IMPERIAL COLLEGE LONDON

DISSERTATION

The Asymptotic Safety Program for Quantum Gravity

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Declaration

Submitted in partial fulfilment of the requirements for the degree of Master of Science of Imperial College London.

Abstract

In this project, we explore asymptotically safe pure gravity as a candidate for a quantum field theory of gravity. We first present Wilson-Kadanoff renormalization before exploring the existence of fixed points on corresponding renormalization group flows. We then study the Asymptotic Safety program which aims at renormalizing gravity in a non-perturbative way using properties of non-Gaussian fixed points. We consider gravity in $2 + \epsilon$ dimensions as original evidence for said program. Then, we compare two methods used to investigate the Asymptotic Safety program. Namely, a Functional Renormalization Group Equation of a gravitational Effective Average Action, and a perturbative one-loop expansion of renormalization group flow equations in higher-derivative quadratic gravity. Both give further evidence to support asymptotically safe gravity as a valid quantum gravity theory. Later, we show that, despite these methods being in one-to-one correspondence, only the latter can be used to address the ultraviolet renormalization problem because it involves less arbitrarily postulated 'initial data' for the coupling flow equation. Finally, we critically assess the Asymptotic Safety program before concluding on potential extensions of this project.

Acknowledgements

I would like to express my gratitude to Pr Kellogg Stelle for kindly supervising this project and accompanying me during the writing process. I am also grateful to my PhD supervisor Pr Daniel Litim for many helpful discussions. In addition, I would like to thank my dear friends Sam and Shaun for their unfailing support during this challenging academic year. Finally, I would like to thank Manon, Arthur, Virgile and my parents for their encouragements and emotional support.

Contents

1 Introduction

Einstein's General Relativity (GR) is our best known theory of gravity [1]. To this day, experimental evidence have confirmed its various foundational principles to very high accuracies. The latest such experiment tested the weak equivalence principle to an astounding accuracy of 10^{-15} [2]. In GR, gravitational interactions are shown to emerge from objects moving on geodesics in a curved Riemannian manifold called 'spacetime', whose curvature is dictated by the presence of mass or energy [3]. The Einstein's field equations describing spacetime as a dynamical entity are given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where we are free to include a cosmological constant on either side, though its interpretation will change depending on which side we introduce it in. The LHS of (1) describes the curvature of spacetime. Indeed, $R_{\mu\nu}$ and R are both constructed from the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$. The metric tensor of spacetime $g_{\mu\nu}$ provides a notion of distance on a general topological space. The RHS of (1) contains the stress energy tensor $T_{\mu\nu}$ which encodes the presence of matter or energy in spacetime.

In 1900, Max Planck's solution to the anachronistically named 'ultraviolet (UV) catastrophe' shifted the scientific paradigm towards what we now know as Modern Physics [4]. In his paper, Planck intuited that electromagnetic radition was only absorbed or emitted in 'bundles', or *quanta* in latin. In this way, many physical quantities, such as energy, can be shown to be mutliples of the fundamental Planck's constant h, or its reduced versions $\hbar \equiv h/2\pi$. Since then, physicists have studied matter through the powerful lense of Quantum Mechanics, which is a special *Quantum Field Theory* (QFT). Today, all known particles and interactions, with the exception of gravity, are described by such theories, in particular the Standard Model of Particle Physics. The RHS of (1) encodes the presence of matter, so it should follow the laws of the Standard Model. Thus, one would want to rewrite the LHS of (1) in a quantum language.

However, the Heisenberg's uncertainty principle makes this impossible [5,6]. It states that the following inequality, $\Delta x \Delta p \geq \hbar/2$, governs the product of the measurement errors of the position x and momentum p of a quantum body. Making a measurement of curvature at small distances implies an uncertainty on the momentum, and therefore the energy, which gives a lower bound on the resolution at which we can measure spacetime [7]. This lower bound is the Planck scale $\ell_p^2 \equiv G\hbar/c^3$ where G is Newton's constant and c is the speed of light. A quantum description of spacetime would imply that spacetime itself undergoes quantum fluctuations below this scale, making it a highly unstable and unpredictable dynamical entity. However, a core concept of GR is asymptotically flat spacetime. Indeed, at the shortest scales, any curved manifold should locally look flat. Hence, any Riemannian spacetime locally looks like Minkowski flat spacetime. Both views cannot be reconciled and reveal a fundamental incompatibility between these theories.

As outlined, quantum fluctuations depend on scale in gravity, but they do not in quantum electrodynamics (QED) [8,9]. This observation can be made into the more technical statement: QED is renormalizable but GR is not. The latter cannot be covariantly quantized as other QFTs. Veltman and t' Hooft showed that the Einstein-Hilbert (EH) action, from which the equations of motion (1) are derived, was perturbatively non-renormalizable at one-loop order [10]. This is known as the **renormalizability problem**. As a special case, pure gravity (GR with no matter content or vacuum energy) has a lucky cancellation of one-loop divergences but the same quantities blow up at 2-loop order and higher [11]. The resulting covariantly quantized theory of gravity is highly non-predictive since it requires an infinity of experiments to fix the coupling constants arising from the finite parts of an ever increasing number of counter-terms introduced into the EH action [12].

For these reasons, GR is considered a low-energy (infrared, IR) Effective Field Theory (EFT), valid below at some momentum $k < \ell_p$. Hence, it should arise from the fundamental theory in which degrees of freedom (dofs) above k are integrated out, and those below k are not included. Importantly, EFTs allows one to treat renormalizable and non-renormalizable theories on equal footing. The quantization of the latter has a perfectly valid intepretation provided we focus on low-energy predictions. The corresponding theory would simply be a low-energy/long-distance, coarse-grained approximation to a more fundamental renormalizable theory in which short distance details are unimportant [13, 14].
EFTs go hand in hand with Wilsonian renormalization [13, 15–21], which lets low-energy theories emerge from coarse-graining dofs by integrating, or averaging over short-scale quantum fluctuations. Iterating this 'changing of the scale' in a theory induces a Renormalization Group (RG) flow in the space of coupling constants. Wilson, observing that physical systems are different depending on the scale at which we probe them, takes seriously the existence of a physical scale above which the EFT is not valid anymore (ℓ_p for GR).

When trying to replace GR by a ”UV completed” fundamental theory of gravity, one can either introduce new dofs and symmetries; or retain the fields and symmetries of GR and postulate that gravity is a fundamental theory at the non-perturbative level. The former strategy is explored by String Theory [22]. A proposal along the lines of the latter strategy is the Asymptotic Safety program [23–29] which, motivated by gravity in $2 + \epsilon$ [30, 31], was proposed by Weinberg in 1979 [32, 33]. This proposal is based on the following non-perturbative Wilsonian renormalization condition: a theory can be UV completed if it lies on the finite-dimensional UV critical surface of a non-trivial fixed point (FP) of the Renormalization Group (RG) flow. The expression 'asymptotic safety' refers to the asymptotic freedom of QCD, which lies on the UV critical surface of the trivial FP at the origin of coupling space, where couplings vanish. This point corresponds to the free theory, hence 'freedom' [34, 35]. Even though ”Matter matters in asymptotically safe quantum gravity”, we restrict our discussion to pure gravity [36].

In this project, we present and motivate the Asymptotic Safety Program before comparing two different methods used to investigate it. Finally, we present counter-arguments to this scenario.
More precisely, in Section 2.1, we discuss the UV renormalization problem encountered in QFTs before motivating the Wilsonian view of renormalization. We then give practical examples of RG transformations used in the Wilsonian framework. In section 2.2, we define the concept of critical surface and linearize the RG flow around a FP. In Section 2.3, we show how FPs can be used in non-perturbatively renormalizing a QFT. Finally, in Section 2.4 we present Weinberg's Asymptotic Safety scenario and his original motivation for proposing it. In Section 3, we apply the methods from previous sections to GR. In particular, we extensively study the Functional Renormalization Group formalism in Section 3.1, and quadratic gravity in Section 3.2, before comparing them in Section 3.3. Finally, we present critical reflections on the program in Section 4, before discussing them and concluding on this work in Section 5.

2 Asymptotic Safety

2.1 Wilson-Kadanoff RG

2.1.1 The UV renormalization problem

To compute observables in a QFT, one might use a perturbative expansion in couplings if these are small enough. In this case, contrary to Quantum Mechanics, one finds UV divergences at each order in the expansion. To deal with these divergences, one can use perturbative renormalization by introducing a set of effective (renormalized) quantities. The divergences are then transferred from the perturbative expansion to the relation between the bare (non-renormalized) and renormalized parameter. Still, perturbative renormalization is merely a mathematical trick: in Wilson's words, using this method amounts to "sweeping divergences under the rug" [21,37]. Instead, Wilson argues that the origin of UV divergences is **physical**: they originate from **quantum fluctuations**. We shall comment on the power of Wilsonian renormalization by discussing its physical interpretation in Section 2.1.2.2.

UV divergences were thought to arise from the infinite number of degrees of freedom (dof) that characterises both Classical Field Theories and QFTs. However, fluctuations are only present in SFTs and QFTs. Hence we only find UV divergences in these theories. The mathematical contribution associated with quantum fluctuations are Feynman diagrams. We can now make two remarks on said fluctuations:

1. Approximations are necessary to compute these fluctuations (otherwise the model would be exactly solvable).

2. According to Wilson, fluctuations are not summed in the right way in perturbation theory. All wavelengths are treated on the same footing and summed over at each order in the perturbation expansion, giving rise to divergent momenta integrals [37,38].

To illustrate remark 2 we consider the 1-loop contribution to 2-2 scattering in ϕ^4, at second order in coupling, after Euclidean continuation, in D-dimensions:

$$J \sim \int^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 - {m_0}^2)^2}. \tag{2}$$

We regularized this integral by a UV momentum cut-off Λ. J is also cut-off in the IR by the non-renormalized mass m_0. Thus, we expect divergences since all momenta should contribute from the UV to the IR cutoff. We don't include $i\epsilon$ terms since the contour integral is along the real axis and does not go through any poles after the Euclidean continuation. Importantly, all momenta scales $\|p\| \leq \Lambda$ in J contribute to the renormalized value of the coupling. In momentum space, the range of J is over

$$\xi^{-1} = m_0 \leq \|p\| \leq \Lambda \tag{3}$$

and in real space, from the uncertainty principle, the range of p is

$$a = \Lambda^{-1} \leq \|x\| \leq \xi = {m_0}^{-1} \tag{4}$$

where we interpret a as a small physical distance like a lattice spacing. ξ is the interaction length scale or the correlation length. Therefore, taking $a \to 0$ corresponds to a continuum limit. Note, since $a = \Lambda^{-1}$, this limit is equivalent to $\Lambda \to \infty$.

Starting from perturbation theory forces us to use a 'bottom-up' approach to renormalization: we start at some energy scale μ, the renormalization scale, and ask if we can take all quantum dofs into account up to some arbitrary high-momentum scale Λ (the UV cutoff). If we can keep the physics at scale μ under control and independent of the choice of cutoff Λ, then we can take the continuum limit $\Lambda \to \infty$ and find a UV complete and renormalized theory [39]. Hence, to define a QFT without a cutoff, one needs to determine in what cases we are allowed to take continuum limits. In this sense, removing the cutoff Λ is similar to defining a real bounded integral as the limit of a Riemannian sum. One may take the area under a curve I as approximately the sum

$$I = \lim_{h \to 0} \Big[h \sum_i f(x_i) \Big]. \tag{5}$$

To ensure the convergence of this sum, we scaled it by an appropriate factor. The limit exists

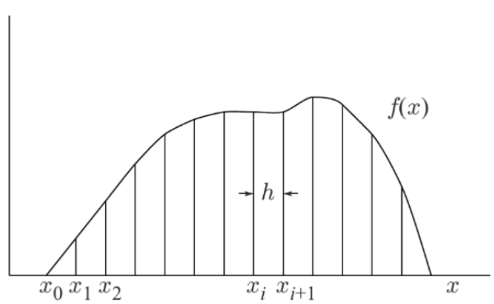

Figure 1: *The ordinate lines are drawn from the x axis to the curve representing $f(x)$, they are at a distance h apart. The integral is approximately the Riemannian sum of h times those ordinates $f(x)$. This approximation gets better as we take the limit $h \to 0$ [40].*

when we take this factor to be h, the spacing between ordinate lines [40]. For QFTs, Wilson argues that the continuum limit exists when one tunes the coupling such that the correlation length, is much larger than the lattice spacing: $\xi \gg a$ [38]. Thus, the problem of defining a QFT is the same as the one of finding a continuous phase transition, where ξ diverges, in Statistical Mechanics.

Continuous phase transitions have notably been studied by Widom [41], Fisher [42, 43], Patashinskii and Pokrovskii [44] and particularly by Kadanoff [45]. At criticality, approaching such phase transition, Kadanoff showed that a system displays an emergent symmetry: **scale invariance**. He argued that long distance (low energy) universal quantities do not sensitively depend on the short distance 'details' of a physical system. UV configurations that vary rapidly average-out and contribute only a small amount to the IR-scale behaviour.

2.1.2 Wilsonian renormalization

To solve the problem raised in remark 2 from Section 2.1.1, Wilson proposed an approximation scheme to sum over fluctuations based on Kadanoff's insight about the decoupling of long and short-distance physics [38]. He related the mathematical use of an effective Hamiltonian in condensed matter and statistical physics and the Feynman path integral formulation of QFTs. Wilson's method consists of integrating-out the fast modes (UV) of the generating functional to focus on the slower modes (IR). Again, we can iterate this procedure down to a scale k to obtain an effective theory for the low momenta modes $p < k$. The long distance physics is found by taking the $p \to 0$ limit. Hence, we can take the $k \to 0$ limit in the effective action, where no fluctuations remain, to find the behaviour of the IR dofs [46]. From the uncertainty principle, a decreasing value of the UV cutoff Λ is equivalent to an increasing value of the lattice spacing a (lowering the number of dofs), hence the iterative process is called **coarse-graining**. Wilson's procedure is called a Renormalization Group (RG) transformation. In statistical physics language, coarse-graining amounts to dividing a critical problem, where quantum fluctuations at all scales are considered, into sub-critical ones where this is not true anymore [24, 47].

In summary, contrary to the perturbation theory approach to renormalization, Wilson RG transformation is a 'top-down' approach. We take seriously the existence of a fundamental UV cutoff Λ (since divergences have a physical origin) and ask which effective theory emerges at a lower energy scale μ, where all quantum fluctuations at intermediate scales $\mu < E < \Lambda$ have been accounted for [39].

Let us now study two examples of RG transformations, the first in real space [45], the second in momentum space [19]. We first quickly explore the main steps of a RG transformation through the real space example. We will then study the momentum space case. The latter example will be more thoroughly and mathematically approached.

2.1.2.1 Real space RG transformation: Kadanoff spin decimation

Consider Euclidean theory in D dimensions regularized on a lattice. The field content of the theory is a real field $\phi(x)$, regularized as ϕ_i, where i labels lattice sites. The action is constructed from monomials $\mathcal{O}$ containing the field and its derivatives ∂_μ, that become finite differences ∇_μ on a discretized lattice. The action is then:

$$\mathcal{S} = \int d^D x \sum_{\alpha}^{\alpha} \mathcal{O}_\alpha = a^D \sum_{i}^{\alpha} \mathcal{O}_\alpha[\phi_i, \nabla_\mu \phi_i, ...], \tag{6}$$

where we used Einstein's summation to make the sum over α implicit in the second equality. u represents all coupling constants compatible with the symmetries of the system. *The key take-away is that the space of all couplings, or all actions, or all theories is defined by fixing the field content and symmetries of the system.* In the context of the Ising model on a square lattice of side-length a, Kadanoff proposed a way to compute the partition function

$$Z = \int \mathcal{D}\phi e^{-\mathcal{S}[\phi]} \tag{7}$$

iteratively. Each spin is a single dof sitting on a single lattice site, interacting with strength u (the coupling) with its nearest neighbours (we could consider next-to-nearest-neighbour interactions, three and four spin interactions...). Each dof has two eigenvalues: $+1$ for up-state and -1 for down state. At each step, a block of spins, say a square lattice of 4 spins, defines a new effective spin as shown in Fig. 2. This is known as spin 'decimation' [39]. The behaviour of a block is dependent on the average behaviour of the rapidly varying spin eigenvalues inside said block. By summing over short distances dofs in this way, one is left with an effective, renormalized picture with less dofs, a coarse-grained picture. The middle picture in Fig. 2 is coarse-grained. Through this coarse-graining process, we double the lattice spacing at each step: $a \to 2a \to 4a \ldots \iff a \to \frac{a}{b} \to \frac{a}{b^2} \ldots$ where $\|b\| < 1$ (here $b = \frac{1}{2}$). We then need to rescale the length scale at each step by b such that the actions are directly comparable

$$x' = bx. \tag{8}$$

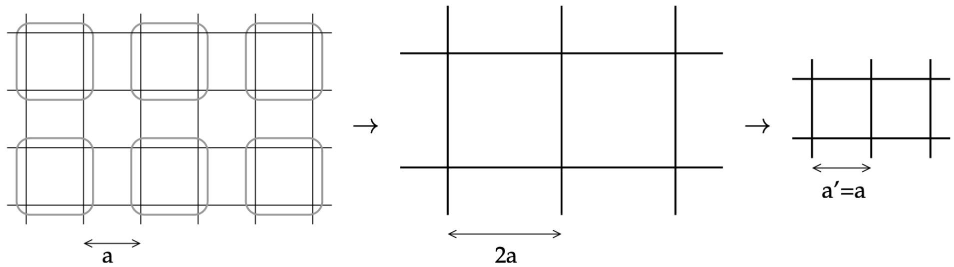

Figure 2: *The 3 steps of Wilson-Kadanoff renormalization in real space for the Ising model. (1) Average over blocks (decimation), (2) compute the effective action with a new lattice spacing, (3) rescale the space x and field ϕ such that the renormalized action is directly comparable to the original one. In this example, the rescaling factor is $b = \frac{1}{2}$, [39].*

The picture on the right in Fig. 2 is rescaled, with new lattice spacing $a' = \frac{a}{b}$, to be comparable with the original system. One would also need to rescale the field by a factor

$$\phi' = b^{\Delta}\phi \tag{9}$$

where Δ is the 'scaling dimension' of ϕ. The choice of Δ usually depends on the action. The field rescaling is required because we want the local fluctuations of ϕ' to look like those of ϕ. Importantly, since the rescaling allows to directly compare the forms of the renormalized action to the original one, the mapping between actions $\mathcal{S} \to \mathcal{S}'$ induced by the RG transformation can be expressed as a mapping between $u \to u'$ in the space of couplings. Each point corresponds to a different theory, so this space is also commonly called **theory space**, denoted $\mathcal{T}$. Taking the continuum limit, allows us to make continuous RG transformations [24, 47]. Therefore, the set of points u_j, for $j \in \mathbb{R}$, becomes a **RG trajectory**. Importantly, different couplings on the same RG trajectory correspond to theories with different micro-physics but similar macro-physics because they have the same generating functional Z [46]. This property is reminiscent of universality in Statistical Mechanics. The set of all such RG trajectories obtained from different initial conditions, i.e. different starting points, is called the **RG flow** [48].

2.1.2.2 Momentum space RG transformation: Wilson-Fisher momentum shell

We now discuss the Wilson-Fisher momentum shell RG transformation [19]. Instead of local blocks of dofs, we focus on iteratively integrating out shells of fast modes in momentum space [13]. We use Euclidean ϕ^4 theory as an example. Our starting point is the following action in D dimensions:

$$S[\phi] = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{{m_0}^2}{2}\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right]. \tag{10}$$

Expressed in terms of the sharp UV cutoff, the mass ${m_0}^2$ and the coupling constant λ are:

$${m_0}^2 = a\Lambda^2 \quad \text{and} \quad \lambda = c\Lambda^\epsilon, \tag{11}$$

where $\epsilon = 4 - D$. This is known as the ϵ expansion [13]. The action becomes

$$S[\phi] = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{a\Lambda^2}{2}\phi(x)^2 + \frac{c\Lambda^\epsilon}{4!}\phi(x)^4 \right]. \tag{12}$$

where a and c are dimensionless quantities.

