A brief history of Massive gravity

Submitted by: **$\mathcal{L}$ucas $\mathcal{F}$ernandez $\mathcal{S}$armiento**

Supervisor: **Prof. Andrew Tolley**

IMPERIAL COLLEGE LONDON

Department of Physics

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Abstract

This thesis reviews the background of Massive gravity, introducing dRGT gravity as originally presented and in the tetrad formalism. Kaluza-Klein (KK) theories are reviewed, together with dimensional deconstruction. The application of these results to the graviton to recover dRGT gravity is explored. The last sections consist of novel work, or modifications of previous work (4.5.1-3,4.8) extending these techniques to all other types of matter with lower spin than the graviton (Scalar, Spinors, Yang-Mills, and Rarita-Schwinger fields). This thesis is meant to serve as a pedagogical introduction to Massive gravity, shaped by the author's own research interests, as such $\sim 95\%$ of calculations on this thesis have been done from scratch to ensure any reader familiar with the mathematical technology can follow every step. As a consequence of this, some parts can be more computation heavy than you would find in usual research publications.

Acknowledgements

I would like to thank Andrew Tolley for kindly supervising this thesis, as it was important for me to acquaint myself with this topic for my future projects, and I had the luck to have two of the world leading experts in massive gravity here at Imperial. I would also like to thank all teaching staff involved in this programme for contributing to create a course that truly makes learning physics a pleasure, and provides excellent preparation for the world of theoretical physics and beyond. I would like to thank my parents, grandparents, and aunt from the bottom of my heart for their support during my studies, and for their future support during my PhD. They have truly been great role models, and taught me to always do what I know to be the right thing to do. I would also like to thank Juliette for her support and understanding, especially, through one of the most turbulent and uncertain periods of my life. I would also like to thank my friends during undergrad and QFFF who made a degree through COVID bearable. Last, but not least, I would like to thank Amihay Hanany for giving me the opportunity to work with him during my undergraduate studies, for always being incredibly generous with his time and extremely kind, and for his advice that stretches far beyond the academic world. Quivers will always be an interest of mine, and I hope to be able to contribute to this topic in any way in the future. I would like to end this prelude with one of my favourite quotes.

"Battle not with monsters, lest ye become a monster, and if you gaze into the abyss, the abyss gazes also into you. But if you gaze into the abyss for long enough, you see the light, not the darkness"

Contents

1 Motivation and introduction

General relativity has been one of the two pillars of modern physics for over a century, geometrically describing the symbiotic interplay between mater and geometry elegantly and succinctly. In fact, the Einstein-Hilbert (EH) action, describing the Einstein field equations (EFE) is close to the simplest equation you can write in $4D$ curved spacetime. Nevertheless, the theory of gravity is one of the most complicated theories to study, as it has an infinite number of self interacting terms. This makes it a very interesting theory to study from a mathematical standpoint, as the linearised Einstein equations would initially appear to have enough breadth to allow for different interacting massless Spin-2 theories to emerge at the non-linear level. However, it has been shown by Deser [1] that the only non-linear extension that respects Diffeomorphism invariance is that described by GR. Thus, there exist few remaining options to extend our current models of gravity. Loss of Poincaré invariance has been suggested as a solution to this [2], however, there has been no evidence to suggest that Poincaré invariance should be abandoned. A further possible extension is the study of interacting massive Spin-2 fields. As such, we expect this fields to propagate 5 degrees of freedom, and demand them not to be plagued by ghosts. These are modes that propagate with the wrong kinetic sign, meaning that when in a Lagrangian with a particle of opposite kinetic sign, one can arbitrarily increase the magnitude of both's kinetic terms without violating conservation of energy, which is clearly unphysical. As a consequence, the Hamiltonian is unbounded from below, which is forbidden physically. The aim of this thesis is to provide an account of some recent advances in this field and how these can be related to other extensions of gravity.

The structure will be threefold; firstly, we will tackle the basics of gravity from a field theory point of view, and summarising the breakthroughs of massive gravity through the years. We will start by deriving Einstein's equations from the Einstein-Hilbert action and linearising them. We will provide some background on why Spin-2 fields make sense to explain gravity from Heuristic arguments and introducing the symmetry of GR; Diffeomorphism invariance. This should be seen as an extension of the introduction as the concepts here are designed to be friendly for general physicists without a theory background. Moving on from here, we will look at results such as the Fierz-Pauli action [3] [4] for a linear massive graviton, which was the first action written with no ghosts at the linear level for a massive Spin-2 field. As we know, adding mass terms breaks the gauge invariance of theories, so we will introduce Stuckelberg fields, so this is preserved and the right degrees of freedom are recovered in the massless limit. We will see that Fierz-Pauli massive gravity experiences a discontinuity (the vDVZ [5] discontinuity) in its propagator in the massless limit, making it irreconcilable with a working linearised theory of gravity. The Vainshtein mechanism [6] will explain that this is due to the radius where non-linearities are important being dependent on the mass of the graviton, and we will see that this radius goes to infinity in the massless limit. We will finish this section by a short note on quantum gravity. This section is heavily inspired by the insightful living reviews of Prof. de Rham [7] and Prof. Hinterbichler [8]. Secondly, will learn about the developments that led to a unique, non-linear, ghost-free theory of massive gravity, namely dRGT gravity [9] [10]. This will be done by generalising the Fierz-Pauli mass term such that the contributing mass terms in the decoupling limit are total derivatives. Additionally, this will raise the cut-off of the theory to a higher scale [11] [9]. Finally, we will delve into different extensions of gravity. We will first introduce the Einstein-Cartan formulation of GR, which uses the tetrad formalism and derive the most important relations. We will see that in this language, dRGT gravity has a very intuitive meaning and can be written succinctly. We will do a short introduction of Kaluza-Klein (KK) [12] theories where we will see how for peculiar metrics, a $5D$ universe with no matter and a compact dimension is equivalent to a $4D$ universe with both matter (Yang-Mills, gravity, and a scalar field) and geometry. We will see how Mach's principle motivated Brans-Dicke [13] gravity, and how this is a special case of KK theories. We will also look at Dvali-Gabadadze-Porrati (DGP) models, where we take this extra dimension to be non-compact, and we see how this gives rise to massive gravitons in $4D$, where the mass is not one, but follows a Källén–Lehmann spectral representation. This digression into extensions of gravity will pay off where we follow the prescription of dimensional deconstruction [14] [11] to recover ghost-free massive gravity from the discretisation of tetrads in $5D$ [15]. Finally, we present some novel (not peer-reviewed) work where we will use this prescription to generalise the construction to other types of matter, starting by a scalar field where we find it will propagate two modes; a massless one and another mode with mass proportional to the graviton mass. Interestingly, these modes mix the fields in both locations in the y dimension. This same approach is used to find similar results for Spin-1/2 fermions, where we see the mass term becomes chiral. For completeness, Yang-Mills and Rarita-Schwinger fields are also studied. These results are obtained in a novel way through dimensional discretisation, but follow what would be expected from KK towers. However, these all display nuisances in the interactions between fields living in different locations in our latticised dimension.

2 History of massive gravity

2.1 EH action and linearised Einstein's equations

The Einstein equations are often introduced in undergraduate level courses the same way they were introduced to the world in 1915 [16]; as the bare field equations with no derivation. Here we show how these can be derived from the variation of a succinct action where the Ricci scalar is minimally coupled to gravity. We then expand this action to find the linearised Einstein equations and introduce the gauge symmetry of general relativity; Diffeomorphism invariance. While some of these calculations are quite straightforward, they are useful to derive some of the tools that will be used in subsequent parts of this report.

2.1.1 EH action

We start by introducing the EH action and explicitly showing that variation with respect to $g_{\mu\nu}$ recovers Einstein's equations. We will follow use [17] as the core of the analysis, but this will deviate in different parts. We will introduce and vary the Einstein-Hilbert action to fnd the equations of motion of GR;

$$S = \frac{1}{\kappa}\int \mathrm{d}^4x\sqrt{-g}[R + \mathcal{L}_{matter}]. \tag{2.1}$$

To vary it, we will need some basic equations that will be useful throughout this dissertation.

The variation of $R_{\mu\nu}$ is A.2 [1]. [18].

$$\delta R_{\sigma\nu} = \nabla_\rho \delta\Gamma^\rho_{\nu\sigma} - \nabla_\nu \delta\Gamma^\rho_{\rho\sigma} \tag{2.2}$$

We now have all the ingredients we need for the variation of the EH action2.1, first ignoring the matter sector: Firstly, the variation of the Ricci tensor vanishes [2]

$$\int \mathrm{d}^4x\Big(\sqrt{-g}(\delta R_{\alpha\beta})g^{\alpha\beta}\Big) = 0 \tag{2.3}$$

up to a boundary term [3]. So

$$\delta S = \frac{1}{\kappa}\int \mathrm{d}^4x\Big(\delta(\sqrt{-g})R + \sqrt{-g}(R_{\alpha\beta}\delta(g^{\alpha\beta}))\Big). \tag{2.4}$$

We have (See A.1):

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \tag{2.5}$$

[1] we will use the notation ∂_μ and $_{,\mu}$ interchangeably according to what is most suitable to each equation

[2] See A.3

[3] Here we assume the universe has no boundary, otherwise we would add a Gibbons–Hawking–York (GHY) [19] [20] boundary term

using A.5 and A.6 we find

$$\delta S=\frac{1}{\kappa}\int \mathrm{d}^4x\Big(\frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta}R-\sqrt{-g}(R_{\rho\sigma}(\delta g_{\alpha\beta})g^{\beta\rho}g^{\alpha\sigma})\Big) \tag{2.6}$$

which yields

$$R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}R=0 \implies R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=0. \tag{2.7}$$

And we are rewarded with Einstein's equations as promised, derived from the variation of a Lagrangian. One can add matter easily by defining

$$T_{\mu\nu}=\frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{matter})}{\delta g_{\mu\nu}}, \tag{2.8}$$

recovering Einstein's equations in their usual form;

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa T_{pv}, \tag{2.9}$$

or using Einstein's tensor

$$G_{\mu\nu}=\kappa T_{pv}. \tag{2.10}$$

2.1.2 Linearised EH action

From a field theory or representation theory viewpoint, the linearised Einstein equations are the equations of motion of a massless Spin-2 particle. Additionally, these have linear Diffeomorphism invariance, as we will see shortly. These equations are ubiquitous in modern physics as they predict gravitational waves [21] and that these will propagate at the speed of light, which was recently verified experimentally by the LIGO collaboration [22] and so these provide deep insight into the inner workings of GR, even at the approximate linear level. To linearise Einstein equations, we consider small fluctuations around a background flat metric (Minkowski in this case);

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} \quad ,h_{\mu\nu}\ll 1. \tag{2.11}$$

We define the following pseudo-tensors;

$$\begin{aligned} &h^{\mu\nu}\equiv h_{\alpha\beta}\eta^{\mu\alpha}\eta^{\nu\beta}\\ &h^{\mu}_{\nu}\equiv h_{\mu\alpha}\eta^{\alpha\nu}\\ &h\equiv \eta^{\mu\nu}h_{\mu\nu}\\ &g^{\mu\nu}=\eta^{\mu\nu}-h^{\mu\nu}\end{aligned} \tag{2.12}$$

to first order

$$\Gamma^{\mu}_{\nu\rho}=\frac{1}{2}\eta^{\mu\sigma}(g_{\sigma\nu,\rho}+h_{\sigma\rho,\nu}-h_{\nu\rho,\sigma}) \tag{2.13}$$

and schematically $\Gamma\Gamma\ h^2$ and so

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma,\rho} - \eta_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho,\sigma} \tag{2.14}$$

which, after some massaging, one can get

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\nu\sigma,\mu\rho} - h_{\mu\rho,\nu\sigma}) \tag{2.15}$$

which is simply all the possible second derivatives of h that respect the symmetries of the Riemann tensor. One also finds

$$R_{\nu\rho} = \frac{1}{2}(2h_{\mu(\sigma,\nu)\mu} - \Box h_{\nu\sigma} - h_{,\nu\sigma}) \quad , R = h^{\mu\nu}{}_{,\mu\nu} - \Box h \tag{2.16}$$

so

$$G_{\mu\nu} = \frac{1}{2}(2h^{\alpha}_{(\mu,\nu)\alpha} - \Box h_{\mu\nu} - h_{,\mu\nu} - \eta_{\mu\nu}(h^{\alpha\beta}{}_{,\alpha\beta} - \Box h))). \tag{2.17}$$

This can also be obtained by considering all possible combinations of the different derivatives of h, requiring that no ghosts are introduced when splitting the metric into a transverse tensor and the derivative of a Spin-1 vector field as shown in [7].

These can also be obtained from varying the FP action:

$$S_{FP} = \frac{1}{\kappa}\int \mathrm{d}^4x(-\frac{1}{4}h_{\mu\nu,\rho}h^{\mu\nu,\rho} + \frac{1}{2}h_{\mu\nu,\rho}h^{\rho\mu,\nu} + \frac{1}{4}h_{,\mu}h^{,\mu} - \frac{1}{2}h_{,\mu}h^{\mu\nu}{}_{,\nu}) \tag{2.18}$$

noting that

$$\delta h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\mu\beta}\delta h_{\alpha\beta} \implies \delta h^{\mu\nu}h_{\mu\nu} = h^{\mu\nu}\delta h_{\mu\nu} \tag{2.19}$$

then

$$\begin{aligned}\delta S_{FP} = \frac{1}{\kappa}\int \mathrm{d}^4(&-\frac{1}{2}(\delta h_{\mu\nu,\rho})h^{\mu\nu,\rho} + \frac{1}{2}((\delta h_{\mu\nu,\rho})h^{\rho\mu,\nu} \\ &+ h_{\mu\nu,\rho}\delta(h^{\rho\mu,\nu})) + \frac{1}{2}\delta(h_{,\mu})h^{,\mu} - \frac{1}{2}((\delta h_{,\mu})h^{\mu\nu}{}_{,\nu} + h_{,\mu}(\delta h^{\mu\nu}{}_{,\nu})),\end{aligned} \tag{2.20}$$

integrating by parts,

$$\begin{aligned}\delta S_{FP} = \frac{1}{\kappa}\int \mathrm{d}^4x(&-\frac{1}{2}\Box h^{\mu\nu}\delta h_{\mu\nu} - \frac{1}{2}h^{\mu\rho,\nu}{}_{\rho}\delta h_{\mu\nu} - \frac{1}{2}h^{\rho\nu,\mu}{}_{\rho}\delta h_{\mu\nu} \\ &- \frac{1}{2}\eta^{\mu\nu}\Box h\delta h_{\mu\nu} + \frac{1}{2}\delta(h^{\mu\nu})h_{,\mu\nu} + \frac{1}{2}h^{\alpha\beta}{}_{,\alpha\beta}\eta^{\mu\nu}\delta h_{\mu\nu})\end{aligned} \tag{2.21}$$

which yields 5.6. It is worth noting we may write this Lagrangian as

$$\mathcal{L} = \frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} \tag{2.22}$$

where the Lichnerowicz symbol $\mathcal{E}$ is defined acting on h as

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} = \frac{1}{2}(2h_{\mu(\alpha,\nu)}{}^{\alpha} - \Box h_{\mu\nu} - h_{,\mu\nu} - \eta_{\mu\nu}(h^{\alpha\beta}{}_{,\alpha\beta} - \Box h))) \tag{2.23}$$

Once again, we can couple matter to our Spin-2 field by simply writting

$$S_{matter} = \int \mathrm{d}^4 h^{\mu\nu}T_{\mu\nu} \tag{2.24}$$

2.1.3 Gauge symmetry

We are familiar with gauge symmetries, that being Abelian such as in electromagnetism (U(1)), or non-Abelian such as in Yang-Mills. As previously anticipated, gauge symmetry of massless linearised GR is linear Diffeomorphism invariance, which physically means that laws are invariant under Diffeomorphisms; the metric and matter distribution change covariantly to give rise to equations of the same form. One of the special things about GR is that the metric is treated dynamically, as opposed to theories such as Yang mills, where the theory is impregnated by an a priori metric that does not change with the matter distribution like GR does. It is to be emphasised that we expect all theories of physics to be Diffeomorphism invariant as we expect physical laws to transform covariantly, as this is one of the postulates of SR, but GR has the extra requirement that the background is dynamical. Linear Diffeomorphism invariance can be written as

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}. \tag{2.25}$$

We can see that linearised GR is explicitly invariant under these Diffeomorphisms,

$$\delta S_{FP} = \frac{1}{\kappa}\int \mathrm{d}^4x(-G_{\mu\nu}\delta h^{\mu\nu}) = \frac{1}{\kappa}\int \mathrm{d}^4x(-G_{\mu\nu}\partial^{\mu}\xi^{\nu}) = \frac{1}{\kappa}\int \mathrm{d}^4x(\partial_{\mu}G^{\mu\nu}\xi_{\nu}) = 0 \tag{2.26}$$

where we have used the conservation of Einstein's tensor (in linearised gravity $\nabla_\mu \rightarrow \partial_\mu$) and its symmetry. Since $h_{\mu\nu}$ is a symmetric 4 dimensional tensor we expect 10 degrees of freedom, nonetheless, we can fix the gauge. To do this, we recall Lorentz gauge in EM:

$$\partial_\mu A^\mu = 0 \tag{2.27}$$

so

$$\partial_\mu F^{\mu\nu} = j_\nu \implies \Box A_\nu = j_\nu. \tag{2.28}$$

The analogue for GR is the de Donder gauge:

$$\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial_\nu h = 0. \tag{2.29}$$

In the full non-linear theory this is equivalent to

$$g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = 0 \tag{2.30}$$

which one can easily check recovers de Donder gauge in the linear limit:

$$\frac{1}{2}(\eta^{\mu\nu} - h^{\mu\nu})((\eta^{\rho\alpha} - h^{\rho\alpha})(h_{\alpha\mu,\nu} + h_{\alpha\nu,\mu} - h_{\mu\nu,\alpha}) = \frac{1}{2}\eta^{\mu\nu}(h^{\rho}{}_{\mu,\nu} + h^{\rho}{}_{\nu,\mu} - h_{\mu\nu}{}^{,\rho}) = h_{\mu}{}^{\rho,\mu} - \frac{1}{2}h^{,\rho} = 0 \tag{2.31}$$

which recovers de Donder gauge 2.29. This is particularly useful as

$$\Box = g^{\mu\nu}\nabla_{\mu}\nabla_{\nu} = \partial^{\mu}\partial_{\mu}. \tag{2.32}$$

Using de Donder gauge we find,

$$h^{\mu\nu}{}_{,\mu\nu} = \frac{1}{2}\Box h \tag{2.33}$$

which we can use to rewrite Einstein's linearised equations as

$$-\Box h_{\mu\nu} + \frac{1}{2}\Box h\eta_{\mu\nu} = 2\kappa T_{\mu\nu} = 16\pi G T_{\mu\nu} \tag{2.34}$$

where G is Newton's constant. We may define the pseudotensor

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} \tag{2.35}$$

satisfying

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{2.36}$$

In this form, the linearised Einstein equations look like a sourced wave equation in $4D$, which we know how to solve. In a vacuum, $T_{\mu\nu} = 0$ and we can then write the vacuum wave equations as

$$\Box\bar{h}_{\mu\nu} = 0 \tag{2.37}$$

which we might solve using the trial solution

$$\bar{h}_{\mu\nu} = Re(H_{\mu\nu}e^{ik_{\rho}x^{\rho}}). \tag{2.38}$$

with the condition that k is a null vector: $k_{\rho}k^{\rho} = 0$. We also require H to be symmetric, and yield a further constraint from the de Donder gauge, $\partial^{\mu}\bar{h}_{\mu\nu} = 0 \implies k^{\mu}H_{\mu\nu} = 0$ which tells us that the polarization is transverse

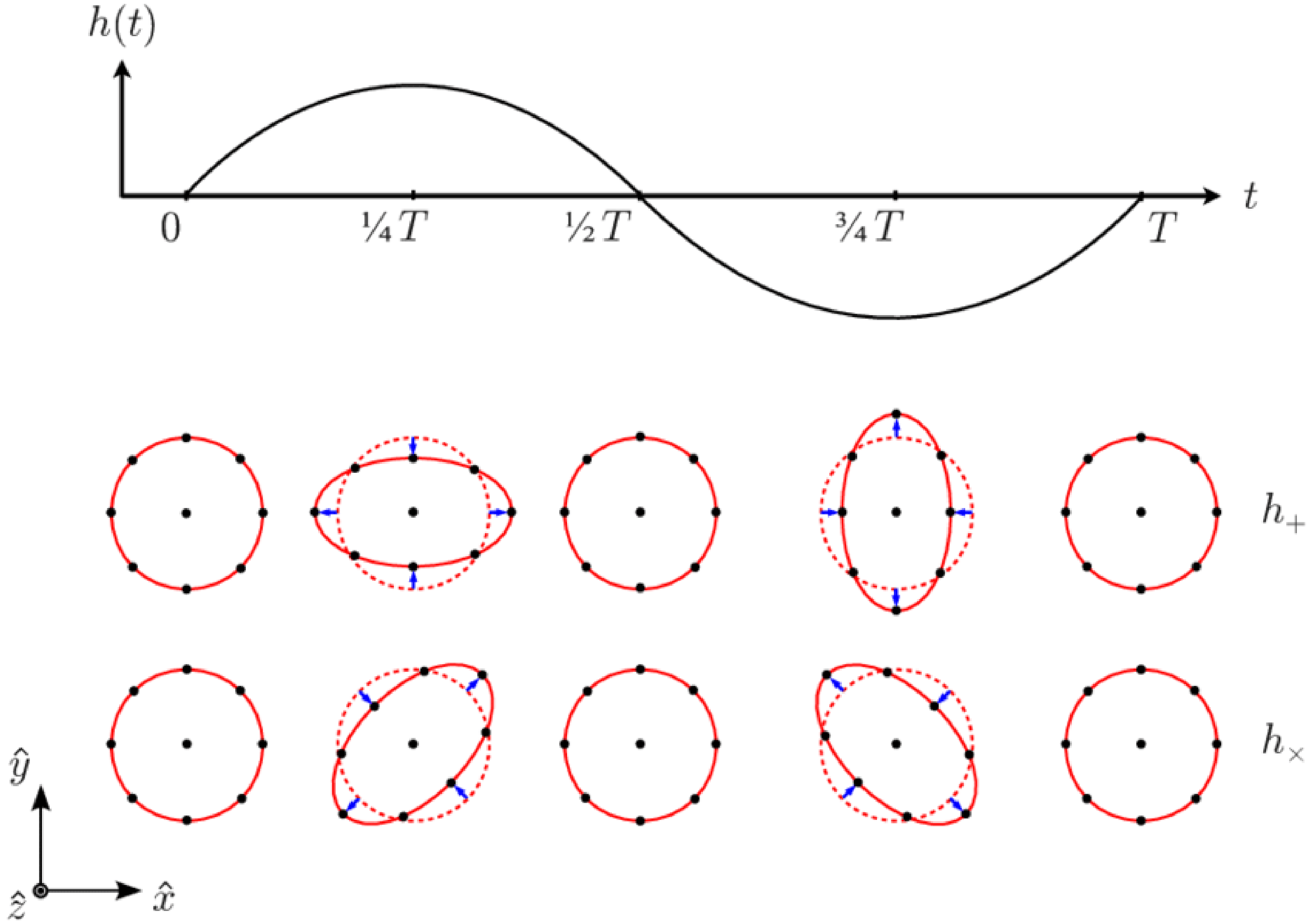

Figure 1: Image taken from [23] showing the polarisations of gravitational waves at different phases. As previously anticipated, we have two polarisations.

to the propagation. We also need to take into account Diffeomorphism invariance:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \implies \bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \partial_\rho \xi^\rho \eta_{\mu\nu}. \tag{2.39}$$

In de Donder gauge we find that ξ satisfies $\Box \xi_\nu = 0$ which following the previous analysis gives us further constraints. Under these transformations we find

$$H_{\mu\nu} \to H_{\mu\nu} + i(2k_{(\mu}\lambda_{\nu)} - k^\rho \lambda_\rho \eta_{\mu\nu}) \tag{2.40}$$

so we can choose λ in such a way such that $H_{0\mu} = 0$ and $H^\mu_\mu = 0$, known as the TT gauge. Our initial tensor H had 10 degrees of freedom, but de Donder Gauge lets us fix four constraints, and λ lets us fix four further degrees of freedom. In total, this leaves us with 2 polarizations, which is the right amount for a Spin-2 massless particle.

2.1.4 Why Spin-2?

We will follow the analysis by Feynman [24] that heuristically lets us find these two polarisations by analogy to electromagnetism, and then check that these correspond to the ones given by Einstein's equations.

First, we ponder on why it is that we need a Spin-2 field to describe gravity? We may consider a simple Lagrangian of the form

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi + g\phi T - \frac{1}{2} m^2 \phi^2 \tag{2.41}$$

where m is small. This is because the range of the force is suppressed by the Yukawa [25] term e^{-rm} and we require gravity to be a long range theory. While this theory might at first seem reasonable, there is an issue that rules it out. The electromagnetic stress energy tensor is

$$T^{\mu\nu} = \frac{1}{\mu_0}\left[F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right] \tag{2.42}$$

from which one can easily check that its trace, T, vanishes in $4D$:

$$T^{\mu}{}_{\nu} = \frac{1}{\mu_0}\left[F^{\mu\alpha}F_{\mu\alpha}(1 - D/4)\right]. \tag{2.43}$$

This implies that the field does not couple to relativistic matter and so light would not bend in this theory, which we know not to be the case, and so we can discard this. Additionally, as pointed out by Feynman pointed out [24] that for Spin-0 fields the energy of a gas goes as γ^{-1}, for Spin-1 we find it is independent of velocity, while for Spin-2 it goes like γ which points at Spin-2 being the correct field. Additionally, in Spin-1 fields, like charges repel, which we know not to be the case in gravity. This leads us to Spin-2 as a possible candidate.

In what follows, we follow Feynman's analysis to see how currents and Spin-1 fields interact in EM, and we draw a comparison to gravity. In Classical EM we have

$$A_\mu = -\frac{1}{k^2} j_\mu \tag{2.44}$$

which connects to a current j' as $-j^\mu \frac{1}{k^2} j_\mu$. One might choose the frame such that $k^\mu = (\omega, \kappa, 0, 0)$ such that

$$-j^\mu \frac{1}{k^2} j_\mu = \frac{1}{\omega^2 - \kappa^2}(j_4' j^4 - j_i' j^i). \tag{2.45}$$

Additionally conservation of charge imposes

$$\omega j^4 - k j^3 = 0 \tag{2.46}$$

so 2.45 reduces to

$$-j^\mu \frac{1}{k^2} j_\mu = \frac{j_4' j^4}{\kappa^2} + \frac{1}{\omega^2 - \kappa^2}(j_1' j^1 + j_2' j^2) \tag{2.47}$$

from where we see the first term is frequency independent, and represents the instantaneous Coulomb potential. One might see this more clearly by taking the inverse Fourier transform:

$$\mathcal{F}^{-1}\left[\frac{j_4' j^4}{\kappa^2}\right] = \frac{e^2}{4\pi r}\delta(t-t') \tag{2.48}$$

and the other terms represent the corrections to the instantaneous potential. This process is pedagogical to extract information about the mediating particles; the photons. We know that the interaction between two particles involves virtual photons, and by looking at the poles of the interaction amplitude ($\omega = \pm\kappa$). The residue of the pole term at $\omega = \kappa$ is nothing but the diagonalised interaction of what looks like two photons interacting with the two different currents: $(j_1' j^1 + j_2' j^2)$, which means photons have two polarisations. Similarly circular polarisation is simply a linear combination of these in a new basis:

$$j_1' j^1 + j_2' j^2 = \frac{1}{\sqrt{2}}(j_1 + ij_2)(j_1' + ij_2')^* + \frac{1}{\sqrt{2}}(j_1 - ij_2)(j_1' - ij_2')^* \tag{2.49}$$

where it is easy to see the photons rotate by $e^{i\theta}$ or $e^{-i\theta}$ so we still have two polarisations. This also splits the photon into helicity modes, as we see each one has a definite helicity; either +1 or -1, from which we can start seeing the quantum properties of EM emerging. Understanding the EM theory is important as many of the techniques used here will translate to the analysis of gravity, including techniques such as Stueckelberg tricks to restore gauge invariance when breaking it explicitly by introducing mass. We now consider the amplitudes for exchanges of gravitons. In this linearised regime, we assume the D'Alambertian operator is simply k^2 and so we write

$$h_{\mu\nu} = \frac{1}{k^2}T_{\mu\nu}. \tag{2.50}$$

We guess that, like in EM, the interaction goes like $T'^{\mu\nu}\frac{1}{k^2}T_{\mu\nu}$. We also require conservation of this tensor, which in momentum space reads $k_{\mu\nu} = 0$. And similarly, using the same convention for the momentum vector as previously, we find that $\omega T_{4\nu} = -\kappa T_\nu$ so one finds

$$\frac{1}{k^2}T_{\mu\nu} = -\frac{1}{\kappa^2}\left[T_{44}'T_{44}(1-\frac{\omega^2}{\kappa^2}) - 2T_{41}'T_{41} - 2T_{42}'T_{42}\right] + \frac{1}{\omega^2-\kappa^2}\left[T_{11}'T_{11}T_{22}'T_{22} + 2T_{21}'T_{21}\right] \tag{2.51}$$

where the first term is akin to the instantaneous force and the second is the retarded part of the interaction. The transverse components are presumably independent, which gives us three polarisations. This means that we have a mixture of Spin-2 and Spin-0 in our theory, and so we can get rid of this by adding a term of the form $T'\frac{1}{k^2}T$ where T is the trace of the stress energy tensor. Tuning this parameter to $-\frac{1}{2}$, the retarded term reduces to

$$\frac{1}{\omega^2-\kappa^2}\left[\frac{1}{2}(T_{11}' - T_{22}')(T_{11} - T_{22}) + 2T_{12}'T_{12}\right] \tag{2.52}$$

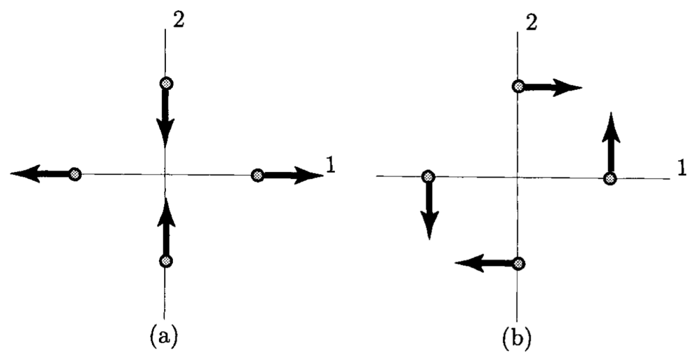

Figure 2: Polarisations of two particles from the tensor components.Image taken from [24]

so that we can see that the polarisations are $\frac{1}{\sqrt{2}}(T_{11} - T_{22})$ and $\sqrt{2}T_{12} = \frac{1}{\sqrt{2}}(T_{12} + T_{21})$ using the symmetry of the tensor. This means in particular that the field $h_{\mu\nu}$ can be written as

$$h_{\mu\nu} = e_{\mu\nu}e^{ik_\alpha x^\alpha} \tag{2.53}$$

and the polarisations are such that $e_{11} = \frac{1}{\sqrt{2}}$,$e_{11} = -\frac{1}{\sqrt{2}}$, $e_{12} = e_{21} = \frac{1}{\sqrt{2}}$. One might visualise these two polarisations as We can see from here that rotating by $\frac{\pi}{2}$ returns us to the same setup but in anti-phase, and by π brings us back to our original setup. This means there are two cycles in 360 and so this is a Spin-2 field as expected. This approach to "come up" with a Spin-2 field is rather instructive, as direct comparison to EM results in linearised GR.

