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Quantum Cosmology and Torsion

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Abstract

The quantum cosmology is a study of quantum theory of the universe. This is especially important when we try to understand the initial state and early stage of our universe. In this dissertation, we introduce the basic concept of the quantum cosmology. Two most important approaches to the boundary value problem are discussed. The relation between these two approaches and their relation to the Chern-Simons state are explored. Then we introduce some new development in this field. Among them, the idea of torsion and theory with torsion is discussed in most details. Several new models with torsion both in classical and quantum cosmology are explored.

Contents

Chapter 1

Introduction

1.1 Classical to quantum cosmology

The Einstein's theory of general relativity gives prediction of our universe to an outstanding precision. Together with the FRW metric constitutes the classical model of cosmology. However, this classical model does have its limitation, and it becomes obvious when we're trying to determine the initial state of the universe. Hence, a quantum approach of cosmology is needed. The quantum model of gravity is also very interesting from a particle physics perspective, as gravity is the only fundamental forces that have not yet been successfully quantized. Cosmology could provide a perfect and perhaps only laboratory for quantum gravity.

So what does a good quantum cosmology model need? It should give a wavefunction that describes the whole universe and this wavefunction need to be interpreted to give actual physical predictions. The prediction made should agree with the classical general relativity in the classical scale. It is worth mention here that the classical general relativity is assumed to be torsion free.

Early attempts to quantize gravity can date back to 1930s where Rosenfeld tried to apply quantum theory into gravitational field [2]. First milestone of this subject is the Wheeler-Dewitt equation.

To derive Wheeler-Dewitt equation, one has to discuss the following Lagrangian:

$$L = \int \alpha \gamma^{1/2} (K_{ij} K^{ij} - K^2 +^{(3)} R) d^3 x \tag{1.1}$$

where K_{ij} is the second fundamental form. This is the classical Lagrangian that is related to the dynamics of gravity. It should be noted that due to the fact that gravity does not vanish outside an arbitrary large distance other than in Minkowski spacetime, the second derivative in the Lagrangian affects the energy. The energy term is given by:

$$E_\infty = \int_\infty \alpha \gamma^{1/2} \gamma^{ij} (\gamma_{ik,j} - \gamma_{ij,k}) dS^k \tag{1.2}$$

This term is known as the canonical energy and this energy term makes gravity very hard to quantize.

One could now represent the metric in the form of

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{1.3}$$

Then one can write momenta conjugate to α, β, γ as:

$$\pi = \frac{\delta L}{\delta \alpha_{,0}} = 0 \tag{1.4}$$

$$\pi^i = \frac{\delta L}{\delta \beta_{i,0}} = 0 \tag{1.5}$$

$$\pi^{ij} = \frac{\delta L}{\delta \gamma_{ij,0}} = -\gamma^{1/2}(K^{ij} - \gamma^{ij} K) \tag{1.6}$$

Note as equation 1.4 and 1.5 hold for all time, $\frac{\partial \pi}{\partial x^0}$ and $\frac{\partial \pi^i}{\partial x^0}$ must be 0. Therefore, we have

$$\mathscr{H} = 0 \tag{1.7}$$

and

$$\mathscr{X}^i = 0 \tag{1.8}$$

This means that in a Ricci-flat universe, both intrinsic and extrinsic curvature vanish. Until this point, one have not yet quantise the gravity. Now one need to change the Poisson brackets to commutators and gives:

$$\pi \Psi = 0 \tag{1.9}$$

$$\pi^i \Psi = 0 \tag{1.10}$$

$$\mathscr{H} \Psi = 0 \tag{1.11}$$

$$\chi^i \Psi = 0 \tag{1.12}$$

Then one can conclude that:

$$H\Psi = 0, \;\; \Psi^\dagger H = 0 \tag{1.13}$$

This is known as Wheeler-DeWitt equation. Explicitly one have

$$H\Psi = [-G_{ijkl} \frac{\delta}{\delta_{ij}} \frac{\delta}{\delta_{kl}} - h^{1/2}({}^3R - 2\Omega) + H^{matter}]\Psi = 0 \tag{1.14}$$

with

$$G_{ijkl} = \frac{1}{2} h^{-1/2} (h_{il} h_{jl} + h_{il} h_{jk} - h_{ij} h_{lk}) \tag{1.15}$$