STEP 1 - Coarse-graining

We now split momentum space into a momentum shell $b\Lambda < \|p\| < \Lambda$ (where $\|b\| < 1$) and the remaining ball of radius $\|p\| < b\Lambda$, such that

$$\phi(x) = \phi_f(x) + \phi_s(x), \tag{13}$$

since

$$\begin{aligned} \phi_s(x) &= \int \frac{d^D x}{(2\pi)^D}\theta(\Lambda^2 - p^2)e^{ipx}\phi(p), \\ \phi_f(x) &= \int \frac{d^D x}{(2\pi)^D}\theta(p^2 - \Lambda^2)e^{ipx}\phi(p), \end{aligned} \tag{14}$$

where $\theta(p^2 - \Lambda^2)$ is the heaviside function. Importantly, we chose Euclidean theory as an example since we cannot impose an upper cutoff on p^2 on a Lorentzian background, where the time component has a sign opposite to the spatial ones. We also split the Fourier transform $\phi(p)$ of the quantum field $\phi(x)$ between fast and slow modes, respectively $\phi_f(p)$ and $\phi_s(p)$. These have different support in momentum space. We use Λ as a sharp UV momentum cutoff, so the fast modes only have support on the momenta shell $b\Lambda < \|p\| < \Lambda$, that we will integrate out, and the slow modes have support on the rest of the space:

$$\phi_f(p) = \begin{cases} 0, & \|p\| > \Lambda \\ \phi(p), & b\Lambda < \|p\| < \Lambda \\ 0, & \|p\| < b\Lambda \end{cases}, \tag{15}$$

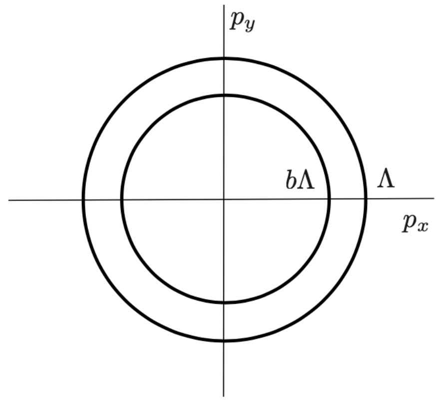

Figure 3: *The momentum shell $b\Lambda < \|p\| < \Lambda$, where $\|b\| < 1$ [37].*

$$\phi_s(p) = \begin{cases} 0, & \|p\| > \Lambda \\ 0, & b\Lambda < \|p\| < \Lambda \\ \phi(p), & \|p\| < b\Lambda \end{cases} . \tag{16}$$

Defining the averaged field as $\phi_s(p)$ with a new cutoff $b\Lambda$ is equivalent to the decimation process of the block-spin approach, shown in step (1) in Fig. 4. From equation (13), we have:

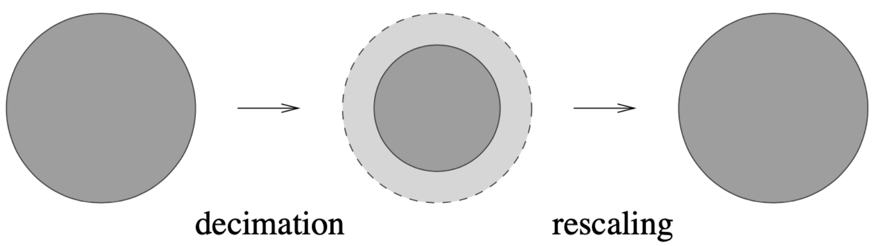

Figure 4: *The 3 steps of the RG procedure in momentum space. Step (1), left picture, corresponds to decimation or coarse-graining, where we define the averaged field as the slow modes ϕ_s for which we want an effective theory. During step (2), the middle picture, we integrate out the momentum shell where the fast modes have support. In step (3) we rescale momentum p, space x and the field ϕ for the actions S and S_{eff} to be directly comparable [39].*

$$S[\phi] = S_{free}[\phi] + S_{int}[\phi] = S_s[\phi_s] + S_f[\phi_f] + S_{int}[\phi_s, \phi_f], \tag{17}$$

and there is no mixing term between fast and slow modes for the quadratic free part of the action. Indeed, Fourier transforming $\phi(x)$

$$\begin{aligned}
S_{free}[\phi] &= \int d^D x \left[\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{a\Lambda^2}{2}\phi(x)^2\right] \\
&= \frac{1}{2}\int d^D x \left[\left(\int \frac{d^D p}{(2\pi)^D}\phi(p)\partial_\mu\left(e^{ip_\nu x^\nu}\right)\right)^2 + a\Lambda^2\left(\frac{d^D p}{(2\pi)^D}\phi(p)e^{ipx}\right)^2\right] \\
&= \frac{1}{2}\int d^D x \left(\int \frac{d^D p}{(2\pi)^D}\phi(p)e^{ipx}\right)\left(\int \frac{d^D q}{(2\pi)^D}\phi(q)e^{iqx}\right)\left[(ip)(iq) + a\Lambda^2\right].
\end{aligned} \tag{18}$$

Integrating over x to get a $(2\pi)^D\delta(p+q)$ factor and using the delta function ($q \to -p$) to integrate over q, we get:

$$\begin{aligned}
S_{free}[\phi] &= \int_{\|p\|<\Lambda} \frac{d^D p}{(2\pi)^D}\frac{1}{2}\left(p^2 + a\Lambda^2\right)\phi(p)\phi(-p) \\
&= S_s[\phi_s] + S_f[\phi_f],
\end{aligned} \tag{19}$$

as stated, where:

$$S_s[\phi_s] = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi_s)^2 + \frac{a\Lambda^2}{2}\phi_s^2 + \frac{c\Lambda^\epsilon}{4!}\phi_s^4\right], \tag{20}$$

and

$$S_f[\phi_f] = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi_f)^2 + \frac{a\Lambda^2}{2}\phi_f^2 + \frac{c\Lambda^\epsilon}{4!}\phi_f^4\right]. \tag{21}$$

Indeed, both terms in the factor $\phi(p)\phi(-p)$ have the same support in momentum space. This kills cross terms of the form $\phi_s(p)\phi_f(-p)$.

STEP 2 - Computing the effective action for the averaged field

Now, we integrate out the fast modes ϕ_f, as explained in step (2) of Fig. 4, by performing a Gaussian integration of ϕ_f. This is possible since there is no mixing between fast and slow modes at the free quadratic level.

$$\begin{aligned}
Z &= \int \mathcal{D}\phi e^{-S^\Lambda[\phi]} \\
&= \int \mathcal{D}\phi_s \mathcal{D}\phi_f e^{-S^\Lambda[\phi_s,\phi_f]} \\
Z &= \int \mathcal{D}\phi_s e^{-S^{b\Lambda}_{eff}[\phi_s]} \int \mathcal{D}\phi_f e^{-S^\Lambda_f[\phi_f] - S^\Lambda_{int}[\phi_s,\phi_f]}
\end{aligned} \tag{22}$$

We write

$$e^{-S^{b\Lambda}_{eff}[\phi_s]} \equiv e^{-S^\Lambda_s[\phi_s]} \int \mathcal{D}\phi_f e^{-S^\Lambda_f[\phi_f] - S^\Lambda_{int}[\phi_s,\phi_f]} \tag{23}$$

which defines the effective action $S^{b\Lambda}_{eff}[\phi_s]$ of the slow modes in the theory cutoff at scale $b\Lambda$. We can generally compute the path integral in equation (23) either by using perturbation theory in c, provided λ is small enough, or by integrating over the high-momentum modes, treating the low momentum ones as external sources [39]. The effective action is now

$$S^{b\Lambda}_{eff}[\phi_s] = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi_s)^2 + \frac{{m_0}^2}{2}\phi_s^2 + \frac{\lambda}{4!}\phi_s^4 \right]. \tag{24}$$

The Wilsonian action at scale $b\Lambda$ is equivalent to the bare action defined in a cutoff scheme since we only integrate modes up to the energy scale Λ to quantize the theory [49]. Once the new Wilsonian action is known, it can be used to compute its corresponding generating functional in the same way as a classical action for which no heavy fields existed. Importantly, the Wilsonian Lagrangian is *local* as long as we work to finite order in its expansion in inverse powers of the fast scales [50].

STEP 3 - Rescaling

There is an implicit dependence of the original action S on the cutoff scale Λ, and the momenta in the effective theory are still to scale with this cutoff. We need a theory where the momenta of the renormalized theory are to scale with the IR cutoff $b\Lambda$ and not the UV one. One must rescale both x, p, and ϕ. The former two rescalings in equation (25) allow to make the new action to scale with the new cutoff $b\Lambda$,

$$p \to p' = \frac{p}{b} \quad \text{and} \quad x \to x' = bx, \tag{25}$$

where $\|b\| < 1$. This induces a rescaling of both the integration measure and the spacetime derivative:

$$\int d^D x' = b^D \int d^D x \quad \text{and} \quad \partial_{\mu'} = \frac{\partial_\mu}{b}. \tag{26}$$

The field rescaling in equation (27) is necessary to make $S' = S^{b\Lambda}_{eff}[\phi_s]$ and S comparable. Accounting for all rescalings above, in order to keep the kinetic term the same in S' and S, we have:

$$\begin{aligned} \int d^D x' (\partial_{\mu'} \phi'(x'))^2 &= \int d^D x (\partial_\mu \phi(x))^2 \\ \int d^D x' (\partial_{\mu'} \phi'(x'))^2 &= \frac{b^2}{b^D} \int d^D x' (\partial_{\mu'} \phi(x))^2 \\ \implies \phi'(x')^2 &= b^{2-D} \phi(x)^2 \\ \phi'(x') &= b^{\frac{2-D}{2}} \phi(x). \end{aligned} \tag{27}$$

As promised, the effective action looks like S:

$$S'[\phi'] = S^{b\Lambda}_{eff}[\phi_s] = \int d^D x' \left[\frac{1}{2}(\partial_\mu \phi')^2 + \frac{{m_0}^2(b\Lambda)}{2}(\phi')^2 + \frac{\lambda(b\Lambda)}{4!}(\phi')^4 \right], \tag{28}$$

As stated before, the RG induced mapping $S \to S'$ can be expressed in terms of the simultaneous mapping $m_0 \to m_0'$ and $\lambda \to \lambda'$ in the, *a priori*, infinite space of couplings. The RG transformation is a scale transformation, which makes the couplings explicitly depend on, or 'run' with, the scale $b\Lambda$ [39].

The theory is now divergence-free. There is no more summation of fluctuations over all length scales since integrations are performed over momenta-shell $p \in [b\Lambda, \Lambda]$ which involve only a finite number of degrees of freedom. We only expect divergences when having integrated all momenta shells, i.e. after an infinite number of iterations of the RG procedure [46, 48].

Now is a good time to assess the power of the Wilsonian approach, especially its advantages over standard perturbative renormalization.

Firstly, rather than performing the complicated continuum limit at once, we break the process in many incremental steps. After one step is well understood, the problem reduces to the iteration of many simple steps. In doing so, we bypass the standard UV renormalization problem to focus on solving a dynamical system [51].

Secondly, the sliding momentum scale $b\Lambda$ enters the action in two ways, but *always* drops out of physical observables. The action's dependence on $b\Lambda$ originates from the running couplings, and from virtual particles which have their momenta cutoff at this scale. It turns out that the $b\Lambda$ dependence in couplings exactly cancels the contribution from virtual particles. This discussion parallels the traditional perturbative treatment of renormalization, in which the regularization dependence of the divergent loop integrals are cancelled by 'manually' introducing counter-interaction terms into the classical action. Thus, one might regard the classical bare action of a non-renormalized theory as the Wilsonian action of a more fundamental theory that applies above the introduced UV cutoff Λ. This view point is particularly powerful because of the insight it gives on the physical nature of the cutoff-dependence ($b\Lambda$) cancellation [50].

2.2 Iteration, RG flow, critical surface and fixed points

By iterating this RG procedure, we build a series of couplings [46].

$$\begin{aligned}
&\Lambda \to u, \\
&b\Lambda \to u', \\
&b^2\Lambda \to u'', \text{etc.}
\end{aligned} \tag{29}$$

As stated previously, the repeated RG action induces a flow in the space of couplings $u \to u' \to u''...$, called the RG flow. If this flow converges to a constant value u^* of the coupling, $\to u' \to u''... \to u^* \to u^*$, we call u^* a **fixed point** (FP), or **critical point** of the RG flow. Despite its misleading name, the set of RG transformations $\{R_b\}$ is not a group in the mathematical sense, but only a 'semi-group', since RG transformations are non-invertible. This is to be expected since, through coarse-graining, we integrate out the microscopic dofs and lose information about the physical system and the structure of the theory [39, 48]. Though, it is possible to invert the transformation in $\mathcal{T}$, $u \mapsto u(b) = R_b(u)$, near a FP, this subtlety can be ignored and we always consider b as a continuous parameter, i.e. we took the continuum limit. Hence, in practice, this RG procedure is performed in a continuous flow. For the Wilson-Fisher method this corresponds to integrating out infinitesimal shells of fast momentum modes.

We define the set of initial conditions that flow to a FP $u \to u' \to \dots u^* \to u^*$ as the **basin of attraction** of the FP. Now, we come back to the block-spin example to illustrate this notion. Take a RG transformation $R_b(\cdot)$ that maps the coupling of the original spin lattice system u onto u'. The dimensionless correlation length is defined as $\xi = \frac{\bar{\xi}}{a}$, with a the lattice spacing [46]. It transforms as $\xi((b)) = b\xi(u)$ under $R_b(\cdot)$. By definition, $R_b(u^*) = u^*$ and so $\xi(u^*) = b\xi(u^*)$, which implies that either $\xi(u^*) = 0$ or $\xi(u^*) = \infty$, where $\|b\| < 1$. In statistical mechanics, a FP with $\xi(u^*) = 0$ is a trivial fixed point. A FP with $\xi(u^*) = \infty$ is a **critical fixed point**. The latter describes the singular behaviour of continuous phase transitions. This type of FP will be our focus. We define the **UV critical hypersurface** or **unstable manifold**, denoted $\mathcal{S}_{UV}$, as the basin of attraction of a critical fixed point. If the starting point of a RG trajectory u_c is on the critical surface, then

$$\lim_{b\to 0} R_b(u_c) = u^*, \tag{30}$$

and in turn $\lim_{b\to 0} \xi((b)) = \infty$. Indeed, if $\xi = \infty$, where the system is at criticality, then $R_b(\xi) = b\xi = \infty$ and in particular $\xi' = b\xi = \infty$. The block-spin system with coupling u' is still critical. In conclusion, the critical surface is also defined as the set of points u_c with infinite correlation length $\xi_c = \infty$ [46, 48].

All theories in the basin of attraction of a FP have the same long distance physics, described by the theory with coupling u^*: they are in the same universality class. Indeed, equation (30) shows that, regardless of where the starting point u_c is on the critical surface, all RG trajectories will end at the FP u^* . However, universality also holds in physical systems asymptotically close to criticality. Hence, we can assume that the RG flow is continuous near the FP. Furthermore, the RG trajectory emanating from a point u close to $\mathcal{S}_{UV}$ stays close to the one emanating from a point u_c on $\mathcal{S}_{UV}$ as long as $\xi(b) = b\xi$ is large enough. When $\frac{1}{b} \sim \xi$ the trajectory starting at u diverges away from u^* and the critical surface [46].

2.2.1 Linear behaviour of RG flows near fixed points

With these relations and the hypothesis of the existence of a FP, we can derive power law behaviours of the RG flow in the viscinity of u^*. We now study all RG trajectories emanating from points close to criticality that stay close to u^* for a while. This allows us to linearise the RG flow equations around u^*, where k denotes a variable mass-scale. We now need an equation from which to derive the RG trajectories and the RG flow. The dimensionful **essential** couplings $\bar{u}_\alpha$, i.e. those that cannot be absorbed by field redefinitions, parametrizing theory space $\mathcal{T}$ give a system of infinitely many coupled partial differential equations:

$$\bar{\beta}_\alpha(\bar{u}_\alpha, k) = k\partial_k \bar{u}_\alpha(k), \tag{31}$$

where $\alpha = 1, 2, \ldots$ indexes the couplings. Equations (31) are called RG equations and they will be derived in Section 3.1.7. The function in the LHS of (31) is the Gell-Mann Low beta function [14], originally used in perturbative renormalization to change the values of the bare coupling u and the UV cutoff Λ while keeping the value of the renormalized coupling, say λ, constant

$$\beta(u) = \Lambda \frac{\partial u}{\partial \Lambda}\Big|_\lambda. \tag{32}$$

Denoting $\bar{u}_\alpha$ as dimensionful couplings with canonical mass dimension d_α, we define

$$u_\alpha = k^{-d_\alpha} \bar{u}_\alpha, \tag{33}$$

where the couplings u_α are now dimensionless. We can rewrite

$$\beta_\alpha(u_\alpha, k) = k\partial_k u_\alpha(k). \tag{34}$$

Now, the β_α have an explicit k dependence and they define a vector field on $\mathcal{T}$, as shown later in Fig. 8. RG trajectories are solutions of equation (34). Following [46], we use these definitions to compare two running couplings on the same RG trajectory, differing by an infinitesimal RG transformation, from scale k to scale $k - dk$ (the direction of increasing coarse-graining is the direction of decreasing momentum-scale k):

$$k\frac{\partial u_k}{\partial k} = \beta(u). \tag{35}$$

This makes the evolution of $\beta(u)$ local in coupling-space, since at each step, $\beta(u)$ only depends on the coupling evaluated at scale k. By definition, in the dimensionless $\mathcal{T}$, a FP is a zero of the β vector field, such that $\beta(u^*) = 0$. In this case, the dimensionless couplings u^* do *not* depend on scale [52, 53]. This is consistent with the scale invariance property of critical systems shown by Kadanoff. Hence, Taylor expanding around the FP,

$$\begin{aligned} k\frac{du_k}{dk} - k\frac{du^*}{dk} &= \beta(u_k) - \beta(u^*) \\ &= \frac{d\beta}{du_k}\Big|_{u^*}(u_k - u^*) + \mathcal{O}\Big((u_k - u^*)^2\Big), \end{aligned} \tag{36}$$

where $\frac{d\beta}{du_k}\Big|_{u^*}$ is the **Jacobi**, or **stability matrix** with explicit indices

$$\mathcal{B}_{ij} = \frac{d\beta_i}{du_{k,j}}\Big|_{u^*} = \partial_j\beta_i(u^*). \tag{37}$$

Thus, near the fixed point where $u_k \simeq u^*$, we approximate to linear order in $(u_k - u^*)$ and we get,

$$k\frac{d(u_k - u^*)}{dk} \simeq \mathcal{B}(u_k - u^*). \tag{38}$$

Having linearized the RG flow near the fixed point, we can rewrite the beta function as

$$\beta(u) = k\partial_k(u(k) - u^*) \simeq \sum_j \mathcal{B}_{ij}(u_j(k) - u_j^*), \tag{39}$$

where we highlight that u_j^* is not dependent on the scale k, or importantly, on the UV cutoff Λ. Solving equation (39), we find in terms of k

$$u_i(k) = u_i^* + \sum_I C_I V_i^I \left(\frac{k}{k_0}\right)^{-\theta_I}. \tag{40}$$

C_I are constants of integration. They are determined by the initial conditions, or starting points, of the RG flow. θ_I are called **stability**, **scaling** or **critical exponents**, since they are related to the critical exponents of second order phase transitions when we apply the RG formalism to critical phenomena [54, 55]. Importantly, they are universal quantities, i.e. they depend neither on the coordinate choice on $\mathcal{T}$, $\{u_\alpha\}$, nor on the cutoff scheme [53]. To derive equation (40) we used

$$\sum_j \mathcal{B}_{ij} V_j^I = -\theta_I V_i^I, \tag{41}$$

i.e. V_I are the eigenvectors of $\mathcal{B}$ with eigenvalues $-\theta_I$. $\{V^I\}$ are sometimes called scaling operators [48, 53]. We assume that $\{V^I\}$ form a complete set of eigenvectors, providing a basis for the tangent spaces to $\mathcal{T}$ at the FP.

We derive equation (40) as follows. We define the difference between couplings

$$g(k) = u(k) - u^*, \tag{42}$$

which is sometimes called the **scaling field** [48, 53]. Expanding in the complete basis $\{V_i^I\}$ formed by the eigenvectors of $\mathcal{B}_{ij}$, we get

$$\sum_i g_i(k) V_i^I = u_i(k) - u_i^*. \tag{43}$$

Now, we can plug this into equation (39) to obtain

$$\begin{aligned} k\frac{\partial g(k)}{\partial k} &= \sum_i \mathcal{B}_{ij} g_i(k) V_i^I, \\ k\frac{\partial g(k)}{\partial k} &= \sum_{i,I} (-\theta_I) g_i(k) V_i^I, \\ k\frac{\partial(\sum_i g_i(k) V_i^I)}{\partial k} &= -\sum_{i,I} \theta_I g_i(k) V_i^I, \end{aligned} \tag{44}$$

where we used equation (41). Dropping the indices and the basis vectors [56], we get

$$k\frac{\partial g(k)}{\partial k} = -\theta_I g(k). \tag{45}$$

We can try solving this partial differential equation using the following general ansatz

$$g(k) = C_I \left(\frac{k}{k_0}\right)^{-\theta_I}, \tag{46}$$

where k_0 is a fixed reference scale and C_I are constants of integration. As expected, equation

(45) is solved by the ansatz (46):

$$
\begin{aligned}
\frac{\partial g(k)}{\partial k} &= \frac{\partial}{\partial k}\left(C_I\left(\frac{k}{k_0}\right)^{-\theta_I}\right), \\
&= (-\theta_I)k^{-\theta_I-1}C_I\left(\frac{1}{k_0}\right)^{-\theta_I}, \\
\therefore k\frac{\partial g(k)}{\partial k} &= -\theta_I C_I\left(\frac{k}{k_0}\right)^{-\theta_I}, \\
k\frac{\partial g(k)}{\partial k} &= -\theta_I g(k).
\end{aligned}
\tag{47}
$$

We used the fact that k_0 is a constant with respect to k. The ansatz (46) is exactly equivalent to equation (40), which is indeed the general solution to (39):

$$
\begin{aligned}
u_i(k) &= u_i^* + \sum_I C_I V_i^I\left(\frac{k}{k_0}\right)^{-\theta_I}, \\
u_i(k) - u_i^* &= \sum_I C_I V_i^I\left(\frac{k}{k_0}\right)^{-\theta_I}, \\
\sum_i g_i(k)V_i^I &= \sum_I C_I V_i^I\left(\frac{k}{k_0}\right)^{-\theta_I}, \\
g(k) &= C_I\left(\frac{k}{k_0}\right)^{-\theta_I}.
\end{aligned}
\tag{48}
$$

We now introduce a general scale variable s parametrizing RG trajectories in the direction of increasing coarse-graining, such that $s = \frac{1}{b}$ [46]. For instance, the momentum shell previously defined in Fig. 3 now has bounds $p \in [\frac{\Lambda}{s}, \Lambda]$. Clearly, $\|s\| > 1$ since $\|b\| < 1$. We replace the fixed reference momentum-scale k_0 by the UV cutoff, and the RG trajectory's scale parametrization is now carried by s rather than k:

$$
\begin{aligned}
k_0 &= \Lambda, \\
k &= \frac{\Lambda}{s}.
\end{aligned}
\tag{49}
$$

We can then rewrite the general solution (40) in terms of this new parameter s:

$$
\begin{aligned}
u_i(s) &= u_i^* + \sum_I C_I V_i^I\left(\frac{\frac{\Lambda}{s}}{\Lambda}\right)^{-\theta_I}, \\
u_i(s) &= u_i^* + \sum_I C_I V_i^I\left(\frac{1}{s}\right)^{-\theta_I}, \\
u_i(s) &= u_i^* + \sum_I C_I V_i^I s^{\theta_I}.
\end{aligned}
\tag{50}
$$

As mentioned before, $\mathcal{B}$ is not symmetric in general and its eigenvalues may be complex [57]. Still, couplings like $g(s)$ are observables, and we expect to experimentally determine such quantities. Hence, C_I should cancel out contributions from the oscillatory behaviour of s^{θ_I} when $\mathrm{Im}(\theta_I) \neq 0$ [53]. We can then determine the stability of the fixed point under RG perturbations, $s \to s + ds$, solely from $\mathrm{Re}(\theta_I)$. We can rewrite

$$g(s) = C_I s^{\mathrm{Re}(\theta_I)}, \tag{51}$$

or

$$\boxed{u_i(s) = u_i^* + \sum_I C_I V_i^I s^{\mathrm{Re}(\theta_I)}}. \tag{52}$$

In conclusion, around the FP u^*, the RG flow behaves as power laws along its eigendirections V^I [24, 37, 46–48, 53, 54, 56, 58–60]. We assume that the flow starts at couplings near the FP but not necessarily in its basin of attraction [48]. We distinguish between three cases:

- $\underline{\mathrm{Re}(\theta_I) > 0}$: since $\|s\| > 1$, $s^{\mathrm{Re}(\theta_I)} \nearrow$ when $s \nearrow$. Hence, $g(s) \nearrow$ when $s \nearrow$ and the coupling is greater at the coarse-grained scale $\frac{\Lambda}{s}$. Since $g(s) = u(s) - u^*$, then the RG flow in the direction V^I gets away from the FP. We call V^I a linearly **relevant direction** and $g(s)$ a linearly **relevant coupling**. Indeed, in the direction of increasing coarse-graining, we go from UV to IR physics so we say $g(s)$ is 'relevant' to low-energy physics.

