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Orbifolds and Orientifolds in String Theory and Gauge/String Dualities

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Abstract

We review various examples of gauge/string dualities between superconformal field theories (SCFTs) in four dimensions and Type IIB superstring theories compactified on orbifolds and orientifolds of the background $AdS_5 \times S^5$ in the near-horizon limit. In particular, we consider Type IIB theory on $N|\mathbb{Z}_k|$ D3-branes at orbifold singularities of the form $\mathbb{R}^4/\mathbb{Z}_k$ and $\mathbb{R}^6/\mathbb{Z}_k$, and review the degree of supersymmetry preserved in their SCFT dual description by considering different actions of the orbifold group on the coordinates of the transverse space. We also consider orientifold theories on $AdS_5 \times \mathbb{RP}^5$ with different amounts of discrete torsion and match them to the dual SCFT with appropriate gauge group.

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Contents

1 Introduction

The AdS/CFT correspondence is ubiquitous in modern-day theoretical physics, and its applications are many. The aim of this dissertation is to take the reader through a pedagogical review of the tools required to motivate and understand various examples of gauge/string dualities between four-dimensional superconformal field theories (SCFTs) and Type IIB superstring theories compactified on different orbifolds and orientifolds of the background $AdS_5 \times S^5$ in the near-horizon limit.

To this end, Section 2 reviews basic concepts in superstring theory including the RNS formalism and the construction of open/closed superstring spectra. The next section uses these spectra to build two different consistent superstring theories in ten dimensions. Of particular interest to this dissertation is Type IIB theory, which is crucial to understand the different examples of gauge/string dualities to be presented further on in the text. Section 4 introduces the concept of compactification and the extra structure this adds to the field theory being examined.

We then introduce the ideas of orbifolding and orientifolding within the context of string theory, to later construct Type I theory as a $\mathbb{Z}_2$ orientifold projection of Type IIB theory. The next section is devoted to the study of D-branes. In particular, we review the dual nature of D-branes both as charged BPS states, and as solutions to the supergravity equations of motion. This is at the heart of the AdS/CFT correspondence. Section 5 concludes by motivating how non-Abelian gauge theories emerge as worldvolume theories on different D-brane configurations.

Section 8 introduces the general idea of dualities within superstring theory and provides some explicit examples. The next sections (9 and 10) build up to the statement of correspondence between $\mathcal{N} = 4$ super Yang-Mills and Type IIB string theory on $AdS_5 \times S^5$. This is verified by considering Maldacena's decoupling argument and by matching the global symmetries of both theories. Further arguments are presented heuristically for its validity. The end of Section 11 motivates the existence of a larger class of dualities between four-dimensional SCFTs and Type IIB string theories on different AdS_5 backgrounds by considering the large N limit of gauge theories.

Finally, we present some examples of dualities in this class, corresponding to different orbifolds and orientifolds of the AdS_5 background, and find that different degrees of supersymmetry are preserved by the dual SCFT. In particular, we construct $\mathcal{N} = 2$, $\mathcal{N} = 1$ and $\mathcal{N} = 0$ SCFTs as worldvolume theories on a stack of D3-branes at different orbifold singularities in the near-horizon limit. Furthermore, the $AdS_5 \times \mathbb{RP}^5$ orientifold of Type IIB theory is also explored and matched to its dual SCFT. Possible extensions to the work presented here are briefly discussed in Section 13.

2 A Review of Superstrings

The theory of bosonic strings proves to be a good toy model to grasp the power and beauty of string theory. However, while a lot can be learnt from it, one must note that it is ultimately flawed. This is mainly due to two reasons: Firstly, it only describes bosonic degrees of freedom, and it is clear that to describe our Universe we need fermions. Secondly, it contains a tachyonic ground state, which implies an unstable vacuum and this, in turn, hints at the fact that the theory itself needs to be adapted [1]. We thus introduce the idea of superstrings.

2.1 RNS Formalism

One can equip the worldsheet (WS) of the bosonic theory with a fermionic sector. This can be done via the RNS formalism, which we follow here, where fermions are introduced via supersymmetry (SUSY) as spacetime vectors on the WS.

Following [2], we can write the full RNS action in conformal gauge as

$$S_{RNS} = -\frac{1}{4\pi}\int d^2\xi \ \frac{1}{\alpha'}\partial_\alpha X^\mu \partial^\alpha X_\mu + i\bar{\psi}^\mu_A \gamma^\alpha_{AB}\partial_\alpha \psi_{\mu B} \tag{1}$$

which corresponds to a two-dimensional free field theory, $\xi^\alpha = (\tau, \sigma)$. The indices $\alpha = 0, 1$ correspond to vector indices on the two-dimensional WS while latin indices are spinor indices, and $\mu = 0, 1 \ldots D-1$ are spacetime coordinates for a string propagating in D flat spacetime dimensions. The first term is the usual Polyakov action in the bosonic theory, and $X^\mu(\xi)$ correspond to the embedding of bosonic fields on the WS. We further identify ψ^μ_A as Majorana-Weyl spinors[1] which are Lorentz vectors on the WS. $\bar{\psi} = \psi^\dagger\gamma^0$ are the dual spinors and γ^α are the usual gamma matrices obeying the Clifford algebra condition $\{\gamma^\alpha, \gamma^\beta\}_{AB} = 2\eta^{\alpha\beta}\mathbb{1}_{AB}$. α' is related to the string length, l_s, as $l_s = \sqrt{\alpha'}$. Note further that the relative factor of $\frac{1}{\alpha'}$ between both terms is due to the difference in dimensions between bosonic fields, $[X] = 1$, and fermionic fields, $[\psi] = \frac{1}{2}$.

The *RNS* action (1) is invariant under the following pair of transformations:

$$\sqrt{\frac{2}{\alpha'}}\delta X^\mu = i\bar{\epsilon}\psi^\mu \tag{2a}$$

$$\delta\psi^\mu = \frac{1}{\sqrt{2\alpha'}}\epsilon\gamma^\alpha\partial_\alpha X^\mu \tag{2b}$$

[1]These are real spinors with definite chirality (i.e. they have a definite eigenvalue under $\gamma = \gamma^0\gamma^1$).

where ϵ is a Grassman-odd parameter. These transformations relate bosonic and fermionic degrees of freedom, which is characteristic of a SUSY. The conditions on ϵ required for invariance of the action take the form $\gamma^\beta \gamma_\alpha \partial_\beta \epsilon = 0$. In light-cone coordinates, where $\xi^\pm = \tau \pm \sigma$, the invariance conditions become $\partial_\pm \epsilon_\pm = 0$ and we clearly see that the above symmetry of the action is chiral. One must note that these conditions only hold *on shell*, assuming the equations of motion for the fields are satisfied.

We now detail the mode expansions and boundary conditions (BCs) for each sector of the worldsheet. In the case of the bosonic sector, we follow the standard literature (e.g. [2]) and the results are as usual (we can have Neumann/Dirichlet BC combinations depending on the string nature). For the fermionic sector, we rewrite the fermionic part of S_{RNS} in light-cone gauge as

$$S_F = \frac{i}{2\pi} \int d^2\xi \; (\psi_+ \partial_- \psi_+ + \psi_- \partial_+ \psi_-) \tag{3}$$

varying this one obtains

$$\delta S_F = \int d\tau \; [\psi_+ \delta\psi_+ - \psi_- \delta\psi_-]_{\sigma=0}^{\sigma=l} \tag{4}$$

plus terms which vanish using the equations of motion. We then distinguish between two cases. The first one is referred to as *closed* sector and corresponds to both terms in (4) cancelling. Imposing Poincaré invariant BCs we have, compactly,

$$\psi^\mu_\pm(\sigma + l_s) = e^{i2\pi\phi_\pm} \psi^\mu_\pm(\sigma) \tag{5}$$

where $\phi = 0$ corresponds to the Ramond (R) sector (spinor fields are periodic) and $\phi = \frac{1}{2}$ corresponds to the Neveu-Schwarz (NS) sector, where the spinor fields are antiperiodic. This distinction is possible since the spinor fields enter quadratically in the constraint (4). The mode expansions for each sector thus take the form [1]:

- R-sector: periodic BCs with $n \in \mathbb{Z}$

$$\psi^\mu_\pm(\tau, \sigma) = \sum_{n\in\mathbb{Z}} \sqrt{\frac{2\pi}{l_s}} b^\mu_n e^{-i\frac{2\pi}{l_s} n(\tau \pm \sigma)} \tag{6}$$

- NS-sector: antiperiodic BCs with $r + \frac{1}{2} \in \mathbb{Z}$

$$\psi^\mu_\pm(\tau, \sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \sqrt{\frac{2\pi}{l_s}} b^\mu_r e^{-i\frac{2\pi}{l_s} r(\tau \pm \sigma)} \tag{7}$$

We note the Fourier coefficients b^μ_r vary between periodic/antiperiodic cases. Thus, within the closed sector of the fermionic part of the WS, we can construct four distinct sectors which we denote as:

$$\begin{aligned} (\phi_+, \phi_-) &= (0,0) \leftrightarrow R\text{–}R \\ (\phi_+, \phi_-) &= (\tfrac{1}{2},0) \leftrightarrow NS\text{–}R \\ (\phi_+, \phi_-) &= (\tfrac{1}{2},\tfrac{1}{2}) \leftrightarrow NS\text{–}NS \\ (\phi_+, \phi_-) &= (0,\tfrac{1}{2}) \leftrightarrow R\text{–}NS \end{aligned} \tag{8}$$

For the *open* case, we now require that the terms in (4) vanish individually. This means that we have

$$\psi^\mu_+(\sigma)\Big|^{\sigma=l_s}_{\sigma=0} = \pm\psi^\mu_-(\sigma)\Big|^{\sigma=l_s}_{\sigma=0} \tag{9}$$

as the relation between spinor components. The mode expansions in this case follow closely those in (6) and (7) for Neumann-Neumann (NN) BCs, except that we replace the factors of 2π by π only. Note for the open case we can impose also Dirichlet-Dirichlet (DD) BCs since open strings are allowed to end on a D-brane. The mode expansions now are the same except $X_- \to -X_-$ and thus by WS SUSY we also must have $\psi_- \to -\psi_-$. Here, we also distinguish between 4 different sectors corresponding to $\sigma = 0, l_s$ and the periodicity of the fields ($\pm$).

Lastly, we can group the sectors according to the spacetime boson-fermion parity of their respective states. Thus, we have the $R-R$ and $NS-NS$ sectors as bosonic (even parity) and then the $NS-R$ and $R-NS$ sectors as fermionic (odd parity).

The coefficients of the bosonic modes X^μ, obey the usual commutation relations $[\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0}$ [2]. The fermions obey the equal-time anti-commutation relations

$$\{\psi^\mu_\pm(\tau,\sigma), \psi^\nu_\pm(\tau,\sigma')\} = 2\pi\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{10a}$$

$$\{\psi^\mu_+(\tau,\sigma), \psi^\nu_-(\tau,\sigma')\} = 0 \tag{10b}$$

provided that $\{b^\mu_m, b^\nu_n\} = \{\tilde{b}^\mu_m, \tilde{b}^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0}$ holds for all components. Note the $\tilde{b}$ coefficients correspond to the ψ_- solutions while b are for ψ_+.

We construct states by acting the corresponding oscillators on the appropriate Fock vacuum. For the NS sector, the vacuum is constructed such that

$$\alpha^\mu_m |0\rangle_{NS} = 0 = b^\mu_r |0\rangle_{NS} \ \forall m, r > 0, m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2} \tag{11}$$

and is annihilated by the right-mover oscillators in the open case[2]. The key is to

[2]For the closed superstring, $|0\rangle_{NS}$ is annihilated by $\tilde{\alpha}^\mu_m$ and $\tilde{b}^\mu_r$ in an analogous manner.

note that $|0\rangle_{NS}$ is unique and also a spacetime scalar, thus $\tilde{\alpha}^{\mu}_{-|m|}$ and $\tilde{b}^{\mu}_{-|r|}$ create bosonic Fock states by acting on the scalar vacuum.

For the R-sector, recall that we only have integer mode expansions of the fields, thus the Fock vacuum here is defined such that

$$\alpha^{\mu}_{m}\,|0\rangle_{R} = 0 = b^{\mu}_{r}\,|0\rangle_{R}\ \forall m, r > 0 \in \mathbb{Z} \tag{12}$$

However, $|0\rangle_{R}$ is degenerate, since $b^{\mu}_{0}\,|0\rangle_{R} \neq 0$ but it is annihilated by any of the oscillators in (12). From the condition below equation (10b), the 0-modes obey $\{b^{\mu}_{0}, b^{\nu}_{0}\} = \eta^{\mu\nu}$. Confronting this with the Clifford algebra condition for Γ^{μ}, we can thus make the identification that

$$b^{\mu}_{0} = \frac{1}{2}\Gamma^{\mu} \tag{13}$$

i.e. that the Fourier 0-modes of the fermion mode expansion are d-dim gamma matrices. Thus, it is clear that $|0\rangle_{R}$ is a d-dim spinor, as it furnishes a d-dim representation of the Clifford algebra (for more details see Ch.6 of [3]).

In d-dim a Dirac spinor has $2^{\frac{d}{2}}$ components. Further, for $d = 2 + 2k$, $k \in \mathbb{Z}$, we can decompose the spinor representation as [4]

$$[2^{\frac{d}{2}}]_{Dirac} \to [2^{\frac{d}{2}-2}]_{Weyl} \oplus [2^{\frac{d}{2}-1}]_{Weyl} \tag{14}$$

where the Weyl spinors have opposite chirality. This result will be useful later. We conclude that the R-sector states are fermionic and are obtained by acting $\alpha^{\mu}_{-|m|}$ and $b^{\mu}_{-|r|}$ respectively on $|0\rangle_{R}$. Due to the Majorana reality condition, these states have $2^{\frac{d}{2}}$ real components.

We now impose the constraints from the super-Virasoro algebra which are of the form:

$$T_{\pm\pm} = 0 = J_{\pm} \tag{15}$$

where the extra constraint with respect to the bosonic theory is due to the presence of a supercurrent $J_{\pm}$, whose Fourier modes (G_r) generate the odd part of the super-algebra. The Fourier modes of the stress-energy tensor (L_m) generate the remaining (even) part (see Appendix 14.1 for more details).

2.2 Open Superstring Spectrum in Light-Cone Quantisation

We assume that we have NN BCs along all dimensions. Light-cone quantisation (LCQ) accounts for ghost contributions by exploiting the residual superconformal symmetry of S_{RNS} to solve for the super-Virasoro constraints explicitly. Following the constraints (15), we impose the physical state conditions for the NS and R sectors. For the NS-sector the mass-shell condition becomes:

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^i b_r^i - a_{NS} \tag{16}$$

Counting the dynamical degrees of freedom in LCQ, the normal-ordering constant is $a_{NS} = (D-2)(\frac{1}{24} + \frac{1}{48})$ where the $\frac{1}{24}$ comes from a periodic boson and the $\frac{1}{48}$ from an antiperiodic fermion. For a Poincaré-invariant theory, states must form irreducible unitary representations under a subgroup of $SO(1, D-1)$ [5]. The first excited state (FES) forms a representation of $SO(D-2)$, the little group of massless states, hence $a_{NS} = \frac{1}{2}$ is required, and thus by comparing both results we see that $D_{crit} = 10$ for the open superstring. Thus, the FES forms the $\mathbf{8_v}$ of $SO(8)$, while higher excited states are massive. The ground state (GS) is tachyonic.

For the R-sector, we had periodic BCs with $n \in \mathbb{Z}$, hence the normal-ordering constant $a_R = (D-2)(\frac{1}{24} - \frac{1}{24}) = 0$ vanishes due to periodicity of the fields. The mass-shell condition here is the same as (16) but with no normal-ordering constant and with both $n, r \in \mathbb{Z}$. The GS is a spinor in 10-dim and therefore has 32 real components. For D even, we apply (14) to decompose $|0\rangle_R$ as $\mathbf{32} = \mathbf{16} \oplus \mathbf{16'}$. Furthermore, light-cone gauge induces the decomposition $SO(1,9) \to SO(1,1) \otimes SO(8)$, where $SO(1,1)$ corresponds to $x^{\pm}$ and the $SO(8)$ to the dynamical x^i. Hence, under this, the Weyl spinors further decompose as

$$\mathbf{16} \to [\frac{1}{2}, \mathbf{8}] \oplus [-\frac{1}{2}, \mathbf{8'}] \tag{17a}$$

$$\mathbf{16'} \to [\frac{1}{2}, \mathbf{8'}] \oplus [-\frac{1}{2}, \mathbf{8}] \tag{17b}$$

In the end, the GS reduces to $|0\rangle_R = [\frac{1}{2}, \mathbf{8}] \oplus [\frac{1}{2}, \mathbf{8'}]$, where only the spin-$\frac{1}{2}$ parts are kept since $|0\rangle_R \neq 0$ and this must satisfy the Dirac equation. It is worth noting that since $a_R = 0$, the GS is massless so $[\frac{1}{2}, \mathbf{8}]$ and $[\frac{1}{2}, \mathbf{8'}]$ form the $\mathbf{8_s}$ and $\mathbf{8_c}$ spinor representations of $SO(8)$ respectively, which have opposite chirality.

2.3 Closed Superstring Spectrum in LCQ

Accounting for G-parity[3] and the fact that up to level-matching the right $(\tau + \sigma)$ and left $(\tau - \sigma)$ movers are independent, we now have 10 independent sectors that

[3] The G-parity operator is $G_{NS} = (-1)^{F+1}$ and $G_R = \Gamma(-1)^F$ where $\Gamma = \Gamma_0\Gamma_1 ... \Gamma_9$ and F refers to the fermion number in each of the NS and R sectors respectively.

can produce a consistent 10-dimensional theory. These take the form (R_+, R_-), (NS_+, R_-)...,etc, and we note that the NS_- sector cannot pair with any of the others due to level-matching. The mass-shell condition is now

$$\alpha' M^2 = 4(N - a) = 4(\tilde{N} - a) \tag{18}$$

We examine the massless FES of the various allowed combinations of sectors up to interchange of left/right-movers. Compactly these are given as:

Sector	$SO(8)$ representation	Nature
(NS_+, NS_+)	$\mathbf{8_v} \otimes \mathbf{8_v}$	Bosonic
(R_+, R_+)	$\mathbf{8}_s \otimes \mathbf{8}_s$	Bosonic
(R_+, R_-)	$\mathbf{8}_s \otimes \mathbf{8}_c$	Bosonic
(R_-, R_-)	$\mathbf{8}_c \otimes \mathbf{8}_c$	Bosonic
(NS_+, R_+)	$\mathbf{8}_v \otimes \mathbf{8}_s$	Fermionic
(NS_+, R_-)	$\mathbf{8}_v \otimes \mathbf{8}_c$	Fermionic

The (NS_-, NS_-)-sector contains a tachyonic GS with $\alpha' M^2 = -2$, where we used that $a_{NS} = \frac{1}{2}$. We now proceed to decompose these states into irreducible representations of $SO(8)$. For the (NS_+, NS_+) sector, the decomposition is as for the purely bosonic theory

$$\mathbf{8}_v \otimes \mathbf{8}_v = [0] \oplus [2] \oplus (2) \tag{19}$$

i.e. into a trace part, an antisymmetric 2-form and a traceless symmetric rank-2 tensor which have the interpretation of the dilaton scalar ϕ, the Kalb-Ramond field $B_{\mu\nu}$, and the graviton $G_{\mu\nu}$ respectively.

The $(R_\pm, R_\pm)$ sectors involve spinor bilinears, which are decomposed using the Fierz decomposition (see Appendix 14.2). Then, we have

Sector	$SO(8)$ representation	$SO(8)$ irrep
(R_+, R_+)	$\mathbf{8_s} \otimes \mathbf{8_s}$	$[0]\oplus[2] \oplus [4]_+$
(R_+, R_-)	$\mathbf{8}_s \otimes \mathbf{8}_c$	$[1]\oplus[3]$
(R_-, R_-)	$\mathbf{8}_c \otimes \mathbf{8}_c$	$[0]\oplus[2] \oplus [4]_-$

where the $[n]$ denote n-forms in $D = 8$-dim, and $[n]_\pm$ refer to the self (antiself)-dual parts of the form with respect to Hodge duality. These are spacetime bosons.

Lastly, the mixed NS/R sectors have states which contain spinor-vector bilinears and hence are spacetime fermions. These decompose under a Fierz-type decomposition as

Sector	$SO(8)$ representation	$SO(8)$ irrep
(NS_+, R_+)	$\mathbf{8_v} \otimes \mathbf{8_s}$	[8]'⊕[56]
(NS_+, R_-)	$\mathbf{8}_v \otimes \mathbf{8}_c$	[8]⊕[56]$'$

In the (NS_+, R_+) sector, the $[8]'$ is identified with a spin-$\frac{1}{2}$ dilatino, λ_a, and the $[56]$ is a spin-$\frac{3}{2}$ gravitino, ψ^i_a, of opposite chirality. The (NS_+, R_-) sector has the same field content but with opposite chirality with respect to the previous sector.

3 Type IIA/IIB Theory

We now have included fermions in the picture via the RNS formalism and we seek to construct a consistent theory of closed superstrings by combining the different allowed sectors available (a priori one would have 2^{10} possible theories). Furthermore, we note that we still have a tachyonic GS, which we must GSO-project out of the spectrum.

To understand how to refine the sectors which are to be included in a consistent theory, we note that (3) is conformally invariant if the RNS fields $\psi^\mu_\pm$ are taken to be primary fields (see Appendix 14.3) of conformal weight $h(\psi) = \bar{h}(\tilde{\psi}) = \frac{1}{2}$ and $\bar{h}(\psi) = h(\tilde{\psi}) = 0$. Indeed, the full *RNS* action is conformally invariant, so S_{RNS} defines an $\mathcal{N} = 1$ super-conformal field theory (SCFT), to which we can impose the following requirements [1; 2]:

- Vertex operators must be *mutually local* in pairs. Vertex operators with branch-cuts in their OPEs are thus not mutually local.
- *Modular invariance* of the one-loop amplitudes implies that we must have at least one R_+ and one R_- sector present in the theory.
- No monodromies[4] can be present (i.e. the OPEs need to be *singled-valued*).