The general interaction is

$$T'^{\mu\nu}\frac{1}{k^2}T_{\mu\nu} - \frac{1}{2}T'\frac{1}{k^2}T \tag{2.54}$$

which can be written with the propagator $P_{\sigma\tau\mu\nu} = \frac{1}{2k^2}(\eta_{\mu\sigma}\eta_{\nu\tau} + \eta_{\mu\tau}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\tau\sigma})$ as

$$T'^{\sigma\tau}P_{\sigma\tau\mu\nu}T^{\mu\nu} \tag{2.55}$$

which will be an important operator later when comparing massive and massless gravity.

2.2 Stuckelberg trick for EM

This section and is based on the living reviews of massive gravity by [7] and [8]. Adding mass terms to the equations of motion of a field with a certain symmetry often explicitly breaks the latter. To mend this, in 1938 Stuckelberg introduced a way of preserving gauge invariance after adding a mass term by means of introducing an extra field so the degrees of freedom in the massless limit would match. Recent reviews include [26]. To begin, we consider Maxwell theory, and we see how introducing a mass breaks gauge invariance explicitly. The generic kinetic term for Maxwell is

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.56}$$

and in Lorentz gauge we have that the components of the vector potential satisfy the massless KG equation $\Box A_\nu = 0$. Additionally, we have the gauge invariance, $\delta A_\nu = \partial_\nu \chi$ which also satisfies the massless KG equation. This lets us fix one more degree of freedom, meaning that a massless Spin-1 field has two propagating degrees of freedom in Maxwell's theory. This theory may be promoted to a massive Spin-1 field by introducing a mass term and a current:

$$\mathcal{L}_{MassiveEM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A^\mu A_\mu + A_\mu J^\mu \tag{2.57}$$

which is also known as the Proca Lagrangian [27] this mass term explicitly breaks the gauge symmetry and so this theory now has 3 propagating degrees of freedom. Even though the Proca equation can be smoothly deformed into the Maxwell Lagrangian, a discontinuity emerges when considering the number of degrees of freedom, as these are different in both theories regardless of how small the mass of the particle is. To reconcile this, we introduce a Stuckelberg field:

$$A_\mu \to A_\mu + \partial_\mu \phi \tag{2.58}$$

then the action changes to

$$S = \int \mathrm{d}^4x - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2(A_\mu + \partial_\mu\phi)^2 + A_\mu J^\mu - \phi\partial_\mu J^\mu \tag{2.59}$$

integrating the last term by parts. We can now define a new gauge transformation

$$\delta A_\mu = \partial\xi, \quad \delta\phi = -\xi \tag{2.60}$$

and we see the new Lagrangian is gauge invariant. One can always recover the initial Lagrangian by setting ϕ to zero, which is called unitary gauge. This manoeuvre of introducing a new field and a new gauge symmetry is called the Stuckelberg trick. This is a clear illustration of Gauge freedom simply being the exemplification of inefficient theories in the sense that they contain redundancies, as we can take any theory and make it a gauge theory by introducing degrees of freedom, or removing the gauge freedom by constraining these fields. This tells us that gauge symmetry is not a true symmetry of our theory, like rotational invariance would be, which is an important distinction. What gauge symmetry is good for, however, is to make the symmetries of the theory explicit, as for example in EM theory we see Lorentz invariance explicitly in the gauged theory, but removing the gauge invariance also removes the explicit gauge invariance, though it remains in the underlying theory, but is not as intuitive. Thus, keeping redundancy can often be helpful to see the symmetries. Coming back to our previous Lagrangian, one may perform a field redefinition $\phi \to \frac{\phi}{m}$, so the action transforms into

$$S = \int \mathrm{d}^4x - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2(A_\mu A^\mu + \frac{2}{m}A_\mu\partial^\mu\phi + \frac{1}{m^2}\partial_\mu\phi\partial^\mu\phi) + A_\mu J^\mu - \frac{1}{m}\phi\partial_\mu J^\mu \tag{2.61}$$

where we see that in the massless limit, requesting that the current is conserved or that the combination $\frac{1}{m}\phi\partial_\mu J^\mu$ goes to zero in the massless limit yields a Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + A_\mu J^\mu \tag{2.62}$$

with a gauge symmetry,

$$\delta A_\mu = \partial_\mu\xi, \quad \delta\phi = 0 \tag{2.63}$$

which clearly has two degrees of freedom, as one can fix ξ and ϕ freely. Thus, introducing the Stuckelberg field ϕ has allowed us to get rid of the discontinuity in degrees of freedom between the massive and massless theories.

2.3 Ghosts, Cauchy problems, and fake interactions

Massive gravity theories are often plagued by what is coined as a ghost. Before we delve into the intricacies of massive gravity models it is important to understand what ghosts are, where they come from and how to properly define a problem for which our theories behave as we would expect them to

2.3.1 Ghosts

In the context of classical mechanics, ghosts refer to fields with a kinetic term entering the Lagrangian with the wrong sign. The issue arises as quantum mechanically the vacuum is unstable as, if we have two particles with different kinetic term signs, we could create an infinite number of pairs without violating energy conservation due to the different signs. The simplest schematic example of a ghost is

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\pi)^2 - V(\pi, \phi) \tag{2.64}$$

where ϕ and π are scalar fields. This can also be seen by noticing that the wrong term of the kinetic term means that the Hamiltonian is unbounded from below, which even at a classical level means these two particles would be able to attain arbitrarily large kinetic energies without violating energy conservation.

Ostrogradsky's [28] ghosts are a special kind of ghosts that arise from Lagrangians with higher derivative terms such as in:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2m_g^2}\phi\Box^2\phi \tag{2.65}$$

for which we may make the substitution $\frac{1}{m_g^2}\Box\phi = \psi$ so that the Lagrangian is

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \partial_\mu\phi\partial^\mu\psi + \frac{1}{2}m_g^2\psi^2 = -\frac{1}{2}(\partial\hat{\phi})^2 + \frac{1}{2}(\partial\psi)^2 + \frac{1}{2}m_g^2\psi^2 \tag{2.66}$$

with $\hat{\phi} = \phi + \psi$ We see that this Lagrangian has a ghostly propagating scalar due to the wrong sign, and is an explicit example of Ostrogradsky's ghosts.

2.3.2 Cauchy Problems and fake interactions

Definition Let $\Omega \subset \mathbb{R}^d$ be an open subset of $\mathbb{R}^d$, for d > 1 a positive integer (called the dimension), and denote by $x = (x^1, x^2, ..., x^d)$ a vector in Ω. Let (unknown) function $u : \Omega \to \mathbb{R}^d$ whose partial derivatives up to order k exist. A partial differential equation (PDE) of order k in Ω in d dimensions is an equation of the form

$$F(x, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}, ..., \frac{\partial u}{\partial x^d}, ..., \frac{\partial^2 u}{\partial {x^1}^2}, ..., \frac{\partial^k u}{\partial x^{d-1}\partial {x^d}^{k-1}}) = 0 \tag{2.67}$$

where F is a given function.

Definition Consider a PDE of the above form and let S be a Hypersurface on $\mathbb{R}^d$ let $n(x)$ be the unit normal vector to S at a point x. Suppose that on any point p of the surface the values of the solution u and of all its derivatives up to order k-1 in the direction of n are given. The Cauchy problem consists of finding the unknown function(s) u that satisfy simultaneously the PDE and all the conditions.A well posed Cauchy problem is one such that a solution exists, this solution is unique, and additionally the solution is continuous with respect to the initial data. To see an example of a non-well-defined Cauchy problem, we look at [29]. We start with a Lagrangian.

$$\mathcal{L} = \frac{1}{2}\chi\Box\chi + \frac{1}{\Lambda^3}\chi(\Box\chi)^2 + \frac{1}{2\Lambda^6}\chi(\Box\chi)\Box(\chi(\Box\chi)) \tag{2.68}$$

where Λ is our EFT cut-off scale. This model has higher derivatives, and so we suspect ghosts might be present. One may expand about a background solution $\chi \to \mu + \delta\chi$ where μ is a constant, such that to quadratic order in the variation:

$$\mathcal{L} = \frac{1}{2}\delta\chi\Box\delta\chi + \frac{1}{\Lambda^3}\mu\delta\chi\Box^2\delta\chi + \frac{1}{2\Lambda^6}\mu^2\delta\chi\Box^3\delta\chi \tag{2.69}$$

we see that the equation of motion w.r.t the perturbation is

$$\Box(\delta\chi + \frac{2}{\Lambda^3}\mu\Box\delta\chi + \frac{1}{\Lambda^6}\mu^2\Box^2\delta\chi) = 0 \implies \Box(1 + \frac{\mu}{\Lambda^3}\Box)^2\delta\chi. \tag{2.70}$$

We may perform one more field redefinition $\bar{\chi} = \frac{\chi + \chi\Box\chi}{\Lambda^3}$ so the Lagrangian reduces to

$$\mathcal{L} = \frac{1}{2}\bar{\chi}\Box\bar{\chi} \tag{2.71}$$

which looks like a free Lagrangian, but it is clearly fake as this exhibits no ghosts and so some solutions (dof) have been masked by the field redefinitions. We now consider a new theory, also containing higher derivatives

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi + \frac{1}{2}\psi\Box\psi + \frac{\Box\psi(\partial_\alpha\partial_\beta\phi)^2}{\Lambda^5} + \frac{(\partial_\alpha\partial_\beta\phi)^2\Box(\partial_\mu\partial_\nu\phi)^2}{2\Lambda^{10}} \tag{2.72}$$

varying w.r.t to ψ yields

$$\Box\psi + \frac{\Box(\partial_\alpha\partial_\beta\phi)^2}{\Lambda^5} = 0 \tag{2.73}$$

meaning we can write $\psi = -\frac{\Box(\partial_\alpha\partial_\beta\phi)^2}{\Lambda^5} + \psi_0$ where ψ_0 solves the massless KG equation. One may integrate out the field ψ to get

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi + \frac{1}{2}\Box\psi\psi + \frac{\Box(\partial_\alpha\partial_\beta\phi)^2}{\Lambda^5}) + \frac{1}{2}(\frac{(\partial_\alpha\partial_\beta\phi)^2}{\Lambda^5})(-\frac{\Box(\partial_\alpha\partial_\beta\phi)^2}{\Lambda^5}) + \frac{(\partial_\alpha\partial_\beta\phi)^2\Box(\partial_\mu\partial_\nu\phi)^2}{2\Lambda^{10}} \tag{2.74}$$

which reduces to

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi + \frac{1}{2}(\Box\psi)\psi_0 = \frac{1}{2}\phi\Box\phi + \frac{1}{2}(\psi)\Box\psi_0 = \frac{1}{2}\phi\Box\phi \tag{2.75}$$

which is just free theory on-shell. This means we need 2 degrees of freedom to determine each field, and thus, no ghosts appear. One may also work out the equations of motion for the ϕ field, which yield

$$\Box\phi + \frac{2}{\Lambda^5}\partial_\mu\partial_\nu\left[\partial^\mu\partial^\nu\phi(\Box\psi + \frac{\Box(\partial_\alpha\partial_\beta\phi)^2}{\Lambda^5})\right] \tag{2.76}$$

however, we see that on shell, this reduces to the massless KG equation for ϕ, meaning only 2 d.o.f are needed. Since all the Cauchy data is specified by the equations of motion, no ghostly degree of freedom propagates, in spite of the higher derivatives.

2.4 Fierz-Pauli massive gravity model

To add mass to our linearised Einstein equations, we consider the different ways in which we can add a Lorentz scalar term of the form of $\sim h^2$. The result is that we can either trace our spin-2 field and square it, or square it and then trace over it. This generates one constant to fix, as one of the two can always be absorbed into the mass term;

$$S = \int d^4x - \frac{1}{2}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{8}m^2(h^{\mu\nu}h_{\mu\nu} - \alpha h^2) + \frac{1}{2}\kappa h_{\mu\nu}T^{\mu\nu}. \tag{2.77}$$

This theory was first introduced by Fierz and Pauli (FP) [30], where it was shown that avoiding a ghostly sixth degree of freedom requires $\alpha = -1$, which we will adopt here onwards. We want to preserve linear Diffeomorphism invariance so, akin to the EM case, we can use a Stuckelberg vector field

$$h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)} \tag{2.78}$$

our gauge transformations here are

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}, \quad \chi_\mu \to \chi_\mu - \frac{1}{2}\xi_\mu \tag{2.79}$$

such that the mass term transforms as

$$\begin{aligned} \mathcal{L}_{mass} &= -\frac{1}{8}\Big((h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)})^2 - (h + 2\partial_\alpha\chi^\alpha)^2\Big) \\ \mathcal{L}'_{mass} &\to -\frac{1}{8}\Big((h^{\mu\nu} + \partial_{(\mu}\xi_{\nu)} + 2\partial_\mu(\chi_{\nu)} - \frac{1}{2}\xi_{\nu)}) - (h_{\mu\nu} + \partial_\alpha\xi^\alpha + 2\partial_\alpha(\chi^\alpha - \frac{1}{2}\xi^\alpha)\Big) = \mathcal{L}_{mass} \end{aligned} \tag{2.80}$$

and so we see this Lagrangian preserves linear Diffeomorphism invariance. Similarly, it is clear that the original FP term is recovered in unitary gauge.

2.4.1 Helicity decomposition

We follow the prescription presented in [29] and [7] to decompose the massive Spin-2 field into helicity two, helicity one, and helicity zero modes, which add up to the correct number of degrees of freedom which is five for a massive Spin-2 field. We decompose the Stuckelberg field as

$$\chi^\mu = \frac{1}{m}A^\mu + \frac{1}{m^2}\eta^{\mu\nu}\partial_\nu\pi \tag{2.81}$$

then the massive FP action splits into

$$\mathcal{L}_{FP} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{2}h^{\mu\nu}(\Pi_{\mu\nu} - [\Pi]\eta_{\mu\nu}) - \frac{1}{8}F_{\mu\nu}F^{\mu\nu} - \frac{1}{8}m^2(h^{\mu\nu}h_{\mu\nu} - h^2) - \frac{1}{2}m(h^{\mu\nu} - h\eta^{\mu\nu})\partial_{(\mu}A_{\nu)} \tag{2.82}$$

where $\Pi_{\mu\nu}$ is defined as $\partial_\mu\partial_\nu\pi$ and $[\Pi]$ is its trace with respect to the background metric ($\eta^{\mu\nu}$). One can diagonalise this metric by making the substitution $\bar{h_{\mu\nu}} = h_{\mu\nu} - \pi\eta_{\mu\nu}$ one finds

$$\begin{aligned} \mathcal{L}_{\rm FP} = &-\frac{1}{4}\tilde{h}^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} - \frac{3}{4}(\partial\pi)^2 - \frac{1}{8}F^2_{\mu\nu} \\ &- \frac{1}{8}m^2\left(\tilde{h}^2_{\mu\nu} - \tilde{h}^2\right) + \frac{3}{2}m^2\pi^2 + \frac{3}{2}m^2\pi\tilde{h} \\ &- \frac{1}{2}m\left(\tilde{h}^{\mu\nu} - \tilde{h}\eta^{\mu\nu}\right)\partial_{(\mu}A_{\nu)} + 3m\pi\partial_\alpha A^\alpha \end{aligned} \tag{2.83}$$

we now see the massive gravity field decomposes into a helicity-2 mode $h_{\mu\nu}$, a helicity-1 A_μ and a helicity-0 mode π which in total sum to 5 d.o.f . Diagonalising also shows us that $h_{\mu\nu}$ couples to the stress energy tensor, $T_{\mu\nu}$ while the longitudinal mode π couples to its trace T.

2.4.2 The vDVZ discuntinuity

Massive gravity was long thought to be inconsistent with a working theory of gravity because of the following calculation. We formally recover 2.55 directly from Einstein equations: in de Donder gauge, the linearised Einstein equations read

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -\kappa T_{\mu\nu} \tag{2.84}$$

taking the trace of this equation we find that

$$\Box h = \kappa T \tag{2.85}$$

where h and T are the traces of $h_{\mu\nu}$ and $T_{\mu\nu}$ respectively. This means we can write the linearised equations in reverse trace form as

$$\Box h_{\mu\nu} = -\kappa(T_{\mu\nu} - \eta_{\mu\nu}\frac{T}{2}) = -\frac{2}{M_{pl}}(T_{\mu\nu} - \eta_{\mu\nu}\frac{T}{2}) \tag{2.86}$$

so can write the interaction between two currents for linearised GR as

$$T'^{\mu\nu}\frac{f^{massless}_{\mu\nu\alpha\beta}}{\Box}T^{\alpha\beta} \tag{2.87}$$

with

$$f^{massless}_{\mu\nu\alpha\beta} = \eta_{\mu(\alpha}\eta_{\nu\beta)} - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}. \tag{2.88}$$

Now we look at the massive gravity analogue, for which the calculations are similar. first, we calculate the trace of the Lichnerowicz operator

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta}\eta^{\mu\nu} = \Box h - h^{\alpha\beta}{}_{,\alpha\beta} \tag{2.89}$$

Additionally, we see this is conserved:

$$\partial^\mu\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} = \frac{1}{2}(2h_{\mu(\alpha,\nu)}{}^{\alpha\mu} - \Box h_{\mu\nu}{}^{,\mu} - h_{,\mu\nu}{}^{\mu} - \eta_{\mu\nu}(h^{\alpha\beta,\mu}{}_{\alpha\beta} - \Box h^{,\mu}))) = 0 \tag{2.90}$$

where the first part of the first term cancels with the second, the second part with the penultimate, and the remaining two terms also cancel. We saw that the equation of motion can be written as

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + \frac{1}{2}m^2(h_{\mu\nu} - h\eta_{\mu\nu}) = \frac{1}{M_{Pl}}T_{\mu\nu}. \tag{2.91}$$

Taking two derivatives of the equation of motion one finds

$$\partial^\mu\partial^\nu\Big(\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta}+\frac{1}{2}m^2(h_{\mu\nu}-h\eta_{\mu\nu})\Big)=\frac{1}{M_{Pl}}T^{\mu\nu}{}_{,\mu\nu}\implies \Box h-h_{\mu\nu}{}^{,\mu\nu}=\frac{-2}{M_{Pl}}T^{\mu\nu}{}_{,\mu\nu}=\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta}\eta^{\mu\nu} \tag{2.92}$$

where we have used that the divergence of the Lichnerowicz symbol is zero and 2.89. Subbing back into the equation of motion and tracing, we find

$$\begin{aligned}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta}\eta^{\mu\nu}+\frac{1}{2}m^2(h-h\eta^{\mu\nu}\eta_{\mu\nu})&=\frac{1}{M_{Pl}}T\\ \implies \frac{-2}{M_{Pl}}T^{\mu\nu}{}_{,\mu\nu}-\frac{3}{2}m^2h&=\frac{1}{M_{Pl}}T\\ \implies h&=\frac{-2}{3m^2M_{Pl}}(T+\frac{2}{m^2}T^{\mu\nu}{}_{,\mu\nu})\end{aligned} \tag{2.93}$$

and similarly

$$\partial^\mu\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta}+\partial^\mu\Big(\frac{1}{2}m^2(h_{\mu\nu}-h\eta_{\mu\nu}\Big)=\frac{1}{M_{Pl}}T^\mu{}_{\nu,\mu} \tag{2.94}$$

[4] which we can rearrange to find

$$h_{\mu\nu}{}^{,\mu}=\frac{2}{M_{pl}m^2}(T_{\mu\nu}{}^{,\mu}-\frac{1}{3}T_{,\nu}-\frac{2}{3m^2}T^{\alpha\beta}{}_{,\alpha\beta\nu}). \tag{2.95}$$

Writing the equation of motion out in full one has

$$\frac{1}{2}(2h_{\mu(\sigma,\nu)\mu}-\Box h_{\nu\sigma}-h_{,\nu\sigma}-\eta_{\mu\nu}(h_{\mu\nu,\mu\nu}-\Box h))+\frac{1}{2}m^2(h_{\mu\nu}-h\eta_{\mu\nu})=\frac{1}{M_{Pl}}T_{\mu\nu} \tag{2.96}$$

we want to have the propagator in the form $\frac{f_{\alpha\beta\mu\nu}}{\Box-m^2}$ so we separate the equation into

$$\begin{aligned}-\frac{1}{2}(\Box-m^2)h_{\mu\nu}=&\frac{-2}{M_{pl}m^2}(T^\alpha{}_{(\nu,\alpha\mu)}-\frac{T_{,\mu\nu}}{3}-\frac{2}{3m^2}T^{\alpha\beta}{}_{,\alpha\beta\mu\nu})\\ &-\frac{1}{3m^2}(T_{,\mu\nu}+\frac{2}{m^2}T^{\alpha\beta}{}_{,\alpha\beta\nu})+\eta_{\mu\nu}\frac{T^{\alpha\beta}{}_{,\alpha\beta}}{m^2M_{pl}}-\frac{\eta_{\mu\nu}}{3M_{pl}}(T+\frac{2}{m^2}T^{\alpha\beta}{}_{,\alpha\beta})+\frac{1}{M_{pl}}T_{\mu\nu}.\end{aligned} \tag{2.97}$$

Which we can simplify to

$$(\Box-m^2)h_{\mu\nu}=-\frac{2}{M_{pl}}\Big[T_{\mu\nu}-\frac{1}{3}(\eta_{\mu\nu}-\partial_\mu\partial_\nu)T-\frac{2}{m^2}T^\alpha{}_{\mu,\alpha\nu}+\frac{1}{3m^2}(\eta_{\mu\nu}+\frac{2}{m^2}\partial_\mu\partial_\nu)T^{\alpha\beta}{}_{,\alpha\beta}\Big]. \tag{2.98}$$

One can write this as

$$(\Box-m^2)h_{\mu\nu}=-\frac{2}{M_{pl}}\Big[\eta_{\mu\alpha}\eta_{\nu\beta}-\frac{1}{3}(\eta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{m^2})-\frac{1}{m^2}(\partial_\alpha\partial_\nu\eta_{\mu\beta}+\partial_\mu\partial_\alpha\eta_{\beta\nu})-\frac{1}{3m^2}(\eta_{\mu\nu}+\frac{2}{m^2}(\partial_\mu\partial_\nu))\partial_\alpha\partial_\beta\Big]T^{\alpha\beta} \tag{2.99}$$

[4] It is important to note that depending on the normalisation convention we use, these answers will differ. For example, [7] and [8] use different conventions, resulting in slightly different answers

Being careful, we may write

$$(\Box - m^2)h_{\mu\nu} = -\frac{2}{M_{pl}}\Big[\tilde{\eta}_{\mu(\alpha}\tilde{\eta}_{\beta\nu)} - \frac{1}{3}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta}\Big]T^{\alpha\beta} \tag{2.100}$$

with $\tilde{\eta}_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu$ and so we can write the propagator of this Spin-2 massive field as

$$G^{massive}_{\alpha\beta\mu\nu}(x, x') = \frac{f^{massive}_{\alpha\beta\mu\nu}}{\Box - m^2} \tag{2.101}$$

with the polarisation tensor being

$$f^{massive}_{\alpha\beta\mu\nu} = \tilde{\eta}_{\mu(\alpha}\tilde{\eta}_{\beta\nu)} - \frac{1}{3}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta} \tag{2.102}$$

which one may translate into Fourier space by interchanging $\partial_\mu \leftrightarrow p_\mu$ as

$$\begin{aligned} f^{massive}_{\alpha\beta\mu\nu} &= \frac{2}{3m^4}p_\mu p_\nu p_\alpha p_\beta + \eta_{\mu(\alpha}\eta_{\beta\nu)} - \frac{1}{3}\eta_{\mu\nu}\eta_{\alpha\beta} \\ &+ \frac{1}{m^2}\left(p_\alpha p_{(\mu}\eta_{\nu)\beta} + p_\beta p_{(\mu}\eta_{\nu)\alpha} - \frac{1}{3}p_\mu p_\nu \eta_{\alpha\beta} - \frac{1}{3}p_\alpha p_\beta \eta_{\mu\nu}\right). \end{aligned} \tag{2.103}$$

While this may seem divergent at first, we realise that we have the requirement that sources are conserved in the massless limit and so all the terms with derivatives vanish in this limit. Thus, the remaining polarisation tensor can be written as

$$f^{m\to 0}_{\alpha\beta\mu\nu} = \eta_{\mu(\alpha}\eta_{\beta\nu)} - \frac{1}{3}\eta_{\mu\nu}\eta_{\alpha\beta} \tag{2.104}$$

and so the scattering amplitude between two sourcing stress energy tensors is

$$\mathcal{A}^{m\to 0}_{TT'} = \int \mathrm{d}^4x h_{\mu\nu}T'^{\mu\nu} = \int \mathrm{d}^4x T'^{\alpha\beta}\frac{f^{m\to 0}_{\alpha\beta\mu\nu}}{\Box}T'^{\mu\nu} = \int \mathrm{d}^4x T'^{\alpha\beta}\frac{1}{\Box}(T^{\alpha\beta} - \frac{1}{3}T\eta_{\alpha\beta}) \tag{2.105}$$

which comparing to 2.88 we see that a discontinuity appears as the Spin-2 field couples to the trace of the source with a factor of a third instead of half in the massless limit. This is known as the van Dam, Veltman and Zakharov discontinuity [5]. This is resolved by the Vainshtein mechanism [6] where it is pointed that after a certain limit, non-linearities become important and we can no longer trust the linear theory. We will see more about this in the upcoming sections.

2.5 Quantum gravity?

It is often said that quantum gravity is an inconsistent theory, and here we go through [31] to see why this is the case from the point of view of EFTs. In EFTs we work in the low energy regimes where the physics is understood, taking a coarse grain approach towards unknown phenomena at higher energies. This approach has been extremely successful. As an example, GR is an EFT with cut-off scale M_Pl of a full theory of quantum gravity, where we may have String theory or Loop quantum gravity as candidates of a UV complete theory. Nonetheless, we know GR to be an extremely suited theory, able to make predictions to many significant figures in most circumstances, and there are many other examples where ETFs can be very accurate models of the physics. For a pedagogical review on ETFs, see for example [32].

2.5.1 Classical EFT

GR is built from an action that depends exclusively on the Ricci scalar R, and the metric determinant g, but why are other terms not allowed? One could build a coordinate invariant theory with the form

$$S = \int \mathrm{d}^4x\sqrt{-g}\left\{\Lambda + \frac{2}{\kappa^2}R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + \ldots + \mathcal{L}_{\mathrm{matter}}\right\} \tag{2.106}$$

with an infinite number of terms following the dots, where the terms are ordered by the number of derivatives involved to construct them, so $\Lambda\ \partial^0$, $R\ \partial^2 \ldots$ the first term is simply the Ricci scalar as this is the only way of fully contracting the Riemann tensor without it being identically zero. One must be careful when constructing higher order terms as, for example, for a Lagrangian of the form

$$\mathcal{L} = \frac{2}{\kappa^2}R + cR^2 \tag{2.107}$$

where c is a constant. The equations of motion for this theory are

$$\Box h + \kappa^2 c^2 \Box\Box h = 8\pi G T \tag{2.108}$$

where the propagator in Fourier space clearly has the form $\frac{1}{p^2+\kappa^2 cq^4}$ which we can split into

$$\frac{1}{p^2+\kappa^2 cq^4} = \frac{1}{p^2} - \frac{1}{p^2 + 1\kappa^2 c} \tag{2.109}$$

but the second has the wrong sign. This would lead to a short range Yukawa potential that goes like

$$V(r) = -Gm_1m_2\left[\frac{1}{r} - \frac{e^{-r/\sqrt{\kappa^2 c}}}{r}\right] \tag{2.110}$$

as worked out by K. Stelle [33], who using empirical data gives bounds for $c_1, c_2 < 10^{74}$, and thus this would be irrelevant for all observable physics as the curvature is so small that these terms would bear no significance physically.

2.5.2 Gauss-Bonnet theorem

The Euler Characteristic is a well known topological invariant in topology, as it is a topological invariant of polyhedra. This was extended to manifolds by Gauss and Bonnet in a formula that reads as

$$\int_{\mathcal{M}} KdA + \int_{\partial\mathcal{M}} k_g ds = 2\pi\chi(M) \tag{2.111}$$

for $\mathcal{M}$ a compact manifold, K is the Gaussian curvature, k_g is the geodesic curvature, and $\chi(\mathcal{M})$ is the Euler characteristic of the manifold. Thus, this connects topology to differential geometry. Intuitively, this formula can be used to work out the geometry of a spacetime from the sum of angles of triangles, or given a manifold, do the inverse calculation. To see this, we notice that the theorem can be written as

$$\int_{\mathcal{M}} KdA = 2\pi - \sum \alpha - \int_{\partial\mathcal{M}} k_g ds \tag{2.112}$$

for geodesic triangles, with interior angles α. As a result, in flat space where the geodesics are straight lines and the Gaussian curvature is zero, we find the usual $\sum \alpha = \pi$. This only applies to two-dimensional manifolds, but can be generalised to any compact, orientable, 2n-dimensional Riemannian manifold without boundaries through the Chern-Gauss-Bonnet Theorem [34]:

$$\chi(\mathcal{M}) = \frac{1}{32\pi^2}\int_{\mathcal{M}} \left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right). \tag{2.113}$$

These terms are however not often added to the equations of motions as they can be seen to vanish identically [35] in 4 dimensions and lower, merely giving a total derivative which of course is not of interest to us. This kind of theories were studied by Lovelock [36] with the shape

$$\mathcal{L} = \sqrt{-g}\alpha_j \mathcal{R}^j \tag{2.114}$$

with

$$\mathcal{R}^j \equiv \frac{1}{2^j}\delta^{\mu_1\nu_1\ldots\mu_i\nu_j}_{\alpha_1\beta_1\ldots\alpha_j\beta_j}\prod_{i=1}^{j} R^{\alpha_i\beta_i}_{\mu_i\nu_i}, \quad \text{and} \quad \delta^{\mu_1\nu_1\ldots\mu_j\nu_j}_{\alpha_1\beta_1\ldots\alpha_j\beta_j} \equiv j!\,\delta^{\mu_1}_{[\alpha_1}\delta^{\nu_1}_{\beta_1}\ldots\delta^{\mu_j}_{\alpha_j}\delta^{\nu_j}_{\beta_j]} \tag{2.115}$$

where it was found that the Gauss-Bonnet term appears at second order in R, and additionally there are no further terms for $2j+1 > D$ given the definition of δ as antisymmetrising on these indices will lead to identically zero terms. This fact will be useful to keep in mind later on when discussing the unique ghost free theory of massive gravity.

