density operator.

From the von Neumann entropy we can also define the entanglement entropy. Consider a bipartite system with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We can find the density matrix of $\mathcal{H}_B$ from the density matrix of $\mathcal{H}$, ρ_{total}, by tracing over all states in subsystem A:

$$\rho_B = tr_A(\rho_{total}). \tag{2.7}$$

The entanglement entropy of subsystem B is then defined as

$$S_E(B) = -tr_B(\rho_B \, log \, \rho_B). \tag{2.8}$$

It can be trivially shown that for systems in a pure state, $S_E(A) = S_E(B)$. Another useful property of entanglement entropy is strong subadditivity, which states that for some system with three subsystems A, B, and C with no overlaps, we have the inequality relations:

$$S_E(A+B+C) + S_E(B) \leq S_E(A+B) + S_E(B+C), \tag{2.9}$$

$$S_E(A) + S_E(C) \leq S_E(A+B) + S_E(B+C). \tag{2.10}$$

Though highly involved to prove using algebra [30], these inequalities can be seen as a requirement for any entropy to be an entanglement entropy. This point will be useful in later discussion.

2.4.2 Coarse-grained Entropy

We now turn to a different definition of entropy. Consider a system with density operator ρ, where we do not measure all variables, but a subset of macroscopic "coarse-grained" variables A_i such that $\langle A_i \rangle = tr(\rho A_i)$. We then find all density matrices $\tilde{\rho}$

which give the same result as the overall system for variables A_i, i.e. $tr(\tilde{\rho}A_i) = tr(\rho A_i)$. We then find the $\tilde{\rho}$ that gives the maximum von Neumann entropy and take this entropy value to be the coarse grained entropy [2].

This is our usual thermodynamic entropy, which obeys the second law of thermodynamics - it increases under unitary time evolution, where fine-grained entropy stays the same. The Bekenstein-Hawking entropy of a black hole, (2.1), is a thermodynamic formulation of entropy, so is the coarse-grained entropy of a black hole.

A useful result from the definition of coarse-grained entropy is that since by definition ρ is the $\tilde{\rho}$ which minimises the coarse-grained entropy, we have the relation

$$S_{vN} \leq S_{coarse}. \tag{2.11}$$

The coarse-grained entropy is dependent on the "coarse-grained" variables chosen, and so will always be higher than the variable-independent fine-grained entropy.

Chapter 3

The Black Hole Information Paradox

Let us now formulate the problem central to this dissertation: the black hole information paradox. We will begin by deriving Hawking radiation, a key component of the information paradox. We will then consider this in the context of our understanding of black hole mechanics and entropy from the previous chapter.

3.1 Quantum Field Theory in a Curved Spacetime

In order to be able to derive Hawking radiation, we must formulate Quantum Field Theory in a curved background. This is non-trivial since QFT relies on many of the symmetries of Minkowski spacetime. For example, a general curved background no longer generally has Poincare symmetry. In our treatment we will be neglecting the back-reaction of any fields on the metric.

3.1.1 Quantizing the Free Scalar

Let us begin with the most simple quantum field: the real Klein-Gordon scalar. We consider a general globally hyperbolic spacetime $(\mathcal{M}, g)$, i.e. a spacetime which admits a Cauchy surface Σ.

The general significance of the existence of a Cauchy surface is that the spacetime has a clear causal structure: for some initial data defined on Σ, it is possible to determine how this data evolves across all of $\mathcal{M}$ [44]. In the context of quantum fields, specifying some field solution ϕ on Σ uniquely defines solutions across the whole spacetime.

We now consider a scalar $\phi = \phi(t, \boldsymbol{x})$ on $(\mathcal{M}, g)$ with Lagrangian

$$\mathcal{L} = \frac{1}{2} g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} m^2 \phi^2. \tag{3.1}$$

Using the Euler-Lagrange equation, we obtain the equation of motion

$$(g^{\alpha\beta} \nabla_\alpha \nabla_\beta - m^2)\phi = 0, \tag{3.2}$$

the Klein-Gordon equation for some general metric g. We may use the definition of the covariant derivative to re-express this as

$$\left(\frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} g^{\alpha\beta} \partial_\beta) - m^2 \right) \phi = 0, \tag{3.3}$$

The canonically conjugate momentum to ϕ is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\nabla_t \phi)} = \sqrt{-g} \nabla_t \phi. \tag{3.4}$$

We now proceed to quantise in the same manner as for QFT in Minkowski spacetime - by promoting ϕ and π to operators $\hat{\phi}$ and $\hat{\pi}$ respectively, and imposing the equal time commutation relations:

$$[\hat{\phi}(t, \vec{x}), \hat{\phi}(t, \vec{y})] = 0, \tag{3.5}$$

$$[\hat{\pi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = 0, \tag{3.6}$$

$$[\hat{\phi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = \frac{i}{\sqrt{-g}} \delta^{(d-1)}(\vec{x} - \vec{y}), \tag{3.7}$$

It is now useful to define a Hilbert space of states that these operators act on. Define $\mathcal{K}$ to be the Hilbert space of complex solutions of our above generalised Klein-Gordon equation. For $\alpha, \beta \in \mathcal{K}$, we define inner product

$$(\alpha, \beta) \equiv \int_\Sigma d\Sigma \, \sqrt{-g} (\alpha^* \nabla_\mu \beta - \beta \nabla_\mu \alpha^*), \tag{3.8}$$

for some Cauchy surface Σ defined on the manifold. We see that

$$(\alpha, \alpha) = -(\alpha^*, \alpha^*), \tag{3.9}$$

$$(\alpha, \alpha^*) = 0. \tag{3.10}$$

Using the above we see that letting $\{f_i\}$ be the complete set of positive norm solutions of the Klein Gordon equation, $\{f_i^*\}$ is manifestly the complete set of negative norm solutions, and hence we form a complete set of solutions $\{f_i, f_i^*\}$ [14].

In Minkowski space, the set of positive norm solutions are positive frequency plane wave modes, and negative norm solutions are negative frequency plane wave modes. Hence the Hilbert space can be decomposed into subspaces,

$$\mathcal{K} = \mathcal{K}_p \oplus \mathcal{K}_p^*, \tag{3.11}$$

with $\mathcal{K}_p$ corresponding to positive frequency modes and $\mathcal{K}_p^*$ corresponding to negative frequency modes [40].

We can now generalise this result for stationary spacetimes. Recall that a spacetime which is asymptotically flat at null infinity, $\mathcal{I}^\pm$, is stationary if it admits a Killing vector field k which is timelike at $\mathcal{I}^\pm$. The Killing vector k generates a symmetry of the spacetime, so the Lie derivative of the scalar ϕ with respect to k, $\mathcal{L}_k\phi$, is a solution to the Klein-Gordon equation provided that ϕ is a solution to the Klein-Gordon equation. In addition, using (3.11) it can be shown that the Lie derivative operator $\mathcal{L}_k$ is anti-Hermitian. This implies that $\mathcal{L}_k$ has purely imaginary eigenvalues [48]. From here we may fix a basis such that positive norm states f_i obey

$$\mathcal{L}_k f_i = -i\omega_i f_i. \tag{3.12}$$

However, for some general space, we have no set positive or negative frequency plane wave modes so there is no preferred way to decompose $\mathcal{K}$. Instead we just chose some basis of solutions such that

$$\begin{aligned}
(\chi_i, \chi_j) &= \delta_{ij}, &(3.13)\\
(\chi_i, \chi_j^*) &= 0, &(3.14)\\
(\chi_i^*, \chi_j^*) &= -\delta_{ij}, &(3.15)
\end{aligned}$$

where we have replaced our momentum indices from Minkowski space with continuous indices i, j. We can now expand the scalar in our chosen basis and quantise to give

$$\phi = \sum_i a_i\chi_i(x) + a_i^\dagger\chi_i^*(x),$$

where a_i and $a_i^\dagger$ are creation and annihilation operators obeying the usual commutation relations and defined for our choice of basis by

$$\begin{aligned}
a_i &= (\chi_i, \phi), &(3.16)\\
a_i^\dagger &= -(\chi_i^*, \phi). &(3.17)
\end{aligned}$$

From here we may construct a Fock basis in the usual way. It is worth noting that due to the arbitrary nature of choosing a basis of solutions to the equations of motion in curved spacetime, the spacetime does not admit a preferred vacuum state. Hence the concept of a particle is not universal, and depends on the chosen basis of solutions.

Our freedom in choosing the basis gives us a tool which is useful for problems concerning QFT in curved spacetimes. Let us choose a new basis of solutions $\{\tilde{\chi}_i, \tilde{\chi}_i^*\}$. In this basis, we have new creation and annihilation operators, $\tilde{a}_i$ and $\tilde{a}_i^\dagger$, so our scalar can be expanded as

$$\phi = \sum_i \tilde{a}_i \tilde{\chi}_i(x) + \tilde{a}_i^\dagger \tilde{\chi}_i^*(x),$$

noting that in this basis we have a new vacuum.

Both $\{\chi_i, \chi_i^*\}$ and $\{\tilde{\chi}_i, \tilde{\chi}_i^*\}$ span $\mathcal{K}$, so we can map between the two choices of basis by what is known as a Bogoliubov transformation:

$$\tilde{\chi}_i(x) = \sum_j A_{ij}\chi_j(x) + B_{ij}\chi_j^*(x), \tag{3.18}$$

$$\tilde{\chi}_i^*(x) = \sum_j B_{ij}^*\chi_j(x) + A_{ij}^*\chi_j^*(x), \tag{3.19}$$

where A_{ij} and B_{ij} are known as Bogoliubov coefficients [40]. We may similarly relate the annihilation operators in the two bases using

$$\tilde{a}_i = \sum_j A_{ij}^* a_j - B_{ij}^* a_j^\dagger. \tag{3.20}$$

Requiring that the expansion of $\tilde{\chi}_i(x)$ and $\tilde{\chi}_i^*(x)$ satisfies (3.13) and (3.14), we obtain the conditions

$$\sum_k (A_{ik}^* A_{jk} - B_{ik}^* B_{jk}) = \delta_{ij}, \tag{3.21}$$

$$\sum_k (A_{ik} B_{jk} - B_{ik} A_{jk}) = 0. \tag{3.22}$$

3.1.2 Particle Creation in a Gravitational Background

Let us now consider a globally hyperbolic "sandwich" spacetime $(\mathcal{M}, g)$, where $\mathcal{M} = \mathcal{M}_- \cup \mathcal{M}_0 \cup \mathcal{M}_+$ with $\mathcal{M}_0$ non-stationary and $\mathcal{M}_\pm$ stationary, arranged as in Fig. 3.2.

