Generalised Geometry and Type II Supergravity

Peiyan Hu

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Professor Daniel Waldram

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Abstract

We construct the $O(d,d)$ generalised geometry on a generalised tangent bundle which is isomorphic to the sum of a tangent bundle and a cotangent bundle. We find that the generalised version of Lie bracket known as the Courant bracket on the generalised tangent bundle is preserved by the diffeomorphism and gauge transformations. We then construct the generalised metric, connection, torsion and curvature on the generalised tangent bundle. We find that the generalised metric encodes the metric and the 2-form B-field, and there is no unique torsion-free connection or a unique curvature. We then use the extended $O(d,d) \times \mathbb{R}^+$ generalised geometry to reformulate 10-dimensional type II supergravity, where the fields in the NSNS sector namely the metric g, the 2-form B field and the dilaton ϕ are all encoded in the extended generalised metric, and the bosonic symmetries are encoded in the symmetry group that preserves the Courant bracket. Then by using the spinor fields, we find a unique curvature scalar which can be used to rewrite the bosonic action and derive equations of motion which are found to be just like a generalised geometric version of Einstein gravity. However the torsion-free connection is still not unique. Finally, we define generalised parallelisable manifold as analogue to local group manifold to explain consistent truncations on spheres. We show that all spheres are generalised parallelisable, and the generalised Scherk-Schwarz reduction on d-sphere gives a gauge group defined by the Lie algebra on the frames of generalised tangent bundle on the sphere, namely $\mathfrak{so}(d+1)$. This is direct analogue to the usual Scherk-Schwarz reduction. We then perform generalised Scherk-Schwarz reduction on S^3 which gives the gauge group with Lie algebra $\mathfrak{so}(4)$ and ansatz which are the same as usual Scherk-Schwarz reduction.

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Contents

1 Introduction

Geometry is very important to physics for its applications in theories of gravities such as general relativity and manifestation of symmetries through Lie groups which are differentiable manifolds. It is particularly useful in string theory and the low energy limit theory supergravity, as string theory is a theory of gravity with many symmetries. In this dissertation, we give a review on a new geometry called generalised geometry and its applications in type II supergravity.

Generalised geometry developed by Hitchin and Gualtieri [1, 2] is a new type of geometry defined on fibre bundles. It has many structures similar to the usual differential geometry. Firstly, one can define the generalised version of Lie derivative and Lie bracket called Dorfman derivative and Courant bracket which encode symmetries. Then the generalised geometric objects namely the generalised metric, connection, torsion and curvature can also be constructed on it as in the usual Riemannian geometry, although there are some problems with them such as the uniqueness of a torsion-free connection and curvature scalar [2, 3, 4]. Generalised geometry is not specially designed for physics or string theory, but a key property is that it describes diffeomorphism and gauge transformation geometrically at the same time, whereas usual geometry only describes diffeomorphism. In the original work [2], the generalised geometry is developed on $T\mathcal{M} \oplus T^*\mathcal{M}$ for a d-dimensional differentiable manifold $\mathcal{M}$. The symmetry transformations that preserve the Courant bracket are diffeomorphism and a 2-form gauge transformation called B-transform, which generate the geometric subgroup, and mathematically this structure leads to the Courant algebroid [2, 5, 6, 7]. The B-transform then defines a generalised tangent bundle E as an exact sequence through splitting [1, 2, 8, 9]

$$0 \to T^*\mathcal{M} \to E \to T\mathcal{M} \to 0 \tag{1.1}$$

and gives the isomorphism $E \simeq T\mathcal{M} \oplus T^*\mathcal{M}$. The generalised metric constructed on E encodes a symmetric 2-tensor and a 2-form field which can be naturally identified to the metric and the 2-form B field in the NSNS sector of type II supergravity [2, 3, 4]. And by extending the generalised space to $\det T\mathcal{M} \otimes (T\mathcal{M} \oplus T^*\mathcal{M})$, the structure group is extended to $O(d,d) \times \mathbb{R}^+$ which provides one more degree of freedom for dilaton [10, 11], hence all the fields in NSNS sectors are included. The geometric subgroup is also found to be the group that encodes the bosonic symmetries in type II supergravity. Moreover, the structure group $O(d,d)$ of the generalised geometry has another transformation parameterised by an anti-symmetric bi-vector which describes the T-duality and non-geometric background in string theory [10]. These all motivate the reformulation of type II supergravity using generalised geometry.

In fact, supergravity was already being described using two copies of tangent bundles by Siegel [12, 13] and in the double field theory developed by Hall and Zwiebach [14, 15]. The generalised metric, structure group and connection given by them are equivalent to those from generalised geometry [4]. However, the power of generalised geometry is that one can define other generalised tangent bundle from other spaces, for example, an anti-symmetric products of tangent bundles. This will be used for the exceptional geometries and hence the geometries for M-theory and the $E_{(d)d} \times \mathbb{R}^+$ 11-dimensional supergravity [3, 16, 17]. Using the generalised geometry reformulation for supergravity, one can find the generalised versions of bosonic action and equations of motion are [4, 16, 17]

$$S_{\mathrm{B}} = \int \mathrm{vol}_G R, \quad R_{MN} = 0 \tag{1.2}$$

where G is the generalised metric, vol_G is the volume form associated to G, R is generalised curvature scalar and R_{MN} is generalised Ricci tensor. Hence, this is simply a generalised version of Einstein gravity theory.

Generalised geometry can do more than reformulation, it can also be used for consistent truncations. In order to cancel the conformal anomalies, string theories live in either 10 or 11 dimensional spacetime [18]. For it to describe the physical world, the extra dimensions must be compactified, and through consistent truncation, one can obtain a theory in the lower dimension. For supergravity, there are some consistent truncations on spheres which are truncation on S^3 near-horizon NS-fivebrane background [19], $AdS_7 \times S^4$ for 11-dimensional supergravity [20, 21], $AdS_5 \times S^5$ for type IIB supergravity [22, 23], $AdS_4 \times S^7$ for 11-dimensional supergravity [24, 25]. Although there is no systematic way of finding consistent truncations, it is known that a local group manifold $\mathcal{M} \simeq G/\Gamma$ gives consistent truncations [26, 27], where Γ is a freely acting discrete subgroup of G and defines a global frame $\{\hat{e}_a\}$ on $\mathcal{M}$ so $\mathcal{M}$ a parallelisable manifold. There is then a Lie algebra on the frame

$$[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c \tag{1.3}$$

and if the structure constant $f_{ab}{}^c$ satisfies the unimodular condition $f_{ab}{}^b = 0$ which is satisfied by compact Lie group, there is a consistent truncation with gauge group defined by the Lie algebra above [28, 29]. This implies that a parallelisable manifold which is also a compact Lie group gives consistent truncation. It explains the truncations on S^1 and S^3 for they are both parallelisable and are compact Lie group. But S^4, S^5 are not parallelisable, and although S^7 is parallelisable it is not a Lie group, so they remain mysterious. However, as analogue to the local group manifold, one can define generalised parallelisable manifold in generalised geometry as

$$[\![\hat{E}_A, \hat{E}_B]\!] = X_{AB}{}^C \hat{E}_C \tag{1.4}$$

where $\hat{E}_A$ is global frame on the generalised tangent bundle, ${X_{AB}}^C$ is constant and $[\![,]\!]$ is the Courant bracket which defines a Lie algebra on the frame [10]. And a conjecture [30, 31, 10, 32] states that a generalised parallelisable manifold gives consistent truncation that preserves same number of supersymmetries, and the gauge group of the truncated theory is given by the Lie algebra defined by ${X_{AB}}^C$. It is then proven that all spheres are generalised parallelisable [32], hence the conjecture explains the consistent truncations on S^4, S^5, S^7. However, one should note that the conjecture is not proven, but the Scherk-Schwarz reduction on spheres in generalised geometry indeed give the same ansatz as those from usual Scherk-Schwarz reduction [32].

Furthermore, there are some other applications of generalised geometry in string theory such as the supersymmetric flux compactification, calibrations [33] and it may also be able to describe mirror symmetry [2]. These will not be included in this dissertation, but they all suggest that generalised geometry is a very powerful tool in string theory.

The dissertation is structured as following. In Chapter 1, we construct the $O(d,d)$ generalised geometry on the $T\mathcal{M} \oplus T^*\mathcal{M}$ with a canonical inner product and Dorfman derivative and Courant bracket. The generalised tangent bundle will also be defined at the end. In Chapter 2, we construct the generalised metric, vielbein, connection, torsion and curvature. Chapter 3 will reformulate the bosonic part of type II supergravity with a slight modified $O(d,d) \times \mathbb{R}^+$ generalised geometry. In Chapter 4, we define the generalised parallelisability and show that spheres are generalised parallelisable, then we give a connection between generalised parallelisability and consistent truncation to explain the truncations on spheres.

2 $O(d,d)$ Generalised Geometry

Generalised geometry was first introduced by Hitchin [1] and Gualtieri [2] by considering a generalised space $T\mathcal{M} \oplus T^*\mathcal{M}$ constructed from the tangent bundle $T\mathcal{M}$ and co-tangent bundle $T^*\mathcal{M}$ of a d-dimensional differentiable manifold. It admits a natural canonical inner product which defines some symmetry structures on the generalised space. The symmetries will be described using generalised versions of Lie derivative and Lie bracket which are also known as the Dorfman derivative and Courant brackets, and their similarities and differences with the Lie derivative and Lie bracket will be discussed. At the end of the chapter, the generalised tangent bundle will be formally defined from a 'twisted' structure and will be shown to be isomorphic to the generalised space. The construction mainly follows [2] and other references are [3, 34, 33].

2.1 G-structure

Before introducing the generalised geometry, we need the concept of G-structure which will be used to define geometric structures throughout the dissertation. It is defined using fibre bundles for which the informations can be found in Appendix C or refer to [35]. The main reference for this section is [33].

For a d-dimensional manifold $\mathcal{M}$, its tangent frame bundle $\mathcal{FM}$ has the structure group $GL(d,\mathbb{R})$ (Definition A.30). If the tangent frame bundle can be reduced such that the structure group is reduced to a proper subgroup $G \subset GL(d,\mathbb{R})$, then there exists a G-structure.

Definition 2.1. A G-structure of a manifold $\mathcal{M}$ is the principle G-sub-bundle of the tangent frame bundle $\mathcal{FM}$.

Example 2.1. If the manifold is parallelisable (Definition A.33), then from Theorem A.19 and A.18, its tangent bundle is trivial, so by Definition A.32, the structure group of the tangent bundle reduces to the trivial group. Since the tangent frame bundle is the associated principal bundle of the tangent bundle, then from Theorem A.16, the structure group of the tangent frame bundle also reduces to the trivial group. Therefore, there exists a G-structure on a parallelisable manifold.

The G-structure can be described by a globally defined non-degenerate tensor (section of tensor product of fibre bundles) that is invariant under the transformation by G. This tensor may not be unique, so there can be several different tensors describing the same G-structure. An example of the G-invariant tensor is the metric in the usual geometry.

Example 2.2. The general structure group on a tangent bundle $T\mathcal{M}$ is $GL(d,\mathbb{R})$, so the basis $\{e_\mu\}$ transforms as $e_\mu = A_\mu{}^\nu e_\nu$ where $A \in GL(d,\mathbb{R})$. If one defines a positive definite symmetric 2-tensor $g \in \Gamma(S^2T^*\mathcal{M})$ i.e. a metric, and requires that the metric is invariant under the structure group, then one has the condition

$$g_{\mu\nu} = g(e_\mu, e_\nu) = g(e'_\mu, e'_\nu) = A_\mu{}^{\mu'} A_\nu{}^{\nu'} g_{\mu'\nu'} \tag{2.1}$$

so A needs to be an element of $O(d) \subset GL(d, \mathbb{R})$ hence the structure group reduces to $O(d)$. This can be understand as that the metric breaks the GL symmetry, and the orthogonal group that preserve the metric is the residue symmetry group. Hence, the metric g parametrise the coset space

$$g \in GL(d, \mathbb{R})/O(d). \tag{2.2}$$

If one also requires an orientation by defining a volume form globally, then to preserve the orientation, the determinant of transformation matrix needs to be positive so the reduced structure group can be further restricted to $SO(d)$.

One can also introduce more tensors to further reduce structure group to a sub-group of G. Some common structure are the complex structure, symplectic structure which are discussed in [33, 2]. Here we introduce the almost product structure which will be used to define the generalised metric for the generalised geometry.

Definition 2.2. An almost product structure is a globally defined non-degenerate tensor $S \in \Gamma(T\mathcal{M} \otimes T^*\mathcal{M})$ satisfying $S^2 = \mathbf{1}$.

The almost product structure has both $+1$ and -1 eigenvalues, and the bundle splits into two sub-bundles C_+ for $+1$-eigen-space and C_- for -1-eigenspace with dimensions p and $d-p$ respectively [33]. The almost product structure reduces the structure group from $GL(d, \mathbb{R})$ to $GL(p, \mathbb{R}) \times GL(d-p, \mathbb{R})$, and $GL(p, \mathbb{R})$ and $GL(d-p, \mathbb{R})$ are structure groups of C_+ and C_- respectively. Furthermore, if there is also a metric g, and the almost product structure satisfies the orthogonality condition

$$S^T g S = g \tag{2.3}$$

then the structure group reduces to $O(p) \times O(d-p)$ [33].

2.2 Generalised Space and Canonical Inner Product

For a d-dimensional manifold $\mathcal{M}$, the generalised space[1] is defined to be the direct sum of its tangent bundle and cotangent bundle $T \oplus T^*$ which is also a fibre bundle (Definition

[1]The term 'generalised tangent bundle' will not be used at this stage, it will be formally defined in Section 2.7. And the tangent bundle will simply be detonated as T in this chapter, so is T^*.

A.26). Locally, for a vector $X \in \Gamma(T)$ and a 1-form $\xi \in \Gamma(T^*)$, the section of $T \oplus T^*$ is

$$V = X + \xi = \begin{pmatrix} X \\ \xi \end{pmatrix} \in \Gamma(T \oplus T^*) \tag{2.4}$$

which is called the generalised vector. One can also define the rank-n generalised tensor as section of tensor product of generalised space $T \oplus T^*$

$$J \in \Gamma((T \oplus T^*)^{\otimes n}) \tag{2.5}$$

where $(T \oplus T^*)^{\otimes n} = (T \oplus T^*) \otimes \cdots \otimes (T \oplus T^*)$ (Definition A.27).

For $V, V' \in \Gamma(T \oplus T^*)$, there is a natural symmetric bilinear form defined on $T \oplus T^*$

$$\langle V, V' \rangle = \langle X + \xi, X' + \xi' \rangle = \frac{1}{2}(i_X \xi + i_{X'} \xi') = \frac{1}{2}(X^i \xi_i + X'^i \xi'_i) \tag{2.6}$$

which is non-degenerate and can be interpreted as the inner product with signature (d, d) and is maximally indefinite[2] [2, 3, 33].

Similarly to the ordinary case, at each $x \in \mathcal{M}$, there is a generalised basis $\{\hat{E}_I\}$ where $\hat{E}_I$ are linearly independent sections of $\Gamma(T \oplus T^*)$ so that a generalised vector V can be written as $V = V^I \hat{E}_I$ with components [4]

$$V^I = \begin{cases} X^i, & I = i, \\ \xi_i, & I = i + d. \end{cases} \tag{2.7}$$

The inner product can then be written in components as

$$\langle V, V' \rangle = \frac{1}{2} \begin{pmatrix} X & \xi \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} X' \\ \xi' \end{pmatrix} = \eta_{IJ} V^I V'^J \tag{2.8}$$

where $I, J = 1, ..., 2d$, $\mathbf{1}$ is the d-dimensional identity matrix, and η_{IJ} is the metric components given by

$$\eta = \frac{1}{2} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{2.9}$$

And the inverse metric is

$$\eta^{-1} = 2 \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{2.10}$$

These can be used to raise or lower index of generalised vectors. This indefinite metric is called the canonical metric and is to be distinguished from the generalised metric defined later. There is also a natural volume form associated with the metric which defines a

[2] Indefinite means that the metric can take positive or negative value.

canonical orientation [2].

There is a natural basis for the generalised space defined by the coordinates $\{x^\mu\}$ of $\mathcal{M}$ as [4]

$$\hat{E}_M = \begin{cases} e_\mu = \partial_\mu = \frac{\partial}{\partial x^\mu}, & M = \mu, \\ e^\mu = dx^\mu, & M = \mu + d, \end{cases} \tag{2.11}$$

satisfying

$$\langle \hat{E}_M, \hat{E}_N \rangle = \eta_{MN}. \tag{2.12}$$

A generalised vector V can now be written as

$$V = V^M \hat{E}_M = X^\mu \frac{\partial}{\partial x^\mu} + \xi_\mu dx^\mu. \tag{2.13}$$

And one can define the generalised frame bundle

$$\mathcal{F} = \{(x, \{E_M\}) : x \in \mathcal{M}, \{E_M\} \text{ is basis of } (T \oplus T^*)_x\}. \tag{2.14}$$

2.3 Structure Group

The structure group of $T \oplus T^*$ is naturally $GL(d, \mathbb{R})$ which is the same as that of the tangent bundle T [3], but as mentioned in Section 2.1, the existence of a metric reduces the structure group to the symmetry group that preserves the indefinite metric, so the reduced structure group is the non-compact indefinite orthogonal group $O(d, d)$ defined as

$$O(d, d) = \{M \in GL(2d, \mathbb{R}) : M^T \eta M = \eta\}. \tag{2.15}$$

The reason for it being non-compact will be shown soon. The canonical orientation defined by the volume form associated with the inner product can further reduce the symmetry group to the subgroup $SO(d, d)$ [2]. Hence, the generalised vector $V = V^I \hat{E}_I$ and the basis transform under the structure group as

$$\begin{matrix} V^I \to V'^I = M^I{}_J V^J \\ \hat{E}_I \to \hat{E}'_I = (M^{-1})^J{}_I \hat{E}_J \end{matrix}, \quad M \in O(d, d). \tag{2.16}$$

The Lie algebra of $SO(d, d)$ is derived in Appendix E and is as usual,

$$\mathfrak{so}(d, d) = \{M \in GL(2d, \mathbb{R}) : M^T \eta + \eta M = 0\}. \tag{2.17}$$

The generator M is also derived in Appendix E and is

$$\begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix} \tag{2.18}$$

where B and β are anti-symmetric i.e. $B = -B^T, \beta = -\beta^T$. A, B, β generates three subgroups and gives the following three transformations [2]:

- GL-action: There is no constraints on A, so A is an element of $GL(d, \mathbb{R})$ which is the structure group of the tangent bundle, and A generates the $GL(d, \mathbb{R})$ subgroup e^A on the tangent bundle T. This can be extended into the $T \oplus T^*$ as

$$e^A \to e^A_{T\oplus T^*} = \begin{pmatrix} e^A & 0 \\ 0 & e^{-A^T} \end{pmatrix}. \tag{2.19}$$

  A generalised vector $V \in \Gamma(T \oplus T^*)$ transforms under the GL-action as

$$V = \begin{pmatrix} X \\ \xi \end{pmatrix} \to e^A_{T\oplus T^*} V = \begin{pmatrix} e^A X \\ e^{-A^T} \xi \end{pmatrix}, \tag{2.20}$$

  so A is an endomorphism $A : T \to T$. With a slight abuse of notation, $e^A_{T\oplus T^*}$ will be denoted as e^A.

- B-transform: B can be interpreted as a 2-form i.e. $B \in \wedge^2 T^*$, generating the group of

$$e^B = \begin{pmatrix} \mathbf{1} & 0 \\ B & \mathbf{1} \end{pmatrix}. \tag{2.21}$$

  Then the generalised vector $V = X + \xi$ transforms as $V \to e^B V$ and in components as

$$\begin{pmatrix} X \\ \xi \end{pmatrix} \to \begin{pmatrix} X \\ \xi + i_X B \end{pmatrix} \tag{2.22}$$

  so B is a map from T to T^* that maps X to $i_X B$.

- β-transform: β can be considered as a bi-vector i.e. $\beta \in \wedge^2 T$, generating the group of

$$e^\beta = \begin{pmatrix} \mathbf{1} & \beta \\ 0 & \mathbf{1} \end{pmatrix}. \tag{2.23}$$

  It acts on $V = X + \xi$ as $V \to e^\beta V$ and in components as

$$\begin{pmatrix} X \\ \xi \end{pmatrix} \to \begin{pmatrix} X + i_\xi \beta \\ \xi \end{pmatrix} \tag{2.24}$$

  where the interior product is generalised so that i_ξ is a map from bi-vector space $\wedge^2 T$ to vector space T i.e. $i_\xi : \wedge^2 T \to T$. Hence β is a map from T^* to T that maps ξ to $i_\xi \beta = \beta\xi$.

The B and β transforms turn vectors into forms and forms into vectors, so they are similar to the boost transformation of the Lorentz group except that the vectors and forms are both null with respect to the canonical metric. Since they are non-compact transformations and are subgroups of the $O(d, d)$ group, then the $O(d, d)$ group is also non-compact.

The structure group describes how generalised vectors transform. The GL-action corresponds to the diffeomorphism, B-transform corresponds to gauge symmetry and β-transform is related to T-dualities [10]. The subgroup generated by A and B is particularly interesting because they contain both of diffeomorphism and the gauge symmetry that appear in supergravity. This will be discussed in Section 2.6.

2.4 The Dorfman Derivative

Dorfman derivative is an operations on two generalised vectors which is defined as a generalisation of the Lie derivative. It combines the diffeomorphism and gauge symmetry, which will be shown explicitly in the Section 2.6 in terms of the Courant bracket.

Definition 2.3. For generalised vectors $V = X + \xi, W = Y + \eta \in \Gamma(T \oplus T^*)$, with $X, Y \in \Gamma(T)$ and $\xi, \eta \in \Gamma(T^*)$, the Dorfman derivative is

$$L_V W = \mathcal{L}_X Y + (\mathcal{L}_X \eta - i_Y d\xi) \tag{2.25}$$

where $\mathcal{L}$ is the Lie derivative on vectors and $\mathcal{L}_X Y = [X, Y]$.

Unlike the Lie derivative on vectors, the Dorfman derivative is not anti-symmetric in the two generalised vectors, but it still satisfies some properties similar to the Lie derivative. It can be easily shown that the Dorfman derivative is bi-linear: for $U, V, W \in \Gamma(T \oplus T^*), a, b \in \mathbb{R}$,

$$\begin{aligned} L_U(aV + bW) &= aL_U V + bL_U W \\ L_{aU+bV} W &= aL_U W + bL_V W \end{aligned} \tag{2.26}$$

Proposition 2.1. *The Dorfman derivative satisfies a Leibniz rule. For $U, V, W \in \Gamma(T \oplus T^*)$,*

$$L_U(L_V W) = L_{L_U V} W + L_V(L_U W). \tag{2.27}$$

To see it more clearly, denote the Dorfman derivative $L_V W$ as $V \circ W$ then

$$U \circ (V \circ W) = (U \circ V) \circ W + V \circ (U \circ W) \tag{2.28}$$

Proof. Let $U = X + \xi$, $V = Y + \eta$, $W = Z + \zeta$, then

$$\begin{aligned} L_{L_U V} W + L_V (L_U W) &= [[X,Y],Z] + \mathcal{L}_{[X,Y]}\zeta - i_Z d(\mathcal{L}_X \eta - i_Y d\xi) \\ &= [Y,[X,Z]] + \mathcal{L}_Y(\mathcal{L}_X \zeta - i_Z d\xi) - i_{[X,Z]} d\eta \end{aligned} \tag{2.29}$$

The anti-symmetric property and the Jacobi identity of Lie bracket gives

$$[[X,Y],Z] + [Y,[X,Z]] = [X,[Y,Z]] \tag{2.30}$$

and using (A.18) and (A.33) gives

$$\mathcal{L}_{[X,Y]}\zeta = \mathcal{L}_X \mathcal{L}_Y \zeta - \mathcal{L}_Y \mathcal{L}_Z \zeta, \quad i_{[X,Z]} d\eta = \mathcal{L}_X i_Z d\eta - i_Z \mathcal{L}_X d\eta \tag{2.31}$$

so some terms cancel and leaving

$$\begin{aligned} L_{L_U V} W + L_V (L_U W) &= [[X,Y],Z] + \mathcal{L}_X(\mathcal{L}_Y \zeta - i_Z d\eta) - \mathcal{L}_Y i_Z d\xi + i_Z d i_Y d\xi \\ &= [[X,Y],Z] + \mathcal{L}_X(\mathcal{L}_Y \zeta - i_Z d\eta) - i_{[Y,Z]} d\xi \\ &= L_U(L_V W). \end{aligned} \tag{2.32}$$

And since it is not anti-symmetric, there is no Jacobi identity. □

It is useful to define a projection map [2]

$$\begin{aligned} \pi: \quad T \oplus T^* &\rightarrow T \\ X, \xi &\rightarrow \pi(X + \xi) = X \end{aligned} \tag{2.33}$$

so that the Dorfman derivative can be extended to the tangent bundle and reduced to Lie derivative as

$$\pi(L_V W) = \mathcal{L}_X Y \tag{2.34}$$

where $V = X + \xi$. And similar to Lie derivative, for a function f, $V, W \in \Gamma(T \oplus T^*)$, the Dorfman derivative satisfies

$$L_V(fW) = f L_V W + \pi(V)[f] W \tag{2.35}$$

or acts on function f purely as

$$L_V f = \pi(V)[f] = X[f] = \mathcal{L}_X f \tag{2.36}$$

where $V = X + \xi$.

Proposition 2.2. *The canonical inner product and the Dorfman derivative satisfy*

$$\pi(U)[\langle V, W\rangle] = \langle L_U V, W\rangle + \langle V, L_U W\rangle \,. \tag{2.37}$$

Proof. Let $U = X + \xi, V = Y + \eta, W = Z + \zeta$, the right hand side is

$$\begin{aligned}
&\langle L_U V, W\rangle + \langle V, L_U W\rangle \\
&= \langle [X,Y] + (\mathcal{L}_X \eta - i_Y d\xi), Z + \zeta\rangle + \langle Y + \eta, [X,Z] + (\mathcal{L}_X \zeta - i_Z d\xi)\rangle \\
&= \frac{1}{2}(i_{[X,Y]}\zeta + i_Z(\mathcal{L}_X \eta - i_Y d\xi)) + \frac{1}{2}(i_{[X,Z]}\eta + i_Y(\mathcal{L}_X \zeta - i_Z d\xi))
\end{aligned} \tag{2.38}$$

using (A.33) and (A.30) give

$$\begin{aligned}
\langle L_U V, W\rangle + \langle V, L_U W\rangle &= \frac{1}{2}([\mathcal{L}_X, i_Y]\zeta + i_Z\mathcal{L}_X\eta - i_Z i_Y d\xi + [\mathcal{L}_X, i_Z]\eta + i_Y\mathcal{L}_X\zeta - i_Y i_Z d\xi) \\
&= \frac{1}{2}(\mathcal{L}_X i_Y \zeta + \mathcal{L}_X i_Z \eta) = \frac{1}{2}(L_X i_Y\zeta + L_X i_Z \eta) \\
&= \frac{1}{2}(-L_{i_Y\zeta}X - L_{i_Z\eta}X) = \frac{1}{2}(i_X d i_Y \zeta + i_X d i_Z \eta) \\
&= i_X d\frac{1}{2}(i_Y\zeta + i_Z\eta) \\
&= i_X d\,\langle V, W\rangle
\end{aligned} \tag{2.39}$$

and finally, since $\langle V, W\rangle$ is a scalar, using (A.35) gives

$$\langle L_U V, W\rangle + \langle V, L_U W\rangle = i_X d\,\langle V, W\rangle = X[\langle V, W\rangle] = \pi(U)[\langle V, W\rangle] \tag{2.40}$$

□

The Dorfman derivative can also be written in components in the coordinate basis (2.11). For two generalised vectors $V = X + \xi, W = Y + \eta$, terms in the RHS of Dorfman derivative (2.25) can be calculated in components,

$$\begin{aligned}
\mathcal{L}_X Y^\mu &= X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu \\
\mathcal{L}_X \eta_\mu &= X^\nu \partial_\nu \eta_\mu + \eta_\nu \partial_\mu X^\nu \\
i_Y d\xi_\mu &= Y^\nu \partial_\nu \xi_\mu - Y^\nu \partial_\mu \xi_\nu
\end{aligned} \tag{2.41}$$

and writing in column gives

$$
\begin{aligned}
L_V W^M &= \begin{pmatrix} \mathcal{L}_X Y^\mu \\ \mathcal{L}_X \eta_\mu - i_Y d\xi_\mu \end{pmatrix} \\
&= \begin{pmatrix} X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu \\ X^\nu \partial_\nu \eta_\mu + \eta_\nu \partial_\mu X^\nu - Y^\nu \partial_\nu \xi_\mu + Y^\nu \partial_\mu \xi_\nu \end{pmatrix} \\
&= \begin{pmatrix} X^\nu \partial_\nu Y^\mu \\ X^\nu \partial_\nu \eta_\mu \end{pmatrix} - \begin{pmatrix} Y^\nu \partial_\nu X^\mu \\ Y^\nu \partial_\nu \xi_\mu \end{pmatrix} + \begin{pmatrix} 0 \\ \eta_\nu \partial_\mu X^\nu + Y^\nu \partial_\mu \xi_\nu \end{pmatrix}
\end{aligned} \tag{2.42}
$$

The partial derivative can also be embedded into the generalised space $T \oplus T^*$ as

$$
\partial_M = \begin{cases} \partial_\mu, & M = \mu, \\ 0, & M = \mu + d. \end{cases} \tag{2.43}
$$

so $X^\nu \partial_\nu = V^N \partial_N$. Then using the canonical metric and its inverse metric to raise and lower the indices in the last term gives the components of Dorfman derivative

$$
L_V W^M = V^N \partial_N W^M - W^N \partial_N V^M + W_N \partial^M V^N. \tag{2.44}
$$

This expression is the same as the double field theory formulation [12, 13, 36]. The action of Dorfman derivative can also be generalised to a rank-n generalised tensor $J \in \Gamma((T \oplus T^*)^{\otimes n})$ as [4]

$$
\begin{aligned}
L_V J^{M_1 \dots M_n} = V^N \partial_N J^{M_1 \dots M_n} &+ (\partial^{M_1} V^N - \partial^N V^{M_1}) J_N{}^{M_2 \dots M_n} \\
&+ \dots + (\partial^{M_n} V^N - \partial^N V^{M_n}) J^{M_1 \dots M_{n-1}}{}_N
\end{aligned} \tag{2.45}
$$

which is an analogue of the Lie derivative action on a usual tensor (A.11), and implies that a generalised tensor can be viewed as sections of different representations of the structure group $O(d, d)$.

2.5 The Courant Bracket

Similar to the Dorfman derivative, Courant bracket is the analogue of Lie bracket on the generalised space $T \oplus T^*$ [1, 2]. The motivation for its definition is to construct a bracket that is anti-symmetric, so it is defined by anti-symmetrising the Dorfman derivative.

Definition 2.4. For generalised vectors $V = X + \xi, Y = Y + \eta \in \Gamma(T \oplus T^*)$, $X, Y \in \Gamma(T)$ and $\xi, \eta \in \Gamma(T^*)$, the Courant bracket is

$$
[\![V, W]\!] = \frac{1}{2}(L_V W - L_W V). \tag{2.46}
$$

The Dorfman derivatives are explicitly written as

$$\begin{aligned} L_V W &= [X,Y] + \mathcal{L}_X \eta - (d i_Y \xi - \mathcal{L}_Y \xi) \\ L_W V &= -[X,Y] + \mathcal{L}_Y \xi - (d i_X \eta - \mathcal{L}_X \eta) \end{aligned} \tag{2.47}$$

where $\mathcal{L}_X Y = [X,Y] = -[Y,X]$, and the identity (A.32) gives $i_X d = d i_X - \mathcal{L}_X$. Hence substitute the Dorfman derivatives back into the definition of Courant bracket gives

$$[\![V,W]\!] = [\![X+\xi, Y+\eta]\!] = [X,Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi). \tag{2.48}$$

Claim 2.3. *A similar calculation shows that*

$$\frac{1}{2}(L_V W + L_W V) = d \langle V, W \rangle, \tag{2.49}$$

hence

$$[\![V,W]\!] = L_V W - d \langle V, W \rangle, \tag{2.50}$$

where d is exterior derivative and $\langle , \rangle$ is the canonical inner product on $T \oplus T^*$.

Proposition 2.4. *The Dorfman derivative and the Courant bracket has a relation similar to (A.18)*

$$L_U(L_V W) - L_V(L_U W) = L_{[\![U,V]\!]} W. \tag{2.51}$$

Proof. Using (2.50) and bi-linearity of Dorfman derivative, RHS can be rewritten as

$$L_{[\![U,V]\!]} W = L_{L_U V - d\langle U,V \rangle} W = L_{L_U V} W - L_{d\langle U,V\rangle} W = L_{L_U V} W \tag{2.52}$$

where $L_{d\langle U,V\rangle} W = 0$ using the definition of Dorfman derivative and $d^2 = 0$. Hence using the Leibniz rule for Dorfman derivative (2.27) one obtains (2.51). □

One can easily see that the Courant bracket inherits bi-linearity from the Dorfman derivative and from its definition (2.46) that it is anti-symmetric. However, the Courant bracket does not satisfy the Jacobi identity which means it is not a Lie bracket [2, 34].