2.5.3 Quantisation of gravity

To try to quantise gravity, we combine the quantisation developed by Feynman and De Witt [37] and the background field method pioneered by 't Hooft and Veltman [38]. Using Feynman's method can be problematic in theories with as much gauge freedom as gravity as some divergences might not be able to be absorbed into the coefficients of the theories displaying the gauge symmetry, but the background field approach, as we have seen through Stuckelberg's trick, preserves the gauge invariance in the theory. We start by rewriting our metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \tag{2.116}$$

we may expand the most general Lagrangian in terms of our Spin-2 field $h_{\mu\nu}$:

$$\begin{aligned}
\frac{2}{\kappa^2}\sqrt{-g}R =& \sqrt{\bar{g}}\left\{\frac{2}{\kappa^2}\bar{R} + \mathcal{L}_g^{(1)} + \mathcal{L}_g^{(2)} + \cdots\right\} \\
\mathcal{L}_g^{(1)} =& \frac{h_{\mu\nu}}{\kappa}\left[\bar{g}^{\mu\nu}\bar{R} - 2\bar{R}^{\mu\nu}\right] \\
\mathcal{L}_g^{(2)} =& \frac{1}{2}D_\alpha h_{\mu\nu}D^\alpha h^{\mu\nu} - \frac{1}{2}D_\alpha h D^\alpha h + D_\alpha h D_\beta h^{\alpha\beta} \\
&- D_\alpha h_{\mu\beta}D^\beta h^{\mu\alpha} + \bar{R}\left(\frac{1}{2}h^2 - \frac{1}{2}h_{\mu\nu}h^{\mu\nu}\right) \\
&+ \bar{R}^{\mu\nu}\left(2h^\lambda{}_\mu h_{\nu\alpha} - h h_{\mu\nu}\right)
\end{aligned} \tag{2.117}$$

where D is the covariant derivative with respect to the background field $\bar{g}_{\mu\nu}$. The first order equations will vanish on-shell for Einstein equations, so we are left with a quadratic Lagrangian and higher order interaction terms. First we must gauge fixing, for which, like in other gauge theories, we must introduce Fadeev-Popov ghost fields [39] [40]. To build renormalizable theories from Lagrangians with gauge symmetries, we must divide by the gauge group "volume" to mend this over counting. One defines the functional delta function

$$\int \mathcal{D}\chi\delta(\chi - \chi_*(A)) = 1 \tag{2.118}$$

where $\chi_*(A)$ is the solution to the gauge fixed field [40]. Then we can extend the functional delta function to transform as

$$\delta(F[A]) = \frac{1}{\left|Det[\frac{\delta F[A]}{\delta A}]\right|}\delta(A - A_*) \tag{2.119}$$

and so

$$\int \mathcal{D}\chi\delta(\chi - \chi_*(A)) = 1 = \int \mathcal{D}\chi\delta(F[A'])Det[\frac{\delta F[A']}{\delta\chi}] \tag{2.120}$$

and so we can insert this into our generating function in the case of gravity:

$$Z = \int \mathcal{D}\chi \mathcal{D}h_{\mu\nu}\delta(F[A'])Det[\frac{\delta F[A']}{\delta\chi}]e^{iS} \tag{2.121}$$

from which we can bring the ghost field by using the identity

$$det(M) = \int d\eta d\bar{\eta} e^{i\int \mathrm{d}^4 x \eta M \bar{\eta}}. \tag{2.122}$$

As usual, we work in harmonic/ de Donder gauge, imposing this constraint. The following analysis follows [31]

$$G^{\alpha} = g^{\frac{1}{4}}(D^{\nu}h_{\mu\nu} - \frac{1}{2}D_{\mu}h)t^{\nu\alpha} \tag{2.123}$$

where we may view t as similar to vierbein[5]:

$$\eta_{\alpha\beta}t^{\mu\alpha}t^{\nu\beta} = \bar{g}^{\mu\nu} \tag{2.124}$$

leading to the gauge fixing Lagrangian [38]

$$\mathcal{L}_{gf}\sqrt{\bar{g}}\Big[(D^{\nu}h_{\mu\nu} - \frac{1}{2}D_{\mu}h)(D^{\nu}h^{\mu}_{\nu} - \frac{1}{2}D^{\mu}h) \tag{2.125}$$

and a ghost Lagrangian carrying free Lorentz indices, meaning we have fermionic fields

$$\mathcal{L}_{gh} = \sqrt{\bar{g}}\eta^{*\mu}\left[D_{\lambda}D^{\lambda}\bar{g}_{\mu\nu} - R_{\mu\nu}\right]\eta^{\nu} \tag{2.126}$$

so the full Lagrangian is

$$\begin{aligned} S = \int \mathrm{d}^4 s\sqrt{\bar{g}}\Big\{&\frac{2}{\kappa^2}\bar{R} - \frac{1}{2}h_{\alpha\beta}D^{\alpha\beta,\gamma\delta}h_{\gamma\delta} \\ &+\eta^{*\mu}\left\{D_{\lambda}D^{\lambda}\bar{g}_{\mu\nu} - \bar{R}_{\mu\nu}\right\}\eta^{\nu} + \mathcal{O}\left(h^3\right)\Big\} \end{aligned} \tag{2.127}$$

where $D^{\alpha\beta,\gamma\delta}$ encapsulates the covariant version of the Lichnerowicz symbol and the gauge fixing terms. Around a flat metric, we recover the GR described in earlier sections.

2.5.4 Issues with quantisation and renormalisation

There are two prominent issues when quantising gravity. Firstly, the coupling constant κ is dimensionful. This in itself is not necessarily a huge problem as theories such as ϕ^3 in $4D$ have this property too, but additionally, we have interactions at all orders in $h_{\mu\nu}$ in the full non-linear theory. Loop diagrams will generate divergences that cannot be renormalised into G, but instead will require from an infinite number of parameters to renormalise when

[5] These will be explained in more detail later.

going to higher loop orders.

We do an energy expansion with the cosmological constant set to zero

$$\mathcal{L}_g = \sqrt{-g}\Big[\frac{2}{\kappa^2}R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + \mathcal{O}(R^3)\Big] \tag{2.128}$$

with dimensionless couplings c_i and a matter Lagrangian

$$\begin{aligned}\mathcal{L}_m =& \sqrt{-g}\left\{\frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)\right.\\ &\left. + d_1 R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + R\left(d_2 g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + d_3 m^2\phi^2\right) + \ldots\right\}\end{aligned} \tag{2.129}$$

but here, the d_i play a more subtle role, proportional to $\frac{1}{m^2}$ where these play the role analogous to charge radius in QED [31].Traditionally, if only one matter field were to be involved, one could use the Euler-Lagrange equations to eliminate some terms, as these relate the matter fields to the curvature, but having several possible interacting matter fields makes this more complicated.

We expand the action in terms of the background metric, and fix the gauge, keeping only those terms quadratic in the Spin-2 quantum field $h_{\mu\nu}$ and the ghost fields and so schematically we have

$$\begin{aligned}Z[\bar{g}] &= \int [dh_{\mu\nu}]\exp\left\{i\int \mathrm{d}^4x\sqrt{\bar{g}}\left\{\frac{2}{\kappa^2}\bar{R} + h_{\mu\nu}D^{\mu\nu\alpha\beta}h_{\alpha\beta}\right\}\right.\\ &= \det D^{\mu\nu\alpha\beta}\\ &= \exp \mathrm{Tr}\ln\left(D^{\mu\nu\alpha\beta}\right)\end{aligned} \tag{2.130}$$

where we $[h_{\mu\nu}]$ means it is gauge fixed, and the curvature terms gets absorbed into the generating function as a result of the Gellman-Low theorem [40], and for the following lines we have used the usual tricks introduced in previous sections and the fact that for bosonic path integrals we have

$$\int \mathcal{D}\phi e^{-\frac{1}{2}^d x\ \mathrm{d}^d y\phi(x)M(x,y)\phi(y)} = \frac{\mathrm{const}}{\sqrt{\det M}}. \tag{2.131}$$

't Hooft and Veltman were able to calculate the one loop renormalisation using dimensional regularisation to be

$$\mathcal{L}^{(\mathrm{div})}_{1loop} = \frac{1}{8\pi^2\epsilon}\left\{\frac{1}{120}\bar{R}^2 + \frac{7}{20}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}\right\} \tag{2.132}$$

in d=$4-\epsilon$, and with additional R^2 contributions from the matter fields at one loop. We notice that $\mathcal{O}(\partial^4 h)$ operators are needed to renormalise divergences at $\mathcal{O}(\partial^2 h)$. Similarly, $\mathcal{O}(\partial^6 h)$ will be needed to renormalise the $\mathcal{O}(\partial^4 h)$ terms, and this process will continue ad infinitum. This is the characteristic of non-renormalizable theories. Pure gravity with $R_{\mu\nu} = 0$ is renormalizable as the higher order terms in R vanish, meaning they don't have to be renormalised and so it is finite at one loop [31].

2.5.5 Quantum predictions

As we know, GR is an EFT of a quantum gravity theory, and as such it does not predict quantum divergences as these arise when considering loop diagrams. In renormalizable theories, these do not affect the theory physically as they get absorbed into the renormalized parameters, that are set by empirical data, and so these are not predictions of the ETF either. In GR, we find couplings with negative mass dimension at a scale, M_{pl} and so this should be treated as the cut-off of this theory [8]. The higher order derivative terms would normally give rise to Ostrogradsky's ghosts, nonetheless, their masses are near or above the energy threshold, and so the EFT treatment lets us ignore them as UV completion might cure them. Each derivative power carries with them a factor of $\frac{1}{M_{pl}}$ so

$$\frac{\partial}{M_{pl}} \sim \frac{1}{M_{pl} r} \tag{2.133}$$

and so these quantum effects only come into play at distances on the scale, or smaller distances than $r = \frac{1}{M_{pl}}$, and thus can be ignored for longer distances.

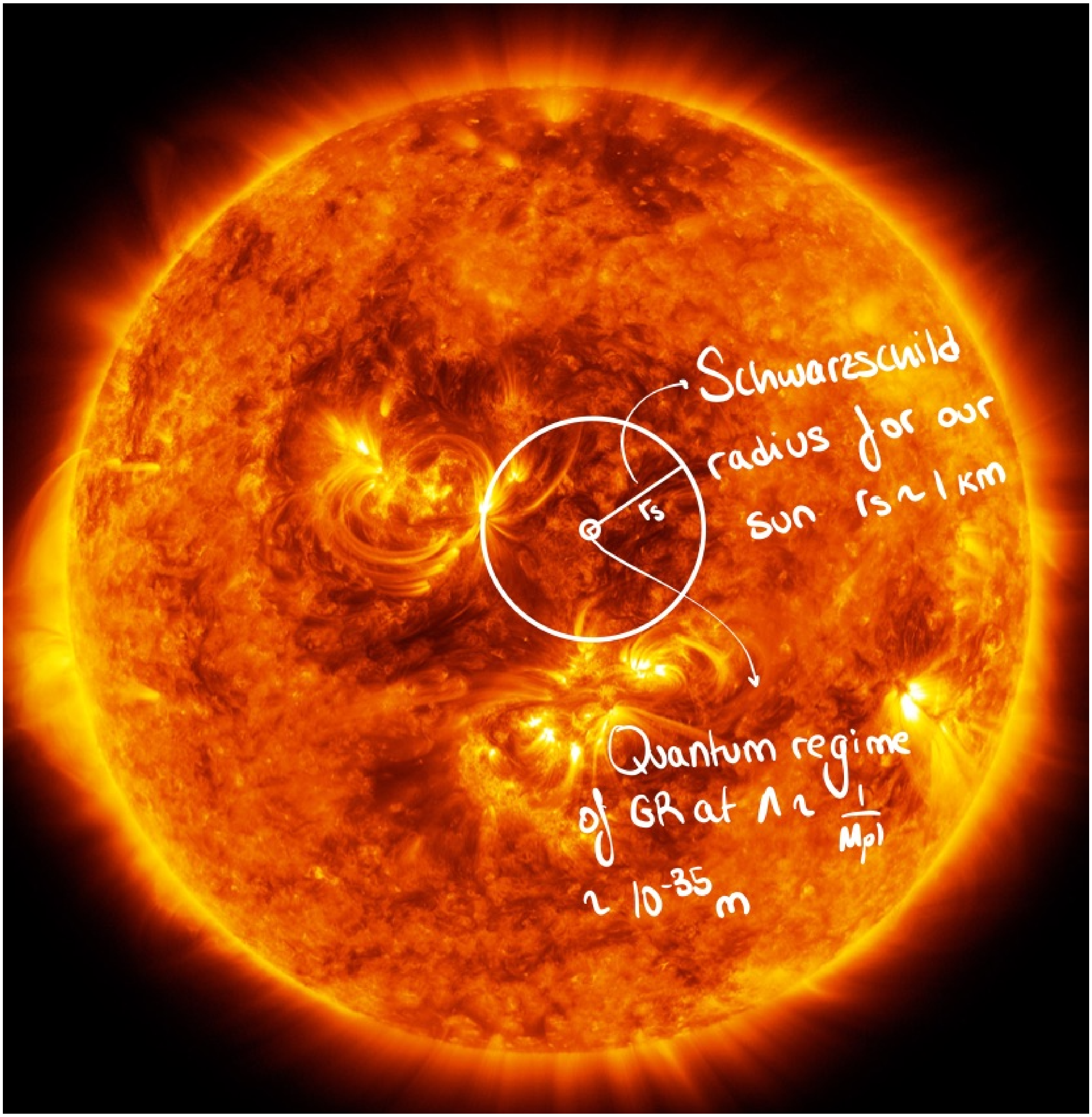

Figure 3: Photo credit NASA's Solar Dynamics Observatory. Here we see the different regimes of GR (not to scale).This is a well behaved theory in the sense that the gap between the quantum regime and the classical linear limit is orders of magnitude away, meaning we can have a non-linear classical theory of gravity.

3 Modern massive gravity

3.1 Extending Massive Gravity

The terms in GR are constrained by the Diffeomorphism invariance to yield GR uniquely [7] [8] as shown by Deser [41]. Nonetheless, there is no obvious continuation to pursue in massive gravity, and so we have some more freedom when adding terms. We have previously introduced the FP mass term around the Minkowski metric, but this can in fact be done around any background metric $g^{(0)}_{\mu\nu}$.Unlike in GR, a solution of our metric cannot be altered via a Diffeomorphism[6] to obtain a second solution to the same theory, as the mass term breaks this invariance explicitly.

3.1.1 Vainshtein radius

Vainshtein set to find the radius at which non-linearities would enter a theory of massive gravity. We know that non-linear solutions are needed to accurately explain the results in General relativity close to the Schwarzschild radius, setting the cut-off scale for the linear theory of linearised GR. The hope was that this calculation would shine light on the nature of the vDVZ discontinuity, but the result was more surprising than one would initially expect. The calculations are not particularly insightful or complicated, so we will simply state the results here. One can work out the spherically symmetrical solution to the free Einstein equations in powers of non-linearity by writing

$$g_{\mu\nu}dx^\mu dx^\nu = -B(r)dt^2 + C(r)dr^2 + A(r)r^2 d\Omega^2 \tag{3.1}$$

one can go through the analysis at every order as prescribed in [6] or [8], but the end result for massless gravity is

$$B(r) - 1 = -\frac{2GM}{r}\left(1 - \frac{GM}{r} + \cdots\right), \quad C(r) - 1 = \frac{2GM}{r}\left(1 + \frac{3GM}{4r} + \cdots\right). \tag{3.2}$$

where gauge freedom allows us to choose A(r)=C(r) [8] This same analysis can be performed in massive gravity, where the calculation yields

$$\begin{aligned} B(r) - 1 &= -\frac{8}{3}\frac{GM}{r}\left(1 - \frac{1}{6}\frac{GM}{m^4r^5} + \cdots\right) \\ C(r) - 1 &= -\frac{8}{3}\frac{GM}{m^2r^3}\left(1 - 14\frac{GM}{m^4r^5} + \cdots\right) \\ A(r) - 1 &= \frac{4}{3}\frac{GM}{4\pi m^2r^3}\left(1 - 4\frac{GM}{m^4r^5} + \cdots\right) \end{aligned} \tag{3.3}$$

These two results are clearly different, and we see that the massive gravity case is an expansion in the Vainshtein radius r_V:

$$r_V \equiv (\frac{GM}{m^4})^{\frac{1}{5}}. \tag{3.4}$$

[6]The two theories are related as their metrics differ by a Diffeomorphism

This is an alarming result, as the massless limit has a radius that goes to infinity, meaning that non-linearities are nowhere trustworthy. This solves the problem with the vDVZ discontinuity; taking the massless limit means that the theory being studied is bad at all possible radii, so one would not expect it to be smoothly deformed to linearised GR, which is good at radii larger than the Schwarzschild radius.

3.1.2 A short note on the ADM formalism and the Boulware-Deser ghost

The ADM formalism [42] is a technique by which one can analyse the equations of motion of the metric field by deconstructing it into functions of its components. This can be used to simplify the analysis in some particular cases and in fully non-linear massless gravity we still find two degrees of freedom, however, in massive gravity we have one extra ghostly degree of freedom [43] as compared to the linearised analysis. This is known as the Boulware-Deser ghost.

3.1.3 Non-linear Stuckelberg formalism

This method was pioneered in the context of string theory [44] [45] and the formalism is covered in detail in [7] [8]. The first ingredient we require in this formalism is a reference metric that might or might not be Minkowski space. In massless theory, we have both Poincaré symmetry and Diffeomorphism invariance, and when adding mass to the graviton, Diffeomorphism invariance is explicitly broken. As a result, similarly to when the VEV of the Higgs impregnates the Lagrangian in the broken phase, here an explicit Higgs mechanism for Lorentz invariant massive gravity has not been found yet. The aforementioned reference metric can be viewed as the VEV of a Spin-2 field that breaks the Diffeomorphism invariance. It is easy to see why this breaks Diffeomorphism invariance: once the numbers are fixed in a reference metric, any non-trivial transformation will change it. we start by writing the gauge symmetry of gravity, which is

$$g_{\mu\nu} \rightarrow \frac{\partial g^{\alpha}}{\partial x^{\mu}} \frac{\partial g^{\beta}}{\partial x^{\nu}} g_{\alpha\beta}(g(x)) \tag{3.5}$$

where g is an arbitrary Diffeomorphism. This invariance is broken by the mass term only, as the EH term is gauge invariant. Thus, we can apply a Diffeomorphism to the metric without changing this part of the action, additionally, we can promote the reference metric f to a full tensor by

$$f_{\mu\nu} \rightarrow \tilde{f}_{\mu\nu} = \frac{\partial \phi^{\alpha}}{\partial x^{\mu}} \frac{\partial \phi^{\beta}}{\partial x^{\nu}} f_{\alpha\beta} \tag{3.6}$$

where ϕ is our Stuckelberg field, and we see that our reference metric transforms now as a tensor under coordinate transforms as long as the Stuckelberg fields transform as scalars. We also see that in unitary gauge (setting $\phi = x$), $G_{\mu\nu} = g_{\mu\nu}$. We also construct

$$\mathbb{X}^{\mu}_{\nu} = G_{\alpha\nu} g^{\mu\alpha} = \frac{\partial \phi^{\alpha}}{\partial x_{\mu}} \frac{\partial \phi^{\beta}}{\partial x^{\nu}} g_{\alpha\beta} \tag{3.7}$$

and in unitary gauge we have $\mathbb{X} = g^{-1}f$. We also see that when the reference metric is Minkowski, the fields will carry Lorentz indices and their derivatives will be equivalent to tetrads.

3.1.4 Helicity decomposition

As previously, we split the Stuckelberg field into $\phi =^a -\frac{1}{M_{pl}\chi^a}$ where "a" is a Lorentz index. One may further decompose χ into its Spin-1 and Spin-0 components: $\chi^a = \frac{1}{m}A^a + \frac{1}{m^2}\partial^a\pi$ and so the Minkowski metric transforms into

$$\begin{aligned}\eta_{\mu\nu} \longrightarrow \tilde{f}_{\mu\nu} &= \eta_{\mu\nu} - \frac{2}{M_{\rm Pl}}\partial_{(\mu}\chi_{\nu)} + \frac{1}{M_{\rm Pl}^2}\partial_\mu\chi^a\partial_\nu\chi^b\eta_{ab} \\ &= \eta_{\mu\nu} - \frac{2}{M_{\rm Pl}m}\partial_{(\mu}A_{\nu)} - \frac{2}{M_{\rm Pl}m^2}\Pi_{\mu\nu} \\ &+ \frac{1}{M_{\rm Pl}^2m^2}\partial_\mu A^\alpha\partial_\nu\mathcal{A}_\alpha + \frac{2}{M_{\rm Pl}^2m^3}\partial_\mu A^\alpha\Pi_{\nu\alpha} + \frac{1}{M_{\rm Pl}^2m^4}\Pi^2_{\mu\nu}\end{aligned} \tag{3.8}$$

and so we can now define the tensor version of $h_{\mu\nu}$:

$$h_{\mu\nu} = M_{Pl}(g_{\mu\nu} - \eta_{\mu\nu}) \to H_{\mu\nu} = M_{Pl}(g_{\mu\nu} - \tilde{f}_{\mu\nu}) \tag{3.9}$$

and so we see that $H_{\mu\nu}$ reduces to

$$\begin{aligned}H_{\mu\nu} &= h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)} - \frac{1}{M_{\rm Pl}}\eta_{ab}\partial_\mu\chi^a\partial_\nu\chi^b \\ &= h_{\mu\nu} + \frac{2}{m}\partial_{(\mu}A_{\nu)} + \frac{2}{m^2}\Pi_{\mu\nu} \\ &- \frac{1}{M_{\rm Pl}m^2}\partial_\mu A^\alpha\partial_\nu\mathcal{A}_\alpha - \frac{2}{M_{\rm Pl}m^3}\partial_\mu A^\alpha\Pi_{\nu\alpha} - \frac{1}{M_{\rm Pl}m^4}\Pi^2_{\mu\nu}\end{aligned} \tag{3.10}$$

This helicity decomposition remains valid in the limit $M_{pl} \to \infty$ as shown in [29] and the degrees of freedom are as usual two from a Spin-2 massless field, a Spin-1 vector and a scalar.

3.1.5 Non-linear FP and BD ghost

We notice that given $\mathbb{X} = g^{-1}f$ we can construct $\mathbb{1} - \mathbb{X} = \mathbb{1} - g^{-1}f = g^{-1}(g-f)$ which looks like the tensor version of h we need, and so we can extend the FP term to a non-linear version

$$\mathcal{L}^{nl}_{FP} = -m^2M^2_{pl}\sqrt{-g}([(\mathbb{1}-\mathbb{X})^2] - [\mathbb{1}-\mathbb{X}]^2) \tag{3.11}$$

which is invariant under non-linear Diffeomorphisms. We notice that $\mathbb{X}^\mu_\nu = \delta^\mu_\nu - 2\tilde{\Pi}^\mu_\nu + \tilde{\Pi}^\mu_\alpha\tilde{\Pi}^\alpha_\nu$ where $\tilde{\Pi} = \frac{1}{m^2M_{pl}}$ and so

$$\mathbb{1} - \mathbb{X} = 2\tilde{\Pi}^\mu_\nu - \tilde{\Pi}^\mu_\alpha\tilde{\Pi}^\alpha_\nu \tag{3.12}$$

which leads to

$$[(\mathbb{1}-\mathbb{X})^2] = 4[\tilde{\Pi}^2] - 4[\tilde{\Pi}^3] + [\tilde{\Pi}^4], \quad [(\mathbb{1}-\mathbb{X})]^2 = 4[\tilde{\Pi}]^2 - 4[\tilde{\Pi}^2][\tilde{\Pi}] + [\tilde{\Pi}^2]^2 \tag{3.13}$$

and so

$$\mathcal{L}^{nl}_{FP} = -m^2 M^2_{pl}\sqrt{-g}(4([\tilde{\Pi}^2] - [\tilde{\Pi}]^2) - 4([\tilde{\Pi}^3] - [\tilde{\Pi}^2][\tilde{\Pi}]) + [\tilde{\Pi}^4] - [\tilde{\Pi}^2]^2). \tag{3.14}$$

The quadratic term is special as it is a total derivative; one can integrate by parts and cancel it. Nonetheless, the other terms do not possess this desirable property, and so reminding ourselves of the form of Π this means that this Lagrangian contains higher derivative terms, which propagate an extra degree of freedom, which by Ostrogradsky's theorem will be a ghost, which is the BD ghost. The resolution to this issue, and ultimately of a working theory of massive gravity, will be to construct a mass term that when expanded yields total derivatives of π, as shown in [9].

3.2 Scale analysis

In this section we consider the scales that come in play in different theories and how they arise naturally from considering the lowest interactions terms. Additionally, we see how the de Rham Gabadadze Tolley model (dRGT) raises the cut-off to previous theories considered from Λ_5 to Λ_3.

3.2.1 Galilean symmetry

By considering the expansion of the terms in powers of the helicity decomposition of our Spin-2 field, we see that requiring that the terms come in at least at order 3, so that they represent interactions, and that $m < M_{pl}$, one arrives to the conclusion that the term suppressed by the lowest scale is the cubic scalar term at a cut-off of $\Lambda_5 = (M_{pl}m^4)^{\frac{1}{5}}$. It is easy to see that since these terms come in with two derivatives, there exists a symmetry

$$\pi \to \pi + c + b^\mu x_\mu \tag{3.15}$$

where b is a constant vector and c is a scalar. Similarly, we can shift the vector field A_μ by a constant vector, and we will also have a symmetry, since this enters with one derivative. These are called Galilean symmetries.

3.2.2 Decoupling limit and Vainshtein screening

The decoupling limit allows us to focus on the cut-off scale, and it is set up by sending $m \to 0$, $M_{Pl} \to \infty$ and fixing Λ_5 and $\frac{T}{M_{pl}}$. In this regime, we are left with only the cubic term in π considered above. The potential in 3.14 fails to be a total derivative by exactly a factor of $([\Pi^3] - [\Pi][\Pi^2])$, which shines light on the origin of the Vainshtein radius:

$$S_\phi = \int \mathrm{d}^4x - 3(\partial\hat{\pi})^2 + \frac{2}{\Lambda_5^{\frac{5}{5}}}\left[(\Box\tilde{\pi})^3 - (\Box\hat{\pi})\left(\partial_\mu\partial_\nu\hat{\pi}\right)^2\right] + \frac{1}{M_P}\hat{\pi}T \tag{3.16}$$

where we are only considering the π terms, but the free graviton also survives. Consider the scalar around a symmetric source of mass M, so $\pi \sim \frac{M}{M_{pl} r}$. Similarly, the non-linear term will be suppressed by a factor of Λ_5^5 [8]:

$$\frac{\partial^4 \pi}{\Lambda_5^5} \sim \frac{M}{M_{pl} \Lambda_5^5 r^5} \tag{3.17}$$

and so the non-linear terms will become important when the scale is of order unity, at

$$r_V \sim \frac{M}{M_{pl}} \sim (\frac{GM}{m^4}) \tag{3.18}$$

which is exactly the Vainshtein radius. In [46] it is shown that after a field redefinition $\pi \to \psi + \chi$ the quartic in derivatives, non-linear part of the Lagrangian can be written as

$$\mathcal{L}_\varphi = -\left(\partial\varphi\right)^2 + \frac{\left(\partial^2\Phi\right)}{\Lambda_5^5}\left(\partial^2\varphi\right)^2 \tag{3.19}$$

which signs to the presence of a ghost given the higher derivative terms, with a mass of scale

$$m_{\text{ghost}}^2\,(x) \sim \frac{\Lambda_5^5}{\partial^2\Phi^c(x)}, \tag{3.20}$$

but since we are dealing with ETFs, we should not worry about ghosts until they enter the regime in which the ETF is valid, which is below the Λ^5 scale. This happens at a distance $\partial^2\Phi \sim \Lambda_5^3$ [46], which is even greater than the Vainshtein radius;

$$R_{\text{ghost}} \sim \frac{1}{\Lambda_5}\left(\frac{M_*}{M_P}\right)^{1/3} \gg R_V^{(5)} \sim \frac{1}{\Lambda_5}\left(\frac{M_*}{M_P}\right)^{1/5} \tag{3.21}$$

this means that at a radius lower than r_V the field will carry a factor of

$$\frac{M\Lambda_5^5}{M_{pl}} \tag{3.22}$$

which after power-counting gives us $\sim (\frac{m^4}{M})^{\frac{1}{2}}$, and since mass and distance have opposite units we must have a $r^{\frac{3}{2}}$ profile. We can see from this that at distances smaller than the Vainshtein radius, the second derivative of the potential goes as $r^{-\frac{1}{2}}$ and so the mass of the ghost is proportional to $r^{\frac{1}{4}}$ and so the mass tends to zero as we get closer to the central mass. Correspondingly, the Yukawa suppression eases as we move closer and the longitudinal mode starts mediating a long range force, but due to the sign of the kinetic term, this force is repulsive. Our Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}\pi\Box\pi + \frac{1}{\Lambda^5}(\Box\pi)^3 - \frac{1}{M_P}\pi T \tag{3.23}$$

which upon varying π gives us

$$\Box\pi + \frac{3}{\Lambda_5^5}\Box((\Box\pi)^2) - \frac{T}{M_{Pl}} = 0, \tag{3.24}$$

which shows us we need two phase space degrees of freedom to fix $\Box\pi$ and two further to fix the d'Alembertian operator of this quantity, which gives us 4 phase degrees of freedom in total which corresponds to two propagating degrees of freedom. One can redefine the Lagrangian by introducing a new field that absorbs this degree of freedom, so each field carries a propagating degree of freedom by introducing a new function into the Lagrangian that limits the derivatives to second order as shown in [47]

$$\mathcal{L}_{\text{eq}} = \frac{1}{2}\pi\Box\pi + \frac{1}{\Lambda^5}(\Box\pi)^3 + F(\lambda, \Box\pi) - \frac{1}{M_P}\pi T, \tag{3.25}$$

with the equations of motion given by

$$\Box\pi + \frac{3}{\Lambda^5}\Box\left((\Box\pi)^2\right) + \Box F^{(0,1)} - \frac{T}{M_P} = 0, \quad \text{for the respective fields, with } F^{(1,0)} = 0, \tag{3.26}$$

where $F^{(i,j)}$ corresponds to the i^{th} and j^{th} variations of the function with respect to the corresponding fields. The correct ansatz for this problem is

$$F(\lambda, \Box\pi) = \frac{2}{3\sqrt{3}}\Lambda^{5/2}\Lambda^3 + \Lambda^2\Box\pi - \frac{1}{\Lambda^5}(\Box\pi)^3 \tag{3.27}$$

for which we may redefine the field as $\pi = \varphi - \lambda^2$ to remove the cross terms, at the cost of one kinetic term in λ. Additionally, we may redefine $\lambda^2 \to \psi$ to obtain the Lagrangian

$$\mathcal{L}_{\text{eq}} = \frac{1}{2}\varphi\Box\varphi - \frac{1}{2}\psi\Box\psi - \epsilon\frac{2}{3\sqrt{3}}\psi^{3/2}\Lambda^{5/2} - \frac{1}{M_P}\varphi T + \frac{1}{M_P}\psi T \tag{3.28}$$

where $\epsilon = \pm 1$ is the sign of $\Box(\varphi - \psi)$ as per [47]. The important finding here is that the new ghost field couples to the trace of the stress-energy tensor in precisely the same way as our longitudinal field, but with a negative sign. This happens to exactly cancel the attractive force of the longitudinal field, meaning we obtain no extra forces beyond the Vainshtein radius. This mechanism is known as the Vainshtein or screening mechanism [8] [7].

