An Analysis of Time in Quantum Mechanics

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Declaration

I herewith certify that all material in this dissertation which is not my own work has been properly acknowledged.

Abstract

An Analysis of Time in Quantum Mechanics.

(Under the supervision of Professor Jonathan J. Halliwell)

Temporal questions in quantum mechanics have been an active topic of discussion and research for several decades. This thesis explores the temporal questions in quantum mechanics, focusing on the time of arrival, namely, finding the time of arrival probability distribution for a quantum particle crossing a particular spatial point. We first discuss the construction of a time of arrival operator, considering the limitation given by Pauli's theorem. Then, we review previous studies on the non self-adjoint time of arrival operator via positive operator valued measure. We continue this review by analysing self-adjoint variants of the time of arrival operator, considering both non-relativistic and relativistic quantum mechanics. We then follow various approaches to derive the time of arrival distribution. Subsequently, we investigate the notion of time measurement linked to the backflow regime. Next, we review the notion of quantum time measurement and its difficulties, considering different models. Finally, following previous work, we approach the arrival time problem through the Leggett-Garg inequalities.

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I would like to express gratitude to my supervisor Professor Jonathan J. Halliwell, my family and friends, in particular Selwyn G Simsek and Adam J Callison.

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†: Dagger (viz. conjugate transpose)

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Abbreviations

APCTOA: Algebra Preserving Confined Time of Arrival

CCR(s): Canonical Commutation Relation(s)

CTOA: Confined Quantum Time of Arrival

IND: Induction

LGI: Leggett-Garg Inequalities

LTOA: Local Time of Arrival

MR: Macroscopic Realism

MRps: Macroscopic Realism Per Se

NIM: Non-invasive Measurability

NSIT: No Signalling in Time

PDX: Path Decomposition Expansion

POVM(s): Positive Operator Valued Measure(s)

PVM(s): Projector Valued Measure(s)

QCTOA: Quantised Confined Quantum Time of Arrival

QM: Quantum Mechanics

QTOA: Quantum Time of Arrival

QUIPP Quantum First-passage Problem

QZE: Quantum Zeno Effect

SQUID: Superconducting Quantum Interference Device

TLS(s): Two-Level System(s)

TOA: Time of Arrival

TOF: Time-of-flight

WKB: Wentzel–Kramers–Brillouin

Chapter 1

Introduction

Time in the Schrödinger equations acts as an independent external parameter rather than an operator. However, quantum mechanical principles are based on the concept of observables associated with measurable quantities. How can time be considered merely a parameter? Is it possible to construct a time operator? What is the arrival time probability distribution for a quantum particle crossing a particular point in space ?

Time in quantum mechanics is a broad subject in physics that captivated philosophers and physicists throughout history [1, 2, 3, 5, 6, 7, 8, 9, 10, 11]. The focal point of this thesis is the quantum arrival time problem, also known as the quantum first passage problem (QUIPP). The goal of QUIPP is to find the arrival time distribution of a particle at a point in configuration space.

In classical mechanics the time of arrival is

$$T_x(q,p) = -\mathrm{sgn}(p)\sqrt{\frac{M}{2}}\int_x^q \frac{dq'}{\sqrt{H(q,p)-V(q')}}. \tag{1.1}$$

One might think that in order to find the quantum analogous and therefore compute an arrival time probability distribution, it is enough to use the correspondence principle and quantise equation (1.1). However, quantising equation (1.1) is non-trivial since it is not real and single-valued everywhere. Furthermore, there exist limitations in the quantisation of Euclidean space. Attempting quantisation would lead to a quantum arrival time operator not canonically conjugate to the Hamiltonian [1]. In addition to this, Pauli's theorem asserts that no self-adjoint time operator exists. Despite these clear obstacles, many physicists have made an effort to overcome them [7, 13, 15, 16, 18, 20, 21, 22, 23].

In Chapter 2, we analyse and derive the different time of arrival operators. In Section 2.2, we focus on the non self-adjoint positive operator valued measure formalism, followed by different self-adjoint variants in Section 2.3. Next, we generalise and compute the time of arrival operator in the presence of a potential. Finally, we consider the time operator for a free relativistic particle.

In Chapter 3, we review different approaches to the time of arrival distribution, starting from Kijowski's probability distribution and continuing with the operator approach and Bohmian mechanics. Finally, we conclude with probability linked to the current density and a more general approach to the computation of the time of arrival probability distribution. In Chapter 4, we briefly discuss quantum backflow, focusing on the backflow constant and links between the time of arrival measurements and the backflow regime.

In Chapter 5, we dive into the realm of time measurement in quantum mechanics. We analyse different models: toy models, the detector model, a theoretical model inspired by time-of-flight (TOF) experiments and the decoherent history model. In Chapter 6, we approach the time of arrival problem through the Leggett-Garg inequalities (LGIs). We initially introduce the construction of the LGIs and continue by discussing the tests of macrorealism, LGI violation of two-level systems (i.e. qubits) and weak and semi-weak measurements, in particular. Finally, we discussed the relationship between the LGIs and the arrival time problem. To conclude, in Chapter 7, we summarise and conclude this thesis.

Chapter 2

Time of Arrival Operator

In a footnote of his second Encyclopedia of Physics article of 1933, Pauli abandoned the existence of a self-adjoint time operator $\hat{T}$ [2]. His thought process was as follow, let the commutation relation between the Hamiltonian and $\hat{T}$, a time operator, be

$$[\hat{H}, \hat{T}] = i\hbar\mathbb{I} \tag{2.1}$$

where $\mathbb{I}$ is the identity operator, and $\hat{H}$ and $\hat{T}$ are canonical conjugate. Pauli argued that if such an operator exists, a unitary operator $\hat{U} = e^{\frac{-i\epsilon\hat{T}}{\hbar}}$ applied to the energy eigenstate $|E\rangle$ produces an energy shift $(E + \epsilon)$

$$\hat{H}\hat{U}\psi_E = (E + \epsilon)\psi_{\epsilon+E} \tag{2.2}$$

where ϵ is an arbitrary real number. Equation (2.2) implies the Hamiltonian and the time operator have a continuous spectrum encompassing the entire real-line. As a result, Pauli concluded that no self-adjoint time operator exists such that equation (2.1) is satisfied for a semibound or discrete Hamiltonian [3]. Initially, researchers discarded the existence of the self-adjoint time operator due to its inconsistency with the semi-bounded character of the energy spectrum [1]. If a time observable cannot be defined, then a particle's quantum arrival time cannot be measured. Therefore, it seemed impossible to formalise the notion of time observable, such as a time operator associated with the arrival of a particle (or the centre of mass of a composite system) at a detector. The time of arrival of a particle is generally defined as the time taken for the particle to reach a fixed location.

The time operator was considered incongruent with the standard formalisation of quantum mechanics developed by Von Neumann [4].

2.1 Early Studies on the Time of Arrival Operator

On the contrary, in 1961, Aharonov and Bohm showed that it is possible to construct a time observable. Their work closely analysed the time-energy uncertainty interpretations by Landau and Peierls [5], and Fock and Krylov [6]. Aharonov and Bohm found their conclusions inaccurate; in other words, it is incorrect to conclude that uncertainty is introduced when energy is measured within a short time. The error lies in the wrong interpretation of Bohr's concept of measurement. In addition, both papers selected an unsuitable example of a measurement process to prove their argument. Before introducing the Aharonov-Bohm time operator, it is essential to understand the measurement example used in [7]. The paper considers two particles colliding. The test particle, which is freely moving, serves as a clock. The other particle is the observed system. The clock's state defines the time of the collision. The time of collision, according to Aharonov and Bohm, is represented by the Hermitian operator $\hat{T}_c$

$$\hat{T}_c = \frac{1}{2}M\left(\hat{y}\frac{1}{\hat{p}_y} + \frac{1}{\hat{p}_y}\hat{y}\right) = \frac{\hat{y}}{v_y} \tag{2.3}$$

with $\left[\hat{H}_c, \hat{T}_c\right] = i\hbar\mathbb{I}$, where $\hat{H}_c = \frac{\hat{p}_y^2}{2M}$ is the Hamiltonian of the clock. $\hat{T}_c$ is maximally symmetric and is canonically conjugate with its Hamiltonian; therefore, it is possible to measure the system's energy with arbitrary accuracy within a short period. However, doing so will leave time undefined as per the uncertainty relations. Note that changing the sign of equation (2.3) for a particle with position q and momentum p leads to the Aharonov-Bohm arrival time operator of the test particle at the origin $q = 0$. [11].

In 1969, G. R. Allcock introduced his signatures papers [8, 9, 10]. They were a turning point in studying time operators in Quantum Mechanics. He attempted to introduce the time of arrival operator. His argument was in agreement with Pauli's theorem about the impossibility of the existence of a self-adjoint time operator. In his first paper, Alcock considered a freely moving particle emitted by a source at a finite distance from the detector. The non-relativistic one-dimensional Schrödinger equation describes the motion of the particle. The particle's wave function has both negative and positive energy components. One has to interpret the resulting wave function to solve the arrival-time problem by computing its probability distribution. Alcock found no solution to such a problem by providing a rigorous and detailed analysis. However, Allcock's arguments contained flaws.

The analysis by A D Baute et al. [12] reevaluates Allcock's work and shows it to be incorrect. In addition, in later work researchers showed that the Aharonov-Bohm time operator can be constructed with associated Positive Operator Valued Measures (POVMs). In this chapter, we look closely at their work and subsequent work on self-adjoint and non self-adjoint time operators.

2.2 Non Self-adjoint Time of Arrival Operator via Positive Operator Valued Measures (POVMs) formalism

Positive Operator Valued Measures (or POVMs) relate to generalising projective measurements in quantum mechanics. Let $\mathcal{H}$ be a finite-dimensional Hilbert space of d dimensions. POVMs is the set of positive-definite operators $\mathcal{M} = \{M_1, ..., M_n\} \in \mathrm{Herm}(\mathcal{H})^{\times n}$ such that

$$\sum_{i=1}^{n} M_i = \mathbb{I} \tag{2.4}$$

and

$$M(i \cup i') = M_i + M_{i'} \tag{2.5}$$

where $\mathbb{I}$ is the identity operator, and i and i' are disjoint intervals [13]. The operators M_i are known as effects. If the effects are orthogonal projectors, a projective measurement is known as projection-valued measure (PVM) [14]. Note that it is important to distinguish POVMs from PVMs, the former is a generalisation of the latter. POVMs aid the generalisation of the concept of observable. Recall that Pauli's theorem states there is no such thing as a self-adjoint time operator such that (2.1) is satisfied. Giannitrapani [15] rephrased this theorem in terms of POVMs formalism, in other words: *if $\mathcal{M}$ is a POVM on $\mathbb{R}$ and covariant with respect to $\mathcal{U} = e^{-\frac{i\lambda \hat{H}}{\hbar}}$, the representation of the one group $\mathcal{G}$ of translation, then $\mathcal{M}$ cannot be PVM.*

Using the same measurement example as in [7], where the clock is represented by a free particle in an Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, we can derive a time of arrival operator using POVMs. Consider equation (2.3) in momentum space, recall $\hat{q} = i\hbar\frac{\partial}{\partial p}$. Therefore

$$\hat{T}(p) = -\frac{1}{2} i\hbar M \left(\frac{\partial}{\partial p} p^{-1} + p^{-1} \frac{\partial}{\partial p} \right). \tag{2.6}$$

Note that we have added a minus sign to describe the time of arrival of a particle at the origin $q = 0$. Although $\hat{T}$ is not defined in the whole of $\mathcal{H}$, it is defined over the infinitely differentiable functions over $\mathbb{R} - \{0\}$. In other words, for each element of $\mathcal{H}$ there exist arbitrarily closed functions for which $\hat{T}$ is defined to act on. $\hat{T}$ cannot be PVM due to Pauli's theorem, but it can be interpreted via the generalisation of POVMs.

Let $|t\rangle$ be an eigenstate of $\hat{T}$ such that

$$\hat{T}\,|t\rangle = t\,|t\rangle \tag{2.7}$$

then if

$$\psi_t(p) = \langle p|t\rangle\,, \tag{2.8}$$

it follows that

$$\hat{T}\psi_t(p) = t\,\langle p|t\rangle \tag{2.9}$$

with solutions

$$\langle p|t,\alpha\rangle = \Theta(\alpha p)\left(\frac{|p|}{2\pi M\hbar}\right)^{1/2}\exp\left(-\frac{itp^2}{2M\hbar}\right) \tag{2.10}$$

where the eigenfunctions are nonsquare integrable, complete, but not orthogonal

$$\sum_\alpha \int_{-\infty}^{\infty} dt\,|t,\alpha\rangle\,\langle t,\alpha| = \mathbb{I} \tag{2.11}$$

$$\begin{aligned}\langle t,\alpha|t',\alpha'\rangle &= \int dp\overline{\psi_{t\alpha}(p)}\psi_{t'\alpha'}(p) &(2.12)\\ &= \int dp\,\langle t,\alpha|p\rangle\,\langle p|t',\alpha'\rangle &(2.13)\\ &= \frac{i}{2}\delta_{\alpha\alpha'}\left(\frac{1}{\pi}\mathrm{P}\frac{1}{t-t'} - i\delta(t-t')\right) &(2.14)\end{aligned}$$

where P is the principal part and $\alpha = \pm$. The eigenvectors of equation (2.10) are quadratically integrable in the limit of $t > 0$ [16]. The positive-operator valued measure must be a positive operator and be covariant with respect to $\hat{\mathcal{U}} = e^{\frac{-i\lambda\hat{H}}{\hbar}}$, where $\hat{H}$ is the generator of time translation. Intuitively the candidate is

$$\mathcal{T}(X) = \sum_\alpha \int_X dt\,|t,\alpha\rangle\,\langle t,\alpha| \tag{2.15}$$

where X is a Borel set. Equation (2.15) is positive, we only have to show that it is covariant with respect to $\hat{\mathcal{U}}$. We see that,

$$\mathrm{e}^{\frac{i\lambda\hat{H}}{\hbar}}\,|t,\alpha\rangle = |t-\lambda,\alpha\rangle \tag{2.16}$$

then using (2.16) we have

$$\langle\phi|\hat{\mathcal{U}}^{\dagger}\mathcal{T}(X)\hat{\mathcal{U}}|\phi\rangle = \langle\phi|\, \mathrm{e}^{\frac{i\lambda\hat{H}}{\hbar}}\mathcal{T}(X)\mathrm{e}^{-\frac{i\lambda\hat{H}}{\hbar}}|\phi\rangle \tag{2.17}$$

$$= \sum_{\alpha}\int_X \langle\phi|\, \mathrm{e}^{\frac{i\lambda\hat{H}}{\hbar}}|t,\alpha\rangle\langle t,\alpha|\, \mathrm{e}^{-\frac{i\lambda\hat{H}}{\hbar}}|\phi\rangle \tag{2.18}$$

$$= \sum_{\alpha}\int_X \langle\phi|t-\lambda,\alpha\rangle\langle t-\lambda,\alpha|\phi\rangle \tag{2.19}$$

$$= \langle\phi|\,\mathcal{T}(X-\lambda)\,|\phi\rangle\,. \tag{2.20}$$

Therefore $\mathcal{T}(X)$ is the POVM of (2.6). POVMs are also helpful in deriving the probability distribution. We will discuss this in Chapter 3.

In addition to equation (2.6), there are alternative forms of $\hat{T}$ one might want to explore. For example

$$\hat{T}(p) = \frac{1}{2}i\hbar M\left(\frac{1}{p^2} - \frac{2}{p}\frac{\partial}{\partial p}\right). \tag{2.21}$$

Equation (2.21) can only act on continuous functions due its singularity at $p = 0$. The singularity is bypassed, if $\psi(p) \sim p^{1/2}$ or $\psi(p)/p^{3/2} \to 0$ for $p \to 0$. The domain on which $\hat{T}$ acts should reflect the same properties as $\hat{T}$ [17]. In other words, if the time operator is symmetric, so is the domain. As a result we must exclude $\psi(p) \sim p^{1/2}$ as an option and it follows that the domain of $\hat{T}$, is the domain of continuous quadratically integrable functions of p, such that $\psi(p)/p^{3/2} \to 0$ for $p \to 0$.

2.3 Self-adjoint Variants of the Time of Arrival Operator

According to Pauli's theorem, a self-adjoint time operator is incompatible with a semi-bounded energy spectrum. However, in 1997, Delgado and Muga [18] found a way to circumvent this theorem by considering an unbounded Hamiltonian and quantising the classical expression $-Mq/|p|$, that was initially studied by Kijowski. The resulting operator is known as T_{KDM} operator. The labels stand for Kijowski, Delgado and Muga, who studied the operator in [25] and [18].

There has been further research on the self-adjoint variants of the time of arrival operator, such as [19, 20, 21, 22, 23]. In particular, in [22], a self-adjoint operator is obtained via the quantisation of $M|q|/|p|$ and therefore, the modification of equation (2.6). A third expression also leads to a self-adjoint operator, $M|q|/p$. This section will examine a few examples of the arrival time operator's non-relativistic and relativistic self-adjoint variants.

2.3.1 Time of Arrival Operator for a Non-Relativistic Particle

In [18], a self-adjoint time operator $\hat{T}$ is constructed using a canonically conjugate self-adjoint and unbounded Hamiltonian $\hat{H}$.