  An action containing relevant couplings will be driven away from $\mathcal{S}_{UV}$. The RG trajectories that these theories follow are shown as black lines in Fig. 5. Starting from the FP action and turning on a relevant coupling generates a **renormalized trajectory**, i.e. the RG flow emanating from the FP. This is the red line in Fig. 5. The black lines evolve along this line in the direction of increasing coarse-graining [52].

- $\underline{\mathrm{Re}(\theta_I) < 0}$: then $s^{\mathrm{Re}(\theta_I)} \searrow$ when $s \nearrow$. The flow in the direction V^I approaches the FP. In this case, V^I is a linearly **irrelevant direction** and $g(s)$ a linearly **irrelevant coupling**. If u describes a trajectory on the critical surface of said FP, we have to set $C_I = 0$ such that $\lim_{s\to\infty} u(s) = u^*$. Irrespective of the initial conditions, if we turn on irrelevant couplings in a given action, the RG flow will drive us back to the FP.

- $\underline{\mathrm{Re}(\theta_I) = 0}$: $g(s)$ is a linearly **marginal coupling**. We need to go beyond the linear approximation to determine if it is relevant or irrelevant. The RG flow in this direction is slow since it is logarithmic instead of a power law.

There are a clear decompositions of $\mathcal{T}$ from the eigenvectors of $\mathcal{B}$ following these definitions.

We say the different V^Is respectively span the relevant, irrelevant and marginal subspaces of $\mathcal{T}$ [53, 56, 58–60]. The irrelevant sub-manifold of $\mathcal{T}$, or the set of points attracted to a FP in along the RG flow, is called the **stable manifold** of that FP.

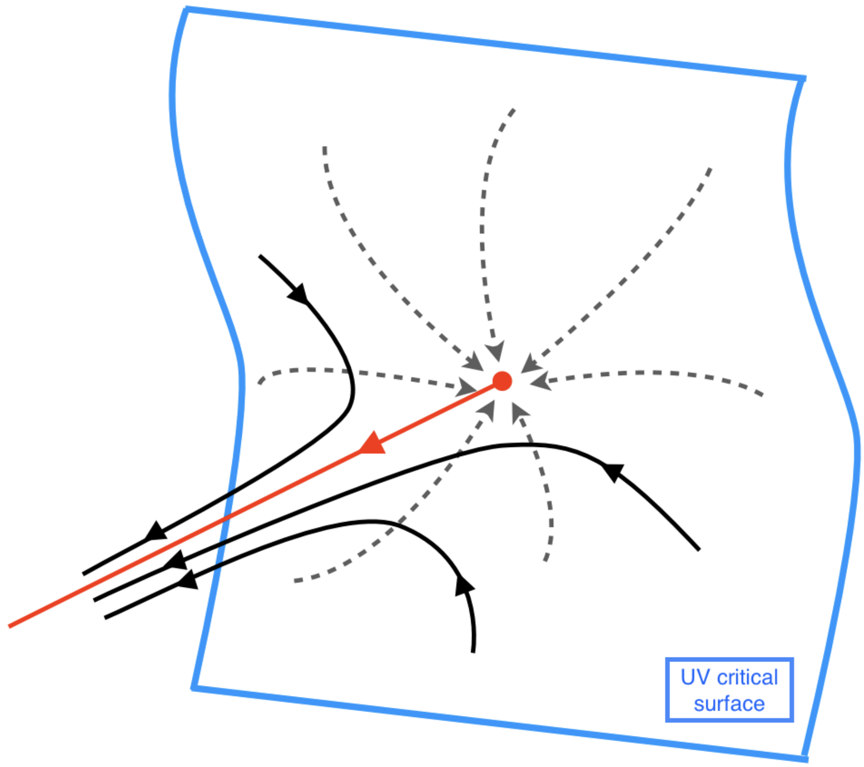

Figure 5: *Arrows point in the direction of increasing coarse-graining. From equation (30), we know that theories on $\mathcal{S}_{UV}$ are attracted to the FP in this direction. The corresponding trajectories appear as grey dotted lines. Black lines correspond to trajectories associated with actions containing relevant couplings. They are driven away from $\mathcal{S}_{UV}$ by the renormalized trajectory, the red line, i.e. the RG trajectory emanating from the FP [60]. Points on such trajectory are called* ***perfect actions*** *since they can be used to compute continuum answers for physical quantities in the presence of a UV cutoff Λ [24, 47]. Theories corresponding to black line trajectories are in the same universality class. That is, they have the same long-distance physics [52].*

2.3 Non-Perturbative renormalizability

2.3.1 Different fixed points

Consider a set of beta-functions behaving as follows

$$\beta_\alpha(u) = -d_\alpha u_\alpha + \mathcal{O}(^2), \tag{53}$$

where we sum over $\alpha = 1, 2, \dots$ and do not account for loop corrections coming from the coupling. As before, the dimensionful couplings $\bar{u}_\alpha$ have canonical mass dimension $[\bar{u}_\alpha] = d_\alpha$. We now want to find a FP, that is, a solution to $\beta_\alpha(u^*) = 0$. An obvious solution is $u^*_\alpha = 0$ for all α, where $d_\alpha \neq 0$. This solution defines a **Gaussian Fixed Point** (GFP), because the FP action becomes quadratic and free in the fields, and the measure becomes Gaussian [24, 47]. There is always a trivial GFP, where all couplings vanish, at the origin of each theory space $\mathcal{T}$. The stability matrix of this FP is given by $\mathcal{B}_{\alpha j} = \partial_j \beta_\alpha(u) = -d_\alpha \delta^\alpha_j$. This matrix is diagonal and we established that its eigenvalues were $-\theta_\alpha$ in equation (41). Hence, cancelling the minus signs on both sides,

$$\theta_\alpha = d_\alpha. \tag{54}$$

This motivates the introduction of a minus sign in equation (41). The eigenvalues of $\mathcal{B}$ are exactly the canonical mass dimension of $\bar{u}_\alpha$. While we can find beta functions behaving differently than in (53), the statement (54) is coordinate-independent, since θ_α are universal for all α, and always defines a GFP. A **Non-Gaussian Fixed Point** (NGFP) is then defined as a point where critical exponents can vary from their canonical value [53, 56, 58, 59].

2.3.2 Dimension of $\mathcal{S}_{UV}$

We must impose that, for all couplings,

$$\lim_{k\to\infty} u_\alpha(k) = u^*_\alpha. \tag{55}$$

This will allow us to determine the dimensionality of the critical surface. Note, taking the $k \to \infty$ limit, provided that $k \leq \Lambda$ as it should, drives the UV cutoff $\Lambda \to \infty$ as well. So removing the cutoff Λ can be done by taking the UV limit $k \to \infty$ [24, 47].

Close to the FP, the general solution (52) to the linearized RG equation (39) provides a valid description of all RG trajectories, whether they run in $\mathcal{S}_{UV}$ or not [53]. Now, letting $k \to \infty$, or equivalently $s \to 0$, in equation (52), makes terms corresponding to relevant directions vanish, independently of the values of C_I. Terms corresponding to irrelevant directions, with

$\mathrm{Re}(\theta_I) < 0$, will diverge in the $k \to \infty$ limit unless their $C_I = 0$. Finally, terms with $\mathrm{Re}(\theta_I) = 0$ will deviate from u^* by a finite amount unless their $C_I = 0$ as well. Therefore, to impose the limit (55), we set $C_I = 0$ for irrelevant and marginal directions. On $\mathcal{S}_{UV}$, we rewrite

$$u_i(s) = u_i^* + \sum_{\mathrm{Re}(\theta_I)>0} C_I V_i^I s^{\mathrm{Re}(\theta_I)}. \tag{56}$$

In conclusion, $\mathcal{S}_{UV}$ is spanned by the RG trajectories emanating from the FP along the relevant directions. The dimensionality of $\mathcal{S}_{UV}$, denoted Δ_{UV}, is given by the dimensionality of the relevant subspace of $\mathcal{T}$, for k decreasing. Equivalently, it is the number of eigenvalues of the stability matrix $\mathcal{B}$ with a positive real part:

$$\dim(\mathcal{S}_{UV}) \equiv \Delta_{UV} = \#\{\theta_I, \mathrm{Re}(\theta_I) > 0\}. \tag{57}$$

This assertion is consistent with the following statement from statistical mechanics: the dimension of the UV critical surface is the number of parameters one needs to fine-tune to make the system critical [46].

Note, even though $\mathcal{B}$ may be an infinite dimensional matrix, Δ_{UV} can be finite. Indeed, the UV critical surface of a critical FP, those associated to second-order phase transitions, are finite dimensional [32, 53]. For instance, there are two relevant directions for the transverse field Ising model since only the temperature and magnetic field are needed to make the system undergo a continuous phase transition [46].

2.3.3 Renormalization using fixed points

The analysis in Section 2.2.1 was done in the direction of increasing coarse-graining, i.e. in the direction of decreasing momentum scale k. This is the direction of the RG flow. We could also work in the *inverse* RG flow direction, i.e. increasing k. In this direction, a relevant direction is a V^I with non-zero C_I and $\mathrm{Re}(\theta_I) < 0$, instead of $\mathrm{Re}(\theta_I) > 0$ under the RG flow. The same logic applies to irrelevant and marginal directions. We can think about the sign (direction) of the vectors V^I changing. This negative sign is absorbed by $\mathrm{Re}(\theta_I)$, making a previously irrelevant direction relevant and vice-versa. We say irrelevant couplings are **IR-attractive** and **UV-repulsive**, since they go towards the FP in the RG flow direction (from UV to IR) and away from it in the direction of inverse RG flow (from IR to UV). On the contrary, relevant couplings are **UV-attractive** and **IR-repulsive**.

By definition, $\mathcal{S}_{UV}$, the unstable manifold of a FP, is the subset of points in $\mathcal{T}$ that are pulled towards the FP under the inverse RG flow, i.e. the relevant subspace of $\mathcal{T}$. Equivalently,

$\mathcal{S}_{UV}$ is the set of points in $\mathcal{T}$ that can be reached by the RG, or *renormalized* trajectories emanating from the FP, as seen in Fig. 5.

What do we mean here by "renormalized"? From the concepts of theory space and FPs, we arrive at a new notion of non-perturbative renormalization. The boundary of $\mathcal{T}$ separates points with well-defined essential couplings u_α (inside), and those with divergent ones (outside). Now, we know removing a UV cutoff from a QFT amounts to integrating all field modes out. To do this, there should exist a well-defined action at all momenta scales $k \in [0, \infty)$. Geometrically, using a dimensionless language, renormalization theory is tasked to construct a **complete RG trajectory**. That is, an infinitely long RG trajectory that never leaves $\mathcal{T}$, i.e. that never develops divergences. This must be true for finite scales from the IR limit $k \to 0$ up to the UV limit $k \to \infty$, and at both limits. Every such RG trajectory corresponds to a renormalized QFT. Thus, non-perturbative renormalization becomes a problem about the behaviour of the trajectories in infinite dimensional $\mathcal{T}$ after enough RG 'time'. RG time is denoted t where $t \equiv \ln\left(\frac{k}{k_0}\right)$ [53, 54].

When looking for complete RG trajectories, it is notoriously difficult to ensure that the trajectories stay in $\mathcal{T}$ at $k \to \infty$. This is the reincarnation of the high-momenta UV divergences met in Section 2.1.1. To overcome this challenge, we can perfom the UV limit $k \to \infty$ at a FP. This point is a zero of the $\vec{\beta} \equiv \beta_\alpha$ vector field where the momentum-scale running of the dimensionless couplings encoded in β_α stops. The 'velocity' of the RG trajectories $\partial_t u_\alpha = \beta_\alpha$ is small in the viscinity of this FP since the β_α are small. Therefore, if the RG trajectory describing a QFT is expected to run into a FP when $k \to \infty$, we can use-up an infinite amount of RG time while staying within an infinitesimally small neighbourhood of the FP. In doing so, we ensure that the RG trajectory stays inside $\mathcal{T}$ when taking the UV limit, i.e. it develops no divergences at high-energies since the FP is an inner point of theory space. We say the corresponding QFT is **asymptotically safe** from UV divergences when removing the cutoff. To ensure that the trajectory of a QFT hits the FP at $k \to \infty$, its couplings must be on the UV critical surface of said FP [32, 53, 54, 56, 58, 59]. Therefore, a trajectory is complete if and only if it lies upon the UV critical hypersurface of the corresponding FP.

2.4 The Asymptotic Safety Program

2.4.1 Weinberg's criteria for asymptotically safe theories

The expression 'asymptotic safety', first introduced by Weinberg in 1979 [32], refers to the 'asymptotic freedom' of non-Abelian gauge theories, such as QCD [34, 35]. Asymptotic freedom is the realisation of the method outlined in Section 2.3.3, restricted to cases where we take the UV limit $k \to \infty$ at a GFP. As required, QCD lies on the UV critical surface of the GFP at the origin of its theory space $\mathcal{T}_{QCD}$, where couplings vanish. We would like to apply this formalism to GR. However, asymptotic freedom fails for the EH action. Requiring that the EH action is dimensionless in non-natural units, we find that the dimensionful gravitational coupling, Newton's constant $\bar{G}_{\mathrm{N}}$, has negative mass dimension $[\bar{G}_{\mathrm{N}}] = -2$ in exactly 4 dimensions. Therefore, it is *irrelevant* in the direction of inverse RG flow. From equation (54) and the analysis in Section 2.2.1, this statement entails that GR is not on the UV critical surface of the corresponding GFP [53, 54].

To tackle this issue, in his seminal 1979 paper [32], Weinberg proposes a conjecture about the existence of the UV limit in quantum gravity. Now, this conjecture is known as the *Asymptotic Safety scenario* or *Asymptotic Safety program.* Weinberg's renormalizability condition based on Wilson-Kadanoff renormalization [15, 16, 20, 21, 38, 45, 61] is the following: ***a QFT is asymptotically safe (UV complete) if it lives on the UV critical surface of a corresponding non-trivial fixed point*** [32]. His conjecture is divided in two non-perturbative renormalization conditions as follows:

- (AS1) The theory space $\mathcal{T}$ contains a NGFP and the corresponding $\mathcal{S}_{UV}$ has low-dimensionality, i.e. Δ_{UV} is finite.

- (AS2) Every trajectory that does not hit said NGFP develops divergences in the UV limit $k \to \infty$, or $\Lambda \to \infty$.

2.4.2 Predictivite power of asymptotically safe theories

The second part of condition (AS1) connects asymptotic safety to phenomenology [56]. Indeed, if $\Delta_{UV} < \infty$ for some FP, the RG trajectory running into it corresponds to a QFT which is as predictive as a perturbatively renormalizable theory with Δ_{UV} 'renormalizable couplings' [54]. We elaborate on these statements now.

Asymptotically safe theories aim to be *fundamental* QFTs, that is theories valid at arbitrarily high momentum scale. In this regard, they are different from EFTs that require, at higher momenta, an ever increasing amount of couplings to be determined by experiments. By requiring the existence of a continuum limit, and presupposing the existence of a FP, we reduce the RG dynamics to the UV critical surface, making finite the number of free parameters of the model. These parameters determine the dynamics at *all scales*, since they are part of complete RG trajectories, i.e. trajectories which correspond to theories that have the same IR physics as that of the FP, a point of scale invariance [59]. Let us now explain how asymptotic safety reduces the number of experimentally determinable parameters.

The limit $k \to \infty$ places all RG trajectories on $\mathcal{S}_{UV}$ since all $C_I = 0$ for irrelevant couplings in this infinite cutoff limit. We determined in Section 2.3.3 that relevant couplings are UV-attractive under the inverse RG flow, so they span $\mathcal{S}_{UV}$. Any point on $\mathcal{S}_{UV}$ can be reached in the IR by a RG trajectory emanating from the FP in the UV.

Hence, IR values of relevant couplings are not predicted and need to be determined by experiments. There is a family of Δ_{UV} free parameters of the model. Of these Δ_{UV}, there are $(\Delta_{UV} - 1)$ RG trajectories approaching u^* in the UV, since we must specify $(\Delta_{UV} - 1)$ constants of integration in equation (52) to identify one specific RG trajectory in $\mathcal{S}_{UV}$. Hence, $(\Delta_{UV} - 1)$ parameters are dimensionless and only one parameter is dimensionful. The latter tells us where we are on the chosen RG trajectory in terms of k [32]. We see that the predictability of the theory increases when Δ_{UV} decreases [54]. Therefore, for the theory to be predictive, we require that Δ_{UV} be finite.

In contrast, all orthogonal directions to the FP (the stable manifold, spanned by irrelevant couplings) are predictable since they are IR attractive. That is, there is no freedom in the IR values of the irrelevant couplings because they are determined by the FP values [56].

In conclusion, the conditions (AS1) and (AS2) of asymptotic safety play a role similar to the usual perturbative renormalizability condition, in QED for instance [53]. That is, they fix all parameters of the theory (irrelevant couplings) but a finite number of them (relevant couplings) [32].

Some couplings do not affect the explicit expressions of some observables. These couplings might diverge at $k \to \infty$ and not be ascribed FP values. Said couplings do not affect the theory's predictivity so we call them **inessential**, following Weinberg [32]. This is why, up to now, we focused on essential couplings to parametrize theory space, as in Section 2.2.1.

2.4.3 The first hint of asymptotic safe gravity: gravity in $2 + \epsilon$ dimensions

As we mentioned in Section 2.4.1, asymptotic freedom fails for GR. Hence, to build a theory of quantum gravity, we might want to look for a NGFP, where $u^*_\alpha \neq 0$, and take the continuum limit at this point, *à la* Asymptotic Safety. However, nothing guarantees that we can use Asymptotic Safety to complete GR at asymptotically high-energies. In particular, there is no reason why there should exist a FP that satisfies the requirements (AS1) and (AS2).

To solve this problem, Weinberg uses results from the study of critical phenomena [32]. The failure of mean field theory shows that some phase transitions are not governed by GFP, outlining the need to look for other FP. This problem was solved by Wilson and Fisher in 1972 by considering a continuous change in the spacetime dimension of the physical system, known as the ϵ expansion [17, 18].

Weinberg studied a toy model inspired by this solution, namely gravity in $D = 2 + \epsilon$ dimensions with $\epsilon \ll 1$, and found it displayed a FP that did satisfy both (AS1) and (AS2). This is the first hint that asymptotic safety may be applied to gravity. It constitutes the original motivation for the presentation of the Asymptotic Safety Program by Weinberg in 1979 [24, 32, 47, 53, 54].

The EH bare action in exactly 2 dimensions is given by

$$S_{EH} = \frac{1}{G_{\mathrm{N}}} \int d^2x \sqrt{-g} R(g), \tag{58}$$

where $g = \det(g_{\mu\nu})$. As usual, the scalar curvature has canonical mass dimension $[R] = -2$ in natural units. This comes from the derivative terms of the Riemann tensor. When $d = 2$, requiring that S_{EH} be dimensionless entails that G_{N} is also dimensionless. Hence, the theory described by (58) is power counting perturbatively renormalizable [24, 47, 53]. However, action (58) is a topological invariant, i.e. it is invariant under homeomorphisms, and is equal to

$$S_{EH} = 4\pi\chi \quad \text{where} \quad \chi = 2 - 2h, \tag{59}$$

where h is the genus of the topological surface. Therefore, the Einstein tensor $G_{\mu\nu} = 0$ and the system is not dynamical, i.e. it has a trivial kinetic term [62]. To solve this problem, we can study the behaviour of beta functions near $D = 2$, i.e. at $D = 2 + \epsilon$ with $\epsilon \ll 1$. This dimensional regularization gives rise to two different types of poles in $1/\epsilon$ [24, 47, 53, 62]. The first one originates from the usual UV divergences. The second arises since the action is purely topological at $\epsilon \to 0^+$. In this case, the graviton propagator gets 'kinematical' poles of order $1/\epsilon$. This pole structure is derived in Appendix A.

Another difficulty, specific to gravity, is encountered. In pure gravity $\bar{G}_{\mathrm{N}}$ is an inessential parameter. That is, it can be absorbed by fields redefinitions. To resolve this issue, we use perturbative renormalization, and start by absorbing the two divergences into renormalized quantities. This turns the bare coupling $\bar{G}_{\mathrm{N}}$ into a running, or floating coupling $\bar{G}_{\mathrm{N}}(\mu)$ where μ is the renormalization mass scale of the dimensional regularization scheme [53]. Since μ is not an observable, only the flow of Newton's constant relative to that of another observable has meaning. This comparison amounts to making $\bar{G}_{\mathrm{N}}(\mu)$ into an essential coupling. The coefficient of the reference operator corresponding to said observable is denoted γ and is fixed to be constant [24, 30–32, 47, 63–65]. γ can be chosen to be:

- a cosmological constant term $\int d^{2+\epsilon}x\sqrt{g}$ [30–32],
- monomials from matter fields that are quantum mechanically scale invariant in $D = 2$,
- monomials from matter fields that are quantum mechanically *not*-scale invariant in $D = 2$,
- the conformal mode ϕ of the metric $g_{\mu\nu}$ in a background field expansion [63–65].