3.2.3 A brief note on quantum corrections

Considering possible terms in a Lagrangian of the form,

$$c_{p,q}\partial^q h^p \tag{3.29}$$

we aim to find a scale for the coefficients of these terms, $c_{p,q}$. Since there is Gaileon symmetry to preserve, we require that derivatives of h come in twos:

$$\sim \frac{\partial^q(\partial^2\hat{\pi}^p)}{\Lambda_5^{3p+q-4}} \tag{3.30}$$

remembering the normalisation $\hat{\pi}$ we can see that

$$c_{p,q} \sim \Lambda_5^{-3p-q+4} M_P^p m^{2p} = \left(m^{16-4q-2p} M_P^{2p-q+4}\right)^{1/5} \tag{3.31}$$

so we can write the effective action

$$S = \int \mathrm{d}^4x \frac{M_P^2}{2}\left[\sqrt{-g}R - \frac{m^2}{4}\left(h_{\mu\nu}^2 - h^2\right)\right] + \sum_{p,q} c_{p,q}\partial^q h^p \tag{3.32}$$

so the radius at which the higher order interaction term becomes of order $(\partial\pi)^2$ is

$$r_{p,q} \sim \left(\frac{M}{M_{Pl}}\right)^{\frac{p-2}{3p+q-4}} \frac{1}{\Lambda_5} \tag{3.33}$$

which on the limit of large p goes as

$$r_Q \sim \left(\frac{M}{M_{Pl}}\right)^{1/3} \frac{1}{\Lambda_5} \tag{3.34}$$

which is larger than the Vainshtein radius. This implies that there is no in between; we cannot have a classical non-linear theory, contrary to GR, where the non-linearities star close to the Schwarszchild radius, far from the quantum regime.

3.3 dRGT Massive Gravity

We have seen some strengths and downfalls of the most straightforward extensions of massive gravity , together with some disheartening and off-putting results such as the BD ghost, the incompatibility of classical gravity with non-linear theories of massive gravity, and other no-go theorems. Now we consider higher order terms in the Lagrangian in powers of h, thus generalising the FP action. In what follows we will look at a special class of theories, defined by two parameters, that were developed in the early 2010s by Claudia de Rham, Gregory Gabadadze, and Andrew Tolley [9] in which a particular choice of terms cures the problems, at least in the decoupling limit. Later it was show by Hassan and R. Rosen [48] that this theory is also ghost free beyond the decoupling limit. It was also shown in [49] that if there were no self interactions between the scalar terms, then the cut-off would be raised from Λ_5 to Λ_3.

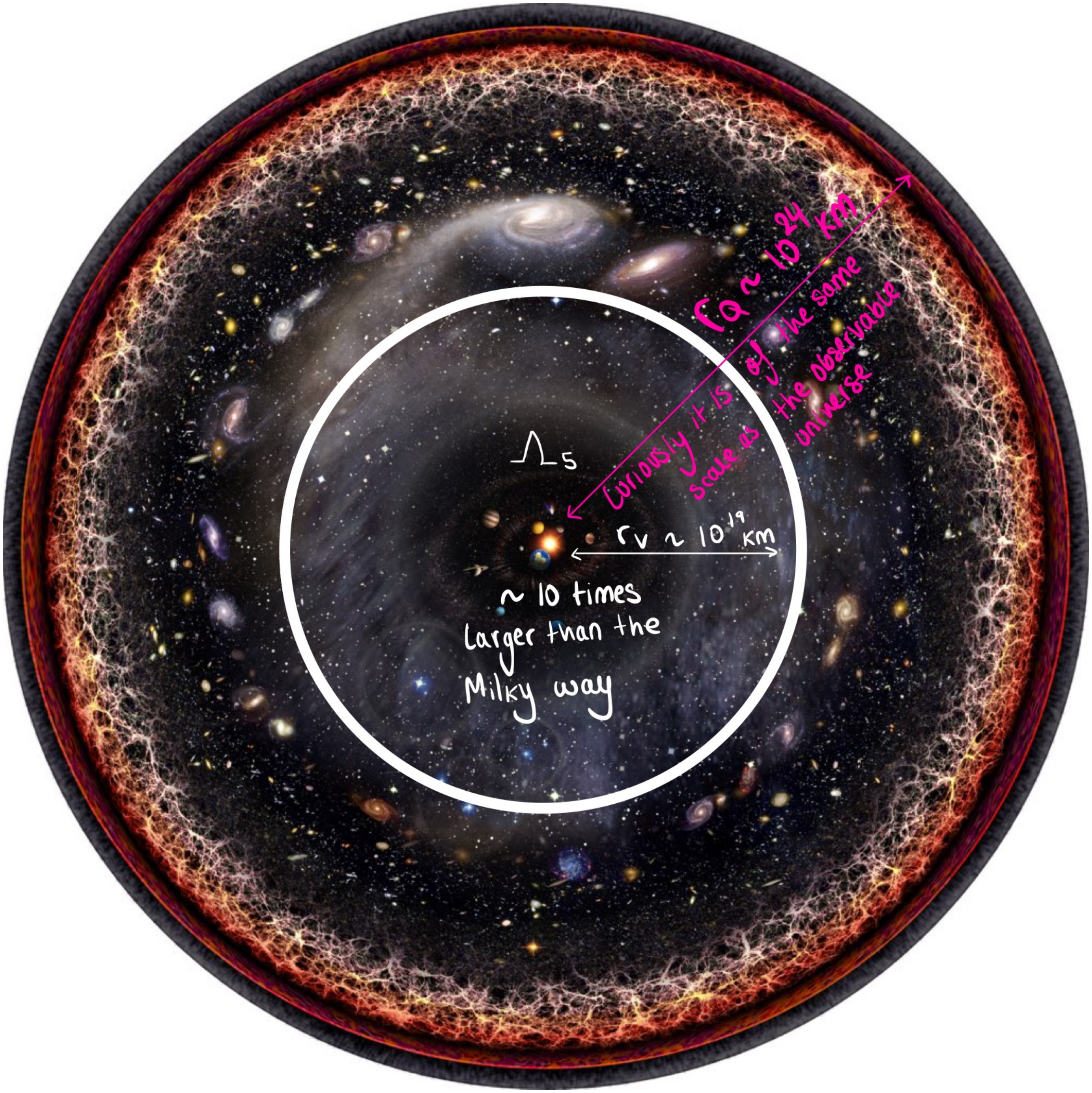

Figure 4: Credits to Pablo Carlos Budassi for the original image which was later edited. Scales for a theory with a Λ_5 cut-off scale. As mentioned in [8] this theory is not of much use empirically, as the quantum regime, beyond where the theory is valid, is larger than the observable universe, and so we have no hope of making predictions of the classical theory. Values for a graviton mass of the Hubble scale ($10^{-33}eV$)

3.3.1 Generalising FP and total derivatives

As just mentioned, it is possible to raise the cut-off by setting the coefficients of higher order terms to particular values by cancelling the scalar self interactions. Since these are the problematic terms in our theory, we can safely ignore the vector field, setting it to zero. Thus, we may write [4]

$$H_{\mu\nu} \to 2\Pi_{\mu\nu} - \Pi_{\mu\alpha}\Pi^{\alpha}_{\nu}. \tag{3.35}$$

A general action may then be written as

$$S = \frac{1}{2\kappa^2}\int d^D x \left[(\sqrt{-g}R) - \sqrt{-g^0}\frac{1}{4}m^2 U\left(g^{(0)}, h\right)\right], \tag{3.36}$$

with

$$U\left(g^{(0)}, h\right) = U_2\left(g^{(0)}, H\right) + U_3\left(g^{(0)}, H\right) + U_4\left(g^{(0)}, H\right) + U_5\left(g^{(0)}, H\right) + \cdots \tag{3.37}$$

and

$$\begin{aligned}
U_2\left(g^{(0)}, H\right) =& \left[H^2\right] - [h]^2 \\
U_3\left(g^{(0)}, H\right) =& + C_1\left[H^3\right] + C_2\left[H^2\right][H] + C_3[H]^3, \\
U_4\left(g^{(0)}, H\right) =& + D_1\left[H^4\right] + D_2\left[H^3\right][H] + D_3\left[H^2\right]^2 + D_4\left[H^2\right][H]^2 + D_5[H]^4, \\
U_5\left(g^{(0)}, H\right) =& + F_1\left[H^5\right] + F_2\left[H^4\right][H] + F_3\left[H^3\right][H]^2 + F_4\left[H^3\right]\left[H^2\right] + F_5\left[H^2\right]^2[H] \\
& + F_6\left[H^2\right][H]^3 + F_7[H]^5 \ldots
\end{aligned} \tag{3.38}$$

where U_2 respects the FP mass term, and the rest are simply all contractions of our Spin-2 field with the background metric, $g^{(0)}$ and the square bracket here means it has been traced by the background metric. We also note that all interactions at $n > D$ are redundant, as they are combinations of characteristic polynomials that are identically zero [8] [4] [9]. These interactions at each order in Π are

$$\begin{aligned}
\mathcal{L}^{(2)}_{\text{der}} &= [\Pi]^2 - \left[\Pi^2\right] \\
\mathcal{L}^{(3)}_{\text{der}} &= [\Pi]^3 - 3[\Pi]\left[\Pi^2\right] + 2\left[\Pi^3\right] \\
\mathcal{L}^{(4)}_{\text{der}} &= [\Pi]^4 - 6\left[\Pi^2\right][\Pi]^2 + 8\left[\Pi^3\right][\Pi] + 3\left[\Pi^2\right]^2 - 6\left[\Pi^4\right]
\end{aligned} \tag{3.39}$$

To find the right coefficients, one can look at the Lagrangian in terms of tensors X to the first order and match them to the polynomials we need:

$$\mathcal{L} = -\frac{1}{2}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + h^{\mu\nu}X^{(1)}_{\mu\nu} - \frac{1}{4\Lambda_5^5}\left((8c_1 - 4)\left[\Pi^3\right] + (8c_2 + 4)[\Pi]\left[\Pi^2\right] + 8c_3[\Pi]^3\right) + \frac{1}{\Lambda_3^3}h^{\mu\nu}X^{(2)}_{\mu\nu} \tag{3.40}$$

with

$$X^{(1)}_{\mu\nu} = [\Pi]\eta_{\mu\nu} - \Pi_{\mu\nu} \tag{3.41}$$

we see that

$$h^{\mu\nu}X^{(1)}_{\mu\nu} = (2\Pi_{\mu\nu} - \Pi_{\mu\alpha}\Pi_{\alpha\nu})([\Pi]\eta^{\mu\nu} - \Pi^{\mu\nu}) = 2([\Pi]^2 - \left[\Pi^2\right]) - ([\Pi]^2[\Pi] - \left[\Pi^3\right]) \tag{3.42}$$

and hence, the cubic coefficients are as in 3.40. Now we want the coefficients to be related in such a way that they are proportional to total derivatives. This requires

$$8c_1 - 4 = 2\alpha, \quad 8c_3 = \alpha \tag{3.43}$$

consequently,

$$c_1 = 2c_3 + \frac{1}{2} \tag{3.44}$$

and similarly

$$c_2 = -3c_3 - \frac{1}{2}. \tag{3.45}$$

We can repeat this calculation for higher order terms, and the careful calculation yields [4]

$$\begin{aligned}
&c_1 = 2c_3 + \frac{1}{2}, \quad c_2 = -3c_3 - \frac{1}{2}\\
&d_1 = -6d_5 + \frac{1}{16}\left(24c_3 + 5\right), \quad d_2 = 8d_5 - \frac{1}{4}\left(6c_3 + 1\right)\\
&d_3 = 3d_5 - \frac{1}{16}\left(12c_3 + 1\right), \quad d_4 = -6d_5 + \frac{3}{4}c_3,\\
&f_1 = \frac{7}{32} + \frac{9}{8}c_3 - 6d_5 + 24f_7, \quad f_2 = -\frac{5}{32} - \frac{15}{16}c_3 + 6d_5 - 30f_7,\\
&f_3 = \frac{3}{8}c_3 - 3d_5 + 20f_7,\\
&f_4 = -\frac{1}{16} - \frac{3}{4}c_3 + 5d_5 - 20f_7,\\
&f_5 = \frac{3}{16}c_3 - 3d_5 + 15f_7,\\
&f_6 = d_5 - 10f_7.
\end{aligned} \tag{3.46}$$

which ensures all the terms are total derivatives [4] [7]. To see why all higher order derivatives are zero we consider

$$\mathcal{L}^{(5)}_{\text{der}} = 24\left[\Pi^5\right] - 30[\Pi]\left[\Pi^4\right] + 20\left[\Pi^3\right]\left([\Pi]^2 - \left[\Pi^2\right]\right) + 15[\Pi]\left[\Pi^2\right]^2 - 10\left[\Pi^2\right][\Pi]^3 + [\Pi]^5 \equiv 0 \tag{3.47}$$

and we consider a diagonal basis. Then, $[\Pi^k] = \sum_{n=1}^{D}\lambda_n^k$ where λ_n are the eigenvalues of these matrices. Plugging this result into $\mathcal{L}^{(5)}_{\text{der}}$ gives us 0 identically. This is just to say that this is simply a statement that the higher order terms are just linear combinations of the lower order terms. This in turn implies that f_7 is redundant. This is to say that there are two free parameters left, which is why this class of theories is referred as a *two parameter family.*

The tensors X are identically conserved [7] [8] and can be written as

$$X_{\mu\nu}^{(n)} = \frac{1}{n+1}\frac{\delta}{\delta\Pi_{\mu\nu}}\mathcal{L}_{n+1}^{\mathrm{TD}} \tag{3.48}$$

It was later recognised that this can be written in a succinct way as

$$\begin{aligned}
X_{\mu'}^{(0)\mu}[\Pi] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu\alpha\beta} \\
X_{\mu'}^{(1)\mu}[\Pi] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha\beta}\Pi_{\nu}^{\nu'} \\
X_{\mu'}^{(2)\mu'}[\Pi] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha'\beta}\Pi_{\nu}^{\nu'}\Pi_{\alpha}^{\alpha'} \\
X_{\mu'}^{(3)\mu}[\Pi] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha'\beta'}\Pi_{\nu}^{\nu'}\Pi_{\alpha}^{\alpha'}\Pi_{\beta}^{\beta'} \\
X_{\mu'}^{(n\geq 4)\mu'[\Pi]} &= 0
\end{aligned} \tag{3.49}$$

or explicitly as

$$\begin{aligned}
X_{\mu\nu}^{(0)}[\Pi] =& 3!\,\eta_{\mu\nu} \\
X_{\mu\nu}^{(1)}[\Pi] =& 2!\,([\Pi]\eta_{\mu\nu} - \Pi_{\mu\nu}) \\
X_{\mu\nu}^{(2)}[\Pi] =& \left([\Pi]^2 - \left[\Pi^2\right]\right)\eta_{\mu\nu} - 2\left([\Pi]\Pi_{\mu\nu} - \Pi_{\mu\nu}^2\right) \\
X_{\mu\nu}^{(3)}[\Pi] =& \left([\Pi]^3 - 3[\Pi]\left[\Pi^2\right] + 2\left[\Pi^3\right]\right)\eta_{\mu\nu} \\
& - 3\left([\Pi]^2\Pi_{\mu\nu} - 2[\Pi]\Pi_{\mu\nu}^2 - \left[\Pi^2\right]\Pi_{\mu\nu} + 2\Pi_{\mu\nu}^3\right)
\end{aligned} \tag{3.50}$$

the former way of writing the X tensors is more intuitive as we see we are contracting Minkowski metrics with Π matrices in every possible way such that there are four contractions. This construction can be extended to Lagrangians [7]

$$\mathcal{L}_{mass} = \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu'\nu'\alpha'\beta'}\left(\underbrace{\frac{\alpha_2}{m^2}\delta_{\mu}^{\mu'}\delta_{\nu}^{\nu'}}_{*} + \underbrace{\frac{\alpha_3}{M_{\mathrm{Pl}}m^4}\delta_{\nu}^{\mu'}\Pi_{\nu}^{\nu'}}_{**} + \underbrace{\frac{\alpha_4}{M_{\mathrm{Pl}}^2m^6}\Pi_{\nu}^{\mu'}\Pi_{\nu}^{\nu'}}_{***} + \right)\Pi_{\alpha}^{\alpha'}\Pi_{\beta}^{\beta'} \tag{3.51}$$

term-wise, we see that [7]

$$* \sim \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu}{}^{\alpha'\beta'}\partial_{\alpha}\partial_{\alpha'}\pi\partial_{\beta}\partial_{\beta'}\pi = \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu}{}^{\alpha'\beta'}\partial_{\alpha}\partial_{\beta}\pi\partial_{\alpha'}\partial_{\beta'}\pi \tag{3.52}$$

up to a total derivative, where we have integrated by parts. We see the partial derivatives are symmetric, and we are antisymmetrising with the ϵ tensor, and so the whole term is a total derivative as this contribution is zero.

[7] Note the original paper [7] contains a typo and writes ν instead of μ in the δ of the first term

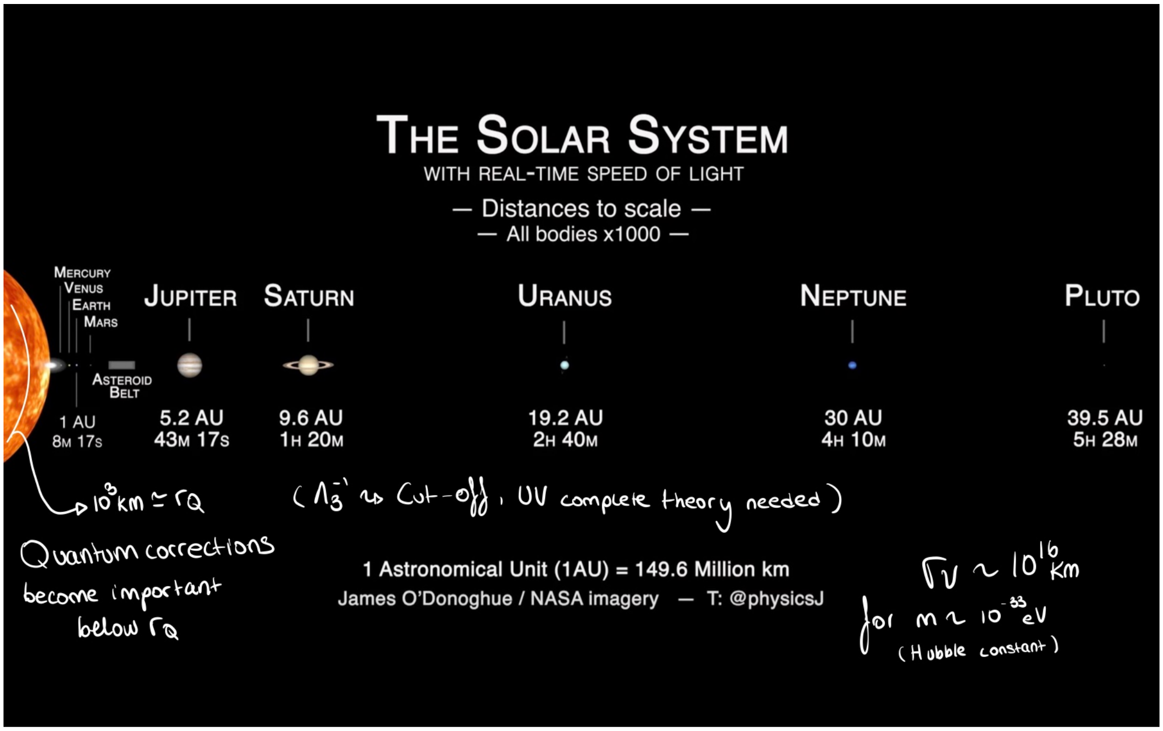

Figure 5: Regimes in massive gravity with a cutoff Λ_3. Image of solar system taken from James O'Donoghue. Here we see that the quantum regime appears at a "short" distance, while the linear theory appears at a solar system scale of 10^{16}km. For reference, the Milky way is approximately 10^{17}km, so this is around a tenth of its size

Similarly,

$$** \sim \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu}{}^{\nu'\alpha'\beta'}\partial_\nu\partial_{\nu'}\pi\partial_\alpha\partial_{\alpha'}\pi\partial_\beta\partial_{\beta'}\pi = -\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu}{}^{\nu'\alpha'\beta'}\partial_\nu\pi(\partial_{\nu'}(\partial_\alpha\partial_{\alpha'}\pi)\partial_\beta\partial_{\beta'}\pi+\partial_\alpha\partial_{\alpha'}\pi\partial_{\nu'}(\partial_\beta\partial_{\beta'}\pi)) = 0 \quad (3.53)$$

where, like before, we have integrated by parts and the remaining term is zero as we are anytisymetrising the prime indices and summing over them. It is easy to show that the remaining term (***) is also zero for the same reasoning (up to a total derivative). Thus, all these terms are total derivatives. This is also why all $\mathcal{L}^{(n)}_{\text{der}} \equiv 0$ for $n > 4$, as we would be antisymmetrising more than four matrices in $4D$, which is trivially zero. Having now removed the scalar self-interactions, we have now found a theory that has a scale $\Lambda_3 = (M_{pl}m^2)^{1/3}$ and the decoupling limit becomes

$$m \to 0, \quad M_{pl\to}\infty, \quad \Lambda_3 \quad \text{Fixed} \qquad (3.54)$$

3.3.2 □ root

In [9], the correct form for the potential so that there are no further terms $h(\partial^2\pi)^n$ beyond n=4 was found. Here, we work through the main results of this paper, giving explicit calculations. The breakthrough was to introduce a new tensor with a square root structure:

$$\mathcal{K}^\mu_\nu(g,H) = \delta^\mu_\nu - (\sqrt{g^{-1}\tilde{f}})^\mu_\nu = \delta^\mu_\nu - \sqrt{\delta^\mu_\nu - H^\mu_\nu} = \sum_{n=1}^{\infty} d_n (H^n)^\mu_\nu, \quad d_n = -\frac{(2n)!}{(1-2n)(n!)^2 4^n} \tag{3.55}$$

where d_n can be found from expanding the square root [8] and $\tilde{f}$ is the reference metric 3.9, and rearranging we get the equality. We also note that $\mathcal{K}$ can be written as

$$\begin{aligned}\mathcal{K}^\mu_\alpha \mathcal{K}^\alpha_\nu = (\delta^\mu_\alpha - \sqrt{\delta^\mu_\alpha - H^\mu_\alpha})(\delta^\alpha_\nu - \sqrt{\delta^\alpha_\nu - H^\alpha_\nu}) &= \delta^\mu_\nu - 2\sqrt{\delta^\mu_\nu - H^\mu_\nu}\delta^\mu_\nu - H^\mu_\nu \\ &= 2(\delta^\mu_\nu - \sqrt{\delta^\mu_\nu - H^\mu_\nu}) - H^\mu_\nu = 2\mathcal{K}^\mu_\nu - H^\mu_\nu\end{aligned} \tag{3.57}$$

Similarly, we have schematically

$$\mathcal{K}_{\mu\nu}|_{h_{\mu\nu=0}} = \eta_{\mu\nu} - \sqrt{(\eta - \Pi)^2}_{\mu\nu} = \Pi_{\mu\nu} \tag{3.58}$$

where a careful calculation shows this is indeed the case. Thus, this property allows us to extend the term found in 4.83, replacing $[\Pi] \leftrightarrow \langle\mathcal{K}\rangle$ where $\langle\mathcal{K}\rangle$ corresponds to taking the trace with the full metric;

$$\mathcal{L}^{(2)}_{\text{dRGT}}(\mathcal{K}) = \langle\mathcal{K}\rangle^2 - \langle\mathcal{K}^2\rangle \tag{3.59}$$

so

$$\mathcal{L}^{(2)}_{\text{dRGT}}(\mathcal{K})|_{h_\mu{}^\nu=0} = [\Pi]^2 - [\Pi^2] \tag{3.60}$$

and the full non-linear Lagrangian is

$$\mathcal{L} = \frac{M_{\text{Pl}}^2}{2}\sqrt{-g}\left(R - \frac{m^2}{4}\mathcal{U}(g,H)\right) \tag{3.61}$$

with

$$\begin{aligned}\mathcal{U}(g,H) &= -4\left(\langle\mathcal{K}\rangle^2 - \langle\mathcal{K}^2\rangle\right) \\ &= -4\left(\sum_{n\geq 1} d_n \langle H^n\rangle\right)^2 - 8\sum_{n\geq 2} d_n \langle H^n\rangle\end{aligned} \tag{3.62}$$

so varying this with respect to the variations we have

$$\frac{\delta\left(\sqrt{-g}\mathcal{L}^{(2)}_{\text{dRGT}}(\mathcal{K})\right)}{\delta h_{\mu\nu}}|_{h_\mu{}^\nu=0} = \left(\sqrt{-g}\left(2\langle\mathcal{K}\rangle\frac{\delta\langle\mathcal{K}\rangle}{\delta h_{\mu\nu}} - \frac{\delta\langle\mathcal{K}^2\rangle}{\delta\tilde{h}_{\mu\nu}}\right) + \left(\langle\mathcal{K}\rangle^2 - \langle\mathcal{K}^2\rangle\right)\frac{\delta\sqrt{-g}}{\delta h_{\mu\nu}}\right)|_{h_\mu{}^\nu=0} \tag{3.63}$$

---

[8]

$$\begin{aligned}\binom{1/2}{k} = \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-k+1)}{k!} &= (-1)^{k-1}\frac{1\times 3\dots\times(2k-3)}{2^k k!} \\ &= -(-1)^k\frac{2k-2!}{2k4^k(k-1)!} = -(-1)^k\frac{2k!}{(1-2k)4^k k!^2}\end{aligned} \tag{3.56}$$

as required

from [9] we have

$$\frac{\delta\langle\mathcal{K}\rangle}{\delta\tilde{h}_{\mu\nu}} = \frac{1}{2}\left(g^{\mu\nu} - \mathcal{K}^{\mu\nu}\right) \tag{3.64}$$

additionally, using 3.57 we have

$$\langle\mathcal{K}^2\rangle = 2\langle\mathcal{K}\rangle - \langle H\rangle \tag{3.65}$$

we easily find

$$\frac{\delta\left\langle\mathcal{K}^2\right\rangle}{\delta\tilde{h}_{\mu\nu}} = H^{\mu\nu} - \mathcal{K}^{\mu\nu} \tag{3.66}$$

and finally, we have [9]

$$\sqrt{-g} = \sqrt{-Det(\eta)}(1 + \frac{1}{2}h) \tag{3.68}$$

and so we can write

$$\begin{aligned}\frac{\delta\left(\sqrt{-g}\mathcal{L}^{(2)}_{\mathrm{dRGT}}(\mathcal{K})\right)}{\delta h_{\mu\nu}}|_{h_{\mu\nu}=0} &= (\eta^{\mu\nu} - \Pi^{\mu\nu})[\Pi] - (2\Pi^{\mu\nu} - \Pi^{\mu\alpha}\Pi^{\nu}_{\alpha} - \Pi^{\mu\nu}) + \frac{1}{2}\eta^{\mu\nu}([\Pi]^2 - (2[\Pi] - 2[\Pi] + [\Pi^2])))\\ &= (\eta^{\mu\nu} - \Pi^{\mu\nu})[\Pi] - \Pi^{\mu\nu} + \Pi^{\mu\alpha}\Pi^{\nu}_{\alpha} + \frac{1}{2}([\Pi]^2 - [\Pi^2])\end{aligned} \tag{3.69}$$

which we can rearrange as

$$\begin{aligned}\frac{\delta\left(\sqrt{-g}\mathcal{L}^{(2)}_{\mathrm{dRGT}}(\mathcal{K})\right)}{\delta h_{\mu\nu}}|_{h_{\mu\nu}=0} &= ([\Pi]\eta^{\mu\nu} - \Pi^{\mu\nu}) + \frac{1}{2}\Big(\left([\Pi]^2 - \left[\Pi^2\right]\right)\eta^{\mu\nu} - 2\left([\Pi]\Pi^{\mu\nu} - \Pi^{\mu\nu 2}\right)\Big)\\ &= \frac{1}{2}(X^{\mu\nu(1)}[\Pi] + X^{\mu\nu(2)}[\Pi]) \equiv \frac{2}{\Lambda_3^3}\bar{X}^{\mu\nu}\end{aligned} \tag{3.70}$$

Where $\hat{X}$ is conserved as it is a linear combination of conserved tensors. In the theory with all the constants, the terms of $\mathcal{O}(h^2)$ carry a factor of $\frac{\Lambda_3^3}{M_{pl}}$, which in the decoupling limit go to zero, meaning the Lagrangian in the decoupling limit is

$$\mathcal{L}^{\lim}_{\Lambda_3} = -\frac{1}{4}\hat{h}^{\mu\nu}(\hat{\mathcal{E}}\hat{h})_{\mu\nu} + \hat{h}_{\mu\nu}\bar{X}^{\mu\nu}. \tag{3.71}$$

This can be extended to contain the other terms in 4.83 to generalise the FP action following a very similar calculation, ensuring all terms appearing in the decoupling limit are total derivatives, and thus elucidating that the decoupling limit is set by Λ_3 and that all terms are healthy. An important point is that the X tensors are conserved off-shell without involving the equations of motion, which is one of the features that makes this Lagrangian structure so especial.