Proposition 2.5. *For $U, V, W \in \Gamma(T \oplus T^*)$, the Courant bracket satisfies the Jacobi identity up to an exact term*

$$\begin{aligned} &[\![[\![U,V]\!],W]\!] + [\![[\![V,W]\!],U]\!] + [\![[\![W,U]\!],V]\!] \\ &\qquad = \frac{1}{3} d(\langle [\![U,V]\!], W\rangle + \langle [\![V,W]\!], U\rangle + \langle [\![W,U]\!], V\rangle), \end{aligned} \tag{2.53}$$

where d is exterior derivative and $\langle , \rangle$ is the canonical inner product on $T \oplus T^*$.

Proof. Firstly, due to bi-linearity and anti-symmetry of Courant bracket, one has

$$\begin{aligned}
[\![[\![U,V]\!],W]\!] &= [\![\frac{1}{2}([\![U,V]\!]-[\![V,U]\!]),W]\!] \\
&= \frac{1}{2}([\![[\![U,V]\!],W]\!]-[\![[\![V,U]\!],W]\!]) \\
&= \frac{1}{4}([\![[\![U,V]\!],W]\!]-[\![W,[\![U,V]\!]]\!]-[\![[\![V,U]\!],W]\!]+[\![W,[\![V,U]\!]]\!])
\end{aligned} \tag{2.54}$$

Also, using (2.50) and (2.51) gives

$$\begin{aligned}
[\![[\![U,V]\!],W]\!] &= L_{[\![U,V]\!]}W - d\left\langle [\![U,V]\!],W\right\rangle \\
&= L_U(L_V W) - L_V(L_U W) - d\left\langle [\![U,V]\!],W\right\rangle
\end{aligned} \tag{2.55}$$

and substitute into (2.54) gives

$$\begin{aligned}
[\![[\![U,V]\!],W]\!] &= \frac{1}{4}(L_U(L_V W) - L_V(L_U W) - d\left\langle [\![U,V]\!],W\right\rangle - [\![W,[\![U,V]\!]]\!] \\
&\quad -L_V(L_U W) + L_U(L_V W) + d\left\langle [\![V,U]\!],W\right\rangle + [\![W,[\![V,U]\!]]\!]) \\
&= \frac{1}{4}(L_U(L_V W) - L_V(L_U W) - d\left\langle [\![U,V]\!],W\right\rangle \\
&\quad -L_W(L_U V) + L_W(d\left\langle U,V\right\rangle) + d\left\langle W,[\![U,V]\!]\right\rangle \\
&\quad -L_V(L_U W) + L_U(L_V W) + d\left\langle [\![V,U]\!],W\right\rangle \\
&\quad +L_W(L_V U) - L_W(d\left\langle V,U\right\rangle) - d\left\langle W,[\![V,U]\!]\right\rangle) \\
&= \frac{1}{4}(2L_U(L_V W) - 2L_V(L_U W) - L_W(L_U V) + L_W(L_V U))
\end{aligned} \tag{2.56}$$

where all terms with inner products cancel since it is symmetric. Therefore,

$$\begin{aligned}
&[\![[\![U,V]\!],W]\!] + [\![[\![V,W]\!],U]\!] + [\![[\![W,U]\!],V]\!] \\
&\quad = \frac{1}{4}(2L_U(L_V W) - 2L_V(L_U W) - L_W(L_U V) + L_W(L_V U) \\
&\qquad +2L_V(L_W U) - 2L_W(L_V U) - L_U(L_V W) + L_U(L_W V) \\
&\qquad +2L_W(L_U V) - 2L_U(L_W V) - L_V(L_W U) + L_V(L_U W)) \\
&\quad = \frac{1}{4}(L_U(L_V W) - L_V(L_U W) + L_V(L_W U) - L_W(L_V U) + L_W(L_U V) - L_U(L_W V)) \\
&\quad = \frac{1}{4}(L_{[\![U,V]\!]}W + L_{[\![V,W]\!]}U + L_{[\![W,U]\!]}V) \\
&\quad = \frac{1}{4}([\![[\![U,V]\!],W]\!] + [\![[\![V,W]\!],U]\!] + [\![[\![W,U]\!],V]\!] \\
&\qquad + d\left\langle [\![U,V]\!],W\right\rangle + d\left\langle [\![V,W]\!],U\right\rangle + d\left\langle [\![W,U]\!],V\right\rangle)
\end{aligned} \tag{2.57}$$

and rearranging the equation gives

$$\frac{3}{4}[\![[\![U,V]\!],W]\!] + [\![[\![V,W]\!],U]\!] + [\![[\![W,U]\!],V]\!] = \frac{1}{4}(d\langle[\![U,V]\!],W\rangle + d\langle[\![V,W]\!],U\rangle + d\langle[\![W,U]\!],V\rangle) \quad (2.58)$$

□

The Courant bracket can also be reduced to Lie bracket using the projection map (2.33) so that

$$\pi([\![V,W]\!]) = [\pi(V),\pi(W)] \quad (2.59)$$

where $V,W \in \Gamma(T \oplus T^*)$, $\pi(V),\pi(W) \in T$ and $[,]$ is the Lie bracket. But unlike the Dorfman derivative case (2.35), the Leibniz rule of Courant bracket on a function has an extra exact term [2],

Proposition 2.6. *For a function f, the Courant bracket satisfies*

$$[\![V,fW]\!] = f[\![V,W]\!] + \pi(V)[f]W - \langle V,W\rangle\, df. \quad (2.60)$$

Proof. For $V = X+\xi, W = Y+\eta$,

$$[\![V,fW]\!] = [X,fY] + \mathcal{L}_X(f\eta) - \mathcal{L}_{fY}\xi - \frac{1}{2}d(i_X(f\eta) - i_{fY}\xi), \quad (2.61)$$

then using Leibniz rules of Lie derivative and Lie bracket and (A.34) yields

$$\begin{aligned}[\![V,fW]\!] =& f([X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)) \\ &+ X[f]Y + X[f]\eta \\ &- i_Y\xi df - \frac{1}{2}(i_X\eta - i_Y\xi)df \\ =& f[\![V,W]\!] + X[f]W - \langle V,W\rangle\, df.\end{aligned} \quad (2.62)$$

□

Mathematically, (2.53), (2.59), (2.60) and (2.37) define a Courant algebroid [5, 6, 7].

2.6 Symmetries of Courant Bracket

The Lie derivative and Lie bracket of vectors encodes diffeomorphism in ordinary geometry. This section will explicitly show how the Courant bracket encodes both of diffeomorphism and gauge symmetry which makes it useful to physics.

Firstly, the Courant bracket is invariant under diffeomorphism i.e. the GL-action generated by A in (2.19).

Proposition 2.7. *For $V, W \in \Gamma(T \oplus T^*)$, the Courant bracket satisfies*

$$e^A [\![V, W]\!] = [\![e^A V, e^A W]\!]. \tag{2.63}$$

Proof. For a diffeomorphism $f = e^A$ on tangent bundle T, its action can be embedded into the $T \oplus T^*$ as (2.19) and acts on generalised vector V as

$$V \to \begin{pmatrix} f_* & 0 \\ 0 & (f^{-1})^* \end{pmatrix} V \tag{2.64}$$

where f_* and $(f^{-1})^*$ push-forward vector and form parts of V respectively. For $V = X + \xi, W = Y + \eta$, using the fact that Lie bracket, Lie derivative, exterior derivative and interior product all preserve diffeomorphisms (see (A.19), (A.36) and (A.37)), the LHS of (2.63) is

$$\begin{aligned}
&f_* [\![V, W]\!] \\
&= f_*([X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)) \\
&= [f_* X, f_* Y] + \mathcal{L}_{f_* X} (f^{-1})^* \eta - \mathcal{L}_{f_* Y} (f^{-1})^* \xi - \frac{1}{2} d(i_{f_* X} (f^{-1})^* \eta - i_{f_* Y} (f^{-1})^* \xi) \\
&= [\![f_* X + (f^{-1})^* \xi, f_* Y + (f^{-1})^* \eta]\!] = [\![f_* V, f_* W]\!]
\end{aligned} \tag{2.65}$$

so $e^A [\![V, W]\!] = [\![e^A V, e^A W]\!]$. □

The Courant bracket is also invariant under the B-transforms for a closed B-form, which is linked to gauge transformation [3].

Proposition 2.8. *For $V, W \in \Gamma(T \oplus T^*)$ and a closed 2-form B i.e. $dB = 0$*

$$e^B [\![V, W]\!] = [\![e^B V, e^B W]\!]. \tag{2.66}$$

Proof. Let $V = X + \xi, W = Y + \eta$ then using the B-transformation (2.22) the RHS becomes

$$\begin{aligned}
[\![e^B V, e^B W]\!] &= [\![e^B (X + \xi), e^B (Y + \eta)]\!] \\
&= [\![X + \xi + i_X B, Y + \eta + i_Y B]\!] \\
&= [X, Y] + \mathcal{L}_X (\eta + i_Y B) - \mathcal{L}_Y (\xi + i_X B) - \frac{1}{2} d(i_X (\eta + i_Y B) - i_Y (\xi + i_X B)) \\
&= [\![X + \xi, Y + \eta]\!] + \mathcal{L}_X i_Y B - \mathcal{L}_Y i_X B - \frac{1}{2} d(i_X i_Y B - i_Y i_X B)
\end{aligned} \tag{2.67}$$

Using the Cartan formula (A.32) and anti-symmetry of interior product, the last term becomes $-i_X i_Y dB - \mathcal{L}_X i_Y B + i_X \mathcal{L}_Y B$, and substitute back gives

$$[\![e^B V, e^B W]\!] = [\![V, W]\!] + i_X \mathcal{L}_Y B - \mathcal{L}_Y i_X B - i_X i_Y dB \tag{2.68}$$

and use (A.33) so that $i_X \mathcal{L}_Y B - \mathcal{L}_Y i_X B = i_{[X,Y]} B$ yielding

$$[\![e^B V, e^B W]\!] = [\![V, W]\!] + i_{[X,Y]} B - i_X i_Y dB \tag{2.69}$$

Since $[X, Y]$ is the vector part of $[\![V, W]\!]$ so the first two terms is $e^B [\![V, W]\!]$ so

$$[\![e^B V, e^B W]\!] = e^B [\![V, W]\!] - i_X i_Y dB \tag{2.70}$$

and if B is closed $dB = 0$, then $[\![e^B V, e^B W]\!] = e^B [\![V, W]\!]$. □

Proposition 2.9. *The β-transform does not preserve the Courant bracket.*

Proof. For $V = X + \xi, W = Y + \eta$,

$$\begin{aligned} [\![e^\beta V, e^\beta W]\!] =& [\![(X + i_\xi \beta) + \xi, (Y + i_\eta \beta) + \eta]\!] \\ =& [X, Y] + \mathcal{L}_{X+\beta\xi} \eta - \mathcal{L}_{Y+\beta\eta} \xi - \frac{1}{2} d(i_{X+\beta\xi} \eta - i_{Y+\beta\eta} \xi) \\ =& [\![V, W]\!] + \mathcal{L}_{\beta\xi} \eta - \mathcal{L}_{\beta\eta} \xi - \frac{1}{2} d(i_{\beta\xi} \eta - i_{\beta\eta} \xi) \end{aligned} \tag{2.71}$$

whereas

$$\begin{aligned} e^\beta [\![V, W]\!] &= [X, Y] + i_{\mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)} \beta + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi) \\ &= [\![V, W]\!] + \beta(\mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)) \end{aligned} \tag{2.72}$$

which are clearly different. □

The propositions above show that both diffeomorphism and the 2-form B-field preserve the Courant bracket, whereas the β-transform represents a symmetry breaking [34], but the β-transform is related to the T-dualities [10]. Hence the only action on $T^* \oplus T$ that preserves the Courant bracket is the semi-direct product of closed 2-forms and the diffeomorphism $\mathrm{diff}(\mathcal{M}) \simeq GL(d, \mathbb{R})$ [2]

$$GL(d, \mathbb{R}) \ltimes \Omega^2(\mathcal{M})_{\text{closed}} \tag{2.73}$$

which is called the geometric subgroup in [3] or the generalised diffeomorphism group in [33]. The element of the group is

$$e^A e^B = \begin{pmatrix} e^A & 0 \\ 0 & e^{-A^T} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ B & \mathbf{1} \end{pmatrix} = \begin{pmatrix} e^A & 0 \\ e^{-A^T} B & e^{-A^T} \end{pmatrix} \tag{2.74}$$

and acts on a generalised vector $V = X + \xi$ as

$$V \to e^A e^B V = \begin{pmatrix} e^A & 0 \\ e^{-A^T} B & e^{-A^T} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} e^A X \\ e^{-A^T}(\xi + i_X B) \end{pmatrix}. \tag{2.75}$$

Semi-direct here indicates that the diffeomorphism acts on both the vector and form parts of the generalised vector whereas the B-transform only transforms the form part as shown in the above equation.

2.7 Twisted Courant Bracket and Generalised Tangent Bundle

In general, the 2-form field B is not closed i.e. $dB \neq 0$, then one needs the H-twisted version of Courant bracket where H is a closed 3-form which is related to the physical 3-form field strength in the NSNS sector [37, 33]. This twisted structure makes B a gerbe connection and can also be used to modify local patching rule which leads to the formal definition of a generalised tangent bundle [3, 2].

Definition 2.5. For a closed 3-form H, the H-twisted Courant bracket is

$$[\![V, W]\!]_H = [\![V, W]\!] + i_X i_Y H \tag{2.76}$$

s.t.

$$[\![e^B V, e^B W]\!]_{H-dB} = e^B [\![V, W]\!]_H. \tag{2.77}$$

If $H = dB$, (2.76) becomes the untwisted Courant bracket again. In this case, for open cover $\{U_i\}$, on $U_i \cap U_j \neq 0$, the 2-form B satisfies[3]

$$dB_{(i)} = dB_{(j)} = H \tag{2.78}$$

so that H is a globally defined closed 3-form ($dH = 0$). But B is not globally defined, and B can be shifted as

$$B_{(j)} - B_{(i)} = d\Lambda_{(ij)} \tag{2.79}$$

where $\Lambda_{(ij)}$ is a 1-form on $U_i \cap U_j$ and $\Lambda_{(ij)} = -\Lambda_{(ji)}$. The shift leaves H unchanged because $d^2 = 0$. This means that B is defined up to a cohomology, and since the shift is a gauge transformation, B can be interpreted as a gauge field. B is also viewed as a potential since its shift does not affect the physical field H [33]. On $U_i \cap U_j \cap U_k \neq 0$, using (2.79) yields

$$B_{(i)} = B_{(j)} + d\Lambda_{(ji)} = B_{(k)} + d\Lambda_{(kj)} + d\Lambda_{(jk)} = B_{(i)} + d\Lambda_{(ik)} + d\Lambda_{(kj)} + d\Lambda_{(jk)} \tag{2.80}$$

[3]The subscript indices in () indicates different charts and not components.

so the gauge transformation needs to satisfy

$$d(\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)}) = 0 \tag{2.81}$$

i.e. $\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)}$ is closed and hence can be written as an exact form

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = d\rho_{(ijk)} \tag{2.82}$$

where $\rho_{(ijk)}$ is a 0-form on $U_i \cap U_j \cap U_k$. Similarly for four overlaps, and so on. In string theory, the cohomological condition $H \in H^3$ implies that [3, 2, 4]

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(kl)} = g^{-1}_{(ijk)} dg_{(ijk)} \tag{2.83}$$

where $g_{(ijk)} = e^{i\rho_{(ijk)}}$ is a $U(1)$ valued function[4] $g_{(ijk)} : U_i \cap U_j \cap U_k \to S^1$ s.t. $g_{(ijk)} = g^{-1}_{(jik)}$ and $g_{(jkl)} g^{-1}_{(ilk)} g_{(ijl)} g^{-1}_{(ijk)} = 1$ on $U_i \cap U_j \cap U_k \cap U_l$. This defines B as a connection structure on a gerbe [38, 1].

For the patching rule, naturally, a section V of $T \oplus T^*$ is locally patched on a chart U_i as $V = X_{(i)} + \xi_{(i)}$, so on $U_i \cap U_j \neq 0$,

$$X_{(i)} + \xi_{(i)} = X_{(j)} + \xi_{(j)}. \tag{2.84}$$

Under the $GL(d, \mathbb{R}) \ltimes \Omega^2_{\text{closed}}$ transformation, the vector part does not change under B-transform so $X_{(i)} = X_{(j)}$ and is globally defined, and for diffeomorphism, the vector part transforms as

$$X^\mu_{(i)} = A_{(ij)}{}^\mu{}_\nu X^\nu_{(j)} \tag{2.85}$$

where $A_{(ij)} \in GL(d, \mathbb{R})$ is the local transformation matrix of the diffeomorphism e^A. The form part transforms under $GL(d, \mathbb{R}) \ltimes \Omega^2_{\text{closed}}$ as

$$\xi_{(i)\mu} = A_{(ij)\mu}{}^\nu \xi_{(j)\nu} + X^\nu_{(j)} \partial_{[\nu} \Lambda_{(ij)\mu]} \tag{2.86}$$

where the B-transform is parametrised by $d\Lambda_{(ij)}$ which is closed as $d^2 = 0$. Hence the overall transformation under $GL(d, \mathbb{R}) \ltimes \Omega^2_{\text{closed}}$ is

$$\begin{pmatrix} X_{(i)} \\ \xi_{(i)} \end{pmatrix} = \begin{pmatrix} A_{(ij)} X_{(j)} \\ A^{-T}_{(ij)} \xi_{(j)} + i_{A_{(ij)} X_{(j)}} d\Lambda_{(ij)} \end{pmatrix} \tag{2.87}$$

which is the patching rule for $T \oplus T^*$.

If the condition (2.79) is satisfied, then by setting the diffeomorphism to identity, the

[4] This structure is similar to the $U(1)$ gauge bundle for electromagnetism and magnetic monopole (see [35]).

patching rule becomes

$$\begin{pmatrix} X_{(i)} \\ \xi_{(i)} \end{pmatrix} = \begin{pmatrix} X_{(j)} \\ \xi_{(j)} - i_{X_{(j)}} d\Lambda_{(ij)} \end{pmatrix} = \begin{pmatrix} X_{(j)} \\ \xi_{(j)} - i_{X_{(j)}} (B_{(j)} - B_{(i)}) \end{pmatrix} \tag{2.88}$$

hence the B-transform can be absorbed into the local patching as

$$\begin{pmatrix} X_{(i)} \\ \xi_{(i)} - i_{X_{(i)}} B_{(i)} \end{pmatrix} = \begin{pmatrix} X_{(j)} \\ \xi_{(j)} - i_{X_{(j)}} B_{(j)} \end{pmatrix}. \tag{2.89}$$

The generalised tangent bundle E is defined such that its section is locally patched as the above equation on $U_i \cap U_j$. The fibre at a point p is still $T_p \oplus T_p^*$ as $X \in \Gamma(T)$, $\xi - i_X B \in \Gamma(T^*)$, but the structure group is restricted to the geometric subgroup $GL(d, \mathbb{R}) \ltimes \Omega^2_{\text{closed}}$ since $d\Lambda$ is a closed 2-form [3].

Formally, the generalised tangent bundle E is defined by an exact sequence implied by the condition (2.81) [1, 2]

$$0 \to T^* \to E \to T \to 0 \tag{2.90}$$

where the map $E \to T$ is the projection map π in (2.33). The map $T^* \to E$ is a natural inclusion sending a 1-form $\lambda \in T^*$ to $\left(\begin{smallmatrix} 0 \\ \lambda \end{smallmatrix}\right) \in \Gamma(E)$. Originally there is no map from T to E. Assume for contraction, if there is a map $T \to E$, on chart $U_i \cap U_j \neq 0$, the map maps a vector $v \in T$ to $\left(\begin{smallmatrix} v_{(i)} \\ 0 \end{smallmatrix}\right)$ on U_i, and by using the original patching rule (2.87), v is mapped to $\left(\begin{smallmatrix} v_{(j)} \\ i_{v_{(i)}} d\Lambda_{(ij)} \end{smallmatrix}\right)$ on U_j where the diffeomorphism is set to identity for simplification. In general, it is clear that

$$\begin{pmatrix} v_{(i)} \\ 0 \end{pmatrix} \neq \begin{pmatrix} v_{(j)} \\ i_{v_{(i)}} d\Lambda_{(ij)} \end{pmatrix} \tag{2.91}$$

which gives the contradiction[5]. However, if the relation (2.79) is satisfied, then the patching rule can be modified to (2.89) so the contradiction disappears, and there exist a map $T \to E$ which is defined by the inverse of B-transform as

$$\begin{aligned} e^{-B}: \quad T &\to E \\ v &\to v - i_v B \end{aligned} \tag{2.92}$$

Then, by the splitting lemma [8, 9], the existence of the map $T \to E$ together with the exact sequence (2.90) imply that the generalised tangent bundle E is isomorphic the direct sum of T and T^* i.e.

$$E \simeq T \oplus T^*. \tag{2.93}$$

As the patching of section changes, the coordinate frame for E also needs to be updated

[5] For the map $T^* \to E$, since the vector part is zero, then under the original patching rule, λ is mapped to $\left(\begin{smallmatrix} 0 \\ \lambda \end{smallmatrix}\right)$ on all charts, so there is no contraction.

to

$$\hat{E}_M = \begin{cases} \hat{E}_\mu = e_\mu + i_{e_\mu} B, & M = \mu \\ \hat{E}^\mu = e^\mu, & M = \mu + d \end{cases} \tag{2.94}$$

where $\{e_\mu\}$ is basis on T, $\{e^\mu\}$ is dual basis on T^*, and B is free to choose so there is no canonical isomorphism [4]. This will be called as split frame and it still satisfies

$$\langle \hat{E}_M, \hat{E}_N \rangle = \eta_{MN} \tag{2.95}$$

where in the calculation, one needs to use that $i_{e_\mu} i_{e_\nu} B = 0$ because B is anti-symmetric whereas $i_{e_\mu} i_{e_\nu}$ is symmetric. The section of E, $V \in \Gamma(E)$ is now

$$V = V^M \hat{E}_M = X^\mu \hat{E}_\mu + \xi_\mu \hat{E}^\mu = \begin{pmatrix} X \\ \xi - i_X B \end{pmatrix} \tag{2.96}$$

In fact, by using the exact sequence and splitting lemma, one can define a general generalised tangent bundle as

$$0 \to \mathcal{E} \to E \to T \to 0 \tag{2.97}$$

so that

$$E \simeq T \oplus \mathcal{E} \tag{2.98}$$

which will be useful for M-theory and other geometry such as exceptional geometry [3, 39, 17, 16]. The generalised vector $V \in \Gamma(E)$ is then

$$V = v + \lambda \tag{2.99}$$

where $v \in \Gamma(T)$ and $\lambda \in \Gamma(\mathcal{E})$. An example will be given in Section 5.2.

One can also define the dual generalised bundle E^* using the dual of the exact sequence (2.97)

$$0 \to T^* \to E^* \to \mathcal{E}^* \to 0 \tag{2.100}$$

and the splitting gives the isomorphism

$$E^* \simeq T^* \oplus \mathcal{E}^*. \tag{2.101}$$

For the $O(d,d)$ generalised geometry, the dual bundle is simply

$$E^* \simeq T^* \oplus T \simeq E. \tag{2.102}$$

3 Generalised Metric, Connection, Torsion and Curvature

As suggested by [2, 34], one can define generalised versions of geometric objects in the Riemannian geometry, namely the metric, vielbein, connection, torsion and curvature. By introducing the generalised metric, the structure group shall be further reduced as will be shown. The construction of generalised objects in this chapter follows [4].

3.1 Generalised Metric

Apart from the canonical metric defined in Section 2.2, one can also define a generalised metric G on $T \oplus T^*$ that is symmetric and positive definite as an analogue of the usual Riemannian metric [2]. The generalised metric is also required to be compatible with the canonical metric η [2, 3]. The requirement is that for two generalised vectors $V, W \in \Gamma(E)$,

$$\langle GV, GW \rangle = \langle V, W \rangle \tag{3.1}$$

and in components gives

$$\eta_{MN} G^M{}_I V^I G^N{}_J W^J = \eta_{IJ} V^I W^J \tag{3.2}$$

where $G^M{}_N$ is a map $E \to E$ sending V to GV, hence we have

$$\eta_{MN} G^M{}_I G^N{}_J \quad \Leftrightarrow \quad G^T \eta G = \eta \tag{3.3}$$

which means that $G \in O(d,d)$. If index of G is lowered using η so $G_{MN} = \eta_{MN} G^{\mathcal{M}}{}_N$, then G can be viewed as an symmetric 2-tensor i.e. a metric $G \in \Gamma(S^2 E^*)$, and the compatible requirement can be equivalently written as

$$\eta_{MN} G^M{}_I G^N{}_J = G_{NI} \eta^{MN} G_{MJ} = \eta_{IJ} \tag{3.4}$$

and multiplying the inverse metric on both sides gives the matrix form

$$\eta^{-1} G \eta^{-1} G = \mathbf{1}. \tag{3.5}$$

Since the inverse canonical metric η^{-1} (2.10) is in 2×2 block form, the general from of G can be written as

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.6}$$

and the symmetry requirement of G is

$$a^T = a, \quad d^T = d, \quad c^T = b. \tag{3.7}$$

Substituting η^{-1} (2.10) and G into the compatible condition (3.5) yields

$$\begin{pmatrix} c^2 + da & cd + db \\ ac + ba & ad + b^2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{3.8}$$

From $ac + ba = 0$ and (3.7), one has

$$ac = ab^T = (ba^T)^T = (ba)^T = -ba \tag{3.9}$$

and similarly, from $cd + db = 0$,

$$cd = -(cd)^T = -dc^T \tag{3.10}$$

so ba and cd are anti-symmetric.

We now define $d = g^{-1}$ where g^{-1} is symmetric, and $B = d^{-1}c$ is anti-symmetric as

$$B = d^{-1}c = d^{-1}cdd^{-1} = -d^{-1}dc^T d^{-1} = -c^T d^{-1} = -(d^{-1}c)^T = -B^T \tag{3.11}$$

Then

$$\begin{aligned} c &= dB = g^{-1}B \\ b &= c^T = (g^{-1}B)^T = B^T g^{-1} = -Bg^{-1} \end{aligned} \tag{3.12}$$

and from $ad + b^2 = \frac{1}{4}\mathbf{1}$ or $c^2 + da = \frac{1}{4}\mathbf{1}$, one has

$$a = \frac{1}{4}g - Bg^{-1}B \tag{3.13}$$

Substitute a, b, c, d into G and rescale $g \to 2g$, $g^{-1} \to \frac{1}{2}g^{-1}$ gives

$$G = \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix} \tag{3.14}$$

One can also set $a = g^{-1}$ and define B using ab, and the result will just be the inverse of the above matrix. Contract G with two generalised vectors $V = X + \xi, W = Y + \eta$ gives

$$\begin{pmatrix} X & \xi \end{pmatrix} \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix} \begin{pmatrix} Y \\ \eta \end{pmatrix} = XgY + (\xi - i_X B)g^{-1}(\eta + i_Y B) \tag{3.15}$$

If we view g as a metric with signature (p, q) on $T\mathcal{M}$ and g^{-1} is metric with signature

(q,p) on $T^*\mathcal{M}$ where $p+q=d$, then the above equation can be written as

$$G(V,W) = g(X,Y) + g^{-1}(\xi + i_X B, \eta + i_Y B) \tag{3.16}$$

so G can be interpreted as a metric with signature $(2p, 2q)$ on $T \oplus T^*$, i.e. the generalised metric [3]. And since B is anti-symmetric, it will be interpreted as the 2-form gauge field in type II theory. Hence G encodes information of both the metric g on $\mathcal{M}$ and the gauge field B and is parametrised by them.

The generalised metric transform under $O(d,d)$ as

$$G \to G' = O^T G O, \quad O \in O(d,d) \tag{3.17}$$

If O is the GL-action (2.19), the transformation is parameterised as

$$g \to A^T g A, \quad B \to A^T B A \tag{3.18}$$

where A is the matrix of diffeomorphism. And if O is the B-transform, then g is invariant, but B is shifted

$$B \to B + \Theta \tag{3.19}$$

where Θ is used as the generator of B-transform to avoid confusion. However, if O is not in the geometric subgroup diff $\ltimes \Omega^2_{\text{closed}}$, it will mix g and B [3].

3.2 Splitting Frame and $O(p,q) \times O(q,p)$ Structure

One can also define

$$S = \eta^{-1} G \tag{3.20}$$

so that the compatible condition (3.5) becomes

$$S^2 = \mathbf{1} \tag{3.21}$$

which defines an almost product structure (see Definition 2.2). For the generalised space $T \oplus T^*$ and a d-dimensional manifold $\mathcal{M}$ with signature (p,q) where $p+q=d$, the generalised tangent bundle E splits into (see Section 2.1)

$$E \simeq C_+ \oplus C_- \tag{3.22}$$

where C_+ and C_- are d-dimensional sub-bundles with signature (p,q) and (q,p) respectively. The almost product structure reduces the structure group from $O(d,d)$ to its maximal compact subgroup $O(p,q) \times O(q,p)$ [33, 3, 2], and $O(p,q) \times O(q,p)$ is the group that preserve both the canonical metric and the generalised metric. Also, $O(p,q)$ and

$O(q,p)$ are structure groups of C_+ and C_- respectively (see Definition A.26 and (A.69)), and C_+, C_- are orthogonal complement to each other [3]. Then the generalised metric G parametrises the coset space

$$G \in \frac{O(d,d)}{O(p,q) \times O(q,p)} \tag{3.23}$$

as an analogue of that the usual metric g parametrise the coset $GL(d,\mathbb{R})/O(d)$ (Example 2.2). One can also check that the difference between the dimensions of $O(d,d)$ and $O(d) \times O(d)$

$$\frac{1}{2}(2d)(2d+1) - 2 \times \frac{1}{2}d(d+1) = d^2 \tag{3.24}$$

equals to the sum of degrees of freedom of symmetric tensor $g_{\mu\nu}$ and the anti-symmetric tensor $B_{\mu\nu}$ on a d-dimensional manifold

$$\frac{1}{2}d(d+1) + \frac{1}{2}d(d-1) = d^2. \tag{3.25}$$

Since the canonical inner product of any generalised vectors that consist of pure vector or pure form is zero, the sections of C_+ or C_- must be a mix of vector and form. For C_+, its section V_+ has a general form [10]

$$V_+ = X + MX \tag{3.26}$$

where $X \in \Gamma(T\mathcal{M})$ and M is a general metric s.t. $M_{\mu\nu}X^\nu$ gives the form part. This also defines an isomorphism from C_+ to $T\mathcal{M}$ since X is the only parameter. As any type $(0,2)$ tensor can be decomposed into symmetric and anti-symmetric parts, M can be written as

$$M_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu} \tag{3.27}$$

where $g_{\mu\nu}$ is symmetric, $B_{\mu\nu}$ is anti-symmetric, so

$$V_+ = X + (g+B)X. \tag{3.28}$$

Using (2.86) and g is symmetric, one has that on the overlap of two charts $U_i \cap U_j \neq 0$,

$$g_{(i)} = g_{(j)}, \quad B_{(i)} = B_{(j)} - d\Lambda_{(ij)} \tag{3.29}$$

so g is interpreted as the metric and B is the 2-form B-field in the NSNS sector. Similarly, one finds that the section of C_- is [10]

$$V_- = X + (B-g)X \tag{3.30}$$

so C_- is also isomorphic to $T\mathcal{M}$. One can check that the inner product of sections from

C_+ and C_- is

$$\langle V_+, V_- \rangle = \frac{1}{2}[i_X(g+B)X + i_X(B-g)X] = i_X i_X B = 0 \tag{3.31}$$

since B is anti-symmetric whereas $i_X i_X$ is symmetric, so it agrees with that C_+ and C_- are orthogonal complement. Then the section of the generalised tangent bundle E is

$$V = V_+ + V_- = X + i_X B. \tag{3.32}$$

We now define orthonormal frames $\{\hat{E}^+_a\}$ for C_+ and $\{\hat{E}^-_{\bar{a}}\}$ for C_- satisfying [4]

$$\begin{aligned} \langle \hat{E}^+_a, \hat{E}^+_b \rangle &= \eta_{ab} \\ \langle \hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}} \rangle &= -\eta_{\bar{a}\bar{b}} \\ \langle \hat{E}^+_a, \hat{E}^-_{\bar{b}} \rangle &= 0 \end{aligned} \tag{3.33}$$

where $a, \bar{a} = 1, ..., d$ are used to distinguish the two sub-bundles, and both of $\eta_{ab}, \eta_{\bar{a}\bar{b}}$ are flat metrics with signature (p, q). The first two conditions indicate that $O(p, q)$ associated with the metric η_{ab} acts on $\{\hat{E}^+_a\}$ and is the structure group of C_+ whereas $O(q, p)$ associated with the metric $-\eta_{\bar{a}\bar{b}}$ acts on $\{\hat{E}^-_{\bar{a}}\}$ and is the structure group of C_-. The third condition means that C_+, C_- are orthogonal complement to each other, and $O(p, q)$, $O(q, p)$ acts on C_+, C_- separately. Thus, the overall structure group is $O(p, q) \times O(q, p)$. The sections of C_+ and C_- are written in components as

$$V_+ = V^a_+ \hat{E}^+_a, \quad V_- = V^{\bar{a}}_- \hat{E}^-_{\bar{a}} \tag{3.34}$$

where indices are raised and lowered using $\eta_{ab}, \eta_{\bar{a}\bar{b}}$ respectively.