While the exact value of γ differs depending on what reference observable we choose, it is agreed in the literature that $\gamma > 0$ in pure gravity [24, 47, 53, 63–65]. We choose to derive the Callam-Symanzik equation at one-loop order in perturbation theory using the cosmological constant term as a reference operator [30–32]. Since $[\bar{G}_{\rm N}] = -\epsilon$, in $D = 2+\epsilon$ dimensions, we start by using the bare dimensionless coupling

$$G_0 = \mu^\epsilon \bar{G}_0, \tag{60}$$

We can expand these couplings in a Laurent series

$$G_0(\mu) = G(\mu) + \sum_{\nu=1}^{\infty} \epsilon^{-\nu}\gamma_\nu(G(\mu)), \tag{61}$$

where we denote G as the finite part of G_0. Now we perform $\mu\partial_\mu$ on both sides of the equation to get

$$\epsilon G(\mu) + \gamma_1(G) + \sum_{\nu=1}^{\infty} \epsilon^{-\nu}\gamma_{\nu+1}(G) = \beta(G) + \sum_{\nu=1}^{\infty} \epsilon^{-\nu}\frac{\partial \gamma_\nu}{\partial G}\beta(G), \tag{62}$$

where we used the renormalization group equation $\mu\partial_\mu G(\mu) = \beta(G, \epsilon)$. We rearrange this into

$$\beta(G) = \left(\epsilon G(\mu) + \gamma_1(G) + \sum_{\nu=1}^{\infty} \epsilon^{-\nu}\gamma_{\nu+1}(G)\right)\left(1 + \sum_{\nu=1}^{\infty} \epsilon^{-\nu}\frac{\partial \gamma_\nu}{\partial G}\right)^{-1}. \tag{63}$$

We can take a binomial expansion of the second term on the RHS to obtain,

$$\begin{aligned}
\beta(G) &= \epsilon G(\mu) - G(\mu)\frac{\partial \gamma_1}{\partial G} - \sum_{\nu=2}^{\infty} \epsilon^{-\nu}\frac{\partial \gamma_\nu}{\partial G} + \gamma_1(G) + \gamma_1(G)\sum_{\nu=1}^{\infty} \epsilon^{-\nu}\frac{\partial \gamma_\nu}{\partial G} + \sum_{\nu=1}^{\infty} \epsilon^{-\nu}\gamma_{\nu+1}(G) \\
&\quad - \left(\sum_{\nu=1}^{\infty} \epsilon^{-\nu}\gamma_{\nu+1}(G)\right)\left(\sum_{\nu=1}^{\infty} \epsilon^{-\nu}\frac{\partial \gamma_\nu}{\partial G}\right) + \ldots, \\
\beta(G) &= \epsilon G(\mu) + \gamma_1(G) - G(\mu)\frac{\partial \gamma_1}{\partial G} + \mathcal{O}\left(\frac{1}{\epsilon}\right).
\end{aligned} \tag{64}$$

Ultimately we want to come back to exaclty 2 dimensions, so we take the limit $\epsilon \to 0^+$. We are left with

$$\beta(G) = \epsilon G(\mu) + \gamma_1(G) - G(\mu)\frac{\partial \gamma_1}{\partial G}. \tag{65}$$

For small G we expect $\gamma_1 = \gamma G^2 + \mathcal{O}(G^3)$. Discarding terms of order $\mathcal{O}(G^3)$, we are left with

$$\beta_{\text{pert}} \equiv \mu \frac{\partial}{\partial \mu}(G(\mu)) = \epsilon G - \gamma G^2. \tag{66}$$

It is possible to show that all choices of reference observable lead to the flow equation (66) [24, 47]. We check that it is true for at least one other reference observable in Appendix A. For all $\gamma > 0$, this RG flow contains a NGFP in the UV at

$$\boxed{G^*(\mu) = \frac{\epsilon}{\gamma}}. \tag{67}$$

The derivation presented in this section is only valid for $\epsilon \ll 1$ and it is not at all clear whether we can extend this to the physical case $\epsilon = 2$. At Weinberg's time [32], this perturbatively accessible NGFP was the only evidence for the non-perturbative renormalizability of pure gravity, in the sense outlined in Section 2.4.1. This changed when the EAA for gravity was introduced [53, 66]. We will come back to this in Section 3.1.6.

3 Methods for the Asymptotic Safety program

We now compare two methods used to derive a quantum theory of gravity using the formalism of Asymptotic Safety. The first is the Functional Renormalization Group (FRG) formalism which is based on the concept of the EAA [53,55,67–78]. The second, introduced by Nierdemaier [79, 80], is based on the use of perturbation theory and results from higher-derivative gravity theories.

There is additional evidence to support the Asymptotic Safety scenario from methods such as dimensional reduction [24, 47]. This is out of the scope of this dissertation.

3.1 The Functional Renormalization Group

The most commonly used non-perturbative method in the Asymptotic Safety scenario is based on the Functional Renormalization Group Equation (FRGE), first introduced by Wetterich [67] and adapted to gravitational scenarios by Reuter [66]. Originally, it was used to probe the scale dependence of a QFT, to extract β-functions and look for asymptotic safety [56, 67].

3.1.1 Motivation

In Section 2.1.1, we argued that we could define real integrals as limits of Riemannian sums. Yet, if we think about path integrals like (7) as weighted sums over all histories, ensuring the convergence of such sums is much harder than for real integrals. This is because path integrals usually involve a high-order of infinity of paths to sum over. We could try to find a normalizing factor, similar to h in Fig. 1, to define (7) as the limit of a sum. However, Feynman says: "Unfortunately, to define such a normalizing factor seems to be a very difficult problem and we do not know how to do it in general terms." [40].

As we shall see, the power of the FRGE approach lies in 'taking the derivative' of the gravitational path integral. Indeed, rather than studying the integral *per se*, we interpret it as the solution of a differential equation called the FRGE. The FRGE can be considered as a closed 'evolution equation' taking place in an infinite dimensional dynamical system where the RG scale plays the role of time [53, 58]. Contrary to the path integral, the evolution equation itself is well-defined since it describes infinitesimal changes in the IR cutoff k (see Appendix B) [24, 47, 56, 58, 59, 66].

3.1.2 Background independence

We focus on the possibility of constructing a QFT of gravity in which the spacetime metric carries the dof associated with 'space', i.e. the background is dynamical, and the symmetry is given by diffeomorphism invariance. Such theory is generically called **Quantum Einstein Gravity** (QEG). Such quantization is quite difficult, since it is in contradiction with the usual construction of other QFT where we perform calculations on a rigid, non-dynamical background structure, e.g. Minkoswki spacetime [24, 47, 53]. In the same sense as GR, **background independence** is required to construct a quantum theory of gravitation. More precisely, none of the theory's rules, assumptions and predictions should depend on a specific metric chosen *a priori*. The physically relevant metrics should arise from the intrinsic gravitational dynamics. However, in many ways, this impairs FRG methods when applying them to gravity. Firstly, regularization schemes depend heavily on the metric given by the background spacetime. Secondly, when requiring background independence, non-perturbative methods suffer from the concept of coarse-graining becoming ill-defined. Even more so when trying to use Wilson-type methods that rely on iterative coarse-graining. For instance, coming back to Section 2.1.2.1, (i) what metric should we use to measure a physical block of spins? (ii) How then should we rescale? (iii) How to discriminate between IR and UV dof, (iv) and how should we integrate out modes?

To answer these questions and comply with the requirement of background independence in Asymptotic Safety, we adopt the following strategy. We use a classical background metric $\bar{g}_{\mu\nu}$ at intermediate steps of the quantization and check at the end that observables do not depend on the choice of said metric. This *background field method* is similar in spirit to the method introduced by DeWitt [81, 82]. It is akin to the one used to investigate gravity in $2+\epsilon$ dimensions in Appendix A, and is central to the gravitational EAA approach [66]. Even though the background independence is not explicit at intermediate quantization steps, this strategy is interesting because it allows us to use the tools from conventional background-dependent QFT [53]. We define the quantum operator $\hat{g}_{\mu\nu}$, i.e. the quantum metric, as

$$\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{h}_{\mu\nu}. \tag{68}$$

Its expectation value is the full spacetime metric of quantum gravity $g_{\mu\nu}$ such that

$$\langle \hat{g}_{\mu\nu} \rangle \equiv g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{69}$$

Here, $\langle \hat{h}_{\mu\nu} \rangle \equiv h_{\mu\nu}$, defines the fluctuation from the background classical metric $\bar{g}_{\mu\nu}$. These definitions correspond to the 'bi-metric' approach to the background independence problem, where we discriminate between $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ [83–87]. We have answered questions (i) and (ii) raised earlier in this section.

3.1.3 Asymptotic Safety reformulated

The goal of the asymptotic safety program is to give mathematical meaning, and to compute the functional integrals, over all off-shell geometries, of the form

$$Z = \int \mathcal{D}\hat{g}_{\mu\nu} e^{-S[\hat{g}_{\mu\nu}]}, \tag{70}$$

from which physical quantities can be deduced [24, 47, 53]. This approach for quantizing gravity was originally adopted by Feynman and Misner [88, 89]. Though this viewpoint is not shared in the asymptotic safety community, some [24, 47] claim that overcoming the difficulties of a functional integral picture is preferable to losing the intuitive physical premise based on past matter-interaction quantizations [8, 9]. We accept this point of view in this dissertation.

We now address three important issues that will constrain (70).

- To address the background independence problem we use results from Section 3.1.2 and replace the integration over $\hat{g}_{\mu\nu}$ by an integration over $\hat{h}_{\mu\nu}$, i.e. $\mathcal{D}\hat{g}_{\mu\nu} \to \mathcal{D}\hat{h}_{\mu\nu}$. This makes our task easier, since the problem is now akin to the quantization of a matter field $\hat{h}_{\mu\nu}$ in a fixed classical background $\bar{g}_{\mu\nu}$ [53, 66].

- As in every QFT, we want to quantise infinitely many dofs. Hence, we introduce IR and UV cutoffs, respectively called k and Λ, at intermediate steps of quantization [53].

- To quantize gravity, a gauge theory, the bare classical action $S[\hat{g}_{\mu\nu}]$ must be diffeomorphism invariant. Diffeomorphic metrics are all equivalent gauge field configurations. To avoid over-counting in each 'gauge orbit', or solution of the path integral, we follow [53] in using the Fadeev-Popov gauge fixing method [90]. To S, we add the gauge-fixing term $S_{\text{gf}} \propto \int \sqrt{\bar{g}}\bar{g}^{\mu\nu}F_\mu F_\nu$, where $F_\mu \equiv F_\mu(\hat{h}, \bar{g})$. The condition $F_\mu = 0$ picks one representative per gauge orbit, i.e. class of equivalent diffeomorphic metrics. The Fadeev-Popov determinant is expressed as a functional integral over the Grassmannian ghost fields C^μ and $\bar{C}_\mu$. This functional integral is governed by $\exp\{-S_{\text{gh}}\}$.

Equation (70) becomes

$$\tilde{Z}[\bar{\Phi}] = \int \mathcal{D}\hat{\Phi} e^{-S_{\text{tot}}[\bar{\Phi}, \hat{\Phi}]}, \tag{71}$$

where the total bare action is $S_{\text{tot}} = S + S_{\text{gf}} + S_{\text{gh}}$. It depends on the dynamical field $\hat{\Phi} \equiv (\hat{h}_{\mu\nu}, C^\mu, \bar{C}_\mu)$, on the background field $\bar{\Phi} \equiv (\bar{g}_{\mu\nu})$, and sometimes on other matter fields, not included for simplicity. If we did add matter fields, $\hat{\Phi}$ may have been altered in this redefinition.

3.1.4 The Effective Average Action

We define the ordinary effective action as Γ, a functional depending on $\Phi \equiv \langle\hat{\Phi}\rangle$, which reduces to $S[\Phi]$ in the classical limit. It yields the solution to the quantum mechanical field equation $(\delta\Gamma[\langle\hat{\Phi}\rangle]/\delta\Phi) = 0$. From the previous section, this action depends on both cutoffs such that $\Gamma \equiv \Gamma_{k,\Lambda}[\Phi, \bar{\Phi}]$. The EAA is obtained from the following functional integral

$$Z_{k,\Lambda}[J, \bar{\Phi}] \equiv \int \mathcal{D}\hat{\Phi} e^{-S^J_{\text{tot}}[\bar{\Phi},\hat{\Phi}]} e^{-\Delta S_k[\hat{\Phi},\bar{\Phi}]}, \tag{72}$$

where we replaced S_{tot} by $S^J_{\text{tot}} \equiv S_{\text{tot}} - \int dx \hat{\Phi}(x)J(x)$ to couple the dynamical fields to a classical external source $J(x)$ [24, 47, 53, 56, 58]. The second exponential, containing the cutoff action $\Delta S_k[\hat{\Phi}, \bar{\Phi}]$ achieves the IR regularization [67, 68, 68].

We now show how this regularization is realised. We expand the integration variable $\hat{\Phi}$ in terms of the eigenfunctions φ_p of the covariant Laplacian operator related to the background metric $\bar{D}^2 \equiv \bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$. We have $-\bar{D}^2\varphi_p = p^2\varphi_p$ and so we write symbolically $\hat{\Phi}(x) = \sum_p \alpha_p\varphi_p(x)$. We replace the integration over $\hat{\Phi}(x)$ by an integration over the generalized Fourier coefficients

$$Z_{k,\Lambda}[J, \bar{\Phi}] = \prod_{p^2\in[0,\Lambda^2]} \int_{-\infty}^{\infty} d\alpha_p \exp\Big\{-S^J_{\text{tot}}[\{\alpha_p\}, \bar{\Phi}] - \Delta S_k[\varphi_p, \{\alpha_p\}, \bar{\Phi}]\Big\}. \tag{73}$$

We retained only the squared momenta modes, i.e. $-\bar{D}^2$ eigenvalues, smaller than Λ^2. The cutoff action is given by $\Delta S_k[\hat{\Phi}, \bar{\Phi}]$

$$\Delta S_k[\varphi] = \frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\varphi(-p)\mathcal{R}_k(p^2)\varphi(p) \tag{74}$$

where $\mathcal{R}_k \propto k^2R^{(0)}(-\bar{D}^2/k^2)$ is the **cutoff operator**, or mode-suppressing operator. $R^{(0)}$ is a dimensionless function. In the $-\bar{D}^2$ basis, $\Delta S_k[\varphi] \propto k^2\sum_p R^{(0)}(p^2/k^2)\alpha_p^2$ contains a p^2-dependent 'mass' term. We want $R^{(0)}(p^2/k^2)$, the mode suppressing part of the cutoff operator, to be like a smeared step function which kills terms satisfying $p^2 \in [0, k^2]$. We then write

$$R^{(0)}\left(\frac{p^2}{k^2}\right) = \begin{cases} 0, & \frac{p^2}{k^2} \gtrsim 1, \\ 1, & \frac{p^2}{k^2} \lesssim 1. \end{cases} \tag{75}$$

One example of such mode-suppressing function is shown in Fig. 6. In explicit calculations,

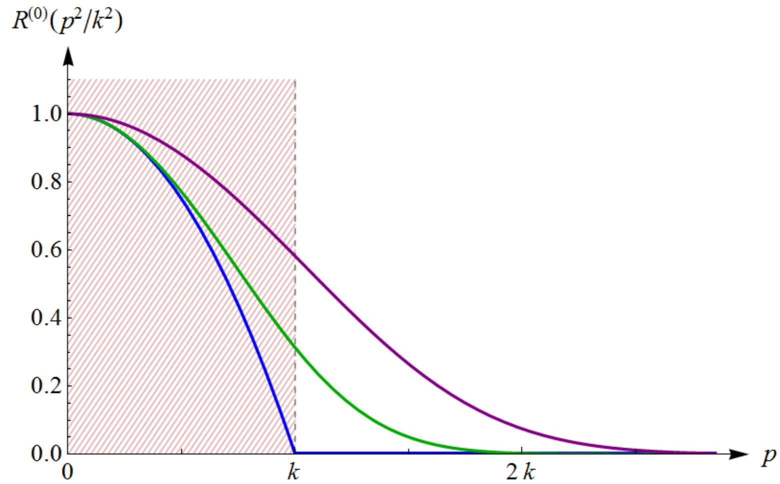

Figure 6: *Typical cutoff shape function separating momentum modes p above and below the squared IR cutoff k. The hatched red area highlights an area of significant mode suppression [91].*

we often use the 'exponential cutoff'

$$\mathcal{R}_k(p^2) = p^2 \left(\exp\left\{ \frac{p^2}{k^2} \right\} - 1 \right)^{-1}, \tag{76}$$

even though other cutoff operators are available [55, 92–95]. The low-momentum modes ($p^2 < k^2$) in (73) are indeed suppressed by a Gaussian exponential factor $e^{-k^2 \alpha_p^2}$ whereas high-momentum ones, $p^2 \in [k^2, \Lambda^2]$, are unaffected by $\Delta S_k[\hat{\Phi}, \bar{\Phi}]$.

The background metric, via the covariant Laplacian $\bar{D}^2$, defines which modes are high or low-momentum when tuning the scale k between Λ and 0. Lowering the IR cutoff from $k = \Lambda$ to $k = 0$ 'un-suppresses' modes of increasingly small momenta. This process, called 'integrating out' in FRGE language, is shown in Fig. 7. Importantly, encoding the contributions of momentum modes to $Z_{k,\Lambda}[J, \bar{\Phi}]$ in a 'running', or scale-dependence, of the cutoff action functional (74) is precisely a *Wilsonian view of renormalization* [24, 47, 53, 56, 58]. Thus, we answered questions (iii) and (iv) raised in Section 3.1.2.

In Appendix B, we derived the Wetterich equation (167) [67], i.e. the FRGE for the EAA defined in (161). We now outline the important properties of this FRGE and its corresponding EAA [24, 47, 53].

The EAA $\Gamma_{k,\Lambda}$, scale dependent solution of the FRGE, forms a RG trajectory in the corresponding theory space [91]. Its end points are the bare microscopic action $S_{\text{tot},\Lambda}$, reached at $k \to \Lambda$ for large Λ, and the UV regularized ordinary effective action Γ_Λ reached when $k \to 0$.

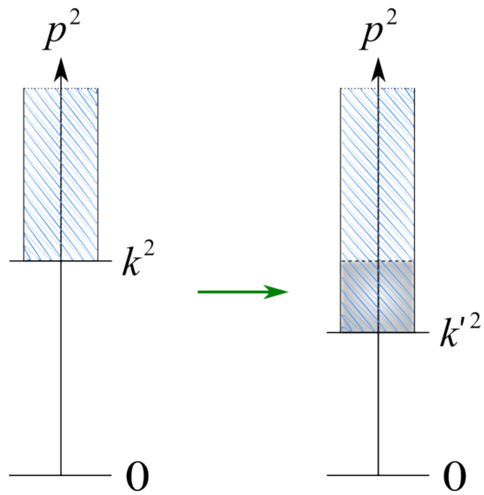

Figure 7: *Modes with covariant (squared) momentum above k^2, appearing as blue hatched areas, are integrated out following the method devised in Section 2.1.2.2. Modes below k^2 are suppressed by functions such as (76), which typical shape appears in Fig. 6. Lowering the IR cutoff to k' amounts to integrating out the fields in the interval $p \in [k', k]$. This corresponds to coarse-graining in a Wilsonian sense [91]. Solving the FRGE from some very high fixed UV scale $k = \Lambda$, where the initial condition $\Gamma_{k,\Lambda} = S_{tot,\Lambda} \equiv S$ is imposed, down to k amounts to integrating out modes with $\|p\| \in [k, \Lambda]$. Going as far as $k = 0$ yields the ordinary Wilsonian effective action $\Gamma_{k,\Lambda} = \Gamma_\Lambda$ [53].*

This is shown in Fig. 8. This interpolation of the EAA is schematically

$$\Gamma_\Lambda \xleftarrow{k\to 0} \Gamma_{k,\Lambda} \xrightarrow{k\to\Lambda} S_{\text{tot},\Lambda}. \tag{77}$$

The latter statement, $\lim_{k\to 0}\Gamma_{k,\Lambda} = \Gamma_\Lambda$, follows from equation (75). Indeed, $\mathcal{R}_k(p^2) = 0$ for all $p^2 > 0$; then, from (161) and (159) the EAA does reduce to the ordinary average action. Justifying the former statement involves (168). We know from (75) that $\mathcal{R}_k(p^2) \to k^2$ for $k \to \Lambda$ and Λ large. Hence, the second exponential on the RHS of (168) becomes $\exp\left\{-k^2 \int d^D x(\hat{\Phi} - \Phi)^2\right\}$ which approaches a delta-functional $\delta[\hat{\Phi} - \Phi]$ up to an irrelevant factor. We can perform the integration over $\hat{\Phi}$ in the rest of (168) and we finally recover $\lim_{k\to\Lambda}\Gamma_{k,\Lambda} \approx S_{\text{tot},\Lambda}$ for Λ large. A more careful treatment of this argument, based on the saddle point approximation of equation (168), is presented in [67].

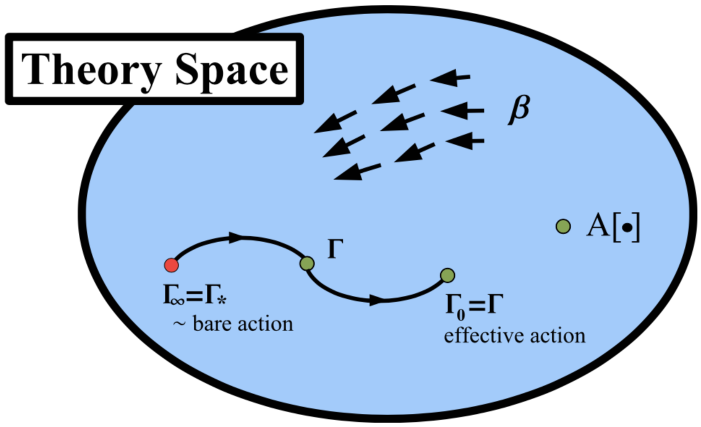

Figure 8: *The blue area denotes a generic theory space, or equivalently a space of all action-functionals $\mathcal{A}[\cdot]$. This hyperspace may be infinite dimensional. It is parameterized by essential couplings. A vector field $\vec{\beta} \equiv \beta_\alpha$ is defined by truncating the generic FRGE (167), i.e. expanding $\mathcal{A}[\cdot]$ in terms of couplings. This is shown explicitly in Section 3.1.7. We relate notations with remark 3.1.4 of Section 3.1.4 as follows. $\Gamma \equiv \Gamma_{k,\Lambda}$ interpolates between the ordinary effective action $\Gamma_0 = \Gamma \equiv \Gamma_\Lambda$ and the bare microscopic fixed point action $\Gamma_* = \Gamma_\infty \equiv S_{tot,\Lambda}$. The arrow on the RG trajectory corresponding to Γ goes in the direction of decreasing mass-scale k, i.e. in the direction of increasing coarse-graining [54].*

3.1.5 The UV renormalization problem reformulated

All previous discussions, including Appendix B, have been done with fixed UV cutoff Λ. However, the central theme of the UV renormalization problem is the removal of the UV cutoff [24, 47]. In this FRG framework, we distinguish between two aspects.