One should note that the physical laws are not changed with the coordinate system. Hence, the theory should be the same for any coordinate system. However, this is not easy to prove in the case of quantum gravity.[3] One way to achieve the invariance under coordinate transformation is through momentum constraint:

$$H_i\Psi = 2iD_j\frac{\delta\Psi}{\delta h_{ij}} + H_i^{matter}\Psi = 0 \tag{1.16}$$

If the configurations only differ by a coordinate transformations in the three surface, the wave function will be the same. If one change the coordinate by $x^i \to x^i - \xi^i$, one have:

$$\Psi[x^i - \xi^i] = \Psi[x^i] + \int d^3x D_{(i}\xi_{j)}\frac{\delta\Psi}{\delta h_{ij}} \tag{1.17}$$

The change in Ψ then can be written as:

$$\delta\Psi = -\int d^3x\xi_j D_i(\frac{\delta\Psi}{\delta h_{ij}}) = \frac{1}{2i}\int d^3x\xi_i H^1\Psi \tag{1.18}$$

Therefore, one can see that any wavefunction that obeys the momentum constraint will be invariant under coordinate transformation.

Theoretically, Ψ contains all the information about one universe. However, it is not easy to actually interpret and subsequently obtain information from the wave-function. Mathematically, Wheeler-DeWitt equation is a second order hyperbolic functional differential equation. Just like any other differential equations, one need boundary condition or initial condition to determine the explicit solution. Unlike other system, however, the boundary condition of a universe can cause confusion, as we assume that nothing is outside the universe and before universe.

Just like quantum field theory, path integral is a different approach towards this issue other than canonical quantization. One can write the wave function as:

$$\Psi = \sum_M \int Dg_{\mu\nu}D\Phi e^{-I} \tag{1.19}$$

The problem is the manifold of which it sums over is very hard to define in practice. One should note that more complicated model are allowed here. The one given here are only the simplest one. Explicitly one have:

$$\Psi = \int DN^\mu \int Dh_{ij}\Phi\delta[\dot{N}^\mu - \chi^\mu]\nabla_\chi \exp(-I[g_{\mu\nu}, \Phi]) \tag{1.20}$$

where ∇_χ is the Faddeev-Popov determinant. It can be shown that only if the path integral is constructed invariant can it fit the Wheeler-DeWitt equation. That is to say that this formulation will also be invariant under diffeomorphism.

The theoretical space where everything we discuss in is called superspace. However, as superspace has infinite dimensions, it is very hard to analyse. In classical cosmology, the trick to deal with this is to use the property of our universe i.e. it is homogeneous and isotropic at large scale. Due to that reason, one can first find a homogeneous and isotropic metric and then study the surrounding perturbation. The same trick can be used in quantum case, one can reduce the degree of freedom to a finite number and suspend the matter fields. This is call minisuperspace. An example is given as follows: consider a four metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(N^2 - N_i N^i)dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j \qquad (1.21)$$

Now one can take this metric to be homogeneous thus $N = N(t)$, and set N^i to be 0. Then one has:

$$ds^2 = -N^2(t)dt^2 + h_{ij}(x,t)dx^i dx^j \qquad (1.22)$$

Note here the three metric is homogeneous and described by finite number of functions of t. Some examples are

1)Roberston-Wakler metric:

$$h_{ij}(x,t)dx^i dx^j = a^2(t)dr^2 + b^2 d\Omega_2^2 \qquad (1.23)$$

2)Bianchi-type metrics:

$$h_{ij}(x,t)dx^i dx^j = a^2(t)(e^\beta)_{ij}\sigma^i\sigma^j \qquad (1.24)$$

where σ^i is the basis of one forms. However, there is still some difficulties that is special to quantum theory. When we are setting everything we do not want to 0, it violates the uncertainty principle. Thus, strictly speaking the minisuperspace model is not an approximation of the full theory but as a toy model that only shows part of the full theory that one want to study.