Consider the one dimensional motion of a free particle along the x axis in the energy representation.

$$\{|E,\alpha\rangle\,; E \geq 0, \alpha = \pm\}. \tag{2.22}$$

The eigenvalue equations are given by

$$\hat{H}_0 |E,\alpha\rangle = \frac{\hat{p}^2}{2M} |E,\alpha\rangle \tag{2.23}$$

$$= E\,|E,\alpha\rangle \tag{2.24}$$

$$\hat{p}\,|E,\alpha\rangle = \pm\sqrt{2ME}\,|E,\alpha\rangle\,. \tag{2.25}$$

The eigenstates are complete and orthogonal,

$$\sum_{\alpha=\pm}\int_0^{\infty} dE\,|E,\alpha\rangle\langle E,\alpha| = \mathbb{I} \tag{2.26}$$

$$\left\langle E,\alpha\middle|E',\alpha'\right\rangle = \int_{-\infty}^{+\infty} dp\,\langle E,\alpha|p\rangle\left\langle p\middle|E',\alpha'\right\rangle \tag{2.27}$$

$$= \delta_{\alpha\alpha'}\delta(E-E') \tag{2.28}$$

where $\int_{-\infty}^{+\infty} dp\,|p\rangle\langle p| = \mathbb{I}$ is used in equation (2.27).

$$\hat{H} = \mathrm{sgn}(\hat{p})\hat{H}_0 \tag{2.29}$$

where $\mathrm{sgn}(\hat{p})$ is a self-adjoint operator

$$\mathrm{sgn}(\hat{p}) = \Theta(\hat{p}) - \Theta(-\hat{p}) \tag{2.30}$$

with $\Theta(\pm\hat{p}) = \int_{-\infty}^{+\infty} |\pm\hat{p}\rangle\langle\pm\hat{p}|$ being the projector acting on a subspace spanned by plane waves with positive and/or negative momenta. The Hamiltonian has unbounded spectrum such that

$$\hat{H}\,|E,\alpha\rangle = \pm E\,|E,\alpha\rangle\,, \text{with } \alpha = \pm. \tag{2.31}$$

By changing notation in (2.32) for the energy eigenstates we can therefore define the time eigenstates $|\tau\rangle$ in terms of $|\epsilon\rangle$.

$$|\epsilon\rangle = \begin{cases} |+E\rangle \equiv |E,+\rangle \text{ if } \epsilon \geq 0 \\ |-E\rangle \equiv |E,-\rangle \text{ if } \epsilon < 0 \end{cases} \tag{2.32}$$

$$|\tau\rangle = \frac{1}{h^{1/2}} \int_{-\infty}^{\infty} d\epsilon \, \mathrm{e}^{\frac{i\epsilon\tau}{\hbar}} \, |\epsilon\rangle \, . \tag{2.33}$$

Remembering that $\{|\epsilon\rangle\}$ is a set of orthonormal and complete basis we can then compute

$$\langle\tau|\tau'\rangle = \int_{-\infty}^{+\infty} d\epsilon \, \langle\tau|\epsilon\rangle \, \langle\epsilon|\tau'\rangle \tag{2.34}$$

$$= \frac{1}{h} \int_{-\infty}^{+\infty} d\epsilon \, \mathrm{e}^{\frac{i\epsilon(\tau-\tau')}{\hbar}} \tag{2.35}$$

$$= \delta(\tau - \tau') \tag{2.36}$$

and using equation (2.33) we have

$$\int_{-\infty}^{+\infty} d\tau \, |\tau\rangle \, \langle\tau| = \frac{1}{h} \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\epsilon \tag{2.37}$$

$$\times \int_{-\infty}^{+\infty} d\epsilon' \mathrm{e}^{\frac{i(\epsilon-\epsilon')\tau}{\hbar}} \, |\epsilon\rangle \, \langle\epsilon'| \tag{2.38}$$

$$= \int_{-\infty}^{+\infty} d\epsilon \, |\epsilon\rangle \, \langle\epsilon| \tag{2.39}$$

$$= \mathbb{I}. \tag{2.40}$$

Hence, one can define the self-adjoint time operator in terms of spectral decomposition via the eigenstate $|\tau\rangle$ and its eigenvalues τ as,

$$\hat{T} = \int_{-\infty}^{+\infty} d\tau \; \tau \, |\tau\rangle \, \langle\tau| \, . \tag{2.41}$$

However, this definition has a caveat. Since $\hat{T}$ depends on the choice of Hamiltonian, if $\hat{H}$ differs from the Hamiltonian of the physical system, the time operator will not represent the actual physical time. Therefore it is necessary to interpret $\hat{T}$ in connection with measurement results. In other words, we need to find the probability distribution of measurement results connected with the eigenvalues and eigenvectors of $\hat{T}$. Note that $|\tau\rangle$ can be decomposed in terms of $|t,\alpha\rangle$, with $\alpha = \pm$. This decomposition leads us to the results of the previous section, equations (2.11) and (2.12). Furthermore, as we saw previously $|t,\alpha\rangle$ is not adequate for constructing a self-adjoint operator.

In 1998, Oppenheim et al. [19] proposed an alternative method by modifying the low momentum behaviour of the time of arrival operator to make it self-adjoint. Equation (2.21) can be re-written as

$$\hat{T}(p) = -i\hbar M \left(\frac{1}{p^{1/2}} \frac{\partial}{\partial p} \frac{1}{p^{1/2}} \right) \tag{2.42}$$

with the eigenstates from equation (2.10) and with $p^{1/2} = i|p|^{1/2}$ for $p < 0$. Equation (2.42) describes the time of arrival operator of a particle whose detector is located at $q = 0$. To modify $\hat{T}(p)$, let us consider a smooth function $f_\epsilon(p)$ and an arbitrary small positive number ϵ. The operator (2.42) takes the following form

$$\hat{T}_\epsilon(p) = -i\hbar M \sqrt{f_\epsilon(p)} \frac{\partial}{\partial p} \sqrt{f_\epsilon(p)}. \tag{2.43}$$

If $f_\epsilon(p) \to 0$ at least at the same rate as p, $\hat{T}_\epsilon(p)$ is self-adjoint and quadratically integrable. Hence the eigenstates are

$$\psi_t^\alpha(p) = \Theta(\alpha p) \left(\frac{1}{2\pi\hbar M} \right)^{1/2} \left(\frac{1}{f_\epsilon(p)} \right)^{1/2} \mathrm{e}^{\frac{it}{\hbar M} \int_\epsilon^p f(p')dp'} \tag{2.44}$$

where $\alpha = \pm$. Furthermore, we have $|t, +\rangle$ for $p > 0$ and $|t, -\rangle$ for $p < 0$. As in Grot et al. [24], paper [19] considers the same requirements for $f_\epsilon(p)$, namely

$$f_\epsilon(p) = \begin{cases} p\epsilon^{-2} & |p| < \epsilon \\ \frac{1}{p} & |p| > \epsilon. \end{cases} \tag{2.45}$$

If $f_\epsilon(p) \to 0$ slower than p , $T_\epsilon(p)$ cannot be self-adjoint. On the other hand, if $f_\epsilon(p) \to 0$ faster than p, then a particle in the eigenstate (2.44) will have a higher probability of not reaching the detector at the estimated time. Knowing this, we can decompose the eigenfunctions into two components.

$$\psi_t^\alpha(p) \equiv \psi_t^\alpha(p)_\epsilon + \psi_t^\alpha(p)_o \tag{2.46}$$

where

$$\psi_t^\alpha(p)_\epsilon = \begin{cases} \Theta(\alpha p) \left(\frac{1}{2\pi\hbar M}\right)^{1/2} p^{-1/2} \mathrm{e}^{\frac{it}{\hbar M} \ln k/\epsilon} & |p| < \epsilon \\ 0 & |p| > \epsilon \end{cases} \tag{2.47}$$

and

$$\psi_t^\alpha(p)_o = \begin{cases} 0 & |p| < \epsilon \\ \Theta(\alpha p) \left(\frac{1}{2\pi\hbar M}\right)^{1/2} p^{1/2} \mathrm{e}^{\frac{it}{\hbar M}(p^2 - \epsilon^2)} & |p| > \epsilon \end{cases} \tag{2.48}$$

These eigenstates are orthogonal, and this can be seen by introducing the eigenstates in new coordinates

$$z^{\pm} = \int_{\pm\epsilon}^{p} \frac{dp'}{f_\epsilon(p')} \tag{2.49}$$

it follows that,

$$\langle t, \alpha \big| t', \alpha' \rangle = \delta_{\alpha\alpha'} \delta(t - t'). \tag{2.50}$$

Another technique we present here was proposed by Halliwell et al. [23], where a self-adjoint time operator, with links to two simple measurement models, is derived. The first measurement model consists of spatial measurement onto $q < 0$ and $q > 0$ to determine the probability that a particle reaches the origin in a given time interval $[0, \tau]$. Given the two projectors $P = \Theta(q)$ and $\bar{P} = \Theta(-q)$, the probability is

$$p(0, \tau) = \langle\psi| \bar{P} P(\tau) \bar{P} |\psi\rangle + \langle\psi| P \bar{P}(\tau) P |\psi\rangle . \tag{2.51}$$

Given that $dP(\tau)/dt = \hat{J}(t)$ with $\hat{J}(t)$ being the current, we can rewrite (2.51) as

$$p(0, \tau) = \int_0^{\tau} dt \,\langle\psi| \bar{P} \hat{J}(t) \bar{P} |\psi\rangle + \int_0^{\tau} dt \,\langle\psi| P \hat{J}(t) P |\psi\rangle . \tag{2.52}$$

The second measurement model comprises a stopwatch coupled to a particle via $p_y \Theta(-q)$, where y is the system's coordinate. As the particle travels from the left, a shift in y is measured until the particle reaches the origin at $q = 0$. Classically, the change is then given by

$$y(T) - y(0) = \int_0^{T} dt \;\Theta\left(-q - \frac{p}{M} t\right) \tag{2.53}$$

$$= \int_0^{T} dt \; tJ(t) \tag{2.54}$$

where T is very large and $J(t)$ is the classical current. If we quantise this system, we can represent the interaction between the particle and the stopwatch through the $S-$matrix

$$S = \mathbb{T} \exp\left(-\frac{i}{\hbar}\lambda \int_0^{T} dt \hat{p}_y(t) \Theta\left(-\hat{q} - \frac{\hat{p}}{M} t\right)\right) \tag{2.55}$$

with λ being the coupling constant and $\mathbb{T}$ the time ordering operator. We note that the classical arrival time operator is linked to the classical current,

$$-\frac{Mq}{|p|} = \int_{-\infty}^{\infty} dt \; tJ(t). \tag{2.56}$$

The quantisation of the LHS of equation (2.56) leads us to the Kijowski, Delgado, Muga arrival time operator. On the other hand, if we quantise the RHS we obtain

$$\hat{T} = \int_{-\infty}^{\infty} dt \; t\hat{J}(t). \tag{2.57}$$

Recall that

$$\frac{\hat{p}t}{M}\delta\left(\hat{q}+\frac{\hat{p}t}{M}\right)=-\hat{q}\delta\left(\hat{q}+\frac{\hat{p}t}{M}\right). \tag{2.58}$$

Let $\hat{R}$ be the reflection operator, such that $\hat{R}\left|p\right\rangle=\left|-p\right\rangle$. To calculate the time integral we compute the following

$$\int_{-\infty}^{+\infty}dt\ \left\langle p_1\right|\delta\left(\hat{q}+\frac{\hat{p}t}{M}\right)\left|p_2\right\rangle=\left\langle p_1\right|\delta(\hat{q})\left|p_2\right\rangle\int_{-\infty}^{\infty}dt\exp\left(\frac{it}{2M\hbar}\left(p_1^2-p_2^2\right)\right) \tag{2.59}$$

$$=2M\delta(p_1^2-p_2^2) \tag{2.60}$$

$$=\frac{M}{|p_1|}\left(\delta(p_1^2+p_2^2)+\delta(p_1^2-p_2^2)\right) \tag{2.61}$$

$$=\left\langle p_1\right|\frac{(1+\hat{R})}{|\hat{p}|}\left|p_2\right\rangle. \tag{2.62}$$

Equation (2.57) becomes,

$$\hat{T}=-\frac{M}{2}\left(\hat{q}\frac{1}{|\hat{p}|}(1+\hat{R})+\frac{1}{|p|}(1+\hat{R})\hat{x}\right) \tag{2.63}$$

$$=-\frac{M}{2}\left(\hat{q}\frac{1}{|\hat{p}|}+\frac{1}{|\hat{p}|}\hat{q}\right)-\frac{M}{2}\left(\hat{q}\frac{1}{|\hat{p}|}\hat{R}+\frac{1}{|\hat{p}|}\hat{R}\hat{q}\right) \tag{2.64}$$

$$=\hat{T}_{KDM}+\frac{i\hbar M}{2\hat{p}|\hat{p}|}\hat{R}. \tag{2.65}$$

Where T_{KDM} is the time of arrival operator in the position-space representation we have discussed previously. The first term, $\hat{T}_{KDM}$, is self-adjoint. The second term is the contribution from equation (2.62) and commutes with the Hamiltonian. As a result, the commutation relation between equation (2.65) and the Hamiltonian yields

$$[\hat{T},\hat{H}]=-i\hbar\epsilon(p). \tag{2.66}$$

Finally, equation (2.65) holds interesting properties, similar to the properties of the dwell time operator.

$$\hat{T}_D=\int_{-\infty}^{+\infty}dt\mathrm{e}^{\frac{i}{\hbar}Ht}P_L\mathrm{e}^{-\frac{i}{\hbar}Ht} \tag{2.67}$$

$$=\frac{ML}{|\hat{p}|}\left(1+\mathrm{e}^{-\frac{i}{\hbar}\hat{p}L}\frac{\sin(\hat{p}L/\hbar)}{\hat{p}L/\hbar}\hat{R}\right) \tag{2.68}$$

with $H=p^2/2M$ and $P_L=\int_0^L dq\left|q\right\rangle\left\langle q\right|$. The dwell time operator is the result of the quantisation of $ML/|p|$. For high momenta the quantum version coincides with the classical expression, whilst for low momenta equation (2.67) becomes

$$\hat{T}_D\approx\frac{ML}{|\hat{p}|}(1+\hat{R}) \tag{2.69}$$

$$=\mathrm{e}^{\frac{i}{\hbar}L\hat{p}}\hat{T}\mathrm{e}^{-\frac{i}{\hbar}L\hat{p}}-\hat{T}. \tag{2.70}$$

Where equation (2.69) shows the relationship between the dwell time operator and $\hat{T}$. Note that $\hat{T}_D$ is self-adjoint. Therefore also $\hat{T}$ is self-adjoint.

2.3.2 Confined Quantum Time of Arrival Operator for a Vanishing Potential

The non self-adjointness of the time operator can be addressed by considering a spatially confined particle [26]. In [26], Galapon et al. also argue that the non-selfadjointness of the time operator has nothing to do with the semibound nature of the energy spectrum [27].

Consider the Aharonov-Bohm time of arrival operator.

$$\hat{T} = -\frac{M}{2}\left(\hat{q}\frac{1}{\hat{p}} + \frac{1}{\hat{p}}\hat{q}\right). \tag{2.71}$$

Recall that $\hat{T}$ is canonically conjugate to the Hamiltonian $H = p^2/2M$, maximally symmetric and non self-adjoint. Consider a particle spatially confined within two points such that the length of the confinement is $2l$. The classical time of arrival is given by $T = -Mq/p$ provided that $q < l$ and $p \neq 0$. Its quantisation $\hat{T}$ is canonically conjugate to the Hamiltonian and maximally symmetric. To understand the properties of this operator, let us consider a Hilbert space $\mathcal{H} = L^2[-l, l]$, the position operator is a unique bounded operator and acts on functions in the Hilbert space $L^2[-l, l]$

$$\hat{q}\psi(q) = q\psi(q), \ \forall \ \psi(q) \in \mathcal{H}. \tag{2.72}$$

On the other hand, the momentum operator must be chosen, bearing in mind that we are dealing with a closed system and that the kinetic Hamiltonian generates the evolution of the system. Therefore we require the time evolution to be unitary, and the momentum operator must commute with the Hamiltonian, which is purely kinetic and is equivalent to the Hamiltonian of a freely evolving classical particle satisfying the boundary conditions. Now, consider values for $|\gamma| < \pi$, the associated momentum operator is self-adjoint and is represented by $\hat{p}_\gamma = -i\hbar\frac{\partial}{\partial q}$. Its domain is spanned by the functions $\phi(q) \in \mathcal{H}$, such that $\int |\phi(q')|^2 dq' < \infty$, where $\phi(-l) = \mathrm{e}^{-2i\gamma}\phi(l)$.

$\forall \ \gamma \neq 0 \wedge \hat{p}_\gamma^{-1} \neq 0$ there exists a time operator $\hat{T}_\gamma$. Hence if $\hat{p}_\gamma^{-1}$ exists, it is also bounded and self-adjoint therefore also, $\hat{T}_\gamma$ is self-adjoint. This operator takes the form

$$\hat{T}_\gamma = -\frac{M}{2}\left(\hat{q}\frac{1}{\hat{p}_\gamma} + \frac{1}{\hat{p}_\gamma}\hat{q}\right). \tag{2.73}$$

Equation (2.73) is also known as the non-periodic confined quantum time of arrival (CTOA) operator $\forall\ |\gamma| < \pi$. Switching to position representation leads to the Fredholm integral operator

$$\hat{T}\psi(q) = \int_{-l}^{l} \langle q| \hat{T}_\gamma \left|q'\right\rangle \psi(q')dq' \tag{2.74}$$

$$= \int_{-l}^{l} T(q,q')\psi(q')dq',\ \forall\ \psi(q) \in \mathcal{H} \tag{2.75}$$

with $T(q, q')$ being the kernel,

$$\langle q| \hat{T}_\gamma \left|q'\right\rangle = -M\frac{(q+q')}{4\hbar \sin\gamma}[\mathrm{e}^{i\gamma}\mathrm{H}(q-q') + \mathrm{e}^{-i\gamma}\mathrm{H}(q'-q)], \tag{2.76}$$

where $H(q - q')$ is the Heavyside function. Now, consider a set of positive integers $n = 1, 2, 3....$ For $\gamma \neq 0, \frac{\pi}{2}$ and for each n there exist eigenfunctions $\psi^{\pm}_{n,\gamma}(q)$ that have eigenvalues of same magnitude and opposite sign.

$$\psi^{\pm}_{n,\gamma}(q) = A_n \mathrm{e}^{\mp i r_n (q^2/l^2)} \left\{ J^{\mp}_{3/4,1/4}\left(r_n \frac{q^2}{l^2}\right) [J_{-1/4}(r^n) - \cot\gamma J_{3/4}(r_n)] \pm \frac{2q\sqrt{r_n}}{l} J^{\mp}_{1/4,3/4}\left(r_n \frac{q^2}{l^2}\right) \times [J_{-3/4}(r_n) - \cot\gamma J_{1/4}(r_n)] \right\} \tag{2.77}$$

where $J_\nu(x)$ is the Bessel function of the first kind, $J^{\mp}_{\nu,\rho} = x^\nu[J_{-\nu}(x) \mp iJ_\rho(x)]$, and A_ν is the normalisation constant. The eigenfunctions vanishing for $[-l, l]$ are known as nodal eigenfunctions otherwise they are nonnodal eigenfunctions. The former correspond to an odd quantum number, whilst the latter correspond to a even quantum number. The eigenvalues are $\tau^{\pm}_{n,\gamma} = \pm\left(Ml^2/4\hbar\right) r_n^{-1}$

On the contrary, if we consider $\gamma = 0$, then 0 is an eigenvalue of $\hat{p}_0$, which means $\hat{p}_0^{-1}$ does not exists, and as a result, $\hat{T}_\gamma$ lacks interpretation. To avoid this issue, let us project $\hat{p}_0$ onto a subspace orthogonal to its subspace whose vectors map to zero [26]. This technique leads to a periodic self-adjoint CTOA operator with the kernel,

$$T_0(q,q') = \langle q| \hat{T}_\gamma \left|q'\right\rangle \tag{2.78}$$

$$= \frac{M}{4i\hbar}\left(q+q'\right)\mathrm{sgn}(q-q') - \frac{M}{4i\hbar l}\left(q^2 - q'^2\right) \tag{2.79}$$

where equation (2.78) is quadratically integrable, in other words

$$\int_{-l}^{l}\int_{-l}^{l} \left|T(q,q')\right|^2 dqdq' < \infty. \tag{2.80}$$

Hence T_γ has a set of complete eigenfunctions with discrete energy spectrum. We conclude that the CTOA operator is compact, self-adjoint, and canonically conjugate to its respective Hamiltonian in a nondense subspace of $\mathcal{H}$. It is however not covariant.