The frame of the generalised tangent bundle is then defined to be $\{\hat{E}_A\} = \{\hat{E}^+_a\} \cup \{\hat{E}^-_{\bar{a}}\}$ i.e.

$$\hat{E}_A = \begin{cases} \hat{E}^+_a, & A = a, \\ \hat{E}^-_{\bar{a}}, & A = \bar{a} + d, \end{cases} \tag{3.35}$$

However, the orthonormal condition is now

$$\langle \hat{E}_A, \hat{E}_B \rangle = \eta_{AB}, \quad \eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix} \tag{3.36}$$

where η_{AB} is different from the canonical metric (2.9). This is because $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$ cannot be purely vectors or forms as discussed before, so $\{\hat{E}_A\}$ is different from the one used in Section 2.2 or the coordinate basis $\{\hat{E}_M\}$ in (2.94). In fact, $\{\hat{E}_A\}$ is defined so that the metric η_{AB} is diagonalised as in (3.44) by using the generalised vielbein which will be defined in Section 3.3, so this frame will be called the non-coordinate frame. In the rest of the dissertation, the indices $A, B, ...$ will be used for this frame where as $M, N, ...$ are used for the split frame (2.94) defined using coordinate basis. The dual basis is also defined by raising the indices of $\hat{E}_A$ using η_{AB} giving

$$\hat{E}^A = \begin{cases} \hat{E}^{+a}, & A = a, \\ -\hat{E}^{-\bar{a}}, & A = \bar{a} + d. \end{cases} \tag{3.37}$$

Thus, using the frame $\{\hat{E}_A\}$, a generalised vector $V \in \Gamma(E)$ is

$$V = V^A \hat{E}_A = V_+^a \hat{E}_a^+ + V_-^{\bar{a}} \hat{E}_{\bar{a}}^- = V_+ + V_-. \tag{3.38}$$

The splitting of frames can also be found by defining two projection operators [40]

$$P_\pm = \frac{1}{2}(\mathbf{1}_{2d} \pm S). \tag{3.39}$$

It can be easily shown that

$$\begin{aligned} P_+ + P_- &= \mathbf{1}_{2d} \\ P_\pm^2 &= P_\pm \\ P_+ P_- &= P_- P_+ = 0 \end{aligned} \tag{3.40}$$

so $P_\pm$ are projectors and are maps $P_\pm : E \to C_\pm$ s.t.

$$P_+(V) = V_+ = V_+^a \hat{E}_a^+, \quad P_-(V) = V_- = V_-^{\bar{a}} \hat{E}_{\bar{a}}^-. \tag{3.41}$$

Therefore, the generalised tangent bundle splits into two sub-bundles C_+ and C_-.

3.3 Generalised Vielbein

In usual Riemannian geometry (Appendix D), for a manifold with signature (p, q), the vielbein diagonalises the metric $g_{\mu\nu}$ to the constant flat metric $\eta_{\mu\nu}$ with signature (p, q). The idea can be generalised in the generalised geometry to define a generalised vielbein so that both of the canonical metric and the generalised metric are diagonalised.

Given the split frame $\{\hat{E}_M\}$ defined in (2.94), the non-coordinate frame $\{\hat{E}_A\}$ defined in (3.35) can be written as

$$\hat{E}_A = \hat{E}_A{}^M \hat{E}_M, \tag{3.42}$$

i.e. a change of basis, and the letters A, B are used for the non-coordinate frame whereas M, N are used for the split frame. This defines the generalised vielbein $\hat{E}_A{}^M$ which satisfies

$$\hat{E}^A{}_M \hat{E}_A{}^N = \delta_M{}^N, \quad \hat{E}^A{}_M \hat{E}_B{}^M = \delta^A{}_B \tag{3.43}$$

as an anologue of (A.92). The vielbein is required to diagonalise the canonical metric

$$\eta_{AB} = \langle \hat{E}_A, \hat{E}_B \rangle = \eta_{MN} \hat{E}^M{}_A \hat{E}^N{}_B = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix} \tag{3.44}$$

which gives the condition (3.33), and the generalised metric G is also diagonalised as

$$G_{AB} = G(\hat{E}_A, \hat{E}_B) = G_{MN} \hat{E}^M{}_A \hat{E}^N{}_B = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix} \tag{3.45}$$

so that

$$G(\hat{E}^+_a, \hat{E}^+_b) = \eta_{ab}, \quad G(\hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}}) = \eta_{\bar{a}\bar{b}}, \quad G(\hat{E}^+_a, \hat{E}^-_{\bar{b}}) = 0 \tag{3.46}$$

and

$$G = \eta^{ab} \hat{E}^+_a \otimes \hat{E}^+_b + \eta^{\bar{a}\bar{b}} \hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}}. \tag{3.47}$$

To find the expression of the generalised vielbein, we first introduce two set of ordinary vielbein $\hat{e}^{\pm}$ and dual $e^{\pm}$ for the two $T\mathcal{M}$ spaces isomorphic to $C_{\pm}$ respectively. The usual vielbein satisfy

$$\begin{gathered} \hat{e}^{\pm} e^{\pm} = e^{\pm} \hat{e}^{\pm} = \mathbf{1} \\ g = e^{\pm T} e^{\pm} = \eta_{ab} e^{+a} \otimes e^{+b} = \eta_{\bar{a}\bar{b}} e^{-\bar{a}} \otimes e^{-\bar{b}} \\ g(\hat{e}^+_a, \hat{e}^+_b) = \eta_{ab}, \quad g(\hat{e}^-_{\bar{a}}, \hat{e}^-_{\bar{b}}) = \eta_{\bar{a}\bar{b}} \\ g^{-1} = \hat{e}^{\pm} \hat{e}^{\pm T} \end{gathered} \tag{3.48}$$

Then using the general form of sections of C_+ (3.28) and C_- (3.30), the non-coordinate frame can be explicitly written as

$$\hat{E}_A = \begin{cases} \hat{E}^+_a = \hat{e}^+_a + (e^+_a + i_{\hat{e}^-_a} B), & A = a, \\ \hat{E}^-_{\bar{a}} = \hat{e}^-_{\bar{a}} - (e^-_{\bar{a}} + i_{\hat{e}^-_{\bar{a}}} B), & A = \bar{a} + d. \end{cases} \tag{3.49}$$

Using this expression and the definition (3.42), the expression of the generalised vielbein is found to be

$$\hat{E} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^+ - \hat{e}^{+T} B & \hat{e}^{+T} \\ -e^- - \hat{e}^{-T} B & \hat{e}^{-T} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}^{+T}(g - B) & \hat{e}^{+T} \\ -\hat{e}^{-T}(g + B) & \hat{e}^{-T} \end{pmatrix}. \tag{3.50}$$

It transforms under $M \in O(d) \times O(d)$ as

$$E \to ME, \quad M = \begin{pmatrix} O_+ & 0 \\ 0 & O_- \end{pmatrix}, \quad O_{\pm} \in O(d) \tag{3.51}$$

i.e. each vielbein rotates under $O(d)$ separately. In type II string theory, the sub-bundles $C_{\pm}$ can be interpreted as left and right moving sectors, and one can choose $e^+ = e^- = e$

by the $O(d) \times O(d)$ transformation to have same spin-connections in both sectors [10].

From (3.45), one can also have the matrix version

$$G = \hat{E}^T \hat{E} \tag{3.52}$$

and substitute (3.50) into the above equation gives

$$G = \frac{1}{2} \begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix} \tag{3.53}$$

which is the same expression as (3.14).

Moreover, one can define another generalised vielbein $\hat{\mathcal{E}}$ such that

$$\langle \hat{\mathcal{E}}_A, \hat{\mathcal{E}}_B \rangle = \begin{pmatrix} 0 & \eta_{ab} \\ \eta_{ab} & 0 \end{pmatrix}, \quad G(\hat{\mathcal{E}}_A, \hat{\mathcal{E}}_B) = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix} \tag{3.54}$$

i.e. the canonical metric is the same as its original form. This generalised vielbein is [10]

$$\hat{\mathcal{E}} = \frac{1}{2} \begin{pmatrix} (e^+ + e^-) + (\hat{e}^{+T} - \hat{e}^T)B & \hat{e}^{+T} - \hat{e}^{-T} \\ (e^+ - e^-) - (\hat{e}^{+T} + \hat{e}^{-T})B & \hat{e}^{+T} + \hat{e}^{-T} \end{pmatrix} \tag{3.55}$$

and transforms under $M \in O(d) \times O(d)$ as

$$\hat{\mathcal{E}} \to M\hat{\mathcal{E}}, \quad M = \frac{1}{2} \begin{pmatrix} O_+ + O_- & O_+ - O_- \\ O_+ - O_- & O_+ + O_- \end{pmatrix}, \quad O_\pm \in O(d). \tag{3.56}$$

By making a suitable transformation, one can set $e^+ = e^- = e$ yielding a triangular form

$$\hat{\mathcal{E}} = \begin{pmatrix} e & 0 \\ -\hat{e}^T B & \hat{e}^T \end{pmatrix} \tag{3.57}$$

which is invariant under the geometric subgroup $GL(d, \mathbb{R}) \ltimes \Omega^2_{\text{closed}}$, and gives the same generalised metric G using $G = \hat{\mathcal{E}}^T \hat{\mathcal{E}}$. This actually corresponds to the original split frame (2.94) as the component of canonical metric η is in off-diagonal form.

3.4 Generalised Connection

The generalised connection is constructed as analogue of the usual connection in Riemannian geometry (Definition A.36) following [41].

Definition 3.1. For a generalised tangent bundle E, the generalised connection is a map

$$\begin{array}{rcll} D: & \Gamma(E)\times\Gamma(E) & \rightarrow & \Gamma(E) \\ & V, W & \rightarrow & D_V W \end{array} \tag{3.58}$$

that satisfies the same condition as the usual connection, so it is bi-linear

$$\begin{aligned} D_U(V+W) &= D_U V + D_U W \\ D_{(U+V)} W &= D_U W + D_V W \end{aligned} \tag{3.59}$$

$\forall U, V, W \in \Gamma(E)$. For a function f,

$$D_{fV} W = f D_V W \tag{3.60}$$

and has the Leibniz rule

$$D_V(fW) = V[f]W + f D_V W. \tag{3.61}$$

On the split frame $\{\hat{E}_M\}$ (2.94), the generalised connection is defined as

$$D_M \hat{E}_N = D_{\hat{E}_M} \hat{E}_N = \Gamma_M{}^I{}_N \hat{E}_I \tag{3.62}$$

where $\Gamma_M{}^I{}_N$ is called components of the generalised affine connection. Then D acts on $V = V^N \hat{E}_N \in \Gamma(E)$ as

$$D_M(V^N \hat{E}_N) = (\partial_M V^N + \Gamma_M{}^N{}_I V^I)\hat{E}_N. \tag{3.63}$$

It can be extended to a generalised rank n tensor $\alpha \in \Gamma(E^{\otimes n})$ as

$$D_M \alpha^{N_1 \dots N_n} = \partial_M \alpha^{N_1 \dots N_n} + \Gamma_M{}^{N_1}{}_I \alpha^{I \dots N_n} + \cdots + \Gamma_M{}^{N_n}{}_I \alpha^{N_1 \dots N_{n-1} I}. \tag{3.64}$$

Similarly to the usual connection where we are interested in the metric connection (Definition A.38), the generalised connection is also required to be compatible with both of the canonical metric η and the generalised metric G so

$$D\eta = DG = 0. \tag{3.65}$$

For the canonical metric, since it is the metric of the $O(d,d)$ structure, we can simply use the split frame $\{\hat{E}_M\}$, and the generalised connection acts on $\eta = \eta_{IJ}\hat{E}^I \otimes \hat{E}^J$ as

$$\begin{aligned} D_M\eta &= D_M(\eta_{IJ}\hat{E}^I \otimes \hat{E}^J) \\ &= (\partial_M \eta_{IJ})\hat{E}^I \otimes \hat{E}^J + \eta_{IJ}[(D_M\hat{E}^I) \otimes \hat{E}^J + \hat{E}^I(D_M\hat{E}^J)] \\ &= \eta_{IJ}\Gamma_M{}^I{}_N\hat{E}^N \otimes \hat{E}^J + \eta_{IJ}\Gamma_M{}^J{}_N\hat{E}^I \otimes \hat{E}^N \\ &= (\Gamma_{MJI} + \Gamma_{MIJ})\hat{E}^I \otimes \hat{E}^J = 0 \end{aligned} \tag{3.66}$$

where $\partial_M\eta_{IJ} = 0$ since η is constant, so

$$\Gamma_{MJI} = -\Gamma_{MIJ} \tag{3.67}$$

which means $(\Gamma_M)^I{}_J$ is an element of Lie algebra $\mathfrak{o}(d,d)$ as a result of the $O(d,d)$ structure.

For the compatibility with the generalised metric G, as G reduce the structure group to $O(p,q) \times O(p,q)$, it is necessary to use the non-coordinate frame $\{\hat{E}_A\}$ that describes this structure. The generalised connection on the non-coordinate frame is now defined as

$$D_M\hat{E}_A = \Omega_M{}^B{}_A\hat{E}_B \tag{3.68}$$

where $\Omega_M{}^B{}_A$ is components of generalised spin connection [4] and on each sub-frame

$$D_M\hat{E}_A = \begin{cases} \Omega_M{}^b{}_a\hat{E}^+_b + \Omega_M{}^{\bar{b}}{}_a\hat{E}^-_{\bar{b}}, & A = a \\ \Omega_M{}^{\bar{b}}{}_{\bar{a}}\hat{E}^-_{\bar{b}} + \Omega_M{}^b{}_{\bar{a}}\hat{E}^+_b, & A = \bar{a} \end{cases} \tag{3.69}$$

The actions of D on a generalised vector $V \in \Gamma(E)$ shall take the same form as before but with the generalised connection component in split frame $\Gamma_M{}^I{}_N$ replaced by $\Omega_M{}^A{}_B$ in non-coordinate frame, so

$$D_MV^A = \partial_MV^A + \Omega_M{}^A{}_BV^B \tag{3.70}$$

or write $V = V^a_+\hat{E}^+_a + V^{\bar{a}}_-\hat{E}^-_{\bar{a}}$ gives

$$D_MV^A = \begin{cases} \partial_MV^a_+ + \Omega_M{}^a{}_bV^b_+ + \Omega_M{}^a{}_{\bar{b}}V^{\bar{b}}_-, & A = a, \\ \partial_MV^{\bar{a}}_- + \Omega_M{}^{\bar{a}}{}_{\bar{b}}V^{\bar{b}}_- + \Omega_M{}^{\bar{a}}{}_bV^b_+, & A = \bar{a}. \end{cases} \tag{3.71}$$

And D acts on a generalised tensor $\alpha \in \Gamma(E^{\otimes n})$ as

$$D_M\alpha^{A_1\ldots A_n} = \partial_M\alpha^{A_1\ldots A_n} + \Omega_M{}^{A_1}{}_I\alpha^{I\ldots A_n} + \cdots + \Omega_M{}^{A_n}{}_I\alpha^{A_1\ldots A_{n-1}I}. \tag{3.72}$$

The compatibility with the canonical metric $\eta = \eta^{ab}\hat{E}^+_a \otimes \hat{E}^+_b - \eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}}$ gives

$$\begin{aligned}
D_M\eta &= D_M(\eta^{ab}\hat{E}^+_a \otimes \hat{E}^+_b - \eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}}) \\
&= (\partial_M\eta^{ab})\hat{E}^+_a \otimes \hat{E}^+_b + \eta^{ab}[(D_M\hat{E}^+_a) \otimes \hat{E}^+_b + \hat{E}^+_a(D_M\hat{E}^+_b)] \\
&- (\partial_M\eta^{\bar{a}\bar{b}})\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}} - \eta^{\bar{a}\bar{b}}[(D_M\hat{E}^-_{\bar{a}}) \otimes \hat{E}^-_{\bar{b}} + \hat{E}^-_{\bar{a}}(D_M\hat{E}^-_{\bar{b}})] \\
&= (\eta^{cb}\Omega_M{}^a{}_c + \eta^{ac}\Omega_M{}^b{}_c)\hat{E}^+_a \otimes \hat{E}^+_b - (\eta^{\bar{c}\bar{b}}\Omega_M{}^{\bar{a}}{}_c + \eta^{\bar{a}\bar{c}}\Omega_M{}^{\bar{b}}{}_c)\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}} \\
&+ (\eta^{cb}\Omega_M{}^{\bar{a}}{}_c - \eta^{\bar{a}\bar{c}}\Omega_M{}^b{}_{\bar{c}})\hat{E}^-_{\bar{a}} \otimes \hat{E}^+_b + (\eta^{ac}\Omega_M{}^{\bar{b}}{}_c - \eta^{\bar{c}\bar{b}}\Omega_M{}^a{}_{\bar{c}})\hat{E}^+_a \otimes \hat{E}^-_{\bar{b}} \\
&= (\Omega_M{}^{ab} + \Omega_M{}^{ba})\hat{E}^+_a \otimes \hat{E}^+_b + (\Omega_M{}^{\bar{a}\bar{b}} + \Omega_M{}^{\bar{b}\bar{a}})\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}} \\
&+ (\Omega_M{}^{\bar{a}b} - \Omega_M{}^{b\bar{a}})\hat{E}^-_{\bar{a}} \otimes \hat{E}^+_b + (\Omega_M{}^{\bar{b}a} - \Omega_M{}^{a\bar{b}})\hat{E}^+_a \otimes \hat{E}^-_{\bar{b}} \\
&= 0
\end{aligned} \tag{3.73}$$

so

$$\Omega_M{}^{ab} = -\Omega_M{}^{ba}, \quad \Omega_M{}^{\bar{a}\bar{b}} = -\Omega_M{}^{\bar{b}\bar{a}}, \quad \Omega_M{}^{\bar{a}b} = \Omega_M{}^{b\bar{a}} \tag{3.74}$$

For the generalised metric $G = \eta^{ab}\hat{E}^+_a \otimes \hat{E}^+_b + \eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}}$, there is a sign change, and following the same procedure gives a similar result

$$\begin{aligned}
D_M G &= (\Omega_M{}^{ab} + \Omega_M{}^{ba})\hat{E}^+_a \otimes \hat{E}^+_b + (\Omega_M{}^{\bar{a}\bar{b}} + \Omega_M{}^{\bar{b}\bar{a}})\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}} \\
&+ (\Omega_M{}^{\bar{a}b} + \Omega_M{}^{b\bar{a}})\hat{E}^-_{\bar{a}} \otimes \hat{E}^+_b + (\Omega_M{}^{\bar{b}a} + \Omega_M{}^{a\bar{b}})\hat{E}^+_a \otimes \hat{E}^-_{\bar{b}} \\
&= 0
\end{aligned} \tag{3.75}$$

And the condition on spin connection is

$$\Omega_M{}^{ab} = -\Omega_M{}^{ba}, \quad \Omega_M{}^{\bar{a}\bar{b}} = -\Omega_M{}^{\bar{b}\bar{a}}, \quad \Omega_M{}^{\bar{a}b} = -\Omega_M{}^{b\bar{a}} \tag{3.76}$$

Together with the condition found from η (3.74), one has

$$\Omega_{Mab} = -\Omega_{Mba}, \quad \Omega_{M\bar{a}\bar{b}} = -\Omega_{M\bar{b}\bar{a}} \tag{3.77}$$

which means that $(\Omega_M)^a{}_b$ is element of Lie algebra $\mathfrak{o}(p,q)$ and $(\Omega_M)^{\bar{a}}{}_{\bar{b}} \in \mathfrak{o}(q,p)$. And also that

$$\Omega_{M\bar{a}b} = \Omega_{Mb\bar{a}} = 0 \tag{3.78}$$

indicates that the connection shall act on the two sub-bundles $C_\pm$ separately as a result of the $O(d) \times O(d)$ structure. Therefore, the action of D on $V = V^a_+\hat{E}^+_a + V^{\bar{a}}_-\hat{E}^-_{\bar{a}}$ is simply

$$D_M V^A = \begin{cases} \partial_M V^a_+ + \Omega_M{}^a{}_b V^b_+, & A = a, \\ \partial_M V^{\bar{a}}_- + \Omega_M{}^{\bar{a}}{}_{\bar{b}} V^{\bar{b}}_-, & A = \bar{a}. \end{cases} \tag{3.79}$$

Furthermore, as suggested in [4], the generalised connection can be constructed from an ordinary Levi-Civita connection ∇ with a metric g. A generalised vector V in the split frame $\{\hat{E}_M\}$ (2.94) is

$$V = X + \xi = X^\mu \hat{E}_\mu + \xi_\mu \hat{E}^\mu. \tag{3.80}$$

Since $X \in \Gamma(T\mathcal{M}), \xi \in \Gamma(T^*\mathcal{M})$, ∇ can acts on them and gives a generalised connection denoted as D^∇_M. The action of D^∇_M on V is simply [4]

$$D^\nabla_M V^M \hat{E}_M = \begin{cases} (\nabla_\mu X^\nu)\hat{E}_\nu + (\nabla_\mu \xi_\nu)\hat{E}^\nu, & M = \mu \\ 0, & M = \mu + d. \end{cases} \tag{3.81}$$

For the spin connection, we introduce two sets of vielbein $\hat{e}^\pm$ for the two $T\mathcal{M}$ spaces that are isomorphic to $C_\pm$ as before. Then a vector $v \in \Gamma(T\mathcal{M})$ can be written in two bases as

$$v = v^a \hat{e}^+_a = v^{\bar{a}} \hat{e}^-_{\bar{a}}. \tag{3.82}$$

The Levi-Civita connection acts on v as (A.119)

$$\nabla_\mu v^\nu = (\partial_\mu v^a + \omega^{+}{}_{\mu}{}^{a}{}_{b} v^b)\hat{e}^{+}{}_{a}{}^{\nu} = (\partial_\mu v^{\bar{a}} + \omega^{-}{}_{\mu}{}^{\bar{a}}{}_{\bar{b}})\hat{e}^{-}{}_{\bar{a}}{}^{\nu} \tag{3.83}$$

where $\omega^+_{\mu ab}, \omega^-_{\mu\bar{a}\bar{b}}$ are two spin connections for the $O(d)$ structure, each satisfies

$$\omega^+_{\mu ab} = -\omega^+_{\mu ba}, \quad \omega^-_{\mu\bar{a}\bar{b}} = -\omega^-_{\mu\bar{b}\bar{a}} \tag{3.84}$$

manifesting the $O(d) \times O(d)$ structure. It is now natural to identify

$$\Omega_{\mu ab} = \omega^+_{\mu ab}, \quad \Omega_{\mu\bar{a}\bar{b}} = \omega^-_{\mu\bar{b}\bar{a}}. \tag{3.85}$$

For $M = \mu + d$, since both of D^∇_M and ∂_M vanish, Ω_{Mab} and $\Omega_{M\bar{a}\bar{b}}$ must also be zero. Therefore, the generalised spin connection (3.79) can be written as

$$D_M V^A = \begin{cases} D^\nabla_M V^a_+ = \begin{cases} \nabla_\mu V^a_+, & M = \mu \\ 0, & M = \mu + d \end{cases}, & A = a \\ D^\nabla_M V^{\bar{a}}_- = \begin{cases} \nabla_\mu V^{\bar{a}}_-, & M = \mu \\ 0, & M = \mu + d \end{cases}, & A = \bar{a} \end{cases} \tag{3.86}$$

By performing a $O(d) \times O(d)$ transformation, one can set $e^+ = e^- = e$ as in Section 3.3, so

$$V = V^a_+ \hat{E}^+_a + V^{\bar{a}}_- \hat{E}^-_{\bar{a}} = (V^a_+ + V^a_-)\hat{E}_a + (V_{+a} - V_{-a})\hat{E}^a \tag{3.87}$$

and substitute into (3.81) gives same expression as (3.86), so the two generalised connections match.

3.5 Generalised Torsion and Torsion-free Connection

The generalised connection defined before is not generally torsion-free. For the usual geometry, a torsion free connection i.e. the Levi-Civita connection is uniquely determined by the metric (Definition A.40), so an analogue, we want to construct the generalised connection that is torsion free and preserve the $O(p,q) \times O(q,p)$ structure. The construction follows [12, 13], however, the resultant generalised connection is not unique.

Firstly, we need to define the generalised torsion as an analogue of usual torsion using Definition A.37 and Claim A.26 [4, 41].

Definition 3.2. The generalised torsion tensor is a map

$$\begin{aligned} \mathcal{T}: \quad \Gamma(E) \times \Gamma(E) &\rightarrow \Gamma(E) \\ V, W \quad &\rightarrow \mathcal{T}(V,W) = L^D_V W - L_V W \end{aligned} \tag{3.88}$$

where the partial derivative ∂ in the Dorfman derivative is replaced by the generalised connection D.

The generalised torsion in component is

$$\begin{aligned} \mathcal{T}(V,W)^I = \mathcal{T}_M{}^I{}_N V^M W^N &= (L^D_V W)^I - (L_V W)^I \\ &= V^M D_M W^I - W^M D_M V^I + W_M D^I V^M \\ &\quad - V^M \partial_M W^I + W^M \partial_M V^I - W_M \partial^I V^M \\ &= V^M \Omega_M{}^I{}_N W^N - W^M \Omega_M{}^I{}_N V^N + W_M \Omega^{IM}{}_N V^N \\ &= (\Omega_M{}^I{}_N - \Omega_N{}^I{}_M + \Omega^I{}_{NM}) V^M W^N \end{aligned} \tag{3.89}$$

Using the canonical metric to lower the I index and $\Omega_{IMN} = -\Omega_{INM}$ gives

$$\mathcal{T}_{MIN} = \Omega_{MIN} + \Omega_{NMI} + \Omega_{INM} = 3\Omega_{[MIN]} \tag{3.90}$$

so $\mathcal{T} \in \Gamma(\wedge^3 E)$.

On the non-coordinate frame $\{\hat{E}_A\}$, the components of generalised torsion can also be defined as the usual case (A.107) so

$$\mathcal{T}_{ABC} = \langle \hat{E}_C, \mathcal{T}(\hat{E}_A, \hat{E}_B) \rangle \tag{3.91}$$

where $\langle , \rangle$ is the canonical inner product.

Similar to the Levi-Civita connection, we want the spin connection to be torsion free. The condition $\mathcal{T} = 0$ in the non-coordinate frame $\{\hat{E}_A\}$ is

$$\mathcal{T}_{ABC} = \langle \hat{E}_C, \mathcal{T}(\hat{E}_A, \hat{E}_B) \rangle = \langle \hat{E}_C, L^D_{\hat{E}_A} \hat{E}_B - L_{\hat{E}_A} \hat{E}_B \rangle = 0 \tag{3.92}$$

so

$$\langle \hat{E}_C, L^D_{\hat{E}_A}\hat{E}_B\rangle = \langle \hat{E}_C, L_{\hat{E}_A}\hat{E}_B\rangle \tag{3.93}$$

which shall give constraints on spin connection component Ω_{MAB}. For the LHS, the Dorfman derivative reads

$$\begin{aligned}(L^D_{\hat{E}_A}\hat{E}_B)^M &= \hat{E}_A{}^N D_N \hat{E}_B{}^M - \hat{E}_B{}^N D_N \hat{E}_A{}^M + \hat{E}_{BN} D^M \hat{E}_A{}^N \\ &= \hat{E}_A{}^N \Omega_N{}^D{}_B \hat{E}_D{}^M - \hat{E}_B{}^N \Omega_N{}^D{}_A \hat{E}_D{}^M + \hat{E}_{BN}\Omega^{MD}{}_A \hat{E}_D{}^N \\ &= \Omega_A{}^D{}_B \hat{E}_D{}^M - \Omega_B{}^D{}_A \hat{E}_D{}^M + \Omega^M{}_{BA}\end{aligned} \tag{3.94}$$

where contracting the first index of spin connection with the generalised vielbein simply transfer it from a coordinate index to the non-coordinate index as it is a generalised tensor index. Then contract with $\hat{E}_C$ using (3.44) gives

$$\begin{aligned}\langle \hat{E}_C, L^D_{\hat{E}_A}\hat{E}_B\rangle &= \eta_{MN}\hat{E}_C{}^N (L^D_{\hat{E}_A}\hat{E}_B)^M \\ &= \hat{E}_{CM}(\Omega_A{}^D{}_B \hat{E}_D{}^M - \Omega_B{}^D{}_A \hat{E}_D{}^M + \Omega^M{}_{BA}) \\ &= \Omega_A{}^D{}_B \eta_{CD} - \Omega_B{}^D{}_A \eta_{CD} + \Omega^D{}_{BA}\eta_{CD}\end{aligned} \tag{3.95}$$

where for the last term, $\hat{E}_{CM}\Omega^M{}_{BA} = \eta_{CD}\hat{E}^D{}_M \Omega^M{}_{BA} = \Omega^D{}_{BA}\eta_{CD}$. However, unlike the torsion component in the coordinate basis (3.90), one cannot directly write down an anti-symmetrised expression by lowering indices using η since η in the non-coordinate frame (3.36) has a different sign for the two sub-frames $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$. Explicitly, one finds four different cases

$$\begin{aligned}\langle \hat{E}_c, L^D_{\hat{E}_a}\hat{E}_b\rangle &= \Omega_a{}^d{}_b \eta_{cd} - \Omega_b{}^d{}_a \eta_{cd} + \Omega^d{}_{ba}\eta_{cd} = 3\Omega_{[acb]} \\ \langle \hat{E}_{\bar{c}}, L^D_{\hat{E}_{\bar{a}}}\hat{E}_{\bar{b}}\rangle &= -\Omega_{\bar{a}}{}^{\bar{d}}{}_{\bar{b}}\eta_{\bar{c}\bar{d}} + \Omega_{\bar{b}}{}^{\bar{d}}{}_{\bar{a}}\eta_{\bar{c}\bar{d}} - \Omega^{\bar{d}}{}_{\bar{b}\bar{a}}\eta_{\bar{c}\bar{d}} = -3\Omega_{[\bar{a}\bar{c}\bar{b}]} \\ \langle \hat{E}_c, L^D_{\hat{E}_{\bar{a}}}\hat{E}_b\rangle &= \Omega_{\bar{a}}{}^d{}_b \eta_{cd} - \Omega_b{}^d{}_{\bar{a}}\eta_{cd} + \Omega^d{}_{b\bar{a}}\eta_{cd} = \Omega_{\bar{a}cb} \\ \langle \hat{E}_{\bar{c}}, L^D_{\hat{E}_a}\hat{E}_{\bar{b}}\rangle &= -\Omega_a{}^{\bar{d}}{}_{\bar{b}}\eta_{\bar{c}\bar{d}} + \Omega_{\bar{b}}{}^{\bar{d}}{}_a \eta_{\bar{c}\bar{d}} - \Omega^{\bar{d}}{}_{\bar{b}a}\eta_{\bar{c}\bar{d}} = -\Omega_{a\bar{c}\bar{b}}\end{aligned} \tag{3.96}$$

where we used $\eta_{a\bar{b}} = 0$ and $\Omega_{Mab} = -\Omega_{Mba}, \Omega_{M\bar{a}\bar{b}} = -\Omega_{M\bar{b}\bar{a}}, \Omega_{M\bar{b}c} = \Omega_{Mc\bar{b}} = 0$, and so there is no terms with last two indices from different sub-frames.