[9]

$$\begin{aligned}\sqrt{-g} &= \sqrt{-\eta - h} = e^{(log(\sqrt{-\eta-h}))} = e^{(1/2log(-Det(\eta)Det(1+\eta^{-1}h))}\\ &= \sqrt{-Det(\eta)}e^{(1/2log(Det(1+\eta^{-1}h)))} = \sqrt{-Det(\eta)}e^{(1/2Tr(log(1+\eta^{-1}h)))})\\ &= \sqrt{-Det(\eta)}e^{(1/2lTr(\eta^{-1}h)+\mathcal{O}(h^2))} = \sqrt{-Det(\eta)}(1 + \frac{1}{2}h)\end{aligned} \tag{3.67}$$

4 Extensions of gravity

In this section we do a quick review of the Einstein-Cartan formalism and see how it can be applied to obtain dRGT gravity elegantly, together with other standard results. The tetrad formalism can be found is standard texts such as [50] [51] or in most introductory courses in Differential Geometry, so the results will simply be stated. Additionally, some parts of this review can be found in the talk by R. Rosen [52].

4.0.1 Tetrads

The feature that made $\mathcal{K}$ so special was the square root structure, but there exist objects that do this for us and have been extensively researched; tetrads. These go by different aliases; tetrads, vierbeins, vielbeins... but they all refer to the same object:

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab} \tag{4.1}$$

we can also define e_a^μ such that $e^a{}_\mu e_b{}^\mu = \delta_a^b$ and $e^a{}_\nu e_a{}^\mu = \delta_\nu^\mu$ and similarly we can show that

$$g^{\mu\nu} = e_a{}^\mu e_b{}^\nu \eta^{ab}, \quad \eta_{ab} = g_{\mu\nu} e^\mu{}_a e^\nu{}_b, \quad \eta^{ab} = g^{\mu\nu} e_\mu{}^a e_\nu{}^b. \tag{4.2}$$

We see that the Latin indices do not interact with the Greek ones, and so we can build composite objects such as $e_{a\mu} = \eta_{ab} e^b{}_\mu$ etc. We can also write tetrads as one forms as

$$e^a = e^a{}_\mu dx^\mu \tag{4.3}$$

similarly, we can build vectors:

$$e_a = e_a{}^\mu \partial_\mu \tag{4.4}$$

meaning that the commutator of two dual basis tetrads does not necessarily commute:

$$[e_a, e_b] = C^d_{ab} e_d, \quad C^d_{ab} e_d = \left(-2 e^\mu_a e^\nu_b \partial_{[\mu} e^d_{\nu]}\right) e_d \tag{4.5}$$

where C^d_{ab} are called objects of anholonomy. It can also be shown that

$$de^a = -\frac{1}{2} C^a_{bc} e^b \wedge e^c. \tag{4.6}$$

One can build maps in concepts from Yang-Mills theory to the Einstein-Cartan formulation of gravity by replacing the gauge field with a connection and the field strength with a curvature.

To make sense how to perform this map, we first note that tetrads have 16 degrees of freedom as they are not required to be symmetric like the full metric is (in fact this would not even make sense as their indices are of different types as we will see soon). We also notice that under the Lorentz transform

$$e^a{}_\mu \to e'^a{}_\mu = \Lambda^a{}_b e^b{}_\mu, \tag{4.7}$$

the full metric transforms as

$$g_{\mu\nu} \to g'_{\mu\nu} = e'^a{}_\mu e'^b{}_\nu \eta_{ab} = \Lambda^a{}_c e^c{}_\mu \Lambda^b{}_d e^d{}_\mu \eta_{ab} = e^c{}_\mu e^d{}_\nu \eta_{cd} = g_{\mu\nu}, \tag{4.8}$$

and so the metric is invariant under Lorentz transforms of the tetrads. This means that we can use these to fix a further 6 degrees of freedom, recovering the 10 degrees of freedom typical from a symmetric $4D$ metric. We also see that Latin indices are Lorentz indices, and tetrads are a "dictionary" to move from Lorentz to spacetime indices and vice versa. As we will see, tetrads are more than a mathematical artefact, or curiosity, in that they are ubiquitous when generalizing terms in modified theories of GR, making some results much more intuitive. Additionally, we are forced to introduce them when dealing with spinors in curved space as these are defined as representations of the Poincaré group, and we lose this invariance in curved space-time, so tetrads allow us to set up Local Inertial Frames (LIFs) in which we can promote the Lorentz indices (for example in Dirac's equation) to full spacetime indices. This will be very important later on in this dissertation.

4.0.2 Cartan's structure equations

As mentioned earlier, we treat our connection Γ as the equivalent of a gauge field, and so we can write it as a one-form as

$$\Gamma^a{}_b = \Gamma^a{}_{\mu b} dx^\mu \tag{4.9}$$

requiring antisymmetry in the raised ab indices to satisfy the Lorentz algebra $SO(3,1)$:

$$\Gamma^{ab} = -\Gamma^{ba} \tag{4.10}$$

and we define the curvature two-form as

$$R^a_b = \frac{1}{2} R^a_{b\mu\nu} dx^\mu dx^\nu. \tag{4.11}$$

We can also define the torsion form

$$\hat{T}^a \equiv \frac{1}{2} T^a_{bc} \hat{e}^b \wedge \hat{e}^c \tag{4.12}$$

And finally

$$\Gamma^a_{\mu b} = e^a_\nu \nabla_\mu e^\nu_b \tag{4.13}$$

which can be shown to have the right transformation rules to behave like a connection. With this, one can show Cartan's structure equations:

$$De^a = de^a + \Gamma^a_{\ b} \wedge e^b \equiv T^a \tag{4.14}$$

which is Cartan's 1^{st} equation and

$$\mathcal{R}^a_b = d\Gamma^a_{\ b} + \Gamma^a_{\ c} \wedge \Gamma^c_{\ b} \tag{4.15}$$

which is Cartan's 2^{nd} equation. We can also define area forms as

$$\Sigma^{ab} = e^a \wedge e^b \tag{4.16}$$

which we see span planes when written as two-forms. There exist 6 distinct combinations of these. Similarly, we have the identity,

$$DDe^a = \mathcal{R}^b_a \wedge e^b \tag{4.17}$$

which is equivalent to the way that the Riemann tensor is defined in GR using the commutator of covariant derivatives. The fact that the connection is antisymmetric in its Lorentz indices as a result of being a Lorentz generator has the implication that the connection is also a metric connection:

$$\nabla_\rho g_{\mu\nu} = 0 \leftrightarrow \Gamma^{(ab)} = 0. \tag{4.18}$$

We also have that the torsion free condition associated with the Levi-Civita connection is

$$T^a = 0 = De^a = de^a + \Gamma^a_{\ b} \wedge e^b \tag{4.19}$$

which looks more like the requirement of metricity written with the covariant derivative, and vice versa. Lastly, the Bianchi identity can be translated to simply

$$D\mathcal{R}^{ab} = 0 \tag{4.20}$$

4.0.3 Lagrangians using tetrads.

As a word of caution, we will be using the notation

$$\varepsilon^{\mu\nu\alpha\beta} = \frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta}, \quad \varepsilon_{\mu\nu\alpha\beta} = \sqrt{-g}\epsilon_{\mu\nu\alpha\beta}, \quad \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu_1\nu_1\alpha_1\beta_1} = \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu_1\nu_1\alpha_1\beta_1} \tag{4.21}$$

throughout, with

$$\epsilon_{0123} = 1 = -\epsilon^{0123} \tag{4.22}$$

We can now write the Einstein Hilbert Lagrangian in terms of tetrads a s

$$\mathcal{S}_{EC} = \int e^a \wedge e^b \wedge \mathcal{R}^{cd}\varepsilon_{abcd} = \int \frac{1}{2}\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu R^{cd}{}_{\alpha\beta}dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta \tag{4.23}$$

and since there is only one possible combination of $dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta$ we can write.

$$\mathcal{S}_{EC} = -\frac{1}{2}\int \mathrm{d}^4x \quad \varepsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\gamma e^d{}_\delta R^{\gamma\delta}{}_{\alpha\beta} \tag{4.24}$$

and using

$$\epsilon_{abcd}e^a{}_\mu e^b e^c{}_\gamma e^d{}_\delta = -\varepsilon_{\mu\nu\gamma\delta}Det(e) = -\varepsilon_{\mu\nu\gamma\delta}\sqrt{-g} \tag{4.25}$$

then we have

$$\mathcal{S}_{EC} = \int \mathrm{d}^4x\sqrt{-g}(\delta^\alpha_\gamma\delta^\beta_\delta - \delta^\alpha_\delta\delta^\beta_\gamma)R^{\gamma\delta}{}_{\alpha\beta} = 2\int \mathrm{d}^4x\sqrt{-g}R \sim \mathcal{S}_{EH}. \tag{4.26}$$

This action has the important property of being Lorentz invariant [52]: under the transformation of $e^a \to \Lambda^a{}_b e^b$, $\mathcal{R}^{ab} \to \Lambda^a{}_c\Lambda^c{}_d\mathcal{R}^{cd}$ we see that since $\Lambda \in SO(3,1) \implies Det(\Lambda = 1)$ and so the term is Lorentz invariant. The Diffeomorphism invariance comes from the transformation $e^a \to e^a + D\phi^a$, $\mathcal{R}^{ab} \to \mathcal{R}^{ab}$ where ϕ^a is built out of four zero forms (scalars). We see that

$$\delta(\mathcal{S}_{EC}) \sim \int D\phi^a \wedge e^b \wedge \mathcal{R}^{cd}\varepsilon_{abcd} = 0 \tag{4.27}$$

via integration by parts[10] using the torsion free condition, together with the Bianchi identity.

Similarly, it can be shown that one can obtain the cosmological constant term by simply wedging four tetrads together:

$$S_\Lambda \sim \int e^a \wedge e^b \wedge e^c \wedge e^d \tag{4.28}$$

We can also write the Gauss-Bonnet term as,

$$\mathcal{S}_{GB} = \int \mathcal{R}^{ab} \wedge \mathcal{R}^{cd}\varepsilon_{abcd} \tag{4.29}$$

which in this language it is clear that it is zero in four dimensions due to the Bianchi identity. Similarly, terms such as

$$\mathcal{S} \sim \int R^{ab} \wedge R_{ab}, \quad \int R^{ab} \wedge e_a \wedge e_b ... \tag{4.30}$$

are also topological terms, for the first one we have a simple Bianchi identity trick, while for the second one, we can use the graded Leibniz rule to show it is identically zero.

[10] D only ever hits Lorentz indices, so if all are contracted, $D \to d$ and so we can integrate by parts using this.

The equations of motion can be obtained using the "1^{st} order form" which involves treating the spin-connection as an independent variable with respect to the metric, but this ends up being equivalent to setting the torsion to zero from the get-go, which is the "2^{nd} order form". in first order form,

$$\delta\mathcal{R}^{ab} = d\delta\Gamma^{ab} + \delta\Gamma^{ac} \wedge \Gamma_c^b + \Gamma^{ac} \wedge \delta\Gamma_c^b = D\delta\Gamma^{ab} \tag{4.31}$$

and so

$$\delta\mathcal{S}_{EH} = \int D\delta\Gamma^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd} = 0 \tag{4.32}$$

which for a variation of the action, which implies that the equations of motion are,

$$De^c \wedge e^d \varepsilon_{abcd} = 0 \implies De^a = 0 \tag{4.33}$$

on the other hand, varying with respect to the spin connection yields

$$\delta\mathcal{S}_{EH} \sim \int \mathcal{R}^{ab} \wedge e^c \wedge \delta e^d \varepsilon_{abcd} \implies \mathcal{R}^{ab} \wedge e^c \varepsilon_{abcd} = 0 \tag{4.34}$$

which can be manipulated to obtain Einstein's equations.

4.0.4 dRGT using vierbeins

We review the results in [53] where dRGT gravity was translated into the vierbein language, where some of its properties become more apparent. We start by rewriting the dRGT Lagrangian as [10] [54]

$$\mathcal{S} = \frac{M_P^{D-2}}{2} \int d^D x \sqrt{-g} \left[R - \frac{m^2}{4} \sum_{n=0}^{D} \beta_n S_n \left(\sqrt{g^{-1}\eta} \right) \right] \tag{4.35}$$

as this will be most convenient to us. To begin translating this term, we write [15]

$$e_{1,a}^{\mu} e_{\alpha}^{2,a} e_{1,b}^{\alpha} e_{\nu}^{2,b} = e_{1,a}^{\mu} e_{\alpha,b}^{2} e_1^{\alpha,a} e_{\nu}^{2,b} = g^{\mu\alpha} f_{\alpha\nu} = g^{-1} f \tag{4.36}$$

where the subscripts 1 and 2 refer to the tetrads corresponding to the full metric and the reference metric respectively, which will typically be Minkowski. We can then write

$$\sqrt{g^{-1} f} = (e_2 e_1^{-1})^T. \tag{4.37}$$

So now we can write

$$(Det(e_1)) S_n((e_2 e_1^{-1})^T) = \sqrt{-g} S_n \left(\sqrt{g^{-1}\eta} \right) \tag{4.38}$$

for a flat metric the vierbeins are $\mathbb{I}^a = \delta^a{}_\mu dx^\mu$ so the action can be written as

$$\mathcal{S} = \frac{M_{lP}^2}{2}\left(\int e^a \wedge e^b \wedge \mathcal{R}^{cd}\varepsilon_{abcd} \right. \\ \left. -\frac{m^2}{4}\int\sum_{n=0}^{4}\frac{\beta_n}{n!\,(4-n)!}\varepsilon_{a_1a_2a_3a_4}\mathbb{I}^{a_1}\wedge\ldots\wedge\mathbb{I}^{a_n}\wedge e^{A_{n+1}}\wedge\ldots\wedge e^{a_4}\right) \tag{4.39}$$

even though this Lagrangian has 16 constraints, it can be restricted to the normal 10 constraints coming from a symmetric metric in 4 dimensions if we add the constraint [53]

$$e\eta = \eta e^T \tag{4.40}$$

so we can write the vierbeins as this constrained vierbein times a generic Lorentz transform:

$$e = \bar{e}exp(w) \tag{4.41}$$

which as usual gives us additional constraints. Similarly to this, one can build bi-gravity theories by adding a curvature term for a second metric, $f_{\mu\nu}$:

$$\mathcal{S}_{bi} = \frac{M_{lP}^2}{2}\left(\int\frac{M_{plg}^2}{2}e^a_{(1)}\wedge e^b_{(1)}\wedge\mathcal{R}^{cd}_{(1)}\varepsilon_{abcd} \right. \\ +\frac{M_{plf}^2}{2}e^a_{(2)}\wedge e^b_{(2)}\wedge\mathcal{R}^{cd}_{(2)}\varepsilon_{abcd} \\ \left. -\frac{M_{plgf}^2}{2}\frac{m^2}{4}\int\sum_{n=0}^{4}\frac{\beta_n}{n!\,(4-n)!}\varepsilon_{a_1a_2a_3a_4}e_1^{a_1}\wedge\ldots\wedge e_1^{a_n}\wedge e_2^{a_{n+1}}\wedge\ldots\wedge e_2^{a_4}\right) \tag{4.42}$$

4.1 Kaluza-Klein theories

In this section we look at Kaluza-Klein (KK) theories, which have the remarkable property of describing a matter-rich four dimensional space with both gravity and Yang Mills, starting from a completely empty five dimensional space, with minimal coupling to the five dimensional Ricci scalar. We will follow the notes on this topic that can be found in [55].

4.1.1 Features in Kaluza-Klein theories

The following features are common to KK models, though at least one of the following might at times be relaxed,

1. All nature is pure geometry. All phenomena in $4D$ results purely from the Einstein tensor in $5D$, no stress-energy tensor.

2. Minimal extensions of GR; we simply add one more number to the set of numbers indices may take. No Lovelock terms or interactions.

3. A priori cylindrical, physics does not depend on this fifth dimension.

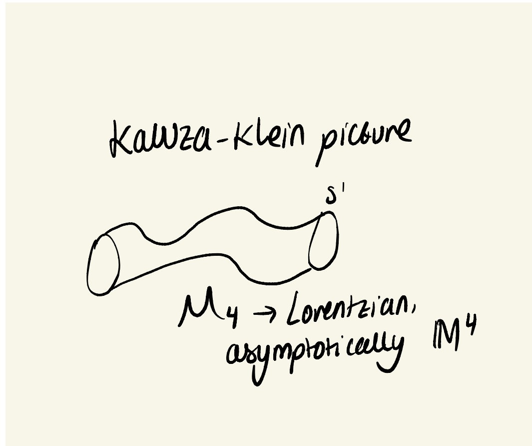

Figure 6: Idea behind KK theories

While the first two conditions appear reasonable and soothing to physicists, the third one seems more convoluted. Klein [56] [57] showed that if the fifth coordinate had a circular topology, one could expand any function in terms of Fourier modes along this coordinate and so if the excited states were heavy enough to not be excited by any conventional physics, the equations would be independent of this fifth coordinate.

4.1.2 D=11 Supergravity and D=10 Superstring theory

KK theories in higher dimensions only give rise to $4D$ gauge bosons, so to include fermionic fields too, these must be put in by hand. D=11 is a natural setting for these theories, as Nahm [58] showed this is the maximum number of dimensions that give rise to a single graviton. Additionally, Witten showed [59] that to unify all forces, that is to have a model with the $SU(3) \times SU(2) \times U(1)$ symmetry such as the standard model, we need a minimum of 11 dimensions. This tremendously constrain the number of dimensions that are available for us to choose, though one may want theories with, for example, more than one graviton. What is more, E.Cremmer, B.Julia, and J.Scherk [60] showed that, unlike in lower dimensions, only one configuration was consistent with supersymmetry for the extra matter fields. To further reinforce the idea that D=11 was the right theory to study, Freund Rubin showed that one could only compactify the d=11 model in either 4 macroscopic dimensions and 7 microscopic dimensions, or vice versa. This was compelling enough for D=11 supergravity to be taken seriously. Unfortunately, several imperfections developed from this theory, firstly, quarks and leptons did not arise from the manifolds Witten had proposed, and the groups SO(7) and $SO(2) \times SO(5)$ were used as replacements [55]. This could be modified by adding more matter, at the expense of our first principle of KK theories. A more concerning one was the fact that chirality could not be recovered in these theories. Additionally, it is difficult to remove the cosmological constant arising in these theories, which is of sizeable dimensions [55]. Lastly, Salam and Sezgin [61] showed that the quantised theory suffers from anomalies. Bleak as the situation might have looked at first, Green, Schwarz, Gross, et al. showed that in D=10 superstring theory there exist two and only two models that cure the anomalies, restoring the uniqueness virtue. These were theories with gauge groups SO(32) and $E_8 x E_8$. These originally comprised five different theories, but Witten [62] showed that they could be unified into what we know as M-theory. Additionally, the low energy limit of this theory is D=11 Supergravity, just as one might have wished for.

4.1.3 Kaluza mechanism

We follow [55] for part of this subsection, but we work out the equations of motion using the method presented in [63] working through the results shown from scratch with the explicit calculations. Einstein's equations in $5D$ are

$$\hat{G}_{AB} = 0 \implies \hat{R}_{AB} = 0 \tag{4.43}$$

where the indices A,B now run from 0 to 4. Similarly, the 5 dimensional Einstein action is

$$S = -\frac{1}{16\pi} \int \sqrt{-g}^{(5)} \hat{R} \mathrm{d}^4 x \mathrm{d}y \tag{4.44}$$

The Christoffel symbols and Ricci tensors are defined analogously to 4 dimensional theory. We can now be imaginative choosing our metric, but the specific choice

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} + \kappa^2 \phi^2 A_\alpha A_\beta & \kappa\phi^2 A_\alpha \\ \kappa\phi^2 A_\beta & \phi^2 \end{pmatrix} \tag{4.45}$$

where A is the EM vector potential and ϕ is a scalar field. We can alternatively write a metric such as

$$ds^2 = e^{2\beta\phi} \gamma_{ij} dx^i dx^j + e^{2\alpha\phi} \left(dx^5 + A_i dx^i\right)^2 \tag{4.46}$$

where now ϕ is a new field. This has invariance under the transformation $A_i \to A_i + \partial_i \Lambda$ and $x^5 \to x^5 - \Lambda$, similar to the transformations performed when dealing with the Stuckelberg trick and in EM, a gauge transform will be the same as a coordinate transformation in $5D$. One can find a tetrad basis in $5D$ defining

$$ds^2 = e^{2\beta\phi} \gamma_{ij} dx^i dx^j + e^{2\alpha\phi} \left(dx^5 + A_i dx^i\right)^2 \tag{4.47}$$

and regard γ_{ij} as the $4D$ metric, which can be decomposed into a normal tetrad basis, further defining $E_i = e^{\beta\phi} e^i$ so we can write the full metric as a sum of tensor products of the tetrads:

$$ds^2 = E^5 \otimes E^5 + \eta_{ij} E^i \otimes E^j \tag{4.48}$$

which we can spell out to ensure it is right as

$$E^5 \otimes E^5 = e^{2\alpha\phi}(dx^5 \otimes dx^5 + 2dx^5 \otimes A + A \otimes A) \tag{4.49}$$

which corresponds with our previous metric. We now proceed to calculate connection one-forms. Since we are working with a theory that we would like to resemble GR in the $4D$ limit, we can set torsion to zero. Firstly, we can calculate

$$\begin{aligned} dE^i &= \beta d\phi e^{\beta\phi} \wedge e^i + e^{\beta\phi} de^i \\ &= \beta d\phi e^{\beta\phi} \wedge e^i - e^{\beta\phi}\Gamma^i_j \wedge e^j \end{aligned} \tag{4.50}$$

where we have used $T^i = de^i + \Gamma^i_j \wedge e^j = 0$ in the second line. Similarly,

$$dE^5 = \alpha d\phi \wedge e^{\alpha\phi}\left(dx^5 + A\right) + e^{\alpha\phi} dA \tag{4.51}$$

which remembering that

$$F = \frac{1}{2}F_{ij}e^i \wedge e^j = \frac{1}{2}\left(\partial_i A_j - \partial_j A_i\right) e^i \wedge e^j \tag{4.52}$$

then writing $d\phi = \partial_i \phi e^i$ and remembering our definitions for $E^i and E^5$ we have

$$\begin{aligned} dE^i &= \beta e^{-\beta\phi}\partial_j\phi E^j \wedge E^i - \hat{\Gamma}^i{}_j \wedge E^j \\ dE^5 &= \alpha e^{-\beta\phi}\partial_j\phi E^j \wedge E^5 + \frac{1}{2}e^{(\alpha-2\beta)\phi}\left(F_{ij}E^i \wedge E^j\right) \end{aligned} \tag{4.53}$$

where $\hat{\Gamma}$ refers to the $4D$ connection one-form. With this, one can find the connection one-form components as follows;

$$dE^5 = -\Gamma^5_i \wedge E^i \quad \Longrightarrow (-\alpha e^{-\beta\phi}\partial_i\phi E^5 \wedge E^i - \frac{1}{2}E^{(\alpha-2\beta)\phi}F_{ij}E^j \wedge E^i) = -\Gamma^5_i \wedge E^i \tag{4.54}$$

and so, one can remove one of the tetrads to obtain

$$\alpha e^{-\beta\phi}\partial_i\phi E^5 + \frac{1}{2}e^{(\alpha-2\beta)\phi}F_{ij}E^j = \Gamma^5_i \tag{4.55}$$

one can similarly obtain

$$dE^i = -\Gamma^i_j \wedge E^j - \Gamma^i_5 \wedge E^5 = -\Gamma^i_j \wedge E^j + \frac{1}{2}e^{(\alpha-2\beta)\phi}F^i_j E^j \wedge E^5 \tag{4.56}$$

$$\Gamma^i_j = \hat{\Gamma}^i{}_j - \beta e^{-\beta\phi}\left(\partial^i\phi E_j - \partial_j\phi E^i\right) - \frac{1}{2}e^{(\alpha-2\beta)\phi}F^i{}_j E^5 \tag{4.57}$$

where the first part of Γ involves a E^5 and so becomes zero when we wedge it with itself. Now using $dE^i = \beta e^{-\beta\phi}\partial_j\phi E^j \wedge E^i - \hat{\Gamma}^i{}_j \wedge E^j$ we can repackage 4.57into

$$\omega^i{}_j = \hat{\omega}^i{}_j - \beta e^{-\beta\phi}\left(\partial^i\phi E_j - \partial_j\phi E^i\right) - \frac{1}{2}e^{(\alpha-2\beta)\phi}F^i{}_j E^5 \tag{4.58}$$

We can now proceed to calculate the curvature two-form using these results:

$$R^z_i = d\Gamma^z_i + \Gamma^z_\rho \wedge \Gamma^\rho_i \tag{4.59}$$

tackling the first part of this expression we have

$$\begin{aligned} d\Gamma^z_i = &\alpha\partial_i\partial_j\phi e^{-2\beta\phi}E^j \wedge E^5 - \beta\alpha\partial_i\phi\partial_j^{-2\beta\phi}E^j \wedge E^5 \\ &+ \alpha^2\partial_i\phi e^- - 2\beta\phi\partial_j\phi E^j \wedge E^5 + \frac{1}{2}(\alpha-3\beta)e^{(\alpha-3\beta)\phi}\partial_k\phi F_{ij}E^k \wedge E^j \\ &+ \frac{1}{2}e^{(\alpha-3\beta)\phi}\partial_k F_{ij}E^k \wedge E^j + \frac{1}{2}e^{(\alpha-2\beta)\phi}F_{ij}(\beta e^{-\beta\phi}\partial_k\phi E^k \wedge E^j - \hat{\Gamma}^j_k \wedge E^k) \end{aligned} \tag{4.60}$$

from which we can collect terms as

$$\begin{aligned} &= E^j \wedge E^5(\alpha\partial_i\phi\partial_j\phi e^{-2\beta\phi} - \beta\alpha\partial_i\phi\partial_j\phi e^{-2\beta\phi} + \alpha^2\partial_i\phi\partial_j\phi e^{-2\beta\phi}) + \\ &e^{(\alpha-3\beta)\phi}E^k \wedge E^j(\alpha\partial_i\phi F_{kj} + \frac{1}{2}(\alpha-3\beta)F_{i[j)}\partial_{k]}\phi - \frac{1}{2}\partial_{[k}F_{j]i} + \frac{1}{2}\beta F_{i[j}\partial_{k]}\phi \\ &- \frac{1}{2}e^{(\alpha-2\beta)\phi}F_{ij}\hat{\Gamma}^j_k \wedge E^k \end{aligned} \tag{4.61}$$

similarly, for the second component we find

$$\begin{aligned} \Gamma^z_\rho \wedge \Gamma^\rho_i = &(\alpha\partial_j\phi e^{-\beta\phi}E^5 + \frac{1}{2}e^{(\alpha-2\beta)\phi}F_{jk}E^k) \wedge (\hat{\Gamma}^j_i - \beta e^{-\beta\phi}(\partial^j\phi E_i - \partial^j_i) - \frac{1}{2}e^{\alpha-2\beta}\phi F^j_i E^5) \\ &= \alpha\partial_j\phi e^{-\beta\phi}E^5 \wedge \hat{\Gamma} + E^j \wedge E^5(\beta\alpha\partial_k\phi e^{-2\beta\phi}\partial^k\phi\eta_{ji} - \beta\alpha e^{-2\beta\phi}\partial_j\phi\partial_i\phi - \frac{1}{4}e^{-2(\alpha-2\beta)\phi}F_{kj}F^k{}_i) \\ &+ \frac{1}{2}e^{(\alpha-2\beta)\phi}F_{jk}E^k \wedge \hat{\Gamma}^j_i \end{aligned} \tag{4.62}$$

we can put all these together to finally obtain

$$\begin{aligned} R^z_i = &e^{-2\beta\phi}\left(\alpha(\alpha-2\beta)\partial_i\phi\partial_j\phi + \alpha\partial_j\partial_i\phi + \eta_{ij}\alpha\beta\partial_k\phi\partial^k\phi - \frac{1}{4}e^{2(\alpha-\beta)\phi}F_{kj}F^k_i\right)E^j \wedge E^z \\ &+ e^{(\alpha-3\beta)\phi}\left(\frac{1}{2}(\alpha-\beta)\partial_i\phi F_{kj} - \frac{1}{2}(\alpha-\beta)\partial_{[k}\phi F_{j]i} - \frac{1}{2}\partial_{[k}F_{j]i} + \frac{1}{2}\beta\eta_{i[j}F_{k]l}\partial^l\phi\right)E^k \wedge E^j \\ &- \frac{1}{2}e^{(\alpha-2\beta)\phi}\left(F_{ij}\hat{\Gamma}^j{}_k \wedge E^k + F_{jk}\hat{\Gamma}^j{}_i \wedge E^k\right) + \alpha\partial_j\phi e^{-\beta\phi}E^z \wedge \hat{\Gamma}^j{}_i \end{aligned} \tag{4.63}$$

A similarly tedious calculation yields

$$\begin{aligned}
R^i_j =& r^i_j + E^z \wedge E^k e^{(\alpha-3\beta)\phi} \Big((\alpha-\beta) F^i_j \partial_k \phi + \frac{1}{2}\partial_k F^i_j - \frac{1}{2}(\alpha-\beta)\left(\partial^i \phi F_{jk} - F^i_k \partial_j \phi\right) \\
&+ \frac{1}{2}\beta\left(\partial_n \phi F^n_j \delta^i{}_k + F^i{}_l \partial^l \phi \eta_{jk}\right)\Big) + E^k \wedge E^l e^{-2\beta\phi}\left(\beta\left(\partial_j \partial_{[k}\phi \delta^i{}_{l]} - \partial_{[k}\partial^i \phi \eta_{l]j}\right)\right. \\
&- \frac{1}{4}e^{2(\alpha-\beta)\phi}\left(F^i_j{}_j F_{kl} + F^i{}_{[k|}F_{j|l]}\right) + \beta^2\left.\left(\partial^i \phi \partial_{[k}\phi \eta_{l]j} - \partial_p \phi \partial^p \phi \delta^i{}_{kk}\eta_{l]j} - \partial_j \phi \partial_{[k}\phi \delta^i_{l]}\right)\right) \\
&+ \beta e^{-\beta\phi}\left(\partial_k \phi E^i \wedge \hat{\Gamma}^k{}_j - \partial^k \phi \hat{\Gamma}^i{}_k \wedge E_j\right) - \frac{1}{2}\beta e^{(\alpha-2\beta)\phi}\left(F^k_j \hat{\Gamma}^i_k \wedge E^z - F^i_k \hat{\Gamma}^k{}_j \wedge E^z\right)
\end{aligned} \tag{4.64}$$

where r^i_j is the usual $4D$ Riemann tensor in tetrad language. One can use these two results to work out the Ricci scalar;

$$R = R^{\alpha\beta}{}_{\alpha\beta} = R^{55}{}_{55} + R^{5i}{}_{5i} + R^{ij}{}_{ij} + R^{i5}{}_{i5} = 2R^{5i}{}_{5i} + R^{ij}{}_{ij} \tag{4.65}$$

Firstly, for R^{i5}_{i5} we have

$$R^{5i}{}_{5i} = -e^{2\beta\phi}(\alpha(\alpha-2\beta)(\nabla\phi)^2 + \alpha\Box\phi + d\alpha\beta(\nabla\phi)^2 - \frac{1}{4}e^{(\alpha-\beta)\phi}F^2) + \alpha\partial_j\phi e^{-\beta\phi}\hat{\Gamma}^{ij}_i \tag{4.66}$$

meanwhile,

$$\begin{aligned}
R^{ij}{}_{ij} =& re^{-2\beta\phi} + 2e^{-2\beta\phi}\Big[\frac{\beta}{2}(1-d)\Box\phi - \frac{1}{4}e^{2(\alpha-\beta)\phi}\frac{F^2}{2} + (\nabla\phi)^2(\frac{d-1}{2} - \frac{d(d-1)}{2} - \frac{1-d}{2})\Big] \\
&+ 2(d-1)\beta e^{-\beta\phi}\partial_k\phi\hat{\Gamma}^{jk}_j
\end{aligned} \tag{4.67}$$

which yields

$$\begin{aligned}
R =& e^{-2\beta\phi}r - 2e^{-2\beta\phi}(\nabla\phi)^2\left(\alpha^2 + (d-2)\alpha\beta + \frac{1}{2}(d-2)(d-1)\beta^2\right) - 2e^{-2\beta\phi}\Box\phi(\alpha + \beta(d-1)) \\
&+ \frac{1}{4}e^{(2\alpha-4\beta)\phi}F^2 + 2e^{-\beta\phi}\partial_k\phi\hat{\Gamma}^{ik}{}_i(\alpha + \beta(d-1)).
\end{aligned} \tag{4.68}$$

we can remove the α parameter by fixing it as $\alpha = -(d-1)\beta$ which gets rid of the connection factor in the Ricci tensor, which can be resummed to

$$R = e^{-2\beta\phi}r - e^{-2\beta\phi}(d-1)(d+2)\beta^2(\nabla\phi)^2 + \frac{1}{4}e^{(2\alpha-4\beta)\phi}F^2 \tag{4.69}$$

additionally, using $g = e^{2(\alpha+d\beta)\phi}\det\gamma$ we can simplify this expression to

$$\sqrt{-g}R = \sqrt{\gamma}e^{-\beta\phi}\left(r(\gamma) + \frac{1}{4}e^{-\beta(2d+1)\phi}F^2 - (\nabla\phi)^2(d-1)(d+2)\beta^2\right) \tag{4.70}$$

which through a simple field redefinition we may write as

$$\sqrt{-g}R = \sqrt{\gamma}e^{-\beta\phi}\left(r(\gamma) + \frac{1}{4}e^{-\beta(2d+1)\phi}F^2 - (\nabla\phi)^2\right) \tag{4.71}$$

Which we can see looks like the Lagrangian density for a $4D$ GR theory with YM and a scalar field.

$$\begin{aligned} G_{\alpha\beta} &= \frac{\kappa^2\phi^2}{2}T^{EM}_{\alpha\beta} - \frac{1}{\phi}\left[\nabla_\alpha\left(\partial_\beta\phi\right) - g_{\alpha\beta}\Box\phi\right], \\ \nabla^\alpha F_{\alpha\beta} &= -3\frac{\partial^\alpha\phi}{\phi}F_{\alpha\beta}, \quad \Box\phi = \frac{\kappa^2\phi^3}{4}F_{\alpha\beta}F^{\alpha\beta}, \end{aligned} \tag{4.72}$$

where $G_{\alpha\beta}$ is the usual $4D$ Einstein tensor, $T^{EM}_{\alpha\beta} \equiv g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}/4 - F^\gamma_\alpha F_{\gamma\beta}$. Setting ϕ=0 recovers the Einstein equations coupled to EM, and Maxwell's equations. However, this suffers from the issue that the fourth equation can only be satisfied if $F_{\alpha\beta}F^{\alpha\beta} = 0$. In spite of this issue, the remarkable takeaway is that one can build electromagnetism in $4D$ out of pure geometry in one more dimension when the right metric is used.