2.3.3 Confined Quantum Time of Arrival Operator for a Continuous Potential

The previous study can be generalised to a CTOA in the presence of an arbitrary continuous interaction potential [28]. The generalisation is achieved by quantising the classical arrival time for a given potential. The results from paper [28] show that only a particular class of potentials leads to CTOA operators conjugate to their respective Hamiltonian.

Let us consider a particle of mass M subjected to an interaction potential $V(q)$, with x and p being the initial position and momentum at $t = 0$. Then the time of arrival at q is

$$T(p, q) = -\text{sgn}(p)\sqrt{\frac{M}{2}} \int_x^q \frac{dq'}{\sqrt{H(q,p) - V(q')}}. \tag{2.81}$$

Because $T(p, q)$ can be complex valued and multiple valued, its quantisation is only possible if the potential is constant. As a result, the quantisation of equation (2.81) was initially dismissed [30]. However, in [28] Galapon quantises exclusively the trajectories passing through the point of arrival q, in addition quantisation is restricted to the first time the particle arrives. Rather than quantising the arrival time directly the expansion in the neighbourhood of the arrival time point is considered,

$$T(p, q) = \sum_{k=0}^{\infty} (-1)^k T_k(q, p). \tag{2.82}$$

Where $T_k(q, p) = -Mp^{-1} \int_x^q (\partial_{q'} V)(\partial_p T_{k-1}) dq'$ and V is continuous around the arrival point. Equation (2.82) is known as Local Time of Arrival (LTOA). Next, a change of variables is applied to equation (2.81), such that $\tilde{q} = q - x$ and $\tilde{p} = p$ to obtain the time of arrival at the point of origin for a potential $V(\tilde{q} + x)$. Given that the general form for LTOA is $T(q, p) = \sum_{m,n} \alpha_{m,n} q^n p^{-m}$, Weyl quantisation leads to the following time operator

$$\hat{T} = \sum_{m,n \geq 0} \alpha_{m,n} \hat{T}_{m,n} \tag{2.83}$$

where

$$\hat{T}_{m,n} = 2^{-n} \sum_{j=0}^{n} \binom{n}{j} \hat{q}^j \hat{p}^{-m} \hat{q}^{n-j}. \tag{2.84}$$

By projecting $\hat{T}$ onto the Hilbert space, $\mathcal{H} = L^2[-l,l]$, a family of operators $\{T_\gamma\}$, with their respective momenta $\{p_\gamma\}$, is obtained. T_γ is the quantum confined time of arrival (QCTOA). As in section 2.2.2, there are two cases to consider: $\gamma \neq 0$ and $\gamma = 0$. For $\gamma \neq 0$, the QCTOA is given by

$$\hat{T}^{\gamma\neq 0}_{m,n\geq 0} = 2^{-n} \sum_{j=0}^{n} \binom{n}{j} \hat{q}^j \hat{p}^{-m} \hat{q}^{n-j}. \tag{2.85}$$

Equation (2.85) is compact and self-adjoint. On the other hand for $\gamma = 0$

$$\hat{T}^{\gamma=0}_{m,n\geq 0} = 2^{-n} \sum_{j=0}^{n} \binom{n}{j} \hat{q}^j \hat{P}_0^{-m} \hat{q}^{n-j} \tag{2.86}$$

where $P_0 = E p_0^{-1} E$, with E being the projector onto the subspace orthogonal to the subspace of p_0 [28].

A generalisation for an arbitrary continuous potential leads to a LTOA given by

$$T = -\sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!}{k!} \frac{M^k}{p^{2k+1}} \int_0^q (V(q) - V(q'))^k dq'. \tag{2.87}$$

By assuming the potential to be a power series, and with quantisation, equation (2.87) leads to the kernels for $\gamma \neq 0$ and $\gamma = 0$

$$\langle q| T_{\gamma\neq 0} |q'\rangle = -M \frac{T_0(q,q')}{\hbar \sin\gamma} [\mathrm{e}^{i\gamma} \mathrm{H}(q-q') + \mathrm{e}^{-i\gamma} \mathrm{H}(q'-q)] \tag{2.88}$$

$$\langle q| T_{\gamma=0} |q'\rangle = \frac{M}{i\hbar} T_0(q,q') \mathrm{sgn}(q-q') - \frac{M}{i\hbar l} B_0(q,q') \tag{2.89}$$

where $T_0(q,q')$ is symmetric quadratically integrable, whilst $B_0(q,q')$ is asymmetric quadratically integrable. They both depend on the Gaussian analytic function, ${}_pF_q$ [28].

Therefore, it follows that T_γ is a self-adjoint, quadratically integrable operator for finite l. However, T_γ is only canonically conjugate with the Hamiltonian of linear systems. It is not canonically conjugate with the Hamiltonian of non-linear systems. The limitations to the QCTOA are due to constraints in the quantisation of Euclidean space. Introducing the algebra preserving CTOA operator (APCTOA) was found to help derive a canonically conjugate CTOA for non-linear systems [28]. QCTOA and APCTOA are always equivalent for linear systems (e.g. harmonic oscillators).

2.3.4 Time of Arrival Operator for a Relativistic Particle

The natural next step is to compute the arrival time operator for a relativistic particle. In this case, the time of arrival for a relativistic particle of 1/2 spin has a self-adjoint extension due to particle-antiparticle symmetry [29]. Consider Einstein's summation convention by which repeated indices are summed over. Let the metric be $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. In this section, let us set $\hbar = c = 1$. Consider 4 dimensional Dirac matrices with algebra $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$. Then the Hamiltonian is given by

$$\hat{H} = \gamma^0\gamma^i\hat{p} + \gamma^0 M \tag{2.90}$$

with $i = (0, 1, 2, 3)$ and M being the rest mass. Let the momentum $p^\mu = (E, p, 0, 0)$ be parallel to the x-axis, so that

$$\hat{H} = \gamma^0\gamma^1\hat{p} + \gamma^0 M. \tag{2.91}$$

The Dirac equations are

$$i\frac{\partial\psi(x,t)}{\partial t} = \left(\gamma^0\gamma^1\hat{p} + \gamma^0 M\right)\psi(x,t) \tag{2.92}$$

$$= \left(-i\gamma^0\gamma^1\frac{\partial}{\partial x} + \gamma^0 M\right)\psi(x,t). \tag{2.93}$$

We now find the arrival time operator conjugate to the Hamiltonian equation (2.91). To do so, consider the eigenstates in the momentum representation. They are complete and orthogonal.

$$\begin{cases} \langle p, \lambda, s|p', \lambda', s'\rangle = \delta_{\lambda\lambda'}\delta_{ss'}\delta(p' - p) \\ \sum_{\lambda,s}\int_{-\infty}^{+\infty} |p, \lambda, s\rangle\langle p, \lambda, s|\, dp = \mathbb{I} \end{cases} \tag{2.94}$$

with $\lambda, \lambda' = \pm 1$, $s, s' = \pm 1/2$ and $|p', \lambda', s'\rangle = \psi_{\lambda s}(p)\,|p\rangle$. The eigenstates are given by

$$\psi_{\lambda,s}(p) = \sqrt{\frac{M + \lambda E_p}{2\lambda E_p}}\begin{pmatrix} \eta_s \\ \frac{\sigma_1 p}{M+\lambda E_p}\eta_s \end{pmatrix} \tag{2.95}$$

where η_s are orthogonal and complete two components spinors such that, $\eta_s^\dagger\eta_{s'} = \delta_s\delta_s'$, $\sum_s \eta_s\eta_s^\dagger = \mathbb{I}$, and $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We know that classically a freely moving particle has an arrival time $t = -Mq/p$, for a relativistic particle, the expression becomes

$$T = -q\frac{E}{p}. \tag{2.96}$$

Quantising equation (2.96) leads to

$$\hat{T}_{\text{Dirac}} = -\frac{1}{4}\left[\hat{H}\left(\frac{1}{\hat{p}}\hat{q} + \hat{q}\frac{1}{\hat{p}}\right) + \left(\frac{1}{\hat{p}}\hat{q} + \hat{q}\frac{1}{\hat{p}}\right)\hat{H}\right] \tag{2.97}$$

it may be written in momentum representation as

$$\hat{T}_{\text{Dirac}}(p) = -i\frac{1}{4}\left[\hat{H}(p)\left(\frac{1}{\hat{p}}\frac{\partial}{\partial p} + \frac{\partial}{\partial p}\frac{1}{\hat{p}}\right) + \left(\frac{1}{\hat{p}}\frac{\partial}{\partial p} + \frac{\partial}{\partial p}\frac{1}{\hat{p}}\right)\hat{H}(p)\right] \tag{2.98}$$

$$= -i\frac{1}{4}\left[\left(\gamma^0\gamma^1\hat{p} + \gamma^0 M\right)\left(\frac{1}{\hat{p}}\frac{\partial}{\partial p} + \frac{\partial}{\partial p}\frac{1}{\hat{p}}\right) + \left(\frac{1}{\hat{p}}\frac{\partial}{\partial p} + \frac{\partial}{\partial p}\frac{1}{\hat{p}}\right)\left(\gamma^0\gamma^1\hat{p} + \gamma^0 M\right)\right]. \tag{2.99}$$

From this we can compute the canonical commutation relations (Appendix A) and this returns $[\hat{T}, \hat{H}] = -i$. It is also very useful to switch to energy representation

$$\begin{cases} \langle E, s|E', s'\rangle = \delta_{ss'}\delta(E' - E) \\ \sum_{\mathcal{R}_s}\int_{-\infty}^{+\infty} |E, s\rangle\langle E, s|\, dE = \mathbb{I} \end{cases} \tag{2.100}$$

where $\langle E, s| \equiv [E^2/(E^2 - m^2)]^{-1/4}\,|p', \lambda', s'\rangle$, and $E \in \mathcal{R}_m \equiv (-\infty, -M) \cup (M, +\infty)$. It can be shown that

$$\hat{T}_{\text{Dirac}}(E) = -i\frac{\partial}{\partial E}. \tag{2.101}$$

The operator from equation (2.97) is symmetric and hermitian; therefore so it is equation (2.101).

Chapter 3

Time of Arrival Probability Distribution

This chapter reviews some approaches to the time of arrival probability distribution. In particular, we focus on Kijowski's distribution in section 3.1, followed by the operator approach to the probability distribution in section 3.2. The first derives the probability distribution from the classical expression by imposing a set of axioms. The second derives the probability distribution from the regulated time operator discussed in Chapter 2, section 2.3.1. Section 3.3 dives into the probability distribution via Bohmian mechanics and briefly compares it to Kijowski's distribution. Section 3.4 discusses how the arrival time probability distribution is computed from the probability current. Section 3.5 addresses the results for the probability distribution in the presence of a complex potential. Finally, section 3.5 focuses on a general approach to the probability distribution for different time observables.

3.1 Kijowski's Probability Distribution

Kijowski starts from the classical analogue and its properties to derive the quantum time of arrival distribution via an axiomatic approach. Let $\Pi_\psi^K(t)$ be the quantum analogue of the classical time of arrival distribution. We require the quantum version to be normalised, positive, have minimum variance, and symmetric with respect to the arrival point q [31].

Consider a freely moving wave function $\psi(t)$ [32]. Let F be a set of positive continuous bilinear functionals of $\psi(t)$. If $\psi(t)$ is normalised then $\int dtF[\psi(t)] = 1$. Furthermore, the finite dispersion is defined as

$$\int_{-\infty}^{\infty} dtt^2 F[\psi(t)] - \left(\int_{-\infty}^{\infty} dttF[\psi(t)]\right)^2 \tag{3.1}$$

and

$$F[\bar{\psi}] = F[\psi]. \tag{3.2}$$

There exists a functional F_0 for which equation (3.1) is minimum

$$F_0[\psi] = \int \frac{dp_1 dp_2}{2\pi M\hbar} \bar{\psi}(p_1)\sqrt{p_1 p_2}\psi(p_2) \tag{3.3}$$

then the Kijowski's distribution for particles arriving from the left is given by

$$\Pi_+^K(t) = F_0[\psi_t] = \left| \int_0^{+\infty} dp \sqrt{\frac{p}{2\pi M\hbar}} \mathrm{e}^{-ip^2t/2M\hbar}\psi(p) \right|^2. \tag{3.4}$$

Similarly to equation (3.4), one can obtain an expression for particles arriving from the right via symmetry

$$\Pi_-^K(t) = \left| \int_{-\infty}^{0} dp \sqrt{\frac{-p}{2\pi M\hbar}} \mathrm{e}^{-ip^2t/2M\hbar}\psi(p) \right|^2. \tag{3.5}$$

The time of arrival probability distribution for a freely moving particle computes the distribution of measurements and is given by

$$\Pi_\psi^K(t) = \Pi_-^K(t) + \Pi_+^K(t) \tag{3.6}$$

$$= \left| \int_{-\infty}^{0} dp \sqrt{\frac{-p}{2\pi M\hbar}} \mathrm{e}^{-ip^2t/2M\hbar}\psi(p) \right|^2 \tag{3.7}$$

$$+ \left| \int_0^{+\infty} dp \sqrt{\frac{p}{2\pi M\hbar}} \mathrm{e}^{-ip^2t/2M\hbar}\psi(p) \right|^2$$

$$= |\langle t, +|\psi\rangle|^2 + |\langle t, -|\psi\rangle|^2. \tag{3.8}$$

Where the operator associated with equation (3.6) is not self-adjoint, and it is represented by the POVM of equations (2.6) [33]. Equation (3.6) has exciting properties. The probability of a particle arriving at t it is the same as the probability of arrival at $t - \tau$ for the same eigenstate. In other words, equation (3.6) is covariant under time evolution.

Note that Kijowski's distribution can also be recovered from CTOA operators. However, this is not immediately obvious [34]. Recall that in the confined case, we are dealing with boundary conditions and a Hilbert space $\mathcal{H} = L^2[-l,l]$. As a result, the energy spectrum is discrete. One can compute the probability of the initial state ψ_0 arriving at the origin before an instant t.

$$F_{\psi_0}(t) = \sum_{\tau_{\gamma,s}\leq t} |\langle\psi_{\gamma,s}|\psi_0\rangle|^2. \tag{3.9}$$

In [34], equation (3.9) is compared to the accumulated probability of equation (3.6) (i.e. $F^k_{\psi_0}(t) = \int_{-\infty}^{+t} \Pi_k[\tau,\psi_0]d\tau$). It is found that the Kijowski's distribution is recovered for large l.

3.2 Probability Distribution via Operator Approach

In this section we discuss the study by Grot et al.[24] in regards to the probability distribution derived from the time operator $\hat{T}_\epsilon$. Consider the regulated time operator from equation (2.43) and its eigenstates from equation (2.48). Recall that the operator depends on a smooth function $f_\epsilon(p)$ that must tend to zero at least at the same rate as p to make $\hat{T}_\epsilon$ self-adjoint. Then the probability distribution of the time of arrival is given by

$$\Pi(t) = |_\epsilon\langle t,+|\psi\rangle|^2 + |_\epsilon\langle t,-|\psi\rangle|^2. \tag{3.10}$$

By using the eigenstates from equation (2.48) we obtain

$$\begin{aligned}\Pi(t) = &\frac{\hbar}{2\pi M}\left|\int_0^{+\infty} dp\sqrt{p}\exp\left(\frac{it\hbar(p^2-\epsilon^2)}{2M}\right)\psi(p)\right|^2 \\ &+\frac{\hbar}{2\pi M}\left|\int_{-\infty}^{0} dp\sqrt{p}\exp\left(\frac{it\hbar(p^2-\epsilon^2)}{2M}\right)\psi(p)\right|^2.\end{aligned} \tag{3.11}$$

Note that the ϵ dependency vanishes when taking the modulus squared. Therefore we are left with

$$\begin{aligned}\Pi(t) = &\frac{\hbar}{2\pi M}\left|\int_0^{+\infty} dp\sqrt{p}\exp\left(\frac{it\hbar p^2}{2M}\right)\psi(p)\right|^2 \\ &+\frac{\hbar}{2\pi M}\left|\int_{-\infty}^{0} dp\sqrt{p}\exp\left(\frac{it\hbar p^2}{2M}\right)\psi(p)\right|^2 \end{aligned} \tag{3.12}$$

$$= \Pi^+(t) + \Pi^-(t). \tag{3.13}$$

Equation (3.12) is the probability of a particle arriving at $q = 0$ coming from both the left and right. For a more general case of a particle arriving at a detector positioned at an arbitrary location q, the probability distribution of the time of arrival is given by

$$\Pi(t,x) = \frac{\hbar}{2\pi M}\left|\int_0^{+\infty} dp\sqrt{p}\exp\left(\frac{it\hbar p^2}{2M} - ipq\right)\psi(p)\right|^2 + \frac{\hbar}{2\pi M}\left|\int_{-\infty}^{0} dp\sqrt{p}\exp\left(\frac{it\hbar p^2}{2M} - ipq\right)\psi(p)\right|^2. \tag{3.14}$$

Because the current density can be negative, the time of arrival probability distribution was not derived from it [35]. However, we will see in the next section that the use of Bohmian mechanics bypasses this obstacle.