1. The first, trivial one, is removing the UV cutoff from the trace on the RHS of the Wetterich equation (166). This can be done since $(\Gamma^{(2)}_{k,\Lambda}[\Phi,\bar{\Phi}] + \mathcal{R}_{k,\Lambda}[\bar{\Phi}])$ defines a bounded operator. Indeed, the trace is finite in the IR and UV (also staying finite when taking the limit $\Lambda \to \infty$). The former property is implemented by the mass-like regulator $\mathcal{R}_{k,\Lambda}$ in the denominator. The latter property is ensured by the fact that, in momentum representation, $(\Gamma^{(2)}_k[\Phi,\bar{\Phi}] + \mathcal{R}_k[\bar{\Phi}]) \propto (p^2 + \mathcal{R}_k(p^2) + \dots)$ and $[k\frac{\partial}{\partial k}\mathcal{R}_k(p^2)]$ are only non-zero around $p^2 \approx k^2$. A more careful discussion is presented in Appendix B.

2. The second, non-trivial one, is sending the UV cutoff to infinity in the EAA itself. This

directly relates to the *traditional UV renormalization problem*. Truly, since $\Gamma_{k,\Lambda}[\Phi,\bar{\Phi}]$ comes from a regularized functional integral, it will develop the usual UV divergences outlined in Section 2.1.1. To solve this issue, we can fine tune the bare action S, in $S_{\text{tot},\Lambda} \equiv S + S_{\text{gf}} + S_{\text{gh}}$, so that the functional integral is asymptotically independent of Λ. As shown in Section 3.1.8, the FRGE is precisely solved from some very high fixed UV scale $k = \Lambda$, where the initial condition $\Gamma_{k,\Lambda} = S_{\text{tot},\Lambda}$ is imposed, down to k. However, the FRGE itself does *not* provide any means of identifying how to adjust the bare action [24, 47, 79, 80, 96].

Still, there are ways to investigate the EAA's RG flow in a generic theory space $\mathcal{T}$ to find a NGFP without specifying the bare action. If this NGFP satisfies both (AS1) and (AS2), we can UV-complete a gravity theory in the 'asymptotically safe' way. Said FP then *becomes the bare action*. The task of finding a bare classical action that reproduces a given effective one is known as the **reconstruction problem** [97,98]. A 'reconstructive' strategy is used to investigate QEG for instance [53, 54].

This represents one of the strengths of the Asymptotic Safety program. The bare actions, corresponding to UV fixed points actions of the RG flow, can be considered as *predictions* of Asymptotic Safety rather than required inputs [24, 47, 53, 91]. The Asymptotic Safety program based on the EAA then amounts to a research process among quantum theories, rather than the quantization of a given classical system.

We will come back to this in Section 3.3.

3.1.6 A FRGE for the gravitational EAA

We now focus on the examination of QEG by investigating the RG flow of a **gravitational EAA** (GEAA), defined by a corresponding FRGE. That is, we study a gravitational version of the path integral (72) *as a solution of an evolution equation*, the **gravitational FRGE**. We follow exactly the method outlined in Appendix B, by making explicit the fields denoted as Φ, to derive a gravitational FRGE for the GEAA [54, 66, 91, 99]. For simplicity we drop hats on quantum operators for now. The gravitational generating functional is given by

$$\exp\{W_k[t^{\mu\nu}, \sigma^{\mu}, \bar{\sigma}_{\mu}, \bar{g}_{\mu\nu}, \beta^{\mu\nu}, \tau_{\mu}]\} = \\ \int \mathcal{D}h_{\mu\nu}\mathcal{D}C^{\mu}\mathcal{D}\bar{C}_{\mu} \exp\Big\{-S[\bar{g}+h] - S_{\text{gf}}[\bar{g},h] - S_{\text{gh}}[\bar{g},h,C,\bar{C}] - \Delta S_k[\bar{g},h,C,\bar{C}] - S_{\text{source}}\Big\}. \tag{78}$$

For the gravitational field itself, in Euclidean signature (where the tensor density of the volume element has a positive sign in front of the spacetime metric's determinant inside the square root), the cutoff action is given by

$$\Delta S_k[h, C, \bar{C}, \bar{g}] = \frac{1}{2}\kappa^2 \int d^D x \sqrt{\bar{g}} h_{\mu\nu} R_k^{\text{grav}}[\bar{g}]^{\mu\nu\rho\sigma} h_{\rho\sigma} + \sqrt{2} \int d^D x \sqrt{\bar{g}} \bar{C}_\mu R_k^{\text{gh}}[\bar{g}] C^\mu, \tag{79}$$

where $\kappa \equiv (32\pi \bar{G}_N^{\text{bare}})^{-1/2}$. The cutoff operators have the general form $R_k[\bar{g}] = \mathcal{Z}_k k^2 R^{(0)}(-\bar{D}^2/k^2)$. As explained in Section 3.1.4 and shown in Fig. 6, the shape function $R^{(0)}(-\bar{D}^2/k^2)$ is chosen to smoothly interpolate between $R^{(0)}(0) = 1$ and $R^{(0)}(\infty) = 0$. Note, $\mathcal{Z}_k^{\text{gh}}$ is a pure number whereas $\mathcal{Z}_k^{\text{grav}}$ is a tensor defined as

$$(\mathcal{Z}_k^{\text{grav}})^{\mu\nu\rho\sigma} = \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \mathcal{Z}_k^{\text{grav}}. \tag{80}$$

Coupling $h_{\mu\nu}$, C^μ and $\bar{C}_\mu$ to the sources $t^{\mu\nu}$, $\bar{\sigma}_\mu$ and σ^μ, we obtain the following source action

$$S_{\text{source}} = -\int d^D x \sqrt{\bar{g}} [t^{\mu\nu} h_{\mu\nu} + \bar{\sigma}_\mu C^\mu + \sigma^\mu \bar{C}_\mu + \beta^{\mu\nu} \mathcal{L}_C(\bar{g}_{\mu\nu} + h_{\mu\nu}) + \tau_\mu C^\nu \partial_\nu C^\mu], \tag{81}$$

where the last two terms are required by BRST symmetry, which is out of the scope of this dissertation. $\mathcal{L}_C$ is defined as the Lie derivative with respect to the ghost vector field. We define the expectation values of the quantum fields, i.e. classical fields, as follows

$$\bar{h}_{\mu\nu} \equiv \langle h_{\mu\nu} \rangle = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta t^{\mu\nu}} \quad , \quad \xi^\mu \equiv \langle C^\mu \rangle = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta \bar{\sigma}_\mu} \quad , \quad \bar{\xi}_\mu \equiv \langle \bar{C}_\mu \rangle = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta \sigma^\mu}. \tag{82}$$

Finally, the **gravitational EAA** (GEAA) is given by

$$\boxed{\Gamma_k[\bar{h}, \xi, \bar{\xi}, \beta, \tau, \bar{g}] = \int d^D x \sqrt{\bar{g}} \Big(t^{\mu\nu} \bar{h}_{\mu\nu} + \bar{\sigma}_\mu \xi^\mu + \sigma^\mu \bar{\xi}_\mu \Big) - W_k[t, \sigma, \bar{\sigma}, \beta, \tau, \bar{g}] - \Delta S_k[\bar{h}, \xi, \bar{\xi}, \bar{g}]}. \tag{83}$$

Now, using (69) we define

$$\Gamma_k[\bar{h}, \xi, \bar{\xi}, \beta, \tau, \bar{g}] \equiv \Gamma_k[g - \bar{g}, \xi, \bar{\xi}, \beta, \tau, \bar{g}]. \tag{84}$$

It is now easier to see that the GEAA is invariant under general coordinate transformations, when all its arguments transform as tensors of the corresponding rank, including the classical background metric $\bar{g}_{\mu\nu}$ [54, 66, 91]. Indeed, since

$$W_k[\mathcal{J} + \mathcal{L}_v \mathcal{J}] = W_k[\mathcal{J}] \quad , \quad \mathcal{J} \equiv (t^{\mu\nu}, \sigma^\mu, \bar{\sigma}_\mu, \bar{g}_{\mu\nu}), \tag{85}$$

then we can write

$$\boxed{\Gamma_k[\Phi + \mathcal{L}_v \Phi] = \Gamma_k[\Phi]} \quad , \quad \Phi \equiv (g_{\mu\nu}, \bar{g}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu). \tag{86}$$

We now assume that the functional measure in (78) is diffeomorphism invariant. In addition, it is shown in [66] that setting $\xi = \bar{\xi} = 0$ does not change the symmetry of the GEAA. Finally, we succeeded in constructing a diffeomorphism invariant generating functional for gravity, since both $\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu]$ and $\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}, 0, 0]$ are invariant under the general coordinate transformations $\delta g_{\mu\nu} = \mathcal{L}_v g_{\mu\nu}$ [54,66]. Importantly, whilst the physically interesting situation is given by $\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}, 0, 0]$, we must first solve the RG dynamics of (83) before setting the classical ghost fields ξ and $\bar{\xi}$ to zero [53,66].

The GEAA (83) is a solution of the exact non-perturbative **gravitational FRGE**

$$\begin{aligned} \frac{\partial}{\partial t}\Gamma_k =& \frac{1}{2}\operatorname{Tr}\Big[(\Gamma_k^{(2)} + \hat{\mathcal{R}}_k)^{-1}_{\bar{h}\bar{h}}(\partial_t \hat{\mathcal{R}}_k)_{\bar{h}\bar{h}}\Big] \\ &- \frac{1}{2}\operatorname{Tr}\Big[\Big\{(\Gamma_k^{(2)} + \hat{\mathcal{R}}_k)^{-1}_{\bar{\xi}\xi} - (\Gamma_k^{(2)} + \hat{\mathcal{R}}_k)^{-1}_{\xi\bar{\xi}}\Big\}(\partial_t \hat{\mathcal{R}}_k)_{\bar{\xi}\xi}\Big], \end{aligned} \tag{87}$$

where we defined the RG time as $t \equiv \ln k$, such that $k\partial_k = \partial_t$. This equation is exact, since it contains all couplings: we call it an Exact Renormalization Group Equation (ERGE). The GEAA (83) and its corresponding gravitational FRGE (87) have the properties outlined in remark 3.1.4. Now, $\mathrm{Tr}[\dots]$ is thought of as $\int d^D x \sqrt{\bar{g}(x)}$ in position space. Instead, the trace is understood as $\int d^D x$ for a generic, non-gravitational FRGE, as presented in Appendix B. Similarly, the Hessian of the GEAA is, at fixed $\bar{g}$,

$$\Gamma_k^{(2)}(x,y) = \frac{1}{\sqrt{\bar{g}(x)\bar{g}(y)}} \frac{\delta^2 \Gamma_k}{\delta\varphi(x)\delta\varphi(y)}, \tag{88}$$

where we defined $\varphi \equiv (\bar{h}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu)$ as the dynamical fields. The functional derivatives are left-derivatives in the ghost sector [54,66,91,99].

The background gauge invariance of the GEAA expressed in (86) will play a key practical role in Section 3.1.7. Truly, if we know *a priori* that the initial functional does not contain non-invariant terms, then no symmetry-violating terms will be generated during the RG evolution. This reduces the number of operators to be retained in a reliable truncation of theory space, i.e. an invariant combination of the fields. Such approximation is customarily used to solve (87) [24,47,53,54,56,58,59,66,91,99]. We now study this method in more detail.

3.1.7 Truncations of theory space

First, using the results and notations from Appendix B and [53, 54], we investigate general truncations. We start by decomposing a general theory space $\mathcal{T}$. A generic point action functional $\mathcal{A} \in \mathcal{T}$ admits an expansion in terms of infinitely many action functionals $I_\alpha[\cdot]$,

$$\mathcal{A}[\Phi, \bar{\Phi}] = \sum_\alpha \bar{u}^\alpha I_\alpha[\Phi, \bar{\Phi}], \tag{89}$$

where $\alpha = 1, 2, \dots$. Again, $\bar{u}^\alpha$, the components of $\mathcal{A}[\cdot]$ in the basis $\{I_\alpha\}$, are dimensionful essential couplings, sometimes called **generalized couplings** [53,54]. $I_\alpha[\Phi, \bar{\Phi}]$ are monomials of powers of both fields Φ and $\bar{\Phi}$ and their derivatives, all evaluated at the same point, and integrated over all spacetime. Geometrically, the integral curves of (167) are the RG trajectories $k \mapsto \Gamma_k$. They are one-parameter families of actions like the EAA Γ_k, defined in (161), i.e. they are solely parametrized by the scale k. For fixed values of k, we can expand Γ_k as in (89),

$$\Gamma_k[\Phi, \bar{\Phi}] = \sum_\alpha \bar{u}^\alpha(k) I_\alpha[\Phi, \bar{\Phi}]. \tag{90}$$

The k dependence of the EAA is now carried by the running coupling constants $\bar{u}^\alpha(k)$. We can plug (90) in the Wetterich equation (167) to yield

$$\sum_\alpha k\partial_k \bar{u}^\alpha(k) I_\alpha[\Phi, \bar{\Phi}] = \frac{1}{2}\,\mathrm{Tr}\left\{ \left(\sum_\alpha \bar{u}^\alpha(k) I_\alpha^{(2)}[\Phi, \bar{\Phi}] + \mathcal{R}_k[\bar{\Phi}] \right)^{-1} k\partial_k \mathcal{R}_k[\bar{\Phi}] \right\}. \tag{91}$$

In turn, we can expand the $\mathrm{Tr}\{\dots\}$ term on the RHS in terms of $\{I_\alpha[\cdot]\}$ as $\frac{1}{2}\mathrm{Tr}\{\dots\} = \sum_\alpha \bar{\beta}_\alpha(\bar{u}_1, \bar{u}_2, \dots, k) I_\alpha[\Phi, \bar{\Phi}]$. The expansion coefficients on the RHS are interpreted as beta-functions. We then arrive at the following system of infinitely many coupled partial differential equations

$$k\partial_k \bar{u}_\alpha(k) = \bar{\beta}_\alpha(\bar{u}_1, \bar{u}_2, \dots, k). \tag{92}$$

These are precisely the RG equations (31) we promised to derive in Section 2.2.1. They define the vector field $\vec{\beta} \equiv \beta_\alpha$ on $\mathcal{T}$, shown in Fig. 8. The local potential approximation is an example of a truncation of the scalar theory space [24, 39, 47, 53, 54].

3.1.8 The Einstein-Hilbert truncation

With these tools in hand, our goal is to solve (87), subject to the initial condition $\Gamma_* = \Gamma_\infty \equiv S_{\mathrm{tot},\Lambda}$, and search for a NGFP in the RG flow of the GEAA. However, practically, this task is as hard as solving (72). Hence, we need to devise approximation methods. The truncation of theory space makes use of the full power of the FRGE approach and follows the 'reconstructive' philosophy. We first fix the operators entering the GEAA to solve the RG flow before ascribing to the couplings their NGFP values. The GEAA with fixed FP couplings in the continuum limit defines the bare microscopic action.

An example of a very general truncation consists of freezing the ghost sector, i.e. neglecting the evolution of the ghost action [24,47,53,54,56,58,59,66,91,99]. The corresponding 'ghost-freezing' ansatz for the GEAA is

$$\begin{aligned}\Gamma_k[g,\bar{g}] &= \Gamma_k[g,g,0,0,0,0] + \hat{\Gamma}_k[\bar{g},g] + S_{\rm gf}[g-\bar{g},\bar{g}] + S_{\rm gh}[g-\bar{g},\bar{g},\xi,\bar{\xi}] \\ &\quad - \int d^D x\sqrt{\bar{g}}\Big(\beta^{\mu\nu}\mathcal{L}_\xi g_{\mu\nu} + \tau_\mu \xi^\nu \partial_\nu \xi^\mu\Big),\end{aligned} \tag{93}$$

where $\hat{\Gamma}_k[\bar{g},g]$ encodes the deviation of $\bar{g}$ from g. Hence, it vanishes when its two arguments are equal, i.e. $\hat{\Gamma}_k[g,g]=0$. When inserting (93) into (87) we get,

$$\begin{aligned}\partial_t\Gamma_k[g,\bar{g}] &= \frac{1}{2}\,\mathrm{Tr}\Bigg[\Big(\kappa^{-2}\Gamma_k^{(2)}[g,\bar{g}] + \mathcal{R}_k^{\rm grav}[\bar{g}]\Big)^{-1}\partial_t\mathcal{R}_k^{\rm grav}[\bar{g}]\Bigg] \\ &\quad - \mathrm{Tr}\Bigg[\Big(-\mathcal{M}[g,\bar{g}] + \mathcal{R}_k^{\rm gh}[\bar{g}]\Big)\partial_t\mathcal{R}_k^{\rm gh}[\bar{g}]\Bigg],\end{aligned} \tag{94}$$

where the Fadeev-Popov operator is

$$\mathcal{M}[\hat{g},\bar{g}]^\mu{}_\nu = \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda(\hat{g}_{\rho\nu}D_\sigma + \hat{g}_{\sigma\nu}D_\rho) - 2\omega\bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda\hat{g}_{\sigma\nu}D_\rho. \tag{95}$$

Here, ω is a free parameter and D is the usual covariant derivative defined from the metric $\hat{g}_{\mu\nu}$. To not confuse the latter with the spacetime dimension, we now work in d dimensions. Equation (94) is known as the **reduced FRGE**. The first term on its RHS corresponds to the graviton term, induced by metric fluctuations $h_{\mu\nu}$, whereas the second one arises from the ghosts. In (94), the RG flow dynamics have been projected to the space spanned by the set of all action functionals $\{\mathcal{A}[g,\bar{g},\xi,\bar{\xi}]\}$. However, this truncation is still too general for practical purposes so we will further constrain the ansatz (93) using the well-known single-metric EH truncation [24,47,53,54,56,58,59,66,91,99].

We start from the EH action at the UV cutoff Λ and evolve it to lower momenta scales $k<\Lambda$. We want to project the RG flow defined by (87) onto the finite-dimensional subspaces of $\mathcal{T}_{\text{QEG}}$ spanned by a suitable truncation ansatz for the GEAA (83). This ansatz is given by

$$\begin{aligned}\Gamma_k^{\text{trunc}}[g,\bar{g}] &= \frac{1}{16\pi\bar{G}_k}\int d^dx\sqrt{g}(-R+2\bar{\lambda}_k) + \text{classical gauge term},\\ &= 2\kappa^2 Z_k\int d^dx\sqrt{g}(-R+2\bar{\lambda}_k) + \kappa^2 Z_k\int d^dx\sqrt{\bar{g}}\bar{g}^{\mu\nu}(\mathcal{F}_\mu^{\alpha\beta}g_{\alpha\beta})(\mathcal{F}_\nu^{\rho\sigma}g_{\rho\sigma}),\end{aligned} \tag{96}$$

where

$$\mathcal{F}_\mu^{\alpha\beta} = \delta_\mu^\beta\bar{g}^{\alpha\gamma}\bar{D}_\gamma - \frac{1}{2}\bar{g}^{\alpha\beta}\bar{D}_\mu. \tag{97}$$

Note, $\bar{G}_k \equiv \bar{G}_k^{\text{renorm}} = (Z_k)^{-1}\bar{G}_k^{\text{bare}}$ where Z_k is the renormalization factor of Newton's constant. By inserting (96) into (94), we project the reduced evolution equation onto the subspace of $\mathcal{T}_{\text{QEG}}$ spanned only by the operators $\int d^dx\sqrt{g}$ and $\int d^dx\sqrt{g}R$. Taking a derivative expansion on the RHS of (94), we arrive at a RG flow equation for Z_k and the dimensionful running cosmological constant $\bar{\lambda}_k$. The EH truncation is a 'single-metric' truncation so we now set $g_{\mu\nu} = \bar{g}_{\mu\nu}$ which makes the classical gauge term vanish, i.e. $\hat{\Gamma}[g,g] = 0$ [53]. We renormalize $\bar{G}_k$ and $\bar{\lambda}_k$ by solving their respective differential equations and imposing that we recover the EH action at Λ. This is done by using the following initial conditions

$$Z_\Lambda = 1 \quad , \quad \bar{\lambda}_\Lambda = \bar{\lambda}. \tag{98}$$

We then obtain RG flow equations for $\bar{G}_k$ and $\bar{\lambda}_k$. We can define the dimensionless Newton's and cosmological couplings g_k and λ_k as follows,

$$g_k \equiv k^{d-2}\bar{G}_k \quad , \quad \lambda_k \equiv k^{-2}\bar{\lambda}_k. \tag{99}$$

Their RG flow equations are given by

$$\begin{aligned} k\partial_k g_k &= \beta_g(g_k,\lambda_k),\\ k\partial_k\lambda_k &= \beta_\lambda(g_k,\lambda_k).\end{aligned} \tag{100}$$

These are solved by the following beta-functions,

$$\beta_g(g,\lambda) = (d-2+\eta_N)g, \tag{101}$$

and

$$\beta_\lambda(g,\lambda) = -(2-\eta_N)\lambda + \frac{1}{2}g(4\pi)^{1-\frac{d}{2}}\Big[2d(d+1)\Phi^1_{d/2}(-2\lambda) - 8d\Phi^1_{d/2}(0) - d(d+1)\eta_N\tilde{\Phi}^1_{d/2}(-2\lambda)\Big]. \tag{102}$$