Next, for the RHS of (3.93), the Dorfman derivative has component

$$(L_{\hat{E}_A}\hat{E}_B)^M = \hat{E}_A{}^N \partial_N \hat{E}_B{}^M - \hat{E}_B{}^N \partial_N \hat{E}_A{}^M + \hat{E}_{BN}\partial^M \hat{E}_A{}^N \tag{3.97}$$

and contract with $\hat{E}_{CM}$ gives

$$\begin{aligned}\langle \hat{E}_C, L_{\hat{E}_A}\hat{E}_B\rangle &= \hat{E}_A{}^N(\partial_N \hat{E}_B{}^M)\hat{E}_{CM} - \hat{E}_B{}^N(\partial_N \hat{E}_A{}^M)\hat{E}_{CM} + \hat{E}_{BN}(\partial^M \hat{E}_A{}^N)\hat{E}_{CM} \\ &= \hat{E}_A{}^N(\partial_N \hat{E}_B{}^M)\hat{E}_{CM} + \hat{E}_B{}^N(\partial_N \hat{E}_C{}^M)\hat{E}_{AM} + \hat{E}_C{}^M(\partial_M \hat{E}_A{}^N)\hat{E}_{BN} \\ &= \hat{E}_{[A}{}^N(\partial_{|N|}\hat{E}_B{}^M)\hat{E}_{C]M}\end{aligned} \tag{3.98}$$

where for the second term, we used $\partial_N \eta_{AC} = 0 = (\partial_N \hat{E}_C{}^M)\hat{E}_{AM} + \hat{E}_B{}^N(\partial_N \hat{E}_A{}^M)\hat{E}_{CM}$. Hence the RHS is also cyclic A, B, C and can be anti-symmetrised. As the LHS has four cases, the RHS shall also have and only have four cases. To avoid redundant calculation, we only consider the easiest case with $A = a, B = b, C = c$ as an example. Using the expression for $\hat{E}_A$ (3.49) and (3.48), one has

$$\begin{aligned}\hat{E}_a{}^N(\partial_N \hat{E}_b{}^M)\hat{E}_{cM} &= \begin{pmatrix} \hat{e}^+{}_a{}^\nu \partial_\nu \hat{e}^+{}_b{}^\mu \\ \hat{e}^+{}_a{}^\nu \partial_\nu (e^+{}_{b\mu} + \hat{e}^+{}_b{}^\rho B_{\rho\mu}) \end{pmatrix}^T \frac{1}{2}\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} \hat{e}_c{}^\mu \\ e^+{}_{c\mu} + \hat{e}^+{}_c{}^\rho B_{\rho\mu} \end{pmatrix} \\ &= \frac{1}{2}[2\hat{e}^+{}_a{}^\nu(\partial_\nu \hat{e}^+{}_b{}^\mu)e^+{}_{c\mu} + \hat{e}^+{}_a{}^\nu(\partial_\nu g_{\rho\mu})e^+{}_b{}^\rho \hat{e}^+{}_c{}^\mu + (\partial_\nu B_{\rho\mu})\hat{e}^+{}_a{}^\nu \hat{e}^+{}_b{}^\rho \hat{e}^+{}_c{}^\mu]\end{aligned} \tag{3.99}$$

Then, by anti-symmetrising a, b, c, the middle term with the metric vanishes since $g_{\mu\nu}$ is symmetric in ρ, μ whereas $e^+{}_b{}^\rho \hat{e}^+{}_c{}^\mu$ is anti-symmetric in ρ, μ. The last term becomes

$$\frac{1}{2}(\partial_\nu B_{\rho\mu})\hat{e}^+{}_{[a}{}^\nu \hat{e}^+{}_b{}^\rho \hat{e}^+{}_{c]}{}^\mu = \frac{1}{2}(\frac{1}{3}dB_{\nu\rho\mu})\hat{e}^+{}_{[a}{}^\nu \hat{e}^+{}_b{}^\rho \hat{e}^+{}_{c]}{}^\mu = \frac{1}{6}H_{abc} \tag{3.100}$$

where $H = dB$ is interpreted as the 3-form field strength. The first term will become the Levi-Civita connection on a vielbein after anti-symmetrisation. Consider

$$\begin{aligned}\hat{e}^+{}_{[a}{}^\nu(\partial_\nu \hat{e}^+{}_b{}^\mu)e^+{}_{c]\mu} = \frac{1}{6}[&\hat{e}^+{}_a{}^\nu(\partial_\nu \hat{e}^+{}_b{}^\mu)e^+{}_{c\mu} - \hat{e}^+{}_b{}^\nu(\partial_\nu \hat{e}^+{}_a{}^\mu)e^+{}_{c\mu} \\ &+\hat{e}^+{}_b{}^\nu(\partial_\nu \hat{e}^+{}_c{}^\mu)e^+{}_{a\mu} - \hat{e}^+{}_c{}^\nu(\partial_\nu \hat{e}^+{}_b{}^\mu)e^+{}_{a\mu} \\ &+\hat{e}^+{}_c{}^\nu(\partial_\nu \hat{e}^+{}_a{}^\mu)e^+{}_{b\mu} - \hat{e}^+{}_a{}^\nu(\partial_\nu \hat{e}^+{}_c{}^\mu)e^+{}_{b\mu}] \\ = \frac{1}{6}[&\hat{e}^+{}_a{}^\nu(\partial_\nu \hat{e}^+{}_b{}^\mu + \Gamma_{\rho\nu}{}^\mu \hat{e}_b{}^\rho)e^+{}_{c\mu} - \hat{e}^+{}_b{}^\nu(\partial_\nu \hat{e}^+{}_a{}^\mu + \Gamma_{\rho\nu}{}^\mu \hat{e}_a{}^\rho)e^+{}_{c\mu} \\ &+\hat{e}^+{}_b{}^\nu(\partial_\nu \hat{e}^+{}_c{}^\mu + \Gamma_{\rho\nu}{}^\mu \hat{e}_c{}^\rho)e^+{}_{a\mu} - \hat{e}^+{}_c{}^\nu(\partial_\nu \hat{e}^+{}_b{}^\mu + \Gamma_{\rho\nu}{}^\mu \hat{e}_b{}^\rho)e^+{}_{a\mu} \\ &+\hat{e}^+{}_c{}^\nu(\partial_\nu \hat{e}^+{}_a{}^\mu + \Gamma_{\rho\nu}{}^\mu \hat{e}_a{}^\rho)e^+{}_{b\mu} - \hat{e}^+{}_a{}^\nu(\partial_\nu \hat{e}^+{}_c{}^\mu + \Gamma_{\rho\nu}{}^\mu \hat{e}_c{}^\rho)e^+{}_{b\mu}]\end{aligned} \tag{3.101}$$

where the extra Levi-Civita connection terms Γ cancel in each line as it is symmetric in the first two indices, hence, using (A.121), one has

$$\begin{aligned}
\hat{e}^{+}{}_{[a}{}^{\nu}(\partial_{\nu}\hat{e}^{+}{}_{b}{}^{\mu})e^{+}{}_{c]\mu} = &\frac{1}{6}[\hat{e}^{+}{}_{a}{}^{\nu}(\nabla_{\nu}\hat{e}^{+}{}_{b}{}^{\mu})e^{+}{}_{c\mu} - \hat{e}^{+}{}_{b}{}^{\nu}(\nabla_{\nu}\hat{e}^{+}{}_{a}{}^{\mu})e^{+}{}_{c\mu} \\
&+\hat{e}^{+}{}_{b}{}^{\nu}(\nabla_{\nu}\hat{e}^{+}{}_{c}{}^{\mu})e^{+}{}_{a\mu} - \hat{e}^{+}{}_{c}{}^{\nu}(\nabla_{\nu}\hat{e}^{+}{}_{b}{}^{\mu})e^{+}{}_{a\mu} \\
&+\hat{e}^{+}{}_{c}{}^{\nu}(\nabla_{\nu}\hat{e}^{+}{}_{a}{}^{\mu})e^{+}{}_{b\mu} - \hat{e}^{+}{}_{a}{}^{\nu}(\nabla_{\nu}\hat{e}^{+}{}_{c}{}^{\mu})e^{+}{}_{b\mu}] \\
= &\frac{1}{6}[\hat{e}^{+}{}_{a}{}^{\nu}\omega_{\nu bc} - \hat{e}^{+}{}_{b}{}^{\nu}\omega_{\nu ac} + \hat{e}^{+}{}_{b}{}^{\nu}\omega_{\nu ca} - \hat{e}^{+}{}_{c}{}^{\nu}\omega_{\nu ba} + \hat{e}^{+}{}_{c}{}^{\nu}\omega_{\nu ab} - \hat{e}^{+}{}_{a}{}^{\nu}\omega_{\nu cb}] \\
= &\frac{1}{6}(\omega_{abc} - \omega_{acb} + \omega_{bca} - \omega_{bac} + \omega_{cba} - \omega_{cab})
\end{aligned} \tag{3.102}$$

where ω is the Levi-Civita spin connection component with metric g. Since Levi-Civita connection is torsion-free, so $\omega_{abc} = -\omega_{acb}$ then

$$\hat{e}^{+}{}_{[a}{}^{\nu}(\partial_{\nu}\hat{e}^{+}{}_{b}{}^{\mu})e^{+}{}_{c]\mu} = \frac{1}{3}(\omega_{acb} + \omega_{bac} + \omega_{cba}) = \omega^{+}_{[acb]} \tag{3.103}$$

where the spin connection is denoted as ω^{+} for it is calculated on $\{\hat{e}_a\}$. Then anti-symmetrise (3.99) and substitute the results of (3.100) and (3.103) into it gives

$$\langle \hat{E}_c, L_{\hat{E}_a}\hat{E}_b\rangle = 3\hat{E}_{[a}{}^{N}(\partial_{|N|}\hat{E}_{b}{}^{M})\hat{E}_{c]M} = 3\omega^{+}_{[acb]} + \frac{1}{2}H_{abc} \tag{3.104}$$

Hence by comparing with (3.96), the torsion-free condition (3.93) for $A = a, B = b, C = c$ becomes

$$\Omega_{[abc]} = \omega^{+}_{[abc]} - \frac{1}{6}H_{abc}. \tag{3.105}$$

Note there is a switching in indices b, c giving the extra minus sign in front of H and the whole equation is divided by 3. The other cases shall be considered in a similar way by brutal calculations giving [42]

$$\begin{aligned}
\Omega_{[\bar{a}\bar{b}\bar{c}]} &= \omega^{-}_{[\bar{a}\bar{b}\bar{c}]} + \frac{1}{6}H_{\bar{a}\bar{b}\bar{c}} \\
\Omega_{\bar{a}bc} &= \omega^{+}_{\bar{a}bc} - \frac{1}{2}H_{\bar{a}bc} \\
\Omega_{a\bar{b}\bar{c}} &= \omega^{-}_{a\bar{b}\bar{c}} + \frac{1}{2}H_{a\bar{b}\bar{c}}
\end{aligned} \tag{3.106}$$

where ω^{-} is calculated in the $\{\hat{E}^{-}_{\bar{a}}\}$ frame.

However, for the two conditions with indices all from the same sub-frame, the anti-symmetrisation implies that one can freely add another rank 3 tensor which vanishes under anti-symmetrisation i.e. for $A^{+}_{[abc]} = A^{-}_{[\bar{a}\bar{b}\bar{c}]} = 0$, the following conditions still hold

$$\begin{aligned}\Omega_{[abc]} &= \omega^{+}_{[abc]} - \frac{1}{6}H_{abc} = \omega^{+}_{[abc]} - \frac{1}{6}H_{abc} + A^{+}_{[abc]} \\ \Omega_{[\bar{a}\bar{b}\bar{c}]} &= \omega^{-}_{[\bar{a}\bar{b}\bar{c}]} + \frac{1}{6}H_{\bar{a}\bar{b}\bar{c}} = \omega^{-}_{[\bar{a}\bar{b}\bar{c}]} + \frac{1}{6}H_{\bar{a}\bar{b}\bar{c}} + A^{-}_{[\bar{a}\bar{b}\bar{c}]}\end{aligned}. \tag{3.107}$$

Also, for the connection to be compatible with the generalised metric, $A^{\pm}$ also need to satisfy $A^{+}_{abc} = -A^{+}_{acb}$, $A^{-}_{[\bar{a}\bar{b}\bar{c}]} = -A^{-}_{[\bar{a}\bar{c}\bar{b}]}$. Therefore, unlike the Levi-Civita connection, the compatible torsion-free generalised connection is not unique due to these extra tensors.

In conclusion, the conditions for the compatible and torsion-free generalised connection are

$$\begin{aligned}\Omega_{[abc]} &= \omega^{+}_{[abc]} - \frac{1}{6}H_{abc} = \omega^{+}_{[abc]} - \frac{1}{6}H_{abc} + A^{+}_{[abc]} \\ \Omega_{[\bar{a}\bar{b}\bar{c}]} &= \omega^{-}_{[\bar{a}\bar{b}\bar{c}]} + \frac{1}{6}H_{\bar{a}\bar{b}\bar{c}} = \omega^{-}_{[\bar{a}\bar{b}\bar{c}]} + \frac{1}{6}H_{\bar{a}\bar{b}\bar{c}} + A^{-}_{[\bar{a}\bar{b}\bar{c}]} \\ \Omega_{\bar{a}bc} &= \omega^{+}_{\bar{a}bc} - \frac{1}{2}H_{\bar{a}bc} \\ \Omega_{a\bar{b}\bar{c}} &= \omega^{-}_{a\bar{b}\bar{c}} + \frac{1}{2}H_{a\bar{b}\bar{c}}\end{aligned} \tag{3.108}$$

where

$$A^{+}_{[abc]} = A^{-}_{[\bar{a}\bar{b}\bar{c}]} = 0, \quad A^{+}_{abc} = -A^{+}_{acb}, \quad A^{-}_{\bar{a}\bar{b}\bar{c}} = -A^{-}_{\bar{a}\bar{c}\bar{b}}. \tag{3.109}$$

Furthermore, from the calculation for $\langle \hat{E}_C, L_{\hat{E}_A}\hat{E}_B\rangle$ i.e. (3.101) to (3.102), one can see that

$$\begin{aligned}\langle \hat{E}_C, L_{\hat{E}_A}\hat{E}_B\rangle &\sim \hat{E}_A{}^{\mu}(\nabla_\mu \hat{E}_B{}^M)\hat{E}_{CM} + \text{cyclic terms} + H\text{term} \\ &\sim \hat{E}_A{}^{M}(D^{\nabla}_M \hat{E}_B{}^M)\hat{E}_{CM} + \text{cyclic terms} + H\text{term}\end{aligned} \tag{3.110}$$

using the definition of D^{∇} (3.86). Then one can find the torsion of D^{∇} denoted as Σ is

$$\begin{aligned}\Sigma_{ABC} &= \langle \hat{E}_C, L^{D^{\nabla}}_{\hat{E}_A}\hat{E}_B - L_{\hat{E}_A}\hat{E}_B\rangle = \langle \hat{E}_C, L^{D^{\nabla}}_{\hat{E}_A}\hat{E}_B\rangle - \langle \hat{E}_C, L_{\hat{E}_A}\hat{E}_B\rangle \\ &\sim [\hat{E}_A{}^{M}(D^{\nabla}_M \hat{E}_B{}^M)\hat{E}_{CM} - \hat{E}_A{}^{M}(D^{\nabla}_M \hat{E}_B{}^M)\hat{E}_{CM}] + \text{cyclic terms} - H\text{term} \\ &\sim -H\text{term}\end{aligned} \tag{3.111}$$

Explicitly, it is [4]

$$\Sigma = -4H, \quad H = dB \tag{3.112}$$

and in component is

$$3\Sigma_{[ABC]} = -4H_{ABC}. \tag{3.113}$$

Hence, by subtracting the torsion from D^{∇}, one can construct a torsion-free connection

$$D_M^{\text{free}} V^A = D_M^{\nabla} V^A + \Sigma_M{}^A{}_B V^B \tag{3.114}$$

where the index of Σ is raised using η so there are some changes in sign.

The torsion-free conditions can also be written using Σ. Firstly, the 3-form H can be written in components as

$$H = \frac{1}{3!} H_{\mu\nu\rho} dx^{\mu} \wedge dx^{\mu} \wedge dx^{\rho} \tag{3.115}$$

and using the non-coordinate basis,

$$dx^{\mu} = \frac{1}{2}(\hat{e}_a^{+\mu} \hat{E}^{+a} - \hat{e}_{\bar{a}}^{-\mu} \hat{E}^{-\bar{a}}) \tag{3.116}$$

we have

$$\begin{aligned} H = \frac{1}{6}\frac{1}{8}(&H_{abc}\hat{E}^{+a} \wedge \hat{E}^{+b} \wedge \hat{E}^{+c} - 3H_{\bar{a}bc}\hat{E}^{+\bar{a}} \wedge \hat{E}^{+b} \wedge \hat{E}^{+c} \\ &+3H_{a\bar{b}\bar{c}}\hat{E}^{+a} \wedge \hat{E}^{+\bar{b}} \wedge \hat{E}^{+\bar{c}} - H_{\bar{a}\bar{b}\bar{c}}\hat{E}^{+\bar{a}} \wedge \hat{E}^{+\bar{b}} \wedge \hat{E}^{+\bar{c}}) \end{aligned} \tag{3.117}$$

where minus signs come from switching indices. Σ is also decomposed to $\Sigma_{abc}, \Sigma_{\bar{a}bc}, \Sigma_{a\bar{b}\bar{c}}$ and $\Sigma_{\bar{a}\bar{b}\bar{c}}$, and using (3.113) one has

$$\begin{aligned} \Sigma_{[abc]} &= -\frac{1}{6}H_{abc}, \quad \Sigma_{[\bar{a}\bar{b}\bar{c}]} = \frac{1}{6}H_{\bar{a}\bar{b}\bar{c}} \\ \Sigma_{\bar{a}bc} &= -\frac{1}{2}H_{\bar{a}bc}, \quad \Sigma_{a\bar{b}\bar{c}} = \frac{1}{2}H_{a\bar{b}\bar{c}} \end{aligned} \tag{3.118}$$

and similar to Ω, one can add $A^{\pm}$ satisfying (3.109) to the two terms in the first line, so the connection is not unique. These conditions on Σ differ from (3.108) by the usual spin connection ω which comes from ∇ and is absorbed into D^{∇} as

$$D^{\text{free}} = \partial + \Omega = \partial + \omega + \Sigma = \nabla + \Sigma = D^{\nabla} + \Sigma. \tag{3.119}$$

Therefore, one can find a torsion-free compatible connection but is not unique. In terms of D^{∇}, it is given by

$$\begin{aligned} D_a V_+^b &= \nabla_a V_+^b - \frac{1}{6}H_a{}^b{}_c V_+^c + A^+{}_a{}^b{}_c V_+^c \\ D_{\bar{a}} V_+^b &= \nabla_{\bar{a}} V_+^b - \frac{1}{2}H_{\bar{a}}{}^b{}_c V_+^c \\ D_a V_-^{\bar{b}} &= \nabla_a V_-^{\bar{b}} + \frac{1}{2}H_a{}^{\bar{b}}{}_{\bar{c}} V_-^{\bar{c}} \\ D_{\bar{a}} V_-^{\bar{b}} &= \nabla_{\bar{a}} V_-^{\bar{b}} + \frac{1}{6}H_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} V_-^{\bar{c}} + A^-{}_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} V_-^{\bar{c}} \end{aligned} \tag{3.120}$$

where

$$A^+_{[abc]} = A^-_{[\bar{a}\bar{b}\bar{c}]} = 0, \quad A^+_{abc} = -A^+_{acb}, \quad A^-_{\bar{a}\bar{b}\bar{c}} = -A^-_{\bar{a}\bar{c}\bar{b}}. \tag{3.121}$$

3.6 Generalised Curvature

The Riemann curvature can also be generalised as direct analogue of Definition A.42, with the affine connection replaced by the generalised connection and Lie bracket replaced by Courant bracket.

Definition 3.3. The generalised Riemann curvature tensor is a map

$$\begin{array}{rlcl} R: & \Gamma(E)\times\Gamma(E)\times\Gamma(E) & \to & \Gamma(E) \\ & U, V, W & \to & R(U,V,W) \end{array} \tag{3.122}$$

where

$$R(U,V,W) = D_U D_V W - D_V D_U W - D_{[\![U,V]\!]} W \tag{3.123}$$

and satisfies symmetry properties as the ordinary curvature.

However, the generalised curvature is not tensorial since it is not multi-linear [41]. For functions f, g, h,

$$\begin{aligned} R(fU, gV, hW) &= D_{fU} D_{gV} hW - D_{gV} D_{fU} hW - D_{[\![fU,gV]\!]} hW \\ &= fghR(U,V,W) - \frac{1}{2} h \langle U, V\rangle D_{fdg-gdf} W \end{aligned} \tag{3.124}$$

where $[\![fU, gV]\!]$ is expanded using (2.60). For it to be tensorial, the extra term

$$\frac{1}{2} h \langle U, V\rangle D_{fdg-gdf} W \tag{3.125}$$

needs to vanish. It is zero if $\langle U, V\rangle = 0$ which means U, V need to be elements of two subspaces that are orthogonal with respect to the canonical inner product, for example the sub-bundles $C_\pm$. Therefore, the $O(d)\times O(d)$ structure can define a generalised curvature that is tensorial i.e. for $U \in \Gamma(C_+), V \in \Gamma(C_-)$, then $R \in \Gamma((C_+ \otimes C_-) \otimes \mathfrak{o}(d,d))$ is a tensor [4]. But since the generalised connection is not unique, there is also no unique generalised curvature.

4 Type II Supergravity

It is known that for type II supergravity, the bosonic fields in the NSNS sector are the metric, a 2-form gauge field and a dilaton. From the previous chapters, one sees how a non-degenerate symmetric 2-tensor g and a 2-form B are encoded in the generalised metric of the $O(d,d)$ generalised geometry. It is natural to identify them with the metric and the gauge field for they have same properties correspondingly as discussed before. Also, the Courant bracket or the Dorfman derivative encodes the symmetries of diffeomorphism and gauge symmetry which are the bosonic symmetries in type II supergravity. Then it is natural to use the $O(d,d)$ generalised geometry to formulate type II supergravity, at least for the bosonic NSNS sector. Moreover, the extra dilaton can also be included by adding one more degree of freedom to the generalised tangent bundle. The reformulation of type II supergravity will follow [4], and spinors and RR fields will be briefly introduced at the end for completeness of curvature scalar and the equations of motion.

4.1 Bosonic Symmetries

This section provides some basic informations for type II supergravity. Type II superstring theory is a 10-dimensional theory with $\mathcal{N}=2$ supersymmetry [18]. Using the democratic formalism [43], the bosonic 'pseudo'-action is

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int \sqrt{-\det g}\left[e^{-2\phi}\left(\mathcal{R}+4(\partial\phi)^2-\frac{1}{12}H^2\right)-\frac{1}{4}\sum_n \frac{1}{n!}(F^{(B)}_{(n)})^2\right] \tag{4.1}$$

where $g_{\mu\nu}$ is the metric, $H=dB$ is the 3-form field strength, $B_{\mu\nu}$ is the 2-form potential, ϕ is the dilaton, and $F^{(B)}_{(n)}$ is the n-form RR field strength where explicit expressions can be found in [4]. The NSNS sector only contains the metric g, 2-form B-field and the dilaton ϕ, so the action for NSNS sector is

$$S_{\mathrm{NS}} = \frac{1}{2\kappa^2}\int \sqrt{-\det g}e^{-2\phi}\left(\mathcal{R}+4(\partial\phi)^2-\frac{1}{12}H^2\right) \tag{4.2}$$

which is invariant under diffeomorphism

$$\delta g_{\mu\nu}=\mathcal{L}_X g_{\mu\nu}, \quad \delta B_{\mu\nu}=\mathcal{L}_X B_{\mu\nu}, \quad \delta\phi=\mathcal{L}_X\phi, \tag{4.3}$$

where X is a vector. It is also invariant under gauge transformation as the field strength $H=dB$ is not changed after shifting the B-field by an exact term

$$B_{(i)} = B_{(j)} - d\Lambda_{(ij)} \tag{4.4}$$

which is same as the patching (2.79) for the B field in $O(d,d)$ generalised geometry and $\Lambda_{(ij)}$ is a 1-form satisfying (2.82). Hence, B is a gauge field and is only locally defined. As the gauge transformation is parameterised by the 2-from $d\Lambda$ which is closed, and gauge transformations do not commute with the diffeomorphism, so the overall symmetry group is [4]

$$\mathrm{Diff}(\mathcal{M}) \ltimes \Omega^2_{\text{closed}}(\mathcal{M}) \tag{4.5}$$

which is the geometric subgroup that preserves the Courant bracket. Then, combine the diffeomorphism and gauge transformation gives

$$\delta_{X+\xi} g_{\mu\nu} = \mathcal{L}_X g_{\mu\nu}, \quad \delta_{X+\xi} B_{\mu\nu} = \mathcal{L}_X B_{\mu\nu} - d\xi, \quad \delta_{X+\xi}\phi = \mathcal{L}_X \phi, \tag{4.6}$$

where ξ is a 1-form, and the patching of B (4.4) implies that

$$\xi_{(i)} = \xi_{(j)} - i_{X_{(j)}} d\Lambda_{(ij)} \tag{4.7}$$

which is same as in (2.88) for the patching of the form part of the generalised vector in $O(d,d)$ generalised geometry.

Therefor, one sees clearly that the bosonic symmetries of NSNS sector of type II supergravity namely the diffeomorphism and gauge transformation can be described using the $O(d,d)$ generalised geometry.

4.2 $O(p,q) \times O(q,p) \times \mathbb{R}^+$ Generalised Geometry

As mentioned before, diffeomorphism and gauge transformations are encoded in the Courant bracket, and the metric g with signature (p,q) and B-field are encoded in the generalised metric of the $O(p,q)\times O(q,p)$ generalised geometry. However, to describe the full NSNS sector of type II supergravity, we also need to include the dilaton by extending the structure group from $O(d,d)$ to $O(d,d)\times\mathbb{R}^+$ where $\mathbb{R}^+$ provides one more degree of freedom for the dilaton [10, 11].

Firstly, the tangent bundle E defined before is extended by weighting by $\det T^*\mathcal{M}$, where $\mathcal{M}$ is a d-dimensional spin manifold [4] (see Definition A.34). Then the extended tangent bundle is

$$\tilde{E} \simeq \det T^*\mathcal{M} \otimes E \simeq \det T^*\mathcal{M} \otimes (T\mathcal{M} \oplus T^*\mathcal{M}). \tag{4.8}$$

The significance of $\mathcal{M}$ being a spin manifold is that it is orientable, so from Claim A.21, $\det T^*\mathcal{M}$ is a trivial bundle so $\det T^*\mathcal{M} \simeq \mathbb{R}\times\mathcal{M}$ where $\mathbb{R}$ provides the degree of freedom for the dilaton, and $\det T^*\mathcal{M}$ can have any power i.e. $(\det T^*\mathcal{M})^p$ is well-defined where

p gives the weight for the element. Thus, a rank-n generalised tensor is now a section of

$$E^{\otimes n}_{(p)} = (\det T^*\mathcal{M})^p \otimes E \otimes \cdots \otimes E \tag{4.9}$$

with weight p [4].

The split frame of $\tilde{E}$ needs to be a conformal basis $\{\hat{E}_M\}$ including a conformal factor $\Phi \in \Gamma(\det T^*\mathcal{M})$ for the weighting, so that

$$\langle \hat{E}_M, \hat{E}_N \rangle = \Phi^2 \eta_{MN} = \Phi^2 \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \tag{4.10}$$

where $\Phi^2 \in \mathbb{R}^+$. This condition manifests the structure group $O(d,d) \times \mathbb{R}^+$. Under a change of frame, the generalised vector V and basis $\hat{E}_A$ transform as before

$$V^A \to V'^A = M^A{}_B V^B, \quad \hat{E}_A \to \hat{E}'_A = (M^{-1})^B{}_A \hat{E}_B, \tag{4.11}$$

where $M \in O(p,q) \times O(q,p) \times \mathbb{R}^+$ satisfying $(M^{-1})^C{}_A (M^{-1})^D{}_B \eta_{CD} = \sigma^2 \eta_{AB}$ for some σ from the extra weighting and this means that the transformation preserves the metric up to a conformal factor. The transform that preserve the Courant bracket is

$$M = (\det A)^{-1} \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ \Theta & \mathbf{1} \end{pmatrix} \tag{4.12}$$

where $A \in GL(d,\mathbb{R})$ generates the diffeomorphism and Θ is generator of the B-transform and is closed, so this still defines the geometric subgroup $GL(d,\mathbb{R}) \ltimes \Omega^2_{\text{closed}} \subset O(d,d) \times \mathbb{R}^+$ as before. The extra factor $(\det A)^{-1}$ is to cancel the $\det A$ term from transformation of $V \in \Gamma(\tilde{E})$ under diffeomorphism $V^M \to (\det A) A^M{}_N \tilde{V}^N$ where $\tilde{V} \in \Gamma(E)$ is the unweighted section.

As $\tilde{E} \simeq (\det T^*\mathcal{M}) \otimes (T\mathcal{M} \oplus T^*\mathcal{M})$, we can use the generic basis $\{\hat{e}_\mu\}$ for $T\mathcal{M}$, dual basis $\{e^\mu\}$ for $T^*\mathcal{M}$ and $\det e$ for basis of $\det T^*\mathcal{M}$ to construct the conformal split frame for $\tilde{E}$, and it is also useful to rescale it by a function ϕ to include the dilaton [4], so

$$\hat{E}_M = \begin{cases} \hat{E}_\mu = e^{-2\phi}(\det e)(\hat{e}_\mu + i_{\hat{e}_\mu} B), & M = \mu \\ E^\mu = e^{-2\phi}(\det e) e^\mu, & M = \mu + d \end{cases} \tag{4.13}$$

satisfying

$$\langle \hat{E}_M, \hat{E}_N \rangle = e^{-4\phi}(\det e)^2 \eta_{MN}, \quad \Phi = e^{-2\phi}(\det e) \tag{4.14}$$

and the patching of a generalised vector $V \in \Gamma(\tilde{E})$ in this conformal split frame is also weighted from (2.89), so on a chart U_i,

$$V = e^{2\phi} \begin{pmatrix} X_{(i)} \\ \xi_{(i)} - i_{X_{(i)}} B_{(i)} \end{pmatrix} \tag{4.15}$$

For the Dorfman derivative, although the definition remains the same as before, but since the components of generalised vectors are weighted, the component of the Dorfman derivative changes. Following the procedure in Section 2.4, we first need the Lie derivative on a weighted vector $X \in \Gamma((\det T^*\mathcal{M})^p \otimes T\mathcal{M})$ and a weighted 1-form $\xi \in \Gamma((\det T^*\mathcal{M})^p \otimes T\mathcal{M})$ each with weight p, which are [4]

$$\begin{aligned} \mathcal{L}_v X^\mu &= v^\nu \partial_\nu X^\mu - X^\nu \partial_\nu v^\mu + p(\partial_\nu v^\nu) X^\mu \\ \mathcal{L}_v \xi_\mu &= v^\nu \partial_\nu \xi_\mu + (\partial_\mu v^\nu)\xi_\nu + p(\partial_\nu v^\nu)\xi_\mu \end{aligned} \tag{4.16}$$

where $v \in \Gamma(T\mathcal{M})$ is an unweighted vector. Then for a unweighted generalised vector $V \in \Gamma(E)$ and a weighted generalised vector $W \in \Gamma((\det T^*\mathcal{M})^p) \otimes E$, substituting the above equations into the definition of Dorfman derivative (2.25) as in (2.42) gives the component of Dorfman derivative with an extra term

$$L_V W^M = V^N \partial_N W^M - W^N \partial_N V^M + W_N \partial^M V^N + p(\partial_N V^N) W^M \tag{4.17}$$

and extend the action to a weighted tensor $J \in \Gamma(E_{(p)}^{\otimes n})$ also gives an extra term

$$\begin{aligned} L_V J^{M_1 \ldots M_n} = V^N \partial_N J^{M_1 \ldots M_n} &+ (\partial^{M_1} V^N - \partial^N V^{M_1}) J_N{}^{M_2 \ldots M_n} \\ &+ \cdots + (\partial^{M_n} V^N - \partial^N V^{M_n}) J^{M_1 \ldots M_{n-1}}{}_N + p(\partial_N V^N) W^M \end{aligned} \tag{4.18}$$

After introducing the generalised metric G, the structure group reduces to $O(p,q) \times O(q,p) \times \mathbb{R}^+$, and as for the $O(p,q) \times O(q,p)$ case, we can define the conformal non-coordinate frame $\{\hat{E}_A\} = \{\hat{E}_a^+\} \cup \{\hat{E}_{\bar{a}}^-\}$ where $\{\hat{E}_a^+\}$ and $\{\hat{E}_{\bar{a}}^-\}$ are bases for C_+ and C_- respectively, satisfying

$$\begin{aligned} \langle \hat{E}_a^+, \hat{E}_b^+ \rangle &= \Phi^2 \eta_{ab} \\ \langle \hat{E}_{\bar{a}}^-, \hat{E}_{\bar{b}}^- \rangle &= -\Phi^2 \eta_{\bar{a}\bar{b}} \\ \langle \hat{E}_a^+, \hat{E}_{\bar{b}}^- \rangle &= 0 \end{aligned} \tag{4.19}$$

This fixes the choice of $\Phi \in \Gamma(\det T^*\mathcal{M})$ and gives an isomorphism between the weighted bundle $\tilde{E}$ and the unweighted bundle E so apart from the canonical metric η and the generalised metric G, we have another structure associated with Φ. Then from (4.19) the conformal non-coordinate frame satisfies

$$\langle \hat{E}_A, \hat{E}_B \rangle = \Phi^2 \eta_{AB} = \Phi^2 \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix}, \tag{4.20}$$

and with respect to the generalised metric as

$$G(\hat{E}_A, \hat{E}_B) = \Phi^2 \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix} \tag{4.21}$$

By using the usual vielbein $\hat{e}^{\pm}$ and dual $e^{\pm}$ that satisfy conditions in (3.48), the conformal non-coordinate frame is

$$\hat{E}_A = \begin{cases} \hat{E}^+_a = e^{-2\phi}\sqrt{-\det g}(\hat{e}^+_a + e^+_a + i_{\hat{e}^-_a} B), & A = a, \\ \hat{E}^-_{\bar{a}} = e^{-2\phi}\sqrt{-\det g}(\hat{e}^-_{\bar{a}} - e^-_{\bar{a}} + i_{\hat{e}^-_{\bar{a}}} B), & A = \bar{a} + d. \end{cases} \tag{4.22}$$

i.e. conformally scaled version of (3.49), and

$$\Phi = e^{-2\phi}\sqrt{-\det g} \tag{4.23}$$

where $\sqrt{-\det g}$ is the new basis for $\det T^*\mathcal{M}$. The generalised metric is now

$$G = \Phi^{-2}(\eta^{ab}\hat{E}^+_a \otimes \hat{E}^+_b + \eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}}). \tag{4.24}$$

Hence the generalised metric G is parameterised by g, B, ϕ and so the coset space is also parametrised by

$$\{g, B, \phi\} \in \frac{O(d,d)}{O(p,q) \times O(q,p)} \times \mathbb{R}^+ \tag{4.25}$$

where $d = p + q$.