Now, let us go back to the metric that we are discussing. The Einstein action is :

$$S[h_{ij}, N, N^i] = \frac{m_p^2}{16\pi}\int dt d^3x N h^{1/2}[K_{ij}K^{ij} - k^2 +^3 R - 2\Lambda] \qquad (1.25)$$

Constrain it to minisuperspace gives:

$$S[q^\alpha(t), N(t)] = \int_0^1 dt N[\frac{1}{2N^2} f_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta - U(q)] \qquad (1.26)$$

This is not the only way to get the minisuperspace model. Another option is to consider a metric that is not homogeneous but is of strict types.

1.2 Path Integral

Another way to obtain the wavefunction is through path integral. One should note that the action is invariant under the transformation:

$$\delta_\eta q^\alpha = \eta(t)\{q^\alpha, H\},\ \delta_\eta p_\alpha = \eta(t)\{p_\alpha, H\},\ \delta_\eta N = \dot{\eta}(t) \qquad (1.27)$$

The change in action is then:

$$\delta S = [\eta(t)(p_\alpha \frac{\partial H}{\partial p_\alpha} - H)]_0^1 \qquad (1.28)$$

Hence, as long as $\eta(0) = 0 = \eta(1)$, the action will be unchanged under the transformation. This invariance can be broken by setting:

$$G \equiv \dot{N} - \chi(p_\alpha, q^\alpha, N) = 0 \tag{1.29}$$

where χ is a any function.

The path integral is then:

$$\Psi(q^{\alpha''}) = \int Dp_\alpha Dq^\alpha DN\delta[G]\Delta_G e^{iS[p,q,N]} \tag{1.30}$$

where Δ_G is the Faddeev-Popov measure. The path integral is independent of G because of the Faddeev-Popov measure. The boundary conditions are $q^\alpha(1) = q^{\alpha''}$ at t=1 with p_α and N free. Now let us consider the gauge $\dot{N} = 0$. One have $\Delta_G = constant$ and subsequently:

$$\Psi(q^{\alpha''}) = \int dN \int Dp_\alpha Dq^\alpha e^{iS} \tag{1.31}$$

One advantage of using this formula is that one can evaluate the wavefunction directly with it. First one need to rotate to Euclidean space-time with $\tau = it$. Hence, one have:

$$\Psi(q^{\alpha''}) = \int dN \int Dq^\alpha \exp(-I[q^\alpha(\tau), N]) \tag{1.32}$$

with

$$I[q^\alpha(\tau), N] = \int_0^1 d\tau N[\frac{1}{2N^2} f_{\alpha\beta}(q)\dot{q}^\alpha \dot{q}^\beta + U(q)] \tag{1.33}$$

One can call the $I[q^\alpha(\tau), N]$ as minisuperspace Euclidean action. The gravitation part of this action is indefinite. Hence, the kinetic term is indefinite and the potential $\int 2\Lambda -^3 R$ is not positive definite. Here we use the lowest order semi-classical approximation. It is similar to WKB approximation in Wheeler-DeWitt equation.

Now one can see providing that one is careful, the classical solution can be the full solution to the theory. The lowest order semi-classical approximation to the wave function is the lowest order wavefunction to the full theory.

The Wheeler-DeWitt equation is associated with a conserved current:

$$J = \frac{i}{2}(-\Psi\nabla\Psi^* + \Psi^*\nabla\Psi) \tag{1.34}$$

One have:

$$\nabla \cdot J = 0 \tag{1.35}$$

Similar to Klein-Gorden equation, the Wheeler-DeWitt equation can exert negative probabilities. One way to fix this is to use:

$$dP = |\Psi|^2 dV \tag{1.36}$$

as the measure. However, this measure cause another problem, that is, it causes confusion of the nature of time in quantum mechanics when trying to interpret the result.

As mentioned at the very beginning, the prediction of quantum cosmology must agree with classical cosmology in the classical regime. However, one might ask when one can regard a quantum system as classical. The common requirements are:

1) The wavefunction must peak at one or more classical result.

2) The wavefunction interference should be very small.