Here, η_N is the anomalous dimension of Newton's constant. Its value is

$$\eta_N = \frac{gB_1(\lambda)}{1 - gB_2(\lambda)}, \tag{103}$$

where $B_1(\lambda)$ and $B_2(\lambda)$ are complicated functions of the cosmological constants, $\Phi^p_n(x)$ and $\tilde{\Phi}^p_n(x)$. The threshold functions $\Phi^p_n(x)$ and $\tilde{\Phi}^p_n(x)$ depend on the cutoff shape function $R^{(0)}$ [24,47,53,54,56,59,66,91,99]. The beta functions (101) and (102) give rise to RG flows that can be analyzed for any cutoff shape and in general dimensions d. For instance, QEG was shown to be non-perturbatively renormalizable in the 'asymptotically safe' sense for the sharp cutoff function

$$\mathcal{R}_k(p^2) \equiv \hat{\mathcal{R}}\theta(k^2 - p^2). \tag{104}$$

Indeed, for $d = 4$, both RG flows yield a UV-attractive NGFP at numerical values $g^* = 0.403$ and $\lambda^* = 0.330$ called the 'Reuter FP' [66,99]. This is shown in Fig. 9 and Fig. 10. We

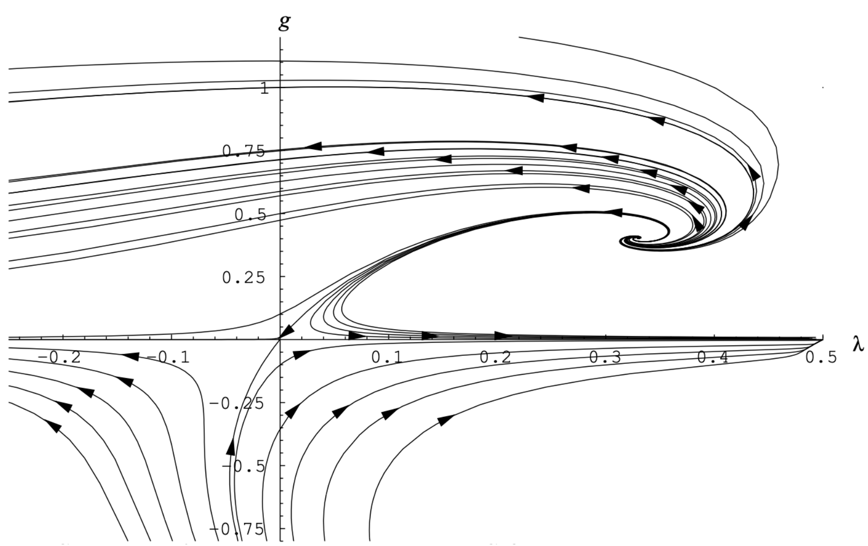

Figure 9: *This graph presents the phase portrait of the EH truncation. That is, the RG flow defined by (100) in a selected part of the g-λ-plane. The arrows point in the direction of increasing coarse-graining, i.e. of decreasing k. This flow is dominated by a NGFP at $g^* = 0.403$ and $\lambda^* = 0.330$ and a GFP at the origin of this truncated 2-dimensional subspace of $\mathcal{T}_{QEG}$. The NGFP is UV attractive in both the g and λ direction [99].*

recover the NGFP derived in Section 2.4.3 and Appendix A when taking $d = 2+\epsilon$. However, the result derived in the present Section is more general. Its derivation does not involve any analytic continuation, and is valid in any arbitrary continuous dimension d. We find

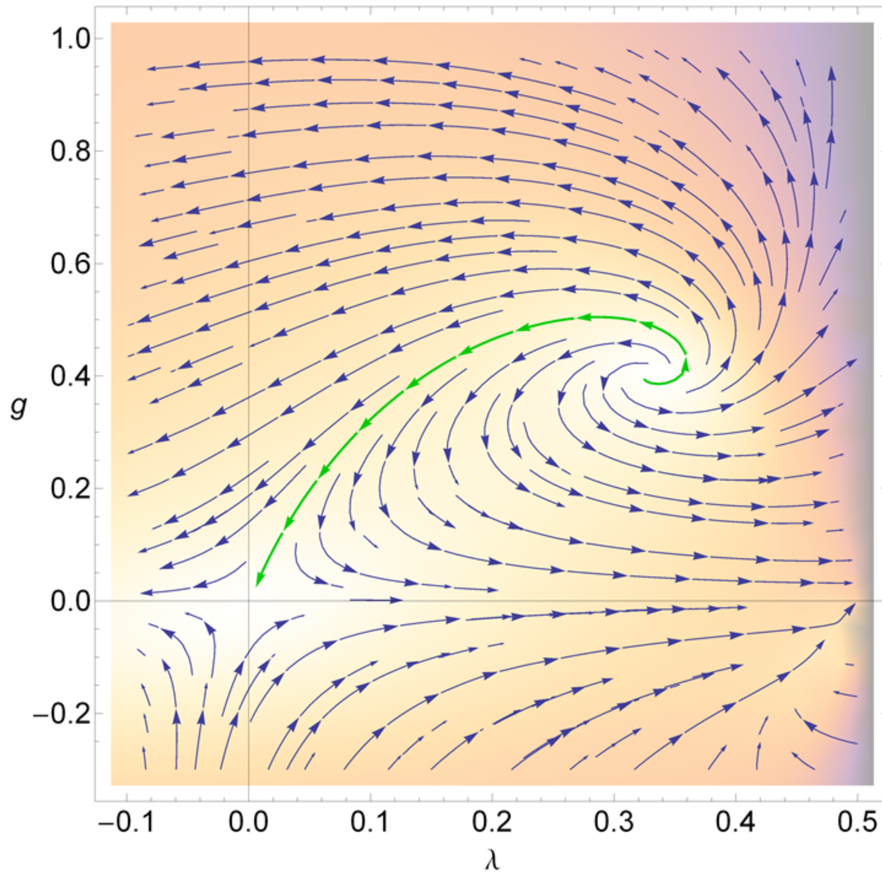

Figure 10: *This reproduction of Fig. 9 contains additional information about the velocity of the RG flow. The flow is slower in darker regions and faster in lighter regions. There is no RG flow in white regions, i.e. the running stops completely. There are two such regions, at the GFP and the NGFP. The green RG trajectory is called the 'separatrix' [100]. It joins the two FPs [91, 99].*

particularly interesting results in the physically relevant situation where $d = 4$. One can show that gravity is 'anti-screening' at such dimension. That is, Newton's coupling grows with the interaction distance,

$$G_k = G_0(1 - w\bar{G}_k k^2 + \mathcal{O}(\bar{G}_k^2 k^4)), \tag{105}$$

where w is a positive parameter. This is rather intuitive since we expect the gravitational mass *not* to be screened by quantum fluctuations but instead to receive positive contributions from the virtual particles surrounding it [24, 47, 53, 54, 56, 59, 66, 91, 99].

3.1.9 Discussion on the validity of the truncation approximation scheme

We may question the generality of the results derived in Section 3.1.8, together with the reliability of truncations. Is the NGFP we found a projection of a true FP in the full untruncated theory space or an artefact of our approximation [54,96]? We respond to this in two steps: first we argue that truncations are a valid approximation scheme. Second, we show that a NGFP that can be used in an Asymptotic Safety construction is derived for every known alternative to this truncation [101]: both within the EH one for different cutoff functions and backgrounds, but also for different truncations.

- In theory space truncations, operators beyond the truncation can be generated by the RG evolution on the right-hand-side of equation (87). This contribution are set to zero in the beta functions (101,102). We seem to lose important information in this process. However, results on interacting FPs in various QFTs obtained from truncations (mostly in $d < 4$) were compared with other techniques such as the ϵ-expansion, Monte Carlo simulations or the conformal bootstrap. The results were all shown to agree, highlighting the power of the truncation method.
  To know if the FP found from a truncation is a true FP in the exact theory, we study its stability under extensions of the truncation used, specifically robustness under changes of cutoff scheme [96]. Actual FPs are convergent whereas truncation-induced ones are unstable [67, 102–106]. In our case, the quality of approximate solutions to (87), including truncations, is also given by their consistency with the BRS Ward identities (170) (see Appendix C) [107, 108].

- Within the EH truncation, a considerable number of truncations with increasingly large subsets I_α, different cutoff functions and different backgrounds have been investigated. The publication [101] contains an exhaustive list of the results obtained from such extensions. These include quadratic gravity (a term like R^2 appears in the truncation ansatz) [109], gravity with the two-loop counterterm, actions containing up to 71 powers of the Ricci scalar, or actions containing a single trace of up to 35 Ricci tensors. This has further been extended by accounting for polynomials in R ("$f(R)$ truncations") [96], polynomials in $R_{\mu\nu\rho\sigma}$ ("$f(R_{\mu\nu\rho\sigma})$ truncations") [110], and the squared Weyl curvature tensor C^2 [111]. In addition, the impact of matter fields has been investigated in [56, 59, 112].
  Moreover, other truncations have been studied. Examples include the Hilbert-Palatini-Holst type [53], implemented in the Einstein–Cartan gravity setting by [113, 114].
  Importantly, all extensions of the EH truncation and other truncations agree with the

existence of at least one NGFP for Newton's coupling, satisfying both (AS1) and (AS2) [24, 47, 53, 54, 56, 91]. This **universal existence** in 4 dimensions is non-trivial since we can find cutoff functions that destroy the NGFP in $d \gtrsim 5$ in the EH truncation [53].

Although a complete proof is not yet within reach, it seems that the NGFP derived in the present section exists in the un-truncated theory space $\mathcal{T}_{\text{QEG}}$. That is, it is not an artifact of our approximation scheme [58, 96]. Within the EH truncation, there is even further evidence hinting at the asymptotic safety of QEG in exactly 4 dimensions [53, 54, 58]. Namely,

- All the different cutoff schemes find positive values for both g^* and λ^*. This is important for stability reasons.
- The NGFP is always UV attractive. Indeed, linearizing the RG flow, as done in Section 2.2.1, yields positive real parts of the critical exponents for all cutoffs.
- The product $g^*\lambda^*$ takes universal values as expected from FP coupling values.

Since truncations seem to be reasonable approximation schemes, we may now ask: what constitutes a good truncation? We want the exact position of the NGFP in the un-truncated theory space *not* to depend on the truncation, or within a given truncation on different backgrounds and different cutoffs. A valid truncation scheme should capture relevant physics at low order in the truncation. For a given FP this is given by the relevant directions. For a GFP, eigenvectors of the Jacobi matrix with positive real valued eigenvalues point exactly in the direction of theory space's axis. Following Section 2.3.1, at the free FP we truncate according to canonical power counting.
For a NGFP, relevant eigenvectors can be linear combination of many couplings, making it difficult to know if a given truncation contains enough information about the RG flow near a NGFP. To devise a good non-perturbative truncation scheme in this case, we assume '*near-perturbativity*', i.e. critical exponents exhibit small deviations from the canonical spectrum of scaling dimensions. In principle, including higher-order terms in such truncations will make all NGFPs *converge* to the true FP in the full infinite-dimensional theory space. Results in this direction seem encouraging thus far [101].

3.2 Perturbative quadratic gravity

In addition to non-perturbative treatments, we can use methods from higher-derivative gravity, such as perturbation theory, to investigate the Asymptotic Safety program [24, 32, 47, 96]. By higher-derivative gravity theories, we mean gravitational theories whose bare actions contain terms proportional to the Riemann tensor and its covariant derivatives [24, 47].

3.2.1 Motivation

The motivation for taking this route comes from the fact that higher derivative gravity in four dimensions is close to being a renormalizable quantum theory of gravity. Indeed, motivated by [115–117], Stelle showed in 1977 [118] that adding terms like quadratic products of the curvature tensor to the EH action

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\Big(\frac{R}{\kappa^2} + \alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2\Big), \tag{106}$$

defines a power-counting perturbatively renormalizable theory of gravity to all loop orders. In (106), $\kappa \equiv (16\pi \bar{G}_N^{\text{bare}})^{1/2}$. As in the original paper [118], we omitted total derivative terms like $\nabla^2 R$ and the integrand of the Gauss-Bonnet term E in (106) . Herein, we focus on Stelle's treatment of higher-derivative gravity theories [118] rather than on the one advocated by Gomis and Weinberg [119]. We can rewrite (106) using the square of the Weyl tensor in terms of the topological invariant term E,

$$C^2 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = E + 2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2. \tag{107}$$

We reabsorb the term quadratic in the Ricci tensor into a term proportional to (107) and find with general signature $\varepsilon = (\pm 1, 1, 1, 1)$,

$$S = \varepsilon \int d^4x\sqrt{g}\Big(\tilde{\Lambda} - \frac{R}{\kappa^2} + \frac{1}{2s}C^2 - \frac{\omega}{3s}R^2\Big), \tag{108}$$

where ω and s are couplings, $\kappa \equiv (16\pi \bar{G}_N^{\text{bare}})^{1/2}$ and $\tilde{\Lambda} \equiv \frac{2\Lambda}{\kappa^2}$ [79, 80].

The field equations arising from the variation of (106) include quartic derivative terms. These are used to absorb the $1/p^4$ UV divergences appearing in the one-loop scalar corrections to the free graviton propagator. The corresponding theory is 'strictly renormalizable' according to the definition in [24, 47]. That is, a formal continuum limit exists in which we can remove the UV cutoff such that observables are independent of the regularization scheme in the sense of power series in the loop counting parameter. The successor problem to the perturbative non-renormalizability of gravity is twofold:

1. Firstly, perturbation theory is presumed to cover only a small part of a theory's physics content, so we must find a formulation of the theory that is renormalizable in the Kadanoff-Wilson sense.

2. Secondly, we must find observables obeying unitarity [79, 80]. Indeed, the $1/p^4$ falloff of the free propagator arises from $1/p^2 - 1/(p^2 + s/\kappa^2)$, where the second term has a negative norm in the Fock space of the theory [120]. Thus, they are problematic from the point of view of unitarity and causality [24, 47, 118]. We will come back to this in Section 4.2.

The asymptotic safety program offers a solution to both problems at once [24, 32, 47].

3.2.2 Does perturbation theory see gravitational fixed points?

The contact from the Asymptotic Safety program to higher-derivative gravity and perturbation theory has been established by Niedermaier [24, 47, 79, 80]. As outlined in Section 2.4.1, two elements are crucial to this program.

1. The first is the existence of a UV-attractive NGFP for the dimensionless positive Newton coupling constant, $g^* > 0$, and potentially for the dimensionless cosmological constant $\lambda^* \neq 0$. In analogy with (99), the dimensionless couplings are defined as $g = \mu^2\kappa^2$ and $\lambda = \mu^{-2}\Lambda$. As usual, μ is the renormalization mass scale of the regularization scheme.

2. The second is that the flow of all four gravitational couplings must be asymptotically safe, that is, bounded for all μ and finite at both limiting values $\mu \to 0$ and especially $\mu \to \infty$.

As explained by Weinberg in his 1979 paper [32], these properties are dependent on the theory space parametrization chosen. One should try to define "the coupling constants as coefficients in a power series expansion of the reaction rates themselves" [32]. If this is achieved, the UV regime of an asymptotically safe theory should be accessible by standard perturbation theory [80]. In what follows, we show that a NGFP for Newton's constant is indeed visible from perturbation theory.

In quantizing the EH action [120, 121], the fluctuation around the background metric ($h_{\mu\nu}$) is rescaled by a factor of $\kappa \equiv \sqrt{8\pi\bar{G}_N^{\text{bare}}}$ such that the coupling constant is absorbed away from the kinetic term of the Lagrangian relevant to the graviton propagator ($h_{\mu\nu} \to \kappa h_{\mu\nu}$). The corresponding gravitational vertices then carry factors of κ.
Applying this reasoning to the higher-derivative gravity defined by (108), we expect vertices

to contain terms in positive powers of s. Since s is asymptotically free in perturbation theory [122, 123], *perturbative expansions are expansions in powers of s.* The perturbation expansion is asymptotically convergent (it reaches its GFP), and hence describes well the theory in its UV limit. Note, s and g are of order 1 in the loop counting parameter $\hbar$, whereas other couplings in (108) are of order 0 [79, 80]. Since Newton's constant may appear in the perturbative expansion in ratios s/g of order $\mathcal{O}(\hbar^0)$, a non-zero NGFP of g is within the reach of perturbative theory.

This argument is analogous to the one demonstrating the ability of perturbation theory to describe UV interactions in QCD because the theory's coupling is asymptotically free. This originally motivated Niedermaier to attempt to build a conceptual bridge between the antisymmetric field strength tensor $F_{\mu\nu}$ in QCD, and the contracted curvature Weyl tensor $C_{\mu\nu}$ in gravity [24, 47].

3.2.3 Finding the NGFP

Now, we follow [79] to compute the gravitational coupling flow to lowest order in perturbation theory. In this derivation, we focus on the Euclidean signature to rewrite (108) as follows

$$S = \int d^4x\sqrt{q}\Big(\tilde{\Lambda} - \frac{R}{\kappa^2} + \frac{1}{2s}C^2 - \frac{\omega}{3s}R^2\Big), \tag{109}$$

where $q_{\mu\nu}$ is the full metric from which the Ricci scalar and Weyl tensor are constructed. Recall, $q = \det(q_{\mu\nu})$. Since $\tilde{\Lambda}$ is the cosmological constant, the UV cutoff is denoted Λ_{UV}.

Once again, we expand the full metric $q_{\mu\nu}$ around a static background $g_{\mu\nu}$ such that $q_{\mu\nu} = g_{\mu\nu} + f_{\mu\nu}$. Raising and lowering indices is done with the background static metric $g_{\mu\nu}$. We perform the functional integral over the quantum dynamical metric $f_{\mu\nu}$ to obtain the divergent part of the one-loop effective action for pure gravity [124]. It contains logarithmic *and* powerlike divergences, and is given by

$$\Gamma_1^{\text{div}} = -\frac{1}{(4\pi)^2}\int d^4x\sqrt{q}\Big[\Lambda_{UV}^4\Upsilon_1+\Lambda_{UV}^2(\Upsilon_2 R+\mu^2\Upsilon_3)+\ln\left(\frac{\Lambda_{UV}}{\mu}\right)(\zeta_1 C^2+\zeta_2 R^2+\mu^2\zeta_4 R+\mu^4\zeta_5)\Big], \tag{110}$$

where we introduced the dimensionful sliding scale μ to make the argument of the logarithm dimensionless. The loop counting parameters Υ and ζ are real valued functions of ω, λ and the ratio s/g, that we expected. Note, the gauge independence of the effective action (110) is automatic for the coefficients of the logarithmic divergences but is non-trivial otherwise [80]. Studying the non-trivial cases is not in the scope of this dissertation.

From remark 1 in Section 3.2.1, and the dimensional nature of the gravitational couplings in (108), we expect them to have a power-law running. This is confirmed by equation (110). Hence, the standard scheme of dimensional regularization and minimal substraction, which only 'sees' logarithmic divergences, is no longer valid. Instead, we use a background covariant operator cutoff which will keep track of powerlike divergences [125]. This allows one to make contact with the non-perturbative results outlined in Section 3.1 [79, 80]. The *non-minimal* subtraction ansatz for the bare couplings κ_0 and $\tilde{\Lambda}_0$, used to absorb the divergences in (110), is

$$\begin{aligned}\tilde{\Lambda}_0 &= \mu^4 \frac{2\lambda}{g}\Big\{1 + \frac{\hbar}{(4\pi)^2}\Big[a_{10} + a_{11}\ln\Big(\frac{\Lambda_{UV}}{\mu}\Big) + a_{12}\Big(\frac{\Lambda_{UV}}{\mu}\Big)^2 + a_{13}\Big(\frac{\Lambda_{UV}}{\mu}\Big)^4\Big] + \mathcal{O}(\hbar^2)\Big\},\\ \kappa_0^2 &= \mu^{-2} g\Big\{1 + \frac{\hbar}{(4\pi)^2}\Big[b_{10} + b_{11}\ln\Big(\frac{\Lambda_{UV}}{\mu}\Big) + b_{12}\Big(\frac{\Lambda_{UV}}{\mu}\Big)^2\Big] + \mathcal{O}(\hbar^2)\Big\}.\end{aligned} \tag{111}$$

g and λ are now the *renormalized* gravitational and cosmological couplings. Standard minimal subtraction suffices for the dimensionless couplings s and ω [80], such that

$$\begin{aligned}s_0 &= s\Big\{1 + \frac{\hbar}{(4\pi)^2}\Big[c_{11}\ln\Big(\frac{\Lambda_{UV}}{\mu}\Big)\Big] + \mathcal{O}(\hbar^2)\Big\},\\ \omega_0 &= \omega\Big\{1 + \frac{\hbar}{(4\pi)^2}\Big[d_{11}\ln\Big(\frac{\Lambda_{UV}}{\mu}\Big)\Big] + \mathcal{O}(\hbar^2)\Big\}.\end{aligned} \tag{112}$$

The field renormalization is given by

$$q^0_{\mu\nu} = q_{\mu\nu} + \frac{\hbar}{(4\pi)^2}\ln\Big(\frac{\Lambda_{UV}}{\mu}\Big) g\xi q_{\mu\nu} + \mathcal{O}(\hbar^2), \tag{113}$$

where ξ can be a function of s/g, λ, ω. We define $u_0 \equiv (\tilde{\Lambda}_0, \kappa_0^2, s_0, \omega_0)$ as the bare couplings, and $u \equiv (\lambda, g, s, \omega)$ as the dimensionless renormalized ones. The bare action can be written as $S_0[g^0, u_0] = S[g, u] + \Delta S[g, u]$, where g^0 is the bare background metric. $\Delta S[g, u]$ corresponds to the divergent terms in $S_0[g^0, u_0]$, whereas $S[g, u]$ is the finite part of the bare action. By plugging (113,111) into $S_0[g^0, u_0]$, we obtain

$$\begin{aligned}\Delta S[g,u] = \frac{\hbar}{(4\pi)^2}\int d^4x\sqrt{q}\Big[&\Lambda_{UV}^4\frac{2\lambda}{g}a_{13} + \Lambda_{UV}^2\Big(\mu^2\frac{2\lambda}{g}a_{12} + \frac{b_{12}}{g}R\Big)\\ &+ \ln\Big(\frac{\Lambda_{UV}}{\mu}\Big)\Big(\mu^4\frac{\lambda}{g}(2a_{11} + 4g\xi) + \frac{\mu^2}{g}R(b_{11} - g\xi) - \frac{c_{11}}{2s}C^2 - \frac{\omega}{3s}(d_{11} - c_{11})R^2\Big)\Big] + \mathcal{O}(\hbar^2).\end{aligned} \tag{114}$$

The counter-term '*cancellation condition*' $\Delta S[g, u] = -\Gamma_1^{\text{div}}$ fixes a_{11}, a_{12}, a_{13}, b_{11}, b_{12}, b_{13}, c_{11} and d_{11} in terms of the Υs and ζs but leaves a_{10} and b_{10} unconstrained. In standard PT, bare couplings are assumed to be independent of μ. By constrat, in Wilsonian renormalization, when the running renormalization scale μ reaches the UV cutoof Λ_{UV}, the renormalized parameters should match their bare values. Thus, we require

$$\begin{aligned}\tilde{\Lambda}_0 &= \Lambda_{UV}^4 \frac{2\lambda}{g} \quad \text{when} \quad \mu = \Lambda_{UV}, \\ \kappa_0^2 &= \Lambda_{UV}^{-2} g \quad \text{when} \quad \mu = \Lambda_{UV}.\end{aligned} \tag{115}$$

Replacing μ by Λ_{UV} in (112) fixes the 'initial' values of the yet unfixed a_{10} and b_{10} coefficients as follows

$$\begin{aligned}a_{10} + a_{12} + a_{13} &= 0, \\ b_{10} + b_{12} &= 0,\end{aligned} \tag{116}$$

since $\ln(\Lambda_{UV}/\Lambda_{UV}) = \ln(1) = 0$. Together, (115,116) form the '*matching condition*' [79, 80]. Essentially, we determined a_{10} and b_{10} in terms of the Υs and ζs by deriving conditions that relate them to the fixed a_{11}, a_{12}, a_{13}, b_{11}, b_{12}, b_{13}. Thus, we arrived at a uniquely defined non-minimal subtraction ansatz, fixed by the *matching* and *cancellation* conditions. Importantly, we have fixed the dependence of all the latin letters parameters (a_{1i}, b_{1n}, c_{11} and d_{11} for $i = 0, 1, 2, 3$ and $n = 0, 1, 2$) in terms of all the greek letters ones (Υ_o, ζ_t and ξ for $o = 1, 2, 3$ and $t = 1, 2, 3, 4, 5$).