4.3 Extended Generalised Connection and Torsion

The introduction of Φ and the conformal frame also modifies the constructions of other generalised geometric objects. These objects associated with the extended bundle $\tilde{E}$ will be called 'extended generalised'.

Starting with the connection, for $W \in \Gamma(\tilde{E})$, the extended generalised connection D is defined as

$$D_M W^A = \partial_M W^A + \tilde{\Omega}_M{}^A{}_B W^B \tag{4.26}$$

and on the conformal frame $\{\hat{E}_A\}$ as

$$D_M \hat{E}_A = \tilde{\Omega}_M{}^B{}_A \hat{E}_B. \tag{4.27}$$

As the structure group is extended to $O(p,q) \times O(q,p) \times \mathbb{R}^+$, it is reasonable to require that the extended generalised connection is not only compatible with η and G but also with Φ, so the compatible conditions are now

$$D\eta = DG = D\Phi = 0. \tag{4.28}$$

Since components of both η and G have the extra Φ factor (4.20) and (4.21), this gives extra terms from Φ when calculating $D\eta$ and DG as in (3.73) and (3.75). However, by imposing $D\Phi = 0$ ensures these extra terms from Φ vanishes so that the conditions on the spin connections remain the same as before (3.77) (3.78) i.e.

$$\tilde{\Omega}_{Mab} = -\tilde{\Omega}_{Mba}, \quad \tilde{\Omega}_{M\bar{a}\bar{b}} = -\tilde{\Omega}_{M\bar{b}\bar{a}}, \quad \tilde{\Omega}_{M\bar{a}b} = \tilde{\Omega}_{Mb\bar{a}} = 0 \tag{4.29}$$

For the extended generalised torsion, as the Dorfman derivative includes an extra term from weighting (4.17), the component of torsion (A.107) takes a different form. The Dorfman derivative $L_V W$ in (4.17) has $V \in \Gamma(E)$ whereas $W \in \Gamma(\tilde{E})$, so V, W are in different basis. Given $\{\hat{E}_A\}$ is the frame of $\tilde{E}$, the frame of E is simply $\{\Phi^{-1}\hat{E}_A\}$, so we have the Dorfman derivative acts on basis as $L_{\Phi^{-1}\hat{E}_A}\hat{E}_A$. Then the components of generalised torsion is defined as

$$\mathcal{T}_{ABC} = \Phi^{-2} \langle \hat{E}_C, L^D_{\Phi^{-1}\hat{E}_A}\hat{E}_B - L_{\Phi^{-1}\hat{E}_A}\hat{E}_B \rangle \tag{4.30}$$

where the factor Φ^{-2} is introduced to cancel the Φ^2 factor from the inner product (see later in (4.35)). For $L_{\Phi^{-1}\hat{E}_A}\hat{E}_B$, since $\partial_N\Phi \neq 0$, we need to write $\hat{E}_B = \Phi\Phi^{-1}\hat{E}_B$ and using Leibniz rule gives

$$\begin{aligned} L_{\Phi^{-1}\hat{E}_A}\hat{E}_B &= (L_{\Phi^{-1}\hat{E}_A}\Phi)\Phi^{-1}\hat{E}_B + \Phi L_{\Phi^{-1}\hat{E}_A}(\Phi^{-1}\hat{E}_B) \\ &= [(\Phi^{-1}\hat{E}_A)^N\partial_N\Phi + \partial_N(\Phi^{-1}\hat{E}_A)^N\Phi]\Phi^{-1}\hat{E}_B + \Phi L_{\Phi^{-1}\hat{E}_A}(\Phi^{-1}\hat{E}_B) \\ &= (\partial_N \hat{E}_A{}^N)\Phi^{-1}\hat{E}_B + \Phi L_{\Phi^{-1}\hat{E}_A}(\Phi^{-1}\hat{E}_B) \end{aligned} \tag{4.31}$$

where $L_{\Phi^{-1}\hat{E}_A}\Phi$ is calculated using (4.17) with weight $p = 1$ and Φ does not contract with index N, and from second line to third, we used Leibniz rule again on $\partial(\Phi\Phi^{-1}\hat{E})$. Contracting with $\hat{E}_C$ gives

$$\langle \hat{E}_C, L_{\Phi^{-1}\hat{E}_A}\hat{E}_B \rangle = \Phi(\partial_N \hat{E}_A{}^N)\eta_{BC} + \Phi^2 \langle \Phi^{-1}\hat{E}_C, L_{\Phi^{-1}\hat{E}_A}(\Phi^{-1}\hat{E}_B) \rangle \tag{4.32}$$

note that $\langle \hat{E}_C, \hat{E}_B \rangle = \Phi^2\eta_{BC}$ has extra Φ^2 factor. We also moved one factor Φ^{-1} into the inner product to indicate that the inner product part is the same as (3.97) except that the notation of the frame is changed, so, as before, this term can be written as an anti-symmetrisation of a rank 3 object i.e. $\langle \Phi^{-1}\hat{E}_{[C}, L_{\Phi^{-1}\hat{E}_A}(\Phi^{-1}\hat{E}_{B]}) \rangle$. For the $L^D_{\Phi^{-1}\hat{E}_A}\hat{E}_B{}^M$

term, as $D\Phi = 0$, we can simply expand it

$$
\begin{aligned}
&L^D_{\Phi^{-1}\hat{E}_A}\hat{E}_B{}^M \\
&= (\Phi^{-1}\hat{E}_A)^N D_N \hat{E}_B{}^M - \hat{E}_B{}^N D_N(\Phi^{-1}\hat{E}_A)^M + \hat{E}_{BN} D^M(\Phi^{-1}\hat{E}_A)^N + D_N(\Phi^{-1}\hat{E}_A{}^N)\hat{E}_B{}^M \\
&= \Phi^{-1}(\hat{E}_A{}^N \tilde{\Omega}_N{}^I{}_B \hat{E}_I{}^M - \hat{E}_B{}^N \tilde{\Omega}_N{}^I{}_A \hat{E}_I{}^M + \hat{E}_{BN}\tilde{\Omega}^{MI}{}_A \hat{E}_I{}^N + \tilde{\Omega}_N{}^I{}_A \hat{E}_I{}^N \hat{E}_B{}^M)
\end{aligned}
\tag{4.33}
$$

where the weight of $\hat{E}_B$ is 1. Then contract with $\hat{E}_C$ gives

$$
\begin{aligned}
\langle \hat{E}_C, L^D_{\Phi^{-1}\hat{E}_A}\hat{E}_B\rangle &= \Phi(\hat{E}_A{}^N \tilde{\Omega}_N{}^I{}_B \eta_{IC} - \hat{E}_B{}^N \tilde{\Omega}_N{}^I{}_A \eta_{IC} + \tilde{\Omega}^{MI}{}_A \eta_{BI}\hat{E}_{CM} + \tilde{\Omega}_N{}^I{}_A \hat{E}_I{}^N \eta_{BC}) \\
&= \Phi^2(\tilde{\Omega}_{ACB} + \tilde{\Omega}_{BAC} + \tilde{\Omega}_{CBA} + \tilde{\Omega}_I{}^I{}_A \eta_{BC}) \\
&= \Phi^2(3\tilde{\Omega}_{[ACB]} + \tilde{\Omega}_I{}^I{}_A \eta_{BC})
\end{aligned}
\tag{4.34}
$$

also note that $\langle \hat{E}_C, \hat{E}_B\rangle = \Phi^2\eta_{BC}$, $\tilde{\Omega}_{MAB} = -\tilde{\Omega}_{MBA}$, and the generalised vielbein transfers the coordinate indices M, N to the conformal non-coordinate indices A, B, C and gives an extra factor Φ. Therefore, the component of generalised torsion is

$$
\begin{aligned}
\mathcal{T}_{ABC} &= \Phi^{-2}\langle \hat{E}_C, L^D_{\Phi^{-1}\hat{E}_A}\hat{E}_B - L_{\Phi^{-1}\hat{E}_A}\hat{E}_B\rangle \\
&= \Phi^{-2}\langle \hat{E}_C, L^D_{\Phi^{-1}\hat{E}_A}\hat{E}_B\rangle - \Phi^{-2}\langle \hat{E}_C, L_{\Phi^{-1}\hat{E}_A}\hat{E}_B\rangle \\
&= -3\tilde{\Omega}_{[ABC]} + \tilde{\Omega}_I{}^I{}_A \eta_{BC} - \Phi^{-1}(\partial_I \hat{E}_A{}^I)\eta_{BC} - 3\langle \Phi^{-1}\hat{E}_{[C}, L_{\Phi^{-1}\hat{E}_A}(\Phi^{-1}\hat{E}_{B]})\rangle
\end{aligned}
\tag{4.35}
$$

where we switched indices BC for the first term and get a minus sign and the last term can be anti-symmetrised as discussed above. We then see that the component has two terms that are anti-symmetric in A, B, C and two terms that are symmetric in B, C. Writing out explicitly, they are

$$
\begin{aligned}
-3\tilde{\Omega}_{[ABC]} - 3\langle \Phi^{-1}\hat{E}_{[C}, L_{\Phi^{-1}\hat{E}_A}(\Phi^{-1}\hat{E}_{B]})\rangle &\in \Gamma(\wedge^3\tilde{E}) \\
\tilde{\Omega}_I{}^I{}_A - \Phi^{-1}(\partial_I \hat{E}_A{}^I) &\in \Gamma(\tilde{E})
\end{aligned}
\tag{4.36}
$$

therefore, the torsion can be seen as

$$
\mathcal{T} \in \Gamma(\wedge^3\tilde{E} \oplus \tilde{E}). \tag{4.37}
$$

For the torsion-free condition, as the torsion component splits into two sub sections, we have two equations

$$
\tilde{\Omega}_{[ABC]} = -\langle \Phi^{-1}\hat{E}_{[C}, L_{\Phi^{-1}\hat{E}_A}(\Phi^{-1}\hat{E}_{B]})\rangle \tag{4.38}
$$

$$
\tilde{\Omega}_B{}^B{}_A = \Phi^{-1}(\partial_B \hat{E}_A{}^B) \tag{4.39}
$$

The first one is exactly the same as the un-extended one, giving conditions (3.108). For the second one, using the vielbein expression (4.22) gives

$$\begin{aligned}\tilde{\Omega}_b{}^b{}_a &= \Phi^{-1}(\partial_\mu \hat{E}^{+\mu}_a) = \Phi^{-1}\partial_\mu(e^{-2\phi}\sqrt{-\det g}\hat{e}^+{}_a{}^\mu) \\ &= \Phi^{-1}[(-2\partial_\mu\phi)e^{-2\phi}\sqrt{-\det g}\hat{e}^+{}_a{}^\mu + e^{-2\Phi}\partial_\mu(\sqrt{-\det g}\hat{e}^+{}_a{}^\mu)\end{aligned} \tag{4.40}$$

where $\partial_{\mu+d} = 0$ so only vector part of $\hat{E}^+_a$ is considered, and due to $\tilde{\Omega}_{Mb\bar{a}} = 0$ the other index on $\tilde{\Omega}$ must be in the same sub-frame. Then using (A.123) for the second term gives

$$\begin{aligned}\tilde{\Omega}_b{}^b{}_a &= \Phi^{-1}[(-2\partial_\mu\phi)e^{-2\phi}\sqrt{-\det g}\hat{e}^+{}_a{}^\mu + e^{-2\Phi}\sqrt{-\det g}\nabla_\mu\hat{e}^+{}_a{}^\mu] \\ &= \Phi^{-1}e^{-2\phi}\sqrt{-\det g}[-2(\partial_\mu\phi)\hat{e}^+{}_a{}^\mu + \nabla_\mu\hat{e}^+{}_a{}^\mu] \\ &= -2\partial_a\phi + \omega_\mu{}^b{}_a\hat{e}^+{}_b{}^\mu \\ &= \omega_b{}^b{}_a - 2\partial_a\phi\end{aligned} \tag{4.41}$$

where ω is the usual Levi-Civita spin connection and note that $\Phi = e^{-2\phi}\sqrt{-\det g}$ by definition (4.23) and the vielbein transfer coordinate indices to non-coordinate indices. Similarly one has

$$\tilde{\Omega}_{\bar{b}}{}^{\bar{b}}{}_{\bar{a}} = \omega_{\bar{b}}{}^{\bar{b}}{}_{\bar{a}} - 2\partial_{\bar{a}}\phi. \tag{4.42}$$

One may also follow the method at the end of Section 3.5 using (3.114). The torsion component of D^∇ also splits into two parts $\Sigma_{[ABC]}$ and $\hat{\Sigma}_B{}^B{}_A$ where $\Sigma_{[ABC]}$ satisfies the same condition as before (3.118) and [4]

$$\hat{\Sigma}_B{}^B{}_A = -4(d\phi)_A = \begin{cases}\frac{1}{2}\partial_a\phi, & A = a \\ \frac{1}{2}\partial_{\bar{a}}\phi, & A = \bar{a} + d\end{cases} \tag{4.43}$$

implying

$$\hat{\Sigma}_b{}^b{}_a = -2\partial_a\phi, \quad \hat{\Sigma}_{\bar{b}}{}^{\bar{b}}{}_{\bar{a}} = -2\partial_{\bar{a}}\phi. \tag{4.44}$$

Again, the usual Levi-Civita spin connection ω that comes from ∇ is absorbed into D^∇.

However, same as the un-extended case, the two anti-symmetrised conditions are free to add a rank-3 tensor that vanishes under anti-symmetrisation. Therefore, the torsion-free compatible connection is still not unique. Also, the new condition (4.44) needs to be added to the connection in the form $\partial_a\phi\eta_{bc}$ (see the second term in the last line of (4.35)), and with two of indices anti-symmetrised so that the connection component satisfies (4.29). Since the metric η is symmetric, the only way to anti-symmetrise indices is

$$\partial_{[b}\phi\eta_{c]a} = \frac{1}{2}(\partial_b\phi\eta_{ca} - \partial_c\phi\eta_{ba}) \tag{4.45}$$

so setting

$$\hat{\Sigma}_{abc} = k(\partial_b\phi\eta_{ca} - \partial_c\phi\eta_{ba}) \tag{4.46}$$

gives the term that is anti-symmetric in last two indices b, c, and we introduce a coefficient k so that when the first two indices of $\hat{\Sigma}$ are contracted, it gives correct result as (4.44). Explicitly, we have

$$\hat{\Sigma}_a{}^a{}_c = k\eta^{ab}(\partial_b\phi\eta_{ca} - \partial_c\phi\eta_{ba}) = k(\partial_c\phi - \partial_c\phi\delta_a{}^a) = k(1-d)\partial_c\phi = 2\partial_c\phi \tag{4.47}$$

where d is the dimension of flat metric η_{ab} so $\delta_a{}^a = d$, then

$$k = \frac{2}{1-d}. \tag{4.48}$$

For type II supergravity, $d = 10$, so

$$\hat{\Sigma}_{abc} = -\frac{2}{9}(\eta_{ab}\partial_c\phi - \eta_{ac}\partial_b\phi). \tag{4.49}$$

Therefore, the component of connection is updated to

$$\begin{aligned}
D_aV_+^b &= \nabla_aV_+^b - \frac{1}{6}H_a{}^b{}_cV_+^c - \frac{2}{9}(\delta_a{}^b\partial_c\phi - \eta_{ac}\partial^b\phi)V_+^c + A^+{}_a{}^b{}_cV_+^c \\
D_{\bar{a}}V_+^b &= \nabla_{\bar{a}}V_+^b - \frac{1}{2}H_{\bar{a}}{}^b{}_cV_+^c \\
D_aV_-^{\bar{b}} &= \nabla_aV_-^{\bar{b}} + \frac{1}{2}H_a{}^{\bar{b}}{}_{\bar{c}}V_-^{\bar{c}} \\
D_{\bar{a}}V_-^{\bar{b}} &= \nabla_{\bar{a}}V_-^{\bar{b}} + \frac{1}{6}H_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}}V_-^{\bar{c}} - \frac{2}{9}(\delta_{\bar{a}}{}^{\bar{b}}\partial_{\bar{c}}\phi - \eta_{\bar{a}\bar{c}}\partial^{\bar{b}}\phi)V_-^{\bar{c}} + A^-{}_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}}V_-^{\bar{c}}
\end{aligned} \tag{4.50}$$

where the middle index of $\hat{\Sigma}$ is raised using η^{ab} or $\eta^{\bar{a}\bar{b}}$. Apart from the condition (3.121) on $A^\pm$, it is also required that

$$A^+{}_a{}^a{}_b = A^-{}_{\bar{a}}{}^{\bar{a}}{}_{\bar{b}} = 0 \tag{4.51}$$

so that when the indices a, b contract, $A^\pm$ does not give any contribution which violates the condition (4.44).

4.4 Generalised Ricci Tensor, Scalar and Equations of Motion

Even though the generalised curvature is not unique, one can still construct a unique Generalised Ricci tensor, hence gives the equations of motion [4]. From (4.50) we see that

$$\begin{aligned}
D_{\bar{a}}V_+^b &= \nabla_{\bar{a}}V_+^b - \frac{1}{2}H_{\bar{a}}{}^b{}_cV_+^c \\
D_aV_-^{\bar{b}} &= \nabla_aV_-^{\bar{b}} + \frac{1}{2}H_a{}^{\bar{b}}{}_{\bar{c}}V_-^{\bar{c}}
\end{aligned} \tag{4.52}$$

are uniquely defined. For the other two cases, due to the new condition (4.51), by contracting the a, b indices, one finds that

$$\begin{aligned} D_a V_+^a &= \nabla_a V_+^a - 2(\partial_a \phi) V_+^a \\ D_{\bar{a}} V_-^{\bar{a}} &= \nabla_{\bar{a}} V_-^{\bar{a}} - 2(\partial_{\bar{a}} \phi) V_-^{\bar{a}} \end{aligned} \tag{4.53}$$

are also uniquely defined. Then using (4.52) and (4.53) one can construct the generalised Ricci tensor which are unique [4]

$$\begin{aligned} R_{a\bar{b}} V_+^a &= [D_a, D_{\bar{b}}] V_+^a \\ R_{\bar{a}b} V_-^{\bar{a}} &= [D_{\bar{a}}, D_b] V_-^{\bar{a}} \end{aligned}. \tag{4.54}$$

But since the indices are from different sub-frames, we cannot construct a curvature scalar from it. In fact, we must introduce spinors. The $O(p, q) \times O(q, p)$ structure defines the splitting $E \simeq C_+ \oplus C_-$, and implies the $Spin(p, q) \times Spin(q, p)$ structure for $Spin(p, q)$ spinors. $C\pm$ gives spin bundles $S(C_\pm)$ with corresponding gamma matrices $\gamma^a, \gamma^{\bar{a}}$. For $\varepsilon^\pm \in \Gamma(S(C_\pm))$, following same procedure as before, we have Ricci tensors [4]

$$\begin{aligned} \frac{1}{2} R_{a\bar{b}} \gamma^a \varepsilon^+ &= [\gamma^a D_a, D_{\bar{b}}] \varepsilon^+ \\ \frac{1}{2} R_{\bar{a}b} \gamma^{\bar{a}} \varepsilon^- &= [\gamma^{\bar{a}} D_{\bar{a}}, D_b] \varepsilon^- \end{aligned} \tag{4.55}$$

which are also equivalent to (4.54) and the Ricci tensor is unique. These two equations can be used to define the generalised curvature scalar

$$\begin{aligned} -\frac{1}{4} R \varepsilon^+ &= (\gamma^a D_a \gamma^b D_b - D^{\bar{a}} D_{\bar{a}}) \varepsilon^+ \\ -\frac{1}{4} R \varepsilon^- &= (\gamma^{\bar{a}} D_{\bar{a}} \gamma^{\bar{b}} D_{\bar{b}} - D^a D_a) \varepsilon^- \end{aligned} \tag{4.56}$$

and again are two equivalent definitions and gives a unique curvature scalar.

By setting $e_a^+ = e_a^-$, one finds that

$$R_{ab} = \mathcal{R}_{ab} - \frac{1}{4} H_{acd} H_b{}^{cd} + 2\nabla_a \nabla_b \phi + \frac{1}{2} e^{2\phi} \nabla^c (e^{-2\phi} H_{cab}) \tag{4.57}$$

so it is a tensor, and

$$R = \mathcal{R} + 4\nabla^2 \phi - 4(\partial\phi)^2 - \frac{1}{12} H^2 \tag{4.58}$$

is a scalar [4]. Substitute S into the bosonic 'pseudo'-action (4.1) gives

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2} \int \left(\Phi R - \frac{\sqrt{-\det g}}{4} \sum_n \frac{1}{n!} (F_{(n)}^{(B)})^2 \right) \tag{4.59}$$

where $\Phi = e^{-2\phi}\sqrt{-\det g}$, and ignoring the RR fields, the action for NSNS sector (4.2) is

$$S_{\mathrm{NS}} = \frac{1}{2\kappa^2}\int \Phi R \tag{4.60}$$

where $\Phi \sim \mathrm{vol}_G$, so it is in the same form as the Einstein-Hilbert action (A.131). The equations of motion for the metric g and 2-form B is then found by varying the action with respect to G giving

$$R_{a\bar{b}} = 0 \tag{4.61}$$

which looks like the vacuum field equation (A.132), and if the RR fields are also included, then the equations of motion becomes

$$R_{a\bar{b}} + \frac{1}{16}\Phi^{-1}\left\langle F, \Gamma_{a\bar{b}}F\right\rangle = 0 \tag{4.62}$$

so the RR fields are like source terms. And for dilaton, the equation of motion is

$$R = 0. \tag{4.63}$$

Hence, the reformulation of type II supergravity using $O(d,d)\times\mathbb{R}^+$ generalised geometry can be thought as a generalised geometric Einstein gravity theory. See [4] for a detailed discussion.

5 Generalised Parallelisability and Consistent Truncation

String theories have either 10 or 11 dimensions whereas our physical world only has 4 dimensions, so the extra dimensions need to be compactified. For string theories to describe the physical world, we need to be able to extract a low dimensional theory from it, which is called dimensional reduction. A very strong condition is consistent truncation where truncation means that one requires the fields to be independent of higher dimensions hence give a reduced theory in low dimensions, and consistent requires that the solution of the reduced theory is still the solution of the full theory. The first example is Kaluza-Klein reduction on a 5-dimensional theory which unifies electromagnetism and gravity [44, 45]. The fifth dimension is compactified on a circle S^1, and after consistent truncation, the 5-dimensional diffeomorphism becomes a 4-dimensional diffeomorphism for gravity and a $U(1)$ gauge symmetry from the compactified dimension S^1 for electromagnetism. This can be generalised to d-dimensional compactified manifold i.e. a d-dimensional torus T^d, giving gauge group $[U(1)]^d$. But the KK reduction truncates all the massive mode, hence by modifying it, one has the Scherk-Schwarz reduction which gives the gauged and massive supergravities [28, 29]. Later on, Pauli developed the coset reduction on $\mathcal{M} = G/H$ which is a coset space and the gauge group of reduced theory is G, and De Witt developed the group reduction on a compact Lie group manifold $\mathcal{M} \simeq G$ which is also the gauge group of reduced theory [26].

However, there is no systematic way of finding consistent truncation, although some of them can be explained by parallelisability of a manifold. We first notice that the coset reduction and group reduction can be combined into the consistent truncations on a local group manifold [27] which is defined below.

Definition 5.1. A manifold $\mathcal{M}$ is a local group manifold if $\mathcal{M} \simeq G/\Gamma$ where G is a Lie group and Γ is a discrete subgroup of G that acts freely (Definition A.12).

Note that a Lie group itself is a local group manifold, and there is a relation between local group manifold and parallelisable manifold.

Claim 5.1. *For a parallelisable manifold $\mathcal{M}$, if the Lie bracket of its frame $\{\hat{e}_a\}$*

$$[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c \tag{5.1}$$

has constant $f_{ab}{}^c$ then this is a Lie algebra, and the manifold is a local group manifold with the discrete subgroup Γ defining the left or right invariant vector fields which give the globally defined frame.

If the structure constant of Lie algebra (5.1) satisfies the unimodular condition $f_{ab}{}^b = 0$ which is satisfied by compact Lie groups, then there is a consistent truncation on $\mathcal{M}$

[28, 29]. This implies that if a parallelisable manifold is also a compact Lie group then there is consistent truncation. This quickly explains the consistent truncation on S^1 for it is parallelisable and also a compact Lie group $S^1 \simeq U(1)$, and also explains the consistent truncation on S^3 for the NSNS sector of type II supergravity [19] as $S^3 \simeq SU(3)$ and is parallelisable. However, there are other consistent truncations on spheres which cannot be explained which are

- S^4 reduction on $AdS_7 \times S^4$ for 11-dimensional supergravity [20, 21]
- S^5 reduction on $AdS_5 \times S^5$ for type IIB supergravity [22, 23]
- S^7 reduction on $AdS_4 \times S^7$ for 11-dimensional supergravity [24, 25]

since S^4, S^5 are not parallelisable, and although S^7 is parallelisable, it is not a Lie group. But as an analogue to the relation between parallelisability and consistent truncation, one may guess if there is a generalised parallelisability which gives consistent truncations. In fact, this conjecture is implied in [30, 31, 10], and from [32] that all round sphere S^d are generalised parallelisable, hence may explain consistent truncations on S^1, S^3, S^4, S^5, S^7.

In this chapter, we first define generalised parallelisability as in [10], then construct the generalised geometry for spheres and and show that all spheres are generalised parallelisable following procedures in [32]. Finally, we sketch the link between generalised parallelisability and Scherk-Schwarz reduction, and use the consistent truncation on S^3 as an example [32].

5.1 Generalised Parallelisability

In ordinary geometry, if a manifold $\mathcal{M}$ is parallelisable, then there is a globally defined frame $\{e^a\}$ for $T\mathcal{M}$, and $T\mathcal{M}$ is trivial (see Theorem A.19). This then implies a G-structure as discussed in Example 2.1, and the G-structure can be described by a globally defined non-degenerated tensor (see Section 2.1), hence the manifold $\mathcal{M}$ can have a metric $g = g_{ab} e^a \otimes e^a$ where g_{ab} is constant and the frame $\{e^a\}$ is known as consistent absolute parallelism [46, 47, 48]. As an analogue, to define the generalised parallelisability, one requires that the generalised tangent bundle E has a global frame $\{\hat{E}_A\}$ and the canonical metric can be written as

$$\eta = \eta^{AB} \hat{E}_A \otimes \hat{E}_B. \tag{5.2}$$

where η_{AB} is constant and $\langle \hat{E}_A, \hat{E}_B \rangle = \eta_{AB}$ [10]. Moreover, in order to use generalised parallelisability in consistent truncation, we require the generalised parallelisable manifold to be an analogue to the local group manifold, so we impose a further condition from Claim 5.1, and gives the definition below.

Definition 5.2. Given a differentiable manifold $\mathcal{M}$ and its generalised tangent bundle has a global frame $\{\hat{E}_A\}$, $\mathcal{M}$ is generalised parallelisable if

$$L_{\hat{E}_A}\hat{E}_B = X_{AB}{}^C\hat{E}_C \tag{5.3}$$

where $X_{AB}{}^C$ is constant. And $\{\hat{E}_A\}$ is called generalised parallelisation.

One should notice that unlike Lie derivative and Lie bracket, the Dorfman derivative and Courant bracket are not the same, but differed by an exact term as in (2.50). However, for basis $\{\hat{E}_A\}$, since we require that $\langle\hat{E}_A, \hat{E}_B\rangle = \eta_{AB}$ then (2.50) gives

$$[\![\hat{E}_A, \hat{E}_B]\!] = L_{\hat{E}_A}\hat{E}_B + d\langle\hat{E}_A, \hat{E}_B\rangle = L_{\hat{E}_A}\hat{E}_B + d\eta_{AB} = L_{\hat{E}_A}\hat{E}_B \tag{5.4}$$

as $d\eta_{AB} = 0$ for constant η_{AB}. Hence the condition for generalised parallelisability can be equivalently written as

$$[\![\hat{E}_A, \hat{E}_B]\!] = L_{\hat{E}_A}\hat{E}_B = X_{AB}{}^C\hat{E}_C. \tag{5.5}$$

Proposition 5.2. *For a d-dimensional generalised parallelisable manifold $\mathcal{M}$, with D-dimensional generalised tangent bundle E, the Courant bracket on its generalised parallelism $\{\hat{E}_A\}$ defines a D-dimensional Lie algebra $\mathfrak{g} \subset \mathfrak{o}(D)$.*

Proof. Substitute (5.5) into (2.53) gives

$$[\![[\![\hat{E}_A, \hat{E}_B]\!], \hat{E}_C]\!] + [\![[\![\hat{E}_B, \hat{E}_C]\!], \hat{E}_A]\!] + [\![[\![\hat{E}_C, \hat{E}_A]\!], \hat{E}_B]\!] \sim d\langle\hat{E}_D, \hat{E}_C\rangle = 0 \tag{5.6}$$

since $d\langle\hat{E}_A, \hat{E}_B\rangle = 0$, so the Courant bracket satisfies Jacobi identity on $\{\hat{E}_A\}$. Therefore, together with the fact that $X_{AB}{}^C$ is constant as $\mathcal{M}$ is generalised parallelisable, the generalised parallelisable condition

$$[\![\hat{E}_A, \hat{E}_B]\!] = X_{AB}{}^C\hat{E}_C \tag{5.7}$$

defines a Lie algebra $\mathfrak{g}$ with dimension D. Moreover, from (2.37) one has

$$\langle L_{\hat{E}_A}\hat{E}_B, \hat{E}_C\rangle + \langle\hat{E}_B, L_{\hat{E}_A}\hat{E}_C\rangle = \pi(\hat{E}_A)[\langle\hat{E}_B, \hat{E}_C\rangle] = i_{\hat{E}_A}d\langle\hat{E}_B, \hat{E}_C\rangle \tag{5.8}$$

where $\pi : E \to T\mathcal{M}$ is the projection map defined in (2.33), and again using $d\langle\hat{E}_A, \hat{E}_B\rangle = d\eta_{AB} = 0$ we have

$$\langle L_{\hat{E}_A}\hat{E}_B, \hat{E}_C\rangle + \langle\hat{E}_B, L_{\hat{E}_A}\hat{E}_C\rangle = 0. \tag{5.9}$$

Using the condition (5.3) and $\langle\hat{E}_A, \hat{E}_B\rangle = \eta_{AB}$ gives

$$X_{AB}{}^D\eta_{DC} + X_{AC}{}^D\eta_{BD} = 0 \tag{5.10}$$

which means that $(X_A)^B{}_C$ is an element of the Lie algebra $\mathfrak{o}(D)$ of the structure group $O(D)$ defined by the canonical metric η, hence the Lie algebra $\mathfrak{g}$ generated by $X_{AB}{}^C$ is a sub-algebra $\mathfrak{g} \subset \mathfrak{o}(D)$. For the $O(d,d)$ generalised geometry, we have $(X_A)^B{}_C \in \mathfrak{o}(d,d)$ and $\mathfrak{g} \subset \mathfrak{o}(d,d)$. □

Proposition 5.3. *A generalised parallelisable manifold $\mathcal{M}$ is a coset space.*

Proof. Use the projection map $\pi : E \to T\mathcal{M}$, the Courant bracket reduces to Lie bracket as in (2.59), so

$$\pi([\![\hat{E}_A, \hat{E}_B]\!]) = [v_A, v_B] = X_{AB}{}^C v_C \tag{5.11}$$

where $v_A = \pi(\hat{E}_A) \in T\mathcal{M}$, and $\{v_A\}$ forms a basis for $T\mathcal{M}$. However, since the dimension of $T\mathcal{M}$ is d, there must be d non-vanishing v_A that are linearly independent, whereas the other $D-d$ vectors are either linearly dependent or zero. Thus, the set $\{v_A\}$ is linearly dependent, and one can always construct $V = V^A\hat{E}_A \in \mathfrak{g}$ such that $\pi(V) = V^A\pi(\hat{E}_A) = V^A v_A = 0$. The set of these V forms a $(D-d)$-dimensional Lie sub-algebra

$$\mathfrak{h} = \{V \in \mathfrak{g} : \pi(V) = 0\}. \tag{5.12}$$

Also, if $\pi(V) = \pi(W) = 0$, them $[\pi(V), \pi(W)] = 0$, hence $\mathfrak{h}$ is closed. Therefore, by exponentiating the Lie algebra, one has

$$\mathcal{M} \simeq G/H, \quad H \subset G \subset O(D) \tag{5.13}$$

where G is D-dimensional Lie group with Lie algebra g, H is Lie sub-group of G with Lie sub-algebra $\mathfrak{h} \subset \mathfrak{g}$. □

This proposition provides a constraint on possible generalised parallelisable manifold. As the spheres are all coset spaces

$$S^n \simeq SO(n+1)/SO(n) \tag{5.14}$$

it is natural to make the conjecture that at least some spheres are generalised parallelisable. In Section 5.3, we will prove that all spheres are generalised parallelisable. Furthermore, there are other coset spaces in string theory such as de Sitter and Anti-de Sitter space with isomorphisms given in Example A.7 which may also be considered for generalised parallelisability.