Such requirements are not only for quantum cosmology but also for other quantum mechanical systems too. In many other quantum mechanical system, one can achieve this by construct the coherent states. This is not very easy in quantum cosmology. The analogue wavefunction is:

$$\Psi(q^\alpha) = e^{i\phi(q^\alpha)} \exp(-f^2(q^\alpha)) \tag{1.37}$$

where $f(q^\alpha) = 0$. This function can only work at very simple case. It does not arise very naturally and boundary conditions are needed. One should note if a wavefunction can predict the classical solution, they usually peaked about an entire history and provide a notion of time. In the above case, the notion of time is provided by the affine parameter. In this sense, the notion of space-time are only derived concepts. The most common wavefunction in quantum cosmology are WKB wavefunction. Moreover, those which corresponds to classical space-time are oscillatory wavefunction. The reason is that wave-functions which can be regarded as predictions need to be peaked at certain configurations of coordinate and momenta so that one can derive the classical solution from that peak. Exponential function will not give a peak around certain configuration. One example of possible wave-functions is the Wigner function. It can be proven that a wavefunction with form e^{-I} will indicate there is no relation between momenta and coordinates and those with form e^{iS} gives:

$$p_\alpha = \frac{\partial S}{\partial q^\alpha} \tag{1.38}$$

This equation is equivalent to the first integral of the equations of motion and thus corresponds to a group of classical solutions. One should note that S is a solution to the Hamilton-Jacobi equation.

Now let us explicitly verify this using a one-dimensional case. Let us consider a generating function G_0

$$p = \frac{\partial G_0}{\partial q},\ \tilde{q} = \frac{\partial G_0}{\partial \tilde{p}} \tag{1.39}$$

Then a transformation can be defined as:

$$\tilde{\Psi}(\tilde{p}) = \int dq e^{iG(q,\tilde{p})} \Psi(q) \tag{1.40}$$

One should note that G is defined as:

$$G_0(q, \tilde{p}) = q\tilde{p} + S(q) \tag{1.41}$$

where

$$\tilde{p} = p - \frac{\partial S}{\partial q}, \ \tilde{q} = q \tag{1.42}$$

and Ψ is defined as:

$$\Psi(q) = e^{iS} \tag{1.43}$$

Combining these equations, one can see that wavefunction has the form:

$$\tilde{\Psi}(\tilde{p}) = \delta(\tilde{p}) + O(\tilde{p}^2) \tag{1.44}$$

Therefore it peaked at a configuration just as one needed.

Chapter 2

Basic formulation of Quantum Cosmology

Now we shall consider Lagrangian $\mathscr{L} = l_p^{-2}R+\frac{1}{2}(\partial_\mu\phi)^2-V(\phi)$. This is originally considered in Vilenkin's paper [4]. The wave function is then determined by the three-metric $h_{ij}$and the scalar fields ϕ. In a close universe the wave function would obey:

$$H^i\psi = 0, \tag{2.1}$$

$$H^0\psi = 0, \tag{2.2}$$

where

$$H^i = 2iD_j\frac{\delta}{\delta h_{ij}} - ih^{ij}\phi_{,j}\frac{\delta}{\delta\phi}, \tag{2.3}$$

$$H^0 = -l_p^{-2}\nabla^2 + h^{\frac{1}{2}}[-l_p^{-2(3)}R + \frac{1}{2}h^{ij}\phi_{,i}\phi_{,j} + V] \equiv -l_p^{-2}(\nabla^2 - U) \tag{2.4}$$

$$\nabla^2 = G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} + \gamma_{ij}\frac{\delta}{\delta h_{ij}} + \frac{1}{2}l_p^{-2}h^{-1/2}\frac{\delta^2}{\delta\phi^2} \tag{2.5}$$

These were given by the paper mentioned above [4]. Using equation 2.4 and equation 2.5, the Wheeler-DeWitt equation can be given in an analogous form to Klein-Gordon equation:

$$(\nabla^2 - U)\psi = 0 \tag{2.6}$$

2.1 General Behaviour of the solutions

One can now compare equation 2.6 with a one-dimensional equation

$$[\frac{d^2}{dx^2} + U(x)]\Psi(x) = 0 \tag{2.7}$$

In the one-dimensional case, one can see that the wavefunction is oscillatory in the region $U \ll 0$. The case in 2.6 is more complicated.

One can divide the variables into two sets. One time-like variable and a set of space-like variables. Equation 2.6 then can be written as:

$$[\frac{\partial^2}{\partial {q^0}^2} - \frac{\partial^2}{\partial q^2} + U(q^0, \mathbf{q})]\Psi(q) = 0 \tag{2.8}$$

Assuming one has a space-like surface U in the mini-superspace, one can perform a rotation in a local region so that the surface depends only on time-like coordinate. One can then separate the variables and use the result in the one-dimensional case. Similar technique can be used to a time-like surface.