We can now derive the flow equations for all renormalized dimensionless couplings. As an example, we derive the one for s explicitly and simply state the others. For instance, the *cancellation* condition fixed $c_{11} = -2s\zeta_1$. We plug this into the first line of (112) and apply $\mu\frac{d}{d\mu}$ on both sides of the equation

$$\begin{aligned}\mu\frac{ds_0}{d\mu} &= \mu\frac{ds}{d\mu} + \mu\frac{d}{d\mu}\left[-\frac{2\zeta_1 s^2\hbar}{(4\pi)^2}\ln\left(\frac{\Lambda_{UV}}{\mu}\right)\right], \\ 0 &= \mu\frac{ds}{d\mu} + \mu\frac{d}{d\mu}\left[-\frac{2\zeta_1 s^2\hbar}{(4\pi)^2}\ln(\Lambda_{UV}) + \frac{2\zeta_1 s^2\hbar}{(4\pi)^2}\ln(\mu)\right], \\ 0 &= \mu\frac{ds}{d\mu} + \frac{\mu}{\mu}\left[\frac{2\zeta_1 s^2\hbar}{(4\pi)^2}\right], \\ \therefore \mu\frac{ds}{d\mu} &= -\frac{2\zeta_1 s^2\hbar}{(4\pi)^2},\end{aligned} \tag{117}$$

where we used the fact that bare couplings are independent of μ. We now easily see that s is indeed asymptotically free [122, 123]. By applying this method to the other couplings, we recover the universal flow equations for both s and ω [122, 123, 126], and obtain the ones for

g and λ:

$$
\begin{aligned}
\mu\frac{d\omega}{d\mu} &= -\frac{s\hbar}{(4\pi)^2}(3\zeta_2 + 2\omega\zeta_1),\\
\mu\frac{dg}{d\mu} &= 2g + \frac{g^2\hbar}{(4\pi)^2}(\zeta_4 + \xi + 2\Upsilon_2),\\
\mu\frac{d\lambda}{d\mu} &= -2\lambda + \frac{g\hbar}{2(4\pi)^2}[\zeta_5 + 4\lambda\zeta_4 + \Upsilon_3 + 4\lambda\Upsilon_2 + 4\Upsilon_1 - (2\lambda\xi + 2\lambda\zeta_4 - \Upsilon_3)].
\end{aligned}
\tag{118}
$$

Note, $\zeta_4 = \zeta_5 = \xi = \Upsilon_3 = 0$ at the GFP $s^* = 0$. Hence, the position of the FPs for g and λ are only determined by Υ_1 and Υ_2. The non-trivial ones are located at

$$
g^* = -\frac{(4\pi)^2}{\Upsilon_2(\omega^*)} \quad , \quad \lambda^* = -\frac{\Upsilon_1(\omega^*)}{2\Upsilon_2(\omega^*)}. \tag{119}
$$

Here, the numerical value for the ω FP is $\omega^* \approx -0.022864$. While the ζs have been determined in many gauges [122, 123, 126–128], the Υs have not been computed before Niedermaier [79]. It is possible to determine the numerical values of g^* and λ^*, by computing the Υ coefficients in the three parameter harmonic gauge [79, 80]:

$$
\begin{aligned}
S_{\text{gf}} &= \frac{1}{2s}\int d^4x\sqrt{-g}\chi_\mu Y^{\mu\nu}\chi_\nu,\\
\chi_\mu &= \nabla^\nu f_{\mu\nu} + b_1\nabla_\mu f,\\
Y^{\mu\nu} &= -\frac{1}{a}(g^{\mu\nu}\nabla^2 + (b_2 - 1)\nabla^\mu\nabla^\nu - R^{\mu\nu}),\\
b_1 &= -\frac{1}{4c_1}\frac{1+4\omega}{1+\omega},\\
b_2 &= -\frac{2c_2}{3}(1+\omega).
\end{aligned}
\tag{120}
$$

$a = c_1 = c_2 = 1$ defines the minimal gauge. The ghost action in this gauge is

$$
S_{\text{gh}} = \int d^4x\sqrt{-g}\bar{C}_\mu\Delta^\mu{}_\nu C^\nu, \tag{121}
$$

with kernel

$$
\Delta^{\mu\nu} = -g^{\mu\nu}\nabla^2 - (1+2b_1)\nabla^\mu\nabla^\nu - R^{\mu\nu}. \tag{122}
$$

Using heat kernel methods [79, 80, 127, 129], in minimal gauge, on a flat background, we can determine Υ_1 and Υ_2. This derivation is out of the scope of this dissertation, so we only state the final result. We can rewrite (118) as one-loop gauge independent RG flows:

$$\begin{aligned}\mu\frac{dg}{d\mu} &= 2g + 2g^2(u_2 + \frac{s\lambda}{g}u_3),\\ \mu\frac{d\lambda}{d\mu} &= -2\lambda + 2g(u_1 + \lambda u_2) + 2s(u_3 + \lambda u_5) + \frac{s^2}{2g}u_4.\end{aligned} \tag{123}$$

Taking $s_0 = 1$ and $\omega_0 = -1/2$, which corresponds to a smooth cutoff, we find one NGFP for each flow in (123). Their numerical values are given by

$$g^* \approx 1.3697 \quad , \quad \lambda^* \approx 0.9451. \tag{124}$$

Importantly, these arise from a gauge-independent effective action which in turn gives rise to the gauge-independent RG flows (123) [79, 80]. The corresponding UV-attractive NGFP is shown in Fig 11.

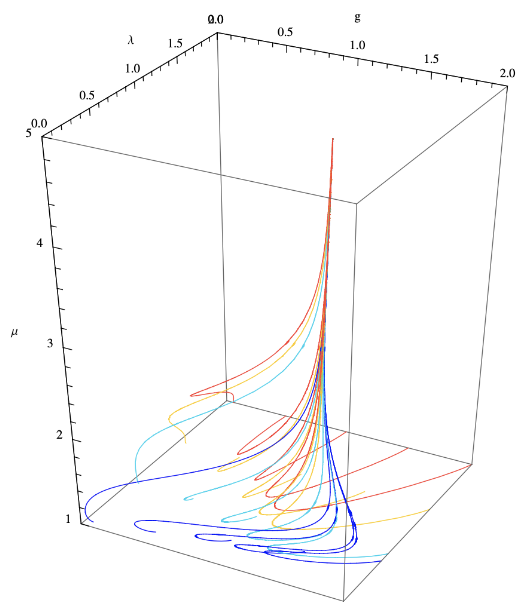

Figure 11: *This graph presents the gauge-independent Wilsonian RG flow in quadratic gravity (109) for g and λ at one-loop order in perturbation theory, in minimal gauge, with a smooth cutoff. We reach the NGFP (124) when increasing the sliding renormalization mass scale μ, i.e. in the direction of inverse RG flow. Hence, the NGFP is UV-attractive under the inverse RG flow, as required by the Asymptotic Safety program [79].*

3.3 Comparing the two methods

The methods of the FRGE and of perturbation theory used in higher-derivative quadratic gravity are in one-to-one correspondence [96]. However, only the latter is able to tackle the traditional UV renormalization problem [79, 80, 96].

3.3.1 FRG and perturbative quadratic gravity are equivalent methods

There are many ways to draw a connection between FRG and perturbative formalisms:

- The Exact FRGE (87) (or ERGE) can be interpreted as an RG improvement of a perturbative one-loop equation [96]. Indeed, the one-loop effective action corresponding to the bare action S for a bosonic field ϕ, is given by

$$\Gamma^{(1)} = S + \frac{1}{2}\,\mathrm{Tr}\left\{\ln\left[\frac{\delta^2 S}{\delta\phi\delta\phi}\right]\right\}. \tag{125}$$

  Now, adding the cutoff term

$$\Gamma^{(1)}_k = S + \frac{1}{2}\,\mathrm{Tr}\left\{\ln\left[\frac{\delta^2 S}{\delta\phi\delta\phi} + \mathcal{R}_k\right]\right\}. \tag{126}$$

  This is now the one-loop effective *average* action. It satisfies the one-loop equation

$$k\frac{d\Gamma^{(1)}_k}{dk} = \frac{1}{2}\,\mathrm{Tr}\left\{\left[\frac{\delta^2 S}{\delta\phi\delta\phi} + \mathcal{R}_k\right]^{-1} k\frac{d\mathcal{R}_k}{dk}\right\}, \tag{127}$$

  which is of the same form as (167). The only difference with the Wetterich equation is that the RHS of equation (127) contains bare couplings through the Hessian of the bare action. Instead, the Wetterich equation contains renormalized couplings. In this sense, we can say that the ERGE (167) is an RG improved version of the one-loop equation (127).
  We can choose to approximate the cutoff function $\mathcal{R}_k$ so that it only depends on k explicitly, and not through its implicit dependence on running couplings. Then, the entire content of the ERGE is contained within the RG improved one-loop beta-functions [79, 80, 96, 129].

- The flow equations have the same structure in the EH truncation and in perturbation theory treatments [96]. The former contain extra-terms compared to those obtained from one-loop calculations. This is a result of the renormalization group improvement discussed in the previous remark.

- We recover the one-loop perturbative UV divergences, originally derived by Veltman and 't Hooft, from the ERGE [10]. In the EH truncation of the ERGE (87) without a cosmological constant, we recover the cutoff-scheme-dependent divergences. Including the cosmological constant, we recover the scheme-independent ones [96].

Since the two methods are seemingly equivalent, one can even use one method in concordance with the other, as in [130]. Indeed, in a reconstructive spirit, one can use the ansatz (109) for the running effective action. This is known as the "Higher-Derivative Gravity truncation" [130]. Using this method, it can be shown that one of the NGFPs found for Newton's coupling corresponds to a theory free of tachyons.

In addition, one can confirm that the dimension of the critical surface is $\Delta_{UV} = 3$ for pure gravity [131]. The correspond asymptotically safe theory is predictive as understood in Section 2.4.2. Yet, this result is not general since it is possible to find 4-dimensional UV critical surfaces in other cases. This is out of the scope of the present work.

3.3.2 Advantage of perturbation theory

In Section 3.1.5 we established that the FRGE is solved from a fixed UV scale $k = \Lambda$, where the initial condition $\Gamma_{k,\Lambda} = S_{\text{tot},\Lambda}$ is imposed, down to k. For instance, in Section 3.1.8, we approximately solved the gravitational FRGE (87) by using the EH truncation ansatz (96) as an approximation to the exact GEAA (83). We required that (96) obeys the limit $\lim_{k=\Lambda\to\infty} \Gamma_k = S_{\text{tot}}$, where the bare action S appears in the total action $S_{\text{tot},\Lambda} = S + S_{\text{gf}} + S_{\text{gh}}$. In the FRG formalism, we essentially trade the traditional UV renormalization problem for the identification of a fine-tuned initial bare action [47]. However, we explained that the FRGE itself gives no means to find such adjusted action. Hence, the UV renormalization problem is not properly adressed in the FRG framework.

Beyond PT, the only hope for FRG, and only known non-perturbative method to identify the bare action, is **constructive renormalization** [132, 133]. This approach treats the instabilities of the RG-induced-non-polynomial action at large fields [51]. Yet, this method cannot be used because the 4-dimensional QFTs of interest are beyond constructive control [47]. Hence, only perturbation theory allows to address the traditional UV renormalization problem, i.e. the continuum limit $\Lambda \to \infty$ is not rigorously defined in FRG [79, 80]. The FRG flow matches PT by construction only *after* said problem is solved [134].

4 Critical reflections on the Asymptotic Safety Program

On top of the technical difficulties raised in Section 3.1.9, some foundational questions still pose serious challenges to the Asymptotic Safety program. We present two of them: the running of gravitational couplings and unitarity.

4.1 Running gravitational couplings

As mentioned in Section 3.1.9, in theory space truncations, the RG evolution causes dimensionful running couplings to receive contributions from operators that were turned off in the original truncation. For example, this is the case of GR, where each loop-order added in a computation induces terms two derivative orders higher in the counterterm action. In Section 3.1.9, we argued that these contributions were set to zero in the beta functions. These contributions are dependent on the physical process used to probe how the couplings run with momentum scale. Hence, the definition of dimensionful running couplings is non-universal and ambiguous [101]. Concretely, Donoghue, Anber, Dunbar and Norridge showed this by calculating the perturbative one-loop order of Newton's constant's running [135–137]. They computed it for the graviton vacuum polarisation, graviton-graviton scattering and the gravitational scattering of identical massless scalars and non-identical relativistic scalars. The calculated one-loop runnings are different for each process.

As an example, we follow [137] and consider scattering of non-identical scalars $A+B \to A+B$. We introduce the usual Mandelstam variables s, t and u and neglect particle masses by taking the relativistic limit $s \gg m^2$. The tree level reduced matrix element of the scattering amplitude is given by

$$\mathcal{M}_{\text{tree}} = \frac{i\kappa^2 su}{4t}, \tag{128}$$

where $\kappa^2 \equiv 32\pi G$. The one-loop amplitude is given by

$$\begin{aligned}\mathcal{M}_{\text{1-loop}} = \frac{i\kappa^4}{(4\pi)^2}\Big[&\frac{1}{16}(s^4 I_4(s,t) + u^4 I_4(s,t)) + \frac{1}{8}(s^3+u^3+tsu)I_3(t) - \frac{1}{8}(s^3 I_3(s) + u^3 I_3(u)) \\ &- \frac{1}{240}(71us - 11t^2)I_2(t) + \frac{1}{16}(s^2 I_2(s) + u^2 I_2(u))\Big].\end{aligned} \tag{129}$$

$I_2(s)$, $I_3(s)$ and $I_4(s,t)$ are the scalar bubbled, triangle and box diagrams respectively. They take the form

$$
\begin{aligned}
I_2(s) &= \left(\frac{1}{\epsilon} - \ln(-s) + \text{finite}\right), \\
I_3(s) &= -\frac{1}{s}\left(\frac{1}{\epsilon^2} - \frac{\ln^2(-s)}{\epsilon} + \frac{\ln(s)^2}{2}\right), \\
I_4(s,t) &= \frac{1}{st}\left(\frac{4}{\epsilon^2} - \frac{2\ln(-s) + 2\ln(-t)}{\epsilon} + 2\ln(-s)\ln(-t) + \text{finite}\right),
\end{aligned}
\tag{130}
$$

where $\epsilon \equiv (4-d)/2$ is introduced in the regularization scheme. We can define the amplitude corresponding to IR divergences as

$$
\mathcal{M}_{\text{IR}} = \frac{\kappa^4}{2(4\pi)^2} \frac{((-s)^{1-\epsilon} + (-u)^{1-\epsilon} + (-t)^{1-\epsilon})}{\epsilon^2} \mathcal{M}_{\text{tree}}. \tag{131}
$$

We absorb the divergences of the one-loop amplitude (129) by subtracting (131) from it

$$
\mathcal{M}_h = \mathcal{M}_{\text{1-loop}} - \mathcal{M}_{\text{IR}}, \tag{132}
$$

to be left with $\mathcal{M}_h$, the 'hard part' of the one-loop amplitude. To compute the one-loop running of Newton's constant requires to calculate said hard amplitude at the renormalization center kinematic point $s = 2E^2$, $t = u = -E^2$. We obtain the total amplitude

$$
\mathcal{M}_{\text{tot}} = \frac{i\kappa^2 E^2}{2}\left[1 - \frac{\kappa^2 E^2}{10(4\pi)^2}\left((19 + 10\ln 2)\ln\left(\frac{E^2}{\mu^2}\right) + 5(\pi^2 - (\ln 2 - 1)\ln 2)\right)\right]. \tag{133}
$$

This can be used to renormalize the bare Newton's constant as follows

$$
G(E) = G_0\left[1 - \frac{\kappa^2 E^2}{10(4\pi)^2}\left((19 + 10\ln 2)\ln\left(\frac{E^2}{\mu^2}\right) + 5(\pi^2 - (\ln 2 - 1)\ln 2)\right)\right]. \tag{134}
$$

We can consider the amplitude of the slightly different process $A + A \to B + B$ which is given by (129) under the exchange $s \leftrightarrow t$. Following the previous derivation, we obtain the following total amplitude

$$
\mathcal{M}_{\text{tot}} = \frac{i\kappa^2 E^2}{8}\left[1 + \frac{\kappa^2 E^2}{10(4\pi)^2}\left(9\ln\left(\frac{E^2}{\mu^2}\right) - 5\pi^2 + (19 + 5\ln 2)\ln 2)\right)\right]. \tag{135}
$$

Clearly, (133) and (135) are different. This rather general result applies to other amplitudes of gravitational processes that receive contributions from implicit higher-order derivative operators. This is known as the **non-universality problem** of gravitational amplitudes [135, 137]. It especially impairs the interpretation of equations such as (105), which we mentioned as evidence for anti-screening of gravitational interactions in the extreme UV.

4.2 Unitarity and causality

In truncations of theory space that include more terms than the EH truncation, higher-derivatives terms are necessarily present in the microscopic classical action. This induces a violation of unitarity, which corresponds to an instability of the theory, through Ostrogradsky's theorem [138]. This states that any non-degenerate Lagrangian with higher time derivative of finite order leads to an unstable Hamiltonian with at least one term linear in a conjugate momentum. Said momentum can be made arbitrarily small which results in a Hamiltonian unbounded from below [101]. Therefore, a truncation to finite order in momenta is not well suited to study the unitarity of a theory since it generates truncation-induced instabilities.

In QFTs, the instability problem manifests as propagating ghost states that violate unitarity. As mentioned in remark 2 of Section 3.2.1, Stelle showed that graviton propagators with dependence $\propto p^{-4}$ could absorb divergences arising in quadratic gravity [118]. However, this theory, defined by (106), has spin-2 ghost states with mass m_g propagating according to the following partial-fraction decomposition

$$D(p) = \frac{1}{p^2(p^2 + m_g^2)} = \frac{1}{m_g^2}\left(\frac{1}{p^2} - \frac{1}{p^2 + m_g^2}\right). \tag{136}$$

It is possible for m_g to be negative, implying the existence of tachyons. Using Feynman ($+i\epsilon$) prescription, this propagator can further be rewritten in the Källén-Lehmann spectral representation

$$D(p) = \frac{1}{p^2 - m^2 + i\epsilon} + \int_{4m^2}^{\infty} dM^2 \frac{\rho(M^2)}{p^2 - M^2 + i\epsilon}, \tag{137}$$

where m is the mass of the free theory, and $\rho(M^2)$ is the spectral density of the theory

$$\rho(M^2) = \sum_n (2\pi)\delta(M^2 - m_n^2)\Big| \langle\Omega|\, \hat{\phi}(0)\, |n\rangle \Big|^2. \tag{138}$$

We define $\Big| \langle\Omega|\, \hat{\phi}(0)\, |n\rangle \Big|$ as the norm of a Poincaré invariant state constructed from the adjoint of the true vacuum state $|\Omega\rangle$. The positivity of this function allows it to be interpreted as a density of states. Problematically, the negative sign on the rightmost fraction in (136) induces a negative residue of the propagator, and in turn a negative spectral density which violates unitarity. There are three general ways of fighting Ostrogradsky instabilities, or violations of unitarity [101]:

1. Avoiding a negative residue in (136) by finding a propagator consisting of an entire

function with a single zero at vanishing momentum. This is done in Non-local ghost-free gravity [139–142].

2. Giving up on Lorentz invariance by introducing higher-order spatial derivatives, like in Hořava-Lifshitz gravity [143].

3. Allowing causality to be violated microscopically and interpreting the ghost dofs as propagating backwards in time. If these ghosts are sufficiently heavy, they are undetectable [144–147].