5.2 $GL^+(d+1,\mathbb{R})$ Generalised Geometry

For a d-dimensional theroy with a metric g and d-from field strength $F = dA$, the equations of motion (4.62) are written as [32]

$$R_{\mu\nu} = \frac{1}{d-1} F^2 g_{\mu\nu}, \quad F = \frac{d-1}{R} \mathrm{vol}_g \tag{5.15}$$

where $R_{\mu\nu}$ is Ricci tensor, $R = g^{\mu\nu} R_{\mu\nu}$ is Ricci scalar vol_g is volume form associated with g and $F^2 = \frac{1}{d!} F^{\mu_1 \ldots \mu_d} F_{\mu_1 \ldots \mu_d}$. A solution is a round sphere S^d with radius R [32].

To construct the geometry with the d-sphere background, we notice that the d-form F and $(d-1)$-form A correspond to the H flux and 2-form B field in the $O(d,d)$ case respectively, so as an analogue, the generalised tangent space is now defined as

$$T\mathcal{M} \oplus \wedge^{d-2} T^* \mathcal{M} \tag{5.16}$$

so that its section $V = v + \lambda \in \Gamma(T\mathcal{M} \oplus \wedge^{d-2} T^* \mathcal{M})$ in coordinate basis is

$$V^M = \begin{pmatrix} v^\mu \\ \lambda_{\mu_1 \ldots \mu_{d-2}} \end{pmatrix} \tag{5.17}$$

where $v \in \Gamma(T\mathcal{M})$ is a vector and $\lambda \in \Gamma(\wedge^{d-2} T\mathcal{M})$ is a $(d-2)$-form. One can easily check that the dimension is

$$\dim(T\mathcal{M}) + \dim(\wedge^{d-2} T^* \mathcal{M}) = d + \frac{1}{2} d(d-1) = \frac{1}{2} d(d+1). \tag{5.18}$$

Formally, the generalised tangent bundle E is defined by the exact sequence as (see Section 2.7)

$$0 \to \wedge^{d-2} T^* \mathcal{M} \to E \to T\mathcal{M} \to 0. \tag{5.19}$$

The splitting defined by the B-transformis now parametrised by the $(d-1)$-form A, and will be called A-shift. On chart $U_i \cap U_j$, A needs to be patched similar to (2.79) as

$$A_{(j)} = A_{(i)} + d\Lambda_{(ij)} \tag{5.20}$$

so that the section $\tilde{V}$ of E is patched similar to (2.89) as

$$v_{(i)} + \lambda_{(i)} - i_{v_{(i)}} A_{(i)} = v_{(j)} + \lambda_{(j)} - i_{v_{(j)}} A_{(j)}. \tag{5.21}$$

Then one can see a relation between the section $V \in \Gamma(T\mathcal{M} \oplus \wedge^{d-2} T^* \mathcal{M})$ and $\tilde{V} \in \Gamma(E)$

$$\tilde{V} = e^{-A} V = v + \lambda - i_v A \tag{5.22}$$

hence the A-shift defines an isomorphism

$$E \simeq T\mathcal{M} \oplus \wedge^{d-2} T^*\mathcal{M}. \tag{5.23}$$

To find the structure group, we notice that the section of E can be written as a $\frac{1}{2}d(d+1)$-dimensional bi-vector [49]. An example of $d = 4$ case is given in [3]. To see this explicitly, we first use the isomorphism from (A.85)

$$\wedge^2 T\mathcal{M} \times \det T^*\mathcal{M} \simeq \wedge^{d-2} T^*\mathcal{M} \tag{5.24}$$

to rewrite the form part as (A.86)

$$\lambda^{mn} = \frac{1}{(d-2)!} \varepsilon^{mn\mu_1\dots\mu_{d-2}} \lambda_{\mu_1\dots\mu_{d-1}}, \tag{5.25}$$

so that the form part of sections of E can be equivalently written as a section of $\wedge^2 T\mathcal{M} \otimes \det T^*\mathcal{M}$ i.e. anti-symmetric rank 2 tensor density. Then, the section of E can be written as

$$V^M = V^{\underline{mn}} = \begin{cases} V^{m,d+1} = v^m & \in T\mathcal{M} \\ V^{mn} = \lambda^{mn} & \in \wedge^2 T\mathcal{M} \otimes \det T^*\mathcal{M} \simeq \wedge^{d-2} T^*\mathcal{M} \end{cases} \tag{5.26}$$

where $\underline{mn}$ is an anti-symmetric pair, so the section is a $\frac{1}{2}d(d+1)$-dimensional bi-vector. Or roughly in a 'matrix' form, it is

$$\begin{pmatrix} v^\mu \\ \lambda_{\mu_1\dots\mu_{d-2}} \end{pmatrix} \sim \begin{pmatrix} 0 & \lambda & \cdots & \lambda & v^1 \\ -\lambda & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \lambda & \vdots \\ -\lambda & \cdots & -\lambda & \ddots & v^d \\ -v^1 & \cdots & \cdots & -v^d & 0 \end{pmatrix} \tag{5.27}$$

and it is clear that the degrees of freedom of both sides are $\frac{1}{2}d(d+1)$. Then, as a $(d+1)$-dimensional matrix, the natural action on the section $V^{\underline{mn}}$ is $GL(d, \mathbb{R})$ group, and since the background manifold is sphere which is orientable, the determinant of the transformation needs to be positive to preserve orientation, therefore, the structure group is $GL^+(d+1, \mathbb{R})$.

For the partial derivative, we need to define the dual generalised tangent bundle by the dual of the exact sequence (5.19) as in the end of Section 2.7

$$0 \to T^*\mathcal{M} \to E^* \to \wedge^{d-2} T\mathcal{M} \to 0 \tag{5.28}$$

giving the isomorphism

$$E^* \simeq T^*\mathcal{M} \oplus \wedge^{d-2}T\mathcal{M} \tag{5.29}$$

and the partial derivative is embedded into E^* using the map $T^*\mathcal{M} \to E^*$ so

$$\partial_M = \partial_{\underline{mn}} = \begin{cases} \partial_{m,d+1} = \partial_m, \\ \partial_{mn} = 0. \end{cases} \tag{5.30}$$

The contraction between elements of E and E^* is given by

$$V \cdot W = V^M W_M = \frac{1}{2} V^{\underline{mn}} W_{\underline{mn}} \tag{5.31}$$

where the $\frac{1}{2}$ comes from when contracting p anti-symmetric indices, there is a weight $\frac{1}{p!}$.

Since both of the sections of E and E^* are in the $\frac{1}{2}d(d+1)$-dimensional bi-vector representation, the structure group acts on them in the adjoint representation, hence we have the adjoint generalised frame bundle. By calculating the Lie algebra of $GL^+(d+1,\mathbb{R})$ group, one finds that [32]

$$\mathrm{ad}\mathcal{F} \simeq \mathbb{R} \oplus (T\mathcal{M} \otimes T^*\mathcal{M}) \oplus \wedge^{d-1}T\mathcal{M} \oplus \wedge^{d-1}T^*\mathcal{M}. \tag{5.32}$$

Also, as

$$\begin{aligned} &E \otimes E^* \\ &\simeq (T\mathcal{M} \oplus \wedge^{d-2}T^*\mathcal{M}) \otimes (T^*\mathcal{M} \oplus \wedge^{d-2}T\mathcal{M}) \\ &\simeq (T\mathcal{M} \otimes T^*\mathcal{M}) \oplus (T\mathcal{M} \otimes \wedge^{d-2}T\mathcal{M}) \oplus (T^*\mathcal{M} \otimes \wedge^{d-2}T^*\mathcal{M}) \oplus (\wedge^{d-2}T^*\mathcal{M} \otimes \wedge^{d-2}T\mathcal{M}) \\ &\simeq (T\mathcal{M} \otimes T^*\mathcal{M}) \oplus (T\mathcal{M} \otimes \wedge^{d-2}T\mathcal{M}) \oplus (T^*\mathcal{M} \otimes \wedge^{d-2}T^*\mathcal{M}) \oplus \det T^*\mathcal{M} \end{aligned} \tag{5.33}$$

where we used the isomorphism (5.24) for the last term so it equals $\wedge^2 T\mathcal{M} \otimes \det T^*\mathcal{M} \otimes \wedge^{d-2}T\mathcal{M} = \det T^*\mathcal{M}$, then we have

$$\mathrm{ad}\mathcal{F} \subset E \otimes E^*. \tag{5.34}$$

Then the section R of $\mathrm{ad}\mathcal{F}$ can be written as

$$R = R^A{}_B \hat{E}_A \otimes \hat{E}^B \tag{5.35}$$

where $\{\hat{E}_A\}$ is basis of E and $\{\hat{E}^B\}$ is the dual basis for E^*, and there is a projection map [17]

$$\times_{\mathrm{ad}} : E \otimes E^* \to \mathrm{ad}\mathcal{F} \tag{5.36}$$

so that for $V \in \Gamma(E), W \in \Gamma(E^*)$ [32]

$$(V \times_{\text{ad}} W)^{\underline{m}}{}_{\underline{n}} = V^{\underline{mp}} W_{\underline{np}} - \frac{1}{4} V^{\underline{pq}} U_{\underline{pq}} \delta^{\underline{m}}{}_{\underline{n}}. \tag{5.37}$$

Therefore the Dorfman derivative can now be written down as [17]

$$L_V W^M = (V \cdot \partial) W^M - (\partial \times_{\text{ad}} V)^M{}_N W^N \tag{5.38}$$

where the second term indicates that the transformation is in adjoint representation of the structure group. The Courant bracket is still defined as anti-symmetrisation of Dorfman bracket, and as the $O(d,d)$ case, the Dorfman derivative can be extended to any generalised tensors i.e. on any representations of $GL^+(d+1, \mathbb{R})$.

Similar to the $O(d,d) \times \mathbb{R}^+$ case, one can introduce the generalised metric G which is parametrised by the metric g, the $(d-1)$-form field A and a scale factor Δ which comes from compactification of extra dimensions i.e. for dilaton [17, 16]. By introducing the generalised metric, the structure group reduces to the maximal compact subgroup $SO(d+1) \subset GL^+(d+1, \mathbb{R})$ (see Example 2.2) and agrees with the sphere background.

By definition, the generalised metric on E is given by

$$G(V,V) = G_{MN} V^M V^N \tag{5.39}$$

where V is in its original form (5.17) as a section of E. However, similar to the $O(d,d) \times \mathbb{R}^+$ case, the basis for the generalised bundle E is different from the basis for $T\mathcal{M} \oplus \wedge^{d-2} T^*\mathcal{M}$ by A-shift and a scaling e^{Δ}, so, in order to express G using the usual metric g on $T\mathcal{M}$, the section V needs to be transferred back to the basis of $T\mathcal{M} \oplus \wedge^{d-2} T^*\mathcal{M}$ by

$$\tilde{V} = e^{-\Delta} e^{-A} V \tag{5.40}$$

and in component is

$$\begin{pmatrix} \tilde{v}^m \\ \tilde{\lambda}_{m_1 \dots m_{d-2}} \end{pmatrix} = e^{-\Delta} \begin{pmatrix} v^m \\ \lambda_{m_1 \dots m_{d-2}} + i_{v^m} A_{m m_1 \dots m_{d-2}} \end{pmatrix} \tag{5.41}$$

so

$$
\begin{aligned}
G(V,V) &= G_{MN}\tilde{V}^M\tilde{V}^N \\
&= g_{mn}\tilde{v}^m\tilde{v}^n + \frac{1}{(d-2)!}g^{m_1n_1}\cdots g^{m_{d-2}n_{d-2}}\tilde{\lambda}_{m_1\ldots m_{d-2}}\tilde{\lambda}_{n_1\ldots n_{d-2}} \\
&= e^{-2\Delta}\Big(g_{mn}v^mv^n + \frac{1}{(d-2)!}A_m{}^{p_1\ldots p_{d-2}}A_{np_1\ldots p_{d-2}}v^mv^n \\
&\qquad + v^mA_m{}^{n_1\ldots n_{d-2}}\lambda_{n_1\ldots n_{d-2}} + v^nA_n{}^{m_1\ldots m_{d-2}}\lambda_{m_1\ldots m_{d-2}} \\
&\qquad + \frac{1}{(d-2)!}g^{m_1n_1}\cdots g^{m_{d-2}n_{d-2}}\lambda_{m_1\ldots m_{d-2}}\lambda_{n_1\ldots n_{d-2}}\Big) \\
&= V^T\cdot e^{-2\Delta}\begin{pmatrix} g_{mn}+\frac{1}{(d-2)!}A_m{}^{p_1\ldots p_{d-2}}A_{np_1\ldots p_{d-2}} & A_m{}^{n_1\ldots n_{d-2}} \\ A_n{}^{m_1\ldots m_{d-2}} & \frac{1}{(d-2)!}g^{m_1n_1}\cdots g^{m_{d-2}n_{d-2}}\end{pmatrix}\cdot V
\end{aligned}
\tag{5.42}
$$

One can also find the generalised metric for the bi-vector representation. We first define another bundle whose transformation is in the $(d+1)$-dimensional fundamental representation of $GL^+(d+1,\mathbb{R})$, and anti-symmetrise this bundle shall give the $\frac{1}{2}d(d+1)$-dimensional bi-vector representation. This bundle is defined as[1] [16, 32]

$$
\begin{aligned}
W &\simeq (\det T^*\mathcal{M})^{\frac{1}{2}}\otimes(T\mathcal{M}\oplus\wedge^dT\mathcal{M}) \\
&\simeq (\det T^*\mathcal{M})^{\frac{1}{2}}\otimes T\mathcal{M}\oplus(\det T^*\mathcal{M})^{-\frac{1}{2}}
\end{aligned}
\tag{5.43}
$$

where we used the isomorphism $(\det T^*\mathcal{M})^{-\frac{1}{2}}\simeq(\det T^*\mathcal{M})^{\frac{1}{2}}\otimes\wedge^dT\mathcal{M}$ which is a generalisation of (A.85) with weight. Then the section $K=q+t\in\Gamma(W)$ is given by [32]

$$
K^{\underline{m}} = \begin{cases} V^m = q^m & \in(\det T^*\mathcal{M})^{\frac{1}{2}}\otimes T\mathcal{M} \\ V^{d+1}=t & \in(\det T^*\mathcal{M})^{-\frac{1}{2}}\end{cases}
\tag{5.44}
$$

which is $(d+1)$-dimensional. By considering the basis for W, roughly, one has

$$
\wedge^2[(\det g)^{\frac{1}{4}}\otimes e\oplus(\det g)^{-\frac{1}{4}}]\sim[(\det g)^{\frac{1}{2}}\otimes(e\wedge e)]\oplus e\oplus 0
\tag{5.45}
$$

where $(\det g)^{-\frac{1}{4}}, e, (\det g^{-\frac{1}{4}})$ are bases of $(\det T^*\mathcal{M})^{\frac{1}{4}}, T\mathcal{M}, (\det T^*\mathcal{M})^{-\frac{1}{4}}$ respectively, and $\wedge^2\det g=0$ since $\det g$ is a density. Therefore

$$
\wedge^2W\simeq\det T^*\mathcal{M}\otimes\wedge^2T\mathcal{M}\oplus T\mathcal{M}\simeq E
\tag{5.46}
$$

where we used the isomorphism (5.24) again. Thus, from anti-symmetrisation of W, one has that the degrees of freedom of sections of E is $\frac{1}{2}d(d+1)$ which agrees with the bi-vector

[1] The advantage of this bundle is that the weight $(\det T^*\mathcal{M})^{\frac{1}{2}}$ cancels the weight of the spin bundle of original tangent bundle $S^{\pm}(E)\simeq(\det T^*\mathcal{M})^{-\frac{1}{2}}\otimes\wedge^{\text{even/odd}}T^*\mathcal{M}$ [4]. This will not be discussed further here.

representation.

However, unlike the section $V \in \Gamma(E)$, the corresponding 'A-shift' transformation for $K \in \Gamma(W)$ is not clear, so instead of using $G(K,K)$, one can find the frame $\{\tilde{E}_i\}$ of W, and use $G = \tilde{E}^T\tilde{E}$ to find the generalised metric on W. Since $E \simeq \wedge^2 W$, the frame for W is defined as [16]

$$\hat{E}_{ij} = \tilde{E}_i \wedge \tilde{E}_j \tag{5.47}$$

where $\{\hat{E}_{ij}\}$, $i,j = 1,...,d+1$ is the frame of E in the bi-vector representation, and is labelled by an anti-symmetric pair as the indices of bi-vectors are anti-symmetric. The orthonormal condition implied by the structure group $SO(d+1)$ is

$$G(\hat{E}_{ij}, \hat{E}_{kl}) = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{5.48}$$

And similar to the $O(d,d)$ splitting frame, as $E \simeq T\mathcal{M} \oplus \wedge^{d-2}T^*\mathcal{M}$, one can construct two sub-frames for $T\mathcal{M}$ and $\wedge^{d-2}T^*\mathcal{M}$. For $T\mathcal{M}$ the basis is as usual $\{\hat{e}_a\}$ with its dual $\{\hat{e}^a\}$ and the usual metric g, and the basis transform under $SO(d) \subset SO(d+1)$. The wedge product of $\{\hat{e}^a\}$ gives the basis for the form part. Then as analogue to (4.13), the split frame is given as [16, 3, 32]

$$\hat{E}_{ij} = \begin{cases} \hat{E}_{a,d+1} = e^{\Delta}(\hat{e}_a + i_{\hat{e}_a}A) \\ \hat{E}_{ab} = \frac{1}{(d-2)!}e^{\Delta}\varepsilon_{abc_1\ldots c_{d-2}}\hat{e}^{c_1} \wedge \cdots \wedge \hat{e}^{c_{d-2}} \end{cases} \tag{5.49}$$

where the form part is identified to a 2-form density using (5.24). Hence, the frame of W is defined using (5.47), and the dual frame is given by [32]

$$\tilde{E}^i = \begin{cases} \tilde{E}^a = e^{-\frac{1}{2}\Delta}(\det g)^{-\frac{1}{4}}(\hat{e}^a + \hat{e}^a \wedge A) \\ \tilde{E}^{d+1} = e^{-\frac{1}{2}\Delta}(\det g)^{-\frac{1}{4}}\mathrm{vol}_g \end{cases} \tag{5.50}$$

where vol_g can be identified to $\sqrt{\det g}$ using the isomorphism in (A.85). Also, note that the scaling factor is now $\frac{1}{2}\Delta$ and also $(\sqrt{\det g})^{\frac{1}{2}}$ because the weighting of W is $(\det T^*\mathcal{M})^{\frac{1}{2}}$ instead of $\det T^*\mathcal{M}$ for the $O(d,d) \times \mathbb{R}^+$. For $\hat{e} \wedge A$, it is a d-from, hence can be identified to a tensor density using (A.85) so

$$\hat{e}^a \wedge A = \hat{e}^a{}_m A^m \in \Gamma(\det T^*\mathcal{M}) \tag{5.51}$$

where A^m is a vector density defined as

$$A^m = \frac{1}{(d-1)!}\varepsilon^{mp_1\ldots p_{d-1}}A_{p_1\ldots p_{d-1}} \in \Gamma(\det T^*\mathcal{M} \otimes T\mathcal{M}) \tag{5.52}$$

using the isomorphism $\det T^*\mathcal{M} \otimes T\mathcal{M} \simeq \wedge^{d-1}T\mathcal{M}$ and $A_{m_1\ldots m_{d-1}}$ is the $(d-1)$-form field of E. Then by definition, the generalised metric is given by

$$G_{\underline{mn}} = \delta_{ij}\tilde{E}^i{}_{\underline{m}}\tilde{E}^i{}_{\underline{n}} = \frac{e^{-\Delta}}{\sqrt{\det g}}(\hat{e}_{am}\hat{e}^a{}_n + \hat{e}_{am}\hat{e}^a{}_n A^n + \hat{e}_{an}\hat{e}^a{}_m A^m + \hat{e}_{am}\hat{e}^a{}_n A^m A^n + \det g) \tag{5.53}$$

and using $g = \hat{e}^T\hat{e}$, one has

$$G = \frac{e^{-\Delta}}{\sqrt{\det g}}\begin{pmatrix} g_{mn} & g_{mp}A^p \\ g_{np}A^p & \det g + g_{pq}A^pA^q \end{pmatrix} \tag{5.54}$$

Thus, the generalised metric acts on $K \in \Gamma(W)$ as

$$G(K, K) = G_{\underline{mn}}K^{\underline{m}}K^{\underline{n}} \tag{5.55}$$

and using that $E \simeq \wedge^2 W$, one also has the generalised metric on $V \in \Gamma(E)$ as

$$G(V, V) = \frac{1}{2}G_{\underline{mp}}G_{\underline{nq}}V^{\underline{mn}}V^{\underline{pq}} \tag{5.56}$$

where the $\frac{1}{2}$ factor comes from summing anti-symmetric indices.

Finally, both of (5.42) and (5.54) show explicitly that the generalised metric is parameterised by the degrees of freedoms g, A, Δ, and hence the coset space is parameterised by

$$\{g, A, \Delta\} \in \frac{GL^+(d+1, \mathbb{R})}{SO(d+1)}. \tag{5.57}$$

5.3 Generalised Parallelisability of Spheres

In this section, we will show all spheres are generalised parallelisable following [4]. A d-dimensional sphere S^d with radius R can be embedded into $\mathbb{R}^{d+1}$ as

$$x^i = Ry^i, \quad \delta_{ij}y^iy^j = 1 \tag{5.58}$$

where x^i is coordinate of $\mathbb{R}^{d+1}$ and y^i is coordinate for a unit d-sphere. The metric g on the sphere is

$$ds^2 = R^2\delta_{ij}dy^idy^j \tag{5.59}$$

and the volume form on S^d is given by

$$\mathrm{vol}_g = \frac{R^d}{d!}\varepsilon_{i_1\ldots i_{d+1}}y^{i_1}dy^{i_2} \wedge \cdots \wedge dy^{i_{d+1}}. \tag{5.60}$$

By definition, S^d has symmetry group $SO(d+1)$, so there are $d+1$ conformal Killing vectors k_i given by [32]

$$k_i(y_j) = i_{k_i} dy^j = \delta_{ij} - y_i y_j, \quad \mathcal{L}_{k_i} g = -2 y_j g \tag{5.61}$$

giving $\frac{1}{2}d(d+1)$ rotation Killing vectors

$$v_{ij} = R^{-1}(y_i k_j - y_j k_i) \tag{5.62}$$

which generates Lie algebra $\mathfrak{so}(d+1)$

$$[v_{ij}, v_{kl}] = R^{-1}(\delta_{ik} v_{lj} - \delta_{il} v_{kj} - \delta_{jk} v_{li} + \delta_{jl} v_{ki}) = X_{[ij][kl]}{}^{[mn]} v_{mn} \tag{5.63}$$

where

$$X_{[ij][kl]}{}^{[mn]} = R^{-1}(\delta_{ik}\delta_l{}^m \delta_j{}^n - \delta_{il}\delta_k{}^m \delta_j{}^n - \delta_{jk}\delta_l{}^m \delta_i{}^n + \delta_{jl}\delta_i{}^m \delta_i{}^n). \tag{5.64}$$

The Killing vectors can be used to define the global frame $\hat{E}_{ij}$ according to (5.49) as

$$\hat{E}_{ij} = \begin{pmatrix} v_{ij} \\ \sigma_{ij} - i_{v_{ij}} A \end{pmatrix} \tag{5.65}$$

where

$$\sigma_{ij} = *(R^2 dy_i \wedge dy_j) = \frac{R^{d-2}}{(d-2)!} \varepsilon_{ijk_1 \dots k_{d-1}} y^{k_1} dy^{k_2} \wedge \cdots \wedge dy^{k_{d-1}} \tag{5.66}$$

and satisfies [32]

$$\mathcal{L}_{v_{ij}} \sigma_{kl} = R^{-1}(\delta_{ik}\sigma_{lj} - \delta_{il}\sigma_{kj} - \delta_{jk}\sigma_{li} + \delta_{jl}\sigma_{ki}) \tag{5.67}$$

Since $v_{ij} = 0$ when $y_i = y_j = 0$ whereas $dy_i \wedge dy_j = 0$ when $y_i^2 + y_j^2 = 1$, the upper part and lower part of $\hat{E}_{ij}$ cannot be zero at same time, hence $\hat{E}_{ij}$ is globally defined. Also, one has from [32] that

$$G(\hat{E}_{ij}, \hat{E}_{kl}) = v_{ij} v_{kl} + \sigma_{ij}\sigma_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \tag{5.68}$$

so it is indeed a frame for the $SO(d+1)$ structure.

Using the definition of Doefman derivative (2.25), one find explicitly that

$$\begin{aligned} L_{\hat{E}_{ij}} \hat{E}_{kl} &= [v_{ij}, v_{kl}] + \mathcal{L}_{v_{ij}}(\sigma_{kl} + i_{v_{kl}} A) - i_{v_{kl}} d(\sigma_{ij} + i_{v_{ij}} A) \\ &= [v_{ij}, v_{kl}] + \mathcal{L}_{v_{ij}}\sigma_{kl} + \mathcal{L}_{v_{ij}} i_{v_{kl}} A - i_{v_{kl}}(d\sigma_{ij} + \mathcal{L}_{v_{ij}} A - i_{v_{ij}} dA) \\ &= [v_{ij}, v_{kl}] + \mathcal{L}_{v_{ij}}\sigma_{kl} + i_{[v_{ij}, v_{kl}]} A - i_{v_{kl}}(d\sigma_{ij} - i_{v_{ij}} F) \end{aligned} \tag{5.69}$$

where we used (A.32) from first to second line, then $dA = F$ and (A.33) to the third line. Then from [32] one has

$$i_{v_{ij}} \mathrm{vol}_g = \frac{R}{d-1} d\sigma_{ij} \tag{5.70}$$

so substitute into F (5.15) gives

$$i_{v_{ij}}F = \frac{d-1}{R} i_{v_{ij}} \mathrm{vol}_g = \frac{d-1}{R}\frac{R}{d-1} d\sigma_{ij} = d\sigma_{ij} \tag{5.71}$$

hence the last term vanishes and substitute (5.63) (5.67) into the equation gives

$$\begin{aligned}
[\![\hat{E}_{ij}, \hat{E}_{kl}]\!] = L_{\hat{E}_{ij}} \hat{E}_{kl} &= R^{-1}[\delta_{ik}(v_{lj} + \sigma_{lj} + i_{v_{lj}}A) - \delta_{il}(v_{kj} + \sigma_{kj} + i_{v_{kj}}A) \\
&\qquad - \delta_{jk}(v_{li} + \sigma_{li} + i_{v_{li}}A) + \delta_{jl}(v_{ki} + \sigma_{kj} + i_{v_{kj}}A)] \\
&= R^{-1}(\delta_{ik}\hat{E}_{lj} - \delta_{il}\hat{E}_{kj} - \delta_{jk}\hat{E}_{li} + \delta_{jl}\hat{E}_{ki}) \\
&= X_{[ij][kl]}{}^{[mn]}\hat{E}_{mn}
\end{aligned} \tag{5.72}$$

where $X_{[ij][kl]}{}^{[mn]}$ is given in (5.64) and is the structure constant that defines the Lie algebra $\mathfrak{so}(d+1)$ of the structure group $SO(d+1)$ which agrees with Theorem 5.2. Therefore, by Definition 5.2, all round spheres are generalised parallelisable.

5.4 Generalised Scherk-Schwarz Reduction on S^d

Consider for example, the 10-dimensional type II supergravity with d dimensions being compactified, the spacetime is

$$\mathcal{M}_{10} \simeq \mathcal{N} \times \mathcal{M} \tag{5.73}$$

where $\mathcal{M}$ is the d-dimensional compactified manifold and $\mathcal{N}$ is a $(10-d)$-dimensional manifold for reduced theory. We assign coordinate x^μ with greek indices for $\mathcal{N}$ and use latin or capital letters for indices on $\mathcal{M}$. After Scherk-Schwarz reduction, the fields in the NSNS sector g, B, ϕ give three types of fields for the reduced theory on $\mathcal{N}$

1. g^{mn}, B^{mn}, ϕ which are encoded in the generalised metric

   $$G^{MN} = \phi^{IJ}(x)\hat{E}_I{}^M \hat{E}_J{}^N \tag{5.74}$$

   and transform as a scalar $\phi(x)$ on $\mathcal{N}$ for they have no indices of $\mathcal{N}$

2. $g_\mu{}^m, B_\mu{}^m$ transform as vectors on $\mathcal{N}$, and can be gathered into a single vector $V_\mu{}^N$

   $$V_\mu{}^N = A_\mu{}^I(x)\hat{E}_I{}^N \tag{5.75}$$

   which gives the gauge field $A_\mu(x)$

3. $g_{\mu\nu}(x)$ which is a metric on $\mathcal{N}$

so the action for the reduced theory on $\mathcal{N}$ generally depends on $\phi(x), A_\mu(x), g_{\mu\nu}(x)$ as $S = S[\phi, A, g]$. One can find the field strength for the gauge field ${A_\mu}^I$ is

$$F_{\mu\nu}{}^I = 2\partial_{[\mu} A_{\nu]}{}^I + X_{JK}{}^I A_\mu{}^J A_\nu{}^K \tag{5.76}$$

where $X_{AB}{}^C$ is the structure constant that defines the Lie algebra

$$[\hat{E}_A, \hat{E}_B] = X_{AB}{}^C \hat{E}_C \tag{5.77}$$

of the gauge symmetry group from compactified manifold $\mathcal{M}$ after Scherk-Schwarz reduction. Similar to the usual Scherk-Schwarz reduction (Appendix F) where the Lie algebra of the gauge group in the reduced theory is the same as the Lie algebra on the global frame of the manifold, [32] suggests that for the generalised version, the structure constant $X_{AB}{}^C$ for the gauge group is also equivalent to the structure constant that defines the Lie algebra of the structure group of the generalised tangent bundle

$$[\![\hat{E}_A, \hat{E}_B]\!] = X_{AB}{}^C \hat{E}_C \tag{5.78}$$

where $\{\hat{E}_A\}$ is the global frame of the generalised tangent bundle. Hence, for $\mathcal{M}$ is a d-sphere background, the Lie algebra of the gauge group after Scherk-Schwarz reduction is $\mathfrak{so}(d+1)$ from (5.72), and the gauge group is $SO(d+1)$ i.e. the structure group of generalised tangent bundle.

Then we define the generalised Scherk-Schwarz reduction on spheres as analogue to the usual Scherk-Schwarz reduction in Appendix F. We first define a new global rotated generalised frame. For the bi-vector representation, the new frame is [32]

$$\hat{E}'_{ij} = U_i{}^k(x) U_j{}^l(x) \hat{E}_{kl}, \quad U_i{}^j(x) \in GL^+(d+1, \mathbb{R}) \tag{5.79}$$

giving the inverse of generalised metric

$$G'^{MN} = \frac{1}{2} T^{ik}(x) T^{jl}(x) \hat{E}^M_{ij} \hat{E}^N_{kl} \tag{5.80}$$

where $T^{kl} = \delta^{ij} U_i{}^k U_j{}^l \in GL^+(d+1, \mathbb{R})$ and $\frac{1}{2}$ is to cancel the factor from (5.48). By setting the determinant of transformation matrix to 1, the structure group can be restricted to $SL(d+1, \mathbb{R})$. The scalar field of reduced theory is then

$$\phi(x) = \frac{1}{2} T^{ik}(x) T^{jl}(x) \tag{5.81}$$

and parameterises the coset space

$$\frac{SL(d+1,\mathbb{R})}{SO(d+1)} \tag{5.82}$$

where $SO(d+1)$ is the gauge group of reduced theory.