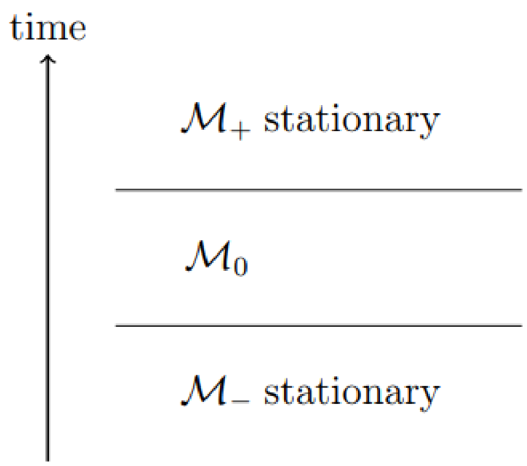

Figure 3.1: Sandwich spacetime with a fixed direction of time. Image source: [40]

The stationarity of $\mathcal{M}_-$ and $\mathcal{M}_+$ means that we have a preferred decomposition of the Hilbert spaces of solutions to the Klein-Gordon equation, and hence preferred positive frequency mode bases, on these sub-manifolds. This spacetime is globally hyperbolic, so we can have a Cauchy surface with initial data on $\mathcal{M}_\pm$ so that any solution of the Klein-Gordon equation on $\mathcal{M}_\pm$ extends to all of $\mathcal{M}$ [40].

Let the preferred positive frequency mode bases admitted by $\mathcal{M}_\pm$ be $\{f_i^\pm(x)\}$, with associated annihilation operators $a_i^\pm$. The two bases and annihilation operators are not generally the same, and so we can relate them by a Bogoliubov transformation:

$$f_i^+ = \sum_j A_{ij} f_j^- + B_{ij} f_j^{-*}), \tag{3.23}$$

$$a_i^+ = \sum_j A_{ij}^* a_j^- - B_{ij}^* a_j^{-\dagger}. \tag{3.24}$$

The two bases have different vacua, denoted $|0^\pm\rangle$. We may also define number operator for the ith mode in our two bases as

$$N_i^\pm = a_i^{\pm\dagger} a_i^\pm. \tag{3.25}$$

Let us now consider what happens if we act with the number operator for $\mathcal{M}_+$ on the vacuum of region $\mathcal{M}_-$. We can think about this explicitly in terms of the vacuum expectation value of N_i^+ with respect to $|0^-\rangle$:

$$\begin{aligned}
\langle 0^- | \, N_i^+ \, |0^-\rangle &= \langle 0^- | \, a_i^{+\dagger} a_i^+ \, |0^-\rangle && (3.26)\\
&= \langle 0^- | \sum_j (A_{ij} a_j^{-\dagger} - B_{ij} a_j^-) \sum_k (A_{ik}^* a_k^- - B_{ik}^* a_k^{-\dagger}) \, |0^-\rangle && (3.27)\\
&= \sum_{j,k} \langle 0^- | \, (-B_{ij} a_j^-)(-B_{ik}^* a_k^{-\dagger}) \, |0^-\rangle && (3.28)\\
&= \sum_{j,k} B_{ij} B_{ik}^* \langle 0^- | \, a_j^- a_k^{-\dagger} \, |0^-\rangle && (3.29)\\
&= \sum_j B_{ij} B_{ij} = tr(BB^\dagger). && (3.30)
\end{aligned}$$

We see that we have non-zero number of particles at late times for non-zero B, i.e. for the case where $\{f_i^+\}$ and $\{f_i^-\}$ have different bases, which is not generally the case. Hence we arrive at the statement that dynamic gravitational fields produce particles.

3.1.3 Hawking Radiation

We are now in a position to recover Hawking's famous result from 1973 [20]. To begin, we consider a massless scalar ϕ in the spacetime of a radially collapsing body. By Birkhoff's theorem, we may describe the spacetime outside the radially collapsing body by the Schwarzschild solution [44]. For this case let us use the form

$$ds^2 = \left(1 - \frac{2M}{r}\right) du\, dv + r^2 d\Omega^2, \qquad (3.31)$$

with $u = t - r_*$ and $v = t + r_*$, where r_* are tortoise coordinates. We now consider some massive scalar ϕ on this metric, satisfying Eq. 3.3. The spherical symmetry of the Schwarzschild metric admits spherical harmonic solutions to the Klein-Gordon equation, of the form

$$\phi = e^{-i\omega t} R_{\omega lm}(r_*) Y_{lm}(\theta, \phi), \qquad (3.32)$$

where ω is the frequency of the mode, $Y_{lm}(\theta, \phi)$ are spherical harmonics, and $R_{\omega lm}(r_*)$ is a radial component satisfying the Schrodinger-style equation

$$\left[\frac{d^2}{dr_*^2} + \omega^2\right] R_{\omega l}(r_*) = V_l(r_*) R_{\omega l}(r_*), \qquad (3.33)$$

with potential V_l given by

$$V_l(r_*) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right). \qquad (3.34)$$

Using this form, we see that the potential vanishes at null infinities, $\mathcal{I}^{\pm}$, so here the scalar has plane wave solutions. The spherical symmetry of the spacetime allows us to decompose into ingoing and outgoing solutions, which for $\mathcal{I}^-$ we define to be

$$f^{(out)}_{lm\omega} = \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega u} \frac{Y_{lm}}{r}, \tag{3.35}$$

$$f^{(in)}_{lm\omega} = \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega v} \frac{Y_{lm}}{r}, \tag{3.36}$$

and for $\mathcal{I}^+$ we define to be

$$g^{(out)}_{lm\omega} = \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega u} \frac{Y_{lm}}{r}, \tag{3.37}$$

$$g^{(in)}_{lm\omega} = \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega v} \frac{Y_{lm}}{r}. \tag{3.38}$$

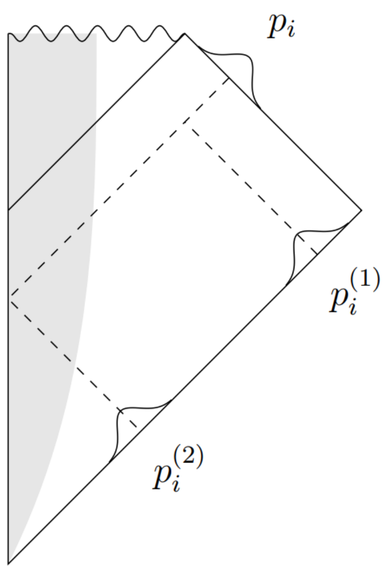

Figure 3.2: Carter-Penrose diagram for a wavepacket travelling inwards from $\mathcal{I}^+$ in the spacetime of a radially collapsing body. Image source: [40]

Let us now consider a wavepacket $p_i \equiv g^{(out)}_{lm\omega}$ coming from $\mathcal{I}^+$. As it approaches the potential $V_l(r_*)$, a part of the wavepacket $p_i^{(1)}$ is reflected off the potential, and another part $p_i^{(2)}$ is transmitted. $p_i^{(1)}$ does not experience any of the spherical collapse geometry so only consists of positive frequency modes, with Bogoliubov coefficients $A^{(1)}_{ij}$. The transmitted part, $p_i^{(2)}$, does enter the time-dependent collapse geometry, before ending up at $\mathcal{I}^-$. This part does have positive and negative frequency modes on $\mathcal{I}^+$, with Bogoliubov coefficients $A^{(2)}_{ij} + B^{(2)}_{ij}$.

Let $\gamma_{\mathcal{H}}$ be a generator of $\mathcal{H}^+$ which is extended to $\mathcal{I}^-$, where we have defined v so that $\gamma_{\mathcal{H}}$ hits $v = 0$ at $\mathcal{I}^-$. The wavepacket will be localised at some $v_0 < 0$ on $\mathcal{I}^-$, with infinitely many oscillations in the range $v \in (v_0, 0)$. Hence wavepackets originating at v oscillates rapidly in the vicinity of a generator γ from $\mathcal{I}^-$ to $\mathcal{I}^+$. This allows us to use the geometric optics approximation that the scalar field has the form

$$\phi = A(x)e^{i\lambda S(x)}, \tag{3.39}$$

for $\lambda \gg 1$. To leading order in λ the Klein-Gordon equation gives

$$(\nabla S)^2 = 0, \tag{3.40}$$

which is the statement that surfaces of constant S are null hypersurfaces, and their generators are null geodesics. Consider the null congruence of hypersurfaces satisfying (3.40) and $\mathcal{H}^+$, which is at $S = \infty$. Let l be a tangent vector to the congruence of null geodesics, and introduce null vector n such that $n \cdot l = -1$ and $l^\mu \nabla_\mu n^\nu = 0$. Spherical symmetry implies that $n^\theta = n^\phi = 0$.

Outside the collapsing body we have Schwarzschild spacetime, so in Kruskal-Szekeres coordinates the affinely parametrised generator of $\mathcal{H}^+$ is given by

$$l = \frac{\partial}{\partial V}, \tag{3.41}$$

and so from our condition that $n \cdot l = -1$ we obtain

$$n = C\frac{\partial}{\partial U}, \tag{3.42}$$

for some positive constant C. So outside matter we have deviation vector $-\epsilon n$ which connects γ_H to some null geodesic γ, which is located at $U = -C\epsilon$. Using the definition of the Kruskal-Szekeres coordinate U, we can now say that at late times γ hits $\mathcal{I}^+$ with $u = -\frac{1}{\kappa}log(C\epsilon)$, and the phase of the wavepacket p_i is hence $\frac{i\omega}{\kappa}log(C\epsilon)$ everywhere on corresponding geodesic γ.