4.1.4 Mach's principle and Brans-Dicke theory

This model arose when considering Mach's principle; Newtonian mechanics had three basic absolute pillars ; time, space, and motion [64]. Nonetheless, Mach would criticize this view, ascertaining that "No one is competent to predicate things about absolute space and absolute motion; they are pure things of thought, pure mental constructs, that cannot be produced in experience", holding the view that it is through the synergies between physical objects that we develop our theories. It is this principle that guided Einstein to the principles of relativity, which would ultimately demolish two of Newton's absolute pillars; time and space are no longer absolute. Nonetheless, there was still one pillar standing: motion. To argue why motion must be absolute, Newton came up with the following thought experiment: Take a bucket of water and rotate it. Due to the "centrifugal" force, the water gets pushed against the walls of the bucket and rises on the edges, causing the water surface to become curved. It is easy to see why the height of the water would go as roughly the inverse of the gravitational force; in an extremely heavy gravitational field the water would be pushed downwards so vigorously that high revolutions per second would be needed to exert enough force to resist gravity and rise as high as it would have risen in a weak field. Conversely, a small gravitational field would have the opposite effect, and taking the limit as we go in the vacuum, where no gravitational field is present, we would see the height of the water rise to infinity, in other words, it would all eventually leave the bucket. Careful analysis of this basic problem shows this heuristic view to be correct. In turn, this implies that one can tell the difference between a bucket that is not in motion, and one that is by simply looking at the shape of the water. Nonetheless, this falls prey to Mach's principle, in that this system is intertwined with earth; you cannot fail to account for the effect that being in Earth's gravitational field would have since this is a phenomenon mediated by gravity, and so to solve this, one should decouple gravity from the problem. A corollary of this problem is that of skaters in a circular rink. As the rink starts spinning, you will tangentially slide towards the edge of the rink and hit the wall. On the other hand, for a bird's-eye point of view, you would be rotating while the rest of the world would be stationary. You would see the stars rotating as you spun in circles and feel the "centrifugal force" pushing you against these walls. Alternatively, one might imagine the whole world rotating together with the stars. This is, physically, quite a different picture, and indeed, we would expect the observer in the rink to remain stationary. These views are, therefore, not equivalent, and Einstein was adamant that Mach's principle, which is that the small scale dynamics are determined by the large scale mass distribution. This can in fact be seen in GR when considering rotating black holes; the Kerr metric demands that the ϕ coordinate changes, meaning that we are forced to rotate along the black hole if we do not accelerate in the opposite dimension. This can be thought of as the black hole twisting the fabric of space and dragging it while doing so, which results in the name "frame-dragging". Newton offered a further example, of two masses linked together by a piece of rope (or spring, as this is more visual). As the objects started spinning, the rope (spring) would stretch, meaning that by measuring the distance it has stretched, one would be able to tell whether one finds oneself in either a rotating frame or not. Mach's response to this argument is that one would need to first measure the unstretched length of the rope (spring) in a laboratory, where you know that the spring is in an inertial frame, as there would be no other way to tell what this length is in an empty vacuum as you could not check whether the spring was rotating. Thus, to draw any conclusions we would need knowledge from an inertial frame elsewhere, meaning that one still has to take into account the "greater picture". Because of this, this experiment also fails to be proof of an experiment that shows absolute motion as it requires from an external system. There seems to be a field communicating information about the matter distribution in the universe, a transmitter of causality, which to avoid coordinate dependent issues, should also be a scalar field, coupled in some way to gravity. Similarly, in the 1930s, Dirac pointed out that $\frac{GM}{R} \sim 1$ where M and R are the masses of the universe respectively. While this observation would not immediately bear a great significance, Dicke observed that if written as $\frac{1}{G}\frac{M}{R} = \rho$ we could treat $\frac{1}{G} = \Box\phi$ where ϕ is a scalar field, leading to a theory where the gravitational constant is treated as an extra field. Brans considered the Lagrangian

$$\mathcal{S} \int \sqrt{-g}\left(R\phi + 16\pi\mathcal{L}_{\text{matter}} + \mathcal{L}_{\phi}\left(\phi, \phi_{,\mu}\right)\right) \mathrm{d}^4 x \tag{4.73}$$

on dimensional grounds, we require two derivatives and a factor of ϕ, leading to the trial Lagrangian [13] [55]

$$S_{BD} = \int \mathrm{d}^4 x \sqrt{-g}\left(R\phi + \omega\frac{\partial^{\alpha}\phi\partial_{\alpha}\phi}{\phi}\right) + S_m \tag{4.74}$$

setting $\omega = 0$ recovers the action we would get if we wrote the Kaluza-Klein model out in components and let A=0. This attempt to specify one of the fundamental constants of nature, however, introduces yet another constant that one must experimentally measure ω and cannot be predicted theoretically, meaning that the efforts of introducing ϕ to remove a constant simply introduce another one. Nonetheless, what this theory does allow us to do is to explain the gravitational constant as the value of a field in a position and at a time. Varying the equation of motion with respect to ϕ yields

$$\Box\phi - \frac{\partial_\mu\phi\partial^\mu\phi}{2\phi} + \frac{\phi}{2\omega}R = 0 \tag{4.75}$$

while doing so with respect to the metric yields

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{\omega}{\phi^2}(\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g_{\mu\nu}\phi\partial^\alpha\partial_\alpha\phi) - \frac{1}{\phi}(\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi) = \frac{8\pi}{\phi}T'_{\mu\nu} \tag{4.76}$$

where T' is the usual stress energy tensor but now containing no matter. This equation can be rearranged as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}) + \frac{1}{\phi}(\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi). \tag{19}$$

While all we have done has been to change which side of the equation the scalar terms appeared, conceptually there is a great difference between the two equations. We know how GR beautifully discerns between geometry and matter, and relates them on both sides of Einstein's equation. Here, we have the dilemma of whether the scalar field ϕ should be seen as matter or as geometry, which henceforth changes some assumptions made, such as restricting the sign of the energy term of the field to be positive. Taking the trace of this equation yields

$$-R = \frac{8\pi}{\phi}T_{\mu\nu} - \frac{\omega}{\phi^2}(\phi_{,\mu}\phi^{,\mu}) - \frac{3}{\phi}\Box\phi \tag{4.77}$$

plugging back yields

$$\Box\phi = \frac{8\pi}{2\omega+3}T \tag{4.78}$$

which now resembles the $\frac{1}{G} = \rho = \Box\phi$ equation. Conformal rescaling While the Lagrangian arising from dividing $5D$ in 4+1D resembles the gravitational potential, the ϕ field either implies that we do not get a minimal coupling to gravity, or that the ϕ field is not dynamical; many authors set it to one canonically, so it is constant through spacetime, and we can achieve minimal coupling. Nonetheless, we can mend this issue by introducing a Weyl or conformal rescaling

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{4.79}$$

using $\omega^2 = \phi^{-1/3}$ we have for the Brans-Dicke action with A=0. This gives a Lagrangian

$$S' = \int \mathrm{d}^4x\sqrt{-g'}\left(\frac{R'}{16\pi G} + \frac{1}{6\kappa^2}\frac{\partial'^\alpha\phi\partial'_\alpha\phi}{\phi^2}\right) \tag{4.80}$$

and transforming to the dilaton field $\sigma \equiv \frac{ln(\phi)}{\sqrt{3}\kappa}$ we might rewrite it as

$$S' = \int \mathrm{d}^4x\sqrt{-g'}\left(\frac{R'}{16\pi G} + \frac{1}{2}\partial'^\alpha\sigma\partial'_\alpha\sigma\right). \tag{4.81}$$

The original unrescaled metric is referred to as the Jordan metric, while the one after the conformal transformation is dubbed the Pauli metric, as he was the first to question whether we should consider the original or the rescaled metric as the "real" metric. It has also been suggested [65] [55] that normal matter could couple to the Jordan metric, while dark matter couples to the rescaled metric. As we can see, different variations of KK theories bring about exciting and thought-provoking frameworks for new physics to be explored, at least theoretically.

4.1.5 Compactification of the extra dimension

The assumption that no physics depends on the fifth dimension as put forward by Kaluza is a rather peculiar and puzzling one. What is the point of this extra fifth dimension, if for all practical purposes it is as though it did not exist? Klein came up with an explanation for this under two assumptions,

1. The extra dimension had a circular topology.
2. The scale of this manifold was to be small (e.g. a circle with a small radius).

Property 1 implies that we can expand quantities in the fifth dimension using Fourier modes as they will be periodic; $f(x,y) = f(x, y+2\pi r)$ where r is the radius of this extra dimension

$$\begin{aligned} g_{\alpha\beta}(x,y) &= \sum_{n=-\infty}^{n=\infty} g^{(n)}_{\alpha\beta}(x)e^{iny/r} \quad , \quad A_\alpha(x,y) = \sum_{n=-\infty}^{n=\infty} A^{(n)}_\alpha(x)e^{iny/r}, \\ \phi(x,y) &= \sum_{n=-\infty}^{n=\infty} \phi^{(n)}e^{iny/r}, \end{aligned} \tag{4.82}$$

one can quantise the momentum in the y direction on the order of $|n|\ r$ (akin to the angular momentum quantisation of the hydrogen atom). If the radius of this dimension is small enough, it will be impossible to excite any mode other than n=0 as the energy will be too large, and this mode is independent of the y coordinate, meaning we would recover the desired y coordinate independence for the low energy effective theory. The empirical evidence constraints this radius to be of a very small scale, smaller than 10^{-18} [66] [55] although it is often assumed to be of Planck scale. This mechanism, however, gives rise to a natural way of charge quantisation, and actually gives a "close" estimate of the fine structure constant, something that Dirac often emphasised was imperative of any GUT. One can imagine a matter field in five dimensions, minimally coupled to the metric as

$$S_{\hat\psi} = -\int \mathrm{d}^4x dy \sqrt{-\hat g}\partial^A\hat\psi\partial_A\hat\psi \tag{4.83}$$

and then one could expand the field $\hat\psi$ in Fourier modes as

$$\hat\psi(x,y) = \sum_{n=-\infty}^{n=\infty} \hat\psi^{(n)}e^{iny/r} \tag{4.84}$$

one can invert the metric to obtain

$$\hat{g}^{\alpha\beta} = \phi^{1/3} \begin{bmatrix} g^{\alpha\beta} & -\kappa A^{\beta} \\ -\kappa A^{\alpha} & \phi^{-1} + \kappa^2 A^2 \end{bmatrix} \tag{4.85}$$

plugging this into 4.83 we obtain

$$S_{\hat{\psi}} = -\left(\int dy\right) \sum_n \int \mathrm{d}^4x\sqrt{-g}\Big[\left(\partial^{\alpha} + \frac{in\kappa A^{\alpha}}{r}\right)\hat{\psi}^{(n)}\left(\partial_{\alpha} + \frac{in\kappa A_{\alpha}}{r}\right)\hat{\psi}^{(n)} - \frac{n^2}{\phi r^2}\hat{\psi}^{(n)2}\Big] \tag{4.86}$$

which we see looks like a charged particle in electrodynamics, together with a mass term, which we can read off as

$$q_n = \frac{n\kappa}{r}\left(\phi\int dy\right)^{-1/2} = \frac{n\sqrt{16\pi G}}{r\sqrt{\phi}} \tag{4.87}$$

where we have used $\kappa = 4\sqrt{\pi G}$ and $G = \frac{\hat{G}}{\int \mathrm{d}y}$ Additionally, the mass can be read off from the Lagrangian as $\frac{|n|}{\sqrt{\phi}r}$. If we take $\sqrt{\phi}r \sim l_p$ where l_p is the Plank mass. The problem with this theory is that it predicts that for the electron (n=1) we will have a mass of $\sim 10^{19}GeV$ while the actual value is $0.5MeV$ [55]. This can be solved by setting n=0 so the Kaluza mechanism give electrons zero mass, and the small mass observed is explained by the Higgs mechanism and charge can be given by extending the gauge group in higher dimensions so that the massless representations are no longer singlets of the gauge group [55]. This can be seen directly more easily by looking at the KG equation [63];

$$\left[\Box_{D+1} - m^2\right]\phi = \left[\Box_D + \left(\frac{\partial}{\partial y}\right)^2 - m^2\right]\phi = 0 \tag{4.88}$$

as we have just seen, we can expand the solutions in Fourier modes, resulting in infinite towers of states with different masses corresponding to the n^{th} Fourier mode. This means we can rewrite 4.88 as

$$\left[\Box_D + \left(\frac{n}{r}\right)^2 - m^2\right]\phi = 0 \tag{4.89}$$

4.1.6 Dimensional reduction in spinors

We want to extend the results we just found to more fields. For this, we consider the option of a Spinor field as vector fields are straightforward to derive. We know our field should follow the Dirac equation for a Spin-$\frac{1}{2}$ field;

$$[\partial\!\!\!/_D - m]\Psi = 0 \tag{4.90}$$

Additionally, we can redefine our field as $\tilde{\Psi} = e^{-\beta\gamma_*}\Psi$ where γ_* acts as the equivalent of γ^5 in $4D$ but in D dimensions. First we note that since $\Gamma_*^2 = \mathbb{I}$ we can expand $e^{\beta\gamma_*} = \mathbb{I}cos\beta + i\gamma_* sin\beta$ and noting that γ_* anticommutes with all other matrices we find this is equivalent to

$$[\partial\!\!\!/_D(\mathbb{I}cos\beta - i\gamma_* sin\beta) - m(\mathbb{I}cos\beta + i\gamma_* sin\beta)]\Psi = 0 \tag{4.91}$$

and extracting a factor of $e^{-\beta\gamma_*}\Psi$ we are left with

$$\left[\partial\!\!\!/_D - m\left(\mathbb{I}\cos 2\beta + \mathrm{i}\gamma_*\sin 2\beta\right)\right]\tilde{\Psi} = 0 \tag{4.92}$$

we see that by simply choosing $\beta = \frac{\pi}{2}$ we can change the sign of the mass term without affecting the equations of motion! [67] This will also be important when realising that the fermionic field doesn't have to be 2π periodic as we can have $\Psi \to -\Psi$ but the observables will always come as bilinear quantities [67] such as $T^{00} = -\bar{\Psi}\gamma^0\partial^0\Psi$ which is 2π periodic. That means that our Fourier expansion will change slightly to

$$\Psi\left(x^\mu, y\right) = \sum_k \mathrm{e}^{\mathrm{i}ky/L}\Psi_k\left(x^\mu\right) \tag{4.93}$$

where k are now half-integers. Plugging this back into the original Dirac equation, we find

$$\left[\partial\!\!\!/_D - \left(m - \mathrm{i}\gamma_*\frac{k}{L}\right)\right]\Psi_k\left(x^\mu\right) = 0 \tag{4.94}$$

We thus see that we simply have an extra mass term for the Dirac spinor, with a γ_* contribution. Through this section, we have seen how the power of KK theories; one can almost miraculously obtain a theory with matter and geometry in $4D$ starting from a theory with only geometry in $5D$. The power of KK theories doesn't stop there, expanding the extra dimension in modes means that one can make a prediction, albeit not an accurate one, of the fine structure constant from a theoretical standpoint. Dirac would famously ask anyone brave enough to claim they have a theory of everything whether their theory could predict the value of the fine structure constant theoretically, going as far as calling this "the most fundamental unsolved problem of physics". One can't help but wonder what his reaction would be if he knew this still stands as one of the mysteries in physics, with new, perhaps even shocking mysteries such as the eery fine-tuning of the cosmological constant. In addition to this, KK theories also build infinite towers (or truncated, depending on the energy threshold) of massive states for different particles, and obtains this mass with a process other than the Higgs mechanism, though, as we will shortly see this could perhaps be seen as the discretisation, and thus symmetry breaking of a fifth dimension. However, KK theories make several wrong predictions such as the mass of electrons, and so, as they stand, need to be modified, or further explored to be able to accurately describe the world we live in.

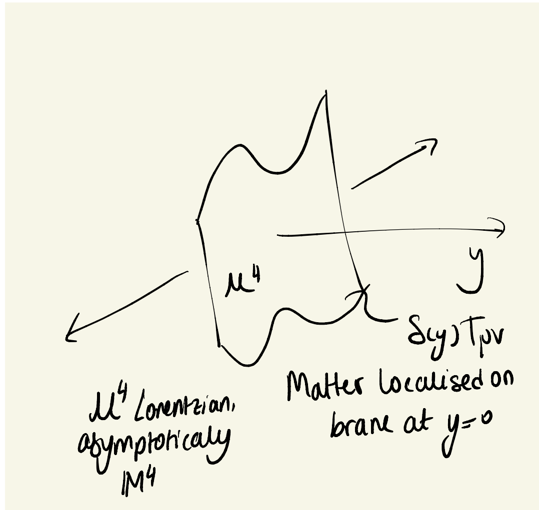

Figure 7: In the DGP set-up a brane containing matter partitions the continuous fifth dimension into two equal halves.

5 Dvali–Gabadadze–Porrati (DGP) model

A section on extensions of gravity would not be complete without a short review of DGP models [68] [69], which are a beautiful construction of gravity in 4+1D where matter is localised in a 3+1D brane. Their initial aim was to find an explanation for cosmic acceleration without introducing dark energy. As we will see, this model has very particular properties, including a Källén–Lehmann spectral representation for the mass of the graviton. The idea behind the model is that the curvature induced by the matter in the 3+1D brane would make gravity appear $4D$ at short distances, but on cosmic scales we would see gravity leak through this extra dimension, diminishing its strength and therefore making it seem as though there was an accelerating force. We will follow the cited papers and [7] through this section.

The set-up for this model, as previously anticipated, consists of a 3+1D (4D here onwards) brane embedded in $5D$. the location in the 5^{th} dimension can be set at y=0, with a symmetry between the regions in the extra dimension where $y > 0$ and $y < 0$. This extra dimension, contrary to the case in KK theories, is non-compact and extends infinitely. The Lagrangian for this model is

$$S = \int \mathrm{d}^4x \; \mathrm{d}y \left(\frac{M_5^3}{4} \sqrt{-^{(5)}g}\,^{(5)}R + \delta(y) \left[\sqrt{-g} \frac{M_{\mathrm{Pl}}^2}{2} R[g] + \mathcal{L}_m \left(g, \psi_i\right) \right] \right) \tag{5.1}$$

where M_5 is the 5 dimensional Plank scale and ψ_i are the matter fields living in our $4D$ brane. We see the equations of motion are

$$M_5^{3(5)} G_{AB} = 2\delta(y)^{(5)} T_{AB} \tag{5.2}$$

with

$$^{(5)}T_{AB} = \left(-M_{\mathrm{Pl}}^2 G_{\mu\nu} + T_{\mu\nu} \right) \delta_A^\mu \delta_B^\nu \tag{5.3}$$

so we see there is no matter for $y \neq 0$ and regular GR in the $4D$ brane. The metric takes the same form as the linearised metric and the de Donder gauge in $5D$ is also imposed ;

$$\partial_A h^A{}_B = \frac{1}{2}\partial_B h^A{}_A \tag{5.4}$$

In analogy to the $4D$ case 2.34 (and since these equations were derived without reference to dimensions) we have

$$^{(5)}G_{AB} = -\frac{1}{2}\Box_5\left(h_{AB} - \frac{1}{2}h^C_C\eta_{AB}\right) \tag{5.5}$$

with $\Box_5 = \Box + \partial_y^2$. Additionally, as we have no source along the y direction, this imposes $^{(5)}T_{\mu y} = {}^{(5)}T_{yy} = 0$ Taking the μy component of the equations we have that $\Box_5(h_{\mu y}) = 0$ as $\eta_{\mu y} = 0$. This means that $h_m u_y$ satisfies the wave equation in $5D$, and we can set it to zero up to a homogeneous solution. Similarly, for the yy component we find that $\Box_5(h_{yy} - \frac{1}{2}(h + h_{yy})) = 0$ so $\Box_5(h_{yy} - h) = 0$ where h is the trace of over the $4D$ spin-2 field $h_{\mu\nu}$. This in turn implies that $h_{yy} - h = 0$ up to a homogeneous solution.

In $5D$ de Donder gauge, choosing $B = \mu$ we have $\partial_y h^y_\mu + \partial_\nu h^\nu{}_\mu = \frac{1}{2}\partial_\mu(h + h_{yy}) \implies \partial_\nu h^\nu{}_\mu = \partial_\mu h$ due to the relations just derived. From this we can immediately see that $\Box h = h^{\mu\nu}{}_{,\mu\nu}$. Thus, the Einstein tensor in $4D$ becomes

$$G_{\mu\nu} = \frac{1}{2}(2h^\alpha_{(\mu,\nu)\alpha} - \Box h_{\mu\nu} - h_{,\mu\nu} - \eta_{\mu\nu}(h^{\alpha\beta}{}_{,\alpha\beta} - \Box h)) = \frac{1}{2}(2h_{,\mu\nu} - \Box h_{\mu\nu} - h_{,\mu\nu}) = \frac{1}{2}(h_{,\mu\nu} - \Box h_{\mu\nu}) \tag{5.6}$$

and so now it is easy to see that the $\mu\nu$ components of5.2 become

$$\frac{1}{2}M_5^3\left[\Box + \partial_y^2\right](h_{\mu\nu} - h\eta_{\mu\nu}) = -\delta(y)\left(2T_{\mu\nu} + M_{\mathrm{Pl}}^2\left(\Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right)\right) \tag{5.7}$$

which admits a solution

$$h_{\mu\nu}(x,y) = e^{-|y|\sqrt{-\Box}}h_{\mu\nu}(x) \tag{5.8}$$

as we require this solution to tend to zero as $y \to \infty$. It is clear this is a solution, as when $y \neq 0$ the left-hand side vanishes as $\left[\Box + \partial_y^2\right] \to [\Box - \Box] = 0$ and the right-hand side vanishes due to the delta function. When $y = 0$, the left-hand side vanishes all the same, but the right-hand side satisfies Einstein's equations on shell, so it's equal to zero. Integrating over y from $y = -\epsilon$ to $y = \epsilon$ sets y=0 on the right hand side and we must use Israel [70] [7] [71] matching conditions on the right-hand side which lead to $\partial_y(h_{\mu\nu} - h\eta_{\mu\nu})\Big|_{-\epsilon}^{\epsilon} \to -2\sqrt{\Box}(h_{\mu\nu} - h\eta_{\mu\nu})$ meaning that we can rewrite 5.7 as [11]

$$M_{\mathrm{Pl}}^2\left[(\Box h_{\mu\nu} - \partial_\mu\partial_\nu h) - m_0\sqrt{-\Box}\left(h_{\mu\nu} - h\eta_{\mu\nu}\right)\right] = -2T_{\mu\nu} \tag{5.9}$$

[11] For a brane embedded in a higher dimensional space such as in DGP models, the Israel matching conditions reduce to the derivative in the extra dimension along the boundary being equal to the extrinsic curvature of the brane

where we define $m_0 = M_5^3/M_{Pl}^2$. Additionally, the FP massive gravity emerges in the form of $(h_{\mu\nu} - h\eta_{\mu\nu})$. Tracing over 5.9 we find

$$M_{pl}^2[\Box h - \Box h - m_0\sqrt{-\Box}(h - 4h)] = -2T \implies h = \frac{-2T}{3M_{Pl}^2 m_0\sqrt{-\Box}} \tag{5.10}$$

and so we can rewrite 5.9 as

$$M_{pl}^2[\Box - m_0\sqrt{-\Box}]h_{\mu\nu} = -2T_{\mu\nu} - \frac{2}{3m_0\sqrt{-\Box}}T_{,\mu\nu} + \frac{2}{3}T\eta_{\mu\nu} \tag{5.11}$$

so we can finally write our field as

$$h_{\mu\nu} = -\frac{2}{M_{\mathrm{Pl}}^2}\frac{1}{\Box - m_0\sqrt{-\Box}}\left(T_{\mu\nu} - \frac{1}{3}T\eta_{\mu\nu} + \frac{1}{3m\sqrt{-\Box}}T_{,\mu\nu}\right) \tag{5.12}$$

which we recognise as the same relation as that for the FP theory, but with $m^2 = m_0\sqrt{-\Box}$. However, we see that now our mass is not determined but takes a range of different values as it involves the D'Alambertian operator. This gives rise to a Källén–Lehmann spectral representation instead of the definite single mass for the graviton that we had before for FP models.

6 Dimensional deconstruction in dRGT

We have seen that gravity models can be written in terms of tetrads in a natural and elegant way that allow us to see the properties that different theories might have in an intuitive sense. We have also seen that we can construct higher dimensional theories with just geometry that give rise to theories in lower dimensions with both matter and geometry. This begs the question, can we relate these two concepts in any way? As it turns out, dRGT gravity emerges naturally when discretising the fifth dimension. In what follows, we will deviate from what was considered in KK style theories, and treat the fifth dimension as a flat extra dimension (not wrapped).This technique was pioneered by Arkani-Hamed et al. in the early 2000s [14] [72] where Moose diagrams (nowadays quivers) were used to show how a four dimensional theory can dynamically generate theories that look five dimensional at bigger scales. Later, it was also shown by in [14] how dimensional deconstruction could be used in the context of gravity to recover FP-like mass terms [14], and additionally, how adding some interactions could raise the cut-off scale to Λ_3, as we saw in previous sections. We will follow the prescription given in [15] and [73].