3.3 Probability Distribution via Bohmian Mechanics

In contrast to section 3.2, here, the time of arrival probability density is given by the current density via Bohmian mechanics. Bohmian mechanics, also known as de Broglie-Bohm theory or pilot-wave theory, is a tool that aids the numerical simulations of the Schrödinger equations. It also assists in understanding how the world works [36]. The equation of motions are

$$v(x,t) = \frac{dr(t)}{dt} \tag{3.15}$$

$$= \frac{J(q,t)}{|\psi(q,t)|^2} \tag{3.16}$$

$$= \frac{J(q,t)}{P(q,t)} \tag{3.17}$$

where $J(q,t)$ is the current density

$$J(q,t) = \frac{\hbar}{M}\text{Im}\left[\psi^*(q,t)\frac{\partial\psi(q,t)}{\partial q}\right] \tag{3.18}$$

and $P(q,t)$ is the single particle probability [37]. Consider an arbitrary potential $V(q)$, the probability distribution of the time of arrival reaching the detector is

$$\Pi_B(q,t) = \Pi^-(q,t) + \Pi^+(q,t) \tag{3.19}$$

$$= \frac{-J(q,t)\Theta(-J(q,t))}{\int_{-\infty}^{+\infty} dt|J(q,t)|} + \frac{+J(q,t)\Theta(+J(q,t))}{\int_{-\infty}^{+\infty} dt|J(q,t)|} \tag{3.20}$$

$$= \frac{|J(q,t)|}{\int_{-\infty}^{+\infty} dt|J(q,t)|}. \tag{3.21}$$

Contrary to Grot, Rovelli and Tate's (GRT) theory, there is no certainty that the particles will reach the detector in this case. GRT starts from the assumption that every particle reaches the detector; however, this assumption is not generally upheld in Bohmian mechanics.

Equation (3.19) is not in the same class as the Kijowski distribution. In fact if that was the case there would exists a quadratic form $q \in \mathcal{Q}$ [38] such that

$$\Pi_B^{\psi}(q,t) = q(\psi_t), \forall\, \psi \in \mathcal{D}(\mathbb{R}) \text{ with } \|\psi\| = 1 \wedge \forall\, t \in \mathbb{R}. \tag{3.22}$$

Since equation (3.19) breaches the quadratic form structure, it also breaches the rules of standard quantum mechanics.

3.4 Probability Distribution linked to Probability Current Density

There is an alternative approach to computing the quantum arrival probability distribution; it requires quantising the modulus of the classical current. In 1998, Delgado considered the work done in his previous paper [18] and re-introduced the quantum arrival time distribution via the classical expression for the average current at X

$$\langle J(q)\rangle = \int\int f(q,p,t)\frac{p}{M}\delta(q-X)dqdp \tag{3.23}$$

with $f(q,p,t)$ being the phase-space distribution or Wigner function [40]. Intuitively, one might want to quantise equation (3.23) to obtain the arrival time probability for a quantum particle via Weyl-Wigner quantisation [39]

$$\hat{J}(x) = \frac{1}{2M}\left(\hat{p}\,|q\rangle\,\langle q| + |q\rangle\,\langle q|\,\hat{p}\right). \tag{3.24}$$

This method would allow us to obtain the expectation value of the current $\hat{J}(x)$ via the correspondence principle. However, equation (3.24) is not positive-definite, which means quantum backflow can arise. Therefore, only if the backflow is insignificant equation (3.24) is a good approximation for the arrival time probability distribution.

Now, consider the Hamiltonian with unbounded spectrum from equation (2.29) where $\text{sgn}(\hat{p})$ is

$$\text{sgn}(\hat{p}) = \int_0^{+\infty} dp\left(|p\rangle\,\langle p| - |-p\rangle\,\langle -p|\right). \tag{3.25}$$

The corresponding self-adjoint time operator is described by equation (2.41) with eigenstates

$$|\tau\rangle = \frac{1}{h^{1/2}}\int_{-\infty}^{+\infty} dp\sqrt{\frac{|p|}{M}}\mathrm{e}^{i[\text{sgn}(p)(p^2/2M)\tau - pX]/\hbar} \tag{3.26}$$

Recall that equation (3.26) can be decompose into two components $|t, \alpha\rangle$ with $\alpha = \pm$. Note that the state vectors for a particle moving on the x axis towards a detector at X is given by

$$|\psi_\alpha(t)\rangle \equiv \Theta(\alpha\hat{p})\,|\psi_\pm(t)\rangle \tag{3.27}$$

where $\Theta(\alpha\hat{p})$ are the projectors with $\alpha = \pm$. From equation (3.27) one can find that for the normalised states $|\psi_\alpha\rangle$ where $|\psi_\alpha\rangle \to 0$ faster that $p \to 0$ the following statement is upheld

$$_\alpha\,\langle\psi_\alpha|\,\hat{T}\,|\psi_\alpha\rangle = \int_{-\infty}^{+\infty} dt\; t\,\langle\psi_\alpha|t,\alpha\rangle\,\langle t,\alpha|\psi_\alpha\rangle \tag{3.28}$$

$$= \frac{\int_{-\infty}^{+\infty} dt\; t\,\langle\psi_\alpha(t)|\,\hat{J}(X)\,|\psi_\alpha(t)\rangle}{\int_{-\infty}^{+\infty} dt\,\langle\psi_\alpha(t)|\,\hat{J}(X)\,|\psi_\alpha(t)\rangle}. \tag{3.29}$$

Equation (3.28) corresponds to the average quantum time of arrival at X. Therefore we conclude that the quantum probability density of the time of arrival at the detector located at X is

$$|\psi_\alpha(t,X)|^2 = \langle\psi_\alpha|t,\alpha\rangle\,\langle t,\alpha|\psi_\alpha\rangle \tag{3.30}$$

$$= \int_0^{+\infty} dp' \int_0^{+\infty} dp \frac{\sqrt{pp'}}{Mh}\,\langle\psi_\alpha|\alpha p\rangle \tag{3.31}$$

$$\times\,\langle\alpha p|\psi_\alpha\rangle\,\mathrm{e}^{i(p^2/2M - p'^2/2M)t/\hbar}\mathrm{e}^{-\alpha i(p-p')X/\hbar}.$$

Further study in the quantum arrival probability distribution, with links to the probability current, has been carried out by Yearsley et al. [41] where they consider an often used measurement model of an idealised clock to compute the arrival time and dwell time ditributions. In [41], the authors focus on understanding the possible similarities between the quantum probability distributions and their classical counterparts via analysing a clock model using path integrals.

Consider a wave packet with negative momenta located at $q > 0$. The arrival time probability distribution is equivalent to the current density $J(t)$

$$\Pi(t) = J(t) \tag{3.32}$$

$$= \frac{i\hbar}{2M}\left(\psi^*(0,t)\frac{\partial\psi(0,t)}{\partial q} - \frac{\partial\psi^*(0,t)}{\partial q}\psi(0,t)\right). \tag{3.33}$$

As previously discussed, this probability distribution is not positive-definite, and backflow may arise. Moreover, expression (3.33) does not hold in the regime of strong measurement,

where most incoming wavepackets are reflected at $q = 0$. This phenomenon is known as the quantum Zeno effect (QZE). The strong measurement regime corresponds to frequent projective measurements. Where taking measurements hinders the state evolution of the quantum system. As $\hbar \to 0$, the QZE vanishes [42].
The correct form for equation (3.33) is

$$\Pi(t) = C \langle\psi_t| \hat{p}\delta(\hat{q})\hat{p} |\psi_t\rangle \tag{3.34}$$

with C being a constant. The realm of frequent measurements used to estimate the arrival time probability was initially abandoned by Allcock due to the QZE. However, it is possible to obtain a probability distribution by normalising part of the norm. Doing so leads to the following equation, which is the normalisation of equation (3.34)

$$\Pi_N(t) = \frac{\hbar}{M|\langle p\rangle|} \langle\psi_t| \hat{p}\delta(\hat{q})\hat{p} |\psi_t\rangle \,. \tag{3.35}$$

Equation (3.35) does not depend on the detector's features and can be obtained by considering a general class of complex potentials. The term $\hat{p}\delta(\hat{q})\hat{p}$ appearing in equation (3.35) reminisce of the current $J(t) = \hbar/2M \langle\psi_f(t)| [\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}] |\psi_f(t)\rangle$ which gives the Kijowski distribution. In actuality, the term from equation (3.35) is more of a kinetic energy density and holds as a generic result for the arrival probability distribution in the Zeno limit.

However note that the proper form of equations (3.35) depends both on the clock properties,

$$\Pi_C(t) = \int_{-\infty}^{\infty} ds R(t,s)\Pi(s) \tag{3.36}$$

and the system's initial state. Where $R(t,s)$ is the response function depending on the clock variable [41]. Equations (3.35) is a semiclassical expressions. We can also find an equation for the probability of arrival or dwell times within a time interval $[t_1, t_2]$ that is greater than the time scale,

$$p(t_1, t_2) = \int_{t_1}^{t_2} dt\Pi_C(t) \approx \int_{t_1}^{t_2} \Pi(t). \tag{3.37}$$

The authors derive the two distributions by coupling the particle with the clock model. Let (q, p) be the the particle's coordinates, and (y, p_y) the clock's coordinates. Let $|\psi\rangle$ be the initial state of the particle, and $|\phi\rangle$ be the initial state of the clock. The

Interaction Hamiltonian H_I is given by $\lambda\chi(\hat{q})H_C$ therefore,

$$H = H_0 + H_I \tag{3.38}$$

$$= H_0 + \lambda\chi(\hat{q})H_C \tag{3.39}$$

with $H_C = H_C(y, p_y)$. Let H_C be self-adjoint so that it can be written in terms of a complete and orthogonal set of basis,

$$H_C = \int_{-\infty}^{+\infty} d\epsilon\epsilon \left|\epsilon\right\rangle \left\langle\epsilon\right| \tag{3.40}$$

The evolution of the particle coupled with the clock is given by

$$\Psi(q, y, \tau) = \langle q, y| \, \mathrm{e}^{-i\frac{H\tau}{\hbar}} \, |\Psi_0\rangle \tag{3.41}$$

$$= \langle q, y| \exp\left[-i\frac{H_0\tau}{\hbar} - i\frac{H_I\tau}{\hbar}\right] |\psi_0\rangle \, |\phi_0\rangle \tag{3.42}$$

$$= \langle q, y| \exp\left[-i\frac{H_0\tau}{\hbar} - i\frac{\lambda\chi(\hat{q})H_C\tau}{\hbar}\right] |\psi_0\rangle \, |\phi_0\rangle \tag{3.43}$$

$$= \int d\epsilon \, \langle y|\epsilon\rangle \, \langle\epsilon|\phi_0\rangle \tag{3.44}$$

$$\times \, \langle q| \exp\left[-i\frac{H_0\tau}{\hbar} - i\frac{\lambda\chi(\hat{q})\epsilon\tau}{\hbar}\right] |\psi_0\rangle \, .$$

Where Ψ is the state of the total system, and τ is taken to be large enough that the wave packet is not in the range of $\chi(\hat{q})$.

The distribution for y is

$$\Pi(y) = \int_{-\infty}^{+\infty} dx |\Psi(q, y, \tau)|^2. \tag{3.45}$$

Then, the propagator is resolved using path decomposition expansion (PDX) such that,

$$\Psi(q, y, t) = \langle q, y| \exp\left\{-i\left[\frac{H_0}{\hbar} + \lambda\frac{\theta(\hat{q})H_C}{\hbar}\right]\tau\right\} |\psi_0\rangle \tag{3.46}$$

$$= \frac{1}{M}\int d\epsilon \, \langle y|\epsilon\rangle \, \langle\epsilon|\phi_0\rangle \tag{3.47}$$

$$\times \int_0^\tau dt \, \langle q| \exp\left\{-i\left[\frac{H_0}{\hbar} + \lambda\epsilon\frac{\theta(\hat{q})}{\hbar}\right](\tau - t)\right\}$$

$$\times \, \delta(\hat{q})\hat{p} \exp\left[-i\left(\frac{H_0}{\hbar} + \frac{\lambda\epsilon}{\hbar}\right)t\right] |\psi_0\rangle \, .$$

Yearsly et al. [41] further simplify equation (3.47) by considering two regimes, the weak-coupling $E \gg \lambda\epsilon$ and strong-coupling regimes $E \ll \lambda\epsilon$.

For $E \gg \lambda\epsilon$, consider equation (3.47). The probability distribution takes the form

$$\Pi(y) = |\Psi(q,y,\tau)|^2 \tag{3.48}$$
$$= \frac{1}{M^2}\int d\epsilon d\epsilon' \langle\phi_0|\epsilon'\rangle \langle\epsilon'|y\rangle \langle y|\epsilon\rangle \langle\epsilon|\phi_0\rangle \tag{3.49}$$
$$\times \int_0^\tau dtdt' \langle\psi_0| \exp\left[i\left(\frac{H_0}{\hbar}+\frac{\lambda\epsilon'}{\hbar}\right)t'\right]\hat{p}\delta(\hat{q})$$
$$\times \exp\left[-i\left(\frac{H_0}{\hbar}(t'-t)\right)\right]\delta(\hat{q})\hat{p}\exp\left[-i\left(\frac{H_0}{\hbar}+\frac{\lambda\epsilon}{\hbar}\right)t\right]|\psi_0\rangle\,.$$

Now recall that the current $J(x) \propto \hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}$. Moreover, the integral $\int_0^\tau dtdt'$ may be rewritten as $\int_0^\tau dt \int_t^\tau dt' + \int_0^\tau dt' \int_t^\tau dt$. It follows that the probability $\Pi(y)$ takes the following form

$$\Pi(y) = (-1)\frac{\hbar}{2M}\int_0^\infty du|\Phi(q,u)|^2 \langle\psi_u|\left[\hat{p}\delta(\hat{q})+\delta(\hat{q})\hat{p}\right]|\psi_u\rangle \tag{3.50}$$
$$= \int_0^{+\infty} dt|\Phi(y,t)|^2 J(t). \tag{3.51}$$

Where $u = t'$ and $\Phi(y,t) = \langle y|\,\mathrm{e}^{i\lambda\frac{H_C}{\hbar}t}\,|\phi_0\rangle$. These results are consistent with equation (3.36). Therefore we found that the arrival time probability is linked to the probability current density in the weak coupling regime.

Similarly, an analysis can be carried out for $E \ll \lambda p_y$, the strong coupling regime. The semiclassical calculations are carried out using WKB approximation [41] and by setting

$$\langle y|p_y\rangle = \frac{1}{\sqrt{2\pi}}\exp(-iyp_y). \tag{3.52}$$

It follows that the arrival time distribution is given by,

$$\Pi(y) \approx \int d\epsilon|\langle\epsilon|\phi_0\rangle|^2\frac{2\hbar}{M^2}\sqrt{\frac{2M}{\lambda\epsilon}}|C(y,\epsilon)|^2 \tag{3.53}$$
$$\times \langle\psi_0|\exp\left\{i\frac{H_0}{\hbar}\frac{1}{\lambda}\frac{\partial S(y,\epsilon)}{\partial\epsilon}\right\}\hat{p}\delta(\hat{q})\hat{p}$$
$$\times \exp\left\{i\frac{H_0}{\hbar}\frac{1}{\lambda}\frac{\partial S(y,\epsilon)}{\partial\epsilon}\right\}|\psi_0\rangle$$

where $t = \frac{1}{\lambda}\frac{\partial S(y,\epsilon)}{\partial\epsilon}$. Therefore equation (3.53) describes an arrival time probability linked to the kinetic energy density in the strong coupling regime.

3.5 Probability Distribution: A More General Approach

3.5.1 Probability Distribution in the Presence of a Complex Potential

In 2008, Halliwell derived a generic expression describing the time of arrival distribution applicable to a broad class of complex potentials in the Zeno limit [46].
The approach entails introducing a complex potential, $V(q) = iV_0\theta(-q)$, in the Schrödinger equations such that the final state is

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}H_0 t - \frac{V_0}{\hbar}\theta(-x)t\right). \tag{3.54}$$

This is done because part of the wave is absorbed as it crosses the origin in the time interval $[0, t]$. As a result, the probability of not crossing is given by

$$N(t) = \langle\psi(t)|\psi(t)\rangle\,; \tag{3.55}$$

whilst the unnormalised probability of crossing is the derivative of $N(t)$ with respect to t,

$$\Pi(t) = -\frac{dN}{dt}. \tag{3.56}$$

Given such a potential, the Zeno effect corresponds to the phenomenon in which the wave function is completely reflected for $V_0 \to \infty$.

In [8, 9, 10], Allcock used a complex, imaginary potential to deduce the time of arrival distribution and noticed the same phenomena. When solving the Schrödinger equations, he observed that for V_0 being much greater than the maximum energy E_{max} the detector response disappeared (i.e. $dN/dt \to 0$) due to what we now know as the Zeno effect. He then dismissed the existence of a quantum arrival time. However his solution was not general enough.

There exists a technique to overcome this obstacle [47], it requires considering a small part of the norm δN and normalising equation (3.56) to obtain

$$\Pi_{Zeno} = \lim_{\delta t \to 0} \frac{\Pi(t)}{1 - N(\infty)} \tag{3.57}$$

Note that equation (3.57) is independent of V_0, and $1 - N(\infty)$ corresponds to the probability of crossing within an interval $[0, \infty)$. Moreover, $1 - N(\infty) \to 0$ as $V_0 \to \infty$, yet the ratio is defined [46]

The following expression gives the normalised arrival time distribution for a scattering wave in the Zeno limit,

$$\Pi_N(t) = \frac{\hbar}{M\langle p\rangle} \langle\psi_f(t)| \hat{p}\delta(\hat{q})\hat{p} |\psi_f(t)\rangle . \tag{3.58}$$

In [46], the author focuses on two main issues; firstly, he tries to understand why the expression (3.58) looks the way it does and secondly, whether or not it depends on the shape of the complex potential.

Given a positive function $f(q)$, it is found that equation (3.58) holds for a class of potentials of the type,

$$V(q) = -iV_0\theta(-q)f(q), \tag{3.59}$$

and the respective arrival time distribution is,

$$\Pi(t) = 2\langle\psi_\tau| V(\hat{q}) |\psi_\tau)\rangle \tag{3.60}$$

$$= \frac{2V_0}{\hbar}\int_{-\infty}^{0} dq f(q)|\psi(q,\tau)|^2. \tag{3.61}$$

In this paper the propagator is computed via sum over paths and path decomposition expansion. The calculations lead to

$$\Pi(t) \approx C\langle\psi| \exp\left(\frac{i}{\hbar}H_0\tau\right) \hat{p}\delta(q)\hat{q} \exp\left(-\frac{i}{\hbar}H_0\tau\right) |\psi\rangle \tag{3.62}$$

which is of the same form as equation (3.34). The constant C is

$$C = \frac{2V_0}{\hbar M^2}\int_0^\tau ds' \int_0^\tau ds \int_{-\infty}^0 dq f(q)\langle 0| \exp\left(-\frac{i}{\hbar}H^\dagger s'\right) |q\rangle \tag{3.63}$$

$$\times\langle q| \exp\left(-\frac{i}{\hbar}s\right) |0\rangle . \tag{3.64}$$

3.5.2 A Generic Probability Distribution for Different Time Observables

It is also helpful to develop a general theory for computing the probability distribution for different time observables, such as the arrival time, tunnelling time and dwell time. In 2021, Jurman and Nikolić [44] proposed a method by considering an interval of time δt sufficiently small that it is not zero. They proposed a general formula for the probability distribution for an event detected at an instant t. Then from this general formula, the arrival time distribution at the detector was derived [45].