Path integral can also be used to analyse the general behaviour of wavefunction. The wavefunction has the form $e^{-I_{cl}}$ in the saddle-point approximation. One need to solve Einstein equation to obtain I_{cl}. If the solution is real, the function will be exponential and if it is imaginary, the wavefunction will be oscillatory. The latter case is more common.

Now we should solve the Wheeler-DeWitt equation more explicitly. We now use the equation in the following form:

$$[-\frac{1}{2m_p^2}\nabla^2 + m_p^2 U(q)]\Psi(q) = 0 \tag{2.9}$$

One would often use the Wheeler-DeWitt equation with the path integral. The solution one should look for is in the from of:

$$\Psi(q) = C(q)e^{-m_p^2 I(q)} + O(m_p^{-2}) \tag{2.10}$$

where I and C are complex. One can therefore obtain:

$$-\frac{1}{2}(\nabla I)^2 + U(q) = 0 \tag{2.11}$$

$$2\nabla I \cdot \nabla C + C\nabla^2 I = 0 \tag{2.12}$$

where ∇ is the co-variant derivative. Writing I in the form of $I = I_R(q) - iS(q)$, we have:

$$-\frac{1}{2}(\nabla I_R)^2 + \frac{1}{2}(\nabla S)^2 + U(q) = 0 \tag{2.13}$$

$$\nabla I_R \cdot \nabla S = 0 \tag{2.14}$$

In order for the wavefunction to correspond to the classical space-time S has to be a solution of Hamilton-Jacobi equation:

$$\frac{1}{2}(\nabla S)^2 + U(q) = 0 \tag{2.15}$$

S is generally not appearing in wavefunction 2.10. If:

$$|\nabla S| \gg |\nabla I_R| \tag{2.16}$$

S will be an approximate solution to equation 2.15. And the wavefunction will be in the form of e^{iS}. Momenta will be in the form of:

$$p_\alpha = m_p^2 \frac{\partial S}{\partial q^\alpha} \tag{2.17}$$

Now we can differentiate equation 2.15 with respect to q^γ and obtain:

$$\frac{1}{2} f^{\alpha\beta}_{,\gamma} \frac{\partial S}{\partial q^\alpha} \frac{\partial S}{\partial q^\beta} + f^{\alpha\beta} \frac{\partial S}{\partial q^\alpha} \frac{\partial^2 S}{\partial q^\beta \partial q^\gamma} + \frac{\partial U}{\partial q^\gamma} = 0 \tag{2.18}$$

Hence, one can define a vector:

$$\frac{d}{ds} = f^{\alpha\beta} \frac{\partial S}{\partial q^\alpha} \frac{\partial}{\partial q^\beta} \tag{2.19}$$

Now one can write equation 2.18 as :

$$\frac{dp_\gamma}{ds} + \frac{1}{2m_p^2} f^{\alpha\beta}_{,\gamma} p_\alpha p_\beta + m_p^2 \frac{\partial U}{\partial q^\gamma} = 0 \tag{2.20}$$

Now one can see that the wavefunction corresponds to a set of classical equations.

The solution to equation 2.17 depends on n parameters, but the solution to original equation depends on 2n-1 parameters. By imposing the boundary conditions, one can have one specific solution.

Let us choose a classical beginning. One can write the function 2.16 and 2.12 as

$$\nabla \cdot (|C|^2 \nabla S) = 0 \tag{2.21}$$

Combining this with 2.14, one have:

$$\nabla \cdot (\exp(-2m_p^2 I_R)|C|^2 \nabla S) = 0 \tag{2.22}$$

One can easily rewrite this as:

$$\nabla \cdot J = 0 \tag{2.23}$$

where

$$J = \exp(-2m_p^2 I_R)|C|^2 \nabla S \tag{2.24}$$

is the current.

One should note that equation 2.24 is just a special case of Wheeler-DeWitt current:

$$J = \frac{i}{2}(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \tag{2.25}$$

It can be shown that the current as a probability measure can only be used in WKB solution.