We consider a new way of characterizing the allowed sectors of the theory instead of using $NS_\pm$ and $R_\pm$. The sectors will be labelled by $(\alpha, F, \tilde{\alpha}, \tilde{F})$ where $\alpha = 1 - 2\phi$ and $\phi = 0, \frac{1}{2}$ as usual for the R and NS sectors respectively. F denotes the fermion number and the tilded quantities refer to the left-movers. Hence, the overall phase a vertex operator acquires when it encircles another is given by

$$\exp[i\pi(F_1\alpha_2 - F_2\alpha_1 - \tilde{F}_1\tilde{\alpha}_2 + \tilde{F}_1\tilde{\alpha}_2)] \tag{20}$$

which must be unity so that the amplitude of both vertex operators can be defined. We consider the effect of each the requirements imposed on the SCFT on (20).

- Mutual locality: $(F_1\alpha_2 - F_2\alpha_1 - \tilde{F}_1\tilde{\alpha}_2 + \tilde{F}_1\tilde{\alpha}_2) \in 2\mathbb{Z}$
- OPE closure: Since α and F are conserved mod-2, if $(\alpha_1, F_1, \tilde{\alpha}_1, F_1)$ and $(\alpha_2, F_2, \tilde{\alpha}_2, F_2)$ are in the spectrum, then $(\alpha_1 + \alpha_2, F_1 + F_2, \tilde{\alpha}_1 + \tilde{\alpha}_2, \tilde{F}_1 + \tilde{F}_2)$ must be too.

[4]Monodromy refers to the behavior of operators when they encircle each other near a branch-cut.

Assuming the presence of at least one $(\alpha, \tilde{\alpha}) = (1, 0)$ sector (i.e. an (R, NS) sector)[5], we find that a closed superstring theory must contain only pairs of the following sector combinations:

$$(NS_+, R_+) \quad (NS_+, R_-) \quad (R_+, NS_+) \quad (R_-, NS_+) \quad (NS_+, NS_+) \quad (R_\pm, R_\pm) \tag{21}$$

For the case of Type IIB theory, the massless spectrum is given by

Sector	$SO(8)$ irrep	Particle Interpretation
(NS_+, NS_+)	$[0] \oplus [2] \oplus (2)$	$\phi \oplus B_{[\mu\nu]} \oplus G_{(\mu\nu)}$
(R_+, R_+)	$[0] \oplus [2] \oplus [4]_+$	$C^{(0)} \oplus C^{(2)}_{[\mu_1\mu_2]} \oplus C^{(4)+}_{[\mu_1 \dots \mu_4]}$
(R_+, NS_+)	$[8]' \oplus [56]$	$\lambda^a \oplus \psi^i_a$
(NS_+, R_+)	$[56] \oplus [8]'$	$\psi^i_a \oplus \lambda^a$

where it is clear that this is a chiral theory since the left/right-movers have the same chirality in all sectors. The so-called Type IIB' theory, which is also chiral, can be defined as above, but by interchanging $R_+ \to R_-$ and adapting the field content.

Now, for the case of Type IIA theory, the massless spectrum is given by

Sector	$SO(8)$ irrep	Particle Interpretation
(NS_+, NS_+)	$[0] \oplus [2] \oplus (2)$	$\phi \oplus B_{[\mu\nu]} \oplus G_{(\mu\nu)}$
(R_+, R_-)	$[1] \oplus [3]$	$C^{(1)}_{[\mu_1]} \oplus C^{(3)+}_{[\mu_1 \dots \mu_3]}$
(R_+, NS_+)	$[8]' \oplus [56]$	$\lambda^a \oplus \psi^i_a$
(NS_+, R_-)	$[8] \oplus [56]'$	$\tilde{\lambda}^a \oplus \tilde{\psi}^i_a$

which defines a non-chiral theory since it is clear that we have 2 gravitinos and 2 dilatinos of opposite chirality. Similarly, Type IIA' theory can be defined by considering the interchange $R_\pm \to R_\mp$. This corresponds to a spacetime reflection about a single axis of the Type IIA theory.

We now consider some remarks regarding the superstring theories we have constructed. The absence of an (NS_-, NS_-) sector means that we do not have a tachyonic GS. This is an automatic consequence of the consistency conditions obtained by projecting the spectrum of the theory into eigenspaces of $e^{i\pi F}$ and of $e^{i\pi \tilde{F}}$. This is known as the GSO projection. For Type IIA, we have taken opposite GSO projections in the $NS - R$ and $R - NS$ sectors, resulting in a non-chiral spectrum (invariant under $[8] \leftrightarrow [8]'$ and $[56] \leftrightarrow [56]'$). For Type IIB theory, we have the same GSO projection in each sector, so the spectrum is chiral. On the worldsheet, this symmetry corresponds to the product of spacetime parity and WS parity.

We note also that Type IIA/IIB theories contain an equal number of bosonic and fermionic degrees of freedom. This is a necessary condition for spacetime SUSY [3].

[5]Dropping this assumption leads to Type 0 theories which contain no fermionic field content.

Each theory contains two massless spin-$\frac{3}{2}$ gravitinos, which are the superpartners of the gravitons in the (NS_+, NS_+) sector, implying the existence of *local* SUSY. This will be relevant later, but for now provides motivation for the fact that the low-energy limit of a Type II superstring theory can describe a model of supergravity (SUGRA).

Furthermore, the presence of two independent gravitinos implies the existence of two distinct SUSY algebras (2 conserved supercurrents) in the theory, hence Type II theories display $\mathcal{N} = 2$ SUSY in 10-dimensions (i.e. maximal SUSY in $d = 10$).

Ultimately, what we have argued here is that the requirements of vacuum stability and consistency of the SCFT are sufficient to show that Type IIA/IIB are consistent, closed, oriented superstring theories with local SUSY in ten dimensions.

4 Compactification

We must introduce the idea of compactification to be able to understand our sought after gauge-string dualities. This concept is also central in the emergence of orbifolds and orientifolds within string theory.

The idea here is to split our 10-dim manifold into a product of a $(10-n)$-dim space and an n-dim internal space as $\mathcal{M}^{10} \to \tilde{\mathcal{M}}^{10-n} \times \Omega^n$. We take the limit such that the size of the internal space is very small, and require Ω to satisfy the equations of motion coming out from the effective field theory of the string theory to which we apply the compactification to. This requires a modification of the WS action (free theory) to an intereacting non-linear σ-model [6].

4.1 Kaluza-Klein Compactification in Field Theory

We begin by considering a field theory in $(d+1)$ flat spacetime dimensions. Introducing the compactification ansatz:

$$\mathbb{R}^{1,d} \to \mathbb{R}^{1,d-1} \times S^1 \tag{22}$$

for S^1 a circle of radius R is equivalent to identifying $x^{d+1} \sim x^{d+1} + 2\pi R$ for the $(d+1)^{th}$ dimension only. Imposing diffeomorphism invariance, we obtain the following 'new' features with respect to a usual field theory in flat spacetime in the same number of dimensions [7]:

- Kaluza-Klein (KK) tower of massive states in d dimensions
- Extra $U(1)$ symmetry in d dimensions
- Modulus fields (massless scalars) emerge in d dimensions

We examine these in turn. Firstly, let $M, N = 0, 1...d, d+1$ and $\mu, \nu = 0, 1..., d$. Then, for a massless scalar field ϕ, we have that $\partial_\mu \partial^\mu \phi(x^\mu) = 0$ must hold. In order to maintain periodicity along x^{d+1} we take the ansatz

$$\phi(x^\mu) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) e^{\frac{in}{R} x^{d+1}} \tag{23}$$

Plugging this into the K-G equation of motion (where $\partial_M = \partial_\mu + \partial_{d+1}$) we obtain

$$\partial_\mu \partial^\mu \phi_n(x^\mu) = \frac{n^2}{R^2} \phi_n(x^\mu) \quad \forall n \tag{24}$$

which from the perspective of the d-dimensional theory, we see that $\phi_n(x^\mu)$ appears as a scalar of mass squared $m_n^2 = \frac{n^2}{R^2}$. These are precisely the states which form the massive KK tower in d-dim (note that the $n = 0$ state is massless and independent on x^d).

In the limit $R \to 0$, $m_1^2 \to \infty$, so the KK tower of massive states collapses in the low-energy spectrum. For $E \ll \frac{1}{R}$, the compactified theory looks d-dimensional.

The emergence of an extra $U(1)$ gauge potential arises from the components $G_{\mu d}^{d+1}$ of the $(d+1)$-dim metric. From (22), it is natural to split the metric of the uncompactified theory as

$$ds^2 = G_{MN}^{(d+1)} dx^M dx^N = G_{\mu\nu}^{(d)} dx^\mu dx^\nu + G_{dd}(dx^d + A_\mu dx^\mu)^2 \tag{25}$$

and hence one can reparametrize the $G_{\mu d}^{d+1}$ components of the metric as $G_{\mu d}^{d+1} = 2G_{dd}A_\mu$. A priori, A_μ is introduced for convenience of the reparametrization, but we will motivate its gauge field nature [8].

Considering the 0-modes of the field expansion only for simplicity, so that $G_{\mu\nu}$, A_μ and G_{dd} depend only on x^μ (x^{d+1} dependence decouples), the subgroup of the $(d+1)$-dimensional diffeomorphisms compatible with (22) has the following action on the compactified product manifold:

- Diffeomorphism invariance on $\mathbb{R}^{1,d-1}$: $x^\mu \to x^{\mu\prime} = x^\mu$ (is invariant)
- Diffeomorphism invariance along S^1: $x^d \to x^d + \lambda(x^\mu)$ which in turn implies that $A_\mu \to A'_\mu = A_\mu - \partial_\mu \lambda(x^\mu)$ is gauge invariant. Thus, A_μ can be interpreted as a gauge potential in d-dimensions.

It is therefore clear that the extra $U(1)$ gauge symmetry emerges from $(d+1)$-dimensional diffeomorphism invariance, so that under (22) we have that

$$GL(d+1, \mathbb{R}) \to GL(d, \mathbb{R}) \times U(1) \tag{26}$$

for the diffeomorphism symmetry group under the compactification ansatz.

Lastly, note that from the perspective of the d-dim theory, G_{dd} is a scalar. The vacuum expectation value (VEV) of G_{dd} gives the volume of the internal space S^1 as [6]

$$\mathrm{Vol}(S^1) = \int_0^{2\pi R} dx^d \sqrt{G_{dd}} = 2\pi R\sqrt{G_{dd}} \tag{27}$$

We say G_{dd} is a modulus field since it is a flat scalar whose VEV gives geometrical properties of the internal (compactified) space. More formally, one can say that this scalar (dilaton) parametrises the geometry of the fiber S^1.

4.2 KK Compactification of the Closed Bosonic String

We choose to work in units where $\frac{2\pi}{l_s} = 1$. The theory of bosonic strings lives in $(d+1) = 26$ dimensions. The mode expansion for the bosonic fields is given in [1] as

$$x^\mu(\tau,\sigma) = \frac{1}{2}(x^\mu + \tilde{x}^\mu) + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + N + \tilde{N} \tag{28}$$

where we note that $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$, $\tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\tilde{p}^\mu$ and $\mu = 0, 1...(d+1)$. N and $\tilde{N}$ refer to the number operators of the right/left-movers respectively. Under a shift of the coordinates such that $\sigma \to \sigma + 2\pi$, then, we have that

$$x^\mu(\tau,\sigma) \to x^\mu(\tau,\sigma) + 2\pi\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu) \tag{29}$$

and hence, imposing periodicity of the fields leads to $\alpha_0^\mu = \tilde{\alpha}_0^\mu \leftrightarrow p^\mu = \tilde{p}^\mu$. If we consider a KK compactification of x^{25} along S^1, such that as before we identify $x^{25} \sim x^{25} + 2\pi R$, then, this implies two new features on the field theory:

- Momentum quantisation along the x^{25} direction, that is:

  $$p^{25} = \frac{n}{R} \leftrightarrow (\alpha_0^{25} + \tilde{\alpha}_0^{25}) = 2\sqrt{\frac{\alpha'}{2}}\frac{n}{R} \quad \forall n \in \mathbb{Z} \tag{30}$$

  which can be understood from the fact that we require the wavefunctions of the bosonic fields ($\sim e^{ip_{25}x^{25}}$) along this direction to be single-valued. In other words, they must have the same periodicity as the compactified coordinate.

- Another possible configuration is that of *winding* strings. These loop ω times around the internal space S^1. This is better illustrated in Figure **1**, where the non-winding string ($\omega = 0$) obeys $x^{25}(\tau, \sigma + 2\pi) = x^{25}(\tau, \sigma)$. The strings with $\omega = \pm 1$ wind once around the compact the dimension and the sign is associated with the directionality of the winding as shown. These obey $x^{25}(\tau, \sigma + 2\pi) = x^{25}(\tau, \sigma) + 2\pi R\omega$, and it is useful to think of the covering space as the quotient $S^1 \simeq \mathbb{R}/2\pi R\omega$.

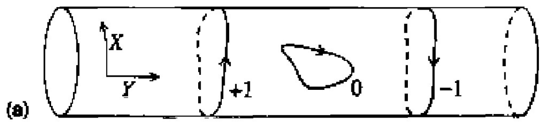

Figure 1: Representation of the compactified space ($\mathbb{R}^{1,24} \times S^1$) for the theory of bosonic strings in $(d+1) = 26$ dimensions. The Y direction corresponds to $\mathbb{R}^{1,24}$ and X to S^1. Adapted from [9].

For the winding case, the periodicity constraint from (29) translates to $\alpha_0^{25} - \tilde{\alpha}_0^{25} = \sqrt{\frac{2}{\alpha'}}\omega R$, and one can express the left/right-moving momenta independently (in the compact dimension) as

$$\alpha_0^{25} = (\frac{m}{R} + \frac{\omega R}{\alpha'})\sqrt{\frac{\alpha'}{2}} = p_L^{25}\sqrt{\frac{\alpha'}{2}} \tag{31a}$$

$$\tilde{\alpha}_0^{25} = (\frac{m}{R} - \frac{\omega R}{\alpha'})\sqrt{\frac{\alpha'}{2}} = p_R^{25}\sqrt{\frac{\alpha'}{2}} \tag{31b}$$

which still respects the momentum quantisation condition in (30). The mass-shell condition following from the Virasoro constraints gives the effective mass of the compactified theory as [6]

$$m^2 = -p^\mu p_\mu = \frac{2}{\alpha'}(N + \tilde{N} - 2) + \frac{n^2}{R^2} + \frac{\omega^2 R^2}{\alpha'^2} \tag{32}$$

where aside from the usual first term, we have a contribution from the quantised momentum and a 'wrapping' energy term. Furthermore, for a closed string theory we have the additional level-matching condition which in this case reads as

$$N - \tilde{N} = m\omega \tag{33}$$

In the winding case therefore, we see the following structure emerging:

- For $\omega = m = 0$ and in the limit $R \to \infty$, we recover the states of the uncompactified closed bosonic theory.
- For $n \neq 0$ and $\omega = 0$, following (32), we note the presence of a KK tower of states with mass $m_n = \frac{n}{R}$ in the point-particle theory along S^1.

- For $\omega \neq 0$, winding states with *extra* mass of $m^2_\omega = \frac{\omega^2 R^2}{\alpha'^2}$ arise.

Similarly as before, in the $R \to 0$ limit the KK tower vanishes from the low-energy spectrum and the winding states become light (since $m^2_\omega \sim R^2$).

We now examine the massless spectrum of the compactified theory for generic R. This corresponds to $n = \omega = 0$ and $N = \tilde{N} = 1$. The possible states are thus[6]:

State	Particle Interpretation
$\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}\ket{0;k}$	$G_{(\mu\nu)} \oplus B_{[\mu\nu]} \oplus \phi \quad$ in $\mathbb{R}^{1,24}$
$\ket{\mathcal{V}^\mu_1} = (\alpha^\mu_{-1}\tilde{\alpha}^{25}_{-1} + \alpha^{25}_{-1}\tilde{\alpha}^\mu_{-1})\ket{0;k}$	Vector of $\mathbb{R}^{1,24}$/ $U(1)$ gauge potential from $G^{(d+1)}_{\mu d} \sim A_\mu$
$\ket{\mathcal{V}^\mu_2} = (\alpha^\mu_{-1}\tilde{\alpha}^{25}_{-1} - \alpha^{25}_{-1}\tilde{\alpha}^\mu_{-1})\ket{0;k}$	Vector of $\mathbb{R}^{1,24}$ (antisymmetric combination) $\sim \tilde{A}_\mu$
$\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}\ket{0;k}$	Scalar of $\mathbb{R}^{1,24}$ corresponding to $\mathrm{G}^{(d+1)}_{dd}$

The states containing combinations of spacetime/internal oscillators have a general $U(1)_L \times U(1)_R$ gauge symmetry corresponding to the left/right isometries of the internal space S^1 where $U(1)_L$ is associated to A_μ and $U(1)_R$ to $\tilde{A}_\mu$. Furthermore, the massless scalar is a compactified degree of freedom of the uncompactified metric in $(d+1) = 26$ dimensions. This scalar has VEV=R [7] and thus is also a modulus field.

We now examine how the above gauge symmetry is enhanced at special radii, namely at $R = \sqrt{\alpha'}$, which is a purely stringy effect. The L/R momenta along the compactified direction take the form $p^{25}_{L/R} = \frac{1}{\sqrt{\alpha'}}(n \pm \omega)$ and, thus, the massless condition now requires that

$$(n+\omega)^2 + 4N = (n-\omega)^2 + 4\tilde{N} = 4 \tag{34}$$

together with level-matching (33). This allows for the following new massless states:

- For $n = \omega = \pm 1$, $N = 0$ and $\tilde{N} = 1 \leftrightarrow 2$ new vectors $\ket{V^\mu_a} = \alpha^\mu_{-1}\ket{\pm 1, \pm 1}$ and two scalars $\ket{\phi_a} = \alpha^d_{-1}\ket{\pm 1, \pm 1}$ for $a = 1, 2$ and $\mu = 2, ...(d-1)$.
- For $n = -\omega = \pm 1$, $N = 1$ and $\tilde{N} = 0 \leftrightarrow 2$ new vectors $\ket{\tilde{V}^\mu_a} = \tilde{\alpha}^\mu_{-1}\ket{\pm 1, \mp 1}$ and two scalars $\ket{\tilde{\phi}_a} = \tilde{\alpha}^d_{-1}\ket{\pm 1, \mp 1}$ for the same value of the indices as above.

Hence, for $R = \sqrt{\alpha'}$, in addition to $\ket{\mathcal{V}}^\mu_1$ and $\ket{\mathcal{V}}^\mu_2$, we have 4 extra massless vectors which altogether form the adjoint representation of the gauge group $SU(2)_L \times SU(2)_R$. Furthermore, states with $\omega = 0$ (considering also, in addition to the ones above, $\alpha^\mu_{-1}\tilde{\alpha}^d_{-1}$ and $\alpha^d_1\tilde{\alpha}^\mu_{-1}$) form the $u(1)_L \times u(1)_R$ Cartan subalgebra of $su(2)_L \times su(2)_R$. Thus, in total, for $R = \sqrt{\alpha'}$, we have a non-abelian gauge symmetry enhancement of the form

$$U(1)_L \times U(1)_R \to SU(2)_L \times SU(2)_R \tag{35}$$

[6] For generality we use d indices as we did in the previous section. Here, $d + 1 = 26$.

One can generalize the compactified space to an n-torus, $T^n \simeq S^1 \times ... \times S^1$, where n dimensions are compactified. The compactification ansatz thus becomes $\mathbb{R}^{(1,d)} \to \mathbb{R}^{(1,d-n)} \times T^n$. In a sense, KK compactification can be regarded as a special case of toroidal compactification on an $n = 1$ torus. The results obtained above also generalize for the case of toroidal compactification, where the geometry of T^n is entirely contained in a non-diagonal internal metric G_{IJ} and a constant 2-form background field B_{IJ} by adapting the ansatz (25) for n compact directions.

The gauge symmetry enhancement due to the compactification procedure for special radii relies on the presence of tachyonic states (e.g. $\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1} |0; k\rangle$) to generate the Cartan subalgebra of the enhanced symmetry group. Thus, this effect is not present in Type II theories where GSO projection removes tachyonic states from the particle spectrum.

5 Orbifolds and Orientifolds

We seek to explore the role of orbifolds and orientifolds within string theory. To this extent, and to construct somewhat more realistic models from superstring theories, we start by focusing on a class of compactification spaces, which are not manifolds, called orbifolds. In this section we will develop this idea, as well as the need to introduce so-called twisted sectors in the theory, and finally present the concept of orientifold projections. The latter will be complemented by a discussion on orientifold planes (On-planes) in Section 7.2. We begin with some definitions.

Definition 1. *For $\mathcal{M}$ a smooth, differentiable manifold with a finite, discrete isometry group G, the quotient space $\mathcal{X} \simeq \mathcal{M}/G$ defines an orbifold.*

A point $x \in \mathcal{X}$ corresponds to the set of points in $\mathcal{M}$ which include the point itself and all other $x' \in \mathcal{M}$ such that $x' \sim x$ are identified via the action of G. The points in $\mathcal{M}$ which are left invariant by the action of a non-trivial group element $g \in G$ are mapped to singular points in the quotient space $\mathcal{M}/G$. At non-singular points, the orbifold and the manifold are locally isomorphic ($\mathcal{M}/G \simeq \mathcal{M}$) and hence one can define a metric as a local structure on the orbifold at such points. For convenience, we assume the orbifold quotient acts only along spatial dimensions of the manifold.

Consider the following examples:

- $S^1/\mathbb{Z}_2 \simeq [0, \pi]$: Quotienting the manifold S^1 by $\mathbb{Z}_2$, that is, introducing antipodal identification on the points of the circle ($x \sim -x$), S^1 is mapped to the closed interval on the real line from 0 to π (i.e. a compact space). This orbifold contains singularities at $x = 0$ and $x = \pi$ since the $\mathbb{Z}_2$ action leaves these points invariant on the original manifold.

- $\mathbb{C}/\mathbb{Z}_2$: Introducing antipodal identification on the complex plane, the orbifold space has the topology of a cone with apex at $(0,0)$ which opens up along the positive direction of the imaginary axis. This space is clearly non-compact since the radius at the base of the cone is not bounded. The group action has precisely one fixed point at $(0,0)$ leading to one isolated quotient singularity in the orbifold space.

Even though the orbifold may contain (conical) singularities, strings are able to propagate consistently on such spaces if we introduce the extra structure of twists on the spectrum of states of the theory [10]. Twisted states make the string CFT on the orbifold consistent and agree with unitarity, which we state somewhat vaguely here. For a more complete discussion the reader is referred to [11] and [12]. For free strings propagating on an orbifold, we thus differentiate between untwisted/twisted states.