6.1 Curvature tensor

To begin we consider the extrinsic curvature, typically given by $K_{\mu\nu} = \frac{1}{2}n_\rho g^{\rho\sigma}\partial_\sigma g_{\mu\nu}$ which in the present case reduces to $K_{\mu\nu} = \frac{1}{2}\partial_y g_{\mu\nu}$. We now replace the new dimension by two sites at y_1 and y_2 such that $g_{\mu\nu}(y_1) = g_{\mu\nu}$ and $g_{\mu\nu}(y_2) = f_{\mu\nu}$. If f is dynamical, the result will be bigravity [15] [48].Introducing this reference metric endows the theory with an extra 6 degrees of freedom coming from the Lorentz invariance that f enjoys, meaning we can use these to gauge fix certain quantities as we shall see, in particular we will fix $\Omega_y^{ab} = 0$ By approximating the derivative as $K_{\mu\nu} = m(f_{\mu\nu} - g_{\mu\nu})$ we introduce the well known BD ghost. What we can instead do is to discretise the tetrads as opposed to the metric in the following way;

$$K_{\mu\nu} = \frac{1}{2}\partial_y g_{\mu\nu} = \frac{1}{2}\left(e^a_\mu\left(\partial_y e^b_\nu\right)\eta_{ab} + \left(\partial_y e^a_\mu\right)e^b_\nu\eta_{ab}\right) \tag{6.1}$$

discretising the tetrads as $\partial_y e^a_\mu \to m(e^{2,a}_\mu - e^{1,a}_\mu)$ the curvature transforms to

$$\begin{aligned} K_{\mu\nu} \to \mathcal{K}_{\mu\nu} &= \frac{m}{2}\left(e^{1,a}_\mu\left(e^{2,b}_\nu - e^{1,b}_\nu\right)\eta_{ab} + e^{1,a}_\nu\left(e^{2,b}_\mu - e^{1,b}_\mu\right)\eta_{ab}\right) \\ &= -m\left(g_{\mu\nu} - \frac{1}{2}\left(e^{1,a}_\mu e^{2,b}_\nu + e^{1,a}_\nu e^{2,b}_\mu\right)\eta_{ab}\right). \end{aligned} \tag{6.2}$$

Using the usual definition of the metric in terms of tetrads and remembering that at the first site the metric is g. Additionally, working in a torsion free environment in the y direction, we have $T^A = de^A + \Gamma^A{}_B \wedge e^B = 0$. As previously anticipated, we can use the remaining Lorentz symmetry to fix the spin connection $\Gamma^{ab}_y = e^{a\mu}\partial_y e^b_\mu - e^{b\mu}\partial_y e^a_\mu = 0$ [15]. Focusing on one site and discretising the tetrads we see, $\Gamma^{ab}_y = \frac{1}{2}(e^{1,a\mu}(e^{2,b}_\mu - e^{1,b}_\mu) - e^{1,b\mu}(e^{2,a}_\mu - e^{1,a}_\mu)) = 0$ this in turn implies $e^{1,a\mu}e^{2,b}{}_\mu = e^{1,b\mu}e^{2,a}{}_\mu$. Additionally, the torsion free condition implies the Deser-van

Nieuvenhuizen condition $e^{1,a}_{\mu} e^{2,b}_{\nu} \eta_{ab} = e^{1,a}_{\nu} e^{2,b}_{\mu} \eta_{ab}$ [15] [74]. With this we then have $\mathcal{K}_{\mu\nu} = -m(g_{\mu\nu} - e^{1,a}_{\mu} e^{2,b}_{\nu} \eta_{ab})$ so we can write $\mathcal{K}^{\mu}_{\nu} = -m(\delta^{\mu}_{\nu} - e^{1,\mu}_{a} e^{2,a}_{\nu})$ we also notice that $g^{\mu\alpha} f_{\alpha\nu} = e^{1\mu}{}_{a} e^{1,\alpha}{}_{b} \eta^{ab} \eta_{cd} e^{2,c}{}_{\alpha} e^{2,d}{}_{\nu} = e^{1\mu b} e^{1\alpha}_{b} e^{2}{}_{\alpha c} e^{2,d}{}_{\nu}$ from this we can compute

$$\mathcal{K}^{2\mu}{}_{\nu} = m^2(\delta^{\mu}_{\nu} - 2e^{1,\mu}_{a} e^{2,a}_{\nu} + g^{\mu\alpha} f_{\alpha\nu}) \tag{6.3}$$

which means that $\mathcal{K}^{\mu}_{\nu} = -m(\delta^{\mu}_{\nu} - \sqrt{g^{-1}f}^{\mu}{}_{\nu})$, which surprisingly corresponds to the $\mathcal{K}$ of dRGT.

6.2 Generating dRGT gravities from $5D$ gravity

Since the discretisation procedure transforms the extrinsic curvature into the dRGT tensor $\mathcal{K}$, it is straightforward to build new theories of gravity:

$$S^{5d}_{\mathrm{ADM}} = \frac{M_5^3}{2} y \, \mathrm{d}^4 x \sqrt{-g} \left({}^{(4)}R[g] + [K]^2 - \left[K^2\right] \right) \tag{6.4}$$

now discretising the vielbeins yields as previously seen,

$$\begin{gathered} y\mathcal{L}(x,y) \longrightarrow m^{-1} \mathcal{L}\left(x, y_1\right) \\ K^{\mu}_{\nu} \longrightarrow m\mathcal{K}^{\mu}_{\nu}(g,f) \end{gathered} \tag{6.5}$$

so

$$S^{4d} = \frac{M_{\mathrm{P}}^2}{2} \int \mathrm{d}^4 x \sqrt{-g} \left({}^{(4)}R[g] + m^2 \left([\mathcal{K}]^2 - \left[\mathcal{K}^2\right] \right) \right) \tag{6.6}$$

and by replacing the equivalent terms to those in the dRGT potential one can translate the $5D$ theory into a $4D$ one.

6.3 GB term interactions

We can now start looking at the interplay between intrinsic curvature given by the Riemann tensors, and extrinsic curvature, given by the dRGT tensors $\mathcal{K}$ by looking at the different combinations of these terms. First we have a look at the Gauss-Bonnet term introduced earlier, allowing the indices to run from 0 to 4 as usual in $5D$ theories.

$$
\begin{aligned}
S_{\rm GB} =& \frac{M_5^3}{m^2}{}^5 x\sqrt{-g}\left({}^{(5)}R^2_{ABCD} - 4^{(5)}R^2_{AB} + {}^{(5)}R^2\right)\\
=&\frac{M_5^3}{4m^2}{}^5 x\sqrt{-g}\mathcal{E}^{\mu\nu\alpha\beta}\mathcal{E}^{\mu'\nu'\alpha'\beta'}\Big[R_{\mu\nu\mu'\nu'}R_{\alpha\beta\alpha'\beta'} - \frac{1}{12}K_{\mu\mu'}K_{\nu\nu'}K_{\alpha\alpha'}K_{\beta\beta'} \\
&+K_{\mu\mu'}K_{\nu\nu'}R_{\alpha\beta\alpha'\beta'}]
\end{aligned} \tag{6.7}
$$

The first term is the $4D$ GB term, while the others are new interactions. In fact, one rewrite this using the dual of the Riemann tensor

$$
{}^\star R^{\mu\nu\mu'\nu'} = \mathcal{E}^{\mu\nu\alpha\beta}\mathcal{E}^{\mu'\nu'\alpha'\beta'}R_{\alpha\beta\alpha'\beta'} \tag{6.8}
$$

and further adding the shift $\mathcal{K}_{\mu\nu} \to g_{\mu\nu} + \mathcal{K}_{\mu\nu}$ we can obtain yet another interaction term

$$
S^{4d}_{\mathcal{K}G} = -\frac{M^2_{\rm Pl}}{4}\int \mathrm{d}^4x\sqrt{-g}g_{\mu\nu}\mathcal{K}_{\alpha\beta}{}^\star R^{\mu\alpha\nu\beta} \tag{6.9}
$$

which we can explicitly compute as

$$
\begin{aligned}
S^{4d}_{\mathcal{K}G} =& -\frac{M^2_{\rm Pl}}{4}\int \mathrm{d}^4x\sqrt{-g}\Big[\delta^{\alpha\beta}[\delta^{\mu'\nu'}\delta^{\alpha'\beta'} - \delta^{\mu'\beta'}\delta^{\nu'\alpha'}]\\
&\qquad - \delta^{\alpha\nu'}[\delta^{\mu'\beta}\delta^{\alpha'\beta'} - \delta^{\mu'\beta'}\delta^{\beta'\alpha'}]\\
&\qquad + \delta^{\alpha\beta'}[\delta^{\mu'\beta}\delta^{\alpha'\nu} - \delta^{\mu'\nu'}\delta^{\beta\alpha'}]\Big]\mathcal{K}_{\alpha\beta}R_{\mu'\alpha'\nu'\beta'}\\
=& -\frac{M^2_{\rm Pl}}{4}\int \mathrm{d}^4x\sqrt{-g}\Big[\mathcal{K}[R^{\mu'\alpha'}{}_{\mu'\alpha'} - R^{\mu'\alpha'}{}_{\alpha'\mu'}]\\
&\qquad - \mathcal{K}^{\mu'\nu'}R_{\mu'}{}^{\nu'}{}_{\nu'\alpha'} + \mathcal{K}^{\alpha'\nu'}R^{\beta'}{}_{\alpha'\nu'\beta'} + \mathcal{K}^{\mu'\beta'}R_{\mu'}{}^{\nu'}{}_{\nu'\beta'} - \mathcal{K}^{\beta'\alpha'}R^{\nu'}{}_{\alpha\nu'\beta'}\Big]\\
=& -\frac{M^2_{\rm Pl}}{4}\int \mathrm{d}^4x\sqrt{-g}\Big[-4\mathcal{K}^{\mu\nu}[R_{\mu\nu} -_{\mu\nu}\frac{R}{2}]\Big]\\
=& -\frac{M^2_{\rm Pl}}{4}\int \mathrm{d}^4x\sqrt{-g}\Big[-4\mathcal{K}^{\mu\nu}G_{\mu\nu}\Big]
\end{aligned} \tag{6.10}
$$

one can analyse this kind of theory in minisuperspace [12] as

$$
g_{00} = -N^2(t), \quad g_{0i} = 0, \quad g_{ij} = a^2(t)\delta_{ij} \tag{6.11}
$$

Then in this metric we have $g = -N^2a^6 \implies \sqrt{-g} = Na^3$ and since $g^{-1} = Diag(-\frac{1}{N^2}, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^2})$ we have $\sqrt{g^-1f} = Diag(\frac{1}{N}, \frac{1}{a}, \frac{1}{a}, \frac{1}{a})$ from this we can easily work out

$$
\mathcal{K}^{\mu\nu} = Diag(\frac{1}{N^3} + N, -\frac{1}{a^3} + a, -\frac{1}{a^3} + a, -\frac{1}{a^3} + a) \tag{6.12}
$$

[12]Minisuperspace is an approximation used in theories with infinite dimensional phase spaces to cut off all modes larger than the size of the observable universe.

while the Ricci tensor is, defining $\kappa = \frac{aa'N'+N\left(-5a'^2+2aa''\right)}{4a^3}$

$$R_{\mu\nu} = Diag\Big(\frac{3\left(aa\prime N' + N\left(-3a'^2 + 2aa''\right)\right)}{4a^2N}, \kappa, \kappa, \kappa\Big) \tag{6.13}$$

from this, one can calculate the Einstein tensor and work out that

$$\mathcal{K}^{\mu\nu}G_{\mu\nu}Na^3 = \frac{3aa'^2}{N^2} + \frac{3a'^2}{N} - \frac{6aa'^2}{N} - \frac{6aa'N'}{N^2} + \frac{6a^2a'N'}{N^2} + \frac{6aa''}{N} - \frac{6a^2a''}{N} \tag{6.14}$$

remembering this is inside of an integral, meaning we can integrate by parts, and doing so in the last term yields

$$\mathcal{K}^{\mu\nu}G_{\mu\nu}Na^3 = \frac{3aa'^2}{N^2} + \frac{3a'^2}{N} + \frac{6aa'^2}{N} - \frac{6aa'N'}{N^2} + \frac{6aa''}{N} \tag{6.15}$$

and integrating by parts again yields

$$\mathcal{K}^{\mu\nu}G_{\mu\nu}Na^3 = \frac{3aa'^2}{N^2} - \frac{3a'^2}{N} + \frac{6aa'^2}{N} \tag{6.16}$$

yielding

$$S^{4d}_{\mathcal{K}G} = -\frac{M^2_{\rm Pl}}{4}\int \mathrm{d}^4x\Big(\frac{3aa'^2}{N^2} - \frac{3a'^2}{N} + \frac{6aa'^2}{N}\Big) \tag{6.17}$$

in agreement with [73]. This is an issue as the Lapse appears non-lineary, and one of the conditions for the theory to propagate 5 degrees of freedom is that the Hessian's ($L_{\mu\nu} = \frac{\delta\mathcal{H}}{\delta N^\mu \delta N^\nu}$) determinant vanishes [73], which requires the Lagrangian to depend on the variables linearly, which this theory fails to display. A similar calculation for the $\mathcal{K}\mathcal{K}$ term yields [73]

$$S_{\mathcal{K}\mathcal{K}^\star R} = 24\int \mathrm{d}t\ \mathrm{d}^3x a^3N\left(\frac{\dot{a}^2}{a^2N^2} - \frac{\dot{a}^2}{a^3N^2} + \frac{\dot{a}^2}{a^2N^3} - \frac{\dot{a}^2}{a^3N^3}\right) \tag{6.18}$$

which suffers from the same pathology. In [73] it was shown that no other kinetic terms are possible for massive gravity, further reinforcing dRGT as an extremely special theory. As we will see, the mathematical curiosities don't end up here, as we will find dRGT models also appeart when discretising extra dimensions.

6.4 dRGT dimensional deconstruction in tetrad language

This section is inspired by [15], which was inspired by [68]. Here, the authors consider discretising an extra dimension into two, or generally N sites. By discretising the tetrads instead of the metric, they recover ghost-free massive gravity. We remind ourselves of the conditions under which we are carrying this calculation;

$$E^a = e^a + N^a\ \mathrm{d}y, \quad E^5 = \mathcal{N}\mathrm{d}y \tag{6.19}$$

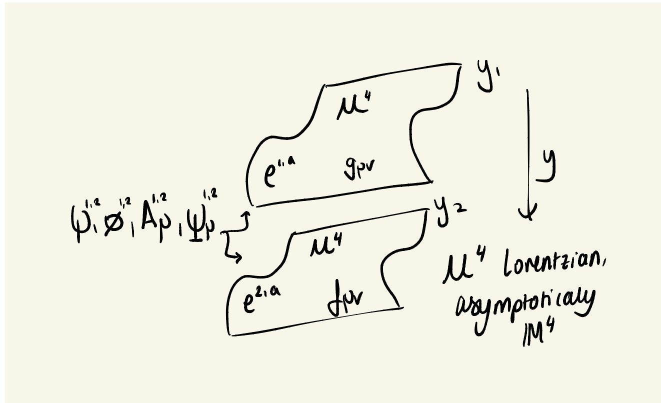

Figure 8: Set up for our theory, one could generalise this to N sites in the lattice. Note that the $4D$ space is represented as a brane here instead of as a line as for KK, but both represent $4D$ space.

we can use our Diffeomorphism gauge symmetry to set $N^a = 0$ and $\mathcal{N} = 1$, fixing 5 degrees of freedom in total. Additionally, we use 4 Local Lorentz transformations to set $E^5_\mu = 0$ and a further 6 to set $\Omega^{ab}_y = 0$, bringing the total to 5 Diffeomorphisms and 10 local Lorentz transforms. In this gauge, $\Omega^{ab} = \Gamma^{ab}$, $\quad \Omega^{5a} = K^a$, $K^a = \frac{1}{2}\left(e^{\nu b}\partial_y e^a_\nu + e^{\nu a}\partial_y e^b_\nu\right) e_{\mu b}\, \mathrm{d}x^\mu$ satisfying $K^a_\mu = e^{a\nu}K_{\mu\nu}$ with $K_{\mu\nu}$ defined as before.

With this we can now write

$$\mathcal{R}^{AB} = d\Omega^{AB} + \Omega^{AC} \wedge \Omega_C{}^B \tag{6.20}$$

and so

$$\mathcal{R}^{ab} = d\bar{\Gamma}^{ab} + \partial_y \Gamma^{ab}{}_c \mathrm{d}y \wedge e^c + \Gamma^{ac} \wedge \Gamma_c{}^b + \Omega^{a5} \wedge \Omega_5{}^b = R^{ab} - \partial_y \Gamma^{ab} \wedge \mathrm{d}y - K^a \wedge K^b \tag{6.21}$$

similarly for $\mathcal{R}^{5a}$;

$$\mathcal{R}^{5a} = \bar{d}K^a - \partial_y K^a \wedge \mathrm{d}y + \Gamma^{5C} \wedge \Gamma_C{}^a = \bar{d}K^a - \partial_y K^a \wedge \mathrm{d}y + K_b \wedge \Gamma^{ba} \tag{6.22}$$

Finally, we have the discretisation procedure as per [15]

$$\begin{aligned} \partial_y e^a_\mu &\to m\left(e^{2,a}_\mu - e^{1,a}_\mu\right) \text{ on site 1} \\ &\to m\left(e^{1,a}_\mu - e^{2,a}_\mu\right) \text{ on site 2} \\ \int f_\mu(x,y)\mathrm{d}x^\mu \wedge \mathrm{d}y &\to \frac{1}{m}\sum_{j=1}^{2}\int f_{j,\mu}(x)\mathrm{d}x^\mu \end{aligned} \tag{6.23}$$

for a theory with two sites, although this procedure can easily be generalised to N different sites [15]. We start by writing the EH action in $5D$:

$$S^{(5)}_{\mathrm{EH}} = \frac{M_5^3}{12}\int \varepsilon_{ABCDE}\mathcal{R}^{AB} \wedge E^C \wedge E^D \wedge E^E \tag{6.24}$$

since we are antisymmetrising five indices, y must appear in one of them. For one of them to appear in $\mathcal{R}$ we have two options, and forces the other four indices to take values in the $4D$ spacetime, so the contribution to the total Lagrangian from this is

$$\mathcal{C}_1 = \frac{M_5^3}{6}\int \varepsilon_{5abcd}(\mathcal{R}^{5a}\wedge E^b\wedge E^c\wedge E^d) = \frac{M_5^3}{6}\int \varepsilon_{5abcd}((\bar{d}K^a - \partial_y K^a\wedge \mathrm{d}y + K_b\wedge\Gamma^{ba})\wedge e^b\wedge e^c\wedge e^d) \tag{6.25}$$

acknowledging that only the dy terms in $\mathcal{R}^{5a}$ contribute, and bearing in mind the order of the tetrads with respect to the epsilon tensor we find

$$\mathcal{C}_1 = -\frac{M_5^3}{6}\int \varepsilon_{abcd}((-\partial_y K^a\wedge \mathrm{d}y)\wedge e^b\wedge e^c\wedge e^d) = \frac{M_5^3}{2}\int \varepsilon_{abcd}(K^a\wedge\partial_y e^b\wedge e^c\wedge e^d\wedge \mathrm{d}y) \tag{6.26}$$

now turning our attention to the term where dy appear as one of the vierbeins, we find

$$\begin{aligned}\mathcal{C}_2 &= \frac{M_5^3}{4}\int \varepsilon_{abcd}(R^{ab}\wedge \mathrm{d}y\wedge e^c\wedge e^d) = \frac{M_5^3}{4}\int \varepsilon_{abcd}((R^{ab} - \partial_y\Gamma^{ab}\wedge \mathrm{d}y - K^a\wedge K^b)\wedge \mathrm{d}y\wedge e^c\wedge e^d)\\ &= \frac{M_5^3}{4}\int \varepsilon_{abcd}((R^{ab} - K^a\wedge K^b)\wedge \mathrm{d}y\wedge e^c\wedge e^d)\end{aligned} \tag{6.27}$$

so the total Lagrangian is

$$S^{(5)}_{\mathrm{EH}} = \frac{M_5^3}{4}\int \varepsilon_{abcd}((R^{ab} - K^a\wedge K^b)\wedge e^c\wedge e^d + 2K^a\wedge\partial_y e^b\wedge e^c\wedge e^d)\wedge \mathrm{d}y \tag{6.28}$$

proceeding with our usual discretisation we see that

$$K^a = \frac{m}{2}[e^{1,b\nu}(e^{2,a}{}_\nu - e^{1,a}{}_\nu) + e^{1,a\nu}((e^{2,b}{}_\nu - e^{1,b}{}_\nu))]e^1_{\mu b}dx^\mu \tag{6.29}$$

which we can simplify to $K^a = m(e^{2,a} - e^{1,a})$ and so we can write the Lagrangian as

$$S^{(5)}_{\mathrm{EH}} = \frac{M_5^3}{4}\int \varepsilon_{abcd}((R^{1,ab} + (e^{2,a} - e^{1,a})\wedge(e^{2,b} - e^{1,b}))\wedge e^{1,c}\wedge e^{1,d}\wedge + 1\leftrightarrow 2 \tag{6.30}$$

which is a specific example of ghost-free massive bi-gravity [15] [53]. In [15] it is explained how this approach is equivalent to dimensional deconstruction in KK theories, with a more sophisticated approach to defining the derivative.

6.4.1 Adding matter

One can add matter through a KG Lagrangian in curved spacetime. The action for this in differential geometry form is simply

$$\mathcal{S}_{matter} = -\frac{1}{2}\int d\phi\wedge\star d\phi \tag{6.31}$$

for which the variation is simply

$$\delta\mathcal{S}_{matter} = -\int d\delta\phi \wedge \star d\phi = \int \delta\phi \wedge \star d^\dagger d\phi = 0 \tag{6.32}$$

or simply $d^\dagger d\phi = 0$ which in components yields

$$-\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) = 0. \tag{6.33}$$

We can write it in a coordinate basis in $4D$ as this will be good practice for what's to follow;

$$\begin{aligned}\mathcal{S}_{matter} &= -\frac{1}{2}\frac{1}{3!}\int \partial_\mu\phi\partial_\nu\phi\varepsilon^\nu{}_{\mu_1\mu_2\mu_3}e^\mu \wedge e^{\mu_1} \wedge e^{\mu_2} \wedge e^{\mu_4} = \frac{1}{12}\int \mathrm{d}^4x\sqrt{-g}\partial_\mu\phi\partial^\nu\phi\varepsilon_{\nu\mu_1\mu_2\mu_3}\varepsilon^{\mu\mu_1\mu_2\mu_3} \\ &= \frac{3!}{12}\int \mathrm{d}^4x\sqrt{-g}\partial_\mu\phi\partial^\nu\phi\delta^\mu_\nu = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\partial_\mu\phi\partial^\mu\phi\end{aligned} \tag{6.34}$$

Now we perform the equivalent calculation in 5 dimensions;

$$4!\, d\phi^\star d\phi = \partial_A\phi\partial^B\phi\varepsilon_{BCDEF}E^A \wedge E^C \wedge E^D \wedge E^E \wedge E^F \wedge \tag{6.35}$$

we can first sum over spacetime indices and the discretised dimension y for b;

$$4!\, d\phi^\star d\phi = \partial_A\phi\partial^b\phi\varepsilon_{bCDEF}E^A \wedge E^C \wedge E^D \wedge E^E \wedge E^F \wedge + \partial_A\phi\partial^y\phi\varepsilon_{yCDEF}E^A \wedge E^C \wedge E^D \wedge E^E \wedge E^F \tag{6.36}$$

and doing the same over A

$$\begin{aligned}4!\, d\phi^\star d\phi = {}& \partial_a\phi\partial^b\phi\varepsilon_{bCDEF}E^a \wedge E^C \wedge E^D \wedge E^E \wedge E^F + \partial_y\phi\partial^b\phi\varepsilon_{bCDEF}\mathrm{d}y \wedge E^C \wedge E^D \wedge E^E \wedge E^F \\ &+ \partial_a\phi\partial^y\phi\varepsilon_{yCDEF}E^a \wedge E^C \wedge E^D \wedge E^E \wedge E^F + E^F \wedge + \partial_y\phi\partial^y\phi\varepsilon_{yCDEF}E^y \wedge E^C \wedge E^D \wedge E^E \wedge E^F\end{aligned} \tag{6.37}$$

the antisymmetry of the wedge products and of the epsilon tensor makes the second and third terms drop out, so the whole action is

$$\mathcal{S}_{matter} = -\frac{1}{2}\frac{1}{4!}\int [\partial_a\phi\partial^b\phi\varepsilon_{bCDEF}E^a \wedge E^C \wedge E^D \wedge E^E \wedge E^F + \partial_y\phi\partial^y\phi\varepsilon_{ycdef}E^y \wedge E^c \wedge E^d \wedge E^e \wedge E^f] \tag{6.38}$$

we must choose one and only one index in the Levi-Civita tensor to be equal to the fifth coordinate for a non-zero contribution. Since the first term is symmetric, we can choose any of them and get the same result, and as there are four to choose from we must add a factor of four to this calculation;

$$\mathcal{S}_{matter} = -\frac{1}{2}\frac{1}{4!}\int [\partial_a\phi\partial^b\phi\varepsilon_{bycde}e^a \wedge \mathrm{d}y \wedge e^c \wedge e^d \wedge e^e + \partial_y\phi\partial^y\phi\varepsilon_{ycdef}\mathrm{d}y \wedge e^c \wedge e^d \wedge e^e \wedge e^f]^{13} \tag{6.39}$$

[13] We ascertain that the remaining tetrads to be summed over must be from the regular $4D$ spacetime so we can change $E \to e$

and we can relabel the dummy indices to have it in a more appealing form

$$\mathcal{S}_{matter} = -\frac{1}{2}\frac{1}{4!}\int \varepsilon_{abcd}[4\partial_e\phi\partial^a\phi e^e \wedge +\partial_y\phi\partial^y\phi e^a] \wedge e^b \wedge e^c \wedge e^d \wedge \mathrm{d}y \tag{6.40}$$

In components, and discretising the vierbeins, we can tackle the first term first;

$$\varepsilon_{abcd}4\partial_e\phi\partial^a\phi e^e \wedge e^b \wedge e^c \wedge e^d \wedge \mathrm{d}y = -4 * 3!\sqrt{-g}\partial_e\phi_1\partial^a\phi_1\delta_a^e\mathrm{d}^4x + 1 \leftrightarrow 2 \tag{6.41}$$

where the 1 superscripts denote the field in one of the two y locations. For the second field, we find under $\partial_y f(y) \to m(f(y_2) - f(y_1))$

$$\varepsilon_{abcd}\partial_y\phi\partial^y\phi e^a \wedge e^b \wedge e^c \wedge e^d \wedge \mathrm{d}y = -4!\sqrt{-g}m^2(\phi_2 - \phi_1)^2\mathrm{d}^4x + 1 \leftrightarrow 2 \tag{6.42}$$

This is an interesting result because the discretisation of the field in $5D$ gives us a mass dynamically, the same as the one we get for the massive graviton. One can diagonalise the mass matrix

$$M = m^2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{6.43}$$

to find that we have two orthogonal propagating modes $\frac{\phi_1+\phi_2}{\sqrt{2}}$ with mass $\sqrt{2}m$ and $\frac{\phi_1-\phi_2}{\sqrt{2}}$ which is massless. One can easily add a mass in $5D$ with a term like $M^2\phi\wedge\star\phi$ which would generate scalars with masses M and $M+\sqrt{2}m$ in $4D$.

6.4.2 Spin-$\frac{1}{2}$

The book Supergravity by Freedman and Van Proeyen [67] was the main resource used for this section and provides a good introduction to spinors in curved spacetime. To add Fermionic matter, we need to first introduce the Dirac equation in curved spacetime. The usual Dirac Lagrangian in Minkowski spacetime is

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \tag{6.44}$$

with $\partial\!\!\!/ = \gamma^\mu\partial_\mu$. This equation can also be gauged by replacing the derivative into a covariant derivative, where we add iqA_μ to the normal partial derivative. These indices are really Lorentz indices and to transform this equation into curved spacetime, we need real spacetime indices. To translate one into the others, as aforementioned, we require from tetrads. Additionally, we need to include the spin connection one-form. This will need some explaining.

One can regard the spin connection one-form as the gauge field generated by local Lorentz transformations. This comes a long way to explaining why an addition to the normal partial derivative is needed in the same way as we do with any other gauge field. On vector fields, the Christoffel symbols are the representation of this, but when dealing with quantities with spinor indices such as Dirac spinors, we need to introduce the spin connection. This is nothing other than what we found before when dealing with tetrads, as these too have a Lorentz index and so it was needed to write the structure equations. We derive some other useful results here that allows us to understand them better, together with their transformation laws. Firstly, we remind ourselves of the definition of a covariant derivative acting on a tetrad;

$$De^a = de^a + \Gamma^a_b e^b = T^a \tag{6.45}$$

if we require the covariant derivative to transform like a Lorentz vector then $De^a \to \Lambda^{-1}{}^a_b De^b$, then one can readily see that the connection Γ must transform as

$$\Gamma'^a{}_b = \Lambda^{-1a}{}_c \, \mathrm{d}\Lambda^c{}_b + \Lambda^{-1a}{}_c \Gamma^c{}_d \Lambda^d{}_d \tag{6.46}$$

which has the usual $A \to MAM^{-1} + (\partial M)M^{-1}$ structure of the transformation of a gauge field. In components, we have

$$\Gamma'^a_\mu = \Lambda^{-1a}{}_c \partial_\mu \Lambda^c{}_b + \Lambda^{-1a}{}_c \Gamma_\mu{}^c{}_d \Lambda^d{}_b \tag{6.47}$$

and so we see that it also transforms as a one form (as the name spoiled) under coordinate transformations (spacetime indices). We now see that for Lorentz vectors such as V^a with transformation $V^a \to V'^a = \Lambda^{-1a}{}_b V^b$, the covariant derivative

$$D_\mu V^a = \partial_\mu V^a + \Gamma_\mu{}^a{}_b V^b \tag{6.48}$$

transforms as a Lorentz vector (this is the exact same calculation as we did to show that De^b has the right transformation properties, but in components!). Similarly, since both Γ, ∂ transform as vectors under coordinate transforms, so do the covariant derivatives. This can easily be extended to Lorentz covectors and tensors in general. Additionally, we have

$$\nabla_\mu V^\rho = e^\rho_a D_\mu V^a = e^\rho_a D_\mu e^a_\nu V^\nu = \partial_\mu V^\rho + e^\rho_a \left(\partial_\mu e^a_\nu + \Gamma_\mu{}^a{}_b e^b_\nu \right) V^\nu \tag{6.49}$$

but also we have, as usual,

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\alpha} V^\alpha \tag{6.50}$$

from which we can see that

$$\Gamma^\rho_{\mu\nu} = e^\rho_a \left(\partial_\mu e^a_\nu + \Gamma^a_{\mu b} e^b_\nu \right) \tag{6.51}$$

which gives us a dictionary between the two connections via tetrads. In curved spacetime, one needs to transform the indices in gamma matrices into spacetime indices. For this, we use tetrads to change these indices; $\gamma^\mu = \gamma^a e^\mu{}_a$.