Consider a classical trajectory with tangent vector ∇S about which the wavefunction is peaked. It will intersect a surface Σ_1 at $B \cap \Sigma_1$ and then intersect with another surface Σ_2 at $B \cap \Sigma_2$. The volume swept out is V. One may write:

$$0 = \int_V dV \nabla \cdot J = \int_{\partial V} J \cdot dA \tag{2.26}$$

and

$$\int_{B\cap\Sigma_1} J \cdot dA = \int_{B\cap\Sigma_2} J \cdot dA \tag{2.27}$$

Hence, one can see that the trajectories across a hypersurface does not depend on the hyperspace. Then one can use the quantity:

$$dP = J \cdot d\Sigma \tag{2.28}$$

as a conserved probability measure.

Now one need to consider the negative probability issue. Just like quantum field theory, one have to choose a proper surface Σ. More specifically, one has to choose a surface of constant q^0. Hence, in the surface the conformal part of three metric does not change. If the sign of the time-like part is negative, it corresponds to the universe that expands and then re-collapses. In that case the line will intersect a surface twice. For any other case the sign will be positive. The surfaces at which the trajectories only intersects once can be found by inspection.

One simple example would be the surface of constant S. This breaks down near the surface $U(q) = 0$. But apart from this, the surface is valid. Hence, one can construct a good probability measure from the current.

One should note that the probability measure is only regional. Thus, the equation

$$\int_\Sigma J \cdot dA = 1 \tag{2.29}$$

does not hold here barring special boundary conditions. It is a good conditional probability given by:

$$P(s_0|s_1) = \frac{\int_{s_0} J \cdot dA}{\int_{s_1} J \cdot dA} \tag{2.30}$$

As mentioned several times above, a boundary condition is needed. There are mainly two approaches here.

2.2 Quantum tunnelling approach

In quantum tunnelling approach[24][4][5], one can obtain the wave function by integrating between a vanishing 3-geometry and (h,ϕ).[31][32][27]

$$\psi(h,\phi) = \int_{\varnothing}^{(h,\phi)} [dg][d\phi]e^{iS} \tag{2.31}$$

One approach that might solve the divergence issue is suggested by Linde [14]. The action is obtained by rotating the contour to $t \rightarrow +i\pi$. However, this approach cannot solve the divergence issue of the model that involves matter field.

Another approach is suggested by Halliwell and Hartle [15] where a contour that is not purely Lorentzian or Euclidean is chosen. The issue with this approach is no particular contour is obviously the preferred one.

The physical picture of quantum tunnelling approach is the universe nucleates in a de-Sitter space and evolves in an inflationary scenario. In this case, the nucleation is a non-singular event. However, this does not mean that singularities cannot develop in the future. Also, the singularity of a three geometry does not mean that there is a singularity in the four geometry. Hence, we can divide the boundary of superspaces into two categories. The first one is the boundary where its singularities are due to slicing of the four geometries and the rest of the boundaries as the second category. We call the first category non-singular boundary of super-space and the second category as the singular boundary of super-space. One should note that only outgoing mode are allowed in ψ at singular boundaries. This is called tunnelling boundary condition.

Ingoing and outgoing modes are analogous to the positive and negative frequency modes. The direction towards boundary are defined as time direction. Hence, we can write

$$\psi = \sum_n C_n e^{iS_n} \tag{2.32}$$

where S_n obeys Hamilton-Jacobi equation

$$(\nabla S_n)^2 + U = 0 \tag{2.33}$$

In addition to the tunnelling condition, we have another condition called regularity condition:

$$|\psi| < \infty \tag{2.34}$$

To illustrate one approach to solve the problem, one can first consider a simpler case. To start with, one need to make the following three assumptions.:

First, the bubble from which the true vacuum bubbles nucleate from is infinitely thin. Second, the tunnelling action is very large hence the bubble can be treated as a sphere, and thirdly the gravitation from the false vacuum are ignored.