- Untwisted states: If one projects the space of string states on the manifold $\mathcal{M}$ onto a subspace of states which are invariant under the action of the isometry group, then, $\psi \in \mathcal{M}$ is an untwisted state of the orbifold if $\psi = g\psi \;\; \forall g \in G$ (i.e. if it is G-invariant).

- Twisted states: For closed strings, recall that $X^\mu(\sigma + 2\pi) = X^\mu(\sigma)$. Hence, a string connecting $x \to gx$ where $x, gx \in \mathcal{M}$ would not be an allowed configuration of the theory for non-trivial $g \in G$ due to the periodicity condition. However, in the orbifold $\mathcal{M}/G$, this would be allowed since points related via the action of G are equivalent. That is, the periodicity condition for closed strings on the orbifold becomes $X^\mu(\sigma + 2\pi) = gX^\mu(\sigma)$ (the value of the fields after a cycle around the closed string is restricted to their original value modulo g). For a twisted sector we require $g \in G$ to be a non-trivial group element. Hence, the number of distinct twisted sectors which can be defined on the orbifold is precisely equal to the number of conjugacy classes of the isometry group G.

We will now introduce the idea of orientifold projections by considering the quotient of a Type II theory by a representative of a general class of operators with elements $\tilde{\Omega}$. In our particular case, it is useful to define Ω as the worldsheet parity operator which acts on the WS coordinates in both open and closed string theory by:

$$\text{Open} \;\; \Omega : (\tau, \sigma) \to (\tau, \pi - \sigma) \tag{36a}$$

$$\text{Closed} \;\; \Omega : (\tau, \sigma) \to (\tau, 2\pi - \sigma) \tag{36b}$$

where (τ, σ) denote the time and spatial coordinates of the WS respectively. We have chosen units such that $l_s = \pi$, and take the length of the closed string to be

$2l_s$. These actions reverse the orientation of the WS (see Section 7.2).

The general operator $\tilde{\Omega} = \Omega\Sigma$ defining the orientifold projection is formed by combining the WS parity operator Ω, with any discrete symmetry the background of the superstring theory we are considering might have, which we denote Σ for Type IIB and $\bar{\Sigma}$ for Type IIA. Furthermore, the WS parity operator as well as the discrete isometry actions of the background obey:

$$\Omega^2 = 1 \tag{37a}$$

$$\Omega\Sigma\Omega^{-1} = \Sigma \quad \Omega\bar{\Sigma}\Omega^{-1} = \bar{\Sigma} \tag{37b}$$

and in the case where Σ and $\bar{\Sigma}$ correspond to a $\mathbb{Z}_2$ symmetry, then $\Sigma^2 = \bar{\Sigma}^2 = 1$ too.

The orientifolds we will be concerned about are built on Type II theories and supersymmetric compactifications to $\mathbb{R}^{(1,3)}$, so that the 10-dimensional space of the theory is given as $\mathcal{M}^{10} \simeq \mathbb{R}^{(1,3)} \times \mathcal{W}$ where $\mathcal{W}$ is 6-dimensional and its topology depends on the degree of supersymmetry [7]. More types of orientifolds exist but we refer the reader to [10] and [13] for further details.

We can further consider other symmetry actions of the group G acting on the internal space, so that one can then think of the full orientifold group of, say, Type IIB theory as:

$$G_{\Omega}^{(IIB)} = \Omega\Sigma G = \tilde{\Omega}G \tag{38}$$

That is, the orientifold is constructed by first taking an orbifold quotient of the internal space by G and then performing a projection on the orbifold space by the general operator $\tilde{\Omega} = \Omega\Sigma$.

We now consider an example of a toroidal orientifold to better illustrate this. Let $\mathcal{W} \simeq T^6$ and $G \simeq \mathbb{Z}_N$ (or more generally $\mathbb{Z}_N \times \mathbb{Z}_M$) be the cyclic group describing the discrete isometry of $\mathcal{W}$, and let Σ be another distinct isometry of the internal space. Introducing a complex structure on T^6, we can parametrize the manifold by a set of complex coordinates z^i for $i = 1, 2, 3$ which describe the bosonic fields on the WS[8]. We can further choose the action of the $\bar{\Sigma}/\Sigma$ isometry such that for:

$$\text{Type IIA:} \;\; \bar{\Sigma}z^i\bar{\Sigma}^{-1} = \pm\bar{z}^i \tag{39a}$$

$$\text{Type IIB:} \;\; \Sigma z^i\Sigma^{-1} = \pm z^i \tag{39b}$$

where for Type IIA, all three coordinates are either positive or negative after conjugation by $\bar{\Sigma}$. For Type IIB, the nature of the conjugation is specified by the

[7] For $\mathcal{N} = 2$, $\mathcal{W}$ is a Calabi-Yau manifold, for $\mathcal{N} = 4$, $\mathcal{W} \simeq K3 \times T^2$ and for $\mathcal{N} = 8$, $\mathcal{W} \simeq T^6$.

[8] Recall for Type II theories these exist only in the (NS,NS) and (R,R) sectors.

number of possible minus signs (even or odd) induced on the coordinates z^i after the involution acts. Indeed, there exists a distinction with respect to the number of complex directions which are reflected by the conjugation with Σ, which leads to different allowed orientifold planes in each case. For an even number of $-$ signs, the theory accepts $O5$ and $O9$-planes, whereas for an odd number of $-$ signs the theory accepts $O3$ and $O7$-planes. We describe this in more detail in the next section. The full orientifold group on the 6-torus is then constructed as per (38). These actions extend to the fermionic fields via WS SUSY. Note that (39a) describes an antiholomorphic involution on the WS coordinates, whereas (39b) describes a holomorphic conjugation of the coordinates in Type IIB theory, explaining our choice of notation.

Later, we will see explicitly that one can construct a Type I theory as an orientifold projection of Type IIB theory, where we have $\Sigma = \mathbb{I}$ be a trivial isometry so that $\tilde{\Omega} \simeq \Omega$ is just the WS parity operator. Thus, schematically, we have Type I $\simeq$ Type IIB/Ω. For now, this is all rather heuristic, but it will be formalized in Section 7.1.

6 Dp-Branes

In this section we introduce Dp-branes and review some of their features. We explore their dynamical nature as well as some tools needed to interpret them as both hyperplanes where open strings end and as solutions to the field equations of supergravity theories. In particular, we focus on the emergence of $U(N_a) \times U(N_b)$ gauge theories for specific configurations of parallel Dp-branes, which provide an important step towards the understanding of gauge/string dualities.

6.1 Motivation and Definition

String theory predicts the presence of higher-dimensional extended objects known as Dp-branes. These are $(p+1)$-dimensional hyperplanes to which open strings are attached (see Figure 2) and arise in the theory when one chooses Neumann boundary conditions, $\partial_\sigma x^\mu|_{\sigma=0,\pi} = 0$, for directions along the hypersurface (i.e. for $\mu = 0, 1...p$) and Dirichlet BCs in the transverse directions, that is, $\delta x^\mu = 0$ for $\mu = (p+1), ...(D-1)$ where D is the dimension of spacetime. The position of the Dp-brane in spacetime is fixed at the boundary coordinates $x^{(p+1)}, ...x^{(D-1)}$, which define the corresponding string theory background. The open string ends are described by the coordinates $x^0, ...x^p$ and are constrained to move along the $(p+1)$-dimensional hypersurface.

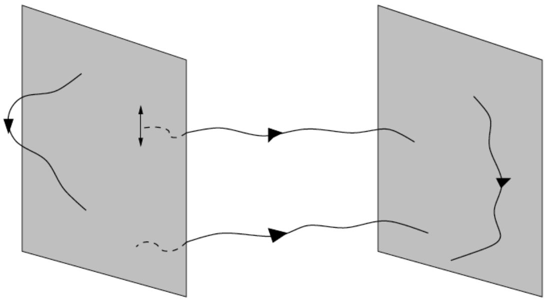

Figure 2: A configuration of two parallel Dp-branes with open strings attached is shown. String ends are constrained to move within the hyperplane but are allowed to end on different Dp-branes if such configuration is allowed by the theory. Adapted from [1].

6.2 Dynamical Nature

Dp-branes are not static in spacetime, but rather, they are dynamical objects which:

- Gravitate by coupling to closed strings in the (NS,NS) sector and thus have mass (by SUSY, coupling to fermionic superpartners also occurs).
- Are charged under R-R p-form potentials $C^{(p+1)}$ [14].

We now delve deeper into the above statements to further motivate the dynamics of Dp-branes. Firstly, in a QFT, the worldvolume swept out by a Dp-brane undergoes quantum fluctuations in directions normal to the hyperplane that defines them [15]. These, in turn, are described by the lowest mass open string states. The excitations take the general form $\psi^n_{-\frac{1}{2}} |0;k\rangle_{NS}$, where the Fock vacuum is a unique scalar in the NS-sector as discussed previously, and hence all the excitations are bosonic. The spacetime spinor $\psi^n_{-\frac{1}{2}}$ of $SO(8)$ is projected along the Dirichlet/Neumann directions defined on the Dp-brane, that is, it is decomposed under $SO(9-p) \times SO(p+1)$. Furthermore, these excitations describe massless scalar fields propagating along the Dp-brane, which happen to be moduli fields since their VEV determines a geometrical property of the internal space, the position of the Dp-brane. The presence of such quantum fluctuations follows by analogy with the closed string case. Closed superstrings in a flat background contain gravitons in their massless spectrum (recall Section 2.3), which correspond to quantum fluctuations of a dynamical metric. For the open string sector, strings propagate along an a priori static surface, however, the massless spectrum exhibits scalar fields (dilatons) which, in turn, represent the quantum fluctuations of the dynamical hyperplane (the Dp-brane).

For the second statement, we first review a crucial observation made in Section 2.4 of [15]. The one-loop amplitude of an open string attached to two distinct Dp-branes is equivalent to the tree-level exchange of closed R-R and NS-NS sector strings between two Dp-branes. In particular, one can compare the exchange amplitude of R-R and NS-NS states with the one described by a low-energy effective action of $(p+1)$-dim hyperplanes charged under an R-R p-form potential. This low-energy effective action on the worldvolume of the Dp-brane is given as [16]:

$$S_{\text{eff}} = S_{\text{DBI}}[\phi, G, B] + S_{\text{CS}}[C_p] \tag{40a}$$

where

$$S_{\text{DBI}}[\phi, G, B] = -T_p \int d^{p+1}\xi \; e^{-\phi}[-\det(G_{ab} + 2\pi\alpha' \mathcal{F}_{ab})]^{\frac{1}{2}} \tag{40b}$$

and

$$S_{\text{CS}}[C_p] = -\mu_p \int ch(2\pi\alpha'\mathcal{F}) \wedge \sqrt{\frac{\hat{A}(\mathcal{R}_T)}{\hat{A}(\mathcal{R}_{\mathcal{N}})}} \wedge \bigoplus C^{(p)} \tag{40c}$$

S_{eff} encodes the dynamics of massless open string modes as well as describing a 10-dimensional gauge field multiplet. S_{DBI} is the Dirac-Born-Infeld action[9] and couples to the NS-NS sector. It describes the Chan-Paton[10] gauge fields on a single Dp-brane at low energies. $G_{ab} = \partial_a x^\mu \partial_b x^\nu G_{\mu\nu}(x(\xi))$ is the pull-back of the ambient space metric onto the worldvolume, where $a \in [0, 1...p]$ are the coordinates along the Dp-brane and $x^\mu(\xi)$ describe the embedding of the brane worldvolume in 10-dimensions. The dilaton pre-factor is related to the string coupling as $g_s = e^\phi$, and implies that S_{eff} describes tree-level processes. The combination $2\pi\alpha'\mathcal{F}_{ab} = 2\pi\alpha' F_{ab} + B_{ab}$ contains the field strength F_{ab} of a $U(1)$ gauge field on a single Dp-brane and the pull-back of the Kalb-Ramond 2-form, $B_{ab} = \partial_a x^\mu \partial_b x^\nu B_{\mu\nu}$, onto the brane worldvolume. This combination is uniquely invariant under the closed string $U(1)$ symmetry transformations ($\delta B_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$ and $\delta A_\mu = -\frac{1}{2\pi\alpha'}\xi_\mu$) due to the WS coupling of the non-linear σ-model for a closed bosonic string on a general background with a potential boundary $\partial\Sigma$. The action of the σ-model takes the form:

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi \;\; \sqrt{h}(\epsilon^{ab}\partial_a x^\mu \partial_b x^\nu B_{\mu\nu} + h^{ab} G_{ab} + \alpha' R(h)\phi(x)) + \int_{\partial\Sigma} d\xi A_\mu \partial_\xi x^\mu \tag{41}$$

The strength of the coupling is controlled by $T_p = \frac{2\pi}{l_s^{(p+1)}}$, where $l_s = 2\pi\sqrt{\alpha'}$. S_{CS} refers to the Chern-Simons action, expressed using standard differential form calculus. Here, $ch(2\pi\alpha'\mathcal{F}) = tr[\exp(2\pi\alpha'\mathcal{F})]$ denotes the Chern character and $\hat{A}(\mathcal{R}_i)$

[9]Expanding the square root to first order gives the kinetic term of the Yang-Mills action.
[10]See section 6.3.

corresponds to the A-roof genus of $\mathcal{R} = l_s^2 R$, where R is a curvature 2-form and the indices on $\mathcal{R}$ specify the curvature of the tangent (or normal) bundles of the D-brane worldvolume. The full form of S_{CS}, (40c), is included only for completeness purposes. We restrict our attention to the massless spectrum of the R-R sector of Type II theories, which contains only p-forms ($C^{(1)} \oplus C^{(3)}$ for IIA and $C^{(0)} \oplus C^{(2)} \oplus C^{(4)+}$ for IIB). In $D = 10$, one can (Hodge) dualize the field strengths as

$$*F^{(q+1)} = *dC^{(q)} = \tilde{F}^{(9-q)} = d\tilde{C}^{(8-q)} \tag{42}$$

where the dualized quantities are tilded. The potentials can be dualized by considering the 8 dynamical (transverse) dimensions of light-cone gauge, where a self-dual 4-form was found. This implies that the forms $C^{(q)}$ and $\tilde{C}^{8-q}$ describe the *same* degrees of freedom. With this in mind, we can equivalently consider the field content for the massless (R_+, R_+) sector of both Type II theories with the added "dual redundancies":

$$\text{IIA} \rightarrow C^{(1)}_{\mu_1} \oplus \tilde{C}^{(7)}_{[\mu_1 \ldots \mu_7]} \oplus C^{(3)}_{[\mu_1 \ldots \mu_3]} \oplus \tilde{C}^{(5)}_{[\mu_1 \ldots \mu_5]} \tag{43a}$$

$$\text{IIB} \rightarrow C^{(0)} \oplus \tilde{C}^{(8)}_{[\mu_1 \ldots \mu_8]} \oplus C^{(2)}_{[\mu_1 \mu_2]} \oplus \tilde{C}^{(6)}_{[\mu_1 \ldots \mu_6]} \oplus C^{(4)+}_{[\mu_1 \ldots \mu_4]} \tag{43b}$$

While the field content appears to have changed, the degrees of freedom described by the p-forms remain the same. We note that only odd p-forms appear in the spectrum of Type IIA whereas only even p-forms appear in the spectrum of Type IIB. The R-R p-forms behave like generalised electromagnetic fields coupling to Dp-branes which act as generalised charged particles.

We now explicitly state how $(p+1)$-forms couple to the worldvolume of Dp-branes. To lowest order, one can expand the Chern-Simons action into [16]:

$$S_{\text{CS}} = -\mu_p \int_{Dp} C^{(p+1)} = -\mu_p \int d\xi^0 \ldots d\xi^p C^{(p+1)}_{[1 \ldots (p+1)]} \tag{44}$$

where $\mu_p = \frac{2\pi}{l_s^{p+1}}$ refers to the charge of the Dp-brane under $C^{(p+1)}$. Knowing the form of the coupling, and the p-form field content, we can thus deduce the spectrum of Dp-branes in Type II theories, which is:

$$\text{IIA} \rightarrow C^{(2p+1)} \sim D(2p)\text{-brane} \quad \text{for } \; p \in (0, 1 \ldots 4) \tag{45a}$$

$$\text{IIB} \rightarrow C^{(2p+2)} \sim D(2p+1)\text{-brane} \quad \text{for } \; p \in (-1, 0 \ldots 4) \tag{45b}$$

Only the above Dp-branes can exist as stable extended objects in each case since they must couple to the available R-R forms in the spectra of the theories (e.g.

in IIB theory a D7-brane is stable as it carries a $\tilde{C}^{(8)}$ charge, whereas in IIA it must decay since it cannot couple to an 8-form). For tree-level processes, we can approximate $T_p \simeq \mu_p$, such that the brane charge and tension coincide. This means that Dp-branes respect the BPS bound $M \geq Z$ between mass and charge, and are thus BPS objects.

6.3 String Orientation, C-P Factors and Wilson Lines

We must make a distinction between oriented and unoriented superstring theories. To this end, we define an *unoriented* string as one which is invariant under an orientation reversal $\sigma \to -\sigma$, corresponding to the conjugation of the fields by the WS parity operator Ω, that is, $\Omega^\dagger x^\mu(\tau, \sigma)\Omega = x^\mu(\tau, l - \sigma)$ where l here refers to the length of a string (can be open or closed). Note that an orientation reversal also interchanges left/right-movers since $f(\tau + \sigma) \to f(\tau - \sigma)$, so the states in the spectrum of an unoriented theory must be symmetric with respect to L/R-mover interchange, or equivalently, they must be symmetric under $\alpha^\mu_{-n} \leftrightarrow \tilde{\alpha}^\mu_{-n}$. Type II theories are consistent closed oriented superstring theories in 10-dimensions. If one were to consider the massless spectrum of an open unoriented bosonic string theory, the K-R 2-form $B_{[\mu\nu]}$ would be projected out of the theory since by definition, it is antisymmetric with respect to $\mu \leftrightarrow \nu$. Oriented theories contain string states which are sensitive to orientation reversal.

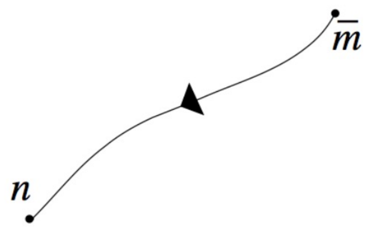

Figure 3: Chan-Paton labels $\bar{m}$ and n attached to the ends of an open oriented string. Adapted from [16].

Chan-Paton (CP) factors are additional non-dynamical degrees of freedom carried by open strings at their endpoints. For the case of oriented open strings, the two ends are inequivalent after an orientation reversal, so one associates the labels $\bar{m}$ and n to the string ends as in Figure **3**. These correspond, respectively, to the antifundamental representation[11] at the $\sigma = 0$ end of the string, and the fundamental

[11]This refers to the complex conjugate of the fundamental representation.

representation at $\sigma = \pi$ of a gauge group $U(N)$ associated to the string via the N non-dynamical degrees of freedom at each end.

The CP labels retain the Poincaré and WS conformal invariance of the theory [17]. We therefore can use the set of $N \times N$ matrices, λ^a_{ij}, as well as the Fock space labels and momenta, to describe the string states. Hence, for $\mathbb{R}^{1,25}$, the states can be expressed in the basis

$$|\phi; k; a\rangle = \sum_{i,j=1}^{N} |\phi; k; ij\rangle \, \lambda^a_{ij} \tag{46}$$

where the λ matrices encode the $U(N)$ charge of string states at each end via the integers $i, j \in [1, ...N]$ respectively. Due to the complex conjugation, the string has opposite $U(N)$ charges at each end. With the CP labels in place, the states become N^2-degenerate (in a background of N coincident D-branes) since we must introduce precisely N^2 hermitian matrices λ^a_{ij} (due to the reality condition on the string fields) to describe the state. These furnish the adjoint representation of the new $U(N)$ gauge group.

Unoriented strings are invariant under orientation reversal, so the CP labels at each end must coincide (i.e. the antifundamental and fundamental representations must be equal) so that the new symmetry group introduced by the CP labels must have a real fundamental representation, reducing the possibilities to $SO(N)$ and $Sp(N)$ for even N. Before being projected out due to symmetry, a general state in the theory can be symmetric or antisymmetric with respect to $\mu \leftrightarrow \nu$. Then, if the massless vector states of the theory correspond to [16]:

- Symmetric states $\rightarrow \frac{N(N+1)}{2}$ of them $\rightarrow Sp(N)$ gauge symmetry.
- Antisymmetric states $\rightarrow \frac{N(N-1)}{2}$ of them $\rightarrow SO(N)$ gauge symmetry.

The general $N \times N$ hermitian matrices introduced to described the basis of states of a general open theory, λ^a_{ij}, are projected, upon introducing orientation, into symmetric/antisymmetric matrices depending on the symmetry of the massless vector states of the theory. That is, they must obey the condition $\lambda^a_{ij} = \lambda^a_{ji}$ up to a sign. In terms of representations, introducing orientation in the theory is equivalent to the adjoint representation of $U(N)$ being decomposed into adjoint representations of $SO(N)$ or $Sp(N)$ depending on the symmetry of the CP matrices.

We further check whether or not the n-point scattering amplitudes are invariant under the new symmetry groups introduced via the CP factors. Consider the 3-point amplitude $\mathcal{A}_3$ in the oriented bosonic string theory. Due to the CP labels, $\mathcal{A}_3$ will contain the extra factor of $\delta^{ii'}\delta^{jj'}\delta^{kk'}\lambda^1_{ij}\lambda^2_{j'k}\lambda^3_{k'i'} = \mathrm{Tr}(\lambda^1\lambda^2\lambda^3)$. Recalling that

λ^a_{ij} form the adjoint representation and using the cyclicity of the trace, we clearly see the emergent $U(N)$ symmetry on the WS at the level of the scattering amplitudes. This argument holds for the other gauge group symmetries.

Hence, if we consider the most general gauge group in a general open string theory (prior to considering orientation) it will be of the form:

$$G = \prod_a U(N_a) \times \prod_b SO(N_b) \times \prod_c Sp(N_c) \tag{47}$$

and the only allowed representations for the fields are the adjoint, symmetric, anti-symmetric and bifundamental representations [17], which is easy to see if we think in terms of Young Tableaux.