Consider a Hamiltonian H such that the evolution of the system is described by,

$$|\psi(t)\rangle = \mathrm{e}^{-i\frac{H}{\hbar}t} |\psi_0\rangle . \tag{3.65}$$

Consider a clock coupled to a perfectly efficient detector and ticking at δt intervals [44]. Let P be a projector acting on the Hilbert space $\mathcal{H}$, such that

$$P = \int_{V_{Det}} d^3q \, |\mathbf{q}\rangle \langle \mathbf{q}| . \tag{3.66}$$

Where V_{Det} is the spatial region of the detector. As the particle is detected by the apparatus, the wave function collapses to the following state

$$|\psi\rangle \to \frac{P |\psi\rangle}{|P |\psi\rangle|} \tag{3.67}$$

$$= \frac{P\mathrm{e}^{-i\frac{H\delta t}{\hbar}} |\psi_0\rangle}{\left|P\mathrm{e}^{-i\frac{H\delta t}{\hbar}} |\psi_0\rangle\right|}. \tag{3.68}$$

If the particle does not reach the detector then the wave function collapses to

$$|\psi\rangle \to \frac{\bar{P} |\psi\rangle}{|\bar{P} |\psi\rangle|} \tag{3.69}$$

$$= \frac{(1-P)\mathrm{e}^{-i\frac{H\delta t}{\hbar}} |\psi_0\rangle}{\left|(1-P)\mathrm{e}^{-i\frac{H\delta t}{\hbar}} |\psi_0\rangle\right|}. \tag{3.70}$$

Let $V = \bar{P}\mathrm{e}^{-i\frac{H\delta t}{\hbar}}$, if the particle has not being detected from δt to $2\delta t$ we can compute the evolution of the system

$$|\psi_0\rangle \to \frac{V |\psi_0\rangle}{|V |\psi_0\rangle|} \tag{3.71}$$

$$\to \frac{VV |\psi_0\rangle}{|VV |\psi_0\rangle|}. \tag{3.72}$$

Therefore if we consider time spanning form δt and $k\delta t$ we have,

$$|\psi_0\rangle \to \frac{V^k |\psi_0\rangle}{|V^k |\psi_0\rangle|}. \tag{3.73}$$

Recall that δt is very small but not zero, and that $P = P^2$, hence we can approximate the expression of V^k to

$$V^k = \bar{P}\mathrm{e}^{-i\frac{H\delta t}{\hbar}} \bar{P}\mathrm{e}^{-i\frac{H\delta t}{\hbar}} ... \bar{P}\mathrm{e}^{-i\frac{H\delta t}{\hbar}} \tag{3.74}$$

$$\simeq \bar{P}\left(1 - i\frac{H\delta t}{\hbar}\right) \bar{P}\left(1 - i\frac{H\delta t}{\hbar}\right) ... \bar{P}\left(1 - i\frac{H\delta t}{\hbar}\right) \tag{3.75}$$

$$= \bar{P}\left(1 - i\frac{H\delta t}{\hbar}\right) \bar{P}\bar{P}\left(1 - i\frac{H\delta t}{\hbar}\right) ... \bar{P}\bar{P}\left(1 - i\frac{H\delta t}{\hbar}\right). \tag{3.76}$$

Then by bearing in mind that $|\psi_0\rangle = \bar{P}|\psi_0\rangle$ and $\bar{H} = \bar{P}H\bar{P}$, it follows

$$V^k|\psi_0\rangle \simeq \bar{P}\left(1 - i\frac{H\delta t}{\hbar}\right)\bar{P}\cdot\bar{P}\left(1 - i\frac{H\delta t}{\hbar}\right)\bar{P}...\bar{P}\left(1 - i\frac{H\delta t}{\hbar}\right)\bar{P}|\psi_0\rangle \tag{3.77}$$

$$= \left(\bar{P} - i\frac{\bar{H}\delta t}{\hbar}\right)\left(\bar{P} - i\frac{\bar{H}\delta t}{\hbar}\right)...\left(\bar{P} - i\frac{\bar{H}\delta t}{\hbar}\right)|\psi_0\rangle \tag{3.78}$$

$$= \left(1 - i\frac{\bar{H}\delta t}{\hbar}\right)\left(1 - i\frac{\bar{H}\delta t}{\hbar}\right)...\left(1 - i\frac{\bar{H}\delta t}{\hbar}\right)|\psi_0\rangle \tag{3.79}$$

$$= \left(1 - i\frac{\bar{H}\delta t}{\hbar}\right)^k|\psi_0\rangle\,. \tag{3.80}$$

However, given that $t = \delta t$ and $k \gg 1$ we have,

$$\lim_{k\to\infty}\left(1 - i\frac{\bar{H}t}{k\hbar}\right) = \mathrm{e}^{-i\frac{\bar{H}t}{\hbar}}, \tag{3.81}$$

hence equation (3.73) can be expressed as

$$V^k|\psi_0\rangle = \mathrm{e}^{-i\frac{\bar{H}t}{\hbar}}|\psi_0\rangle\,. \tag{3.82}$$

Let the state at which the detection has not taken place between $t - \delta t$ be $\left|\bar{\psi}_c(t)\right\rangle \simeq \mathrm{e}^{-i\frac{\bar{H}t}{\hbar}}|\psi_0\rangle$. It follows that the conditional probability of the state being detected at t is

$$\pi(t) = \langle\psi_c(t)|\,P\,|\psi_c(t)\rangle \tag{3.83}$$

$$= \left\langle\bar{\psi}_c(t-\delta t)\right|\mathrm{e}^{i\frac{H\delta t}{\hbar}}P\mathrm{e}^{-i\frac{H\delta t}{\hbar}}\left|\bar{\psi}_c(t-\delta t)\right\rangle \tag{3.84}$$

which is consistent with the fact that

$$\lim_{\delta t\to 0}\pi(t) = 0. \tag{3.85}$$

Finally one can compute the overall probability of the event taking place at t_k

$$\Pi_p(t) = \pi(t_k)\mathrm{e}^{-\sum_{i=1}^{k-1}\pi(t_i)} \tag{3.86}$$

$$\Pi(t) \simeq w(t)\mathrm{e}^{-\int_0^t dt' w(t')}. \tag{3.87}$$

$\Pi(t)$ is the probability density, $w(t) = \frac{1}{\delta t}\langle\psi_c(t)|\,P\,|\psi_c(t)\rangle$, $|\psi_c(t)\rangle = \mathrm{e}^{-i\frac{H\delta t}{\hbar}}\mathrm{e}^{-i\frac{\bar{H}(t-\delta t)}{\hbar}}|\psi_0\rangle$, with $\bar{H} = \bar{P}H\bar{P}$, and $\bar{P} = 1 - P$. Further manipulation of equation (3.79) yields the expression for the arrival time probability distribution, which has been researched in [45].

We have seen how the arrival time probability distribution can be derived. However, different methods yield different results. Despite Allcock's pessimistic view on defining the time of arrival in quantum mechanics and its corresponding probability distribution, we have shown that it is indeed possible to describe such a concept including in the Zeno limit where the wave function is reflected as $V_0 \to 0$. To do so we have considered inverse scattering methods to assemble a class of potentials able to hold the entire wave packet.

Chapter 4

Quantum Backflow

The backflow effect is a counter-intuitive phenomenon in which a wave function with positive momenta produces a negative current. Therefore the probability has the opposite direction to the momentum [48]. This exciting occurrence was first presented by Allcock and studied systematically by Bracken and Melloy [50]. Let us consider a classical particle moving along the x-axis, assuming the only information we have available is that the velocity v is positive and that the particle is not subjected to any force. We can safely say the particle will move towards the positive x-direction. The probability of finding the classical particle in the interval $(-\infty, 0)$ will decrease in time.

On the other hand, in non-relativistic quantum mechanics, the probability of a particle being found in the interval $(-\infty, 0)$ is non-zero and increases in value during a finite time interval for specific values of the x-axis, even though the velocity x-component is positive. A negative Wigner function is sometimes used to describe the existence of the backflow effect [49].

Allcock used the existence of this phenomenon to assert that the arrival time probability distribution cannot be derived from the current. On the other hand, Halliwell et al. [49] focused on identifying cases where the candidate probability is negative and interpreted it as a presence of quantumness where the probability cannot be defined without direct measurement. In this Chapter we discuss the backflow constant and time of arrival in the backflow regime.

4.1 Backflow Constant

To understand better the backflow effect it is useful to consider the eigenvalue problem, whereby the spectrum of the flux operator is studied. Consider a state $|\Phi\rangle$ made of positive momenta exclusively. The goal is to solve the following expression for λ,

$$\theta(\hat{p})\hat{F}(t_1,t_2)\,|\Phi\rangle = \lambda\,|\Phi\rangle \tag{4.1}$$

where $\hat{F}(t_1,t_2)$ is the negative flux defined by

$$\hat{F}(t_1,t_2) = \langle\hat{F}(t_1,t_2)\rangle \tag{4.2}$$

$$= \int_{t_1}^{t_2} dt\,\langle\psi|\,\hat{J}(x)\,|\psi\rangle\,. \tag{4.3}$$

The flux with the most negative magnitude corresponds to the smallest eigenvalue. By solving equation (4.1), it is found that $-c_{bm} \leq \lambda \leq 1$.

The probability for a particles travelling in the interval $(-\infty, 0)$ is bounded by a dimensionless number c_{bm}, also known as the backflow constant. It is sometimes considered a new quantum number because it does not depend on $\hbar$ and the particle's mass. Numerical analysis (e.g. [50], [51]) found the constant to be $c_{bm} \approx 0.038452$.

4.2 Time of Arrival Measurement in the Backflow Regime

Looking at the arrival time problem concerning the backflow effect is essential because it gives an insight in the difficulty of timing quantum events. In [49], the authors consider the quasi-probability to allow negative probabilities to be included. Consider a set of sequential measurements. Consider two projectors P_+ and P_- such that for $\hat{\mathcal{Q}} = \mathrm{sgn}(\hat{x})$ and $s = \pm 1$ the probability is given by

$$p(s_1, s_2, ..., s_n) = \mathrm{Tr}\big[P_{s_n}(t_n)P_{s_{n-1}}(t_{n-1})...P_{s_1}(t_1)\rho P_{s_1}(t_1)...P_{s_{n-1}}(t_{n-1})\big] \tag{4.4}$$

with

$$P_s = \frac{1}{2}(1 + s\hat{\mathcal{Q}}) \tag{4.5}$$

and

$$P_s(t) = \exp\left(\frac{iHt}{\hbar}\right) P_s \exp\left(\frac{iHt}{\hbar}\right). \tag{4.6}$$

Now consider the case for sequential measurements of four possible histories such that

$$p_{12}(s_1, s_2) = \mathrm{Tr}[P_{s_2}(t_2)P_{s_1}(t_1)\rho P_{s_1}(t_1)] \tag{4.7}$$

It is possible to obtain the left and right crossing probabilities from equation (4.7); however, each measurement will disrupt subsequent ones. As a result, the no signalling in time (NSIT) condition does not hold and estimating the arrival time probability is not trivial. Halliwell et al. [49] constructed a two-time probability in the case of nontrivial interference via the quasi-probability satisfying the NSIT condition

$$q(s_1, s_2) = \frac{1}{2}\,\mathrm{Tr}\{[P_{s_2}(t_2)P_{s_1}(t_1) + P_{s_1}(t_1)P_{s_2}(t_2)\rho]\}. \tag{4.8}$$

When equation (4.8) is negative, it reflects the presence of quantumness. Conversely, when positive, it corresponds to a candidate probability.

4.2.1 Properties and Measurement of the Quasi-Probability

Equation (4.8) satisfies the following relations

$$\sum_{s_1} q(s_1, s_2) = \mathrm{Tr}[P_{s_2}(t_2)\rho] = p_2(s_2) \tag{4.9}$$

$$\sum_{s_2} q(s_1, s_2) = \mathrm{Tr}[P_{s_1}(t_1)\rho] = p_1(s_1) \tag{4.10}$$

and can be re-written in terms of the decoherence functional

$$q(s_1, s_2) = p_{12}(s_1, s_2) + \mathrm{Re}\, D(s_1, s_2| - s_1, s_2). \tag{4.11}$$

where $D(s_1, s_2|s_1', s_2) = \mathrm{Tr}\Big[P_{s_2}(t_2)P_{s_1}(t_1)\rho P_{s_1'}(t_1)\Big]$. In order to assign the probability we require

$$q(s_1, s_2) \geq 0 \tag{4.12}$$

Equation (4.11) can be also expressed as

$$q(s_1, s_2) = \frac{1}{4}(1 + \langle\hat{\mathcal{Q}}_1\rangle s_1 + \langle\hat{\mathcal{Q}}_2\rangle s_2 + C_{12}s_1s_2) \tag{4.13}$$

where $C_{12} = \frac{1}{2}\langle\hat{\mathcal{Q}}_1\hat{\mathcal{Q}}_2 + \hat{\mathcal{Q}}_2\hat{\mathcal{Q}}_1\rangle$. It is helpful to compare (4.13) to the probability which is

$$p_{12}(s_1, s_2) = \frac{1}{4}(1 + \langle\hat{\mathcal{Q}}_1\rangle s_1 + \langle\hat{\mathcal{Q}^{(1)}}_2\rangle s_2 + C_{12}s_1s_2) \tag{4.14}$$

where $\langle\hat{\mathcal{Q}^{(1)}}_2\rangle = \langle\hat{\mathcal{Q}}_2\rangle + \frac{1}{2}\langle[\hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2]\hat{\mathcal{Q}}_1\rangle$ and it describes interference. It is evident that equation (4.13) and (4.14) only differ by the term $\frac{1}{2}\langle[\hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2]\hat{\mathcal{Q}}_1\rangle$.

In terms of measurement of the quasi-probability this can be achieved experimentally either from determining the moments of $q(s_1, s_2)$ or via ambiguous measurements.

4.2.2 The Backflow Regime

The backflow effect describes the occurrence in which a wave function $\psi(x,t)$ made of positive momenta leads to a negative current such that

$$p_-(t) = \int_{-\infty}^{0} dx|\psi(x,t)|^2 \tag{4.15}$$

Let $\hat{J}(t)$ be the flux operator such that the degree of increase of the negative current is given by

$$\int_{t_1}^{t_2} \hat{J}(t) = P_+(t_2) - P_-(t_1) \tag{4.16}$$

then the spectrum is found between $[-c_{bm}, 1]$, where c_{bm} is the constant we determined in Section 4.1. Equation (4.1) can be re-written as

$$\frac{1}{2}\theta(\hat{p})(\hat{\mathcal{Q}}_2 - \hat{\mathcal{Q}}_1)\left|\lambda\right\rangle = \lambda\left|\lambda\right\rangle \tag{4.17}$$

hence the quasi-probability can be expressed as

$$\begin{aligned} q(-,+) &= \frac{1}{8}((2\lambda+1)^2 - 1) && (4.18)\\ &= \frac{1}{2}\lambda(\lambda+1) && (4.19)\\ &\approx 0.02 && (4.20) \end{aligned}$$

Recall that the arrival distribution in standard quantum mechanics is given by

$$\Pi(t) = \Pi_+(t) + \Pi_-(t) \tag{4.21}$$

where

$$\pi^{\pm}(t) = \frac{\hbar}{2\pi M}\left|\int_{-\infty}^{+\infty} dp\Theta(\alpha p)|p|^{1/2}\exp(ipq)\phi(p,t)\right| \tag{4.22}$$

with $\alpha = \pm$. Note that equations (4.21) and (4.22) describe the special case of free evolution where $\phi(p,t) = \phi(p)\exp\left(-i\hbar p^2 t/2M\right)$ [52]. Particles travelling to the left (right) are described by $p > 0$ ($p < 0$). There is no interference between the two cases. This can means two things, either the interference does not contribute to the probability distribution, or the distribution is correct only when the measuring apparatus estimates the sign of p. The measurement is described by

$$\psi_\alpha(x,t) = N_\alpha \hat{P}_\alpha \psi(x,t). \tag{4.23}$$

If $\phi(x,t) \neq 0$ for $p > 0$ in the presence of backflow effect, the probability distribution for particle travelling from the right is 0. This is the case despite the probability of the particle being on the left increases with time [52].

Chapter 5

Time Measurement in Quantum Mechanics

Measuring time is not straightforward because time is not a dynamical variable. Using equation (2.6) for this purpose would not work because the eigenfunction associated with the operator $\hat{T}$ does not have any physical meaning [53]. This issue can be bypassed by constructing a quantum clock. This chapter focuses on time measurement, in particular, arrival time measurement. Section 5.1 dives into the different measurement models, such as toy models, decoherence models and detector models. Section 5.2 covers the practical problems linked to quantum time measurement. We found that there is ambivalence in the meaning of time of arrival measurement [18].

5.1 Measurement Models

5.1.1 Toy Models

In 1998, Aharonov et al. [54] proposed four toy models for measuring the arrival time of a particle. However, the authors concluded that the time of arrival could not be measured with more precision than $\Delta t_A \sim \frac{1}{E_k}$, where E_k is the kinetic energy of the particle. Moreover, they argued that the time operators discussed previously in Chapter 2 do not correspond to the measurement studied in [54]. Therefore they assert a time arrival operator does not exist.

Time in classical physics can be measured either directly or indirectly. The first method requires the use of a clock to measure the time of arrival of a particle. The clock is located at $x = x_A$, stopping once the particle reaches it. It is possible to complete this measurement with arbitrary accuracy. The second method requires inverting the equation of motion to compute the time of arrival with respect to position and momentum. These two cases lead to the same results and are, therefore, equivalent.

What about quantum mechanics? Can time be measured with arbitrary accuracy? According to Allcock, it is not possible to measure the quantum first-passage time with any accuracy via direct measurement [9]. Aharonov et al. [54] disagree.

Consider a clock coupled with a detector located at $x = 0$. Consider a beam of particles, as a particle reaches the detector the clock stops. The uncertainty in the location of the measurement apparatus is ignored because the masses of the clock and detector are assumed to be unlimited. Following are the four toy models (Note that in this section $\hbar = 1$).