For $\mathcal{I}^-$ we may follow similar arguments to show that the phase of the wavepacket is given by $\frac{i\omega}{\kappa}log(-CDv)$, for positive constant D. Hence on $\mathcal{I}^-$ we have

$$p_i^{(2)} \sim \begin{cases} 0 & \text{for } v > 0, \\ e^{\frac{i\omega}{\kappa}log(-v)} & \text{for } v < 0. \end{cases} \tag{3.43}$$

We now assume that the wavepacket p_i only has the positive frequency ω, for the sake of simplifying the calculation. We want to find how $p_\omega^{(2)}$ relates to our basis of modes f_ω, f_ω^*. We begin by taking the Fourier transform of $p_\omega^{(2)}$, which gives

$$\tilde{p}_\omega^{(2)} = \int_{-\infty}^{\infty} exp\left[i\omega' v + \frac{i\omega}{\kappa} log(-v)\right] dv. \tag{3.44}$$

We can also get the inverse

$$p_\omega^{(2)}(v) = \int_0^\infty d\omega'\, N_{\omega'} f_{\omega'}(v) \tilde{p}_\omega^{(2)}(\omega') + N_{\omega'}^* f_{\omega'}^*(v) \tilde{p}_\omega^{(2)}(-\omega'). \tag{3.45}$$

Hence comparing coefficients to (3.18) above, we have

$$\begin{cases} A_{\omega\omega'}^{(2)} = N_{\omega'} \tilde{p}_\omega^{(2)}, \\ B_{\omega\omega'} = N_{\omega'}^* \tilde{p}_\omega^{(2)}(-\omega'). \end{cases} \tag{3.46}$$

Using properties of Fourier transforms, we may then show that

$$|B_{\omega\omega'}| = exp\left(\frac{\pi\omega}{\kappa}\right) |A_{\omega\omega'}^{(2)}|, \tag{3.47}$$

and we get an analogous result when looking at the full wavepackets. Looking at the normalisation of $p_i^{(2)}$, we get

$$(p_i^{(2)}, p_i^{(2)}) = \sum_j \left(|A_{ij}^{(2)}|^2 - |B_{ij}|^2\right) = \left(e^{2\omega\pi/\kappa} - 1\right) \sum_j |B_{ij}|^2 \tag{3.48}$$

$$= \left(e^{2\omega\pi/\kappa} - 1\right) Tr(BB^\dagger). \tag{3.49}$$

Hence, for $\Gamma_i \equiv (p_i^{(2)}, p_i^{(2)})$, we see that

$$Tr(BB^\dagger) = \frac{\Gamma_i}{e^{2\omega\pi/\kappa} - 1}. \tag{3.50}$$

If we interpret Γ_i to be the absorption cross-section for the f_i mode, we see that this result is exactly a blackbody spectrum with temperature $T_H = \frac{\kappa}{2\pi}$, known as the Hawking temperature. This result shows that the particle production of black holes is a continuous, radiative process which naturally follows from the starting point of having a scalar field on a curved spacetime. With some work, this derivation can be generalised to include fermions, for collapse to charged and/or rotating black holes, for non-radial collapse among other generalisations [14].

Let us consider some generalised metric

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{3.51}$$

with some function $f(r)$ with horizon at r_s i.e. $f(r_s) = 0$. In order to find the Hawking temperature of this metric, we first employ a rule seen often in considerations of quantum fields - we Wick rotate our time coordinate. We introduce Euclidean time t_E such that $t = it_E$. Our metric then becomes

$$ds^2 = +f(r)dt_E^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2. \tag{3.52}$$

It can be shown from arguments of gravitational path integrals that t_E becomes compact when it is periodic with period β, the inverse Hawking temperature. Let us now go to the near horizon limit, such that $r = r_s + \epsilon$ for ϵ small. In this limit we have the expansion $f(r) = f(r_s) + \epsilon f'(r_s) + \mathcal{O}(\epsilon^2)$. Noting $f(r_s) = 0$, we can re-express our metric,

$$ds^2 \approx \epsilon f'(r_s)dt_E^2 + \frac{1}{\epsilon f'(r_s)}d\epsilon^2 + (r_s + \epsilon)^2 d\Omega^2. \tag{3.53}$$

We now ignore the angular term in the metric, and define new variables,

$$R \equiv 2\sqrt{\frac{\epsilon}{f'(r_s)}} \qquad \Theta \equiv \frac{t_E f'(r_s)}{2}, \tag{3.54}$$

which gives us the metric

$$ds^2 \approx dR^2 + R^2 d\Theta^2, \tag{3.55}$$

which is flat space with polar coordinates R and Θ. For a smooth geometry which avoids conical singularities, which give divergences in gravitational path integrals, we require that Θ is periodic with period 2π. Through our definition of Θ this means that t_E must be periodic with period $4\pi/f'(r_s)$ [17]. However, we have previously argued that t_E has period β, so the Hawking temperature is given by

$$\beta = \frac{4\pi}{f'(r_s)}. \tag{3.56}$$

3.1.4 Entropy on a Curved Background

In the last chapter, we looked at the notion of fine-grained entropy. It is worth reconciling this with our consideration of quantum fields on a curved background. Almheiri et. al. [2] refer to the fine-grained entropy of quantum fields on the curved background on some region of spacetime Σ as the semi-classical entropy, $S_{sc}(\Sigma)$. This is a distinction worth noticing, and in our generalised second law (2.2) replaces $S_{external}$ when we are considering quantum fields on a curved background [2].

3.2 Introducing the Black Hole Information Paradox

We have now formalised two key properties of black holes: entropy and Hawking radiation. Let us now consider how these properties compete.

We have seen that by thinking about quantum fields on black hole spacetimes, there is a natural production of particles by the black hole as a blackbody spectrum with Hawking temperature. We have seen in Chapter 2 that classically all black holes can be characterised by mass, charge, and angular momentum, and the mass-energy equivalence means that the radiation of energy out to infinity from black holes reduces their mass - black holes “evaporate”.

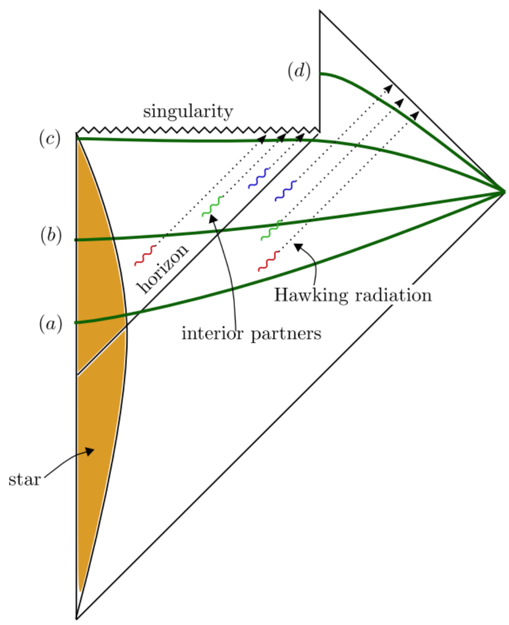

Figure 3.3: A Carter-Penrose diagram of an evaporating black hole. (a) and (b) are slices of the collapsed body producing entangled pairs, with one inside the horizon and one outside the horizon. (c) is the state where the black hole has infinitessimal horizon but still contains a spacetime singularity. (d) shows the smooth spacetime containing no black hole but thermal Hawking radiation. Image source: [2]

Fig. 3.3 shows how the process of black hole evaporation affects the spacetime. We see that for the state d), when the black hole has fully evaporated, there is no evidence that a black hole has ever existed; the only remnant is Hawking radiation, which is indistinguishable from any other thermal radiation. More formally, the surface (a) - (c) are Cauchy surfaces, so for some initial data on these surfaces we would be able to determine the evolution of that data for the entire manifold. However, (d) is not a Cauchy surface, since with initial data on this surface we could not deterministically recover any data about the past black hole spacetime. In this way we see that information has been lost through the evaporation of the black hole [2]. This is what we know to be the black hole information paradox, derived using Hawking's original arguments [19].

3.3 The Page Curve

We will now endeavour to describe the information paradox from the perspective of quantum mechanics and entropy.

We begin by considering a black hole spacetime formed from the gravitational collapse of some pure state. At the moment the event horizon forms (i.e. before Hawking radiation has begun), the fact that the system is in a pure state means that we have fine-grained entropy $S_{vN} = 0$. However, since there is an event horizon with non-trivial area A, the system has coarse-grained entropy given by $S_{BH} = A/4l_p^2$. As the black hole radiates, the area decreases linearly with time and so the coarse-grained entropy also decreases linearly with time, eventually reaching zero when the black hole has fully evaporated.

Let us now follow how Hawking thought about the entropy of the outgoing radiation itself [19]. The black hole radiates like a blackbody with Hawking temperature, so as time progresses and more radiation occurs, the entropy of outgoing radiation increases linearly. When the black hole has fully evaporated, the entropy of outgoing radiation remains constant at some maximum value. This calculation of the entropy outgoing radiation has a clear problem. The process of Hawking radiation was derived just by thinking about QFT on a curved spacetime, and so the process is definitively unitary. Hence we expect the system to evolve unitarily, and so information is expected to be conserved. However, Hawking's calculation violates this. Since Hawking radiation is thermal, even if the infalling matter were in a pure state the entangled radiation from evaporation would be mixed. The quantum information about the state of the original matter that formed the black hole is destroyed, violating the unitarity of the system. There is a related issue that comes by thinking about the entropy of the system. In the previous chapter we showed (2.11), that by definition the fine-grained entropy could never exceed the coarse-grained entropy. However, we clearly see here that the thermodynamic (coarse-grained) entropy will at some point in the evaporation of the black hole be exceeded by the (fine-grained) entropy of outgoing Hawking radiation.