We now switch focus and turn to another gauge-invariant configuration known as the Wilson line. If we consider a spacetime with S^1 topology (corresponding to the compactified coordinate x^{25}) of radius R and a $U(1)$ gauge symmetry, we can choose

$$A_{25}(x^\mu) = -\frac{\theta}{2\pi R} = -i\Lambda^{-1}\partial_{25}\Lambda \quad \text{where} \quad \Lambda = \exp(-\frac{i\theta x^{25}}{2\pi R}) \tag{48}$$

as the constant background gauge field potential where θ is just a phase. This choice corresponds to pure gauge locally. Then, we construct the $U(1)$ Wilson line as [18]

$$W_q = \exp\left[iq \oint dx^{25} A_{25}\right] = e^{-iq\theta} \tag{49}$$

which is an observable of the gauge theory. The integration contour refers to non-contractible cycles on the WS and we have used the usual coupling (second term of (41)) of the gauge field to the WS. With this choice of gauge, all fields moving around the circle with $U(1)$ charge q, pick up a phase equal to $e^{iq\theta}$. Hence, the canonical momentum $p^\mu = i\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = i\dot{x}^\mu$ for a point particle in the compactified dimension is shifted by

$$p^{25} \to \frac{n}{R} + \frac{q\theta}{2\pi R} \tag{50}$$

which shows that A_{25} cannot be fully cancelled, even in pure gauge, unless $W_q = 0$.

Considering now the more general $U(N)$ case for an oriented open string, in the same topology, we make the choice

$$A_{25} = \text{diag}\{\theta_1, ...\theta_N\}/2\pi R = -i\Lambda^{-1}\partial_{25}\Lambda \tag{51}$$

where

$$\Lambda = \text{diag}\left\{\exp(-\frac{i\theta_1 x^{25}}{2\pi R}), ...\exp(-\frac{i\theta_N x^{25}}{2\pi R})\right\} \tag{52}$$

which, again, is pure gauge but locally. This generally breaks $U(N) \to U(1)^N$. The presence of these gauge-invariant observables in an open string theory is equivalent, under T-duality, to a theory of non-coincident D-branes with separations determined by the value of the Wilson line. They also play an important role in orbifold compactifications, but we will see this in more detail when discussing T-duality in Type I theory.

6.4 D-Branes as Charged BPS States

We have seen how D-branes carry conserved antisymmetric charges of topological nature (RR p-form charges) which in a SUSY context one can relate to the central charge of a superalgebra. In particular, recall that the charge of a Dp-brane under $C^{(p+1)}$ is given by $\mu_p = \frac{2\pi}{l_s^{(p+1)}}$, to lowest order, and this alone implied that the Dp-brane satisfied the BPS bound $M \geq Z$. In the case considered in Section 6.2, we had $T_p = \mu_p \to M = Z$, which constitutes a saturation of the BPS bound. The saturated states from the above condition belong to shortened super-multiplets (containing 2^8 states as opposed to the usual 2^{16} states of a generic super-multiplet) due to the extra 0s appearing in the super-algebra if $M = Z$ [19].

These states are stable, since they are fixed so long as SUSY is preserved. We check explicitly the amount of SUSY preserved by D-branes. For this, we consider D9-branes which correspond to open strings in $D = 10$. We denote Q_α and $\tilde{Q}_\alpha$ as the distinct spacetime supercharges carried by L/R-movers in Type IIB theory ($\mathcal{N} = 2$), which are also 10-dimensional Majorana-Weyl spinors, and thus have 16 real components. The open string BCs identify L/R-movers, and preserve only the linear combination $Q_\alpha + \tilde{Q}_\alpha$ of the supercharges, which corresponds to effectively $\frac{1}{2}$ of the original SUSY of Type IIB theory. Thus, D-branes are regarded $\frac{1}{2}$-BPS states since they preserve one half of the original supersymmetry of the theory.

There exists a "no-force" condition [20] between extended objects satisfying a BPS bound and preserving partial SUSY. This applies to configurations where the D-branes are parallel to each other, such that static configurations of these states can exist in the theory due to a cancellation between gravitational/gauge forces arising between them. This cancellation can be interpreted from the fact that the open string one-loop amplitude is equivalent to the tree-level closed exchange of R-R and NS-NS strings, as stated in 6.2. The one-loop amplitude of the open string between two parallel Dp-branes is given by:

$$\mathcal{A}_{1-loop} = \int_0^\infty \frac{dt}{2\pi} \mathrm{Tr}_{NS,R} \left[\frac{1 + (-1)^F}{2} e^{-2\pi t L_0} \right] = 0 \tag{53}$$

where the trace is taken over the whole superstring spectrum. This vanishes from a field theory perspective (see [2]) but also is readily zero due to the BPS nature of the D-branes, since effectively the net forces from the R-R and NS-NS sectors cancel exactly if the D-branes are parallel. To see this, we consider the separate contributions from the NS-NS and R-R sectors respectively (where in the former, terms including $(-1)^F$ refer to periodic fermions in the R-R sector, and terms without the $(-1)^F$ factor refer to antiperiodic fermions corresponding to NS-NS sector exchange). Taking the limit $t \to 0$ as in [15], one finds precisely that $\mathcal{A}_{NS} = -\mathcal{A}_R$. The vanishing of this static force on a p-brane probe due to the gravitational background of a distinct $\tilde{p}$-brane is central in the construction more general composite BPS brane configurations, however, the relevance for us is that D-branes can indeed be regarded as charged BPS states.

6.5 D-Branes as Solutions to Supergravity Field Equations

Now, we present the dual interpretation for D-branes. In particular, we note that a Dp-brane can be interpreted, by WS duality, as a source for *closed* strings (as it carries R-R charges). From the spectrum in (43b) we note that the D3-brane is self-dual, since it couples to $C^{(4)+}$ which gives rise to a self-dual field strength $*F_5 = \tilde{F}_5$ in Type IIB theory. In the low-energy limit of Type IIB theory, one obtains the $\mathcal{N} = 2$ supergravity (SUGRA) action in 10-dimensions via the Type IIB low-energy effective action. This is given by:

$$S_{\text{eff}} = S_{NS} + S_R + S_{CS} \tag{54}$$

where

$$S_{NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \, \sqrt{-G} e^{-2\phi} (R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}) \tag{55a}$$

which coincides with the low-energy effective action of the bosonic theory.
Secondly, we have that

$$S_R = -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-G} (F_1 \wedge *F_1 + \tilde{F}_3 \wedge *\tilde{F}_3 + \frac{1}{2} \tilde{F}_5 \wedge *\tilde{F}_5) \tag{55b}$$

where the tilded field strengths correspond to the gauge invariant combinations $\tilde{F}_3 = F_3 - C_0 \wedge H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$. Finally, S_{CS} is given as

$$S_{CS} = -\frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 \tag{55c}$$

In the above formulae, we have that $F_{n+1} = dC_n$ and $H_3 = dB_2$ corresponding to the exterior derivative of the Kalb-Ramond 2-form [2].

A classical Dp-brane solution to the equations of motion emerging from S_{eff} is invariant under the action of the symmetry group $\mathbb{R}^{(p+1)} \times SO(1,p) \times SO(9-p)$ where one can deduce that:

- Poincaré invariance is preserved in the $(p+1)$-dim hyperplane via the subgroup $\mathbb{R}^{(p+1)} \times SO(1,p)$.
- The remaining $(9-p)$ dimensions exhibit maximal rotational symmetry encoded in $SO(9-p)$.

These solutions are referred to as $\frac{1}{2}$-BPS states as they preserve precisely one half of the SUSY of the theory. Using the above symmetry of the solution, we can take the following ansatz for the metric:

$$ds^2 = \frac{1}{\sqrt{H(\vec{y})}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{H(\vec{y})}d\vec{y}^2 \tag{56}$$

where x^μ refer to coordinates along the Dp-brane ($\mu = 0, 1...p$) and y^i refer to coordinates perpendicular to the Dp-brane ($i = p+1, ..., 10$). Further analysis [21] shows that $e^\phi = H(\vec{y})^{\frac{3-p}{4}}$ and that $H(\bar{y})$ must be a harmonic function of the coordinates y^i taking the form:

$$H(\vec{y}) = 1 + \left(\frac{L}{y}\right)^{(7-p)} \tag{57}$$

in 10-dimensions for L an arbitrary scale factor, after imposing the condition that we must recover flat space away from the bulk (in the limit $y = \sqrt{\vec{y}.\vec{y}} \to \infty$). In particular, for a stack of N coincident Dp-branes, we take the scale factor to be of the form:

$$L^{(7-p)} = (4\pi)^{\frac{5-p}{2}}\alpha'^{(\frac{7-p}{2})}\mathcal{N}g_s\Gamma\left(\frac{7-p}{2}\right) \tag{58}$$

where g_s denotes the string coupling constant and $\mathcal{N}$ refers to the number of units of magnetic 5-form flux $\mathcal{N} = \int_{S^5} F_5$ sourced by the stack of Dp-branes in Type IIB theory, with F_5 being the self-dual field strength. In this way, we can construct a classical Dp-brane solution to the equations of motion from the low-energy effective action of Type IIB theory coupling to the corresponding p-form in the spectrum. For example, we can construct the D3-brane solution coupling to the self-dual 4-form $C^{(4)+}$. Other possibilities exist, such as the D1-brane solution coupling *electrically* to $C^{(2)}$ (with the D5-brane as the corresponding dual brane coupling *magnetically*[12]

[12] The charge of the magnetic dual brane is found by computing the flux $\int_{S^{p+2}} F_{p+2}$, where in $D = 10$, S^{p+2} can surround a $(6-p)$-brane, that is, the magnetic dual of a p-brane [2].

to the 2-form) and more exotic solutions involving the D(-1)-brane and the so-called fundamental string, F1, coupling to the K-R 2-form.

For a stack of N D3-branes, the brane dynamics is precisely described by four-dimensional $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills (SYM) theory [21]. This fact, and the dual interpretation of D-branes will be very useful when introducing the AdS/CFT correspondence via the decoupling argument.

6.6 Configurations of D-Branes

We have motivated how different gauge symmetries can appear in open string theories by considering CP labels. To conclude this section, we consider the effect of having Dp-branes which intersect along some subspace containing $\mathbb{R}^{(1,3)}$. Generally, intersecting D-brane models yield non-supersymmetric, chiral low-energy spectra and consideration of appropriate sets of D-brane stacks can result in the construction of the Standard Model-like gauge theories.

We begin by considering a system of two D6-branes in $\mathbb{R}^{(1,9)}$ (flat, non-compact Minkowski space) in Type IIA theory, such that each D6-brane spans the following dimensions:

$$D6_A \to 0,1,2,3,4,6,8 \quad D6_B \to 0,1,2,3,5,7,9 \tag{59}$$

The pair of D6-branes thus intersects along the space $\mathbb{R}^{(1,3)} \times (x^i = 0)$ for $i = 4,5...9$. We assume that the D-brane angles θ^i_{AB} between the $x^4 - x^5$, $x^6 - x^7$ and $x^8 - x^9$ planes are all $\frac{\pi}{2}$ for simplicity, although general intersection angles can be considered (see Figure **4**).

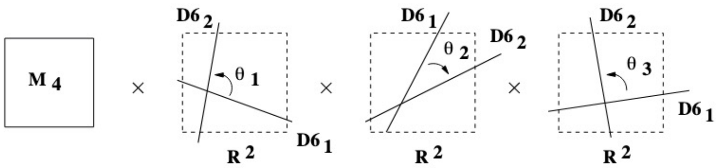

Figure 4: Pair of D6-branes intersecting as in (59) at arbitrary angles θ^i_{ab} in $\mathbb{R}^{(1,9)}$. Angles are to be measured from brane a to brane b as shown. Adapted from [22].

Generally, the pair of D6-branes will be related by a $(SO(8)/SO(p-1)) \times T_{9-r}$ isometry [23], where in our case $p = 6$ and T_{9-r} refers to the translation group in $(9-r)$ dimensions with r being the dimension of the direct sum of the tangent space dimensions (in our case $r = 6$). Although the common dimensions span $\mathbb{R}^{(1,3)}$, the setup is not yet effectively 4-dimensional since states propagating along D6_A and D6_B will generally have components propagating along the remaining dimensions of the full 10-dim space. However, the effect of these is negligible upon a KK compactification to $\mathbb{R}^{(1,3)}$ if the limit of the size of the internal space being very small is taken. At the intersection of Dp-branes, the total amount of SUSY of the 10-dimensional theory preserved, depends on the sum $\sum_{i=1}^{N} \theta^i_{AB}$ of angles between the planes formed by the intersecting directions of the Dp-branes. In the above case, where all angles are $\frac{\pi}{2}$, SUSY is fully broken [24]. In compactified models, D-brane configurations are severely restricted by consistency conditions (e.g. anomaly cancellations). The allowed intersecting models correspond to new effective 4-dimensional vacua of the original 10-dimensional theory and can be used for semi-realistic effective theory model-building.

We now examine the effect of configurations corresponding to stacks of N_A/N_B coincident DA/DB-branes and consider the open string spectra arising from the theory in a background containing these. We can divide the spectrum into different sectors:

- *A-A/B-B Sectors*: These correspond to strings ending on the same type of D-brane stack. They contain $U(N_A)$ and $U(N_B)$ gauge bosons respectively (due to the CP labels) plus their superpartners. Along the intersecting dimensions of the stacks, both bosons propagate and hence, for the space $\mathbb{R}^{(1,3)} \times (x^i = 0)$, the gauge symmetry is of the form $U(N_A) \times U(N_B)$.
- *A→B/B→A Sectors:* These refer to strings which are stretched between different D-brane stacks. The string states are localised at brane stack intersections and thus propagate only along $\mathbb{R}^{(1,3)} \times (x^i = 0)$ [24]. Because of this, adding CP labels results in these states transforming in the bifundamental representation of $U(N_A) \times U(N_B)$, where by convention we choose states in the A→B sector to transform as $(\bar{N}_A, N_B)$ and states in B→A to transform as $(N_A, \bar{N}_B)$, matching the convention in Figure **3**. Given the isomorphism $U(N_A) \simeq SU(N_A) \times U(1)_A$, we choose the antifundamental representation $\bar{N}_A$ to have a normalized $U(1)_A$ charge of -1_A (so then N_A has charge $+1_A$).

For further details on the latter sectors, we quantise the mixed BCs along the compact space dimensions of the stacks. In the A→B sector, if we follow the previous example, these take the form:

$$\partial_\sigma X^n(\tau, \sigma = 0) = 0 \quad n = 0, 1, 2, 3, 4, 6, 8 \tag{60a}$$

$$\partial_\tau X^m(\tau, \sigma = 0) = 0 \quad m = 5, 7, 9 \tag{60b}$$

$$\partial_\sigma X^n(\tau, \sigma = l) = 0 \quad n = 0, 1, 2, 3, 5, 7, 9 \tag{60c}$$

$$\partial_\tau X^m(\tau, \sigma = l) = 0 \quad m = 4, 6, 8 \tag{60d}$$

Where (60a) and (60b) refer to the DA-brane, whereas (60c) and (60d) refer to the DB-brane. We clearly have Dirichlet/Neumann BCs along the dimensions $i = 4, 5, ...9$. Quantising gives:

- *$A \to B$ Sector:* The R-sector contains a fermionic GS along $\mathbb{R}^{(1,3)}$, so the massless spectrum features a single chiral fermion, denoted ψ^{α}_{AB}, where $\alpha = 1, 2$ is a 4-dimensional Weyl spinor index and AB are CP labels. It also contains an antichiral fermion $\psi^{\dot{\alpha}}_{AB}$ of the same properties. In the NS-sector, the mixed BCs of the D-brane stacks give a single boson of $M^2 > 0$. The chiral fermion has no massless superpartner so SUSY is broken by the D-brane intersection.
- *$B \to A$ Sector:* The spectrum here follows by acting the WS parity operator Ω on the previous states. The fermionic states will have the opposite chirality and their CP labels reversed [25].

After a GSO projection, the antichiral fermion is projected out of the spectrum. Then, the total field content is given by ψ^{α}_{AB} transforming as $(\bar{N}_A, N_B)$ in the $A \to B$ sector, and $\psi^{\dot{\alpha}}_{BA}$ transforming as $(N_A, \bar{N}_B)$ in the $B \to A$ sector with opposite chirality. These correspond to a particle-antiparticle pair and so describe the same degrees of freedom. Representations of general open string states are constructed as tensor products of fundamental/antifundamental representations at each end of the string. Thus, we can conclude that a stack of N_A DA-branes and a stack of N_B DB-branes intersecting along $\mathbb{R}^{(1,3)} \times (x^i = 0)$ give rise to a $U(N_A) \times U(N_B)$ (Yang-Mills) gauge theory and one chiral fermion transforming in the bifundamental representation $(\bar{N}_A, N_B)$ of the gauge group.

As a further example, we review the case of $\mathbb{R}^{(1,9)} \simeq \mathbb{R}^{(1,3)} \times T^6$, where we choose to conveniently factorise the internal space as $T^6 \simeq T^2 \times T^2 \times T^2$ and we consider the setup of two stacks of coincident D6-branes as per (59). These will span the entire $\mathbb{R}^{(1,3)}$ space and will intersect non-trivially in the 6-dimensional fiber space. The precise intersection pattern will give the effective particle content in the 4-dimensional model. We consider the case where a stack wraps a non-contractible one-cycle (n^i, m^i) in each of the torii T^2. Here, n^i and m^i refer to the number of times the D6-brane stack wraps around T^2 horizontally and vertically, respectively. Then, the D6$_A$- and D6$_B$-brane will intersect T^6 at precisely three points[13] (for specific examples refer to Figure **5**), where at each point we will have one chiral fermion transforming in the bifundamental representation of the gauge group.

[13]Generally 3-cycles in 6-dim compact spaces intersect at points [22].

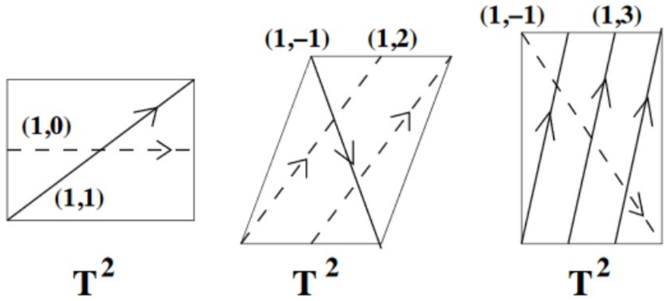

Figure 5: Examples of possible intersecting 3-cycles in T^6. Taken from [22].

Thus, the CP gauge fields on the D6-branes yield the 4-dimensional (Yang-Mills) gauge fields of the effective theory, and while D-brane models do not restrict the number of fermion generations to be exactly 3 (as in the Standard Model) they do induce a finite number of generations of fundamental particles, usually greater than one, which prove useful for semi-realistic model-building.

7 Type I Theory

In this section, we discuss Type I theory, which is the only consistent theory of open unoriented superstrings in 10 dimensions. We firstly construct Type I theory as an orientifold projection of Type IIB theory, which can also be done by considering extra O-planes in the latter. By gauging the symmetry introduced by orientifolding, we review the need to include non-orientable worldsheets in the target space, which add more structure to the theory. Finally, we consider the condition for cancellation of tadpole-like divergences in the theory, from which we can deduce the gauge symmetry group of Type I theory.

7.1 Type I as a Type IIB Orientifold Projection

We begin by describing how the fermionic RNS fields transform upon an orientation reversal. The conjugation by the WS parity operator is analogous to the bosonic field case, giving $\Omega^\dagger \psi^\mu(\tau,\sigma)\Omega = \psi^\mu(\tau, l-\sigma)$. For the bosonic oscillators, the conjugation gave $\Omega^\dagger \alpha_n^\mu \Omega = \tilde{\alpha}_n^\mu$, whereas for the fermionic oscillators we have

$$\Omega^\dagger b_n^\mu \Omega = e^{i2\pi\phi}\tilde{b}_n^\mu \quad \text{where} \quad \phi = \begin{cases} 0 & \to \text{R-sector} \\ \frac{1}{2} & \to \text{NS-sector} \end{cases} \tag{61}$$

Thus, if we consider an orientifold projection of Type IIB theory by $\tilde{\Omega} = \Omega\sigma$, with a trivial $\sigma = \mathbb{I}$ isometry, one can gauge this discrete symmetry since both L/R-moving sectors in Type IIB have the same GSO projection and then, a priori, one can define the closed sector of Type I theory as:

$$(\text{Type I})_{\text{closed}} := \text{Type IIB}/\Omega \tag{62}$$

which we check explicitly now. Firstly we consider the bosonic fields in Type IIB, for which the action of the WS parity operator on the GS gives:

- $\Omega \left|0\right\rangle_L \otimes \left|0\right\rangle_R = \left|0\right\rangle_L \otimes \left|0\right\rangle_R$ for the (NS,NS) sector. This can be easily seen by noting that Ω interchanges L/R-movers and since here, the GS of L/R-movers is bosonic, the product of states must commute.
- For the (R,R) sector this is slightly different since the ground states are spinors and thus their product anticommutes. This gives $\Omega \left|a\right\rangle_L \otimes \left|b\right\rangle_R = \left|a\right\rangle_R \otimes \left|b\right\rangle_L = -\left|b\right\rangle_L \otimes \left|a\right\rangle_R$.

Recall now, from Section 2.3, that the field content in the (NS_+, NS_+) sector was given by $\mathbf{8_v} \otimes \mathbf{8_v} \to [0] \oplus [2] \oplus (2)$. Now, however, it is clear that $[2] \simeq B_{[\mu\nu]} \in SO(8)$ is odd with respect to the action of Ω (since it is antisymmetric in $\mu \leftrightarrow \nu$), so that the state is projected out of the spectrum by the Ω quotient. The states remaining in the spectrum are Ω-even, leaving the multiplet of fields $(\phi, G_{\mu\nu})$ to form the $\mathbf{36}$ of $SO(8)$ (i.e. the symmetric representation).

For the (R_+, R_+) sector we had that $\mathbf{8_s} \otimes \mathbf{8_s} \to [0] \oplus [2] \oplus [4]_+$, where now, a bit more care needs to be taken to work out which states survive the orientifold projection. A priori, one would think that $[2] \simeq C^{(2)}_{[\mu\nu]}$ would be projected out, as it forms the antisymmetric representation of $SO(8)$, however, the extra factor of (-1) from the action of Ω on the fermionic GS in the R-R sector implies that $C^{(2)}_{[\mu\nu]}$ is indeed Ω-even. On the other hand, the states $[0] \oplus [4]_+ \simeq \mathbf{36}$, forming the symmetric representation of $SO(8)$ are Ω-odd due to the extra negative sign from the anticommutation of the ground states and are thus projected out of the spectrum. In summary, the massless level of the bosonic sector of $(\text{Type I})_{\text{closed}}$ theory contains the fields $\phi, G_{\mu\nu}$ and $C^{(2)}_{[\mu\nu]}$.