One can then find the inverse metric from (5.42), and by comparing the two expressions, the ansatz for sphere consistent truncations is found to be [32]

$$\begin{aligned} ds'^2 &= R^2 (T^{kl} y_k y_l)^{-\frac{2}{d-1}} T^{-1}_{ij} dy^i dy^j \\ A' &= -\frac{1}{2}(T^{kl} y_k y_l)^{-1} \frac{R^{d-1}}{(d-2)!} \varepsilon_{i_1 \ldots i_{d+1}} (T^{i_1 j} y_j) y^{i_2} dy^{i_3} \wedge \cdots \wedge dy^{i_{d+1}} + A \\ e^{2\Delta'} &= (T^{kl} y_k y_l)^{\frac{d-3}{d-1}} \end{aligned} \tag{5.83}$$

A final notice is that for a manifold that is generalised parallelisable, the frame of generalised tangent bundle is global and hence it is trivial, so the spinor bundle is also trivial and there are globally defined spinors. Then the truncated theory should have same number of supersymmetries as original theory [32].

5.5 Truncation on S^3

According to [50], the type II supergravity with near-horizon NS fivebrane background has solution of $S^3 \times \mathbb{R}_t \times \mathbb{R}^{5,1}$ where $\mathbb{R}_t$ is a linear dilaton background, and gives

$$\begin{aligned} ds^2 &= ds^2(\mathbb{R}^{5,1}) + dt^2 + R^2 ds^2(S^3) \\ H &= 2R^{-1}\mathrm{vol}_g \\ \phi &= -R^{-1} t \end{aligned} \tag{5.84}$$

where R is the radius of the 3-sphere, and the d-form $F = dA$ becomes 3-form $H = dB$. For 3 dimensions, the generalised tangent bundle is simply $E \simeq T\mathcal{M} \oplus T^*\mathcal{M}$. Note that the dilaton is not in the S^3 background and the factor Δ is set to zero. As S^3 is orientable, the structure group $GL^+(4,\mathbb{R})$ can be restricted to $SL(4,\mathbb{R})$ by setting the determinant of transformation to 1, and the generalised metric is preserved by $SO(4)/\mathbb{Z}_2$ [32], so the structure group which gives the gauge group is

$$SO(4)/\mathbb{Z}_2 \subset SL(4,\mathbb{R}) \tag{5.85}$$

which agrees with the result in [19] that the consistent reduction on S^3 gives the $SO(4)$ gauge group for the reduced theory. Then using the Lie group isomorphism

$$SO(4)/\mathbb{Z}_2 \simeq SO(3) \times SO(3), \quad SL(4,\mathbb{R}) \simeq SO(3,3) \tag{5.86}$$

one see explicitly that the structure of the tangent bundle $E \simeq T\mathcal{M} \oplus T^*\mathcal{M}$ is

$$SO(3) \times SO(3) \subset SO(3,3) \tag{5.87}$$

so the geometry simply becomes the $O(3,3)$ generalised geometry.

From Section 5.3, the Lie algebra on the frame defined by the generalised parallelisable condition is $\mathfrak{so}(4)$, and using Lie algebra isomorphisms one has

$$\mathfrak{so}(4) \simeq \mathfrak{so}(3) \times \mathfrak{so}(3) \tag{5.88}$$

which implies that there are two copies of $\mathfrak{so}(3)$ Lie algebra. To see this explicitly, we notice that the generalised tangent bundle E can be split into two sub-bundles $E \simeq C_+ \oplus C_-$ as in Section 3.2. The sub-frames for C_+ and C_- can be constructed using self-dual and anti-self-dual decomposition of $\hat{E}_{ij} = v_{ij} + \sigma_{ij} - i_{v_{ij}}B$ as in Example A.6 which are

$$\hat{E}^{\pm}_{ij} = \frac{1}{2}\hat{E}_{ij} \pm \frac{1}{4}\varepsilon_{ij}{}^{kl}\hat{E}_{kl}. \tag{5.89}$$

This gives 6 independent basis vectors

$$\begin{aligned}
\hat{E}^{\pm}_1 &= \hat{E}^{\pm}_{12} = \frac{1}{2}(\hat{E}_{12} \pm \hat{E}_{34}) \\
\hat{E}^{\pm}_2 &= \hat{E}^{\pm}_{13} = \frac{1}{2}(\hat{E}_{13} \pm \hat{E}_{24}) \\
\hat{E}^{\pm}_3 &= \hat{E}^{\pm}_{14} = \frac{1}{2}(\hat{E}_{14} \pm \hat{E}_{23})
\end{aligned} \tag{5.90}$$

and the sets of basis vectors labelled by $+$ and $-$ are identified to the bases $\hat{E}^+_a$ for C^+ and $\hat{E}^-_{\bar{a}}$ for C^- respectively where $a, \bar{a} = 1, 2, 3$ in 3 dimensions. Their expressions are given in terms of left and right invariant vector fields on S^3 which can be found in [32], and one can check that they satisfies the orthogonal conditions (3.44) and (3.45) with the canonical metric and generalised metric. Then, under the Courant bracket they give [32]

$$\begin{aligned}
[\![\hat{E}^+_a, \hat{E}^+_b]\!] &= R^{-1}\varepsilon_{abc}\hat{E}^+_c \\
[\![\hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}}]\!] &= R^{-1}\varepsilon_{\bar{a}\bar{b}\bar{c}}\hat{E}^-_{\bar{c}} \\
[\![\hat{E}^+_a, \hat{E}^-_{\bar{b}}]\!] &= 0
\end{aligned} \tag{5.91}$$

so there are two copies of $\mathfrak{su}(2)$ Lie algebra that is closed on each sub-frames $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$. Therefore, the overall Lie algebra is

$$\mathfrak{su}(2) \times \mathfrak{su}(2) \tag{5.92}$$

and using the Lie algebra isomorphism $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$, one obtains (5.88) which is expected from the $O(3) \times O(3)$ structure from splitting frames.

For the Scherk-Schwarz reduction, the frame is transformed to

$$\hat{E}'_A(x) = U_A{}^B(x)\hat{E}_B, \quad U_A{}^B \in SO(3,3) \simeq SL(4,\mathbb{R}) \tag{5.93}$$

giving the new inverse metric

$$G'^{MN} = \phi^{AB}(x)\hat{E}_A{}^M\hat{E}_B{}^N \tag{5.94}$$

where

$$\phi^{AB}(x) = \delta^{CD}U_C{}^A U_D{}^B \tag{5.95}$$

is the scalar field of the reduced theory, and parametrise the coset space

$$\frac{SL(4,\mathbb{R})}{SO(4)/\mathbb{Z}_2} \simeq \frac{SO(3,3)}{SO(3)\times SO(3)}. \tag{5.96}$$

Then using (5.83) gives the ansatz

$$\begin{aligned} ds'^2 &= R^2(T^{kl}y_k y_l)^{-1}T^{-1}_{ij}dy^i dy^j \\ B' &= -\frac{1}{2}R^2(T^{kl}y_k y_l)^{-1}\varepsilon_{i_1 i_2 i_3 i_4}(T^{i_1 j}y_j)y^{i_2}dy^{i_3}\wedge dy^{i_4} + B \\ e^{2\Delta'} &= 1 \end{aligned} \tag{5.97}$$

which matches the results from [23, 19].

However, since we only included bosonic fields and there are more fields in the theory, we cannot prove that the truncation is consistent here.

6 Conclusion

In this dissertation, we constructed the $O(d,d)$ generalised geometry on a generalised space $T\mathcal{M} \oplus T^*\mathcal{M}$ where $\mathcal{M}$ is a d-dimensional differentiable manifold. The generalised space admits a natural canonical inner product which defines the $O(d,d)$ structure group. By considering the Lie algebra of the structure group, we found three different transformations, the diffeomorphism described by the $GL(d,\mathbb{R})$ group, the B-transform where B is a closed 2-form gauge field, and also a β-transform. The diffeomorphism and B-transform are given by the geometric subgroup $GL(d,\mathbb{R}) \ltimes \Omega^2_{\text{closed}}$ and is encoded in the Courant bracket or Dorfman derivative which are the generalised versions of Lie bracket and Lie derivative. The β-transform was not discussed here since it does not preserve the Courant bracket, but it is related to the T-duality in string theory and non-geometrical background [10]. We then defined the generalised tangent bundle E by a short exact sequence with a splitting defined by the B-transform and we identified the isomorphism between E and $T \oplus T^*$. This definition using exact sequence implies that there are more general generalised tangent bundles we can consider [3], for example, replace the cotangent bundle with the anti-symmetrised cotangent bundle gives the $GL^+(d+1,\mathbb{R})$ generalised geometry for d-sphere in Chapter 5. Other geometries such as exceptional generalised geometries can also be defined for 11-dimensional supergravity and M-theory [3].

We then constructed the generalised metric which is compatible with the canonical inner product and further reduces the structure group to $O(d) \times O(d)$. This also splits the generalised tangent bundle into two orthogonal sub-bundles. Then we constructed the generalised vielbein, connection, torsion and curvature as analogue to the usual Riemannian geometry. However, even though we constructed a torsion-free connection that is compatible with both the generalised metric and the canonical metric, it is not unique, so is the curvature scalar. But since the generalised metric encodes the metric and the 2-form B-field which is identified as a gauge field, and the Courant bracket encodes the symmetries of diffeomorphism and gauge transformation, we were still motivated to use generalised geometry to reformulated the type II supergravity. The generalised tangent bundle is extended to $\det T^*\mathcal{M} \otimes (T\mathcal{M} \oplus T^*\mathcal{M})$ so that the structure group is $O(d,d) \times \mathbb{R}^+$ and provides one more degree of freedom for the dilaton. Then we showed that the new generalised metric encodes all of the metric, the 2-form B-field and the dilaton, and the bosonic symmetries are encoded in the Courant bracket. Moreover, with the help of spinor fields, we were able define a unique curvature scalar, although there is still no canonical uniquely defined torsion-free connection. The resultant generalised connection was found to be the equivalent to the results in [12, 13] and the double field theory [14]. The curvature scalar then allows us to write down the action and hence the equations of motion using the generalised geometry language. For the metric and the 2-form B, it is just like the vacuum field equation, and if one includes the RR fields, it can be viewed as a source term to the equations of motion. Hence, one has a generalised version of Einstein gravity theory from the reformulation of type II supergravity using $O(d,d) \times \mathbb{R}^+$ generalised geometry. One remaining task is that the fermionic fields are not included. Furthermore, one can construct the exceptional generalised geometry $E_{d(d)} \times \mathbb{R}^+$ for 11-dimensional supergravity as in [17, 16], and also for geometries of M-theory [3].

In the last chapter, we applied the generalised geometry to the consistent truncation problem. We noticed that the usual parallelisable Lie group manifold gives consistent truncation and can explain the consistent truncations on S^1 and S^3, but there are still truncations on S^4, S^5 and S^7 in supergravity theories which cannot be explained. Hence, we defined the generalised parallelisable manifold as analogue to the local group manifold as

$$[\![\hat{E}_A, \hat{E}_B]\!] = X_{AB}{}^C \hat{E}_C \tag{6.1}$$

where $X_{AB}{}^C$ is constant and $\{\hat{E}_A\}$ is a global frame on the generalised tangent bundle and the Courant bracket on frames defines a Lie algebra. Then, as implied by [30, 31, 10], there is a conjecture that a generalised parallelisable manifold gives a consistent truncation which also preserves the number of supersymmetry, and the gauge group of the reduced theory has the same Lie algebra as (6.1). Even though there is no proof to the conjecture, we showed that all round spheres are parallelisable by constructing a $GL^+(d+1,\mathbb{R})$ generalised geometry for d-sphere, and generalised the Scherk-Schwarz reduction indeed gives a gauge group with the Lie algebra on the frame, namely $\mathfrak{so}(d+1)$. Hence the conjecture may explain the truncations on S^4, S^5 and S^7. We then used the 3-sphere as an example, the generalised geometry of which is simply the $O(d,d)$ one, and the Lie algebra of the gauge group is found to be $\mathfrak{so}(4)$ which agrees with the usual Scherk-Schwarz reduction. The ansatz for generalised Scherk-Schwarz reduction on S^3 was also found to be the same as the usual results in [23, 19]. However, we were not able to show that the truncation is consistent since there are other fields such as fermionic fields which were not included. One can further show that for the truncations on S^4, S^5 and S^7 with exceptional geometries, the ansatz agree with the usual Scherk-Schwarz reduction [32]. In fact, it is proven in [51] that there is consistent truncations on any supersymmetric solutions of 10 or 11-dimensional supergravity with $AdS_D \times \mathcal{M}$ where $\mathcal{M}$ is compactified. However, even if one proved the conjecture, it is still very hard to find a manifold that is generalised parallelisable, although Proposition 5.3 provides a constraint on possible generalised parallelisable manifold. And, one may also expect to explain the Pauli truncation on $\mathcal{M} \simeq G \times G/G$ using generalised parallelisability.

The applications of generalised geometry do not stop here, it can be further extended to M-theory, supersymmetric flux compactification, calibrations for D-branes and potentially the reformulation of AdS/CFT correspondence, and even mirror symmetries.

Appendices

A Differential Geometry

In this section, we consider a m-dimensional smooth manifold $\mathcal{M}$ with coordinate basis $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and dual basis dx^μ. Detailed derivations and proofs can be found in [35].

Definition A.1. For a map $f : \mathcal{M} \to \mathcal{N}$, there is an induced map called push-forward

$$\begin{array}{rccc} f_* : & T_p\mathcal{M} & \to & T_{f(p)}\mathcal{N} \\ & v & \to & f_* v \end{array} \quad , \quad f_* v = v^\mu \frac{\partial x^\nu}{\partial x^\mu}\Big|_p \frac{\partial}{\partial x^\nu} \tag{A.2}$$

and another map called pull-back

$$\begin{array}{rccc} f^* : & T^*_p\mathcal{M} & \leftarrow & T^*_{f(p)}\mathcal{N} \\ & \omega & \leftarrow & \omega \end{array} \quad , \quad f^* \omega = \omega_\mu \frac{\partial x^\mu}{\partial x^\nu}\Big|_{f(p)} dx^\nu \tag{A.3}$$

Definition A.2. For a vector field V, one can define flow which is a map

$$\begin{array}{rccl} \sigma_V : & \mathbb{R} \times \mathcal{M} & \to & \mathcal{M} \\ & (\lambda, p) & \to & p' = \sigma_V(\lambda, p) = \sigma_V(\lambda) p \end{array} . \tag{A.4}$$

Definition A.3. For two vector fields X, Y, The Lie derivative is defined as

$$\mathcal{L}_X Y|_p = \lim_{\varepsilon \to 0} \frac{\sigma_X(-\varepsilon)_* Y|_{p'} - Y|p}{\varepsilon} \in T_p\mathcal{M} \tag{A.5}$$

where $p' = \sigma_X(\varepsilon)p$ and $\sigma_X(\varepsilon) : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism that takes $Y|_{p'} \in T_{p'}\mathcal{M}$ to $Y|_p \in T_p\mathcal{M}$, hence the Lie derivative encodes the infinitesimal diffeomorphism. On a coordinate basis, it can be written in component as

$$\mathcal{L}_X Y^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu. \tag{A.6}$$

Claim A.1. *From (A.6), it is easy to see that the Lie derivative is anti-symmetric*

$$\mathcal{L}_X Y = -\mathcal{L}_Y X. \tag{A.7}$$

Claim A.2. *The Lie derivative satisfies a Leibniz rule on two vectors X, Y,*

$$\mathcal{L}_X(\mathcal{L}_Y Z) = \mathcal{L}_{\mathcal{L}_Y X} Z + \mathcal{L}_Y(\mathcal{L}_X Z). \tag{A.8}$$

And due to anti-symmetry, the above equation can be rewritten as

$$\mathcal{L}_X(\mathcal{L}_Y Z) + \mathcal{L}_Y(\mathcal{L}_Z X) + \mathcal{L}_Z(\mathcal{L}_X Y) = 0 \tag{A.9}$$

which is the Jacobi identity.

Definition A.4. The action of Lie derivative can be extended to a tensor field A as

$$\mathcal{L}_X[A]|_p = \lim_{\varepsilon\to 0}\frac{\sigma(\varepsilon)^* A|_{p'} - A|_p}{\varepsilon}. \tag{A.10}$$

and in coordinate basis, for a (p,q)-tensor A, the components are [4]

$$\begin{aligned}\mathcal{L}_X A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} &= X^\mu \partial_\mu A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} \\ &+ (\partial_\mu X^{\mu_1}) A^{\mu\mu_2\ldots\mu_p}_{\nu_1\ldots\nu_q} + \cdots + (\partial_\mu X^{\mu_p}) A^{\mu_1\ldots\mu_{p-1}\mu}_{\nu_1\ldots\nu_q} \\ &- (\partial_{\nu_1} X^\mu) A^{\mu_1\ldots\mu_p}_{\mu\nu_2\ldots\nu_q} - \cdots - (\partial_{\nu_q} X^\mu) A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_{q-1}\mu}.\end{aligned} \tag{A.11}$$

The terms in the second and third lines can be thought as the transformation under the adjoint representation of Lie algebra $\mathfrak{gl}(d,\mathbb{R})$ with elements $a^\mu{}_\nu = \partial_n X^\mu$ [4], which is the infinitesimal diffeomorphism, hence the Lie derivative generates a diffeomorphism. This also implies that tensors can be viewed as representations of the $GL(d,\mathbb{R})$ group.

Example A.1. The Lie derivative of a co-vector field ω in components is

$$\mathcal{L}_X \omega_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu, \tag{A.12}$$

and for a function f, Lie derivative is the directional derivative

$$\mathcal{L}_X f = X[f]. \tag{A.13}$$

Claim A.3. *The Lie derivative satisfies Leibniz rule. For a function f and tensor field A,*

$$\mathcal{L}_X(fA) = X[f]A + f\mathcal{L}_X A, \tag{A.14}$$

Definition A.5. The Lie bracket is a map that takes two vector fields to another vector field

$$\begin{aligned}[\cdot,\cdot]: \quad T_p\mathcal{M}\times T_p\mathcal{M} &\to T_p\mathcal{M} \\ X, Y &\to [X,Y]\end{aligned} \tag{A.15}$$

s.t. for a function f,

$$[X,Y][f] = X[Y[f]] - Y[X[f]]. \tag{A.16}$$

The Lie bracket can be written in components on a coordinate basis as

$$[X,Y]^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu = \mathcal{L}_X Y^\mu \tag{A.17}$$

so Lie bracket and Lie derivative of two vectors are the same.

Claim A.4. *For $X, Y, Z \in T_p\mathcal{M}$, the Lie bracket satisfies three properties*

- *bi-linear:* $[aX+bY,Z]=a[X,Z]+b[Y,Z]$, $[X,aY+bZ]=a[X,Y]+b[X,Z]$
- *anti-symmetric:* $[X,Y]=-[Y,Z]$
- *Jacobi identity:* $[[X,Y],Z]+[[Y,Z],X]+[[Z,Y],Z]=0$

Claim A.5. *It can be shown from (A.8) that*

$$\mathcal{L}_{[X,Y]}=[\mathcal{L}_X,\mathcal{L}_Y] \tag{A.18}$$

Claim A.6. *For a diffeomorphism $f:\mathcal{M}\to\mathcal{M}$, vector fields X,Y are pushed forwards to f_*X, f_*Y, then Lie bracket is invariant under diffeomorphisms i.e. diffeomorphisms preserve Lie bracket*

$$f_*[X,Y]=[f_*X,f_*Y]. \tag{A.19}$$

Definition A.6. A differential form ω of order r is a totally anti-symmetric tensor in $\mathcal{J}^0_r$ and is given in components is defined as

$$\omega=\frac{1}{r!}\omega_{\mu_1\ldots\mu_r}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}. \tag{A.20}$$

The space of r-form is denoted as $\Omega^r(\mathcal{M})$ with dimension

$$\dim(\Omega^r(\mathcal{M}))=\frac{m!}{(m-r!)r!} \tag{A.21}$$

and hence

$$\dim(\Omega^r(\mathcal{M}))=\dim(\Omega^{m-r}(\mathcal{M})) \tag{A.22}$$

Example A.2. A top form is a m-form

$$V=v(x)dx^1\wedge\cdots dx^m=\frac{1}{m!}v(x)\tilde{\varepsilon}_{\mu_1\ldots\mu_m}dx^{\mu_1}\wedge\cdots dx^{\mu_m} \tag{A.23}$$

and a volume element

$$\mathrm{vol}_g=\sqrt{|g|}dx^1\wedge\cdots\wedge dx^m=\frac{1}{m!}\varepsilon_{\mu_1\ldots\mu_m}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_m} \tag{A.24}$$

where $\varepsilon_{\mu_1\ldots\mu_m}=\sqrt{|g|}\tilde{\varepsilon}_{\mu_1\ldots\mu_m}$, and g is a metric on $\mathcal{M}$ and $|g|$ is its determinent.

Definition A.7. The exterior derivative maps r-form to $(r+1)$-form

$$\begin{array}{rccc} d: & \Omega^r(\mathcal{M}) & \to & \Omega^{r+1}(\mathcal{M}) \\ & \omega & \to & d\omega \end{array} \tag{A.25}$$

In coordinate basis, $\omega=\frac{1}{r!}\omega_{\mu_1\ldots\mu_r}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}$ and

$$d\omega=\frac{1}{r!}(\partial_\alpha\omega_{\mu_1\ldots\mu_r})dx^\alpha\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r} \tag{A.26}$$

Example A.3. For a function $f \in \Omega^0(\mathcal{M})$, $df = \partial_\mu f dx^\mu$ is a 1-form.

Claim A.7. *The exterior derivative is nilpotent i.e.* $d^2 = 0$ *which can be shown by symmetry argument of indices.*

Definition A.8. An r-form ω is

- closed if $d\omega = 0$,
- exact if $\omega = d\beta$, where β is a $(r-1)$-form.

Claim A.8. *Since* $d^2 = 0$*, any exact form is also closed.*

Definition A.9. The interior product is a map induced by a vector field X and is defined as

$$\begin{array}{rccc} i_X : & \Omega^r(\mathcal{M}) & \rightarrow & \Omega^{r-1}(\mathcal{M}) \\ & \omega & \rightarrow & i_X\omega \end{array} \tag{A.27}$$

s.t.

$$(i_X\omega)(\cdots) = \omega(X, \cdots). \tag{A.28}$$

In a coordinate basis, $\omega = \frac{1}{r!}\omega_{\mu_1\ldots\mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$ so

$$i_X\omega = \frac{1}{(r-1)!} X^\nu \omega_{\nu\mu_1\ldots\mu_{r-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{r-1}}. \tag{A.29}$$

Example A.4. If ω is a 1-form, then $i_X\omega = \omega(X) = \langle \omega, X \rangle$ is the inner product.

Claim A.9. *For two vectors* X, Y*, the interior product satisfies*

$$i_X i_Y = -i_Y i_X \tag{A.30}$$

due to symmetry of indices, hence it is also nilpotent

$$i_X^2 = 0. \tag{A.31}$$

Claim A.10. *The Lie derivative or Lie bracket, exterior derivative and interior product satisfy the following identities*

$$\mathcal{L}_X = i_X d + d i_X \tag{A.32}$$

which is called the Cartan formula in this dissertation, and,

$$i_{[X,Y]} = [\mathcal{L}_X, i_Y], \tag{A.33}$$

$$\mathcal{L}_{fX} = f\mathcal{L}_X + df i_X. \tag{A.34}$$

Claim A.11. *For a function* $f \in \Omega^0$*, since there is no* (-1)*-form, so* $i_X f = 0$*, and using (A.32) gives*

$$\mathcal{L}_X f = i_X df + d i_X f = i_X df. \tag{A.35}$$

Claim A.12. *For a diffeomorphism $f : \mathcal{M} \to \mathcal{N}$, an r-form ω can be pulled back using f^*, and satisfies*

$$d(f^*\omega) = f^*(d\omega). \tag{A.36}$$

And a vector X can be pulled to f^*X, and satisfies

$$i_{f^*X} f^*\omega = f^*(i_X \omega). \tag{A.37}$$

And similarly for push forward f_*.

Definition A.10. Hodge star is a map $* : \Omega^r(\mathcal{M}) \to \Omega^{m-r}(\mathcal{M})$, and for a general r-form $\omega = \frac{1}{r!}\omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$

$$*\omega = \frac{1}{r!(m-r)!}\omega_{\mu_1 \ldots \mu_r} \varepsilon^{\mu_1 \ldots \mu_r}{}_{\mu_{r+1} \ldots \mu_m} dx^{\mu_{r+1}} \wedge \cdots \wedge dx^{\mu_m} \tag{A.38}$$

Example A.5. For volume element, $\mathrm{vol}_g = *1$.

Example A.6. On a 4-dimensional manifold, any 2-form $\omega \in \Omega^2$ can be decompose into self-dual and anti-self-dual as

$$\omega = \omega^+ + \omega^- \tag{A.39}$$

where ω^+ is self-dual and ω^- is anti-self-dual i.e.

$$*\omega^+ = \omega^+, \quad *\,\omega^- = -\omega^- \tag{A.40}$$

and one finds that

$$\omega^\pm = \frac{1}{2}(1 \pm *)\omega \tag{A.41}$$

and in components is

$$\omega^\pm_{\mu\nu} = \frac{1}{2}\omega_{\mu\nu} \pm \frac{1}{4}\varepsilon_{\mu\nu}{}^{\rho\sigma}\omega_{\rho\sigma}. \tag{A.42}$$

B Lie Group and Lie Algebra

Definition A.11. For a Lie group G, Lie subgroup $H \subset G$, and an equivalence relation

$$\sim : g \sim g' \text{ if } \exists h \in H \text{ s.t. } g' = gh \tag{A.43}$$

which defines an equivalence class

$$[g] = \{gh : h \in H\} \tag{A.44}$$

then the coset space is group G with its elements identified under $\sim$

$$G/H = \{[g]\} \tag{A.45}$$

which is a manifold but may not be a Lie group. If G/H has differential structure, then it is called homogeneous space. One has a relation for dimensions

$$\dim(G/H) = \dim G - \dim H. \tag{A.46}$$

Example A.7.

$$\begin{aligned} S^n &\simeq SO(n+1)/SO(n) \\ dS_n &\simeq SO(1,n)/SO(1,n-1) \\ AdS_n &\simeq SO(2,n-1)/SO(1,n-2) \end{aligned} \tag{A.47}$$

Definition A.12. An action of group G on a manifold $\mathcal{M}$ is a differentiable map

$$\begin{array}{rccl} \Phi: & G \times \mathcal{M} & \to & \mathcal{M} \\ & g, p & \to & \Phi(g,p) = \Phi_g(p) \end{array} \tag{A.48}$$

s.t. $\Phi(e,p) = p$, $\forall p \in \mathcal{M}$ and $\Phi(g_1, \Phi(g_2, p)) = \Phi(g_1 g_2, p)$ preserve group structure.

Definition A.13. The action $\Phi : G \times \mathcal{M} \to \mathcal{M}$ is transitive if $\forall p \neq p' \in \mathcal{M}$, $\exists g \in G$ s.t. $p' = \Phi_g p$.

Definition A.14. The action $\Phi : G \times \mathcal{M} \to \mathcal{M}$ is free if for all non-trivial $g \neq e \in G$, $\nexists p \in \mathcal{M}$ s.t. $\Phi_g(p) = p$. In other word, $\Phi_p(g) \neq p$, $\forall p \in \mathcal{M}$, so if $\exists p \in \mathcal{M}$ s.t. $\Phi_g(p) = p$, then $g = e$.

Definition A.15. Left and right actions are action of Lie group on itself

$$\begin{array}{lrccl} \text{left action} & L_g: & G \times G & \to & G \\ & & (g,h) & \to & L_g h = gh \\ \text{right action} & R_g: & G \times G & \to & G \\ & & (g,h) & \to & R_g h = hg \end{array} \tag{A.49}$$

they are transitive, as, for example, if $g = g_1 g_2^{-1}$ then $L_g g_2 = g_1$.

Definition A.16. Since left action L_g is diffeomorphism on G and transitive, it can be used to push-forward vector $V \in T_e G$ to $L_{g*} V \in T_g G$, V then defines a vector field over G $X_V|_g = L_{g*} V$. The left-invariant vector field is defined by $V = X|_e \in T_e G$

$$X|_g = L_{g*} X|_e = X^\mu|_e \frac{\partial x^\nu(g)}{\partial x^\mu(e)} \frac{\partial}{\partial x^\nu}\Big|_g = X^\nu \frac{\partial}{\partial x^\nu}\Big|_g \tag{A.50}$$

Theorem A.13 (Lie theorem)**.** *The Lie algebra $\mathfrak{g}$ of a Lie group G is uniquely determined by structure i.e.*

$$[X_a, Y_b]|_g = f_{ab}{}^c Z_c|_g \tag{A.51}$$

where X, Y, Z are left-invariant vector fields on G, and ${f_{ab}}^c$ is structure constant.

C Fibre Bundle

This section gives formal definitions of fibre bundles. More details can be found in [35].

Definition A.17. A coordinate bundle $(E, \pi, \mathcal{M}, F, G, \{U_i\}, \{\phi_i\})$ contains

1. three differentiable manifolds: total space E, base space $\mathcal{M}$ and fibre F.

2. a surjection $\pi : E \to \mathcal{M}$ which is called the projection. The inverse map gives the fibre at $p \in \mathcal{M}$
$$\pi^{-1}(p) = F_p \tag{A.52}$$
and implies that $F \subseteq E$.

3. open covering $\{U_i\}$ of $\mathcal{M}$ with local diffeomorphism
$$\phi_i : U_i \times F \to \pi^{-1}(U_i) \quad \text{s.t.} \quad \pi \circ \phi_i(p, f) = p \tag{A.53}$$
ϕ_i is called local trivialisation as $\pi^{-1}(U_i) \sim U_i \times F$. The map $\phi_i(p, f)$ can also be denoted as
$$\phi_{i,p}(f) : F \to F_p = \pi^{-1}(p) \tag{A.54}$$

4. a Lie group G called structure group acts on F on the left.

5. transition function t_{ij} which is element of the structure group G. For $p \in U_i \cap U_j$ and $f_i, f_j \in F$, the transition function is defined as
$$t_{ji}(p) = \phi^{-1}_{j,p} \circ \phi_{ip} \tag{A.55}$$
so that $f_j = t_{ji}(p) f_i$ and $\pi^{-1}(p) = \phi_i(p, f_i) = \phi_j(p, f_j)$.

For two coordinate bundles $(E, \pi, \mathcal{M}, F, G, \{U_i\}, \{\phi_i\})$, and $(E, \pi, \mathcal{M}, F, G, \{V_i\}, \{\psi_i\})$, if $(E, \pi, \mathcal{M}, F, G, \{U_i\} \cup \{V_i\}, \{\phi_i\} \cup \{\psi_i\})$ is also a coordinate bundle, then they are said to be equivalent. A fibre bundle is then defined as the equivalence class of coordinate bundles, and is denoted as $(E, \pi, \mathcal{M}, F, G)$ or simply E.

Definition A.18. For a fibre bundle E, its local section is a smooth map $s : U \subset \mathcal{M} \to E$ s.t. $\pi \circ S = \mathrm{id}_{\mathcal{M}}$, and $s(p) \in F_p \simeq \pi^{-1}(p)$. Some fibre bundles have global section, and the space is denoted as $\Gamma(E)$.