The Lagrangian of minisuperspace model is:

$$L = -4\pi\sigma\dot{R}^2)^{1/2} + \frac{4\pi}{3}\epsilon R^3 \tag{2.35}$$

where σ is the wall tension and ϵ is the energy density difference. One can see that the following relation is valid:

$$p_R = 4\pi\sigma R^2\dot{R}(1-\dot{R}^2)^{12/3} \tag{2.36}$$

and

$$H = [p_R^2 + (4\pi\sigma R^2)^2]^{1/2} - \frac{4\pi}{3}\epsilon R^3 \tag{2.37}$$

As the nucleation process cannot change the energy, one have $H = 0$ and therefore:

$$p_R^2 + U(R) = 0 \tag{2.38}$$

and

$$U(R) = (4\pi\sigma R^2)^2(1 - R^2/R_0^2) \tag{2.39}$$

where $R_0 = \frac{3\sigma}{\epsilon}$. The equation of motion is subsequently

$$R^2\dot{R}^2 = R^2 = R_0^2. \tag{2.40}$$

with solution:

$$R = \sqrt{t^2 + R_0^2} \tag{2.41}$$

The metric of the world sheet is then:

$$ds^2 = (1 - \dot{R})^2)dt^2 - R^2(t)d\Omega^2 \tag{2.42}$$

One can change this metric into de-Sitter metric by change of variable:

$$\tau = R_0 sinh^{-1}(t/R_0) \tag{2.43}$$

The metric is then:

$$ds^2 = d\tau^2 - R^2(\tau)d\Omega^2 \tag{2.44}$$

with

$$R(\tau) = R_0 cosh(\tau/R_0). \tag{2.45}$$

The bubble wall is essentially a 2-dimensional expanding inflationary universe created at $\tau = 0$.

Now one can promote these equations into quantum equations:

$$H\psi = 0 \tag{2.46}$$

and if $|R_{pR}\psi| \gg |\psi|$

$$[-\partial_R^2 + U(R)]\psi = 0 \tag{2.47}$$

One now need to determine the boundary condition for this universe. The WKB solution for equation 2.47 is

$$\psi_\pm(R) = p(R)^{-1/2} \exp(\pm i \int_{R_0}^{R} p(R')dR' \mp i\pi/4) \tag{2.48}$$

where

$$p(R) = [-U(R)]^{1/2} \tag{2.49}$$

ψ_+ and ψ_- describes a universe that is expanding and contracting respectively. However, only expanding bubble can be present.

Similarly, in classical forbidden region, one have:

$$\tilde{\psi}_\pm = |p_R|^{-1/2} \exp(\pm \int_{R}^{R_0} |p(R)|dR \tag{2.50}$$

The probability distribution is then:

$$|\frac{\psi(R_0)}{\psi(0)}|^2 \sim \exp(-2\int^{R_0} |p(R)|dR) = \exp(-\sigma\pi^2 R_0^3/2) \tag{2.51}$$

Now one need to interpret this result. It seems confusing that there is only one universe, yet one has obtained the result of a probability distribution. And it seems that this wavefunction is not observable to someone who live within that universe.

However, an observer inside the universe can still get some useful information. For all the bubbles that can nucleate, the observer is likely to find themselves inside a typical universe i.e. universe that is near the maxima of the probability distribution.

Also, one should note the bubble is not exactly spherical and the shape of the bubble can be calculated from the wavefunction. The perturbations of the shape of the bubble can be seen as excitation of the field Φ with mass $M^2 = -3R^2$. Such prediction can be tested by the observer.

An external observer can calculate:

$$\psi = \int [dx^\mu e^{iS}]. \tag{2.52}$$

so that they can obtain the amplitude of a given configuration.

An internal observer would treat ξ^0 as the time coordinate. One would expect that perturbation to grow rapidly at small length scales and if allowed, the bubble wall would cross itself, thus generate a daughter universe. Moreover, the observer may find that the universe is not 2-d at all at small enough scales. We, human observer, may experience similar situation in our own universe at Plank scale. Now we will detour into Hartle-Hawking approach, more realistic model describing our own universe will be introduced later.