We now proceed to consider the action of Ω on the fermionic sectors of Type IIB theory, which correspond to the mixed (NS_+, R_+) and (R_+, NS_+) sectors, which are related by WS parity. On the GS, the action of Ω is given by $\Omega \left|0\right\rangle_{L,NS} \otimes \left|a\right\rangle_{R,R} = \left|0\right\rangle_{R,NS} \otimes \left|a\right\rangle_{L,R} = \left|a\right\rangle_{L,R} \otimes \left|0\right\rangle_{R,NS}$ where the last equality follows from commutation of bosonic and fermionic fields. We note that the first state labels characterize L/R-movers while the second state labels distinguish between R/NS-sectors. Hence, one sees that the projection by Ω has the overall effect of interchanging the R/NS sectors. The field content in each of the mixed sectors is given by $[8]' \oplus [56] \simeq \lambda^a \oplus \psi^i_a$, where the dilatino and gravitino have opposite chirality. Quotienting by Ω, only the diagonal combination $\left(\begin{array}{c|c} \mathbf{8'} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{56} \end{array}\right)$ containing one copy of $(\lambda^a, \psi^i_a)_{(NS-R,R-NS)}$ of

$SO(8)$ is kept in the spectrum [26].

The theory we have constructed as an orientifold projection of the Type IIB theory, thus has the following massless spectrum:

$$(\text{Type I})_{\text{closed}} \to \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \oplus \mathbf{8}' \oplus \mathbf{56} \simeq \phi \oplus C^{(2)}_{[\mu\nu]} \oplus G_{(\mu\nu)} \oplus \lambda^a \oplus \psi^i_a \tag{63}$$

Since there is only one independent gravitino in the spectrum, this theory preserves $\frac{1}{2}$ of the SUSY present in Type IIB theory. It therefore exhibits $\mathcal{N} = 1$ SUSY in 10-dimensions.

7.2 Type I as Type IIB with extra O-planes

Before moving further and checking our construction in terms of consistency at the level of interactions, we provide an alternative way of describing the spectrum given in (63).

Consider a path on the closed sector of the WS connecting $\sigma \to 2\pi - \sigma$. Gauging the WS symmetry introduced by Ω, a string carried around a closed path on the WS only needs to come back to itself up to a gauge transformation, hence the path described above forms a closed loop in the orientifold theory (Type IIB/Ω). Gauging WS parity therefore implies the inclusion of unoriented worldsheets, which add extra structure to the string states [27]. The simplest example of a non-orientable Euclidean surface is the projective plane $\mathbb{RP}^2 \simeq S^2/\mathbb{Z}_2$. Further, restricting the string theory WS to $\sigma > 0$ is equivalent to inserting cross-cap states on the WS at $\sigma = 0$, so $\mathbb{RP}^2$ can be thought of as a sphere with a cross-cap insertion as per Figure **6**.

We recall the Euler characteristic for a non-orientable surface, which is a topological invariant, is given by

$$\mathcal{X}(M) = 2 - 2g - b - c \tag{64}$$

where g refers to the genus, b to the number of boundaries and c to the number of cross-caps of the given worldsheet. Topologically, cross-caps can be thought of as boundaries with opposite points identified (boundaries with $\mathbb{Z}_2$ identifications).

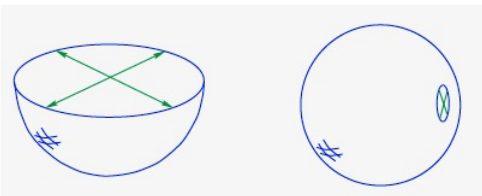

Figure 6: Construction of $\mathbb{RP}^2$ by identifying opposite points on the equator of a hemisphere, which is equivalent to a sphere S^2 with a cross-cap insertion. From [15].

Using the two-fold interpretation of D-branes presented in the last section, introducing D-branes in a closed theory is equivalent to an extension of their CFT by insertions of boundary and cross-cap states. If we think of boundary states as states where the WS ends on a D-brane, cross-cap states correspond to a WS ending on an orientifold plane. The periodicity condition along the cross-cap leads to oscillator mode excitations with Neumann/Dirichlet BCs with extra phase factors, that is:

$$\alpha_n^\mu \pm e^{i\pi n}\tilde{\alpha}_{-n}^\mu = 0 = \psi_r^\mu \pm i\eta e^{i\pi r}\tilde{\psi}_{-r}^\mu \tag{65}$$

where $\eta = \pm 1$ labels the G-parity of the fermions. The cross-cap states $|Cp\rangle$, satisfy these conditions as operator expressions and have an oscillator part given by:

$$|Cp, \eta\rangle_{\text{osc}} = \exp\left[S_{MN}\sum_{n>0}\frac{1}{n}e^{i\pi}\alpha_{-n}^M\tilde{\alpha}_n^N + i\eta S_{MN}\sum_{r>0}e^{i\pi r}\psi_{-r}^M\tilde{\psi}_r^N\right]|Cp, 0, \eta\rangle \tag{66}$$

where $S_{MN} = \text{diag}(-1, ...-1, +1...+1)$ encodes the (+1) Dirichlet and (-1) Neumann BCs of the given state. To define precisely what is meant by an orientifold plane, we consider the image of the path $\sigma \to 2\pi - \sigma$ extending throughout the entire WS in Type I theory. Its endpoints are restricted to a fixed locus of Σ (the discrete isometry) for a closed loop. This fixed locus of Σ is what defines the $(p+1)$-dimensional Op-plane. For Type I theory, where $\Sigma = \mathbb{I}$ is trivial, the locus then refers to the entire target spacetime which can be thought of as an O9-plane. The worldvolume of an Op-plane is defined as the fixed locus of an element of the general orientifold group G_Ω, given in (38) for Type IIB theory. This implies that O-planes cannot fluctuate and that the coordinates describing them, which one can choose to be cross-cap states, are non-dynamical [28].

We can therefore construct the closed sector of an $\mathcal{N} = 1$ unoriented superstring theory by considering a spacetime-filling O9-plane in Type IIB theory, which will effectively project the IIB fields into the Ω-invariant fields of Type I.

7.3 Tadpole Cancellation in Type I Theory

The field content of the closed Type I theory sector we have constructed describes the $\mathcal{N} = 1$ supergravity multiplet in 10-dimensions, which turns out to be anomalous. The full partition function must therefore contain extra contributions from an open unoriented sector for consistency at the level of interactions. For an orientifold theory, these contributions come from amplitudes over non-orientable Riemann surfaces. At $\mathcal{X}(M) = 0$ (tree-level), we thus have the Möbius strip and the Klein bottle to be considered, whose amplitudes will precisely cancel the tadpole-like divergences coming from the closed Type I sector. Prior to this, however, we must think about the allowed configurations of D-branes and O-planes in the full Type I theory.

So far in (Type I)$_{\text{closed}}$, we have defined boundary states as states where the WS ends on a D-brane for the closed string theory. These states have an oscillator part which takes the same form as the crosscap states in (66) (i.e. $|Bp, \eta\rangle_{\text{osc}} = \exp[...]\,|Bp, 0, \eta\rangle$). Further, these must also be compatible with the orientifolding by Ω, thus the only allowed D-branes in this construction are D1- and D5-branes. To see this, we refer back to the massless field content in (63). It is clear that $C^{(2)}_{[\mu\nu]}$ will couple to a D1-brane, so these must be allowed, but it can also couple magnetically to its dual brane as we have seen, which in this case is a D5-brane. These are the only BPS-states that survive the orientifold projection in Type I theory and come from the closed sector.

The emergence of the D9-branes comes from the additional open sector, which we need to include to eliminate divergences, and in a sense, their presence is trivial. This extra sector contains open strings with ends allowed to be anywhere in the 10-dimensional space. The space effectively contains a set of spacetime-filling D9-branes, which one can think of as part of the open sector vacuum rather than as excited states above it, since the theory requires them for consistency. In summary, the D-branes that are kept in the theory after orientifolding are the D1-, D5- and the (non-BPS) D9-branes. By analogy, the only allowed O-planes (non-dynamical, mirror-like hyperplanes) are thus the O1-, 05- and O9-planes.

If we consider a stack of N coincident D9-branes filling the target space, the overall divergent term can be written as [24]:

$$(2^{\frac{D}{2}} \pm N)^2 \int_0^\infty ds \xrightarrow{D=10} (32 \pm N)^2 \int_0^\infty ds \tag{67}$$

where the $\pm$ corresponds to a specific orientifold projection, that is, the possible actions of Ω on the Chan-Paton labels of massless gauge bosons

$$\Omega\,|\phi, k, ij\rangle = \pm\,|\phi, k, ij\rangle \tag{68}$$

with basis $\lambda^a_{ij} = \pm(\lambda^a_{ij})^T$. The choice of orientifold projection will determine the gauge group of the theory. If we choose $\Omega = -1$ (equivalent to the antisymmetry condition of massless vector states in Section 6.3), we generate the gauge group $SO(2\tilde{N}) \simeq Spin(2\tilde{N})/\mathbb{Z}_2$ on the stack of D9-branes. Alternatively, choosing the $\Omega = +1$ projection, the gauge group generated is $Sp(2\tilde{N})$[14]. It is clear that for consistency conditions, if we want our theory to be tadpole-free, we must choose an $N = 32$ D9-brane stack (or more precisely $\tilde{N} = 16$ D9-branes and their images under the mirror symmetry of the O9-plane) with $\Omega = -1$. This analysis assumes

[14]Where the $2\tilde{N} = N$ emphasises the condition of N being even for the symmetry groups generated by the CP labels.

D-branes are coincident with the O-planes, that is, there are massless states in the mixed D-brane/O-plane Möbius amplitude, so that the gauge group is broken into the cases above by the orientifold projection, depending on the nature of the mirror symmetry introduced by the O-planes. For non-coincident D-branes and O-planes, the gauge group is unitary and physical states must be invariant under the mirror symmetry introduced by orientifolding [29]. Thus, the full Type I theory is a theory of unoriented open strings in 10-dimensions formed by the (Type I)$_{\text{closed}}$ sector, as well as by the degrees of freedom coming from the R_+ and NS_+ open sectors coming from 32 D9-branes, subject to the orientifolding with $\Omega = -1$. The gauge group of the theory is $SO(32) \simeq Spin(32)/\mathbb{Z}_2$.

8 Dualities in Superstring Theory

So far, we have seen the idea of duality in passing when mentioning dual branes and self-dual p-forms. In fact, this concept is very powerful in string theory, and allows us to relate the 5 consistent superstring theories (IIA, IIB, I and two heterotic theories) in 10-dim to each other. This hints at the idea that these must all be contained in a larger 11-dimensional *mother* theory known as M-theory. In this section, we briefly explore the geometry of the moduli space of the bosonic theory. We then focus on specific examples of T-duality, namely how Type IIA and Type IIB are T-dual theories and finally provide an alternative view on how orientifold planes emerge under a T-duality transformation of an unoriented theory.

8.1 T-Duality Group of the Closed Bosonic String

We trace back to the example of toroidal compactification of the closed bosonic string, specifically to the mass-shell condition (32). We notice that this formula is invariant under the simultaneous transformations $R \to \frac{\alpha'}{R}$ and $n \to \omega$. That is, a string moving along a circle of radius R exhibits the same particle spectrum as a string moving along a circle of radius $\frac{\alpha'}{R}$ if the winding and momentum numbers are swapped. A theory A is T-dual to another theory B if they are equivalent when A is compactified on a small space and B is compactified on a large space. It is clear that as $R \to 0$, the radius of the T-dual circle becomes very large, thus, the above case constitutes an example of T-duality. These relations state that the physics at $R < \sqrt{\alpha'}$ is equivalent to the physics at $R > \sqrt{\alpha'}$, implying that there exists a minimal scale (self-dual radius) $R = \sqrt{\alpha'}$ at which non-abelian enhancement occurs, explaining the choice at the end of 4.2.

T-duality transformations for more general (non-orthogonal) toroidal compactifications $\mathbb{R}^{(1,d)} \to \mathbb{R}^{(1,d-n)} \times T^n$ $(n > 1)$ are realized via actions of the discrete, finite group $O(n,n;\mathbb{Z}) \subset O(n,n;\mathbb{R})$ [30]. This is the T-duality group and is generated by matrices A satisfying:

$$A^T \begin{bmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{bmatrix} A = \begin{bmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{bmatrix} \tag{69}$$

where we differentiate between two distinct elements

$$\text{Inversions} \rightarrow A = \begin{bmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{bmatrix} \tag{70a}$$

$$\text{Shifts} \rightarrow A = \begin{bmatrix} \mathbb{I}_n & 0 \\ N_{IJ} & \mathbb{I}_n \end{bmatrix} \tag{70b}$$

with N_{IJ} an antisymmetric matrix of integers.

The space of moduli fields in the $(d-n)$ effective theory is parametrized by the n^2-parameters of the linear combination $G_{IJ} + B_{IJ} \in GL(n, \mathbb{R})$ (the metric of the internal space and the constant 2-form background). This space of matrices can be represented as a homogenous space (a topological space where the action of the symmetry group is transitive) and further, using a theorem by Helgason in [31], one can relate this homogeneous space to the coset space G/H for $H \subset G$ a closed subgroup of the symmetry group. The appropriate choice for the moduli space is therefore $M^0_{(n,n)} \simeq O(n,n;\mathbb{R})/[O(n;\mathbb{R}) \times O(n;\mathbb{R})]$ where one can check that $\dim(M^0_{(n,n)}) = \frac{n(2n-1)}{2} - n(n-1) = n^2$, so the dimensions match and that $O(n;\mathbb{R}) \times O(n;\mathbb{R}) \subset O(n,n;\mathbb{R})$ is a closed subgroup. However, points in the moduli space related by the action of T-duality belong to the same equivalence class, which implies the presence of a discrete gauge symmetry, and therefore, we must adapt the above choice to the *physical* moduli space, which is:

$$M_{(n,n)} \simeq M^0_{(n,n)}/O(n,n;\mathbb{Z}) \tag{71}$$

where, while $M^0_{(n,n)}$ is a smooth manifold, $M_{(n,n)}$ contains singularities corresponding to the fixed points of the T-duality group. The quotient ensures the equivalence of states related by an $O(n,n;\mathbb{Z})$ action.

8.2 T-duality for Open Strings

The T-duality transformations we have defined, for a theory compactified on S^1, exchange winding and Kaluza-Klein modes, however, open strings have no such winding modes. This is because, topogically, they correspond to a point. If we consider a spacetime-filling D25-brane, so that open strings[15] have Neumann BCs

[15]The bosonic mode expansion is given as $X(\tau,\sigma) = x + p\tau + i\sum_{n\neq 0} \frac{1}{n}\alpha_n e^{-in\tau} cos(n\sigma)$.

$\partial_\sigma X^\mu(\tau,\sigma)|_{\sigma=0,\pi} = 0$, compactifying $X^{25}(\tau,\sigma) = X_R(\tau-\sigma) + X_L(\tau+\sigma)$ along S^1 with radius R, a T-duality transformation gives [1]:

$$X^{25}(\tau,\sigma) \to \tilde{X}^{25}(\tau,\sigma) = X_L - X_R = \tilde{x} + p\sigma + \sum_{n\neq 0} \frac{1}{n}\alpha_n e^{-in\tau} sin(n\sigma) \tag{72}$$

This result shows that $\tilde{X}^{25}(\tau,\sigma)$ carries no momentum along the compactified dimension, so that the T-dual coordinate features only oscillatory motion. Indeed, the dual coordinate corresponds to an open string coordinate with Dirichlet BCs $(\partial_\tau X(\tau,\sigma)|_{\sigma=0,\pi} = 0)$. One must note that

$$\partial_\tau \tilde{X} = \partial_\sigma X \quad \text{and} \quad \partial_\tau X = \partial_\sigma \tilde{X} \tag{73}$$

so that T-duality interchanges Dirichlet and Neumann BCs. In fact, considering the end-points of (72) we see that $\tilde{X}(\tau,\pi) - \tilde{X}(\tau,0) = 2\pi n\frac{\alpha'}{R} = 2\pi n\tilde{R}$, the string ends are constrained to a fixed position in the compactified dimension by this Dirichlet BC, but are free to propagate along the other 24 spatial directions. The spacetime-filling D25-brane which wraps around the compact dimension X^{25} cannot exist in the T-dual space, since the BC along S^1 becomes Dirichlet, and therefore transforms it into a D24-brane at a specific point along the T-dual circle.

For more general T^n compactifications, the original Dp-brane turns into a D$(p-n)$-brane under T-duality [32]. Recalling that for $U(N)$ we introduced the Wilson line as per (51), then under $X^{25} \to X^{25} + 2\pi R$, the fields pick up a phase equal to $diag\{e^{-i\theta_1}, ..., e^{-i\theta_n}\}$. The open string momenta are shifted and become fractional as before, and since T-duality exchanges $n \leftrightarrow \omega$, then ω must be fractional too. This means that the string endpoints do not lie on the same hyperplane. One such open string with CP labels $|ij\rangle$ will pick up a phase $e^{i(\theta_j - \theta_i)}$, so that the momentum shift is

$$p_{ij} \to \frac{n}{R} + \frac{\theta_j - \theta_i}{2\pi R} \tag{74}$$

so then

$$\tilde{X}^{25}(\tau,\pi) - \tilde{X}^{25}(\tau,0) = (2\pi n + \theta_j - \theta_i)\tilde{R} \tag{75}$$

such that the string end-points in the dual theory are at $\tilde{X}^{25} = \theta_i\tilde{R} = 2\pi\alpha' A_{25,ii}$ and the D-brane position in the dual space is governed by the Wilson line of the Dp-brane wrapping the compactified dimension.

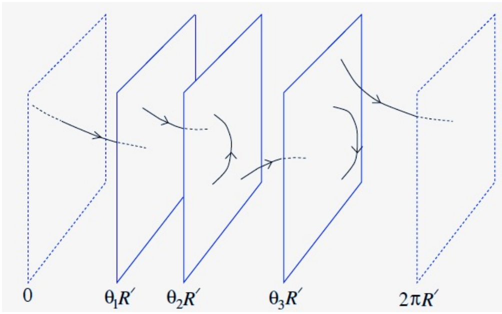

Figure 7: Positions of three D-branes along the T-dual circle, with $R' = \tilde{R}$ the dual radius. Dashed planes are periodically identified. Open strings can be attached to the same or different hyperplanes. From [16].

Generally, we can have N different hyperplanes at different positions if several coordinates (say $X^m = \{X^{25}, ..., X^{p+1}\}$) are periodic, as shown in Figure **7**. The separations between these is given by the difference in their positions (e.g. $\theta_i - \theta_j$) along the T-dual circle, which in turn, are purely determined by the Wilson lines of each D-brane in the original theory.

In the dual theory, compactifying a single coordinate, the effective $(D-1)$-dimensional mass is shifted into

$$M^2 = (p_{25})^2 + \frac{1}{\alpha'}(N-1) \to \left[\frac{(2\pi n + (\theta_i - \theta_j)\tilde{R}}{2\pi\alpha'}\right]^2 + \frac{1}{\alpha'}(N-1) \tag{76}$$

and thus we see that massless states correspond to states with $N = 1$, $n = 0$ and $\theta_i = \theta_j$ (string end-points lying on the same hyperplane). These states are interpreted as gauge fields (since $N = 1$) either on the D-brane or along the compact dimension, and describe fluctuations of the geometry of the hyperplane defined by $\theta_i\tilde{R}$. This is crucial for our later study of gauge-string dualities, since for the first time we see how gauge theories can be used to learn about the geometry and position of D-branes, and, conversely, D-branes and the string theories containing them, can be used to learn about gauge theories. Considering the case of a T^n compactification, with $U(N)$ symmetry, we can have the following setups [33], [34]:

- No coincident D-branes: This gives one massless vector per D-brane, the symmetry breaking pattern due to the Wilson line in the original theory is $U(N) \to U(1)^N$.
- k-coincident D-branes: Extra massless states arise since strings stretching between coincident D-branes automatically satisfy $\theta_i = \theta_j = ... = \theta_k$. We thus have k^2 massless vectors which form the adjoint representation of $U(k)$, which is the subgroup of $U(N)$ left unbroken by the Wilson line in the original theory.

- N-coincident D-branes: This recovers the original $U(N)$ group.

8.3 T-duality for Type II Theories

We have seen that the bosonic string compactified from $D \to (D-n)$ dimensions on some n-torus T^n has a discrete target space duality (T-duality) symmetry action associated with elements of the subgroup $O(n,n;\mathbb{Z}) \subset O(n,n;\mathbb{R})$. For closed strings compactified along a circle, these transformations are equivalent to the momenta transformations

$$p_L^d \to p_L^d \quad \text{and} \quad p_R^d \to -p_R^d \tag{77}$$

which can be easily seen by recalling that $p^d_{L/R} = \frac{n}{R} \pm \frac{\omega R}{\alpha'}$. This is generalized to a full parity transformation of the bosonic string fields, that is:

$$X^d(\tau,\sigma) = X_L^d(\tau+\sigma) + X_R^d(\tau-\sigma) \to \tilde{X}^d(\tau,\sigma) = X_L^d(\tau+\sigma) - X_R^d(\tau-\sigma) \tag{78}$$

which is itself a symmetry of the bosonic CFT. For a T-duality transformation along the compactified coordinate, $X^9(\tau,\sigma)$, of a Type IIA/IIB string, the action on the fermionic fields follows by WS superconformal invariance, so that $\tilde{\psi}_R^9(\tau-\sigma) \to -\tilde{\psi}_R^9(\tau-\sigma)$. For the R-sector, the right-moving 0-modes transform as $\tilde{b}_0^8 \pm i\tilde{b}_0^9 \to \tilde{b}_0^8 \mp i\tilde{b}_0^9$. Recalling the identification made in (13), and the relation for a-dimensional Gamma matrices, $\Gamma^{a\pm} = \frac{1}{2}(\Gamma^{2a} \pm i\Gamma^{2a+1})$, then we see that the chirality of right-moving spinors is flipped under T-duality, since the previous relation implies that for $a=4$, $\tilde{\Gamma}^{4\pm} \to \tilde{\Gamma}^{4\mp}$.

Hence, we have that the different sectors of the theory transform as

$$(R_+, R_\pm) \to (R_+, R_\mp) \quad \text{and} \quad (NS_+, R_\pm) \to (NS_+, R_\mp) \tag{79}$$

under T-duality, so that both chiral and non-chiral sectors become equivalent after compactification of a single spatial dimension along S^1. It is therefore clear that T-duality transforms $IIA \leftrightarrow IIB$ at the level of the different sectors. More precisely, we have that Type IIB theory compactified on S^1 with radius R is T-dual to Type IIA theory compactified on S^1 with radius $\tilde{R} = \frac{\alpha'}{R}$. The value of R corresponds to the classical value of a scalar field in 9-dimensions, and so the compactified IIA/IIB theories are really two distinct 10-dimensional limits of such value. The scalar field has a flat potential and both theories are said to smoothly connect as boundary points of the moduli space of this 9-dimensional theory after compactification [35].