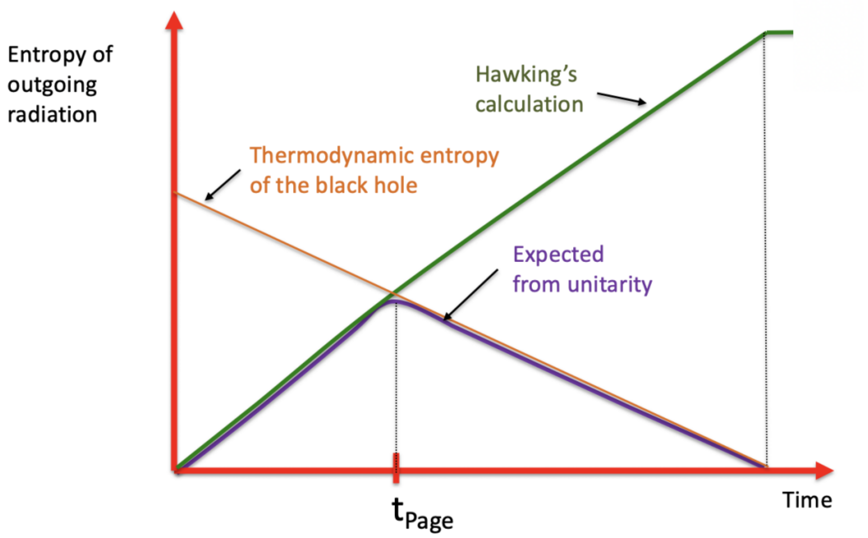

Figure 3.4: A schematic illustration of the evolution of black hole entropy, the fine-grained entropy calculated by Hawking, and the fine-grained entropy expected from the unitarity of the system known as the Page curve. Image source: [2]

The curve that the entropy of outgoing radiation is expected to follow in order to preserve the unitarity of the system is known as the Page curve [36, 37], roughly illustrated in Fig. 3.4. We will now follow Page's suggestion for a qualitative explanation for the shape of the Page curve. After the event horizon is formed, the fine-grained entropy is expected to increase from zero as per Hawking's calculation, emitting thermal Hawking radiation that is entangled with some partner inside the black hole. This continues until the so-called Page time t_{Page}, where the entropy of outgoing radiation and the Bekenstein-Hawking entropy are equal, and the black hole interior no longer contains enough degrees of freedom for the Hawking radiation to be entangled to. At this point, the outgoing Hawking radiation can only be entangled with Hawking radiation that was emitted before t_{Page}, and so the state of the outgoing radiation begins to become more pure and the fine-grained entropy decreases as the upper bound provided by the thermodynamic entropy of the black hole. By the time the black hole has fully evaporated, the outgoing radiation is in a pure state and the fine-grained entropy has decreased to zero.

A non-trivial assumption in our production of the Page curve is that we have required unitarity of our black hole system. This restriction assumes that unitarity is manifest in the underpinning theory of quantum gravity. This is not proven, and is contested by many. However, our understanding that quantum field theories are manifestly unitary provide strong evidence that, at least to semi-classical approximation, any black hole system involving quantum fields would need to evolve unitarily.

3.4 The AMPS Paradox

An important aspect of the black hole information paradox was the AMPS paradox. This paradox was proposed in 2012 by Almheiri, Marolf, Polchinski, and Sully (AMPS) [1]. Generally, they stated that the fact that Hawking radiation is in a pure state violates the principle of black hole complementarity.

Black hole complementarity was a concept first conceived by Susskind, Thorlacius, and Uglum in 1993 [46]. The concept is the combination of three postulates that are widely accepted about black hole systems:

1. Black holes evolve unitarily.
2. At some finite distance away from the event horizon, we may consider the spacetime in a semi-classical approximation.
3. Distant observers view the black hole as a quantum system with discrete energy modes.

In addition, there is another statement that is not as strong, but for our purposes can be viewed as an additional postulate of black hole complementarity:

4. Free-falling observers experience nothing special when crossing the event horizon.

AMPS assert that postulates 1., 2., and 4. cannot all hold at the same time if we take any of our previous discussion of Hawking radiation to be true. This is what is known as the AMPS paradox.

A resolution to the paradox presented by AMPS suggested the entanglement between ingoing and outgoing Hawking partners must be broken immediately, releasing large amounts of energy and creating a black hole “firewall”. However, this violates the equivalence principle of general relativity, and so has been widely contested [11].

Chapter 4

The AdS/CFT Correspondence

4.1 The Holographic Principle

The holographic principle is the principle that all information on the volume enclosed by a surface is encoded on the boundary of the surface. First suggested by Gerard 't Hooft [25], the principle was applied to theories of quantum gravity by Leonard Susskind, who conjectured that any true theory of quantum gravity for some volume of space should admit a full description based purely on the lower-dimensional boundary of the space [45]. Originally formulated due to the the two-dimensional nature of entropy of black holes by Hawking and Bekenstein, the principle proved to have far-reaching consequences beyond this.

The most developed example of the holographic principle to date is the AdS/CFT correspondence. First conjectured by Juan Maldacena in 1997 [32] as a duality between $\mathcal{N} = 4$ Super Yang-Mills in 4 dimensions and type IIB string theory on $AdS_5 \times S^5$ space. It has since been generalised to many other CFTs and AdS spaces. Before discussing the correspondence in more depth, it is worth reiterating that the AdS/CFT duality is to date still conjecture. However, since its conception in 1997, there has been overwhelming evidence to support its arguments. This dissertation will assume the correspondence holds, and will make use of its arguments on the basis of this assumption.

4.2 Conformal Field Theory

Conformal field theories are quantum field theories that are invariant under conformal transformations. A conformal transformation is a coordinate transformation $x^\mu \to x'^\mu(x^\mu)$ which transforms the metric as

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x),$$

or a transformation which rescales lengths but preserves angled between vectors. The group of transformations which satisfy this condition is the conformal group. Conformal transformations are a notable "loophole" to the Coleman-Mandula no-go theorem (the other being Supersymmetry). Restricting to D-dimensional Minkowski space, we can now identify subgroups of the conformal group. It can be seen that for $D = 2$ the conformal group is infinite-dimensional, since, after Euclidean continuation, all holomorphic functions on a plane satisfy the above condition (a property used extensively in string theory). However, for $D \neq 2$ the solutions are at most quadratic in x. Hence, the most general infinitessimal conformal transformation is given by

$$x'_\mu = x_\mu + \delta x_\mu = x_\mu + a_\mu + \omega_{\mu\nu} x_\nu + \lambda x_\mu + (b_\mu x^2 - 2x_\mu bx).$$

Here, we notice the second and third terms to correspond to translations (generated by P_μ) and Lorentz transforms (generated by $J_{\mu\nu}$) respectively, which together form the Poincare group. The fourth term corresponds to scale transformations of the metric, known as dilatations (generated by D). This indicates that all conformal fields must be scale invariant. This scale invariance means that CFTs do not allow for the notions of massive excitations or length scales, and so scattering is no longer relevant. The final term corresponds to a less obvious subgroup, known as the special conformal transformations (generated by K_μ). Finite special conformal transformations are of the form

$$x_\mu \to \frac{x_\mu + c_\mu x^2}{1 + 2cx + (cx)^2}.$$

Given we have already seen that the Poincare group is a subgroup of the conformal group, we can begin with the Poincare algebra and find commutation relations with generators of dilatations and special conformal transformations to construct a full conformal algebra. Given the dilatation and special conformal transformation generators act on a function $f(x)$ as

$$Df(x) = ix^\mu \partial_\mu f(x), \tag{4.1}$$

$$K_\mu f(x) = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) f(x), \tag{4.2}$$

we can explicitly calculate the commutation relations to get the full algebra:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}), \tag{4.3}$$

$$[J_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \tag{4.4}$$

$$[J_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \tag{4.5}$$

$$[P_\mu, K_\nu] = 2i(J_{\mu\nu} + \eta_{\mu\nu}D), \tag{4.6}$$

$$[P_\mu, D] = -iP_\mu, \tag{4.7}$$

$$[K_\mu, D] = iK_\mu, \tag{4.8}$$

with all other commutators vanishing [35]. It can be shown that this algebra is isomorphic to $SO(2, D)$ [49].

Considering how the dilatation generator D acts on scalar $\phi(x)$, we must account for the possibility that the field itself has some intrinsic scale. We denote this using the scaling dimension, Δ. Hence, under some dilatation transformation $x^\mu \to \lambda x^\mu$, the field transforms as

$$\phi(x) \to \lambda^\Delta \phi(\lambda x). \tag{4.9}$$

Using the algebra given in (4.3 - 4.8), one can show that correlation functions of operators are constrained by conformal invariance. For example, the 2-point correlator of a conformally invariant scalar ϕ is given by

$$\langle 0| \phi(x)\phi(y) |0\rangle \propto \frac{1}{(x-y)^{2\Delta}}, \tag{4.10}$$

where both $\phi(x)$ and $\phi(y)$ are required to have the same conformal dimension Δ for there to exist a non-vanishing correlation function.

An important property of conformal field theories is their effect on the stress-energy tensor. It can be shown that conformal invariance restricts the stress-energy tensor to be traceless, i.e. $T^\mu_\mu = 0$. This restriction is purely classical. Introducing some curvature to the manifold gives a conformal anomaly. A simple example of the conformal anomaly is seen in bosonic string theory, where for a 2-dimensional quantum field theory on a generally curved background, the expectation value becomes $\langle T^\mu_\mu \rangle = -\frac{c}{12}\mathcal{R}$, where $\mathcal{R}$ is the Ricci scalar and c is the central charge [47].

4.3 Anti de-Sitter Space

Let us now turn to the other side of the duality.

The n-dimensional Anti de-Sitter Space, AdS_n, is the maximally symmetric solution to Einstein's equations in n dimensions with negative cosmological constant Λ (we will set this to -1). It is best thought of as the Lorentzian analogy to the Euclidean hyperboloid manifold,

$$U^2 + V^2 - X^i X^i = \ell^2, \tag{4.11}$$

where $i = 1, ..., n-1$ and ℓ is the radius of the hyperboloid. Here we are embedding AdS_n in $n+1$ dimensional Euclidean space with signature $\mathbb{R}^{2,n-1}$,

$$ds^2 = -dU^2 - dV^2 + dX^i dX^i, \tag{4.12}$$

where we have 2 timelike dimensions and $n-1$ spatial dimensions. Upon inspecting (4.12) it can be seen that AdS_n has an $SO(2, n-1)$ isometry group.