Toy Model I: Direct Interaction

Let $\theta(x)$ be a step function, with the particle confined in the x-direction. The Hamiltonian describing the interaction between the particle and the clock is given by

$$\hat{H} = \frac{1}{2M}\hat{P}_x^2 + \theta(-\hat{x})\hat{P}_y. \tag{5.1}$$

There are two sets of equations of motion and they are as follow:

$$\dot{\hat{x}} = \hat{P}_x/M, \qquad \dot{\hat{P}}_x = -\hat{P}_y\delta(\hat{x}) \tag{5.2}$$

$$\dot{\hat{y}} = \theta(\hat{x}), \qquad \dot{\hat{P}}_y = 0. \tag{5.3}$$

Time is recorded via the variable $\hat{y}$; in the limit of $t \to 0$ the arrival time is given by

$$\hat{y}_\infty = \hat{y}(t_0) + \int_{t_0}^{\infty} \theta(-\hat{x}(t))dt. \tag{5.4}$$

As we mentioned previously, the classical time of arrival measured directly is equivalent to the value obtained by indirect measurement and it is

$$t_A = y_\infty = \frac{Mx_0(t)}{p_x}. \tag{5.5}$$

However in quantum mechanics indirect and direct measurements are not equivalent. The act of measurement perturbs the system and therefore there is an uncertainty in p_y for $\Delta y = \Delta t_A \to 0$.

There is also a more symmetric expression that does not require knowing the direction of the particles. It entails using two clocks $\hat{P}_x$ ($\forall\, x < 0$) and $\hat{P}_y$ ($\forall\, x > 0$),

$$H = \frac{1}{2M}\hat{P}_x^2 + \theta(-\hat{x})\hat{P}_{y_1} + \theta(\hat{x})\hat{P}_{y_2}. \tag{5.6}$$

Measuring the arrival time for a particle flowing from both directions is difficult. Nonetheless, it is found that for $|x| < L$ one can estimate both $\hat{y}_1$ and $\hat{y}_2$ after the time interval $t \gg L/v$ has passed. Therefore $t_A = \min(\hat{y}_1, \hat{y}_2)$.

Aharonov et al. [54] consider the case of a particle coming from $x < 0$ and a clock located at $x = 0$ for simplicity, where the solutions to the Scrödinger equations are

$$\psi(x, y, t) = N \int_{-\infty}^{\infty} dk \int_{0}^{+\infty} dp f(p) g(k) \phi_{kp}(x, y, t). \tag{5.7}$$

N is the normalisation constant, k and p are the momentum of the particle and clock respectively. While $\phi(x, y, t)$ are the eigenstates of equation (5.6) and $f(p)$ and $g(k)$ are distributions. Note that the clock runs with a probability close to 1.

For example, let $g(k)$ and $f(p)$ be Gaussian distributions such that

$$\begin{cases} f(p) & = \mathrm{e}^{\Delta_y^2 (p-p_0)^2} \\ g(k) & = \mathrm{e}^{\Delta_x^2 (k-k_0)^2 + ikx_0}, \qquad x_0 > 0. \end{cases} \tag{5.8}$$

Hence as a result the particle is at $x < 0$, and $N^2 = \frac{\Delta_x \Delta_y}{2\pi^3}$. If $p_0 \approx 1/\Delta_y$, then the energy of the clock can be chosen to be $0 < p < 2/\Delta_y$. Finally, by using stationary point approximation it is found that the time of arrival for the particle is centred around the classical time of arrival.

On the other hand, if we choose $x > 0$ the quantum time of arrival will peak at the classical time of arrival.

Now let us consider the case in which the clock yields inaccurate measurement, and the back-reaction A_T is $\simeq 1$. The arrival time probability distribution is centred at the time of arrival $t_0 = Mx_0/k_0$. However, if there is a small perturbation due to back-reaction, the clock will stop and measure the arrival time. In order to obtain more accurate measurements it is necessary to consider the transition probability $T = \frac{q}{k}|A_T|^2$ precisely. Hence one can derive the probability to stop the clock that is given by

$$\sqrt{\frac{E_k + p}{E_k}} \left[\frac{2\sqrt{E_k}}{\sqrt{E_k} + \sqrt{E_k + p}} \right]^2 . \tag{5.9}$$

The clock may be triggered with a probability of $\bar{E}_k \Delta t_A > 1$. If the accuracy is higher than $1/\bar{E}_k$ the arrival time probability distribution becomes distorted.

Toy Model II: Two Level Detector

The second toy model for the arrival time measurement is more realistic. Consider a two level spin degree of freedom particle detector. As the particle reaches the detector at $x = 0$ the clock stops. The state of the trigger is inverted from $\uparrow_z$ (on) to $\downarrow_z$ (off) as the particle is detected. The model describing this behaviour is given by

$$\hat{H}_{trigger} = \frac{1}{2M}\hat{P}_x + \lim_{\alpha \to 0} \frac{\alpha}{2}(1 + \hat{\sigma}_x)\delta(\hat{x}). \tag{5.10}$$

Consider the state $|\psi\rangle |\uparrow_z\rangle$ where the particle is incoming and the clock is ticking. Then in the limit of $\alpha \to 0$ the state evolves as follow

$$|\psi\rangle |\uparrow_z\rangle \to \frac{1}{\sqrt{2}} \left[|\psi_R\rangle |\uparrow_z\rangle + |\psi_T\rangle |\downarrow_z\rangle \right] \tag{5.11}$$

$$= \frac{1}{2} |\uparrow_z\rangle (|\psi_R\rangle + |\psi_T\rangle) + \frac{1}{2} |\downarrow_z\rangle (|\psi_R\rangle - |\psi_T\rangle). \tag{5.12}$$

A Hamiltonian for an arbitrary number of detectors N can also be written, noting that as the number increases, the probability of detection tends to 1. This Hamiltonian mimics more closely an ideal detector for which the probability of the trigger being off is 1.

$$\hat{H}_{trigger} = \frac{1}{2M}\hat{P}_x + \lim_{\alpha \to 0} \frac{\alpha}{2}\Pi_{n=1}^{N}(1 + \hat{\sigma}_x^{(n)})\delta(\hat{x}). \tag{5.13}$$

This toy model contradicts the reasoning put forward by Allcock in [9]. Allcock argues that measuring the quantum arrival time with arbitrary precision is impossible. However, the detector he considers, which is represented by an absorber of the type $H_{int} = iV\theta(-\hat{x})$, measures the arrival time by absorbing the particle. Theoretically, when the particle switches from the incident region to the detector region within a short time, it is possible to measure the arrival time. Increasing V means increasing the absorption rate. Nevertheless, the particle is reflected for $V \gg E_k$. Hence according to Allcock, if the particle cannot be absorbed in a short time, the arrival time cannot be measured with arbitrary precision.

Allcock's proposition is refused by Aharonov et al. [54], who show that a two-level detector can be interpreted as a detector whose efficiency allows it to absorb particles in an arbitrarily short time. Let us review such system whose Hamiltonian is described by

$$\hat{H}_{trigger+clock} = \frac{1}{2M}\hat{P}_x + \lim_{\alpha\to 0}\frac{\alpha}{2}(1+\hat{\sigma}_x)\delta(\hat{x}) + \frac{1}{2}(1+\sigma_z)\hat{P}_y. \tag{5.14}$$

Equation (5.14) has the following eigenstates in the σ_z basis

$$\Psi_L(x) = \begin{pmatrix} \mathrm{e}^{ik_\uparrow x} + \phi_{L\uparrow}\mathrm{e}^{-ik_\uparrow x} \\ \phi_{L\downarrow}\mathrm{e}^{-ik_\downarrow x} \end{pmatrix} \mathrm{e}^{ipy}, \qquad \forall\, x<0 \tag{5.15}$$

$$\Psi_R(x) = \begin{pmatrix} \phi_{R\uparrow}\mathrm{e}^{ik_\uparrow x} \\ \phi_{R\downarrow}\mathrm{e}^{ik_\downarrow x} \end{pmatrix} \mathrm{e}^{ipy}, \qquad \forall\, x>0 \tag{5.16}$$

where $k_\uparrow = \sqrt{2ME_k}$ and $k_\downarrow = \sqrt{2M(E_k+p)}$. Finally, for $x=0$ we have $\phi_{R\downarrow} = \phi_{L\downarrow}$ and $\phi_{L\downarrow} = \phi_{R\uparrow} - 1$. From the previous statement it follows that the clock precision improves as $\phi_{L\downarrow} \to 0$ and $\phi_{L\uparrow} \to 1$.

For $\alpha \to \infty$, $\phi_{R\downarrow} = -\phi_{R\uparrow}$. As the particle is reflected, the clock keeps running. The two-level detector model presents the same limitations as the previous framework. Therefore coupling the system with the clock improves precision. However, note that there is no energy exchange between the particle and the clock. In the next section, we describe a method that will help alleviate this problem.

Toy Model III: Local Boost of Kinetic Energy

In this toy model, a local amplification of the kinetic energy is achieved by adding a boosting device to the apparatus. This method is necessary to remove the previous toy model's limitations. If arbitrary high kinetic energy is reached at close range from the clock, ensuring the probability distribution is not tampered with, the arrival time can be measured. The Hamiltonian describing this system is given by

$$\hat{H}_{apparatus} = \frac{1}{2M}\hat{P}_x^2 + \alpha\sigma_x\delta(\hat{x}) + \frac{W}{2}\theta(\hat{x})(1+\sigma_z) + \frac{1}{2}\left[V_1\theta(-\hat{x}) - V_2\theta(\hat{x})\right](1-\sigma_z). \tag{5.17}$$

Where α, W, V_1 and V_2 are positive constants and the eigenstates of equation (5.17) are

$$\Psi_L(x) = \begin{pmatrix} \mathrm{e}^{ikx} + \phi_{L\uparrow}\mathrm{e}^{-ikx} \\ \phi_{L\downarrow}\mathrm{e}^{qx} \end{pmatrix}, \qquad \forall\, x<0 \tag{5.18}$$

$$\Psi_R(x) = \begin{pmatrix} \phi_{R\uparrow} \mathrm{e}^{-\lambda x} \\ \phi_{R\downarrow} \mathrm{e}^{ik'x} \end{pmatrix}, \qquad \forall\, x > 0. \tag{5.19}$$

By looking at $x = 0$ it is found that $\phi_{L\uparrow} = \phi_{R\uparrow} - 1$ and $\phi_{L\downarrow} = \phi_{R\downarrow}$. In [54], the authors make a choice for α and V_1 such that $\alpha = kk' + V_1\lambda$ and $V_1 = \lambda\frac{k}{k'}$. They found that the eigenstate is entirely transmitted.

To make the toy model more realistic they also consider boosting and transmitting only part for the incoming wave. This method yields a higher rate of detection however the form of arrival probability distribution deviates from its original form.

Toy Model IV: Gradual Transition

The fourth and last toy model presented in [54] accomplishes refinement to the previous framework by reducing the reflection by substituting the two-step function with a gradual transition of the interaction between the particle and the clock. The Hamiltonian

$$\hat{H} = \frac{\hat{P}_x^2}{2M} + V(x)\hat{P}_y \tag{5.20}$$

replaces equation (5.1). One can make a choice for $V(x)$ such that the triggering is switched on gradually for $x \to -\infty$ till $x = x_A$. The measured time of arrival will not reflect the actual time of arrival due to errors yields by $V(x)$.

The authors found the error of the clock to be given by

$$C = -\frac{x_A}{\sqrt{2Mp_y}} \left[\ln \frac{1 + \sqrt{1 + \frac{E}{p_y}}}{1 + \sqrt{1 + \frac{E_x^2}{p_y x_A^2}}} + \ln \frac{x_i}{x_A} \right]. \tag{5.21}$$

Equation (5.21) is minimised for large p_y, however, note that this model does not measure the time of arrival for a freely moving particle. Instead, it estimates the arrival time for a particle subjected to a potential $V(x)$. As the accuracy improves, the particle motion is distorted. So far, the toy models we have introduced are unsatisfactory as they do not account for irreversibility. Therefore, they do not reflect natural experimental settings.

5.1.2 Irreversible Detector Model

In 1999, Halliwell [55] proposed an irreversible detector model by coupling a two-level detector with its environment. Introducing irreversibility is necessary to deal with the difficulties introduced by unitary quantum mechanics. Since quantum mechanics is reversible, there is a possibility that the detector returns to the original state once the particle is detected. To compensate for this, Halliwell considered a detector with a large number of degrees of freedom, therefore, making it irreversible. As a result, when the detector detects the particle, its state flips in one direction rather than the reverse. The resultant probability is of the same form as the probability of a particle being detected in the presence of an imaginary potential. This justifies the model we introduced previously in Chapter 3, Section 3.5.1.

Consider two levels $|1\rangle, |0\rangle$ representing the particle being detected and the particle not being detected, respectively. Let σ_+ and σ_- be the raising and lowering operators such that

$$\sigma_+ = |1\rangle\langle 0| \tag{5.22}$$

$$\sigma_- = |0\rangle\langle 1|. \tag{5.23}$$

Let the Hamiltonian of the detector be

$$\begin{aligned} H_d &= \frac{1}{2}\hbar\omega\sigma_z && (5.24)\\ &= \frac{1}{2}\hbar\omega(|1\rangle\langle 1| - |0\rangle\langle 0|) && (5.25)\\ & && (5.26) \end{aligned}$$

such that the eigenstates $|1\rangle, |0\rangle$ have eigenvalues $\frac{1}{2}\hbar\omega, -\frac{1}{2}\hbar\omega$ respectively. The irreversible detector makes transitioning from $|1\rangle$ to $|0\rangle$ possible via coupling of the detector with a large environment of oscillators in the ground state whenever the particle is located at $x < 0$. Otherwise, it stays in the state $|1\rangle$. Therefore if the particle is at state $|1\rangle$ for $x < 0$, there is a high probability the detector will flip its state to $|0\rangle$ if it started with a energy higher than the energy state.

The Hamiltonian of the system takes the following form

$$H = H_s + H_d + H_{\mathcal{E}} + V(x)H_{d\epsilon}. \tag{5.27}$$

The environment Hamiltonian $H_{\mathcal{E}}$ is chosen such that

$$H_{\mathcal{E}} = \sum_n \hbar\omega_n a_n^\dagger a_n \tag{5.28}$$

is the sum of harmonic oscillators, whilst the interaction between the detector and the environment $H_{d\mathcal{E}}$ is given by

$$H_{d\mathcal{E}} = \sum_n \hbar \left(k_n^* \sigma_- a_n^\dagger + k_n \sigma_+ a_n \right). \tag{5.29}$$

The goal is to find the master equation via the computation of the particle and detector's reduced density operator ρ. Let L_m be an operator describing the environment, the most general master equation is given by

$$\dot{\rho} = -\frac{i}{\hbar}[H_s + H_d, \rho] + \sum_m \left(L_m \rho L_m^\dagger - \frac{1}{2} L_m^\dagger L_m \rho - \frac{1}{2} \rho L_m^\dagger L_m \right). \tag{5.30}$$

Equation (5.30) is positive, hermitian and the trace is preserved.

Let γ be the coupling constant expressed by the oscillators' position in the environment. After assuming weak detector-environment approximation coupling, Markov approximation and by setting $L = \gamma^{1/2} V(x)\sigma_-$ in equation (5.30), the following can be derived (see Appendix B for more thorough calculations)

$$\dot{\rho} = -\frac{i}{\hbar}[H_s + H_d, \rho] - \frac{\gamma}{2} \left(V^2(x)\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_- V^2(x) - 2V(x)\sigma_-\rho\sigma_+ V(x) \right). \tag{5.31}$$

The initial state of equation (5.31) tends to $|0\rangle \langle 0|$ at a rate γ^{-1}, for a choice of $V = 1$ and $H_s = 0$.

The solutions to equation (5.31) are given by

$$\rho = \rho_{11} \otimes |1\rangle \langle 1| + \rho_{01} \otimes |0\rangle \langle 1| + \rho_{10} \otimes |1\rangle \langle 0| + \rho_{00} \otimes |0\rangle \langle 0| \tag{5.32}$$

where for $V(x) = \theta(-x)$ we have

$$\dot{\rho}_{11} = -\frac{i}{\hbar}[H_s, \rho_{11}] - \frac{\gamma}{2}(V(x)\rho_{11} + \rho_{11}V(x)) \tag{5.33}$$

$$\dot{\rho}_{01} = -\frac{i}{\hbar}[H_s, \rho_{01}] - \frac{\gamma}{2}\rho_{01}V(x) + i\frac{\hbar\omega'}{2}\rho_{01} \tag{5.34}$$

$$\dot{\rho}_{10} = -\frac{i}{\hbar}[H_s, \rho_{10}] - \frac{\gamma}{2}V(x)\rho_{10} - i\frac{\hbar\omega'}{2}\rho_{10} \tag{5.35}$$

$$\dot{\rho}_{00} = -\frac{i}{\hbar}[H_s, \rho_{00}] - \gamma V(x)\rho_{11}V(x). \tag{5.36}$$

Now let $|\Psi_0\rangle$ be the initial state, equations (5.33-5.36) have initial condition

$$\rho(0) = |\Psi_0\rangle \langle\Psi_0| \otimes |1\rangle \langle 1| \,; \tag{5.37}$$

it follows that the probability of the detector being at $|1\rangle$ at time τ is

$$p_{nd} = \mathrm{Tr}\rho_{11}(\tau) \tag{5.38}$$
$$= \int_{-\infty}^{+\infty} dx \rho_{11}(x, \tau). \tag{5.39}$$

Intuitively, the probability of the detector being in the registered state $|0\rangle$ is

$$p_d = 1 - p_{nd} \tag{5.40}$$
$$= \mathrm{Tr}\rho_{00}(\tau) \tag{5.41}$$
$$= \int_{-\infty}^{+\infty} dx \rho_{00}(x, \tau). \tag{5.42}$$

Equation (5.38) may be written as

$$p_{nd} = \int_{-\infty}^{+\infty} dx |\Psi(x, \tau)|^2. \tag{5.43}$$

There is also a small probability that the detector is not recording the particle at $x < 0$. This probability is linked to the detector efficiency. To address the detector's efficiency, Halliwell introduces a second detector at $x > 0$. Another detector allows us to observe the entire x-axis.