We expand on the latter solution. Using the $(-i\epsilon)$ prescription, instead of the usual $(+i\epsilon)$ one, we can write the Feynman ghost propagator

$$D_F(x,y) = \lim_{\epsilon^+ \to 0} \int_{C^-} \frac{d^4p}{(2\pi)^4} e^{-ip_\mu (x-y)^\mu} \frac{-i}{p^2 - m_g^2 - i\epsilon}. \tag{139}$$

We consider the forward time propagating part of $D_F(x,y)$, characterised by $x^0 > y^0$. Reminding the reader of complex analysis methods, we integrate p_0 in the clockwise direction along the real axis and a semicircle in the lower-half of the complex plane of p_0, called C^-. In this way, the contribution from the integral along the arc vanishes when we take the semicircle's radius to infinity. The contour C^- includes the pole $-E(p)$, so we can use Cauchy's theorem and separate the time and spatial parts of the exponential to obtain

$$D_F(x,y) = \int \frac{d^3\vec{p}}{(2\pi)^3 E(p)} e^{-iE(p)(y_0 - x_0)} e^{i\vec{p}(\vec{x}-\vec{y})}. \tag{140}$$

The residue is positive in this setting, at the cost of a ghost propagating *backwards* in time ($x_0 > y_0$ entails a negative sign in the first exponential) which violates causality. This is equivalent to a ghost particle propagating forwards in time with a negative residue, which violates unitarity [144,145]. We seem to have a choice between violating unitarity or causality.

However, following Donoghue and Menezes [146,147], we can further consider self-energy corrections to (140). From these, the denominator picks up a strictly positive imaginary part γ. We can then replace $E(p)$ by $(E(p) + i\gamma/2E(p))$ in equation (140)

$$D_F(x,y) = \int \frac{d^3\vec{p}}{(2\pi)^3 (E(p) + i\gamma/2E(p))} e^{-iE(p)(y_0 - x_0)} e^{i\vec{p}(\vec{x}-\vec{y})} e^{-\frac{\gamma}{2E(p)}(x_0 - y_0)}. \tag{141}$$

If γ is large enough, and $x_0 > y_0$, causality is only violated in the UV due to the behaviour of the ratio in the rightmost exponential of (141). At low energies the propagator is exponentially suppressed. The ghost's decay timescale is proportional to the Planck scale, so it leaves no detectable trace [101,146,147].

There are also case-specific ways to circumvent the unitarity problem. Still in quadratic gravity, Niedermaier calculated that the Hessian's spectrum eigenvalues are strictly positive near the NGFP [80]. Since the Hessian corresponds to the inverse propagator, there are no ghost states in this case (see Appendix B). One could argue that this is not general since we employ the asymptotic freedom of s, the Weyl tensor's coupling, and have no information about the RG flow far away from the NGFP. However, other methods point at the same result. In particular, in the FRG formalism, all we need to avoid violating unitarity is that $(\Gamma_k^{(2)} + \mathcal{R}_k)$ is a positive operator for all k, such that RG trajectories are defined down to $k = 0$ [24, 47]. This is believed to be the case in the untruncated theory space, as shown in the R^2 truncation [109].

5 Discussion and conclusion

Considering GR as an EFT solves most of the problems raised in this dissertation. Indeed, the perturbative non-renormalizability of GR is not a problem anymore, since EFTs are not required to be UV complete. Similarly, setting the effectiveness of the EFT at the ghost's mass scale resolves the ghost problem in higher-derivative theories.

However, it seems that there is an advantage of the Asymptotic Safety program over EFTs. Not only can we make reliable computations in the gained energy range by UV completing GR, but we can identify 'large' quantum corrections at low energies. Indeed, well-defined low-energy quantum effects are expected to be suppressed by E/m_p in EFTs (where $m_p^2 \equiv \hbar c/G$ is the Planck mass). If these low energy effects arise from UV quantum behaviour (beyond ℓ_p), one would need to computationally propagate their effect through many orders of magnitude down to experimentally accessible energies [24, 47].

Rather than opting for an EFT description of GR, we presented the Asymptotic Safety program, which "takes the degrees of freedom of the gravitational field seriously also in the quantum regime" [24, 47]. To solve the perturbative non-renormalizability, we adopted a Wilsonian view of renormalization, allowing us to make no reference to perturbative theory. We showed that a RG flow arises from a background independent gravitational path integral, in which the metric field carries the gravitational dofs. In a reconstructive spirit, this flow dictates what action we need for a QFT of gravity if there exists a NGFP from which the RG trajectory emanates. All actions on this trajectory are in the same universality class, allowing us to extract physical quantities on all energy scales by following it back to the FP. Importantly, these actions yield quantities that are "asymptotically safe" from UV divergences at all energy scales under the one at which each action is defined.

Summarizing and extending this, we can highlight three main points of the Asymptotic Safety program:

- We can relate UV and IR-physics of the gravitational field, which dofs are carried by the metric field, through a Wilsonian RG flow.
- The resulting QFT of gravity is a quasi-renormalizable theory based on a UV-attractive NGFP and its finite dimensional UV critical surface.
- We find that physical dofs in the extreme UV have antiscreening interactions.

Searching for ways of practically realising the first two points, we explored the possibility that gravity is non-perturbatively renormalizable by outlining two equivalent methods based on

background field expansions.
Namely, the gravitational FRG formalism, presented by Reuter, is based upon the concept of a gauge and background independent GEAA. In a reconstructive spirit, we used the EH truncation to find a non-trivial FP for the dimensionless Newton's constant and cosmological constant in the GEAA's RG flow. In turn, the GEAA at high UV cutoff with FP valued couplings became our bare microscopic action. We showed that truncations results were robust under shifts in cutoff schemes.
The second perturbative method, presented by Niedermaier, extended Stelle's developments in higher-derivative quadratic gravity. By using the asymptotic freedom of the Weyl-tensor-squared term, in minimal gauge and with a smooth cutoff, we derived gauge-independent RG flow equations for all couplings and showed that a NGFP exists for the dimensionless Newton's constant and cosmological constant. Together with the evidence from gravity in $2 + \epsilon$ dimensions, these methods strongly hint at the existence of a NGFP, that can be used to renormalize gravity, in the full un-truncated theory space of 4-dimensional gravity.

Despite the structural equivalence between the two methods, only perturbation theory addresses the traditional UV renormalization problem. That is, the continuum limit $\Lambda \to \infty$ is well-defined. In FRG, there is no means to determine a fine-tuned bare action, as an initial condition for the FRGE, at arbitrarily high Λ. Moreover, other non-perturbative methods such as constructive approaches are out of reach for 4-dimensional gravity.

Finally, we presented some of the difficulties faced by the Asymptotic Safety program. On the one hand, the analysis of the extreme UV behaviour of gravitational dofs (antiscreening) seems to be severely impaired by the non-universality problem faced by dimensionful running gravitational couplings. On the other hand, the unitarity problem arising from the graviton propagators in quadratic gravity can be avoided in many different ways. Notably, we focused on a method which trades unitarity violation for microscopic causality violation. The ghosts are not detectable in this approach.

With more time, I would have liked to extend the present work by discussing Lorentzian quantum gravity and spectral functions for the graviton [101, 148]. Moreover, I would be interested in connecting the discussions from Section 3 to phenomenology by comparing RG improved black holes solutions [149–151] with spherically symmetric and general black hole spacetimes in higher-derivative gravity [152, 153].

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Appendices

A Deriving the kinematical pole and NGFP using the background field method

We decompose the metric into a conformal mode ϕ and a traceless symmetric tensor $h_{\mu\nu}$ as follows

$$\begin{aligned} g_{\mu\nu} &= \hat{g}_{\mu\alpha}(e^h)^\alpha{}_\nu e^{-\phi}, \\ &= \tilde{g}_{\mu\nu}e^{-\phi}, \end{aligned} \tag{142}$$

where $\hat{g}_{\mu\nu}$ is the background metric [63–65]. Reparametrizing the action in terms of this new metric and the conformal field, and taking into account the quantum corrections we get the following EH action:

$$S_{EH}(\hat{g}_{\mu\nu}, h_{\mu\nu}, \phi) = \frac{\mu^\epsilon}{G_0}\int d^Dx\sqrt{\hat{g}}e^{-\frac{\epsilon}{2}\phi}\left[\tilde{R} - \frac{1}{4}\epsilon(D-1)\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \tag{143}$$

where $D = 2+\epsilon$. We can expand in terms of $h^\mu{}_\nu$ and ϕ fields and drop the linear term.

$$\begin{aligned} S_{EH}(\hat{g}_{\mu\nu}, h_{\mu\nu}, \phi) = \frac{\mu^\epsilon}{G_0}\int d^Dx\sqrt{\hat{g}}\Big[&\hat{R} + \frac{1}{4}h^\alpha{}_{\mu,\nu}h^\mu{}_{\alpha,}{}^\nu + \frac{1}{2}\hat{R}^\sigma{}_{\mu\nu\alpha}h^\alpha{}_\sigma h^{\mu\nu} \\ &- \frac{\epsilon}{4}(D-1)\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{\epsilon}{2}\phi h^\mu{}_\nu\hat{R}^\nu{}_\mu + \frac{(\epsilon\phi)^2}{8}\hat{R} + \frac{\epsilon}{2}\phi h^{\mu\nu}{}_{,\mu\nu} \\ &- \frac{1}{2}h^\nu{}_{\mu,\nu}h^{\alpha\mu}{}_{,\alpha}\Big] + \ldots, \end{aligned} \tag{144}$$

where we follow the notations from [10], i.e. $,\mu$ is the covariant derivative ∇_μ with respect to the background metric. We apply the background field method [154,155] to compute quantum corrections and choose the following gauge fixing term

$$\frac{\mu^\epsilon}{G_0}\int d^Dx\sqrt{\hat{g}}\left(\frac{1}{2}h^\nu{}_{\mu,\nu} + \frac{\epsilon}{2}\partial_\mu\phi\right)\left(h^{\alpha\mu}{}_{,\alpha} + \frac{\epsilon}{2}\partial^\mu\phi\right). \tag{145}$$

This choice of gauge fixing cancels the last two terms in equation (144). Finally, the quadratic part of the action is

$$S_{EH}(\hat{g}_{\mu\nu}, h_{\mu\nu}, \phi) = \frac{\mu^\epsilon}{G_0} \int d^{2+\epsilon}x \sqrt{\hat{g}} \left[\frac{1}{4} h^\alpha{}_{\mu,\nu} h^\mu{}_{\alpha,}{}^\nu + \frac{1}{2} \hat{R}^\sigma{}_{\mu\nu\alpha} h^\alpha{}_\sigma h^{\mu\nu} \right. \\ \left. - \frac{\epsilon}{8} D \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{\epsilon}{2} \phi h^\mu{}_\nu \hat{R}^\nu_\mu + \frac{(\epsilon\phi)^2}{8} \hat{R} \right]. \tag{146}$$

We can now expand the background metric around the flat metric

$$\hat{g}_{\mu\nu} = \delta_{\mu\nu} + \hat{h}_{\mu\nu}. \tag{147}$$

We expand the ϕ kinetic term in $\hat{h}_{\mu\nu}$

$$\frac{\epsilon}{8} D \sqrt{\hat{g}} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \frac{\epsilon}{8} D \partial_\mu \phi \partial_\mu \phi - \frac{\epsilon}{8} D \hat{S}_{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \dots, \tag{148}$$

where

$$\hat{S}_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{1}{D} \delta_{\mu\nu} \hat{h}_{\alpha\alpha}. \tag{149}$$

Therefore, the ϕ propagator, where P denotes a generic graviton propagator following the notation from [10], is given by

$$\langle \phi(P) \phi(-P) \rangle = -\frac{1}{P^2} \frac{4}{\epsilon D}. \tag{150}$$

This is precisely the kinematical pole of order $\frac{1}{\epsilon}$ mentioned in Section 2.4.3.

We now evaluate the one-loop divergence to check that we get the same FP as in Section 2.4.3. The one-loop divergence due to one free scalar field is

$$-\frac{1}{24\pi} \sqrt{\hat{g}} \frac{\hat{R}}{\epsilon}. \tag{151}$$

Now, we count the number of dof to determine the total one-loop counterterm considering all our theory's field content. ϕ has 1 dof, $h^\mu{}_\nu$ contributes 2 dof in two dimensions and the complex two component ghost field associated with this QFT contributes -4. The central charges c label the matter fields coupled to gravity. The total one-loop counterterm is therefore

$$-\frac{25 - c}{24\pi} \sqrt{\hat{g}} \frac{\hat{R}}{\epsilon}. \tag{152}$$

Equivalently, defining $\gamma \equiv \frac{25-c}{24\pi}$, we are left with

$$-\gamma\sqrt{\hat{g}}\frac{\hat{R}}{\epsilon}. \tag{153}$$

The bare coupling is then

$$\frac{1}{G_0} = \mu^{\epsilon}\left(\frac{1}{G} - \gamma\frac{1}{\epsilon}\right), \tag{154}$$

where G is the renormalized coupling. The beta function is given by

$$\mu\frac{\partial}{\partial\mu}\left(\frac{1}{G_0}\right) = 0, \tag{155}$$

which yields the following perturbative flow equation, or Callam-Symanzik equation

$$\beta_{\text{pert}} \equiv \mu\frac{\partial}{\partial\mu}(G(\mu)) = \epsilon G - \gamma G^2. \tag{156}$$

The RG flow contains a NGFP in the UV at

$$G^*(\mu) = \frac{\epsilon}{\gamma}. \tag{157}$$

In this case, the dimension of the FP's unstable manifold is 1. This is valid for all $\gamma > 0$, i.e. $c < 25$. We did find the same NGFP as in Section 2.4.3.

B Deriving the Wetterich equation

We now derive a flow equation for the EAA governing its dependence on the IR cutoff scale k [24, 47, 53, 67]. First we recall that

$$
\begin{aligned}
Z_{k,\Lambda} = \exp\Big\{W_{k,\Lambda}[J,\bar{\Phi}]\Big\} &= \int \mathcal{D}\hat{\Phi} e^{-S^J_{\text{tot}}[\bar{\Phi},\hat{\Phi}]} e^{-\Delta S_k[\hat{\Phi},\bar{\Phi}]}, \\
&= \int \mathcal{D}\hat{\Phi} \exp\Big\{-S_{\text{tot}}[\hat{\Phi},\bar{\Phi}] - \Delta S_k[\hat{\Phi},\bar{\Phi}] + \int dx \hat{\Phi}(x) J(x)\Big\}.
\end{aligned} \tag{158}
$$

For simplicity, we drop the $\bar{\Phi}$ dependence for now and reintroduce it at the end of the derivation. We define the cutoff action according to equation (74) but replace p^2 by $-\partial^2$,

$$
\Delta S_k[\hat{\Phi}] = \frac{1}{2}\int d^D x \hat{\Phi}(x) \mathcal{R}_k(-\partial^2)\hat{\Phi}(x). \tag{159}
$$

This allows us to define the cutoff action without any reference to the Fourier decomposition of $\hat{\Phi}$. To obtain the effective action, we take the Legendre transform of $\ln\Big(Z_{k,\Lambda}[J,\bar{\Phi}]\Big)$ or equivalently of $W_{k,\Lambda}[J,\bar{\Phi}]$,

$$
\tilde{\Gamma}_{k,\Lambda}[\Phi] = \sup_{\{J\}}\left(\int dx \Phi(x) J(x) - W_{k,\Lambda}[J]\right). \tag{160}
$$

We then subtract the cutoff action to the effective action to obtain the EAA,

$$
\Gamma_{k,\Lambda}[\Phi] = \tilde{\Gamma}_{k,\Lambda}[\Phi] - \Delta S_{k,\Lambda}[\Phi]. \tag{161}
$$

To derive a FRGE for this EAA, we start by taking the k-derivative of (160) and inserting (158) and (159) which yields

$$
k\frac{\partial}{\partial k}\tilde{\Gamma}_{k,\Lambda}[\Phi] = \frac{1}{2}\int d^D x \int d^D y \langle \bar{\Phi}(x)\bar{\Phi}(y)\rangle k\frac{\partial}{\partial k}\mathcal{R}_{k,\Lambda}(x,y). \tag{162}
$$

Next, we introduce the functional derivative definition of the connected two-point function, associated with the $W_{k,\Lambda}[J]$ generating functional

$$
G_{k,\Lambda}(x,y) \equiv \frac{\delta^2 W_{k,\Lambda}[J]}{\delta J(x)\delta J(y)}, \tag{163}
$$

and the Hessian of $\tilde{\Gamma}_{k,\Lambda}$

$$
\tilde{\Gamma}^{(2)}_{k,\Lambda} \equiv \frac{\delta^2 \tilde{\Gamma}_{k,\Lambda}[\Phi]}{\delta \Phi(x)\delta\Phi(y)}. \tag{164}
$$

By taking two consecutive J-derivatives of (158) we obtain the two-point function $\langle \bar{\Phi}(x)\bar{\Phi}(y)\rangle = G_{k,\Lambda}(x,y) + \Phi(x)\Phi(y)$. We can substitute this into (162) to obtain

$$k\frac{\partial}{\partial k}\tilde{\Gamma}_{k,\Lambda}[\Phi] = \frac{1}{2}\,\mathrm{Tr}\Big\{k\frac{\partial}{\partial k}\mathcal{R}_{k,\Lambda}(x,y)G_{k,\Lambda}(x,y)\Big\} + \frac{1}{2}\int d^D x\Phi(x)k\frac{\partial}{\partial k}\mathcal{R}_{k,\Lambda}(-\partial^2)\Phi(x). \quad (165)$$

In terms of the EAA (161), the trace term on the RHS is actually equal to $k\frac{\partial}{\partial k}\Gamma_{k,\Lambda}[\Phi]$ and the term $\frac{1}{2}\int d^D x\Phi(x)k\frac{\partial}{\partial k}\mathcal{R}_{k,\Lambda}(-\partial^2)\Phi(x)$ cancels. This cancellation is the main motivation of defining the EAA (161). Note, G and $\tilde{\Gamma}^{(2)}$ are related by a Legendre transform, so they are mutually inverse matrices such that $G^{-1}\tilde{\Gamma}^{(2)} = \mathbb{1}$. Also, differentiating (161) twice with respect to $\Phi(x)$ and $\Phi(y)$, we find that $\Gamma^{(2)}_{k,\Lambda} = \tilde{\Gamma}^{(2)}_{k,\Lambda} - \mathcal{R}_{k,\Lambda}$. Therefore, $G_{k,\Lambda} = [\tilde{\Gamma}^{(2)}_{k,\Lambda}]^{-1} = (\Gamma^{(2)}_{k,\Lambda} + \mathcal{R}_{k,\Lambda})^{-1}$. We are left with the **Wetterich equation** [67],

$$k\frac{\partial}{\partial k}\Gamma_{k,\Lambda}[\Phi,\bar{\Phi}] = \frac{1}{2}\,\mathrm{Tr}\Big[(\Gamma^{(2)}_{k,\Lambda}[\Phi,\bar{\Phi}] + \mathcal{R}_{k,\Lambda}[\bar{\Phi}])^{-1}k\frac{\partial}{\partial k}\mathcal{R}_{k,\Lambda}[\bar{\Phi}]\Big]. \quad (166)$$

This exact non-perturbative FRGE is satisfied by the EAA, $\Gamma_{k,\Lambda}[\Phi,\bar{\Phi}]$.

In the following equation, we retain the implicit IR cutoff k which carries the scale dependence of the EAA and tells us that only the thin-shell of momentum $p^2 \approx k^2$ contributes to the FRGE. [24, 47, 53, 56]. We can rewrite

$$\boxed{k\frac{\partial}{\partial k}\Gamma_k[\Phi,\bar{\Phi}] = \frac{1}{2}\,\mathrm{Tr}\Big[(\Gamma^{(2)}_k[\Phi,\bar{\Phi}] + \mathcal{R}_k[\bar{\Phi}])^{-1}k\frac{\partial}{\partial k}\mathcal{R}_k[\bar{\Phi}]\Big]}. \quad (167)$$

The EAA satisfies the following integro-differential equation

$$\exp\Big\{-\Gamma_k[\Phi,\bar{\Phi}]\Big\} = \int \mathcal{D}\hat{\Phi}\exp\Big\{-S[\hat{\Phi}] + \int d^D x(\hat{\Phi}-\Phi)\Gamma^{(1)}_k\Big\}\exp\Big\{-\frac{1}{2}\int d^D x(\hat{\Phi}-\Phi)\mathcal{R}_k(-\partial^2)(\hat{\Phi}-\Phi)\Big\}. \quad (168)$$

C Modified BRS Ward identities

The sum $S[\bar{g}+h]+S_{\text{gf}}+S_{\text{gh}}$ is invariant under the following BRS transformations presented in [66] (ε is anti-commuting):

$$\begin{aligned} \delta_\varepsilon h_{\mu\nu} &= \varepsilon\kappa^{-2}\mathcal{L}_C g_{\mu\nu}, \\ \delta_\varepsilon \bar{g}_{\mu\nu} &= 0, \\ \delta_\varepsilon C^\mu &= \varepsilon\kappa^{-2}C^\nu\partial_\nu C^\mu, \\ \delta_\varepsilon \bar{C}_\mu &= \varepsilon(\alpha\kappa)^{-1}F_\mu. \end{aligned} \tag{169}$$

From this statement, one can derive the modified BRS-Ward identities by taking the BRS variation of the total action, including the cutoff and source actions. These identities are

$$\int d^Dx\frac{1}{\sqrt{\bar{g}}}\left[\frac{\delta\Gamma'_k}{\delta\bar{h}_{\mu\nu}}\frac{\delta\Gamma'_k}{\delta\beta^{\mu\nu}}+\frac{\delta\Gamma'_k}{\delta\xi^\mu}\frac{\delta\Gamma'_k}{\delta\tau_\mu}\right]=Y_k \tag{170}$$

where we defined $\Gamma'_k \equiv \Gamma_k - S_{\text{gf}}[\bar{h},\bar{g}]$. The same action functional is used to derive this equation and the gravitational FRGE. An exact solution of the latter necessarily solves the former. Hence, the GEAA (83) solves the modified BRS-Ward identities (170). This conditions the validity of QEG as a candidate for a QFT of gravity.

Note, we recover the standard gravitational Ward identities [156] from (170) when $Y_k = 0$, i.e. when the cutoff function $R_k[\bar{g}]$ vanishes [66]. Hence, the standard effective action, given by $\lim_{k\to 0}\Gamma_k = \Gamma$ from (77), is guaranteed to obey (170) since $R_k[\bar{g}] \to 0$ in the $k \to 0$ limit.