A peculiar feature of (4.11) is that it has closed timelike curves for $X^i = constant$, which is generally a problem for causality of dynamical spacetimes. An approach to dealing with this problem is to unwrap the closed timelike curves and obtain a "universal cover" for them. We do this by reparametrising the surface in terms of dimensionless coordinates ρ and τ:

$$U = \ell\, cosh(\rho)\, cos(\tau), \tag{4.13}$$

$$V = \ell\, cosh(\rho)\, sin(\tau), \tag{4.14}$$

$$X_i = \ell\, sinh(\rho)\, \Omega_i, \tag{4.15}$$

where Ω_i is the surface for which $\sum_i \ell_i^2 = 1$. It can be shown that $\rho \in \mathbb{R}^+$ and $\tau \in [0, 2\pi]$ cover the manifold exactly once, though still admit closed timelike curves. Hence we take the universal cover where $\tau \in \mathbb{R}$, and generally refer to this as AdS_n. The metric is now

$$ds^2 = \ell^2 \left[-cosh^2(\rho) d\tau^2 + d\rho^2 + sinh^2(\rho) d\Omega_{n-2}^2 \right]. \tag{4.16}$$

We may now consider some limits of this metric. We can deduce that as $\rho \to \infty$,

$$ds^2 \to \ell^2 \left[e^{2\rho}(-d\tau^2 + d\Omega_{n-2}^2) + d\rho^2 \right]. \tag{4.17}$$

Conformally compactifying this metric in the usual way with conformal factor $Z = e^{-\rho}$, we have

$$ds^2 \to \frac{\ell^2}{Z^2}\left[-d\tau^2 + d\Omega_{n-2}^2 + dZ^2\right]. \tag{4.18}$$

We see that the conformal boundary metric has topology $\mathbb{R}\times S^{n-2}$. This AdS boundary is where we may locate the CFT. The topology of AdS spacetime is schematically shown in Fig. 4.1.

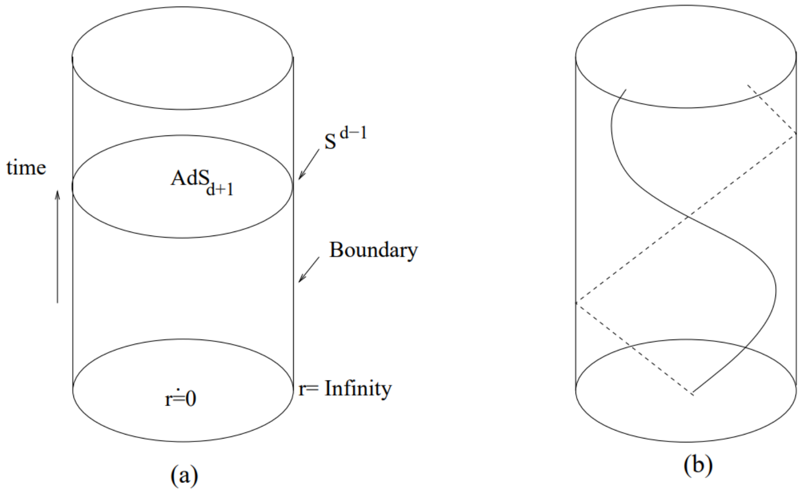

Figure 4.1: Diagram of AdS spacetime. The boundary is a solid cylinder. Timelike geodesics (solid line) and null geodesics (dashed line) are shown in (b). Image source: [31]

Setting $r = \ell\, sinh(\rho)$ and $t = \ell\tau$ gives us

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{n-2}^2, \tag{4.19}$$

where we have defined

$$f(r) \equiv \frac{r^2}{\ell^2} + 1.$$

We see that in the limits $r \to 0$ or $\ell \to \infty$, the metric is Minkowski space. If we transform t to Euclidean time, $t_E = it$, we have the Euclidean AdS metric,

$$ds^2 = f(r)dt_E^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{n-2}^2, \tag{4.20}$$

where we now have the identification

$$t_E \sim t_E + \beta,$$

where β is the inverse temperature. This identification is worth considering further. From quantum field theory, we know that Euclidean path integrals with identification

$t_E \sim t_E + \beta$ make the field thermal. So in the above case Euclidean path integrals on Euclidean thermal AdS makes fields on the Lorentzian AdS thermal.

An alternative coordinate chart used to consider AdS is the Poincare chart $\{V, X^i, U\} \rightarrow \{t, x^a, Z\}$:

$$U = \frac{Z}{2}\left[1 + \frac{1}{Z^2}\left(\ell^2 + (x^a)^2 - t^2\right)\right], \tag{4.21}$$

$$V = \frac{\ell}{2}t, \tag{4.22}$$

$$X^a = \frac{\ell}{Z}x^a, \tag{4.23}$$

$$X^{n-1} = \frac{Z}{2}\left[1 - \frac{1}{Z^2}\left(\ell^2 - (x^a)^2 + t^2\right)\right], \tag{4.24}$$

where $a = 1, ..., n-2$. This chart covers half of the hyperboloid; it is not a global coordinate chart. For this chart, the induced metric is given by

$$ds^2 = \frac{\ell^2}{Z^2}\left(-dt^2 + dx^a dx^b + dZ^2\right), \tag{4.25}$$

which is the $n-1$ dimensional Minkowski metric on the conformal boundary. The Poincare chart makes manifest a group of isometries of the metric: the Poincare group, the group of dilatations, and the special conformal group. It becomes clear that the isometries of the metric are isomorphic to the conformal group described in the previous section.

4.4 The AdS/CFT Correspondence

Having addressed both sides of the duality, we can now state it explicitly. The AdS/CFT correspondence states that any spacetime that is asymptotically AdS_n that can be described by some theory of quantum gravity is dual to (i.e. physically equivalent to) an $n-1$ dimensional conformal field theory living on the conformal boundary of the AdS_n bulk, which has topology $\mathbb{R} \times S^{n-2}$.

The correspondence initially seems very specific, and it might be reasonable to question its practical utility. After all, most field theories we know of in nature are not conformal, and there is no evidence to suggest that we live on an asymptotically AdS universe. Instead, the correspondence can be viewed as an extremely powerful tool.

The nature of the correspondence means that where there is a strongly interacting theory on one side of the duality, the other side is necessarily weakly interacting. This "dictionary" is extremely useful for studying strongly interacting QFTs, as well as strongly interacting gravitational theories, since the correspondence means we need only consider weakly interacting QFTs and gravitational theories respectively to study them. As we will later see, the holographic nature of the correspondence (i.e. the correspondence between theories of different dimensions) provides a promising avenue to understand the fine-grained entropy of black holes.

4.5 AdS/CFT at Finite Temperature

Now we have given a rough overview of AdS/CFT, it is worth looking at the AdS black hole. This is a black hole solution which is asymptotically AdS as opposed to the usual asymptotically flat black holes. For $n \geq 4$, we have a uniquely defined spherically symmetric solution to the Einstein equations with negative vacuum energy, given by

$$ds^2 = -h(r)dt^2 + h^{-1}(r)dr^2 + r^2 d\Omega_{n-2}^2, \tag{4.26}$$

where

$$h(r) = r^2 + 1 - \frac{C}{r^{d-2}}, \tag{4.27}$$

for constant C proportional to the AdS version of the ADM mass [16]. Using (3.56), the temperature of this black hole is computed to be

$$\beta = \frac{4\pi r_s}{d - 3 + (d-1)r_s^2}. \tag{4.28}$$

This is the second time we are seeing thermal properties of metrics in AdS - recall that the Euclidean AdS metric has identification with period β, and Euclidean path integrals in this space give thermal fields. In 1983, Hawking and Page showed using Euclidean path integrals that the stability of AdS black holes depends on the temperature [21]. For temperatures below some critical temperature, $T < T_C$, the black hole solution is unstable, and so generally the metric of the spacetime is the thermal AdS solution given in (4.20). At T_C there is a first order phase transition, such that for $T > T_C$ the AdS black hole solution becomes stable, and is in thermal equilibrium with Hawking radiation. This transition is known as the Hawking-Page transition. An interesting difference to asymptotically flat metrics is that black holes here have positive heat capacity, so become more massive as their temperature increases.

In the context of the AdS/CFT correspondence, a useful point is that any AdS black holes are dual to CFT states. The fact that we have quantum gravity in AdS dual to a unitary quantum field theory means that we may identify the temperature and entropy of AdS black holes with the temperature of the CFT and the number of excited CFT states at said temperature, respectively [28]. This unlocks many possibilities for further study of AdS black holes.

Chapter 5

A Gravitational Description of Entropy

We now have the toolset to begin working towards recovering the Page curve. Let us begin by looking at gravitational formulations of entropy.

5.1 Ryu-Takayanagi Formula

The idea of a gravitational approach to fine-grained entropy was first formalised by Ryu and Takayanagi in 2006 [42, 43]. Using the AdS/CFT correspondence, they related the entanglement entropy of an n dimensional CFT on the conformal boundary of AdS_{n+1} spacetime with an extremal surface on the AdS_{n+1} dual. Restricting to a static spacetime, Ryu and Takayanagi restrict to static AdS spacetimes, and think about some subsystem A of both the CFT conformal boundary of AdS_{n+1} and of the AdS_{n+1} bulk. A is required to be a Cauchy surface. Ryu and Takanagi then propose that in order to find the entanglement entropy of this subsystem, S_A, one must find the minimal area surface γ_A which extends into the bulk with boundary $\partial\gamma_A = \partial A$, the boundary of the subsystem. Then the holographic entanglement entropy (HEE) is given by

$$S_A = \frac{Area(\gamma_A)}{4G_N}, \tag{5.1}$$

where G_N is an $n+1$ dimensional form of Newton's gravitational constant. This is known as the Ryu-Takayanagi (RT) formula for HEE. We may note that A and γ_A are $n-1$ dimensional, and the boundary ∂A is $n-2$ dimensional.

In 2013, the RT formula was proven by Lewkowycz and Maldacena [29] using arguments of gravitational path integrals. The derivation is non-trivial, so is omitted from this dissertation.

5.1.1 Application to AdS_3/CFT_2

Though presented as conjecture, Ryu and Takanagi calculated the HEE for the example of AdS_3/CFT_2, a case for which the entanglement entropy was already known. Let us follow this calculation. For the case of AdS_3/CFT_2, the $1+1$ dimensional field theory has central charge given by

$$c = \frac{3\ell}{2G_N}. \tag{5.2}$$

In AdS_3, referring to (4.16), we have metric given by

$$ds^2 = -cosh^2(\rho)d\tau^2 + d\rho^2 + sinh^2(\rho)d\theta^2, \tag{5.3}$$

where we have $\rho \geq 0$ and θ periodic with period 2π. Since this metric is divergent in the limit $\rho \to \infty$, we introduce some cutoff ρ_0 and restrict our spacetime to $\rho \leq \rho_0$ to regulate the physics. This corresponds to the introduction of a UV cutoff in the corresponding CFT. In the AdS, we have the circumference of the cylinder given by L, corresponding to the length of the CFT system. Since our CFT is divergent in the continuum limit, we also introduce a the lattice spacing of the CFT system as a way of regularizing this divergence. We can now start to see the emergent relation between UV cutoff, length and lattice spacing of the CFT given as

$$e^{\rho_0} \sim L/a. \tag{5.4}$$

The region of subsystem A with length r, is restricted to $0 \leq \theta \leq 2\pi r/L$, where we have set $d\phi = 0$. Our extremal surface γ_A for some time slice t is then simply the geodesic travelling through the AdS bulk which connects $\theta = 0$ to $\theta = 2\pi r/L$. We can find the HEE by finding the length of this geodesic, L_{γ_A}. Taking our UV cutoff ρ_0 into account, we can calculate this geometrically.