Let σ_- and σ_+ be the raising and lowering operator in the range $x > 0$, then equation (5.31) becomes

$$\dot{\rho} = -\frac{i}{\hbar}[H_s, \rho] - \frac{\gamma}{2}(\theta(-x)\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_-\theta(-x) - 2\theta(-x)\sigma_-\rho\sigma_+\theta(-x)) \tag{5.44}$$
$$- \frac{\gamma}{2}(\theta(x)\tilde{\sigma}_+\tilde{\sigma}_-\rho + \rho\tilde{\sigma}_+\tilde{\sigma}_-\theta(x) - 2\theta(x)\tilde{\sigma}_-\rho\tilde{\sigma}_+\theta(x)). \tag{5.45}$$

It is found that the probability of the particle not being detected is Tr ρ_{nd}.
From equation

$$\dot{\rho}_{nd} = -\frac{i}{\hbar}[H_s, \rho_{nd}] - \gamma\rho_{nd} \tag{5.46}$$

it is evident that $\rho_{nd} = \mathrm{e}^{-\gamma t}$. If $\tau \gg \gamma^{-1}$ the detector works efficiently. Recall that Allcock [8, 9, 10] dismissed the possibility of defining the arrival time. However, in his study, he considers a real step function $V(x)$.

In [55], a more general form of the potential $\tilde{V}(x)$ is considered such that the master equation takes the form

$$\dot{\rho} = -\frac{i}{\hbar}[H_s, \rho] - \frac{\gamma}{2}(\tilde{V}^\dagger \tilde{V} \sigma_+ \sigma_- \rho + \rho \sigma_+ \sigma_- \tilde{V}^\dagger \tilde{V} - 2\tilde{V} \sigma_- \rho \sigma_+ \tilde{V}^\dagger) \tag{5.47}$$

where $\tilde{V}(x)$ is usually non hermitian.

To sum up, we have described a detector able to determine if a particle is in $x < 0$ for a time range $[0, \tau]$, for $\tau \gg \gamma^{-1}$. A more precise system may be obtained by using a series of detectors.

5.1.3 Theoretical Model Inspired by Atomic Time-of-Flight Experiments

In 2000, Muga et al. [56] presented an additional theoretical model that draws inspiration from atomic time-of-flight (TOF) experiments. The arrival time is measured as fluorescent photons reach the detector.

Consider a laser whose intensity increases to a steady state in the direction of the propagation of the atom. The atoms are prepared at a particular state and then move freely until they come into contact with an orthogonal laser beam. Note that it is possible to modify the profile of the laser beam to a more realistic profile, such as a Gaussian.

Consider a perfect counter; the first fluorescent photon detected defines the arrival time. Let $|1\rangle$ and $|2\rangle$ be two levels describing the inner structure of the atom, then its interaction Hamiltonian H_{int} is

$$H_{int} = \frac{\omega\hbar}{2}(|2\rangle\langle 2| - |1\rangle\langle 1|), \tag{5.48}$$

while the master equation is given by

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \frac{\gamma}{2}(2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-) \tag{5.49}$$

with

$$H = \frac{p^2}{2M} + \frac{\hbar}{2}\Omega(x)(\sigma_+ + \sigma_-) \tag{5.50}$$

$$\sigma_- = |1\rangle\langle 2| \tag{5.51}$$

$$\sigma_+ = |2\rangle\langle 1|. \tag{5.52}$$

Equation (5.49) is very similar to equation (5.7) in structure, but they are conceptually different. The first represents the inner structure of the atom, whilst the second describes the detector. In [55] the detector starts in an excited state and subsequently decays when coupled with the particle. On the other hand in [56] the atom starts in the ground state and is excited when interacting with the laser beam; finally it decays.

The effective Hamiltonian describing the Schrödinger equations is given by

$$H_{eff} = H - i\hbar\frac{\gamma}{2}\left|2\right\rangle\left\langle 2\right| . \tag{5.53}$$

It follow that the arrival time distribution is

$$\Pi(t) = \frac{dN(t)}{dt}, \tag{5.54}$$

$$\text{with } N = \sum_{j=1,2}\left\langle \psi_{c,j}.(t)|\psi_{c,j}(t)\right\rangle . \tag{5.55}$$

This model can be used for selecting the correct Rabi frequency to carry out the experiment.

5.1.4 Decoherent History Model

There is another model we consider in this thesis. It is known as the decoherent history approach. In 1997, Halliwell and Zafiris [57] attempted to tackle the question of time in quantum mechanics using this model. They focused on measuring a particle's arrival time probability distribution at $x = 0$ within an interval of time $[t_1, t_2]$ rather than at an instant t. More specifically, they find the probability of a particle crossing or not crossing $x = 0$ during the time interval $[0, t]$. The authors argue that to find the probability distribution, one has to consider the complete history of the system during the time interval.

This is a beneficial viewpoint because it has physical relevance since instrumentation may not be able to measure an event at an instant in time with precision. The decoherent history approach has its origin in quantum cosmology. It has been applied to closed quantum systems.

The decoherent histories approach can be constructed using path integrals. Consider an action $S[x(t)]$. The amplitudes are estimated by summing over $\mathrm{e}^{iS[x(t)]}$, and the probabilities are the amplitudes squared as per path integral formulation. One might think that the sum of the probabilities of a particle being in the spacetime region and not being in it is 1. Due to interference, that is not the case. However, the decoherence histories approach provides destruction of interference. Because we are considering the space in which a particle is found during a sequence of times, the problem can be difficult to picture. What we are dealing with is a combination of double slits.

To achieve decoherence, two modifications to the physical situation are considered. First, decoherence is achieved by coupling a point particle to a thermal bath of harmonic oscillators. Secondly, decoherence is obtained by reproducing the system N times. Then in the limit of large N, we estimate the number density of some particles entering the region of spacetime considered [57].

Projection operators describe properties of quantum systems at specific instants in time. If they are of the form $|\alpha\rangle\langle\alpha|$, the projectors are considered to be *fine-grained* otherwise they are *coarse-grained.* Time-dependent projectors are given by

$$P^k_{\alpha_k}(t) = \mathrm{e}^{iH(t_k-t_0)} P^k_{\alpha_k} \mathrm{e}^{-iH(t_k-t_0)}. \tag{5.56}$$

Consider an initial state ρ, then a string of time-dependent projection operators represent the quantum mechanical history of the system with positive candidate probability

$$p(\alpha_1, \alpha_2, ..., \alpha_n) = \mathrm{Tr}\left(P^n_{\alpha_n}(t_n)...P^1_{\alpha_1}(t_1)P^1_{\alpha_1}(t_1)...P^n_{\alpha_n}(t_n)\right). \tag{5.57}$$

Adding up projections at each instant in time,

$$\bar{P}_{\bar{\alpha}} = \sum_{\alpha\in\bar{\alpha}} P_\alpha, \tag{5.58}$$

is the equivalent of assembling coarse-grained histories. Equation (5.58) must follow the additivity requirement to satisfy probability theory. In other words, coarse-grained histories must be the addition of the probabilities of fine-grained histories. However, this is not feasible due to quantum interference. Nevertheless, decoherence functional may be used to obtain histories in which interference is negligible

$$D(\underline{\alpha}, \underline{\alpha'}) = \mathrm{Tr}\Big(P^n_{\alpha_n}(t_n)...P^1_{\alpha_1}(t_1)\rho P^1_{\alpha'_1}(t_1)...P^n_{\alpha'_n}(t_n)\Big) \tag{5.59}$$

where $\underline{\alpha}$ and $\underline{\alpha'}$ are a pair of histories. Note that the additivity requirement is satisfied **iff** $\mathrm{Re}\, D(\underline{\alpha}, \underline{\alpha'}) = 0$ [57].

Decoherence via Quantum Brownian Motion

Before reviewing the quantum Brownian motion model, it is useful to briefly look at the classical counterpart: the classical Brownian motion model. Consider the limit of negligible dissipation, then the phase space probability distribution is

$$\frac{\partial w}{\partial t} = -\frac{p}{M}\frac{\partial w}{\partial x} + 2M\gamma kT\frac{\partial^2 w}{\partial p^2}. \tag{5.60}$$

Where the value $2M\gamma kT$ is taken to be very large. With initial conditions $w(p, x, 0) = w_0(p, x)$ the solution to equation (5.60) is

$$w(p, x, t) = \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dx K(p, x, t|p_0, x_0, 0) w_0(p, x), \tag{5.61}$$

where $K(p, x, t|p_0, x_0, 0)$ is the Fokker-Planck propagator. The probability of not crossing and crossing the origin at $x = 0$ are found to be

$$p_r = \int_{-\infty}^{+\infty} dp \int_{0}^{+\infty} dx \int_{-\infty}^{+\infty} dp_0 \int_{0}^{+\infty} dx_0 K_r(p, x, t|p_0, x_0, 0) w_0(p_0, x_0) \tag{5.62}$$

$$p_c = 1 - p_r \tag{5.63}$$

$$= \int_{-\infty}^{0} dp \int_{-\infty}^{+\infty} dp_0 \int_{0}^{+\infty} dx_0 \frac{p}{M} K_r(p, x = 0, t|p_0, x_0, 0) w_0(p_0, x_0) \tag{5.64}$$

In the quantum analogue we consider a decoherence functional

$$D(\alpha, \alpha') = \mathrm{Tr}(\rho_{\alpha\alpha'}). \tag{5.65}$$

such that

$$\rho_{\alpha\alpha'}(x_f, y_f) = \int_{\alpha} \mathcal{D}x \int_{\alpha'} \mathcal{D}y \exp\left(\frac{i}{\hbar}S[x] + \frac{i}{\hbar}S[y] + \frac{i}{\hbar}W[x, y]\right) \rho_0(x_0, y_0) \tag{5.66}$$

where, in the limit of negligible dissipation, $W[x, y]$ is the functional phase. Let $\alpha = c$ and $\alpha = r$ be the history of crossing or not crossing $x = 0$. Then ρ follows the master equation

$$i\hbar\frac{\rho}{\partial t} = -\frac{\hbar^2}{2M}\left(\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial^2 \rho}{\partial y^2}\right) - \frac{i}{\hbar}D(x - y)^2 \rho. \tag{5.67}$$

Given an initial state ρ_0 and boundary conditions, the solutions to equation (5.67) are

$$\rho_{rr}(x, y) = 0,\ \forall\ x \leq 0 \wedge y \leq 0 \tag{5.68}$$

$$\rho_{rc}(x, y) = 0,\ \forall\ x \leq 0 \tag{5.69}$$

$$\rho_{cr}(x, y) = 0,\ \forall\ y \leq 0. \tag{5.70}$$

Note that $\rho_{rr} + \rho_{rc} + \rho_{cr} + \rho_{cc} = \rho$. Because of the efficiency of the model we find that $\rho_{rc} \approx 0$ and $\rho_{cr} \approx 0$. As a result

$$p_r = \mathrm{Tr}(\rho_{rr}) \tag{5.71}$$

$$p_c = \mathrm{Tr}(\rho_{cc}) \tag{5.72}$$

Decoherence via System Replication

Another technique we consider here is to replicate the system N times. Consider an operator C describing the histories of a particle crossing in the interval $[0, t]$ and an operator $\bar{C}$ describing the histories of not crossing, such that $C + \bar{C} = 1$.
Then the respective probabilities for one particle system are

$$p = Tr(C\rho C^{\dagger} \tag{5.73}$$

$$\bar{p} = \mathrm{Tr}\Big(\bar{C}\rho\bar{C}^{\dagger}\Big) \tag{5.74}$$

Equation (5.73) and equation (5.74) fulfil the following expression

$$p + \bar{p} + \mathrm{Re}\, D = 1 \tag{5.75}$$

where $D = \mathrm{Tr}\big(C\rho C^{\dagger}\big)$.
Let us now increase the system by one particle; we then have three operators equivalent to zero, one and two particle crossing

$$C_0 = \bar{C} \otimes \bar{C} \tag{5.76}$$

$$C_1 = \bar{C} \otimes C + C \otimes \bar{C} \tag{5.77}$$

$$C_2 = C \otimes C. \tag{5.78}$$

For n particles we have

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda \mathrm{e}^{-i\lambda n} (\bar{C} + \mathrm{e}^{i\lambda} C) \otimes (\bar{C} + \mathrm{e}^{i\lambda} C) \otimes ... \tag{5.79}$$

and the decoherence functional takes the form

$$D(n, n') = \mathrm{Tr}\Big(C_n \rho \otimes \rho \otimes ... \otimes \rho C_{n'}^{\dagger}\Big). \tag{5.80}$$

By substituting equation (5.79) into (5.80), and using contour integration we obtain

$$D(n, n') = \frac{1}{2\pi i} \binom{N}{n} \int \frac{dz}{z^{n'+1}} (\bar{p} + D^* z)^{N-n} (D + pz)^n. \tag{5.81}$$

Note that for $z \to \frac{\bar{p}}{D^*} z$ and $\alpha = p\bar{p}/|D|^2$, equation (5.81) becomes

$$D(n, n') = \frac{1}{2\pi i} \binom{N}{n} \bar{p}^{N-n-n'} (D^*)^{n'} D^n \int dz z^{-n'-1} \times (1+z)^{N-n} (1+\alpha z)^n \tag{5.82}$$

For a system of N non-interacting free particles in the limit of large N and large α the probabilities of crossing $x = 0$ between $n - \Delta n$ and $n + \Delta n$ in the interval of time $[0, t]$ are given by

$$p(n) \sim \exp\left(-\frac{N}{2n(N-n)} \left(n - \frac{pN}{(p+\bar{p})} \right)^2 \right) \tag{5.83}$$

$$\sim \exp\left(-\frac{N(p+\bar{p})^2}{2p\bar{p}} \left(\frac{n}{N} - \frac{p}{(p+\bar{p})} \right)^2 \right) \tag{5.84}$$

Therefore the relative frequency of crossing is maximal for $\frac{p}{p+\bar{p}}$ with $\Delta n \ll n$.

5.2 Empirical Problems with Quantum Time Measurement

In [9], Allcock suggests that repeated measurements can be used to probe whether a particle crossed a region or not. However, this proposal is later abandoned due to the quantum Zeno effect (QZE). In the QZE, the particle is reflected and stays in the initial subspace in the realm of frequent measurements.

Echanobe et al. [47] were able to derive a physical time distribution by normalising part of the arriving norm. However, the resultant distribution, given by equation (3.57), is different from Kijowski's.

There exist other techniques to measure arrival time. As previously discussed, a known method relies on a synchronised clock. When doing so is not possible, the time delay between a specific event (e.g. emission or detection of a particle) and the final instant of the particle is detected. When the oscilloscope estimates the delay, the measurement is given by the intensity. The propagation time is computed from the spatial difference on the oscilloscope display.

The limitation in quantum time measurement is due to the difficulty in starting and stopping events. On the other hand, the time resolution difficulties are due to short measurements. A solution to these hurdles is to use the streak camera method [59], where a voltage that steadily rises is used to modify where the image of the particle is created.

Often, the group delay of a wave packet peak, $d\phi/d\omega$, represents arrival time measurements. In other circumstances, the delay may be given by other characteristics; for instance, given a preset amplitude, the group delay may be given by the instant at which the waveform intersects that particular value.

In 2020, Maccone and Sacha [60] proposed a more general technique to measure the time at which an event takes place via *conditioning on a quantum clock* and discussed the time of arrival in particular. In [60], the authors do not consider the arrival time as a property solely belonging to the particle. Instead, the authors identify it as a shared property between the particle and the clock. Page and Wootters formalism [61] is used to avoid previous technical difficulties.

Chapter 6

Leggett-Garg Inequalities

The Leggett-Garg inequality was first introduced in 1985 by Leggett, and Garg [63] to test the notion of macrorealism. It is also a beneficial and innovative method to tackle the temporal questions in quantum mechanics. This doctrine asserts that macroscopic systems satisfy three main assumptions:

- *Macroscopic Realism per se*: Given two or more macroscopically well-defined states, the system will always be in one of these states.
- *Macroscopic Non-invasive Measurability*: In principle, one can determine the state of the system and future dynamics via small perturbations without disturbing the state.

In [63], the authors assert that these statements are incompatible with quantum mechanics deduced at the macroscopic level. The first principle is violated by macroscopic superposition whilst the second is violated by quantum collapse during measurement [67]. A third principle was introduced in later work [64, 65, 66].

- *Induction*: the current state is not affected by future measurements.

If these assumptions are violated, the macroscopic view must be rejected.

Leggett-Garg inequalities may be considered the temporal equivalent of Bell's inequality. Whilst the latter deals with space-like separation between two or more systems, the first concerns repeated measurement of an observable on a system at different times [68].

In this chapter, we review the Leggett-Garg inequalities. In particular, we discuss their construction and tests for macrorealism. Finally, we consider how LG inequalities are linked to the arrival time problem. It is possible to ascertain whether the arrival time quasi-probability is negative or positive via indirect or direct measurement. The LGIs introduce a stricter test on the arrival time quasi-probability that disallows any classical interpretation of the data. If the sequential measurement of a dichotomic variable $\mathcal{Q}$ exists, and the principles are applied to it, the existence of a joint probability distribution and, therefore, a set of LG inequalities describing this probability distribution are guaranteed.

6.1 Construction of LGI

Consider the measurement of a macroscopic quantity $\mathcal{Q}$ on a system S. Let $\mathcal{Q}$ have only two well defined states, -1 and $+1$, then at any time $\mathcal{Q}$ will be exclusively in one of the two states. Now consider pairs of measurements taken time apart such that $\mathcal{Q}$ at t_i is $\mathcal{Q}_i$.

Regardless of whether the measurement of $\mathcal{Q}$ on S takes place at a given t_i we know that S will have a definite value of $\mathcal{Q}$ at any t_i; in other words $\forall\ i\ \in \{1, 2, 3\}$

$$\mathcal{Q} = \mathcal{Q}_1\mathcal{Q}_2 + \mathcal{Q}_1\mathcal{Q}_3 + \mathcal{Q}_2\mathcal{Q}_3. \tag{6.1}$$

The value of $\mathcal{Q}_{LG}$ depends on the value of $\mathcal{Q}_i$ (see Table 6.1).

Considering many copies of S, we would like to measure $\mathcal{Q}$ on these copies in such a way that they are reliable measurements [69]. Note that despite preparing the copies of S in the same way, we still might obtain different values of $\mathcal{Q}$ at measurement. This is because underlying characteristics may exist that were not fixed at preparation [68].