It is important to remember that Gamma matrices also carry suppressed Spinor indices in their rows and columns. The transformation of a Dirac bispinor under infinitesimal local Lorentz transformations can be derived from the canonical example $\Psi_\alpha \bar{\Psi}_\beta$ where $\delta\Psi_\alpha = -\frac{1}{4}\lambda_{ab}\gamma^{ab}\Psi_\alpha$,

$$\delta\bar{\Psi}_\beta = \frac{1}{4}\lambda_{ab}\bar{\Psi}_\beta\gamma^{ab}$$

. We then see (suppressing indices again)

$$\delta(\Psi\bar{\Psi}) = -\frac{1}{4}\lambda_{ab}\gamma^{ab}\Psi\bar{\Psi} + \frac{1}{4}\lambda_{ab}\Psi\bar{\Psi}\gamma^{ab} = -\frac{1}{4}\lambda_{ab}[\gamma^{ab}, \Psi\bar{\Psi}] \tag{6.52}$$

Thus, and bearing in mind gamma matrices have also a spacetime index, we find that the covariant derivative for a Gamma matrix in curved spacetime is

$$\nabla_\mu\gamma_\nu = \partial_\mu\gamma_\nu + \frac{1}{4}\Gamma^{ab}_\mu\left[\gamma_{ab}, \gamma_\nu\right] - \Gamma^\rho_{\mu\nu}\gamma_\rho \tag{6.53}$$

we can use the standard results

$$\frac{1}{2}[\gamma^\alpha, \gamma^\beta] = \gamma^\alpha\gamma^\beta - \eta^{\alpha\beta} \tag{6.54}$$

and

$$\frac{1}{4}[[\gamma^\mu, \gamma^\nu], \gamma^\rho] = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu} \tag{6.55}$$ [14]

which we can use to rewrite

$$\begin{aligned}\nabla_\mu\gamma_\nu &= \gamma^a[\partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu}e_{a\rho}] + \frac{1}{2}\Gamma_\mu{}^{ab}e^c_\nu(\gamma_a\eta_{bc} - \gamma_b\eta_{ac}) \\ &= \gamma^a\left(\partial_\mu e_{a\nu} + \Gamma_{\mu ab}e^b_\nu - \Gamma^\rho_{\mu\nu}e_{a\rho}\right) = 0\end{aligned} \tag{6.56}$$

[15] Meaning that multiplication by gamma matrices commutes with the covariant derivative, which is an extremely useful result when manipulating quantities. We now introduce Dirac's Lagrangian in curved spacetime;

$$\mathcal{L}_{Dirac} = \int \mathrm{d}^4x\sqrt{-g}i\bar{\Psi}\gamma^\mu\nabla_\mu\Psi \tag{6.57}$$

which for our interests can be promoted to five dimensions through the use of a new Gamma matrix $\gamma^4 = -i\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$. Then the $5D$ Lagrangian is

$$\mathcal{L}_{Dirac} = \int d^5x\sqrt{-g}i\bar{\Psi}\gamma^A\nabla_A\Psi \tag{6.58}$$

[14] Here the indices are normal Lorentz indices and not spacetime indices
[15] From 6.51

where $A \in \{0,1,2,3,4\}$ we can start dissecting this equation as follows;

$$\gamma^A \nabla_A = \gamma^\mu \nabla_\mu + \gamma^4(\partial_y + \Gamma_{5ab}\gamma^{ab}) \tag{6.59}$$

Luckily, we previously made the gauge choice of $\Gamma_5 = 0$, tremendously simplifying our quest for a finalised equation. We can therefore rewrite our Lagrangian as

$$\begin{aligned}\mathcal{L}_{Dirac} &= \int \mathrm{d}^4 x e^1 i\bar{\Psi^1}\gamma^{1\mu}\nabla_{1\mu}\Psi^1 + im\bar{\Psi^1}\gamma^{1,4}(\Psi^2 - \Psi^1) + 1 \leftrightarrow 2 \\ &= \int \mathrm{d}^4 x e^1 i\bar{\Psi^1}\gamma^{1\mu}\nabla_{1\mu}\Psi^1 - m\bar{\Psi^1}\gamma^5(\Psi^1 - \Psi^2) + 1 \leftrightarrow 2\end{aligned} \tag{6.60}$$

this interaction splits the Lagrangian into right and left-handed Weyl spinors and generating the same kind of mass term as that of the scalar field[16]. We see that each fermionic field couples to the tetrads at each site, but only minimally for the mass term, as given our choice of gauge, γ^5 is the same in spacetime and Lorentz indices. We see that this also makes the interaction chiral as it separates right and left handed spinors, and also mixes spinors at both sites. This also agrees with our result from dimensional deconstruction. The fact that this theory becomes chiral should not alert us, as when introducing dimensional discretisation for spinors, we saw that we may perform a field redefinition that satisfies a modified equation where the field becomes chiral. Performing a rotation by $\beta = \frac{\pi}{2}$ would get us back to Dirac's equation in its usual form. Nonetheless, it is curious that the chirality is imposed in such a suggestive way through these procedures, and it would be nice to find a physical explanation for this.

6.4.3 Yang-Mills

We can now think of adding the other matter familiar in the standard model; Yang-Mills theories. This computation is more straightforward conceptually, although lengthier in practice. It is well known that we can write the Lagrangian for Yang-Mills theories in curved space in differential form notation, as

$$\mathcal{L}_Y M = -\frac{1}{2}\int Tr(F \wedge \star F) \tag{6.61}$$

Here F is a two form $F = dA + A \wedge A$. $A \wedge A$ would typically be identically zero, but we need to remember that for YM, A is a matrix of forms and so it doesn't vanish. We can get down directly to the $5D$ theory where we can write

$$\mathcal{L}_Y M = -\frac{1}{2}\int Tr\Big([\frac{1}{2}F^a_{AB}T^a e^A \wedge e^b] \wedge [\frac{1}{6}\varepsilon_{CDEFG}F^{FGb}T^b e^C \wedge e^D \wedge e^E]\Big) \tag{6.62}$$

[16]Since both the covariant derivative and the gamma matrices are defined in terms of the tetrads at each site, they also pick up 1 and 2 labels

Using the usual orthogonality relation among generators $Tr(T^aT^b) = \delta^{ab}/2$

$$\begin{aligned}
\mathcal{L}_{YM} &= -\frac{1}{48}\int \Big([F^a_{AB}\varepsilon_{CDEFG}F^{FGa}e^A \wedge e^B \wedge e^C \wedge e^D \wedge e^E]\Big) \\
&= -\frac{1}{48}\int \Big(2F^a_{yb}F^{aFG}\varepsilon_{CDEFG}\mathrm{d}y + F^a_{ab}F^{aFG}\varepsilon_{CDEFG}e^a\Big) \wedge e^b \wedge e^C \wedge e^D \wedge e^E \\
&= -\frac{1}{48}\int \Big(4F^a_{yb}F^{ayg}\varepsilon_{cdeyg}\mathrm{d}y \wedge e^b \wedge e^c \wedge e^d \wedge e^e + 2F^a_{yb}F^{afg}\varepsilon_{CDEfg}\mathrm{d}y \wedge e^b \wedge e^C \wedge e^D \wedge e^E \\
&\quad + 2F^a_{ab}F^{ayg}\varepsilon_{cdeyg}e^a \wedge e^b \wedge e^c \wedge e^d \wedge e^e + F^a_{ab}F^{afg}\varepsilon_{CDEfg}e^a \wedge e^b \wedge e^C \wedge e^D \wedge e^E\Big)
\end{aligned} \tag{6.63}$$

At this point only the first and last terms survive yielding

$$\begin{aligned}
\mathcal{L}_{YM} &= -\frac{1}{48}\int \Big(4F^a_{yb}F^{ayg}\varepsilon_{cdeyg}\mathrm{d}y \wedge e^b \wedge e^c \wedge e^d \wedge e^e + 3F^a_{ab}F^{afg}\varepsilon_{ydefg}e^a \wedge e^b \wedge \mathrm{d}y \wedge e^D \wedge e^E\Big) \\
&= \frac{1}{48}\int \varepsilon_{abcd}\Big(4F^a_{yk}F^{ayd}e^k \wedge e^a \wedge e^b \wedge e^c - 3F^a_{km}F^{acd}e^k \wedge e^m \wedge e^a \wedge e^b\Big) \wedge \mathrm{d}y
\end{aligned} \tag{6.64}$$

we can now do our usual trick to contract the antisymmetric Levi-Civita symbol from the tetrads with the epsilon tensor;

$$\begin{aligned}
\mathcal{L}_{YM} &= \frac{1}{48}\int \varepsilon_{abcd}\Big(4F^a_{yk}F^{ayd}e^k \wedge e^a \wedge e^b \wedge e^c - 3F^a_{km}F^{acd}e^k \wedge e^m \wedge e^a \wedge e^b\Big) \wedge \mathrm{d}y \\
&= -\frac{1}{48}\int \mathrm{d}^4x\mathrm{d}y\sqrt{-g}\Big(-4F^a_{yk}F^{ayd}\epsilon^{kabc}\epsilon_{dabc} - 3F^a_{km}F^{abc}\epsilon_{abcd}\epsilon^{abkm} \\
&= -\frac{1}{48}\int \mathrm{d}^4x\mathrm{d}y\sqrt{-g}\Big(4!\,F^a_{yk}F^{ayk}\epsilon^{kabc}\epsilon_{dabc} + 6F^a_{km}F^{acd}(\delta^k_c\delta^m_d - \delta^m_c\delta^k_d) \\
&= -\frac{1}{4}\int \mathrm{d}^4x\mathrm{d}y\sqrt{-g}\Big(2F^a_{yk}F^{ayk} + F^a_{cd}F^{acd}\Big).
\end{aligned} \tag{6.65}$$

Examining the term $F^a_{yk}F^{ayk}$ we see that

$$F^a_{yk} = \nabla_y A^a_k - \nabla_k A^a_y - gA^b_y A^c_k f^{abc} = \partial_y A^a_k - \partial_k A^a_y - gA^b_y A^c_k f^{abc} \tag{6.66}$$

in a torsion free theory, so the connection is symmetric. Now we gauge fix to set $A_y = 0$ which sets this as a mass term. To do this we look at the usual gauge freedom

$$A^a_A \to A^a_A - f^{abc}\theta^b A^c_A - \frac{i}{g}\partial_A\theta^a \tag{6.67}$$

taking the $A = y$ component of this equation, we find that we have a quantities to set to zero, coming from A. Simultaneously, we decompose θ in the basis of the Lie algebra meaning we can gauge fix a components. This means that we can guarantee that $A^a_y = 0$ at the expense of fixing our gauge. In the case of Maxwell, the constraint reduces to

$$A_y \to A_y - \frac{i}{g}\partial_y\theta = A^1_y - \frac{im}{g}(\theta^2 - \theta^1) = 0. \tag{6.68}$$

It would seem like we have run out of luck as now there are two quantities to fix, A_y^1, A_2^y, however, we see that because we have discretised the 5th dimension, we have also added a second U(1) symmetry that remains intact even after gauge fixing, as this constraint only depends on the difference of θ in both sites, meaning we may arbitrarily add a function to it, and it would still satisfy 6.68. We can now use this second U(1) to make the A_y^2 field zero in the second site too. We could have also first discretised, and then fixed the gauge, in this case our symmetry pattern would go from a U(1) before discretisation, to $U(1)^2$ after, and then fixing our gauge. This is completely equivalent to what we have seen previously in the case of gravity, when $5D$ gravity gets discretised to ghost-free bi-gravity. In the process, we use the gauge (Diff) invariance to fix our vierbeins, retaining one copy of Diffeomorphism invariance ($Diff \times Diff \to Diff$). After all this is said and done, we are rewarded with our mass term,

$$\mathcal{L}_{YM} = \int \mathrm{d}^4x\mathrm{d}y\sqrt{-g}\Big(-\frac{1}{4}F^a_{cd}F^{acd} - \frac{1}{2}\partial_y A^a_k \partial^y A^{ak}\Big) \tag{6.69}$$

which we discretise in our usual way;

$$\mathcal{L}_{YM} = \int \mathrm{d}^4x\mathrm{d}y e^1\Big(-\frac{1}{4}F^{1,a}_{cd}F^{1,acd} - \frac{1}{2}m^2(A_k^{2,a} - A_k^{1,a})^2\Big) + 1 \leftrightarrow 2. \tag{6.70}$$

6.5 The Rarita-Schwinger field

Previously, we have pondered upon the results of performing dimensional discretisation on different every-day fields, inspired by the results of discretising tetrads in $5D$ massless gravity to obtain ghost-free dRGT gravity. Now we take a theoretical leap considering a field that has not yet been observed but that it is predicted by SUSY theories, and completes the classification of fields with half integer spin lower or equal to 2. This section will follow the theory presented in [67]. The Rarita-Schwinger field defines a particle with a spinor index and a spacetime index. This is to say, it describes a Spin-$\frac{3}{2}$ field. This field carries the name Gravitino, as it is the SUSY superpartner of the graviton field. Our field has two indices $\Psi_{\mu\alpha}$ where μ is a spacetime index as usual and α is a spinor index. This field, having a vector index, inherits the gauge transformation of a vector field, but the field it transforms by must be a spinor to preserve the index structure;

$$\Psi_\mu \to \Psi_\mu + \partial_\mu \epsilon(x) \tag{6.71}$$

where ϵ is our spinor field, and we suppress the spinor indices as usual. The restrictions of our action immediately suggest a possible action. Firstly, it must be gauge invariant. Secondly, as usual, it must be Lorentz invariant. Lastly, we are dealing with a Fermionic field, meaning that our Lagrangian should preserve structure of Dirac; space and time derivatives must appear on the same footing, and it should be first order in derivatives. In an effort to make this effort transparent, we introduce some notation for Gamma matrices.

$$\gamma^{\mu_1\mu_2\cdots\mu_r} = \gamma^{[\mu_1}...\gamma^{\mu_r]}. \tag{6.72}$$

So for example

$$\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] \tag{6.73}$$

additionally, we can show that

$$\gamma^{\mu\nu\rho} = \frac{1}{2}\{\gamma^\mu, \gamma^{\nu\rho}\} \tag{6.74}$$

we first consider

$$\begin{aligned}\gamma^{[\mu}\gamma^\nu\gamma^{\rho]} &= \frac{1}{6}[\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\mu\gamma^\rho\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\rho + \gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\rho\gamma^\nu\gamma^\mu] \\ &= \frac{1}{6}[\gamma^\mu[\gamma^\nu, \gamma^\rho] + [\gamma^\nu, \gamma^\rho]\gamma^\mu + \gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\rho] \\ &= \frac{1}{6}[\gamma^\mu[\gamma^\nu, \gamma^\rho] + [\gamma^\nu, \gamma^\rho]\gamma^\mu + \frac{1}{2}[2\eta^{\mu\nu}\gamma^\rho + 2\gamma^\nu\eta^{\mu\rho} - \gamma^\mu\gamma^\rho\gamma^\nu - \gamma^\rho\gamma^\nu\gamma^\mu] \\ &- \frac{1}{2}[2\eta^{\mu\nu}\gamma^\rho + 2\gamma^\nu\eta^{\mu\rho} - \gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\rho\gamma^\mu]] \\ &= \frac{1}{6}[\gamma^\mu[\gamma^\nu, \gamma^\rho] + [\gamma^\nu, \gamma^\rho]\gamma^\mu + \frac{1}{2}[\gamma^\mu[\gamma^\nu, \gamma^\rho] + [\gamma^\nu, \gamma^\rho]\gamma^\mu]] \\ &= \frac{1}{4}[\gamma^\mu[\gamma^\nu, \gamma^\rho] + [\gamma^\nu, \gamma^\rho]\gamma^\mu] = \frac{1}{2}[\gamma^\mu, \gamma^\nu].\end{aligned} \tag{6.75}$$

The Rarita-Schwinger Lagrangian is

$$\mathcal{S} = -\int \mathrm{d}^D x \bar{\Psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \Psi_\rho. \tag{6.76}$$

To see the gauge invariance we expand the action as

$$\mathcal{S}' = -\int \mathrm{d}^D x(\bar{\Psi} + \partial_\mu\bar{\epsilon})\gamma^{\mu\nu\rho}\partial_\nu(\Psi_\rho + \partial_\mu\epsilon) = \mathcal{S} - \int \mathrm{d}^D x \partial_\mu\bar{\epsilon}\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho + \bar{\Psi}\gamma^{\mu\nu\rho}\partial_\nu\partial_\mu\epsilon + \partial_\mu\bar{\epsilon}\gamma^{\mu\nu\rho}\partial_\nu\partial_\mu\epsilon \tag{6.77}$$

the second and third terms drop out because of the antisymmetry of γ while the first can be integrated out and drops out because of this same reason, so the total action is the same up to a total derivative $\partial_\mu(\bar{\epsilon}\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho)$. Varying the action with respect to $\bar{\Psi}_\mu$ yields the equation of motion,

$$\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho = 0. \tag{6.78}$$

This equation can be expressed in several ways. Firstly, one can show that $\gamma_\mu\gamma^{\mu\nu\rho} = (D-2)\gamma^{\nu\rho}$ and that $\gamma^{\mu v\rho} = \gamma^\mu\gamma^{v\rho} - 2\eta^{\mu[v}\gamma^{\rho]}$. Using the former relation, we can immediately see that $\gamma^{\nu\rho}\partial_\nu\Psi_\rho = 0$. Additionally, we also see that because of this

$$(\gamma^\mu\gamma^{v\rho} - 2\eta^{\mu[v}\gamma^{\rho]})(\partial_\nu\Psi_\rho) = 0 \tag{6.79}$$

so

$$\eta^{\mu[\nu}\gamma^{\rho]}\partial_\nu\Psi_\rho = 0 \tag{6.80}$$

and so

$$\gamma^\rho\partial_{[\mu}\Psi_{\rho]} = 0. \tag{6.81}$$

We use the conventional jargon for on-shell (number of propagating degrees of freedom) vs off-shell (number of components - gauge transformations) degrees of freedom. A normal spinor will have $2^{[D/2]}$ degrees of freedom where D is the dimension of the space. On the other hand, a vector has D components. We also have one spinorial gauge symmetry which lets us fix on vector-spinor component or $2^{[D/2]}$ degrees of freedom. This means that in total we have $(D-1)2^{[D/2]}$ off-shell degrees of freedom. It is customary to impose the gauge condition $\gamma^i\Psi_i = 0$, however, as we might expect, this gauge freedom will have to be spent fixing $\Psi_y = 0$.

6.6 Gravitino mass from dimensional deconstruction

As we saw before, scalars, vectors and spinors can all acquire a mass through dimensional reduction. The gravitino is no different. Following the same process as before, we have

$$\gamma^{ABC}\partial_B\Psi_C = 0 \implies \left[\gamma^{\mu\nu\rho}\partial_\nu - \mathrm{i}\frac{k}{L}\gamma_*\gamma^{\mu\rho}\right]\Psi_{\rho k} = 0 \tag{6.82}$$

we can redefine the field like in 4.92 as $\Psi_{\rho k} = \mathrm{e}^{(-\mathrm{i}\pi\gamma_*/4)}\Psi'_{\rho k}$ to remove the gamma matrix and be left with the equation

$$\left(\gamma^{\mu\upsilon\rho}\partial_\nu - m\gamma^{\mu\rho}\right)\Psi_\rho = 0. \tag{6.83}$$

6.7 Gravitino action in curved spacetime and dimensional deconstruction

We are now in a position to extend the Rarita-Schwinger Lagrangian to curved spacetime without many complications. we have

$$S_{RS} = -\frac{1}{2\kappa^2}\int \mathrm{d}^D x e\bar{\Psi}_\mu\gamma^{\mu\nu\rho}D_\upsilon\Psi_\rho. \tag{6.84}$$

Where the gamma matrices now carry spacetime indices as in the Dirac equation in curved spacetime, and the covariant derivative is defined as

$$D_\nu\Psi_\mu \equiv (\partial_\nu\Psi_\mu + \frac{1}{4}\Gamma_{\upsilon ab}\gamma^{ab}\Psi_\mu) + \Gamma^\alpha_{\nu\mu}\Psi_\alpha \tag{6.85}$$

as it carries both vector indices and spinor indices. However, since we are contracting with a gamma matrix which is antisymmetric in the μ,ν indices, the last term drops out for a symmetric connection such as the Levi-Civita connection. Thus, since we are considering torsion-free theories, we will omit this term in what follows. In five dimensions, we follow our now usual prescription;

$$\bar{\Psi}_A\gamma^{ABC}D_B\Psi_C = \bar{\Psi}_\mu\gamma^{\mu\nu\rho}D_v\Psi_\rho + \bar{\Psi}_y\gamma^{y\nu\rho}D_v\Psi_\rho + \bar{\Psi}_\mu\gamma^{\mu y\rho}D_y\Psi_\rho + \bar{\Psi}_\mu\gamma^{\mu\nu y}D_v\Psi_y \tag{6.86}$$

the second and last terms fall out because, as advertised before, we will use our gauge freedom to set the Ψ_y term equal to zero. Additionally, we used the Lorentz invariance from the tetrads to set Ω_y^{ab}=0 and so this reduces to

$$\bar{\Psi}_A\gamma^{ABC}D_B\Psi_C = \bar{\Psi}_\mu\gamma^{\mu\nu\rho}D_v\Psi_\rho + \bar{\Psi}_\mu\gamma^{\mu y\rho}D_y\Psi_\rho \tag{6.87}$$

now

$$\gamma^{\mu y\nu} = \frac{i}{4}\{\gamma^\mu, [\gamma^5, \gamma^\nu]\} = \frac{i}{2}\{\gamma^\mu, \gamma^5\gamma^\nu\} = -\gamma^5\gamma^{\mu\nu} \tag{6.88}$$

and so we find that the Lagrangian becomes

$$S_{RS} = -\frac{1}{2\kappa^2}\int \mathrm{d}^4\mathrm{d}xye\bar{\Psi}_\mu\gamma^{\mu\nu\rho}D_v\Psi_\rho - i\bar{\Psi}_\mu\gamma^5\gamma^{\mu\nu}\partial_y\Psi_\nu \tag{6.89}$$

which matches 6.82 with $\gamma^5 \sim \gamma*$. Finally, we discretise our fifth dimension as usual to obtain

$$S_{RS} = -\frac{1}{2\kappa^2}\int \mathrm{d}^4x\mathrm{d}ye^1\bar{\Psi}^1_\mu\gamma^{1,\mu\nu\rho}D^1_v\Psi^1_\rho - im\bar{\Psi}^1_\mu\gamma^{1,5}\gamma^{1,\mu\nu}(\Psi^2_\nu - \Psi^1_\nu) + 1 \leftrightarrow 2. \tag{6.90}$$

With this, we conclude our effort to explore the effect of discretising a fifth dimension for all matter fields with spin equal or less than 2. Non surprisingly, this resulted in the different fields acquiring different mass terms, as [15] already pointed out the relationship between KK towers and our discretisation procedure. Nonetheless, some non-trivial results were found, such as a massless and a massive propagating mode made out of a linear superposition of fields living in both lattices. Similarly, our fermionic fields couple to the specific tetrads of their respective braneworld through the gamma matrices, which acquire spacetime indices in curved space. In addition to this, there is also mixing between the fermionic fields living at both sites, and the interaction term is now chiral due to the γ^5. Similar results were found for both Yang-Mills and Rarita-Schwinger fields, although the scalar and the Fermionic field are arguably the two archetypal examples when studying these theories, as they in a way encapsulate many of the results in higher spin fields. The author takes a brief moment to remind the reader that this section has not been peer reviewed and these results should be taken with a pinch of salt! All these results can easily be extended to N sites by introducing a $\sum_j$ and using nearest neighbour interactions for the fields; $\partial_y\phi^j \rightarrow m(\phi^{j+1} - \phi^j)$.

7 Conclusion and further scope for research

Through this dissertation, we have looked at the development of massive gravity, from the original FP action, to the breakthrough in ghost-free massive gravity in dRGT gravity. Nonetheless, the journey was far from finished here. We saw how these theories are extremely rich and arise very naturally in the tetrad formalism. This approach allowed us to look at the different terms that could arise in these theories, and to understand their emergence through discretisation of extra dimensions. We also found that this mechanism gives rise to massive particles in $4D$ for all fields with spin lower or equal to 2. It remains to be seen whether these interactions are also ghost-free or whether they might lead to pathologies. This disseration focused on the developments in the field from 2008-2015, nontheless, there have been several exciting developments since, such as using the techniques developed in massive gravity used to generalise the FP theory, to do the equivalent procedure in the Proca Lagrangian [75] [76]. Other recent developments have been pushing the cut-off scale to Λ_2 [77] Research has also been recently carried out on an old concept; partially massless (PM) gravity [78], where we consider gravity in de Sitter space, and for which a specific choice of mass gives extra gauge symmetries and a wide array of appealing properties. With the success of the dRGT mass term, non-linear extensions to the mass term have been investigated [79] [80] and more effort is being put into understanding these theories. Finally, future extensions of the work presented in this dissertation could include finding more sophisticated ways of discretising this extra dimension, making mixed models with extra discretised dimensions, and extra continuous dimensions, or performing this analysis in spacetimes with different kinds of asymptotic spacetimes (dS or AdS for example).

A Useful results

A.1 Variation of $\delta\sqrt{-g}$

Consider

$$e^A = B \implies A = ln(B) \tag{A.1}$$

using the identity

$$Det(e^A) = e^{Tr(A)} Det(B) = Tr(ln(B)) \tag{A.2}$$

schematically,

$$\frac{\delta Det(B)}{Det(B)} = Tr(\frac{\delta B}{B}) \tag{A.3}$$

and so for our specific case

$$\frac{\delta g}{g} = g^{\mu\nu}\delta g_{\mu\nu} \tag{A.4}$$

and so

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \tag{A.5}$$

Another useful relation will be

$$\delta(g^{\mu\alpha}g_{\alpha\nu}) = \delta(\delta^\mu_\nu) = 0 \implies g^{\mu\alpha}\delta(g_{\alpha\nu}) = -\delta(g^{\mu\alpha})g_{\alpha\nu} \tag{A.6}$$

and so

$$\begin{aligned} g^{\rho\nu}g^{\mu\alpha}\delta(g_{\alpha\nu}) &= -g^{\rho\nu}\delta(g^{\mu\alpha})g_{\alpha\nu} \\ g^{\rho\nu}g^{\mu\alpha}\delta(g_{\alpha\nu}) &= -\delta(g^{\mu\rho}) \end{aligned} \tag{A.7}$$

A.2 Ricci tensor variation

$$R^\rho_{\sigma\mu\nu} = \Gamma^\rho_{\nu\sigma,\mu} - \Gamma^\rho_{\mu\sigma,\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{A.8}$$

so

$$\delta R^\rho_{\sigma\mu\nu} = \delta\Gamma^\rho_{\nu\sigma,\mu} - \delta\Gamma^\rho_{\mu\sigma,\nu} + \delta(\Gamma^\rho_{\mu\lambda})\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\delta(\Gamma^\lambda_{\nu\sigma}) - \delta(\Gamma^\rho_{\nu\lambda})\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\nu\lambda}\delta(\Gamma^\lambda_{\mu\sigma}) \tag{A.9}$$

and so

$$\delta R_{\sigma\nu} = \delta\Gamma^\rho_{\nu\sigma,\rho} - \delta\Gamma^\rho_{\rho\sigma,\nu} + \delta(\Gamma^\rho_{\rho\lambda})\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\rho\lambda}\delta(\Gamma^\lambda_{\nu\sigma}) - \delta(\Gamma^\rho_{\nu\lambda})\Gamma^\lambda_{\rho\sigma} - \Gamma^\rho_{\nu\lambda}\delta(\Gamma^\lambda_{\rho\sigma}) \tag{A.10}$$

since a difference of two connections is a tensor, we can calculate its covariant derivative:

$$\nabla_\rho\delta\Gamma^\rho_{\nu\sigma} = \Gamma^\rho_{\nu\sigma,\rho} + \Gamma^\rho_{\rho\lambda}\delta(\Gamma^\lambda_{\nu\sigma}) - \Gamma^\lambda_{\rho\nu}\delta(\Gamma^\lambda\nu\sigma) - \Gamma^\lambda_{\rho\sigma}\delta(\Gamma^\rho_{\lambda\nu}) \tag{A.11}$$

expanding

$$\begin{aligned}\nabla_\rho \delta\Gamma^\rho_{\nu\sigma} - \nabla_\nu \delta\Gamma^\rho_{\rho\sigma} &= \Gamma^\rho_{\nu\sigma,\rho} + \Gamma^\rho_{\rho\lambda}\delta(\Gamma^\lambda_{\nu\sigma}) - \cancel{\Gamma^\lambda_{\rho\nu}\delta(\Gamma^\rho_{\lambda\sigma})} - \Gamma^\lambda_{\rho\sigma}\delta(\Gamma^\rho_{\lambda\nu}) \\ &\quad -\Big(\Gamma^\rho_{\rho\sigma,\nu} + \Gamma^\rho_{\nu\lambda}\delta(\Gamma^\lambda_{\rho\sigma}) - \cancel{\Gamma^\lambda_{\rho\nu}\delta(\Gamma^\rho_{\lambda\sigma})} - \Gamma^\lambda_{\nu\sigma}\delta(\Gamma^\rho_{\lambda\rho})\Big) \\ &= \delta(R_{\sigma\nu})\end{aligned} \tag{A.12}$$

This is also known as the Palatini identity

A.3 Ricci tensor in EH

Consider

$$\int \mathrm{d}^4x\Big(\sqrt{-g}(\delta R_{\alpha\beta})g^{\alpha\beta} = \int \mathrm{d}^4x\Big(\sqrt{-g}(\nabla_\rho\delta\Gamma^\rho_{\alpha\beta} - \nabla_\alpha\delta\Gamma^\rho_{\rho\beta})g^{\alpha\beta} = \int \mathrm{d}^4x\Big(\sqrt{-g}(\nabla_\rho(\Gamma^\rho_{\alpha\beta}g^{\alpha\beta} - \Gamma^\nu_{\nu\alpha}g^{\alpha\rho}) \tag{A.13}$$

we now use

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2}g^{\mu\alpha}(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\nu\mu,\alpha} = \frac{1}{2}g^{\mu\alpha}g_{\alpha\mu,\nu} = \frac{1}{2}Tr(g^{-1}g_{,v}) = \frac{1}{2}Tr(log(g)_{,v}) \tag{A.14}$$

using

$$Tr(log(A)) = log(Det(A)) \tag{A.15}$$

we find

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2g}\partial_\nu g \tag{A.16}$$

we may construct the vector

$$A^\rho = \Gamma^\rho_{\alpha\beta}g^{\alpha\beta} - \Gamma^\nu_{\nu\alpha}g^{\alpha\rho} \tag{A.17}$$

thus we have

$$\nabla_\rho A^\rho = A^\rho_{,}\rho + \Gamma^\rho_{\rho\alpha}A^\alpha = A^\rho_{,\rho} + \frac{1}{2g}(\partial_\alpha g)A^\alpha \tag{A.18}$$

and so

$$\int \mathrm{d}^4x\Big(\sqrt{-g}(\nabla_\rho A^\rho) = \int \mathrm{d}^4x\Big(\sqrt{-g}(A^\rho_{,\rho} + \frac{1}{2g}(\partial_\alpha g)A^\alpha)\Big) = \int \mathrm{d}^4x\Big(\partial_\alpha(A^\alpha\sqrt{-g})\Big) \tag{A.19}$$

which is a total derivative and so it vanishes.

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