In general, geodesics in AdS_n are given by the intersection of 2 dimensional hyperplanes and AdS_n in the surrounding $\mathbb{R}^{2,n-1}$, with the hyperplanes orientated such that such that they include the normal vector at points of intersection.

For the case of AdS_3 we may embed the space into Euclidean space with signature $\mathbb{R}^{2,2}$. For this case, we have the minimum surface as a spacelike geodesic constrained to the hypersurface. In our action, this takes the usual form but we apply the constraint with the inclusion of a Lagrange multiplier. Hence our action is

$$S = \frac{1}{2}\int_{\lambda_0}^{\lambda_1} d\lambda \left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \eta\left(g_{\mu\nu}x^\mu x^\nu + \ell^2\right)\right], \tag{5.5}$$

where λ is some generally non-affine parameter, η is the Lagrange multiplier, $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$, and λ_0 and λ_1 are the boundaries of the geodesic. Varying this action we eventually obtain the equation of motion

$$\ddot{x}^\mu - x^\mu \dot{x}^2 = 0. \tag{5.6}$$

Since our parameter λ is not, in general, affinely parametrised, we are free to transform it such that we have the condition $\dot{x}^2 = 1$, which allows for us to define the length of the geodesic as

$$L_{\gamma_A} = \int ds = \int d\lambda = \lambda_1 - \lambda_0. \tag{5.7}$$

Our equation of motion is then reduced to a simple differential equation, and is generally solved by

$$x^\mu = A^\mu e^{\lambda/\ell} + B^\mu e^{-\lambda/\ell}, \tag{5.8}$$

with constants A^μ and B^μ which are constrained by $A^2 = B^2 = 0$ and $2\,A\cdot B = -\ell^2$ [9]. A useful expression we may gain from this form of the equation of motion:

$$x(\lambda_0)\cdot x(\lambda_1) = 2A\cdot B\,cosh\left(\frac{L_{\gamma_A}}{\ell}\right) = -\ell^2 cosh\left(\frac{L_{\gamma_A}}{\ell}\right) \tag{5.9}$$

In AdS_3 the universal cover can be expressed as

$$U = \ell\,cosh(\rho)\,cos(\tau), \tag{5.10}$$

$$V = \ell\,cosh(\rho)\,sin(\tau), \tag{5.11}$$

$$X = \ell\,sinh(\rho)\,cos(\theta), \tag{5.12}$$

$$Y = \ell\,sinh(\rho)\,sin(\theta). \tag{5.13}$$

Here we have ρ and θ are spatial coordinates and τ is a temporal coordinate. Specifically, ρ is a polar coordinate of the two spatial dimensions as a function of the parameter λ, such that the curve has endpoints at $\theta(\lambda_0) = 0$ and $\theta(\lambda_1) = 2\pi r/L$. Taking some constant time slice τ_0 and plugging these coordinates into our expression (5.9) above, we get the expression

$$cosh\left(\frac{L_{\gamma_A}}{\ell}\right) = 1 + 2\,sinh^2(\rho_0)sin^2\left(\frac{\pi r}{L}\right). \tag{5.14}$$

For some large UV cutoff $e^{\rho_0} \gg 1$, we may plug the above length of the minimal surface into (5.1) to get the holographic entanglement entropy,

$$S_A = \frac{\ell}{4G_N} log\left[e^{2\rho_0} sin^2\left(\frac{\pi r}{L}\right)\right] = \frac{c}{3} log\left[e^{\rho_0} sin\left(\frac{\pi r}{L}\right)\right], \tag{5.15}$$

where we have used the central charge given in (5.2). Noting (5.4), we recover the CFT entanglement entropy that was calculated by Cardy and Calabrese in 2004 [8]. A similar analysis can be done using the AdS metric in Poincare chart.

5.1.2 Recovering Strong Subadditivity

After the proposal of the Ryu-Takayanagi formula, a natural next step was to recover the strong subadditivity condition from it. First proven by Hirata and Takayanagi [24], the arguments were greatly simplified by Headrick and Takayanagi in 2007 [23]. Let us now follow Headrick and Takayanagi's derivation. Noting that the proofs of (2.9) and (2.10) follow very similar arguments, we will focus on recovering (2.9).

We begin by redefining the subsystem configuration from (2.9). Consider two overlapping subsystems, A and B, with $A \cap B \equiv C$. We may then express (2.9) in the form

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B) \tag{5.16}$$

For some AdS spacetime, we now consider two overlapping regions of the boundary, A and B. We then have two minimal generally overlapping hypersurfaces in the bulk m_A and m_B enclosing regions r_A and r_B respectively, intersecting the boundary at ∂A and ∂B respectively. Hence we have $\partial r_A = A \cup m_A$ and $\partial r_B = B \cup m_B$. Defining $r_{A\cup B} = r_A \cup r_B$ and $r_{A\cap B} = r_A \cap r_B$, we may show that by cutting m_A and m_B up into new surfaces $m_{A\cup B}$ and $m_{A\cap B}$, we have

$$\partial r_{A\cup B} = (A \cup B) \cup m_{A\cup B}, \tag{5.17}$$

$$\partial r_{A\cap B} = (A \cap B) \cup m_{A\cap B}. \tag{5.18}$$

These relations can be interpreted as showing that $m_{A\cup B}$ intersects the boundary at $\partial(A \cup B)$. However, there is no information about whether it is the minimal hypersurface with this boundary, and so it can be presented as an upper bound on the area of the minimal hypersurface, and hence on the holographic entanglement entropy. Similar arguments follow for $m_{A\cap B}$ We see that the combined areas of $m_{A\cap B}$ and $m_{A\cup B}$

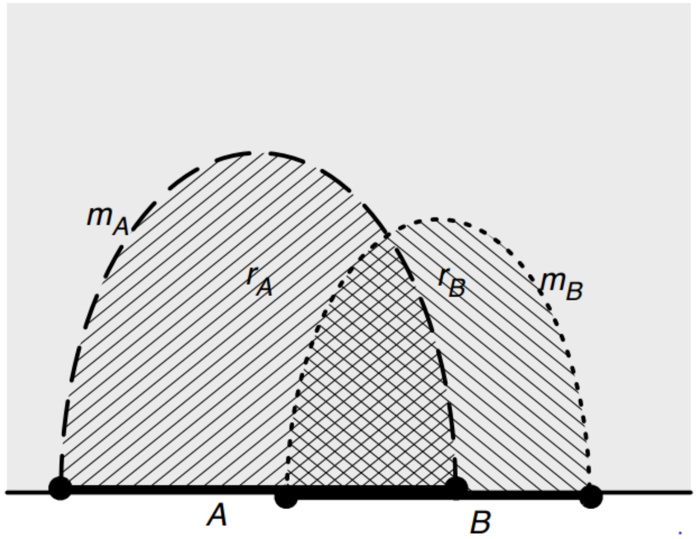

Figure 5.1: Diagram of two overlapping regions A and B on the boundary, with minimal surfaces in the bulk m_A enclosing region r_A and m_B enclosing region r_B respectively. Image source: [23]

are equal to the combined areas of m_A and m_B. Hence (5.16) has been proven, and the strong subadditivity condition has been recovered from geometric arguments that follow from the RT formula.

5.2 Generalised Holographic Entanglement Entropy

In 2007, Hubeny, Rangamani, and Takayanagi (HRT) provided a covariant generalisation to the RT formulation of HEE [26]. This formulation removes the restriction to stationary spacetimes present in the RT formulation, replacing the minimal area surface γ_A with a more general extremal surface Σ_A. In general, Σ_A is not the shortest distance between spatial points, but additionally requires variation in time so results in a spacelike geodesic. The boundaries of γ_A and Σ_A are the same. There are various approaches to finding Σ_A. One intuitive approach, known as the maximin method, was proposed by Engelhart and Wall in 2014 [12]. This roughly works as follows. One begins by choosing some Cauchy surface, and varying to find the minimal surface on the boundary ∂A. This is repeated for many different Cauchy surfaces. Finally, of these minimal surfaces, the one with the maximum surface area is chosen. The maximin method greatly simplifies the process of finding extremal surfaces for many cases.

However, the HRT prescription is only part of the picture of HEE, and is insufficient for the purposes of reproducing the Page curve. In 2013, Faulkner, Lewkowycz and Maldacena (FLM) recognised that the RT and HRT descriptions give only a leading order saddle-point term as part of a more general expansion [13]. They propose that quantum corrections should be added to this expression to account for quantum mechanical effects in the bulk, most notably the coupling of quantum fields to the spacetime. Accounting for quantum processes in the bulk is of great importance when considering Hawking radiation, so these quantum corrections are of great value when working to recover the page curve. If we treat the bulk theory as some perturbative quantum theory of gravity with expansion in G_N, FLM showed the most general HEE expansion is of the form:

$$S_A = S_A^{saddle} + S_A^{1-loop} + \mathcal{O}(G_N) \tag{5.19}$$

The natural issue now concerns where these quantum corrections might come from. FLM proposed that S_A^{1-loop} is given by

$$S_A^{1-loop} = S_{bulk-ent} + \frac{\delta(Area)}{4G_N} + \langle \Delta S_{W-like} \rangle + S_{c.t.}. \tag{5.20}$$

The first term corresponds to the bulk entanglement entropy between the region enclosed by the minimal surface and the rest of the bulk. The next two terms give a much smaller contribution, and correspond to contributions from quantum fluctuations in the background metric and fields. The final term accounts for counter-terms required to make S_A^{1-loop} finite. Of these, only the first term obeys the strong subadditivity condition. This, combined with the small contribution of the second and third terms, means that to $\mathcal{O}(G_N^0)$ we only require the bulk entropy term and counter-terms. FLM considered only systems in a pure state, and so in this case the bulk entropy is the entropy of the spacetime outside the minimal surface, which is treated semi-classically so we label it S_{sc}. Neglecting our counter-terms, we have for some extremal Cauchy surface Σ_A:

$$S_{gen} = \frac{Area(\Sigma_A)}{4G_N} + S_{sc}. \tag{5.21}$$

This expression looks very similar to the generalised second law, with black hole area begin replaced by the area of the extremal Cauchy surface Σ_A.