If three successive measurements are performed we obtain

$$\langle\mathcal{Q}_{LG}\rangle = \langle\mathcal{Q}_1\mathcal{Q}_2\rangle_{M_1M_2M_3} + \langle\mathcal{Q}_1\mathcal{Q}_3\rangle_{M_1M_2M_3} + \langle\mathcal{Q}_2\mathcal{Q}_3\rangle_{M_1M_2M_3}, \tag{6.2}$$

where the label $M_1M_2M_3$ signifies that the expectation value is measured for M_1 of $\mathcal{Q}_1$, M_2 of $\mathcal{Q}_2$ etc... From table (6.1) it follows that

$$-1 \leq \langle\mathcal{Q}_{LG}\rangle_{M_1M_2M_3} \leq 3. \tag{6.3}$$

$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_{LG}$
+1	+1	+1	3
+1	+1	-1	-1
+1	-1	+1	-1
+1	-1	-1	-1
-1	+1	+1	-1
-1	+1	-1	-1
-1	-1	+1	-1
-1	-1	-1	3

Table 6.1: Values for $\mathcal{Q}_{LG}$ [68]

Now, let us include macroscopic non-invasive measurability. This entails that the measurement at M_1, M_2 for $\langle \mathcal{Q}_1 \mathcal{Q}_2 \rangle$ coincides with the measurement performed at M_1, M_2, M_3. Therefore the *Leggett-Garg inequality* takes the following form

$$-1 \leq \langle \mathcal{Q}_1 \mathcal{Q}_2 \rangle_{M_1 M_2} + \langle \mathcal{Q}_1 \mathcal{Q}_3 \rangle_{M_1 M_3} + \langle \mathcal{Q}_2 \mathcal{Q}_3 \rangle_{M_2 M_3} \leq 3. \tag{6.4}$$

One might want to generalise equation (6.2) for n measurements; in that case we would have

$$\begin{aligned} \langle \mathcal{Q}_{LG} \rangle = \langle \mathcal{Q}_2 \mathcal{Q}_1 \rangle_{M_2 M_1} + \langle \mathcal{Q}_3 \mathcal{Q}_2 \rangle_{M_3 M_2} + \langle \mathcal{Q}_4 \mathcal{Q}_3 \rangle_{M_4 M_3} + ... \\ + \langle \mathcal{Q}_n \mathcal{Q}_{n-1} \rangle_{M_n M_{n-1}} - \langle \mathcal{Q}_{n1} \rangle_{M_{n1}}. \end{aligned} \tag{6.5}$$

Also note that the quantum analogue of the quantity $C_{ij} = \langle \mathcal{Q}_i \mathcal{Q}_j \rangle$ does not exists due to operator ordering [67]. Using projective measurements one can find the symmetrised combination [70]

$$C_{ij} = \frac{1}{2} \langle \{ \hat{\mathcal{Q}}_i, \hat{\mathcal{Q}}_j \} \rangle. \tag{6.6}$$

Equation (6.6) will be of use when analysing the LGI violation for qubits.

6.2 Testing Macrorealism

6.2.1 LGI Violation for Two-Level Systems (TLS) via Projective Measurements: Qubits

We now consider the most straightforward quantum mechanical system: a *qubit.* The LGI is violated for this two-level system. Considering projective measurements on the system's initial state, then any time evolution of the system will lead to an LGI regardless of the size of the system [67].

Let the parametrisation of a qubit operator be $\hat{Q} = \mathbf{a} \cdot \hat{\sigma}$. Therefore,

$$(\mathbf{a}_2 \cdot \hat{\sigma})(\mathbf{a}_3 \cdot \hat{\sigma}) = (\mathbf{a}_2 \cdot \mathbf{a}_3)\mathbb{I} + i\hat{\sigma} \cdot (\mathbf{a}_2 \times \mathbf{a}_3). \tag{6.7}$$

As mentioned previously, the correlation function described in the previous section does not have a quantum counterpart. Therefore we must use equation (6.6), which results from projective measurements. This technique leads to

$$C_{ij} = \frac{1}{2}\langle\{\hat{\mathcal{Q}}_i, \hat{\mathcal{Q}}_j\}\rangle \tag{6.8}$$

$$= \mathbf{a}_i \cdot \mathbf{a}_j \langle\mathbb{I}\rangle \tag{6.9}$$

$$= \mathbf{a}_i \cdot \mathbf{a}_j. \tag{6.10}$$

For $n-$measurements on a qubit equation (6.5) becomes

$$\langle\mathcal{Q}_{LG}\rangle = \sum_{m=1}^{n-1} \mathbf{a}_{m+1} \cdot \mathbf{a}_m - \mathbf{a}_n \cdot \mathbf{a}_1 \tag{6.11}$$

$$= \sum_{m=1}^{n-1} cos(\theta_m) - cos\left(\sum_{m=1}^{n-1} \theta_m\right). \tag{6.12}$$

Note that the value for equation (6.12) is at its highest value for $\theta_m = \frac{\pi}{n}$. And the violation takes place for

$$\left[\hat{\mathcal{Q}}_i, \hat{\mathcal{Q}}_j\right] = 2i\hat{\sigma} \cdot (\mathbf{a}_i \times \mathbf{a}_j). \tag{6.13}$$

LGI is maximised when the commutation relations reach their greatest value.

6.2.2 Experimental LGI Violation via Weak and Semi-Weak Measurements

In 2010, Palacios-Laloy et al. [71], implemented the experimental model proposed by Ruskov et al. [72] where weak continuous measurements are employed to probe quantum coherence via correlation functions. This is an attractive alternative to projective measurements, which are very difficult to perform in laboratory settings. By carrying out this experiment, it is possible to understand whether a system exhibits quantumness. Weak continuous measurements is a technique whereby the system is gradually measured, and as a result, information is gathered about its current state. By probing a superconducting two-level system's Rabi oscillations via continuous monitoring, Palacios-Laloy et al. [71] showed an experimental violation of the Leggett-Garg inequalities.

On the other hand, one might want to consider employing semi-weak measurements to generalise the study of LGIs. Semi-weak measurements are defined as measurements between weak and projective measurements. Dressel et al. [73] examined the subject thoroughly by considering a large class of experiments. They proved that different inequalities may be measured all at once via a single-set up [73]. Another important method we will not discuss here is the use of *ideal* non-invasive measurements. In [74], Knee et al. proposed a general protocol that uses an auxiliary system to carry out the measurement. This method is necessary for more robust tests for system of arbitrary size [74].

Finally, another important concept in this subject is known as the *clumsiness loophole*. It asserts that violation or invasive measurements can be due to clumsy experimental methods. In 2010, Wilde and Mize [75] addressed this issue by proposing a variation to the LGI.

6.3 Arrival Time Problem and LGI

Recently, Halliwell et al. [49] approached the arrival time problem for a free particle through the Leggett-Garg inequalities. They formulated the arrival time problem as the joint probability of a particle being on the positive or negative x-axis at two different instants in time.

Measurements of the probability can be indirect or direct. In the first case, by measuring moments. In the second, via weak or ambiguous measurements. The paper [49] analyses two-time quasi probability and asserts that a positive value matches a measurement-independent time of arrival probability. Moreover, in the realm of small intervals the arrival probability approximately match the semi-classical current, as discussed in Chapter 3, section 3.4. On the other hand, a negative value is linked to the backflow effect (see Chapter 5). What we focus on here is that the condition on the quasi probability $q(s_1, s_2) \geq 0$ corresponds to two-time Leggett-Garg inequalities.

Let $P_{s_n}(t)$ be a projector in the Heisenberg picture, and ρ the density operator. The two-time quasi probability is defined as

$$q(s_1, s_2) = \frac{1}{2}\mathrm{Tr}((P_{s_2}(t_2)P_{s_1}(t_1) + P_{s_1}(t_1)P_{s_2}(t_2))\rho). \tag{6.14}$$

Consider values for $\mathcal{Q}_i$ at two instant in time t_1, t_2, such that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ only take values ± 1.

Hence it follows that

$$(1 + s_1\mathcal{Q}_1)(1 + s_2\mathcal{Q}_2) \geq 0. \tag{6.15}$$

The LGIs are given by

$$1 + \langle \mathcal{Q}_1 \rangle + \langle \mathcal{Q}_2 \rangle + \langle \mathcal{Q}_1\mathcal{Q}_2 \rangle \geq 0 \tag{6.16}$$

$$1 - \langle \mathcal{Q}_1 \rangle - \langle \mathcal{Q}_2 \rangle + \langle \mathcal{Q}_1\mathcal{Q}_2 \rangle \geq 0 \tag{6.17}$$

$$1 + \langle \mathcal{Q}_1 \rangle - \langle \mathcal{Q}_2 \rangle - \langle \mathcal{Q}_1\mathcal{Q}_2 \rangle \geq 0 \tag{6.18}$$

$$1 - \langle \mathcal{Q}_1 \rangle + \langle \mathcal{Q}_2 \rangle - \langle \mathcal{Q}_1\mathcal{Q}_2 \rangle \geq 0. \tag{6.19}$$

All quantities must follow the principle of non-invasive measurability. Both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy this principle since they are measured individually. However not the same can be said for $C_{12} = \langle \mathcal{Q}_1\mathcal{Q}_2 \rangle$. The C_{12} measurement requires two sequential measurements. For this case, *ideal negative measurements* are used to achieve non-invasiveness. The techniques entails coupling the detector with the state reading $\mathcal{Q} = -1$, therefore a null results infers $\mathcal{Q} = +1$. Therefore this method is a clear test for quantmness and eliminates any classical description.

Chapter 7

Summary and Conclusion

In this thesis, we analysed the arrival time problem in quantum mechanics. We started Chapter 2 with a brief look at the early studies done in the field [2, 5, 6, 7, 8, 9, 10]. Allcock's seminal papers from 1969 [8, 9, 10] suggested that it is impossible to construct a time of arrival operator. However, subsequent work by Baute et al. [12] showed his work to be incorrect. We then continued by deriving the different time of arrival operators. Finally, we constructed the Aharonov-Bohm time arrival operator with associated POVMs.

Studies from 1997 [18] showed it is possible to derive a self-adjoint variant of the arrival time operator by quantising the expression $Mq/|p|$. However, to circumvent Pauli's theorem, one has to consider an unbounded Hamiltonian. We described a further attempt by Oppenheim et al. [19] where the low momentum behaviour was modified to obtain a self-adjoint arrival time operator. Finally, we introduced another technique developed by Halliwell et al. [23] where a self-adjoint time operator related to two measurement models was derived. We then investigated the realm of confined quantum time arrival operators and concluded Chapter 2 by deriving the time operator for a relativistic particle.

In Chapter 3, we reviewed different approaches to the time of arrival distribution. First, we focused on Kijowski's distribution by deriving the probability distribution from a set of axioms. We then introduced the operator approach, where we computed the probability distribution from the time operator we have derived in Chapter 2, Section 2.3.1. Next, we considered Bohmian mechanics and the associated time of arrival probability distribution. Finally, we derived the arrival probability distribution from the probability current and focused on a general approach to the problem using different time observables.

In Chapter 4, we briefly discussed the phenomena known as quantum backflow. We first derived the backflow constants from the eigenvalue problem. We then looked at the arrival time measurement in the backflow regime. In Chapter 5, we examined time measurement in Quantum Mechanics. We started from the four toy models studied by Aharonov et al. [54]. Each model has additional characteristics to make it most realistic. Of the four models presented by Aharonov et al. [54], the toy model using gradual transition is the more realistic. However, it does not take into account irreversibility and therefore does not entirely reflect natural experimental settings.

In 1999, Halliwell [55] produced a paper including irreversibility in the detector model by coupling a two-level detector with its environment. We found that a more precise system may be obtained using a series of detectors. Next, we considered a theoretical model inspired by time-of-flight experiments by Muga et al. [56]. This model can be used in experimental settings to select the correct Rabi frequency.

Subsequently, we analysed the decoherent history model by Halliwell and Zafiris [57]. The model aims to measure the particle's arrival time probability distribution at $x = 0$ within a time interval. To do so, they considered the complete history of the system during that time interval. First, we coupled the particle with a thermal bath of harmonic oscillators, and then we replicated the system N times to achieve decoherence. We estimated the probability distribution for the spacetime considered. Finally, we looked at the practical problems of quantum time measurement.

In the last Chapter (Chapter 6), we introduced the Leggett-Garg inequalities (LGIs) to test the notion of macrorealism. LGIs are viewed as the temporal Bell's inequalities; If their principles are violated, the macroscopic view must be rejected. After constructing the Leggett-Garg inequalities, we discussed the LGI violations. Finally, we reviewed previous work by Halliwell et al. [49], which approached the arrival time problem through the Leggett-Garg inequalities.

Overall, we have analysed different approaches to the arrival time problem in quantum mechanics. Unfortunately, each method yields different results, which suggests there is still a need for research to be carried out on the topic. Nevertheless, many hurdles have been conquered (e.g. Pauli's theorem). Reaching a consensus will lead to a better understanding of quantum mechanics itself and further development in technological endeavours.

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Appendix A

CCR between Time Operator and Hamiltonian in Relativistic QM

The following will be used

$$[(\hat{A}+\hat{B}),\hat{C}] = [\hat{A},\hat{C}] + [\hat{B},\hat{C}]. \tag{A.1}$$

Recall that

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{A.2}$$

it follows that $(\gamma^0\gamma^1)^2 = \mathbb{I}$.

Consider the Hamiltonian of equation (2.91), let $\beta = \gamma^0$ such that $\alpha_1 = \gamma^0\gamma^1 = \beta\gamma^1$ and $\alpha_1^2 = \mathbb{I}$.

Then the Hamiltonian takes the form

$$\hat{H} = \alpha_1\hat{p} + \beta M. \tag{A.3}$$

Consider equation (2.97), and equation (A.1). Let $\hbar = 1$ then it follows

$$[\hat{T}_{\text{Dirac}},\hat{H}] = -\frac{1}{4}\left[\hat{H}\left(\frac{1}{\hat{p}}\hat{q} + \hat{q}\frac{1}{\hat{p}}\right) + \left(\frac{1}{\hat{p}}\hat{q} + \hat{q}\frac{1}{\hat{p}}\right)\hat{H},\hat{H}\right] \tag{A.4}$$

$$= -\frac{1}{4}\left[\hat{H}\left(\frac{1}{\hat{p}}\hat{q} + \hat{q}\frac{1}{\hat{p}}\right),\hat{H}\right] - \frac{1}{4}\left[\left(\frac{1}{\hat{p}}\hat{q} + \hat{q}\frac{1}{\hat{p}}\right)\hat{H},\hat{H}\right] \tag{A.5}$$

Now let us substitute equation (A.3) in equation (A.5). Note that the commutators including the term βM will vanish. So we are left with

$$\begin{aligned}
[\hat{T}_{\text{Dirac}}, \hat{H}] &= -\frac{1}{4}\left[(\alpha_1\hat{p} + \beta M)\left(\frac{1}{\hat{p}}\hat{q} + \hat{q}\frac{1}{\hat{p}}\right), (\alpha_1\hat{p} + \beta M)\right] \qquad (A.6)\\
&- \frac{1}{4}\left[\left(\frac{1}{\hat{p}}\hat{q} + \hat{q}\frac{1}{\hat{p}}\right)(\alpha_1\hat{p} + \beta M), (\alpha_1\hat{p} + \beta M)\right]\\
&= -\frac{1}{4}\left[(\alpha_1\hat{p})\left(\frac{1}{\hat{p}}\hat{q} + \hat{q}\frac{1}{\hat{p}}\right), (\alpha_1\hat{p})\right] \qquad (A.7)\\
&- \frac{1}{4}\left[\left(\frac{1}{\hat{p}}\hat{q} + \hat{q}\frac{1}{\hat{p}}\right)(\alpha_1\hat{p}), (\alpha_1\hat{p})\right]\\
&= -\frac{2\alpha^2}{4}[\hat{q}, \hat{p}] - \frac{2\alpha^2}{4}[\hat{q}, \hat{p}] \qquad (A.8)\\
&= -\frac{i}{2} - \frac{i}{2} \qquad (A.9)\\
&= -i \qquad (A.10)
\end{aligned}$$

Hence we conclude that

$$[\hat{T}_{\text{Dirac}}, \hat{H}] = -i \qquad (A.11)$$

Appendix B

Derivation of Equation (5.31) from Equation (5.30)

Let $\hat{L} = \gamma^{1/2}\hat{V}(x)\hat{\sigma}_-$ where $\hat{L}$ is a linear combination of Lindbladian operators [62], and $\sigma_-^\dagger = \sigma_+$.

Then we have

$$\begin{aligned}
\dot{\hat{\rho}} &= -\frac{i}{\hbar}[\hat{H}_s + \hat{H}_d, \hat{\rho}] + \sum_m \left(\hat{L}_m\rho\hat{L}_m^\dagger - \frac{1}{2}\hat{L}_m^\dagger\hat{L}_m\rho - \frac{1}{2}\rho\hat{L}_m^\dagger\hat{L}_m\right) && \text{(B.1)}\\
&= -\frac{i}{\hbar}[\hat{H}_s + \hat{H}_d, \hat{\rho}] + \left(\hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\hat{L}^\dagger\hat{L}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}^\dagger\hat{L}\right) && \text{(B.2)}\\
&= -\frac{i}{\hbar}[\hat{H}_s + \hat{H}_d, \hat{\rho}] + \left[\gamma^{1/2}\hat{V}(x)\sigma_-\hat{\rho}(\gamma^{1/2}\hat{V}(x)\hat{\sigma}_-)^\dagger\right] && \text{(B.3)}\\
&- \frac{1}{2}\left[\left(\gamma^{1/2}\hat{V}(x)\hat{\sigma}_-\right)^\dagger\left(\gamma^{1/2}\hat{V}(x)\hat{\sigma}_-\right)\hat{\rho}\right]\\
&- \frac{1}{2}\left[\hat{\rho}\left(\gamma^{1/2}\hat{V}(x)\hat{\sigma}_-\right)^\dagger\left(\gamma^{1/2}\hat{V}(x)\hat{\sigma}_-\right)\right]\\
&= -\frac{i}{\hbar}[\hat{H}_s + \hat{H}_d, \hat{\rho}] + \gamma\left(\hat{V}(x)\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+\hat{V}(x)\right) && \text{(B.4)}\\
&- \frac{\gamma}{2}\left(\hat{V}^2(x)\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}\right)\\
&- \frac{\gamma}{2}\left(\hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\hat{V}^2(x)\right).
\end{aligned}$$

By tidying up equation (B.4) we obtain equation (5.31), that is:

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}[\hat{H}_s + \hat{H}_d, \hat{\rho}] - \frac{\gamma}{2}\left(\hat{V}^2(x)\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} + \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\hat{V}^2(x) - 2\hat{V}(x)\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+\hat{V}(x)\right). \quad \text{(B.5)}$$