To see the duality at the level of the field content more explicitly, we use the results from Section 6. For Type IIB theory compactified on S^1, applying a T-duality transformation to the compactified coordinate we obtain the following map for the D-branes:

$$D(2p+1)\text{-brane} \leftrightarrow D(2p)\text{-brane} \tag{80}$$

where the mapping is two-fold, since equivalently applying a T-duality transformation to a D-brane in the compactified space corresponding to the T-dual theory recovers the D-branes in Type IIB. Recalling that D-branes couple to $(p+1)$-form potentials as specified in Section 6.2, we can deduce the appropriate field content of the R-R sector of the dual theory, which is then matched to that of Type IIA. This transformation will also map the symmetries of one theory to another [36]. In this way, we relate both Type II theories perturbatively.

8.4 T-duality and O-planes

We now explore how compactifying an unoriented theory along S^1 and taking the limit $R \to 0$ leads to the emergence of orientifold planes. Considering the action of the WS parity operator Ω on the T-dual bosonic fields, we have that $\Omega \tilde{X}^d(z, \bar{z}) \to -\tilde{X}^d(\bar{z}, z)$ in complex coordinates. While in the original unoriented theory the coordinates exhibit Ω-invariance, the coordinates of the T-dual theory show a product of WS and spacetime parity invariance. The T-dual compact space in $d+1 = 26$ dimensions corresponds to the line segment $0 \le X^{25} \le \pi \tilde{R}$ (namely $S^1/\mathbb{Z}_2$), where the other *half* of the circle is present as the mirror image of the segment that is reflected by Ω. The end-points of the segment are fixed with respect to the action of Ω and correspond to 24-dimensional hyperplanes in the full space. At these, only one half of the usual states of the theory remain, since the rest are related by the action of Ω and therefore are projected out. Thus we have two O24-planes. More generally, if we have k-compact dimensions the T-dual compact space will be $T^{25-k}/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ reflection acts only along the compact directions. We will have 2^k Ω-invariant points, corresponding to O$(25-k)$-planes arranged as vertices of a hypercube [37]. At the O-planes, the theory remains unoriented, but locally in the bulk, the physics is oriented since the Ω-projection relates string states to their mirror image behind the fixed O-planes.

It is interesting to see that orientifold planes can emerge also as the compact dual spaces of toroidally compactified unoriented theories. Of course, as we have seen, more general On-planes independent on the construction given above can arise from considering more complex orientifold groups with non-trivial discrete isometries Σ, as in Section 5.

8.5 Type I' Theory

We now have the tools required to think about applying a T-duality transformation to Type I $SO(32) \simeq Spin(32)/\mathbb{Z}_2$ superstring theory, which contains 16 D9-branes and their mirror images under the symmetry of a spacetime-filling O9-plane. Consider Type I theory in an $\mathbb{R}^{1,8} \times S^1$ background, where S^1 has compactification radius

R. We have shown that Type I theory is constructed as an orientifold projection of Type IIB, and also that Type IIA/IIB are T-dual theories. Hence, at least heuristically, we expect the T-dual Type I theory (Type I') to be some sort of orientifold projection of Type IIA theory compactified on S^1 with dual radius $\tilde{R} = \frac{\alpha'}{R}$.

To check this educated guess, firstly note that gauging the Ω symmetry gives an orientifold projection of the compact dual space, which topologically is $S^1/\mathbb{Z}_2$. The $\mathbb{Z}_2$ here should be regarded as a projection by the product $\Omega \times \mathcal{I}$, where $\mathcal{I}$ denotes spatial reflections (i.e $\mathcal{I}\tilde{X} = -\tilde{X}$). The reason for this is that Type IIA theory is not invariant under Ω alone, since it is a non-chiral theory, and this is precisely compensated by the involution $\mathcal{I}$. The O9-plane in Type I theory wraps around the compact space, and after T-duality it is isomorphic to the closed interval $0 \leq \tilde{X} \leq \pi\tilde{R}$. The two fixed points under the action of $\Omega \times \mathcal{I}$ correspond to the presence of two O8-planes in Type I' theory and the 16 D9-branes remain untouched. By considering the bulk picture of the T-dual theory (away from the fixed O-planes and D-branes) the local physics is that of a closed[16] oriented theory by the same reasoning as before, that is, a type II theory. Due to this, the cross-cap states are forced to be localised near one of the O8-planes, where the physics is unoriented. The original Type I theory has an equal number of L/R-moving chiralities, thus T-dualising in one dimension makes the theory non-chiral, implying that at least in the bulk the T-dual theory is some projection of Type IIA theory. This projection is precisely the one obtained by gauging Ω. Hence, the statement of T-duality in Type I theory reduces to:

$$\text{Type I} \to \text{Type I'} \simeq \text{Type IIA}/[\Omega \times \mathcal{I}] \quad \text{on} \quad \mathbb{R}^{1,8} \times S^1 \tag{81}$$

Lastly, note that R-R charges must be conserved in the theory since the stack of 16 D9-branes and its image forms a BPS state. While we think of D-branes as sources for such charges, the O-planes act as R-R charge sinks, which carry the opposite BPS charge. More on this can be found in [38] and [39]. There exist other types of dualities such as S-duality (weak/strong coupling duality) and U-duality, which help to further unify the 5 consistent superstring theories in $D = 10$. We will briefly discuss S-duality later and omit U-duality to avoid digression.

9 $\mathcal{N} = 4$ Super Yang-Mills

Gauge theories, and more specifically Yang-Mills theories arise as the low-energy limit of the dynamics of a stack of D-branes and the massless open string modes that live on the D-brane worldvolume. Indeed, D-branes are the starting point to investigate dualities between non-gravitational theories and string theories (gauge/string

[16]Since the open strings are far away from the D-branes we regard them as effectively closed.

dualities) which we will get to later. However, for now, we will define $\mathcal{N} = 4$ Super Yang-Mills (SYM) in $D = 4$ dimensions, which we will need to explore the duality between this theory and Type IIB strings on an $AdS_5 \times S^5$ background as well as subsequent examples.

We consider an $SU(N)$ $\mathcal{N} = 4$ SYM theory in four dimensions. This theory exhibits maximal SUSY in 4-dimensions as it contains the maximum number of allowed non-zero supercharges. One can uniquely define the theory by specifying its gauge group and the field content in the allowed supermultiplets. In the case of $\mathcal{N} = 4$, the only possibility is the gauge multiplet, given by:

$$(A_\mu, \psi^a_\alpha, \phi^i) \tag{82}$$

where A_μ is a spin-1 gauge field and μ is a spacetime index, ψ^a_α are four Weyl fermions ($a = 1, ..., 4$ and $\alpha = 1, 2$) and ϕ^i ($i = 1, ..., 6$) are six real scalars. The gauge multiplet is in the adjoint representation of $SU(N)$. There exists a global symmetry (R-symmetry) realised via the automorphism group $SU(4)_R \simeq SO(6)_R$ under which the fields in the gauge multiplet transform respectively as a singlet, a vector and a rank-2 antisymmetric tensor, that is, $(A_\mu, \psi^a_\alpha, \phi^i) \simeq (\mathbf{1}, \mathbf{4}, \mathbf{6})$. Note the indices a and i denote R-symmetry indices.

The action in Euclidean signature reads [40]:

$$\begin{aligned} S = \frac{1}{g^2_{YM}} \int d^4x \mathrm{Tr}\Bigg(& \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{g^2_{YM}\theta_I}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} + D_\mu \phi^i D^\mu \phi^i + i\bar{\psi}\Gamma^\mu D_\mu \psi \\ & - \frac{1}{2}[\phi^i, \phi^j][\phi^i, \phi^j] + i\bar{\psi}\Gamma^i[\phi^i, \psi] \Bigg) \end{aligned} \tag{83}$$

where the trace is over the gauge group indices which are suppressed. Note we have expressed the Weyl fermions compactly as a single Majorana-Weyl spinor in 10-dimensions, with Γ^i and Γ^μ being Dirac matrices in the same dimensions. The term $\tilde{F} = *F$ denotes the usual Hodge dual field strength, g_{YM} refers to the Yang-Mills coupling constant and θ_I is the instanton angle.

The action (83) is invariant under SUSY transformations which are detailed in [41] and are omitted here. In 4 dimensions the coupling constant and instanton angle have mass-dimension $[g_{YM}] = [\theta_I] = 0$ so the theory exhibits classical scale invariance. It is also invariant under the conformal group in $d = 4$ dimensions, $SO(4, 2) \simeq SU(2, 2)$. In fact, the presence of SUSY and conformal symmetry imply that $\mathcal{N} = 4$ SYM is a SCFT with an enlarged symmetry group denoted $SU(2, 2|4)$ (known as the superconformal group in 4 dimensions[17]). The notation labels the

[17] For more details see Appendix 14.4 and 14.5.

maximal bosonic subgroup $SU(2,2) \times SU(4)_R \subset SU(2,2|4)$ which will be relevant when discussing the map between global symmetries in both sides of the correspondence.

Constructing the parameter $\tau = \frac{\theta_I}{2\pi} + \frac{i4\pi}{g_{YM}^2}$, a further symmetry of the action is revealed. Indeed, (83) is invariant under

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \text{where} \quad a, b, c, d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1 \tag{84}$$

which corresponds to an $SL(2,\mathbb{Z})$ transformation (the S-duality group). This symmetry will appear again in an AdS/CFT context when mapping global symmetries between the theories in both sides of the correspondence.

10 A Brief Account of $AdS_{(d+1)}$ Space

We now understand the SCFT on one side of the yet-to-be formulated gauge/string dualities. To fully understand the statement of correspondence, we must define the background in which the dual superstring theory is to be compactified. The cases of interest involve the product $AdS_5 \times S^5$ (and later $\mathbb{Z}_n$ orbifolds and orientifolds of this space). Hence, we must delve into the nature of $AdS_{(d+1)}$, that is, $(d+1)$-dimensional Anti-de Sitter space. In this section we will briefly review the idea of conformal boundaries in $AdS_{(d+1)}$ and also see that AdS_5 is the only compact space with the same isometries as the four-dimensional conformal group $SO(2,4)$.

10.1 Definition of $AdS_{(d+1)}$ and Isometries

We describe the *Minkowskian* $(d+1)$-dimensional AdS space by the hyperboloid

$$X_0^2 + X_{(d+1)}^2 - \sum_{i=1}^{d} X_i^2 = R^2 \tag{85}$$

embedded in the flat $(d+2)$-dimensional space with intrinsic metric

$$ds^2 = -dX_0^2 - dX_{(d+1)}^2 + \sum_{i=1}^{d} dX_i^2 \tag{86}$$

where R denotes the AdS radius, and one can think of the flat space (86) as having two timelike directions. AdS space is a maximally symmetric (i.e. homogeneous and isotropic) space of constant negative sacalar curvature, and hence has the maximum possible number of Killing vectors for its dimensions, that is, $\frac{(d+1)(d+2)}{2}$ in $(d+1)$-dimensions. The condition of maximal symmetry fully fixes the form of the Riemann tensor, which is given by

$$R_{abcd} = \frac{R}{n(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{87}$$

where in our case $n = d+1$, and g_{ab} denotes the metric tensor defined above. Furthermore, one can solve (85) by setting $X_0 = Rcosh(\rho)cos(\tau)$, $X_{d+1} = Rcosh(\rho)sin(\tau)$ and $X_i = Rsinh(\rho)\Omega_i$ such that $\sum_{i=1}^{d}\Omega_i^2 = 1$, so that then, in these coordinates, (86) becomes

$$ds^2 = -R^2(cosh^2(\rho)d\tau^2 - d\rho^2 - sinh^2(\rho)d\Omega^2) \tag{88}$$

The solution for X_0 covers the entire hyperboloid once with $\rho \geq 0$ and $0 \leq \tau \leq 2\pi$, hence the set of coordinates (ρ, τ, Ω_i) are known as the global coordinates of AdS. Topologically, the hyperboloid is isomorphic to $S^1 \times \mathbb{R}^d$, which one can see by taking the limit $\rho \to 0$ in (88) and keeping the leading terms.

One can equivalently express $AdS_{(d+1)}$ as the coset manifold $SO(2,d)/SO(2,d-1)$ [42]. From this, and the defining equations, it is clear that the isometry group of $AdS_{(d+1)}$ is given by $SO(2,d)$. The maximal compact subgroup of the isometry group is given by $SO(2) \times SO(d) \subset SO(2,d)$, where, using global AdS coordinates, we see that $SO(2)$ corresponds to a constant translation in τ and $SO(d)$ refers to S^{d-1} rotations with $\sum_{i=1}^{d}\Omega_i^2 = 1$.

Wrapping the closed S^1 parametrised by τ and taking the universal cover of the hyperboloid we obtain a causal spacetime. To investigate the causal structure of $AdS_{(d+1)}$ we let $sinh\rho = tan\theta$, where $0 \leq \theta < \frac{\pi}{2}$, so that the line element is expressed in the so-called conformal coordinates as

$$ds^2 = \frac{R^2}{cos^2\theta}(-d\tau^2 + d\theta^2 + sin^2\theta d\Omega^2) \tag{89}$$

Then, performing a Weyl re-scaling ($\Lambda = cos\theta$ such that $ds^2 \to \Lambda^2 ds^2$) one has that the conformal compactification of AdS space corresponds to a region with the same boundary structure as one half of that of the Einstein static Universe (covering one hemisphere of S^{d-1}) where $\theta = \frac{\pi}{2}$ denotes the boundary of S^{d-1}. From this, we extract the useful result which motivates the correspondence between AdS_{d+1}/CFT_d and its holographic nature. That is, the boundary of the conformal compactification of $AdS_{(d+1)}$ is equivalent to the conformal compactification of $\mathbb{R}^{1,d-1} \simeq \mathbb{R} \times S^{d-1}$. More formally, the statement is

$$\partial(AdS_{d+1}) \simeq \mathbb{R}^{1,d-1} \tag{90}$$

and we call spaces where conformal compactification leads to the same boundary structure as that of d-dimensional Minkowski spacetime *asymptotically AdS* spaces.

Lastly, we follow the results given in Appendix 14.4 regarding the conformal group in d-dimensions to conclude that for $d = 4$, the conformal group is given by $SO(2, 4)$, which is precisely the isometry group of AdS_5. Indeed, it turns out that AdS_5 is the only locally compact space that satisfies this property in 4 dimensions. We are now ready to explore the correspondence between AdS_5/CFT_4, namely, we are ready to understand how $\mathcal{N} = 4$ SYM in $d = 4$ dimensions is dual to Type IIB theory compactified on general backgrounds involving the product $AdS_5 \times S^5$.

11 Type IIB on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ Super Yang-Mills Correspondence

In these final sections, we will give further motivation for the AdS/CFT correspondence by following Maldacena's decoupling argument and will state the correspondence formally in its so-called strong form. We will also provide various examples of this family of string/gauge dualities and will show, somewhat explicitly, that the theories on each side of the correspondence are indeed dual. This will be done by first matching their global symmetries, and then briefly discussing the field/operator maps between them.

In this particular section we review the decoupling argument by following Maldacena's reasoning in [21] closely. This leads to a relation between Type IIB string theory compactified on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM in four spacetime dimensions via a duality. This will serve as motivation for further examples of compactification on other AdS_5 backgrounds (i.e. orbifolds and orientifolds of $AdS_5 \times S^5$).

11.1 D3-Branes as Hyperplanes with Open String Modes

Starting with Type IIB theory on $\mathbb{R}^{1,9}$, the decoupling argument is realized by considering a stack of N parallel, coincident D3-branes spanning a $(3+1)$-dimensional hyperplane of the background $\mathbb{R}^{1,9}$. In this configuration, the string theory will contain both open strings (ending on the D3-branes) and closed strings (corresponding to excitations of empty space). In the low-energy limit, $E \ll \frac{1}{l_s}$, only massless modes are excited, which correspond respectively to the 10-dim SUGRA multiplet for the closed excitations, and to an $\mathcal{N} = 4$ vector supermultiplet in the open case. The latter is due to the presence of the stack of D3-branes, noting that the low-energy effective action for the open modes indeed corresponds to that of an $\mathcal{N} = 4$ $U(N)$ SYM theory [43]. Before proceeding further, we recall that D3-branes can be interpreted both as dynamical hyperplanes with open string excitations as well as solutions to the SUGRA field equations (which deform the corresponding string background).

We follow the former interpretation for now, so that the full action for the massless modes takes the form

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}} \tag{91}$$

where S_{bulk} encodes the 10-dimensional Type IIB SUGRA action as well as massive modes of $\mathcal{O}(\alpha')$. S_{brane} contains the $\mathcal{N}=4$ SYM action plus higher order terms (e.g. $(\alpha')^2 Tr(F^4)$ [44]) and lastly, S_{int} describes interactions between the D3-brane modes and bulk modes, which scale as $g_s(\alpha')^2$. Heuristically, then, we see that in the $\alpha' \to 0$ limit (the low-energy limit) we have $S_{\text{int}} \to 0$ and $S_{\text{bulk}} + S_{\text{brane}} \to S_{\text{SUGRA}} + S_{\text{SYM}}$. Hence, in this regime, after the actions simplify, we have free 10-dim SUGRA in the bulk and $(3+1)$-dim $\mathcal{N}=4$ $U(N)$ SYM on the D3-brane stack as two independent decoupled theories. This is shown schematically in Figure **8**.

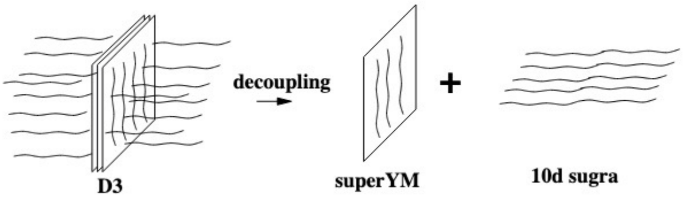

Figure 8: Stack of N coincident D3-branes viewed as hyperplanes where open strings end. The low-energy limit of this configuration decouples into free 10d SUGRA and $\mathcal{N}=4$ SYM gauge field theory. From [22].

11.2 D3-Branes as Solutions to SUGRA Field Equations

Now adopting the latter interpretation for the configuration of D3-branes (recalling also that they are charged BPS states), the explicit solution for the SUGRA field equations for the system is obtained by adapting the general solution given in (56), (57) and (58) for $p=3$. The D3-brane solution therefore takes the form:

$$H(\vec{y}) = 1 + \frac{L^4}{y^4} \tag{92a}$$

where y is the radial coordinate perpendicular to the brane stack, the scale factor is

$$L^4 = 4\pi(\alpha')^2 N g_s \tag{92b}$$

and the magnetic 5-form flux has field strength given by

$$F_5 = (1+*)dx^0 dx^1 dx^2 dx^3 dH(\vec{y})^{-1} \tag{92c}$$

As pointed out by Maldacena, the low-energy limit of the background described by the above equations, as seen by an observer at ∞, corresponds to a theory decoupling into free SUGRA in the bulk ($\mathbb{R}^{1,9}$) corresponding to the limit $y \gg L$, and another theory which we will specify by first considering the near-horizon region of the geometry, which corresponds to the limit $y \ll L$. For this limit, one finds that $H(\vec{y}) \to \frac{L^4}{y^4}$, and the line element (56) becomes

$$ds^2 \to \frac{y^2}{L^2}\left[-(dx^0)^2 + \sum_{i=1}^{3}(dx^i)^2\right] + \frac{L^2}{y^2}\left[dy^2 + y^2 d\Omega_5^2\right] \tag{93}$$

where we have written the six-dimensional metric $d\vec{y}^2 = dy^2 + y^2 d\Omega_5^2$ conveniently in spherical coordinates. Thus, one can identify the geometry in the limit close to the D3-brane stack as corresponding to the geometry of $AdS_5 \times S^5$. Indeed, we find that in the limit $y \gg L$, the system decouples into (10-dim SUGRA in $\mathbb{R}^{1,9}$) $\oplus$ (Type IIB on $AdS_5 \times S^5$).

Taking the low-energy limit of the theory both from the point of view of a field theory of open strings on the D3-brane stack, as well as from the supergravity interpretation, we obtain two independent decoupled theories, where in both cases one of them corresponds to supergravity in 10-dim Minkowski spacetime. Because the decoupled theories are obtained by considering the dual description of D-branes, it is natural to identify the remaining decoupled systems with each other [21]. In other words, one can make the conjecture that $\mathcal{N} = 4$ $U(N)$ *SYM theory in* $(3+1)$-*dimensions is dual to Type IIB superstring theory on* $AdS_5 \times S^5$.

We must point out that this alone doesn't constitute a proof for the correspondence due to various subtleties to do with how the near-horizon limit is precisely taken (for more on this see [21] and references therein). Furthermore, there is another subtlety with respect to the gauge group of the $\mathcal{N} = 4$ SYM dual theory. Firstly, it is useful to note that a $U(N)$ gauge theory is equivalent to an $SU(N)$ gauge theory times a free $U(1)$ vector multiplet (up to $\mathbb{Z}_N$ identifications), which one can understand from the isomorphism $U(N) \simeq [SU(N) \times U(1)]/\mathbb{Z}_N$. Aside from the asymptotically flat space ($y \gg L$) and near-horizon ($y \ll L$) excitations, there also exist zero-modes in the region connecting the bulk and the near-horizon regions. These zero-modes correspond to the aforementioned $U(1)$ degrees of freedom, which also include six scalars related to the centre of mass motion of the D3-branes. These modes are sometimes referred to as singletons (or doubletons) [45], and from the AdS point of view they live at the boundary of the space. Hence, the choice of whether or not they are to be included in the AdS theory directly affects whether the dual gauge theory has a $U(N)$ or an $SU(N)$ symmetry group.

Recall that $\partial(AdS_5) \simeq \mathbb{R}^{1,3}$, matching the background in which the dual $\mathcal{N} = 4$ SYM theory lives. Indeed, one can identify the stack of D-branes as living in the boundary of AdS_5 [46], which in turn contain the dual gauge theory. Hence, one can refer to the correspondence as holographic, in the sense that the dynamics of the Type IIB theory compactified on S^5 (which is effectively 5-dimensional) can be contained on a gauge theory which lives on the boundary of AdS_5 (which is four-dimensional).