The FLM proposal was extended by Engelhardt and Wall (EW) in 2014 [12] to higher order corrections. EW noted that the FLM formulation ignores graviton fluctuations, and that the semi-classical approximation only holds in their case because they restrict to considering terms up to $\mathcal{O}(G_N^0)$. They introduced the notion of the minimal quantum extremal surface (QES), a sort of quantum corrected version of an RT/HRT surface. The minimal QES is defined as the surface with minimal extremal generalised entropy. For some set of QESs Ω, the generalised entropy is given by

$$S_{gen}(R) = min\left\{ext_{\Omega}\left[\frac{Area(\Omega)}{4G_N} + S_{sc}(\Sigma_\Omega)\right]\right\}, \tag{5.22}$$

where Σ_Ω is the region bound by the QES Ω and boundary R. The mechanics of this formula are almost exactly the same as done for our HRT prescription, with the exception that we are now considering QESs instead of Cauchy slices, and in the process are taking the quantum nature of the spacetime into account. We note that the maximin method can also be applied to the above formula to find the minimal QES. We will now interchangably refer to the minimal QES as the quantum RT surface, as it is referred to in the majority of the literature.

5.3 The Entanglement Wedge

The notion of the entanglement wedge is central to reconciling our understanding of generalised HEE with the AdS/CFT framework. So far, we have been considering how we might extremise some surface in the AdS bulk with a given surface on the boundary to get the generalised entropy of the system. In extremising a surface in the bulk, this methodology has for the most part been considering the AdS side of the AdS/CFT correspondence. We know from the nature of the AdS/CFT correspondence that any information in the AdS bulk can be equivalently reconstructed on the CFT boundary, so a natural line of inquiry might be to question how one might reconstruct the bulk operators given some data on the boundary. A promising solution is given by introducing the entanglement wedge.

Let us first recall that the domain of dependence of some Cauchy slice Σ on a manifold $\mathcal{M}$, $D(\Sigma)$, is the set of all points $p \in \mathcal{M}$ for which there exists some timelike curve beginning or ending at p that intersects Σ. Then for some boundary region B and some region Σ_Ω bound by QES Ω and B , the entanglement wedge is given by $D(B) \cap \Sigma_\Omega$. In other words, the entanglement wedge is the domain of dependence of the boundary region, bounded by and homologous to the QES in the bulk [2, 39].

Chapter 6

Recovering the Page Curve

Let us now consider how our treatment of gravitational descriptions of entropy thus far can be applied to our understanding of black hole entropy.

6.1 Entropy of the Evaporating Black Hole

We consider a black hole in an asymptotically flat universe, with a cutoff surface at a suitable distance away from the event horizon that beyond the cutoff surface we may stop thinking of the black hole as a quantum system.

Our first consideration of the system is the time at which the black hole has just been formed from a collapsing shell of matter in a pure state, before any evaporation has begun. In this case, the only QES is of zero size; no deformation of the QES towards the event horizon produces any other extremization, so this is taken to be the minimal QES. Hence it is immediately apparent that the fine-grained entropy of the system at this stage is zero.

As the black hole starts evaporating, the system stays in a pure state since the outgoing Hawking radiation and its ingoing partner are both enclosed by the cutoff surface. Once the outgoing Hawking radiation escapes the cutoff surface, the S_{sc} term in (5.22) begins to increase, but the area term does not increase since the quantum RT surface has not changed through the evaporation process.

Soon after the evaporation process begins, there appears another QES. It has already been shown that by definition the entanglement wedge of a QES is causal, so it is easy to argue that any non-vanishing QES must lie inside the event horizon of the black hole. Penington showed in 2019 that the non-vanishing QES lies inside the black hole close to the event horizon [39]. The argument is that as the black hole radiates, the size of the non-vanishing QES shrinks. At the Page time, the generalised entropy of the non-vanishing QES becomes smaller than that of the vanishing QES, so the non-vanishing QES is the new quantum R-T surface, and so the entropy of the system follows that of the non-vanishing QES.

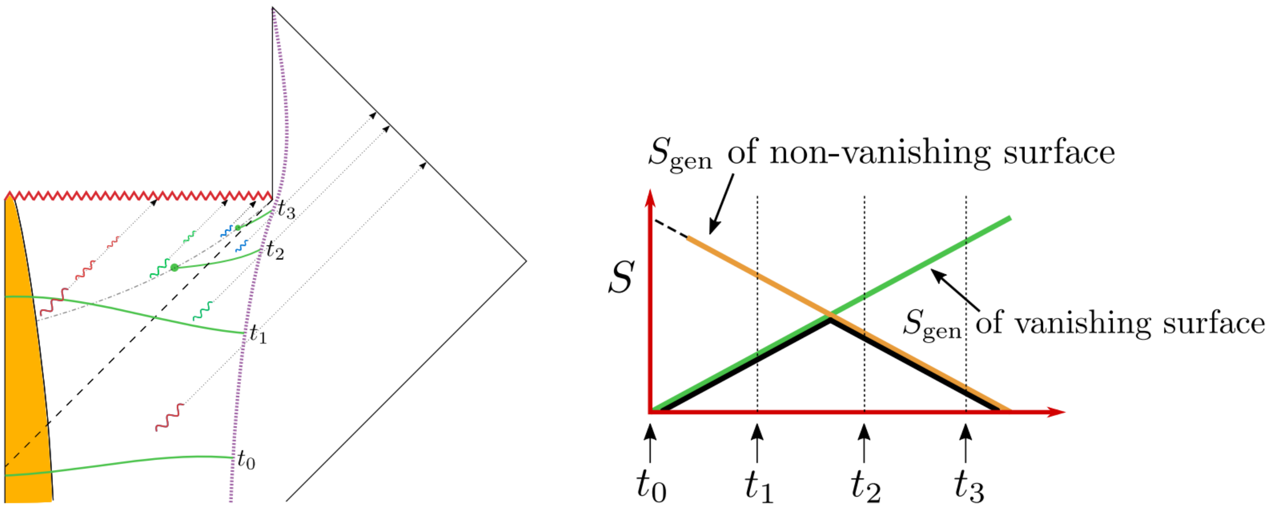

Figure 6.1: Carter-Penrose diagram of the black hole evaporation process with cutoff surface (purple) shown, alongside a graph showing the change in generalised entropy of the vanishing (orange) and non-vanishing (green) QES regimes. The Page curve is shown in black. Image source: [2]

In ingoing Eddington-Finkelstein coordinates, the non-vanishing QES lies at infalling time

$$v = -\frac{\beta}{2\pi} log(S_{BH}) + \mathcal{O}(\beta), \tag{6.1}$$

where β is the inverse temperature of the black hole. The leading order term here is known as the scrambling time, which was first recovered by Hayden and Preskill in 2007 as part of their proposed Hayden-Preskill decoding criterion [22]. Derived based on principles of quantum information and assuming unitary black hole dynamics, Hayden and Preskill proposed that a diary thrown into a black hole before the Page time can be recovered assuming we have sufficient information about the state of the black hole. In addition, any diary thrown after the Page time can be recovered a scrambling time after the diary is thrown. Penington shows that the Hayden-Preskill decoding criterion comes naturally from the QES formulation. Before the Page time, the vanishing QES is the RT surface. Any information thrown into the black hole is immediately enclosed by its entanglement wedge, and so its information is recoverable on the boundary. After the Page time, the non-vanishing QES is the RT surface. Any information thrown into the black hole is not immediately recoverable - time must elapse until the information is enclosed by this non-vanishing QES, which lies a scrambling time inside the event horizon.

6.2 Entropy of Hawking Radiation

In the previous section, we introduced a new framework for how we might consider the fine-grained entropy of black holes to recover the Page curve. However, we have yet to consider a key aspect of the black hole information paradox: the entropy growth of Hawking radiation. We have so far restricted our discussion to the region enclosed by the cutoff surface, ignoring that there is an increasing entropy of Hawking radiation accumulating outside it. By some finite time after the black hole has fully evaporated, we may consider all of the Hawking radiation to be outside of this cutoff surface. From the way the cutoff surface was constructed, we may assume that there is negligible gravity and hence can approximate a static spacetime. However, since our radiation has been obtained through gravitational treatment of fine-grained entropy in (5.22), we may not describe the entropy of this radiation in terms of the usual density matrix.

Our quantum extremal surfaces description of the HEE in (5.22) has until now been used for black hole spacetimes. However, this prescription is formulated more generally - there is no requirement for the presence of black holes for the formula to be valid. The notion of a minimal QES holds for any general spacetime. In addition, we previously only considered connected minimal QESs, since disconnected surfaces by their nature have a larger area, increasing their entropy. However, in the case that $S_{sc}(\Sigma_\Omega)$ decreases, one can see that disconnected QESs could plausibly be minimal. Indeed, since Hawking radiation is entangled to the black hole interior, if we include a portion of the black hole interior in our $S_{sc}(\Sigma_\Omega)$ such that the decrease in $S_{sc}(\Sigma_\Omega)$ is more than the increase in the area term, then we have a new QES. This portion of the black hole interior can be many different disconnected regions, which are known as islands. Hence our formula for the full entropy of Hawking radiation is given by

$$S_{rad} = min\left\{ext_\Omega\left[\frac{Area(\Omega)}{4G_N} + S_{sc}(\Sigma_{rad} \cup \Sigma_{island})\right]\right\}, \tag{6.2}$$

known as the island rule. Here Σ_{rad} is the region outside the cutoff surface out to infinity and Ω is now the boundary surface of the island. The relation to our QES formalism in (5.22) is evident. We are still extremising and minimising, except now we are considering disconnected QESs and we are no longer restricting to the interior of the cutoff surface.

Now that we have a proposed framework for thinking about the entropy of Hawking radiation, let us try to understand how black hole evaporation and the Page curve fit into it.

One possibility that is allowed by (6.2) is to have no islands. This formulation gives the same result as Hawking's original calculation: the entropy of radiation increases linearly as the black hole evaporates due to the increasing mixed state of the outgoing Hawking radiation.