Definition A.19. A vector bundle is a bundle whose fibre is a vector space. For $F = \mathbb{R}^k$, the structure group is $GL(k, \mathbb{R})$. For vector bundles, sections are globally defined. One can define fibre metric $h_{\mu\nu}(p)$ s.t. for sections s, s', there is an inner product

$$(s, s')_p = h_{\mu\nu}(p) s^{\mu}(p) s'^{\nu}(p). \tag{A.56}$$

Definition A.20. For a vector bundle $E \to \mathcal{M}$, with fibre $F \simeq \mathbb{R}^k$, on chart U_i of $\mathcal{M}$,

$$\pi^{-1}(U_i) \simeq U_i \times \mathbb{R}^k \simeq s(U_i) \tag{A.57}$$

hence one can choose k linearly independent sections over U_i: $\{e_1(p), ..., e_k(p)\}$ which defines a local frame over U_i. Given a frame over U_i, there exists a natural map $F_p \to F \simeq \mathbb{R}^k$

$$V = V^{\alpha} e_{\alpha}(p) \in F_p \to \{V^{\alpha}\} \in F \tag{A.58}$$

One can also define a change of frame from $e_1(p), ..., e_k(p)$ to $e'_1(p), ..., e'_k(p)$ under the structure group

$$e'_{\beta}(p) = G^{\alpha}{}_{\beta} e_{\alpha}(p) \tag{A.59}$$

where $G^{\alpha}{}_{\beta} \in GL(k, \mathbb{R})$ is the transition function, and for $V \in \pi^{-1}(p)$ require $V = V^{\alpha} e_{\alpha}(p) = V'^{\beta} e'_{\beta}(p)$ so

$$V'^{\beta} = G^{-1}(p)^{\beta}{}_{\alpha} V^{\alpha}. \tag{A.60}$$

Definition A.21. Dual bundle of a vector bundle E with fibre F is defined as $E^* \to \mathcal{M}$ with fibre F^* which is a set of linear maps from F to $\mathbb{R}$. The projection is still π. Given a basis $\{e_{\alpha}(p)\}$ of F_p, there is a dual basis $\{e^{\alpha}(p)\}$ for F_p^* s.t.

$$\langle e^{\alpha}(p), e_{\beta}(p) \rangle = \delta^{\alpha}{}_{\beta}. \tag{A.61}$$

Definition A.22. Tangent bundle $\mathcal{M}$ on m-dimensional manifold $\mathcal{M}$ is a vector bundle, with fibre $F \simeq \mathbb{R}^m$. It is defined as

$$T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p \mathcal{M}. \tag{A.62}$$

The sections of tangent bundle are vector fields, and its structure group is $GL(m, \mathbb{R})$. For $V \in T_p\mathcal{M}$,

$$V = V^{\mu}_{(i)} \frac{\partial}{\partial x^{\mu}_{(i)}}\Big|_p = V^{\mu}_{(j)} \frac{\partial}{\partial x^{\mu}_{(j)}}\Big|_p \tag{A.63}$$

so the vector components transformation as

$$V^{\mu}_{(j)} = \frac{\partial x^{\mu}_{(j)}}{\partial x^{\nu}_{(i)}} V^{\nu}_{(i)}, \quad \frac{\partial x^{\mu}_{(j)}}{\partial x^{\nu}_{(i)}} \in GL(m, \mathbb{R}). \tag{A.64}$$

Since the fibre of tangent bundle at a point $p \in \mathcal{M}$ is the tangent space $T_p\mathcal{M}$, so there is a natural coordinate basis $\{\frac{\partial}{\partial x^\mu}\}$ over chart U_i which can be used as frame.

Definition A.23. Co-tangent bundle is the dual bundle of tangent bundle. It is defined as

$$T^*\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p^*\mathcal{M} \tag{A.65}$$

so the sections of cotangent bundle are 1-forms. The natural frame for cotangent bundle is the dual basis of $\frac{\partial}{\partial x^\mu}$ i.e. dx^μ s.t.

$$\langle dx^\mu, \frac{\partial}{\partial x^\nu}\rangle = \delta^\mu{}_\nu. \tag{A.66}$$

The structure group is still $GL(m,\mathbb{R})$, so the section $\omega = \omega_\mu dx^\mu$ transform as

$$\omega_{(j)\mu} = \frac{\partial x^\nu_{(i)}}{\partial x^\mu_{(j)}}\Big|_p \omega_{(i)\nu}, \quad \frac{\partial x^\nu_{(i)}}{\partial x^\mu_{(j)}} \in GL(m,\mathbb{R}). \tag{A.67}$$

Definition A.24. For two fibre bundles $E \to \mathcal{M}$ with projection π and $E' \to \mathcal{M}'$ with projection π', a smooth map $\bar{f} : E' \to E$ is a bundle map if $\bar{f}$ maps each fibre F'_p of E' onto F_q of E. On a commutative diagram it is

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathcal{M}' & \xrightarrow{f} & \mathcal{M} \end{array}$$

Definition A.25. Product bundle of two vector bundles $E \to \mathcal{M}$ with fibre F, projction π and $E' \to \mathcal{M}'$ with fibre F', projection π' is defined as

$$E \times E' \xrightarrow{\pi\times\pi'} \mathcal{M}\times\mathcal{M}'$$

with fibre $F \oplus F'$ i.e. $\binom{V}{W} \in F \oplus F'$, $V \in F$, $W \in F'$.

Definition A.26. Whitney sum bundle of two bundles E with projection π_1 and E' with projection π_2 is defined as

$$\begin{array}{ccc} E\oplus E' & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi\times\pi'} \\ \mathcal{M} & \xrightarrow{f} & \mathcal{M}\times\mathcal{M} \end{array}$$

where $E \otimes E' = \{(u, u') \in E \times E' : (\pi \times \pi')(u, u') = (p, p)\}$, and fibre is $F \oplus F'$ so

$$(\pi\times\pi')^{-1}(p,p) \simeq \pi(p) \oplus \pi'^{-1}(p) = F_p \oplus F'_p. \tag{A.68}$$

The transition function T_{ij} of $E \oplus E'$ is

$$T_{ij}(p) = \begin{pmatrix} t_{ij}(p) & 0 \\ 0 & t'_{ij}(p) \end{pmatrix} \tag{A.69}$$

acts on $F \oplus F'$ on the left.

Definition A.27. For two vector bundles $E \to \mathcal{M}$ with fibre F, projection π and $E' \to \mathcal{M}$ with fibre F', projection π, tensor product bundle $E \otimes E'$ is defined by assigning each $p \in \mathcal{M}$ with a fibre $F_p \otimes F'_p$. Given basis $\{e_\alpha\}$ for F, $\{e'_\beta\}$ for F', then the basis of $F \times F'$ is $\{e_\alpha \otimes e'_\beta\}$, so

$$\dim(E \otimes E') = \dim E \times \dim E'. \tag{A.70}$$

Also, using the wedge product on basis of F

$$e_\alpha \wedge e_\beta = e_\alpha \otimes e_\beta - e_\beta \otimes e_\alpha \tag{A.71}$$

one can define anti-symmetric tensor product of vector bundles $E \wedge E'$ s.t. its fibre is spanned by $\{e_\alpha \wedge e_\beta\}$.

Example A.8. An r-form can be defined as the section of anti-symmetric products of co-tangent bundle [33]

$$\Omega^r(\mathcal{M}) = \Gamma(\wedge^r T^*\mathcal{M}) \tag{A.72}$$

where $\wedge^r T^*\mathcal{M} = T^*\mathcal{M} \wedge \cdots \wedge T^*\mathcal{M}$.

Claim A.14. *For vector bundles E, E_1, E_2, $\otimes$ is distributive*

$$E \otimes (E_1 \oplus E_2) = (E \otimes E_1) \oplus (E \otimes E_2). \tag{A.73}$$

Definition A.28. A principal bundle or G-bundle P is a fibre bundle whose fibre F is structure group G. It is denoted as $P(\mathcal{M}, G)$.

Theorem A.15. *For a Lie group G and a closed Lie sub-group H, then G is a principal bundle with fibre H, and the base space is $\mathcal{M} \simeq G/H$.*

Theorem A.16. *A vector bundle naturally induces a principal bundle. For vector bundle $E \to \mathcal{M}$ with fibre $F \simeq \mathbb{R}^k$, the induced principal bundle is $P(E) = P(\mathcal{M}, G)$ with same structure group $G = GL(k, \mathbb{R})$.*

Definition A.29. Given a principal bundle $P(\mathcal{M}, G)$, G acts on the fibre F on left,

$$\begin{aligned} g: \quad P \times F &\to P \times F \\ (u, f) &\to (ug, g^{-1}f) \end{aligned} \tag{A.74}$$

where $g \in G$, which defines an equivalence relation $(u, f) \sim (ug, g^{-1}f)$. The equivalence class $(P \times F)/G$ defines associated fibre bundle $(E, \pi, \mathcal{M}, G, F, P)$. For a vector bundle, fibre F is a k-dimensional vector space V. Assume ρ is k-dimensional representation of G, then G acts on $P \times V$ as

$$\begin{aligned} g: \quad P \times V \quad &\rightarrow \quad P \times V \\ (u, v) \quad &\rightarrow \quad (ug, \rho^{-1}(g)v) \end{aligned} \tag{A.75}$$

where $g \in G$. By identifying $(u, v) \sim (ug, \rho^{-1}(g)v)$, one has associated vector bundle $P \times_\rho V$.

Definition A.30. For a tangent bundle $T\mathcal{M}$ over m-dimensional manifold $\mathcal{M}$, its associated bundle is tangent frame bundle defined as

$$\mathcal{F}\mathcal{M} = \bigcup_{p \in \mathcal{M}} \mathcal{F}_p\mathcal{M} = \{(p, \{e_a\}) | p \in \mathcal{M}, \{e_a\} \text{ is basis of } T_p\mathcal{M}\} \tag{A.76}$$

where $\mathcal{F}_p\mathcal{M}$ is set of frames at $p \in \mathcal{M}$. On chart U_i of $\mathcal{M}$ with coordinate $\{x^\mu\}$, the basis of $T_p\mathcal{M}$ is $\{\frac{\partial}{\partial x^\mu}\}$, the section at p is the frame $u = \{X_1, ..., X_m\}$ where

$$X_\alpha = X_\alpha{}^\mu \frac{\partial}{\partial x^\mu}\Big|_p, \quad X_\alpha{}^\mu \in GL(m, \mathbb{R}) \tag{A.77}$$

so the fibre is $GL(m, \mathbb{R})$. And $a \in GL(m, \mathbb{R})$ acting on u gives a new frame

$$ua = \{Y_1, ..., Y_m\}, \quad Y_\beta = X_\alpha a^\alpha{}_\beta \tag{A.78}$$

therefore, $GL(m, \mathbb{R})$ acts on $\mathcal{M}$ transitively. The structure group is found by considering

$$X_\alpha = X_\alpha{}^\mu \frac{\partial}{\partial x^\mu_{(i)}}\Big|_p = \tilde{X}_\alpha{}^\mu \frac{\partial}{\partial x^\mu_{(j)}}\Big|_p, \quad X_\alpha{}^\mu, \tilde{X}_\alpha{}^\mu \in GL(m, \mathbb{R}) \tag{A.79}$$

and the transition function is

$$t_{ij} = \frac{\partial x^\mu_{(i)}}{\partial x^\nu_{(j)}} \in GL(m, \mathbb{R}) \tag{A.80}$$

so the structure group is $GL(m, \mathbb{R})$ and is the same as that of $T\mathcal{M}$. Therefore, since one has the identification $(u, X) \sim (ua, t_{ij}^{-1}X)$ and both of fibre and structure group are GL, so the tangent frame bundle is an associated principal bundle of tangent bundle.

Definition A.31. Spin bundle of $\mathcal{M}$ is a fibre bundle whose section is spinor field. Since the $GL(k, \mathbb{R})$ group does not have spinor representation, one need to use the orthonormal frame bundle with structure group $SO(k)$ as the spin manifold is orientable. The $SO(k)$ bundle also needs to be lifted to $Spin(k)$ bundle over $\mathcal{M}$ since $Spin(k)$ is universal cover of $SO(k)$.

Definition A.32. A fibre bundle E is trivial if the transition function is the identity map everywhere i.e. the structure group is the trivial group, and hence the fibre bundle can be globally written as direct product of base space and fibre i.e.

$$E \simeq \mathcal{M} \times F. \tag{A.81}$$

Theorem A.17. *A principal bundle is trivial iff it admits a global section.*

Theorem A.18. *A vector bundle E is trivial iff its associated principal bundle $P(E)$ admits a global section. For tangent bundles, since its associated principal bundle is the tangent frame bundle, so if the tangent bundle has global sections, the tangent frame bundle also has global sections, so a tangent bundle is trivial if it has global sections. Similarly for co-tangent bundles.*

Definition A.33. A d-dimensional smooth manifold $\mathcal{M}$ is parallelisable if there exists a global defined smooth vector field $\{V_1, ..., V_m\}$ s.t. at $\forall p \in \mathcal{M}$, $\{V_1(p), ..., V_m(p)\}$ is a basis of $T_p\mathcal{M}$. The set of vector fields is call parallelisation or absolute parallelism.

Example A.9. Lie groups are parallelisable [33]. The only spheres that are parallelisable are S^0, S^1, S^3, S^7 [52]. S^0 is trivial. $S^1 \simeq U(1)$ and $S^3 \simeq SU(2)$ are both Lie group so are parallelisable. For S^7, see [53]. S^2 is not parallelisable due to the hairy ball theorem.

Theorem A.19. *A d-dimensional manifold $\mathcal{M}$ is parallelisable iff its tangent bundle is trivial.*

Proof. For tangent bundle, sections are vector fields which are globally defined on a parallelisable manifold $\mathcal{M}$. Hence its associated principal bundle admits a global section and from Theorem A.18 the tangent bundle is trivial. □

Theorem A.20. *The definition of parallelisable implies that any parallelisable manifold is orientable.*

Definition A.34. The determinant bundle $\det T^*\mathcal{M}$ constructed from cotangent bundle $T^*\mathcal{M}$ of a smooth manifold $\mathcal{M}$ is defined so that the tensor density is an element of it. In usual geometry, for a metric g on d-dimensional manifold $\mathcal{M}$, $\sqrt{-\det g}$ is a tensor density that transforms as $\sqrt{-\det g} \to \det(a)\sqrt{-\det g}$ under $g_{\mu\nu} \to a^\rho{}_\mu a^\sigma{}_\nu g_{\rho\sigma}$. One can use

$$\det e = \sqrt{-\det g} \tag{A.82}$$

as thee basis of $\det T^*\mathcal{M}$, where e is a generic basis on $\mathcal{M}$.

Claim A.21. *If the manifold $\mathcal{M}$ is orientable, such as a spin manifold, then there are non-vanishing global sections, so from Theorem A.18,* $\det T^*\mathcal{M}$ *is a trivial bundle*

$$\det T^*\mathcal{M} \simeq \mathbb{R} \times \mathcal{M} \tag{A.83}$$

then one can take any real power of $\det T^*\mathcal{M}$ *i.e.* $(\det T^*\mathcal{M})^p$*. However, if $\mathcal{M}$ is not orientable, there can be a sign change in the determinant after a transformation, so, for example, it makes no sense taking the square root of* $\det T^*\mathcal{M}$*.*

Claim A.22. *Using the hodge star (A.38), one can find the isomorphism* $\det T^*\mathcal{M} \simeq \wedge^d T^*M$ *given by*

$$\det \lambda = \frac{1}{d!}\varepsilon^{\mu_1\ldots\mu_d}\lambda_{\mu_1\ldots\mu_d} \tag{A.84}$$

where $\det\lambda \in \Gamma(\det T^*\mathcal{M})$ *and* $\lambda_{\mu_1\ldots\mu_d} \in \Gamma(\wedge^d T^*\mathcal{M})$*, hence the bundle is denoted as* $\det T^*\mathcal{M}$*. In general*

$$\det T^*\mathcal{M} \otimes \wedge^r T\mathcal{M} \simeq \wedge^{d-r} T^*\mathcal{M} \tag{A.85}$$

given by

$$\lambda^{\mu_1\ldots\mu_r} = \frac{1}{(d-r)!}\varepsilon^{\mu_1\ldots\mu_r\nu_1\ldots\nu_{d-r}}\lambda_{\nu_1\ldots\nu_{d-r}} \tag{A.86}$$

where $\lambda^{\mu_1\ldots\mu_r} \in \Gamma(\det T^*\mathcal{M} \otimes \wedge^r T\mathcal{M})$ *is a tensor density,* $\lambda_{\nu_1\ldots\nu_{d-r}} \in \Gamma(\wedge^{d-r}T^*\mathcal{M})$ *is a* $(d-r)$*-form, and* $\varepsilon_{\mu_1\ldots\mu_m} = \sqrt{|g|}\tilde{\varepsilon}_{\mu_1\ldots\mu_m}$ *gives the density.*

D Riemannian Geometry

We consider a m-dimensional manifold $\mathcal{M}$ that admits a metric g with signature (p, q). On a chart U for $\mathcal{M}$ with coordinate x^μ, there are coordinate bases $\{e_\mu = \frac{\partial}{\partial x^\mu}\}$ for $T_p\mathcal{M}$ and $\{e^\mu = dx^\mu\}$ for $T_p^*\mathcal{M}$ at a point $p \in U$. Derivations and proofs can be found in [35].

Definition A.35. At $p \in U$, the non-coordinate basis is a set of smooth vector fields $\{\hat{e}_a(p)\}$ that forms a basis for $T_p\mathcal{M}$, where $a = 1, ..., m$, and

$$\hat{e}_a = \hat{e}_a{}^\mu(x)e_\mu, \quad \hat{e}_a{}^\mu(x) \in GL(m, \mathbb{R}). \tag{A.87}$$

$\hat{e}_a{}^\mu$ is called vielbein for a general m-dimensional manifold, and its inverse is defined as

$$(\hat{e}_a{}^\mu)^{-1} := \hat{e}^a{}_\mu. \tag{A.88}$$

If the manifold is orientable, then the non-coordinate basis needs to have same orientation as coordinate basis hence require

$$\det(\hat{e}_a{}^\mu) > 0. \tag{A.89}$$

Unlike the coordinate basis, the Lie bracket of non-coordinate basis is

$$[\hat{e}_a, \hat{e}_b] = c_{ab}{}^c \hat{e}_c \tag{A.90}$$

and does not vanish in general. The dual basis $\{\hat{e}^a\}$ is defined s.t. $\langle \hat{e}^a, \hat{e}_b \rangle = \delta^a{}_b$, so

$$\hat{e}^a = \hat{e}^a{}_\mu e^\mu \tag{A.91}$$

The metric g on a manifold with signature $(p, m-p)$ has component $g_{\mu\nu} = g(e_\mu, e_\nu)$. If the vielbein satisfy

$$\hat{e}^a{}_\mu \hat{e}_a{}^\nu = \delta_\mu{}^\nu, \quad \hat{e}^a{}_\mu \hat{e}_b{}^\mu = \delta^a{}_b \tag{A.92}$$

then the metric g can be diagonalised to a constant flat metric with signature $(p, m-p)$

$$g_{ab} = g(\hat{e}_a, \hat{e}_b) = g_{\mu\nu} \hat{e}_a{}^\mu \hat{e}_b{}^\nu = \eta_{ab} \tag{A.93}$$

and

$$g_{\mu\nu} = \hat{e}^a{}_\mu \hat{e}^b{}_\nu \eta_{ab} \tag{A.94}$$

or in matrix form

$$g = \hat{e}^T \hat{e}. \tag{A.95}$$

To take directional derivative of a vector field on Riemannian manifold, Lie derivative is not enough, and we need to define the covariant derivative.

Definition A.36. Affine connection or covariant derivative is a map

$$\begin{aligned} \nabla : \quad \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) &\to \Gamma(T\mathcal{M}) \\ X, Y \quad &\to \nabla_X Y \end{aligned} \tag{A.96}$$

where $T\mathcal{M}$ is a tangent bundle defined on a d-dimensional manifold $\mathcal{M}$ and $\Gamma(T\mathcal{M})$ is its section i.e. vector field. It is bi-linear so that $\forall X, Y, Z \in \Gamma(T\mathcal{M})$

$$\begin{aligned} \nabla_X (Y + Z) &= \nabla_X Y + \nabla_X Z \\ \nabla_{(X+Y)} Z &= \nabla_X Z + \nabla_Y Z \end{aligned} \tag{A.97}$$

The connection on a coordinate basis $\{e_\mu\} = \left\{\frac{\partial}{\partial x^\mu}\right\}$ is

$$\nabla_\mu e_\nu = \nabla_{e_\mu} e_\nu = \Gamma_{\mu\nu}{}^\alpha e_\alpha \tag{A.98}$$

where $\Gamma_{\mu\nu}{}^\alpha$ is connection components. The connection on a vector X can be found to be

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + X^\alpha \Gamma_{\mu\alpha}{}^\nu. \tag{A.99}$$

Claim A.23. *For a function f,*

$$\nabla_{fX} Y = f \nabla_X Y \tag{A.100}$$

so it is directional i.e. only depends on X, and also has a Leibniz rule

$$\nabla_X (fY) = X[f] Y + f \nabla_X Y \tag{A.101}$$

Claim A.24. *Similar to Lie derivative, the action of affine connection can be extended to tensor, so for a (p, q)-tensor $A = A^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p} \otimes e^{\beta_1} \otimes \cdots \otimes e^{\beta_q}$, the component is*

$$\begin{aligned} \nabla_\mu A = \partial_\mu A^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} &+ \Gamma_{\mu\nu}{}^{\alpha_1} A^{\nu \alpha_2 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} + \cdots + \Gamma_{\mu\nu}{}^{\alpha_p} A^{\alpha_1 \ldots \alpha_{p-1} \nu}_{\beta_1 \ldots \beta_q} \\ &- \Gamma_{\mu}{}^{\nu}{}_{\beta_1} A^{\alpha_1 \ldots \alpha_p}_{\nu \beta_2 \ldots \beta_q} - \cdots - \Gamma_{\mu}{}^{\nu}{}_{\beta_q} A^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_{q-1} \nu} \end{aligned} \tag{A.102}$$

Claim A.25. *The connection can be decompose into symmetric and anti-symmetric parts as*

$$\Gamma_{\mu\nu}{}^{\rho} = S_{\mu\nu}{}^{\rho} + \frac{1}{2} \mathcal{T}_{\mu\nu}{}^{\rho} \tag{A.103}$$

where $\mathcal{T}_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}$ and is called torsion tensor.

Definition A.37. Torsion tensor is a map

$$\begin{aligned} \mathcal{T}: \quad \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) &\rightarrow \Gamma(T\mathcal{M}) \\ X, Y &\rightarrow \mathcal{T}(X, Y) \end{aligned} \tag{A.104}$$

where

$$\mathcal{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{A.105}$$

is

- anti-symmetric $\mathcal{T}(X, Y) = -\mathcal{T}(Y, X)$
- bi-linear $\mathcal{T}(fX, gY) = fg\mathcal{T}(X, Y)$ for all functions f, g

On coordinate basis $\{e_\mu\}$, one can define a (1,2)-tensor with component

$$\mathcal{T}_{\mu\nu}{}^{\rho} = \langle e^\rho, \mathcal{T}(e_\mu, e_\nu) \rangle = \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}. \tag{A.106}$$

On non-coordinate basis $\{\hat{e}_a\}$, the component is

$$\mathcal{T}_{bc}{}^{a} = \langle \hat{e}^a, \mathcal{T}(\hat{e}_b, \hat{e}_c) \rangle = \Gamma_{bc}{}^{a} - \Gamma_{cb}{}^{a} - [\hat{e}_a, \hat{e}_b]^a. \tag{A.107}$$

Claim A.26. *For two vectors X, Y, on coordinate basis,*

$$\mathcal{T}(X,Y) = \mathcal{L}_X^{\nabla} Y - \mathcal{L}_X Y \tag{A.108}$$

where $\mathcal{L}_X^{\nabla}$ is usual Lie derivative with ∂ replaced by ∇ [4]. This can be easily proven using (A.107) and writing RHS out explicitly.

Definition A.38. For a metric g on the manifold $\mathcal{M}$, the connection ∇ is metric connection if it is compatible with the metric i.e. $\nabla_X g = 0$, for any vector X.

Definition A.39. In coordinate basis, $g = g_{\alpha\beta} e^\alpha \otimes e^\beta$, and the compatible condition is

$$\nabla_\mu g = (\partial_\mu g_{\alpha\beta} - \Gamma_{\mu\alpha}{}^{\rho} g_{\rho\beta} - \Gamma_{\mu\beta}{}^{\rho} g_{\alpha\rho}) e^\alpha \otimes e^\beta = 0 \tag{A.109}$$

giving

$$\Gamma_{(\alpha\beta)}{}^{\rho} g_{\rho\mu} - \Gamma_{[\mu\alpha]}{}^{\rho} g_{\rho\beta} - \Gamma_{[\mu\beta]}{}^{\rho} g_{\rho\alpha} = C_{\alpha\beta}{}^{\rho} g_{\rho\mu} \tag{A.110}$$

where $C_{\alpha\beta}{}^{\rho} = \frac{1}{2} g^{\rho\sigma}(\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta})$ is Christoffel symbol determined by g. Denote the symmetric term as $S_{\alpha\beta}{}^{\rho} = \Gamma_{(\alpha\beta)}{}^{\rho}$ and the anti-symmetric terms give the torsion

$$\mathcal{T}_{(\alpha\beta)}{}^{\rho} = \Gamma_{[\alpha\beta]}{}^{\rho} + \Gamma_{[\beta\alpha]}{}^{\rho} \tag{A.111}$$

so we have

$$S_{\alpha\beta}{}^{\rho} = C_{\alpha\beta}{}^{\rho} + \mathcal{T}_{(\alpha\beta)}{}^{\rho}. \tag{A.112}$$

Only the torsion tensor $\mathcal{T}$ is un-constrained, so the metric connection is determined by the torsion tensor $\mathcal{T}$ and the metric g that gives the Christoffel symbol.

Definition A.40. If the torsion tensor $\mathcal{T} = 0$, then the metric connection is completely determined by the metric g and is unique. This connection is known as the Levi-Civita connection

$$\Gamma_{\mu\nu}{}^{\rho} = C_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma}(\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}) \tag{A.113}$$

which is symmetric $\Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\nu\mu}{}^{\rho}$. And since it is torsion-free, using (A.105) one has

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{A.114}$$

On a non-coordinate basis, i.e. using a vielbein $\{\hat{e}_a\}$ and dual $\{\hat{e}^a\}$, the component of connection is defined as

$$\nabla_a \hat{e}_b = \nabla_{\hat{e}_a} \hat{e}_b = \Gamma_{ab}{}^{c} \hat{e}_c. \tag{A.115}$$

Claim A.27. *For metric connection, by imposing the condition* $\nabla_a g = 0$*, one finds that*

$$\begin{aligned}\nabla_a g &= \nabla_a(\eta_{bc}\hat{e}^b \otimes \hat{e}^c) \\ &= \eta_{bc}({\Gamma_{ad}}^b \hat{e}^d \otimes \hat{e}^c + {\Gamma_{ad}}^c \hat{e}^b \otimes \hat{e}^d) \\ &= (\Gamma_{abc} + \Gamma_{abc})\hat{e}^b \otimes \hat{e}^c = 0\end{aligned} \tag{A.116}$$

so

$$\Gamma_{abc} = -\Gamma_{acb} \tag{A.117}$$

hence ${(\Gamma_a)^b}_c$ *is element of Lie algebra* $\mathfrak{o}(d)$ *which matches the fact that to preserve the metric* g*, the structure group of the* d*-dimensional manifold* $\mathcal{M}$ *is* $O(d)$*.*

Definition A.41. One can also define the spin connection as

$$\nabla_\mu \hat{e}_a = {\hat{\omega}_\mu}{}^b{}_a \hat{e}_b \tag{A.118}$$

where ${\hat{\omega}_\mu}{}^b{}_a$ is spin connection component. On a vector field $V = \hat{V}^a \hat{e}_a$,

$$\nabla_\mu V^a = \partial_\mu \hat{V}^a + {\hat{\omega}_\mu}{}^a{}_b \hat{V}^b. \tag{A.119}$$

And using the same procedure as before, it can be easily checked that the torsion-free condition is

$$\hat{\omega}_{\mu bc} = -\hat{\omega}_{\mu cb} \tag{A.120}$$

which is also an element of $\mathfrak{o}(d)$. It can also be written using Levi-Civita connection as

$$\hat{\omega}_{\mu ab} = (\nabla_\mu {\hat{e}_a}{}^\nu)\hat{e}_{b\nu} = (\partial_\mu {\hat{e}_a}{}^\nu + {\Gamma_{\rho\mu}}^\nu {\hat{e}_a}{}^\rho)\hat{e}_{b\nu} \tag{A.121}$$

Claim A.28. *The Levi-Civita connection on tensor density* $\sqrt{-\det g}$ *where* g *is the metric on a manifold satisfies*

$$\Gamma_{\mu\nu}{}^\mu = \frac{1}{\sqrt{-\det g}}\partial_\mu\sqrt{-\det g} \tag{A.122}$$

and for a vector V*,*

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-\det g}}\partial_\mu(\sqrt{-\det g}V^\mu). \tag{A.123}$$

Definition A.42. The Riemann curvature tensor is a map

$$\begin{aligned} R: \quad \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) &\to \Gamma(T\mathcal{M}) \\ X, Y, Z &\to R(X,Y,Z) \end{aligned} \tag{A.124}$$

where

$$R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{A.125}$$

and is

- anti-symmetric $R(X,Y,Z) = -R(Y,X,Z)$
- tri-linear $R(aX,bY,cZ) = abcR(X,Y,Z)$, for any functions a, b, c.

One can define a (1,3)-tensor on a coordinate basis $\{e_\mu\}$ with components

$$R^\alpha{}_{\rho\mu\nu} = \langle e^\alpha, R(e_\mu, e_\nu, e_\rho)\rangle = \partial_\mu \Gamma_{\nu\rho}{}^\alpha - \partial_\nu \Gamma_{\mu\rho}{}^\alpha + \Gamma_{\nu\rho}{}^\beta \Gamma_{\mu\beta}{}^\alpha - \Gamma_{\mu\rho}{}^\beta \Gamma_{\nu\beta}{}^\alpha \tag{A.126}$$

and from the anti-symmetry, one has

$$R^\alpha{}_{\rho\mu\nu} = -R^\alpha{}_{\rho\nu\mu}. \tag{A.127}$$

Definition A.43. Ricci tensor is a (0,2) tensor defined as

$$Ric(X,Y) = \langle e^\mu, R(e_\mu), Y, X\rangle \tag{A.128}$$

and its component is

$$R_{\mu\nu} = Ric(e_\mu, e_\nu) = R^\lambda{}_{\mu\lambda\nu}. \tag{A.129}$$

Definition A.44. For a manifold with a metric g, the Ricci scalar is

$$\mathcal{R} = g^{\mu\nu} R(e_\mu, e_\nu) = g^{\mu\nu} R_{\mu\nu}. \tag{A.130}$$

Using the Ricci scalar, one can write the Einstein-Hilbert action as

$$S = \frac{2}{\kappa} \int d^4x \sqrt{-\det g} R = \frac{2}{\kappa} \int \mathrm{vol}_g R \tag{A.131}$$

and by varying the action, one has the vacuum field equation

$$R_{\mu\nu} = 0. \tag{A.132}$$

E Lie Algebra of $O(d,d)$

The Lie algebra of $O(d,d)$ or $SO(d,d)$ can be found by expending an element $g \in O(d,d)$ near the identity

$$g = \mathbf{1} + \varepsilon M + \cdots, \quad \varepsilon \ll 1 \tag{A.133}$$

where M is the generator. Then imposing the condition in (2.15)

$$g^T \eta g = \eta + \varepsilon(M^T \eta + \eta M) + \cdots = \eta \tag{A.134}$$

leads to

$$M^T \eta + \eta M = 0. \tag{A.135}$$

Since the metric is in block form, it is convenient to write the generator M in block form as well

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{A.136}$$

so (A.135) becomes

$$\begin{pmatrix} a^T & c^T \\ b^T & d^T \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c^T & a^T \\ d^T & b^T \end{pmatrix} + \begin{pmatrix} c & d \\ a & b \end{pmatrix} = 0 \tag{A.137}$$

yielding the conditions

$$a^T = -d, \quad b^T = -b, \quad c^T = -c. \tag{A.138}$$

Relabelling $A = a = -d^T, \beta = b, B = c$ gives the components of the generator

$$M = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix} \tag{A.139}$$

which is the expression in (2.18).

F Usual Scherk-Schwarz Reduction

Consider a theory lives in $\mathcal{M}_{d+k} \simeq \mathcal{M}_k \times \mathcal{M}_d$ where $\mathcal{M}_d$ is compactified and $\mathcal{M}_k$ is uncompactified with coordinate x. If $\mathcal{M}_d$ is a parallelisable manifold with parallelisation $\{\hat{e}_a\}$ satisfying the Lie algebra

$$[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c \tag{A.140}$$

there is consistent truncation if $f_{ab}{}^b = 0$ known as unimodular condition [28, 29]. The truncated theory is gauged by the group defined by the Lie algebra above. The reduction is defined by rotating the frame as

$$\hat{e}'_a(x) = U_a{}^b(x)\hat{e}_a, \quad U_a{}^b \in GL(d, \mathbb{R}) \tag{A.141}$$

so that the inverse metric is

$$g'^{mn} = \delta^{ab} U_a{}^c(x) U_b{}^d(x) \hat{e}_c{}^m \hat{e}_d{}^n \tag{A.142}$$

giving the scalar field in the reduced theory

$$\phi(x) = \delta^{ab} U_a{}^c(x) U_b{}^d(x) \tag{A.143}$$

which parameterises the coset space

$$GL(d,\mathbb{R})/O(d) \tag{A.144}$$

where $O(d)$ is the gauge group of reduced theory. The ansatz for the reduced theory on $\mathcal{M}_k$ is found by comparing the original inverse metric with the scalar ϕ.

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