11.3 Matching the Global Symmetries

After all these considerations, we make the statement that: *Type IIB string theory on $AdS_5 \times S^5$ (both of radius R) with 5-form integer flux N and string coupling $g_s \simeq$ (3+1)-dimensional $\mathcal{N} = 4$ Super-Yang-Mills theory with coupling g_{YM} and gauge group $SU(N)$ (or $U(N)$)* where we identify the parameters of both theories as:

$$L = [4\pi g_s N(\alpha')^2]^{\frac{1}{4}}, \quad 4\pi g_s = g_{YM}^2, \quad \langle \chi \rangle = \theta_I \tag{94}$$

where the first relation is clear from the arguments given previously, and to motivate the second one note that $g_{\text{closed}} \simeq g_{\text{open}}^2$, where open strings live on the D3-brane stack harbouring the SYM theory. The last relation relates the expectation value of an R-R scalar (axion) to the instanton angle. This is the strong form statement of the correspondence, where the *strength* of the statement refers to the conditions imposed on the parameters of the theories and also to the fact that it must hold for all values of the couplings $4\pi g_s = g_{YM}^2$ and for all N.

For the correspondence to hold, if both theories are to be dual, then it is clear that the global unbroken symmetries on both sides must be identical. We will explicitly check this now. Firstly, for $\mathcal{N} = 4$ SYM in $\mathbb{R}^{1,3}$ in (super)conformal phase[18], the continuous global symmetry takes the form of the superconformal group $SU(2,2|4)$. The actions of this supergroup are generated by a conformal symmetry, $SU(2,2) \simeq SO(2,4)$, for the specified background $\mathbb{R}^{1,3}$, an R-symmetry for $\mathcal{N} = 4$, realised via the isometry group $SU(4)_R \simeq SO(6)_R$ and then 32 Poincaré and conformal SUSYs which are generated by the supercharges Q^a_α and $S_{a\alpha}$ and their complex conjugates, respectively. The isometry group of AdS_5 is $SO(2,4)$, which corresponds to the conformal group in four spacetime dimensions. Hence, the low-energy limit of the decoupled theory on the D-brane stack is a CFT, which is seen from the fact that the near-horizon ($y \ll L$) geometry is that of AdS space. The maximal bosonic subgroup takes the form $SU(2,2) \times SU(4)_R \simeq SO(2,4) \times SO(6)_R \subset SU(2,2|4)$, which is precisely the isometry group of $AdS_5 \times S^5$, where $SO(6)$ generates S^5 rotations. The completion into the full supergroup occurs as follows: since the D3-branes are viewed as $\frac{1}{2}$-BPS states, the maximal bosonic subgroup preserves only 16

[18] This refers to the configuration where $\langle \phi^i \rangle = 0$ for all the real scalars in the gauge multiplet, which form the $\mathbf{6}$ of $SU(4)_R$. For this case, the global $SU(2,2|4)$ is unbroken.

out of the 32 SUSYs of the full supergroup [43]. However, the doubling of SUSYs occurs as a consequence of superconformal invariance in the near-horizon region (since the superconformal algebra has twice as many fermionic generators as the Poincaré superalgebra), hence one has the enlargement

$$SO(2,4) \times SO(6)_R \to SU(2,2|4) \tag{95}$$

so that globally, both theories have the same spacetime SUSY.

Furthermore, we also stated that $\mathcal{N} = 4$ SYM had an $SL(2,\mathbb{Z})$ symmetry group, corresponding to transformations of the parameter τ. Using the mapping given in (94), this parameter translates into $\tau \to \frac{i}{g_s} + \frac{\langle\chi\rangle}{2\pi}$ for the dual string theory. In this way, Type IIB theory also has an $SL(2,\mathbb{Z})$ self-duality symmetry which extends to the non-trivial $AdS_5 \times S^5$ background, since all fields defined on it are $SL(2,\mathbb{Z})$-invariant [47]. Hence, at the level of global symmetries, the strong form of the correspondence holds exactly.

11.4 Large N Limit and Further Checks

In this section, we motivate the existence of a family of more general gauge/string dualities to which our previous example belongs to, by considering the large N limit of gauge theories. We also present further *coupling-independent* checks for our stated duality, for which we must first reformulate the statement of correspondence in its weak form. The latter involves the extra conditions that [48]

$$g_{YM} \to 0, \quad N \to \infty \quad \text{and} \quad \lambda \equiv g_{YM}^2 N \to \infty \tag{96}$$

where N refers to an additional parameter of the $SU(N)$ gauge theory being examined, which is introduced so that one can obtain a meaningful perturbative expansion to learn about physics near the QCD scale (Λ_{QCD}) in terms of the parameter $1/N$. This is because SYM theories in four dimensions have no dimensionless parameters, and so, a priori, there isn't an obvious way to perform such an expansion. For the expansion to be valid, one must take the *t'Hooft* limit: $N \to \infty$. Thus, generally, to study the low-energy regime of gauge theories, which is strongly coupled, one needs to rely upon taking this so-called large N limit, for which the theory simplifies. Upon taking this limit, we also need to understand how to scale the coupling constant g_{YM}. For $\mathcal{N} = 4$ SYM (a SCFT), which is the case in hand, the t'Hooft coupling λ can be taken to ∞ consistently as $N \to \infty$, whereas for non-conformal theories, λ is usually kept fixed as $N \to \infty$.

Surprisingly, in the large N limit of non-abelian gauge theories, the perturbative expansion in terms of $1/N$ can be reorganised into a topological expansion in terms of closed oriented surfaces, to which the contributing field theory Feynman diagrams can be embedded into. One can write the perturbative expansion for any diagram in the field theory with fields in the adjoint representation of the gauge group as a double expansion of the form [21]

$$\sum_{g=0}^{\infty} N^{2-2g} \sum_{i=0}^{\infty} c_{g,i} \lambda^{i} = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \tag{97}$$

where in the first and third terms we are summing over genera. The label i on the second term is just a field label (e.g. flavour, spin) with $c_{g,i}$ being expansion coefficients/weights depending on the field label and the genus. The function $f_g(\lambda)$ represents a polynomial in λ which sums over all possible diagrams at a given genus g. We find that for $N \to \infty$, the double expansion is dominated by surfaces of maximal Euler character $\mathcal{X} = 2 - 2g$, or minimal genus g, which correspond to surfaces with the topology of a sphere and are associated to *planar*[19] diagrams in the field theory. These give a contribution of order N^2, with all other diagrams being suppressed by factors of $[\frac{1}{N^2}]^n$ for $n \in \mathbb{Z}$. The relevance of (97), is that it is precisely the same expansion as the one found in a perturbative theory of closed oriented strings, if the identification that $1/N \leftrightarrow g_s$ is made. This therefore motivates the fact that, in general, non-abelian gauge theories and string theories can be dual, where the duality would become more noticeable as $N \to \infty$, in the region where the string theory is weakly coupled. Hence, we see that the example discussed in previous sections is not accidental, and that more examples of gauge/string dualities should exist. We note that this doesn't constitute a formal proof for the existence of such dualities due to the lack of convergence (in general) of the perturbation theory analysis presented. We explore some more examples of dualities belonging to this family in the coming sections.

We now return to the examination of the stated correspondence between Type IIB strings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM in four dimensions. The AdS/CFT correspondence is a strong/weak coupling duality, in the sense that the large N limit relates the region where the t'Hooft gauge theory coupling, $\lambda = g_{YM}^2 N$, is weak (high-energy regime), to the large curvature region in the string theory. Because of this, a direct comparison between the n-point correlators on both sides of the correspondence is generally quite involved[20], since their computation is perturbative (in λ on the gauge theory side, and in $1/\sqrt{\lambda}$ on the string/gravity side). Hence, to test the duality, one must draw upon properties of both theories which are independent on the coupling constants, and thus on the t'Hooft coupling λ. Some of these properties are summarised below:

[19] These are field theory diagrams which can be drawn on the plane.

[20] There exists a prescription by Witten [49] which goes about this, but we omit its discussion.

- In Section 11.3, we saw that both theories have the same $SU(2,2|4)$ global symmetry, as well as a non-perturbative $SL(2,\mathbb{Z})$ self-duality symmetry acting on the parameter τ. Additional global $\mathbb{Z}_n$ isometries emerge when the theory is compactified on non-simply connected surfaces (see following sections). These global symmetries are resistant to (non-extremal) changes in the value of the coupling constants.
- We can also construct what is known as a *Field/Operator* map between the supergravity modes on AdS_5 and the locally gauge-invariant observables of the SYM theory. More specifically, this map relates the representations in which these fields/operators are in for each of the theories in the correspondence. This is constructed explicitly in section 3.1.2 of [21].
- A comparison of the spectrum of chiral differential (primary) operators on either side of the correspondence, which is also coupling-independent, can be performed.
- A priori, one could also relate points in the moduli spaces of each theory, since both are related by a duality. For the $SU(N)$ gauge theory, the moduli space is topologically of the form $\mathbb{R}^{6(N-1)}/S^N$, and it is parametrised by the eigenvalues of six $N \times N$ traceless matrices which commute. The problem arises for the AdS side, where it is not clear how to define such moduli space, which, as a coset superspace, could contain singularities at points of large curvature (in string units). However, in principle this would also be a valid check as we expect the dual theories to have topologically equivalent moduli spaces.

Many more qualitative checks exist, some of which are further outlined in [21], [42] and [48]. We assume that the ones presented here are sufficient to confidently assert that the correspondence between Type IIB strings on an $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ SYM, which constitutes an example of maximal SUSY, holds exactly.

12 Further AdS_5/CFT_4 Duality Examples

We have motivated the existence of a larger class of gauge/string dualities by considering the large N limit of gauge theories, whereby we now assume that a correspondence of similar nature to the one discussed can be formulated between any theory of quantum gravity (where the metric contains an AdS_5 factor) and a SCFT in four dimensions. The theory of quantum gravity must be an AdS theory for consistency, since we want the energy-momentum tensor operator to be mapped to the AdS_5 graviton under the field/operator map [48] and the string theory background must have an $SO(2,4)$ isometry. Thus, the general background need not be of the form $AdS_5 \times X$, where X represents a compact manifold. However, we restrict our discussion to cases of this form for simplicity.

A special class of $AdS_5 \times X$ backgrounds are those arising in the near-horizon limit of D-brane configurations, where one must determine the nature of the dual SCFT to which they correspond to. In the case discussed, we considered the decoupled (non-supergravity) low-energy field theory, and then took the limit $y \ll L$ to construct the sought after dual SCFT. However, if we consider transverse spaces to the D3-brane stack containing orbifold or orientifold singularities, then the corresponding amount of SUSY preserved by the dual SCFT is modified. This section is devoted to studying some of these cases.

12.1 Orbifolds of $AdS_5 \times S^5$

We can use string theory methods to derive the low-energy field theory corresponding to a configuration of D3-branes at an orbifold singularity. In particular, we consider one such configuration at the origin of the transverse space with topology $\mathbb{R}^4 \times \mathbb{R}^6/G$, where G denotes a discrete isometry $G \subset SO(6)_R \simeq SU(4)_R$ of the rotational symmetry of the compact manifold S^5. The dual SCFT is obtained after taking the near-horizon limit, which is of the form $AdS_5 \times S^5/G$, and can have different degrees of SUSY for non-maximally supersymmetric cases which depend on the nature of the isometry G, namely:

- $\mathcal{N} = 2$ SUSY if $G \subset SU(2) \subset SU(4)_R$
- $\mathcal{N} = 1$ SUSY if $G \subset SU(3) \subset SU(4)_R$

In this case, as previously mentioned in Section 5, the orbifolding by G involves more structure than just merely projecting out G-invariant states from the original theory. More specifically, for the dual gauge theory it means an enlargement of its symmetry group, and for the string theory it implies the presence of twisted and untwisted sectors. We distinguish between two cases regarding the quotient by the orbifold group:

- If the G-action has only the origin as its fixed point, then S^5/G is smooth.
- If the G-action produces a set of non-trivial fixed points, then S^5/G contains orbifold singularities.

We now examine briefly the nature of the twisted and untwisted sectors in the orbifold theory for each of the cases above. The *untwisted* sector is not sensitive to orbifold singularities, and contains the remaining states of the theory on $AdS_5 \times S^5$ after the orbifold quotient, as well as G-invariant SUGRA states. However, the *twisted* sector depends on the topology of the orbifold S^5/G. If the orbifold has singularities, one has light twisted states near such singular points, but if S^5/G is smooth, all twisted states are heavy in the sense that they involve strings which are stretched between points in the same equivalence class of the orbifold group G. Heavy twisted states decouple from the low-energy theory in the limit $\lambda \sim g_s N \to \infty$, so that the dual field theory features an additional global discrete symmetry G, under which these twisted states are charged and untwisted states remain neutral.

Taking the t'Hooft limit $N \to \infty$ and $\lambda \sim g_s N \to$ fixed, one finds that all correlators corresponding to operators in the untwisted sectors are equivalent up to a scaling by a factor proportional to $dim(G)$ [50]. We now consider specific examples of orbifolded D3-brane field theories preserving different amounts of SUSY.

12.1.1 $\mathcal{N} = 2$ Supersymmetric Theories

Consider N D3-branes at orbifold singularities of the form $\mathbb{R}^4/G$, where $G \subset SU(2)$ is a discrete group. The worldvolume theory is constructed by taking $N|G|$ D3-branes on the covering space, and then performing a projection on both the Chan-Paton labels and the fields on the worldvolume stack by G. It turns out that the possible forms of the group G fall into an ADE-type classification [21] (where the possible groups G are in one-to-one correspondence with simply-laced Dynkin diagrams). We expect CFT's to arise on the D3-branes when the representation of G acting on the C-P labels is the N-fold copy of the regular representation[21] of the symmetry group. This statement is equivalent to the orbifold quotient acting only on the S^5, leaving the AdS_5 factor untouched [51], so that the unbroken $SO(2,4)$ isometry of AdS_5 becomes the conformal symmetry of the dual gauge theory on the D3-branes. For the S^5, the orbifold quotient causes the breaking pattern of the R-symmetry to be of the form

$$SU(4)_R \to SU(2)_R \times U(1)_R \times G \tag{98}$$

where $SU(2)_R \times U(1)_R$ corresponds to the new R-symmetry group of the boundary SCFT and G becomes an additional discrete global symmetry of the dual SCFT.

We now specify the orbifold group as $G = \mathbb{Z}_k$, which in the ADE language corresponds to the A_{k-1} case. The D3-branes sit at a point in the transverse space $\mathbb{R}^6$, where the locus of fixed points after the orbifold action define a plane, which intersects S^5 along an S^1. Thus, while the untwisted states correspond to $\mathbb{Z}_k$ projections of the $AdS_5 \times S^5$ states, the light twisted states are contained in the fixed locus of the $\mathbb{Z}_k$ action, which topologically corresponds to an $AdS_5 \times S^1$ geometry.

The orbifold breaks spacetime SUSY from $\mathcal{N} = 4$ to $\mathcal{N} = 2$, and the low-energy field theory at non-fixed points in the moduli space features a $U(N)^k$ gauge theory

[21] This refers to the linear representation formed by a group action on itself via translation.

with bifundamental hypermultiplets in the

$$[N, \bar{N}, \mathbf{1}, ..., \mathbf{1}] \oplus [\mathbf{1}, N, \bar{N}, \mathbf{1}, ..., \mathbf{1}] \oplus ... \oplus [\bar{N}, \mathbf{1}, ..., \mathbf{1}, N] \tag{99}$$

representation. The theory with the matter spectrum given above indeed corresponds to an $\mathcal{N} = 2$ SCFT. If we take the near-horizon limit, we have that $U(N)^k \to SU(N)^k$ for the field theory, where the matter content remains the same since the off-diagonal $U(1)$ factors (corresponding to operators from twisted sectors) are IR-free and the diagonal $U(1)$ factors decouple and so can be omitted.

Indeed, the $\mathcal{N} = 2$ gauge theory with field content (99) will have, a priori, k distinct gauge coupling constants which can be tuned independently and define a k-complex-dimensional fixed hypersurface of CFT's. If all the couplings are tuned to the same value, then $g^2_{YM} \to g_s$ as we have seen before. For the string theory, one therefore expects k-complex parameters which can be tuned independently without altering the AdS_5 factor of the background. Some of these k-parameters include the dilaton [21], and the remaining $(k-1)$ parameters correspond to R-R/NS-NS sector Kalb-Ramond 2-form values (marginal operators of the SCFT) which, in turn, also correspond to the so-called blow-up modes[22] of the dual string theory, which for our example turns out to be Type IIB on $AdS_5 \times S^5/\mathbb{Z}_k$. A more detailed analysis of the mapping between marginal operators in the SCFT and fields in the dual string theory shows the duality:

$$\text{Type IIB on } AdS_5 \times S^5/\mathbb{Z}_k \simeq \mathcal{N} = 2 \; SU(N)^k \text{ gauge theory} \tag{100}$$

more explicitly, where the SCFT is constructed on the worldvolume of $N|k|$ D3-branes at an orbifold singularity of the form $\mathbb{R}^4/\mathbb{Z}_k$. The details of the mapping are omitted here, but can be followed in [51].

12.1.2 $\mathcal{N} = 1$ Supersymmetric Theories

We now consider the construction of SCFTs which preserve a smaller amount of SUSY. The general prescription is similar to the one outlined in the previous subsection, with some modifications. Here, we will build the SCFT by considering N D3-branes at orbifold singularities of the form $\mathbb{R}^6/G$, where now $G \subset SU(3)$ is a finite, discrete, abelian group. For the worldvolume theory, we will take $N|G|$ D3-branes on the covering space and then will project both the worldvolume fields and the C-P labels by G, requiring that the representation of G acting on the latter corresponds to the N-fold copy of the regular representation. The action of this representation then translates to an orthogonal action on the D3-branes on $AdS_5 \times S^5$

[22]Blow-up modes allow for the resolution of ADE-type singularities (orbifold fixed points) by replacing them with an n-sphere. For more on these, refer to [21] and references therein.

which, as before, preserves the AdS structure. The $SO(2,4)$ symmetry of AdS_5 is unbroken and constitutes the conformal symmetry of the SCFT on the D3-brane worldvolume. The R-symmetry on S^5 is broken by the orbifold action into:

$$SU(4)_R \to U(1)_R \times G \tag{101}$$

where, again, G corresponds to an additional global symmetry of the theory on the D3-branes (which is also a symmetry of the corresponding quiver diagram) and $U(1)_R$ denotes the R-symmetry of the boundary SCFT. In these cases, we expect the worldvolume theory on the D3-brane configuration to be an $\mathcal{N}=1$ SCFT.

We consider the simplest example, where we take $G=\mathbb{Z}_3$, and follow the general procedure we have outlined. The action of the $\mathbb{Z}_3$ orbifold on the complex coordinates X^i parametrizing the transverse space $\mathbb{R}^6 \simeq \mathbb{C}^3$ is:

$$X^i \to e^{i\frac{2\pi}{3}} X^i \tag{102}$$

for $i=1,2,3$. The $\mathbb{Z}_3$-action has the origin of $\mathbb{R}^6/\mathbb{Z}_3$ as a fixed point, and since S^5 has non-zero volume, then, $S^5/\mathbb{Z}_3$ is a smooth manifold and the orbifold action is free (so we expect no blow-up modes to resolve ADE singularities).

Thus, the low-energy limit of the theory constructed at the orbifold fixed point with $3N$ D3-branes in the covering space gives a $SU(N)^3$ gauge theory with chiral multiplets in the

$$3 \times \left\{ [N,\bar{N},\mathbf{1}] \oplus [\mathbf{1},N,\bar{N}] \oplus [\bar{N},\mathbf{1},N] \right\} \tag{103}$$

representation [52]. Indeed, the theory with such matter content is a SCFT with $\mathcal{N}=1$ SUSY. Taking the near-horizon limit with $\lambda \sim g_s N \to \infty$, the spectrum of untwisted states contains only the $\mathbb{Z}_3$ projection of SUGRA states of the $AdS_5 \times S^5$ theory. Because the compact space is smooth after orbifolding, twisted states are all heavy.

We let the three types of charged matter fields be U_i, V_i and W_i respectively. There is also a classical superpotential of the form

$$W = g_s \epsilon^{ijk} U_i V_j W_k \tag{104}$$

where we have set all the values of the superpotential couplings $h_{ijk} \equiv g_s$ equal to the string coupling. The theory features three distinct gauge couplings, which for the classical case are all tuned to the same value (and equal to g_s too) so that the mapping $g^2_{YM} \to g_s$ follows as usual. In this case, we have four independent parameters which can be changed without any effect on the AdS_5 factor of the string background.

For the quantum case, the procedure is more involved. We do not delve into the precise details here, but it can be shown that in the space spanned by the three gauge couplings and the coupling of the superpotential, there exists a one-dimensional fixed line of superconformal points (for more details, refer to section 3 of [52]). Omitting technical details, the fixed line of this space is parametrised by a parameter which is identified with the dilaton belonging to Type IIB theory on an $AdS_5 \times S^5/\mathbb{Z}_3$ background. Contrary to the $\mathcal{N} = 2$ case, here, there are no remaining $(k-1)$ parameters which define the analogous fixed hypersurface of CFT's described in the $\mathcal{N} = 2$ case. Matching AdS_5 fields to chiral operators of the gauge theory has some additional difficulties in this case (see [21] and [48]). The untwisted states can be matched easily via the field/operator map since they correspond to $\mathbb{Z}_3$ projections of the original SUGRA states and, if one regards the field theory on the D3-brane worldvolume as an $\mathcal{N} = 4$ $SU(3N)$ gauge theory quotiented by $\mathbb{Z}_3$, the states of this theory after orbifolding correspond to a $\mathbb{Z}_3$-projection of those in the original SCFT. For the twisted sector, states are identified with operators of the field theory which are non-trivially charged under the global $\mathbb{Z}_3$ symmetry. Thus, in this case, the AdS/CFT correspondence relates

$$\text{Type IIB on } AdS_5 \times S^5/\mathbb{Z}_3 \simeq \mathcal{N} = 1\ SU(N)^3 \text{ gauge theory} \tag{105}$$

where the dual SCFT is built on a stack of $3N$ D3-branes at an orbifold singularity of the transverse space of the form $\mathbb{R}^6/\mathbb{Z}_3 \simeq \mathbb{C}^3/\mathbb{Z}_3$.