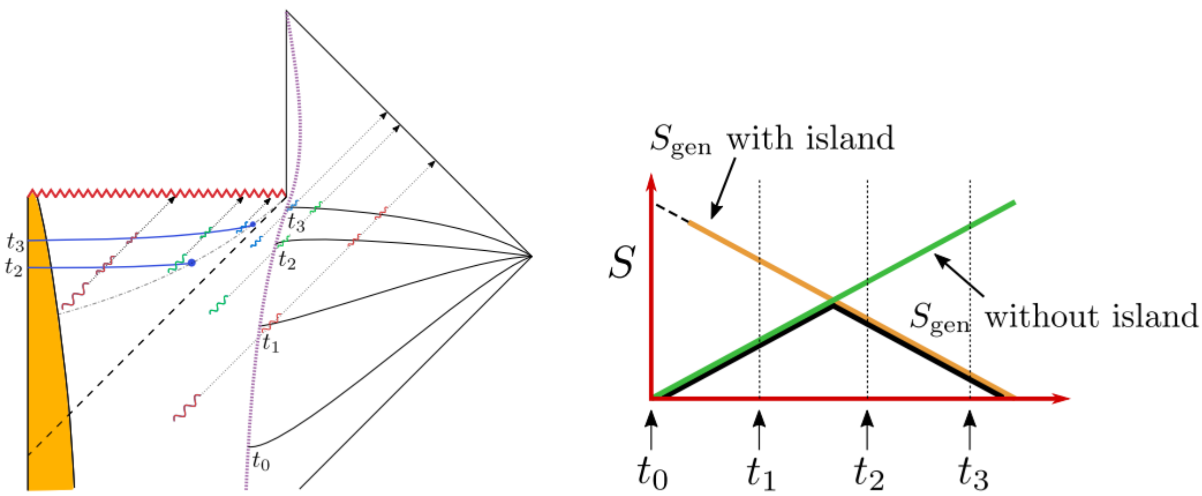

Figure 6.2: (left) Carter-Penrose diagram of the black hole evaporation process for the case of one island. Σ_{island} is represented by blue curves and Σ_{rad} is represented by black curves. (right) Graph showing the change in generalised entropy of the regime with islands (orange) and without islands (green). The Page curve is shown in black. Image source: [2]

The other possibility is non-vanishing islands. A non-vanishing island that gives some contribution to (6.2) appears at some finite time after the evaporation process begins. For this case, the area term in (6.2) dominates, and is roughly equal to our Bekenstein-Hawking entropy. The semi-classical entropy term remains small since it is assumed that most of the outgoing Hawking radiation in Σ_{rad} has its ingoing partner in Σ_{island}, so is purified and so gives trivial contribution to the semi-classical entropy.

The minimising function in (6.2) means that we take the minimum of the regimes with vanishing and non-vanishing islands. This accurately reproduces the Page curve.

6.3 Returning to Entanglement Wedge Reconstruction

So far we have looked at two different ways of thinking about the quantum information of black hole evaporation: the quantum extremal surfaces prescription for the entropy of black holes, and the the island rule for the entropy of Hawking radiation. We have seen that both reproduce the Page curve accurately, and both come from some extremisation of surfaces while exercising the holographic principle. A useful way to consider the interaction of these two principles in the context of the system as a whole is using entanglement wedge reconstruction, introduced earlier.

Fig. 6.3 schematically shows the three cases and the entanglement wedges in each case. Before the Page time, looking at black hole entropy the vanishing QES inside the cutoff surface is the quantum RT surface. Looking at the Hawking radiation entropy the minimal surface does not have any island, so the semi-classical entropy contribution is from Σ_{rad}. After the Page time before the black hole has completely evaporated, the non-vanishing QES that lies a scrambling time behind the event horizon is the minimal QES. In addition, we now have minimal contribution coming from a surface with a non-vanishing island. Finally, after the black hole has fully evaporated, we assume the cutoff surface is just flat space. The entanglement wedge of the radiation now includes the whole black hole interior, and the entanglement wedge of the black hole vanishes [39].

This picture has important implications for our consideration of the degrees of freedom

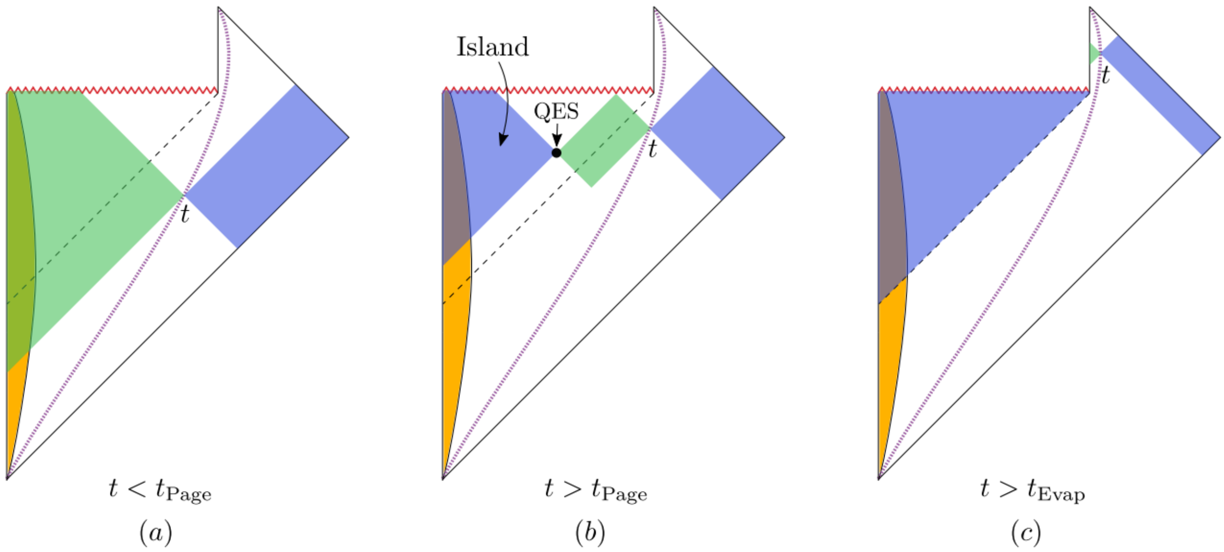

Figure 6.3: Carter-Penrose diagrams showing the entanglement wedges of the black hole (green) and Hawking radiation (blue) systems in the black hole evaporation process. Image source: [2]

in black holes. The entanglement wedges combined should provide a full picture about the information of the original black hole, before evaporation begun. An important insight here is that the "black hole degrees of freedom", shown by the green regions, are usually only given by a section of the black hole interior. This section changes in size, shape, and location as the black hole evaporates. Eventually all the degrees of freedom of the system are transferred to the radiation, but the island rule means that by the end of the evaporation process a large proportion of the radiation degrees of freedom are contained in the original black hole interior. Another key takeaway from this prescription is the effect that causality has on which parts of our surfaces are to be considered.

6.4 Resolving the AMPS Paradox

In our formulation of the black hole information paradox, we introduced the related AMPS paradox. The paradox has one important assumption: that the decrees of freedom of the black hole are described by the entire black hole interior, and no other region. However, the HEE picture views the degrees of freedom of black holes entirely differently. As illustrated in Fig. 6.3, this picture presents the degrees of freedom of the black hole system as much more fluid. The black hole's degrees of freedom are dependent on causal structure in the form of entanglement wedges, and Hawking radiation is considered to be able to contain degrees of freedom. The HEE picture hence contradicts an assumption of the AMPS paradox, which in turn means that the paradox needs not be considered for this formulation.

Chapter 7

Conclusion

We have now shown how a holographic consideration of entanglement entropy is a step towards a solution to the black hole information paradox. As the quantum extremal surfaces formulation is still in its early stages, there are many avenues for further exploration. This dissertation has omitted calculations of QESs for specific cases. For example, the holographic entanglement entropy of the toy model of 2-dimensional Jackiw-Teitelbohm (JT) gravity has been computed explicitly [38]. This theory is a specific case of AdS_3/CFT_2 in which $1 + 1$ dimensional dilaton gravity is coupled to a CFT_2. Although unphysical, this model provides a good continuation of Ryu and Takayanagi's derivation of the classical HEE for AdS_3/CFT_2 as recovered in (5.15). A key issue with this model, as well as all models for which Page curve computations have thus far been carried out, is that they include massive gravitons [15]. In physical situations, gravitons are massless. Hence in order to begin working towards computing Page curves for astrophysical black holes, the theory will need some major modification.

The QES and island formulations of HEE have some problems and limitations which present avenues for further research. We will now present a few of them.

Firstly, it is important to note that this formulation treats gravity as an effective field theory which has an action that may be expanded perturbatively. In order to get an exact picture of QESs and islands, one would require a full theory of quantum gravity.

Another point, mentioned by Almheiri et al. [2], is that the cutoff surface with which we encapsulate our black hole system is ambiguous in its nature. It is unclear where the spacetime begins to be static, or indeed from where we may stop considering the quantum nature of the black hole system, if anywhere. Almheiri et al. [2] posit that a full understanding of this picture would require a formulation for a fully dynamical spacetime, a feature of a theory of quantum gravity.

A central problem of this formulation of HEE is that although able to recreate the Page curve and give a unitary description of the fine-grained entropy of Hawking radiation, it does not provide density operators. This is an active field of research, and work is being done to align ideas about replica wormholes as a way of considering entanglement with the QES and island prescription.

An active avenue of further work is in considering whether the HEE prescription can be applied to cosmological horizons. Hawking and Gibbons showed in 1977 that flat spacetimes expanding with positive Hubble parameter have cosmological horizons which radiate analogous to Hawking radiation from black hole event horizons [18]. They extended the analogy and posited the Gibbons-Hawking entropy for cosmological horizons. It is very possible that the geometric and quantum considerations used by the QES and island formulation could be of use in computing a fine-grained Gibbons-Hawking entropy for cosmological horizons. However, the nature of cosmological horizons is not as well understood as that of black hole event horizons. The field of quantum cosmology is also still in its early stages of research, and so it is unclear whether the consideration of cosmological horizons as quantum systems is valid, or indeed useful. However, by beginning to consider cosmological horizons in these terms there may be some interesting insights to help develop our understanding of the quantum nature of these systems.

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