12.1.3 Non-supersymmetric Theories

As one might imagine, there are also examples of non-supersymmetric orbifolds of $AdS_5 \times S^5$ where the constructed SCFT is an $\mathcal{N} = 0$ theory. These (S)CFT's are built as the low-energy limit of the theory living on a stack of D3-branes which sit at orbifold singularities of the transverse $\mathbb{R}^6$ taking the form $\mathbb{R}^6/G$. In this case, $G \subset SU(4)_R$ is again a discrete subgroup which only quotients the S^5 factor. The isometry of the AdS_5 piece becomes the conformal symmetry of the new CFT on the D3-brane stack's worldvolume upon orbifolding. The R-symmetry is broken by the action of G so that

$$SU(4)_R \to G \tag{106}$$

constitutes a new global symmetry of the boundary CFT.

The example we consider here is for $G = \mathbb{Z}_5$, where this time the orbifold acts only on two of the three complex coordinates parametrising the transverse space to the

D3-branes. Explicitly, the action is of the form:

$$(X^1, X^2, X^3) \to (e^{i\frac{2\pi}{5}} X^1, e^{i\frac{6\pi}{5}} X^2, X^3) \qquad (107)$$

This projection gives an $\mathcal{N} = 0$ gauge theory with $SU(N)^5$ gauge symmetry. The matter content for such theory is summarised below, where we consider the fields emerging from the orbifolding of each X^i [52].

- $X^3 \to X^3 \Rightarrow 1$ complex scalar ϕ_j in the adjoint representation for each $SU(N_j)$ (note the index j labels each of the $SU(N)$ groups) where we omit the trivial representation (the **1**s in the multiplet) with respect to the other four $SU(N)$s.
- $X^2 \to e^{i\frac{6\pi}{5}} X^2 \Rightarrow$ Complex scalars $\phi_{j,j+2}$ in the $[N_j, \bar{N}_{j+2}]$ representation.
- $X^1 \to e^{i\frac{2\pi}{5}} X^1 \Rightarrow$ Complex scalars $\phi_{j,j+1}$ in the $[N_j, \bar{N}_{j+1}]$ representation.

The spectrum also features fermions, which are given by $\psi_{i,i\pm1}$ and $\psi_{i,i\pm2}$, in the $[N_i, \bar{N}_{i+1}]$ and $[N_i, \bar{N}_{i+2}]$ representations respectively. The complex conjugates of the bifundamental fields are also to be included, and their corresponding representations are obtained by complex conjugation. It is clear that the spectrum of this theory is non-supersymmetric.

Similarly to the $\mathcal{N} = 2$ case, the five distinct gauge couplings define a 5-complex-dimensional fixed hypersurface of CFT's. These five expected independent parameters on the string theory are to be matched to Kaluza-Klein supergravity modes on $AdS_5 \times S^5/\mathbb{Z}_5$. However, in this case, the correspondence between primary operators and KK SUGRA modes is not one-to-one [51], and the duality

$$\text{Type IIB on } AdS_5 \times S^5/\mathbb{Z}_5 \simeq \mathcal{N} = 0 \; SU(N)^5 \text{ gauge theory} \qquad (108)$$

with the dual CFT constructed on $5N$ D3-branes at orbifold singularities of the form $\mathbb{R}^6/\mathbb{Z}_5$ has only been checked so far to one-loop order in perturbation theory.

We have provided several examples of gauge/string dualities by constructing four-dimensional (S)CFT's on orbifolds of $AdS_5 \times S^5$ preserving different degrees of SUSY by using string theory methods. In particular, the cases of dualities with $\mathcal{N} = 4, 2, 1$ theories are exact, while the case of $\mathcal{N} = 0$ holds to one-loop.

12.2 Orientifolds of $AdS_5 \times S^5$

The construction of low-energy field theories on different configurations of D3-branes at orientifold planes follows rather similarly to the orbifold case, except that now there are no twisted sectors in the theory, which are projected out upon orientifolding. For completeness, we will briefly discuss the case of N parallel threebranes placed at an O3-plane in the near-horizon limit.

Following [21], without too many details, the above setup generates a four-dimensional $\mathcal{N}=4$ SCFT, where all 32 supercharges are preserved after orientifolding. Recall that in Section 7.3, we distinguished between two types of orientifold projections: the $\Omega = 1$ projection which gave rise to an $Sp(2N)$ gauge theory, and the $\Omega = -1$ projection, which implied a gauge theory with symmetry group $Spin(2N)/\mathbb{Z}_2$. In our case, the symmetry group (symplectic or orthogonal) of the dual $\mathcal{N}=4$ SCFT on the D-brane configuration placed at the orientifold singularity depends on the nature of the orientifold projection that is chosen (in the low-energy limit).

The corresponding dual theory which gives rise to these SCFTs in the near-horizon limit of D-brane configurations is precisely Type IIB string theory on an $AdS_5 \times \mathbb{RP}^5$ background, where $\mathbb{RP}^5 \simeq S^5/\mathbb{Z}_2$ defines the five-dimensional real projective plane formed after introducing an antipodal identification on points of the five-sphere. The $\mathbb{Z}_2$ action leaves only the origin of $\mathbb{R}^6$ (into which S^5 is embedded) as its fixed point, hence $\mathbb{RP}^5$ defines a smooth manifold. The orientifold nature of $AdS_5 \times \mathbb{RP}^5$ is shown by the orientation reversal of the world-sheet upon going around a non-contractible cycle in the target space.

In the orbifold examples considered previously, the string theory topological expansion (in the t'Hooft limit) was in powers of $1/N^2$ as it only included contributions from closed orientable surfaces. However, after orientifolding, the new string theory background allows to sum over non-orientable closed surfaces, which in turn imply the presence of odd powers of $1/N$ in the perturbative expansion. Hence, the dual SCFT will feature Feynman diagrams which admit embeddings into non-orientable Riemann surfaces (with cross-caps) that contribute to the string theory perturbative expansion.

Using this fact, another way to differentiate between the distinct gauge theories that can be produced from orientifolding is given by presenting a useful result shown in [53], which shows the equivalence between $Sp(2N)$ and $SO(-2N)$ gauge theories after taking $N \to -N$ while keeping $\lambda = g_{YM}^2 N$ fixed. Applying this transformation, one finds that matter multiplets in representations given by Young Tableaux M_i for the $Sp(2N)$ theory correspond to the matter fields in the $SO(-2N)$ theory with representations given by the transposed Young Tableaux $(M_i)^T$. Furthermore, all gauge-invariant observables can be shown to coincide. For the given limit, the induced effect of taking $N \to -N$ on the perturbative expansions of the theories inverts the sign of the contribution of field theory diagrams (world-sheets) with $\mathbb{RP}^2$ topology, that is, diagrams with an odd number of cross-caps [54]. The sign difference of $\mathbb{RP}^2$ contributions at the level of the Feynman diagrams corresponds precisely with the main feature differentiating between $SO(2N)$ and $Sp(2N)$ gauge theories for large N. To see this, note that a diagram corresponding to a Riemann surface of genus g with c cross-cap insertions (copies of $\mathbb{RP}^2$) and no boundaries is of order N^{2-2g-c} as per (64) and (97). The effect of taking $N \to -N$ is precisely to include a factor of $(-1)^c$ in the expansion, which reverses the sign of the contribution of the diagram for odd c.

The corresponding effect in the dual string theory on $AdS_5 \times \mathbb{RP}^5$ was found in [55], and translates to the implementation of a *discrete torsion* on the real projective plane. These torsions effectively amount to phenomena which model the B-field on the string theory over an orbifold spacetime. To better grasp this idea, we note an important fact about the Type IIB O3-plane. There exists a supersymmetric O3-plane which is $SL(2,\mathbb{Z})$-invariant, so that, upon orientifolding one can obtain an $SL(2,\mathbb{Z})$-invariant configuration of D3-branes on $\mathbb{R}^4 \times \mathbb{R}^6/\mathbb{Z}_2$. This, in turn, allows to construct an $SL(2,\mathbb{Z})$-invariant compactification on $AdS_5 \times \mathbb{RP}^5$. We assume this choice. To understand the effect of turning on this discrete torsion, we need to see how the 2-forms (B-fields) of the $SL(2,\mathbb{Z})$ theory transform upon orientifolding. We denote these B_{NS-NS} and B_{R-R}. Recall that the orientifold projection reverses the orientation of the WS by exchanging L/R-movers, hence B_{NS-NS} transforms as a 2-form with an extra negative sign due to the $\mathbb{Z}_2$ action induced by the orientifold, it is a twisted two-form. Since the orientifolding is $SL(2,\mathbb{Z})$-invariant, B_{R-R} is also a twisted two-form, since it is related to B_{NS-NS} by the action of $SL(2,\mathbb{Z})$.

Remarkably, this achieves the same effect as the $N \to -N$ operation on the SCFT, in that it reverses the contribution of surfaces with odd number of cross-cap insertions to the perturbative expansion of the string theory. Thus, the above setup leads to four different possible superstring theories on $AdS_5 \times \mathbb{RP}^5$, corresponding to the choices of zero or non-zero discrete torsion for both B_{NS-NS} and B_{R-R} (parametrised by θ_{NS} and θ_R respectively). These in turn, correspond to different dual SCFTs as pointed out in [55]. We state the relevant results for our discussion here:

- String theories with trivial discrete torsion in both sectors (i.e. $\theta_{NS} = \theta_R = 0$) are dual to $\mathcal{N} = 4$ $SO(2N)$ gauge theories which are also $SL(2,\mathbb{Z})$-invariant (self-dual).
- Theories with $\theta_{NS} = 0$ and $\theta_R \neq 0$ are dual to $\mathcal{N} = 2$ $SO(2N+1)$ gauge theories.
- The remaining cases with non-trivial discrete torsion in the NS-NS sector ($\theta_{NS} \neq 0$) correspond to $\mathcal{N} = 2$ $Sp(2N)$ SCFTs.

Further checks of these dualities between SCFTs and the supersymmetric orientifold theories of IIB strings on $AdS_5 \times \mathbb{RP}^5$ can be performed by matching chiral (primary) operators to SUGRA fields, which, as before, we do not detail. It is interesting to note, however, that in the case of trivial discrete torsion in both sectors, $SO(2N)$, there exists an additional chiral superfield known as the *Pfaffian*. This operator is matched to the AdS_5 field corresponding to a D3-brane wrapped around a 3-cycle in $\mathbb{RP}^5$, where the wrapping is only possible when no discrete torsion is turned on [56]. This is consistent with the fact that Pfaffian objects only appear in $SO(2N)$ theories, but not in $SO(2N+1)$ or $Sp(2N)$ SCFTs.

We have outlined more examples of gauge/string dualities by considering orientifold projections of Type IIB theories on $AdS_5 \times S^5$. The resulting dual SCFTs exhibit varying degrees of SUSY too, which depend on the nature and amount of discrete torsion chosen for each of the sectors in the superstring theory.

13 Concluding Remarks

The power of dualities is unquestionable in the field of string theory, since it has allowed us to understand the behaviour of theories in regimes where perturbation theory fails to be valid. This dissertation aims to be a self-contained starting point towards the understanding of different examples of gauge/string dualities between Type IIB string theories on different backgrounds of AdS_5 and super Yang-Mills gauge theories of varying degree of SUSY (different $\mathcal{N}$). For this, we have presented the reader with a pedagogical review of the tools and concepts required, starting with superstrings and compactification, then motivating how to construct gauge theories on stacks of parallel D-branes and finally introducing examples of dualities of the type AdS_5/CFT_4.

We now discuss possible extensions of the work presented in the latter sections of the text. Firstly, note that the methods presented in Section 12.1 can be equivalently used to examine other families of dualities, such as 6-dimensional SCFTs which can be constructed from different orbifolds of $AdS_7 \times S^4$ (i.e. correspondences of the type AdS_7/CFT_6) or 3-dimensional SCFTs, which can be built from $AdS_4 \times S^7$ orbifolds (i.e. correspondences of the type AdS_4/CFT_3) following the general prescription given. Within this section, one can also explore the consequences of the non-injective mapping between primary operators and KK supergravity modes for the non-supersymmetric $\mathbb{Z}_5$-orbifold, and try to understand the consequences of this mismatch between operators of the gauge theory and SUGRA states in terms of the correspondence holding to different orders in perturbation theory.

For Section 12.2, it would be interesting to consider conifold theories, arising from more general types of compactification spaces with Calabi-Yau manifolds. These cases test the AdS/CFT correspondence, where the dual SCFT has reduced degree of SUSY, in a more general setting (with less symmetry). Here, the background of the string theory being examined is of the form $AdS_5 \times M_5$, where M_5 is a five-dimensional compact space that is not locally isomorphic to S^5. The problem arises when trying to perform the equivalent direct analysis of the field theory as we have been doing in our examples, which is not possible for these cases. However, other methods can be used to test the correspondence to some extent (see [21]).

Lastly, one can consider the dual SCFT description of the orientifold theory obtained by orbifolding transversely to a toroidal compact space by $\mathbb{Z}_k$. In particular, we can use string theory methods to see what consistency conditions are required for anomaly cancellation and how these affect the resulting field content and gauge symmetry group of the corresponding dual SCFTs. In [56], it is implied that the precise nature of the SCFT is a result of the choice of complex structure on the compactified torii. Hence, it would be interesting to see whether the choice of complex structure for the orbifold examples introduced in this Section 12.1 (by considering e.g. $\mathbb{R}^4/G \simeq \mathbb{C}^2/G$) is a more fundamental indicator of the properties of its dual gauge theory description.

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14 Appendices

14.1 Super-Virasoro Algebra

This refers to the supersymmetric extension of the Virasoro algebra (the unique central extension of the complexified Lie Algebra generated by elements of Diff(S^1)) to a Lie superalgebra with $\mathcal{N} = 1$ SUSY by including $\mathbb{Z}_2$-grading. Its generators obey

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2 - 2\phi)\delta_{m+n,0} \tag{109a}$$

$$[L_m, G_r] = (\frac{m}{2} - r)G_{m+r} \tag{109b}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2 - 2\phi)\delta_{r+s,0} \tag{109c}$$

where $\phi = 0$ for the R-sector and $\phi = \frac{1}{2}$ for the NS-sector. The central charge is

$$c = \frac{3}{2}D = (1 + \frac{1}{2})D \tag{110}$$

where the bosonic conformal field theory contributes the D and the fermionic CFT contributes the extra factor of $\frac{D}{2}$. Note further that in L_m, $m \in \mathbb{Z}$ always but for G_r, r is an integer in the R-sector and a half-integer in the NS-sector.

14.2 Fierz Decomposition

To decompose spinor bilinears, we construct them of the form $\bar{\xi}\Gamma^{[\mu_1 \cdots \mu_p]}\psi$, where $\bar{\xi}$ is a dual spinor, for suitable p, such that they transform as antisymmetric tensors of $SO(8)$. For a theory in $d = 2k+2$ dimensions with Weyl spinors (2^k) (chiral) and $(2^k)'$ (antichiral), the possible bilinears decompose as [4]

$$(2^k) \otimes (2^k) = \begin{cases} [1] \oplus [3] \oplus ... \oplus [k+1]_+ & \text{for } k\text{-even} \\ [0] \oplus [2] \oplus ... \oplus [k+1]_+ & \text{for } k\text{-odd} \end{cases} \tag{111}$$

$$(2^k) \otimes (2^k)' = \begin{cases} [0] \oplus [2] \oplus ... \oplus [k] & \text{for } k\text{-even} \\ [1] \oplus [3] \oplus ... \oplus [k] & \text{for } k\text{-odd} \end{cases} \tag{112}$$

into direct sums of n-forms. $[n]_+$ corresponds to the self-dual part of the form with respect to the Hodge star $(*)$ operator. Also for reference, an n-form in D-dimensions has $\frac{D!}{r!(D-r)!}$ degrees of freedom.

14.3 Aside on CFT

A *primary field* is a field which under a local conformal transformation $z \to z' = f(z)$, transforms as

$$\phi(z,\bar{z}) \to \phi'(z',\bar{z}') = \left(\frac{\partial f}{\partial z}\right)^{-h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}) \tag{113}$$

where the conformal weights, $(h, \bar{h})$, specify the transformation properties of the fields under the map.

A *quasi-primary field* satisfies condition (113) for $f(z) \in PSL(2,\mathbb{C})$, the automorphism group of the Riemann sphere.

For a general QFT, the *operator product expansion* (OPE), in the case of two operators, is an approximate expansion of the product $\mathcal{O}_i(x_i)\mathcal{O}_j(x_j)$ given by

$$\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) \simeq \sum_k C^k_{ij}(|x_i - x_j|)\mathcal{O}_k(x_k) \tag{114}$$

which holds in the limit $x_i - x_j \to 0$.

To define the RNS theory on a sphere, we perform a Wick rotation and introduce new complex coordinates $w = \tau - i\sigma$ and $\bar{w} = \tau + i\sigma$. This allows to define the periodic coordinate $z = e^{\frac{2\pi}{l}}w$, so that the RNS primary fields can be expanded in terms of a Laurent series as [1]

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+\phi} \frac{b^\mu_r}{z^{r+\frac{1}{2}}} \tag{115a}$$

$$\tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+\phi} \frac{\tilde{b}^\mu_r}{\bar{z}^{r+\frac{1}{2}}} \tag{115b}$$

where again, $\phi = 0, \frac{1}{2}$ for the R and NS sectors respectively. The extra factor of $\frac{1}{2}$ in the denominator exponent is due to the conformal weight. One notices that for the R-sector, $r + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$, which implies that the R sector features a square-root branch cut. This isn't the case for the NS sector since here, $r + \frac{1}{2} \in \mathbb{Z}$ due to the non-zero value of ϕ.

In our case, for the right-moving (holomorphic) RNS fields, the OPE takes the form

$$\psi^\mu(z_1)\psi^\nu(z_2) \simeq \frac{\eta^{\mu\nu}}{z_1 - z_2} \tag{116}$$

which follows identically for the left-moving (antiholomorphic) sector.

14.4 Conformal Group

This refers to the group of transformations preserving the metric up to an overall conformal (scale) factor:

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x) \tag{117}$$

In $\mathbb{R}^{1,d-1}$ the conformal group is generated by linear combinations of the generators of the Poincaré group as well as by dilatations $D : x^\mu \to \lambda x^\mu$ and special conformal transformations $K^\mu : \frac{x^\mu}{\vec{x}^2} \to \frac{x^\mu}{\vec{x}^2} - b^\mu$, where b^μ is a constant d-vector. Following the conformal compactification procedure, one can turn

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2_{d-2} \to d\tilde{s}^2 = -d\tau^2 + d\theta^2 + sin^2\theta d\Omega^2_{d-2} \tag{118}$$

that is, $\mathbb{R}^{1,d-1} \to \mathbb{R} \times S^{d-1}$, which corresponds to the maximal extension of the uncompactified space.

For $\mathbb{R}^{1,d-1}$, the global conformal group is $SO(2,d)$, which corresponds to an enlargement of $SO(1,d-1)$ (the Lorentz group) [1]. The generators of $SO(2,d)$ are given by:

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{\mu,d} = \frac{1}{2}(K_\mu - P_\mu), \quad J_{\mu,d+1} = \frac{1}{2}(K_\mu + P_\mu), \quad J_{d+1,d} = D \tag{119}$$

where $M_{\mu\nu}$ and P_μ generate the Poincaré group. These obey

$$[J_{ab}, J_{cd}] = -i(g_{ac}J_{bd} - g_{bc}J_{ad} + g_{bd}J_{ac} - g_{ad}J_{bc}) \tag{120}$$

for $g_{ab} = diag(-1, +1..., +1)$. After a Wick roatation, $\mathbb{R}^{1,d-1} \to \mathbb{R}^d$, hence field theories on $\mathbb{R}^d$ and S^d are equivalent since there exists a bijective conformal map relating the two.

Global time translations on $\mathbb{R}^{1,d-1} \simeq \mathbb{R} \times S^{d-1}$ are generated by the Hamiltonian

$$H = \frac{1}{2}(P_0 + K_0) = J_{0,d+1} \tag{121}$$

and its existence implies that a correlation function on a CFT on $\mathbb{R}^{1,d-1}$ can be analytically continued to $\mathbb{R} \times S^{d-1}$. One can deduce this by noting that the isometries of AdS space are in one-to-one correspondence with generators of the conformal group [42].

Introducing gradation on the conformal algebra, the symmetry group is enlarged due to the fact that SUSY and special conformal transformations do not commute. One then has, that for $d = 4$

$$SO(2,4) \to SU(2,2|\mathcal{N}) \tag{122}$$

where $SU(2,2|\mathcal{N})$ is the superconformal group (with $\mathcal{N}$ the number of supercharges) and its form can be seen from the isomorphism $\mathfrak{su}(2,2) \simeq \mathfrak{so}(2,4)$ at the level of the Lie algebras.

14.5 Super-Poincaré Algebra and its Representations

One can enlarge the usual Poincaré Algebra (see e.g. [2]) by introducing a graded structure ($\mathbb{Z}_2$-grading) as well as by adding spinor supercharges Q^i_α, where α is a spinor index and $i = 1, ..., \mathcal{N}$ describes the degree of SUSY. These supercharges transform under the spinor representation of the Lorentz group and commute with translations. In $d = 4$, they also obey:

$$\{Q^i_\alpha, \bar{Q}_{\dot{\beta}j}\} = 2\delta^i_j \sigma^\mu_{\alpha\dot{\beta}} P_\mu \tag{123}$$

and

$$\{Q^i_\alpha, Q^j_\beta\} = 2\epsilon_{\alpha\beta} Z^{ij} \tag{124}$$

where P_μ are the generators of translations, $Z^{ij} = -Z^{ji}$ is the antisymmetric central charge which commutes with all generators, σ^μ are Pauli matrices and $\bar{Q}_{\dot{\beta}j} = (Q^\dagger_{\beta j})$. For $\mathcal{N} = 1$ the central charges vanish, but for $\mathcal{N} > 1$ this need not be the case.

The SUSY algebra has an automorphism symmetry group called R-symmetry which takes the form:

$$\begin{cases} U(1)_R & \text{for } \mathcal{N} = 1 \\ SU(\mathcal{N})_R & \text{for } \mathcal{N} > 1 \end{cases} \tag{125}$$

which in both cases corresponds to an (Abelian/non-Abelian) rotational symmetry of the supercharges. For $\mathcal{N} = 4$, the R-symmetry group corresponds to $SU(4)_R \simeq SO(6)_R$, the isometry group of S^5.

Requiring unitarity, choosing $P_\mu = (M, 0, 0, 0)$ and bringing Z^{ij} into block-diagonal form, one arrives to the BPS-bound on the mass:

$$M \geq |Z_{\bar{a}}| \tag{126}$$

Saturation of the bound implies that one or more of the supercharges vanish, leading to a shorter BPS multiplet.

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