2.3 Hartle-Hawking Approach

Another possible solution is No-Boundary solution. There are some new development in this field but here we will only discuss the fundamental theory [25][26][28][29][30]. The Hartle-Hawking wave function is defined using path integral:

$$\psi_H = \int [dg_{\mu\nu}][d\phi]exp[-S_E(g_{\mu\nu}, \phi)] \tag{2.53}$$

The integral is taking over all compact Euclidean histories ends in this configuration. All the histories between nothing and this configuration is a compact 4-geometry. One important point to note that the gravitational part of this is unbounded below hence the integral is divergent. If one tries to define the path integral for gravity action:

$$I[g] = \frac{1}{16\pi G}\int_M R(-g)^{1/2} d^4x + \frac{1}{8\pi G}\int_{\partial M} [K](-h)^{1/2} d^3x \tag{2.54}$$

one will encounter some issues. Namely, if one does a Wick rotation, one will get the following Euclidean action:

$$\hat{I}[g] = \frac{-1}{16\pi G}\int R(g)^{1/2}d^4x - \frac{1}{8\pi G}\int [K](h)^{1/2}d^3x \tag{2.55}$$

. This action is not positive semi-definite. If one does a conformal transformation $\tilde{g}_{ab} = \Omega^2 g_{ab}$, one will get:

$$\hat{I}[\tilde{g}] = \frac{-1}{16\pi G}\int_M \Omega^2 R + 6\Omega_{,a}\Omega^{,a}(g)^{1/2}d^4x - \frac{1}{8\pi G}\int_{\partial M}[\Omega^2 K](h)^{1/2}d^4x \tag{2.56}$$

. One can see that $\hat{I}$ can be negative for a suitable conformal factor. It is also easy to see from a physics point of view. As the existence of black hole implies the canonical ensemble is not well-defined. Thus, the integral cannot be convergent.[11][12]

It is not clear whether this is actually physically meaningful. However, we can use this path integral to partially solve the boundary condition issue for Wheeler-DeWitt equation.[4]

Attempts have been made to fix divergence issue. [10] One can introduce a conformally invariant scalar wave operator on the space of all positive definite metrics:$\Delta = -\square + \frac{1}{6}R$.Let $\{\lambda_n, \phi_n\}$ be the eigenvalue and eigenfunction of this operator under the following boundary condition:

$$\Delta\phi_n = \lambda_n\phi_n \tag{2.57}$$

and

$$\phi_n = 0 \tag{2.58}$$

Equation 2.57 and 2.58 are known as Dirichlet boundary condition. One should note that $\{\lambda_n, \phi_n\}$ change smoothly under the conformal transformation and 0 eigenvalue are invariant under conformal transformation. This means that the eigenvalue will not change signs because of the conformal transformation. Now let assume that $\Omega = 1 + y$ where y is zero one the boundary. One then have:

$$\hat{I}[\tilde{g}] = \hat{I}[g] - \frac{1}{16\pi G}\int (y\Delta y + 2Ry)(g)^{1/2}d^4x \tag{2.59}$$

One can then divide this into two terms:

$$\hat{I}^1 = I[g] + \frac{1}{16\pi G}\int R\Delta^{-1}R(g)^{1/2}d^4x \tag{2.60}$$

$$\hat{I}^2 = -\frac{1}{16\pi G}\int z\Delta z(g)^{1/2}d^4x, \tag{2.61}$$

where $z = y - \Delta^{-1}R$ One can consider a conformal transformation $g'_{ab} = \omega^2 g_{ab}$. $\hat{I}[g] = \hat{I}^1[g']$ if one choose $\omega = 1$ on the boundary. One can assume that $\hat{I} \geqslant 0$ for all asymptotically Euclidean positive definite metrics. This is called positive action conjecture. The dominant part of the path integral is near the minimum of I on $\mathscr{H}$. This means one can approximate $\hat{I}$ as:

$$\hat{I}[g] = \hat{I}[g.] + I_2[g_0, \phi] \tag{2.62}$$

where

$$I_2[g_0, \phi] = \frac{1}{32\pi G}\int \phi^{ab} A_{abcd} \phi^{cd} (g_0)^{1/2} d^4x \tag{2.63}$$

$$g^{ab} = g_0^{ab} + \phi^{ab} \tag{2.64}$$

$$\begin{aligned} A_{abcd} = &\frac{1}{4} g_{cd} \nabla_a \nabla_b - \frac{1}{4} g_{ac} \nabla_d \nabla_b + \frac{1}{8}(g_{ac}g_{bd} + g_{ab}g_{cd})\nabla_e \nabla^e + \frac{1}{2} R_{ad} g_{bc} \\ &- \frac{1}{4} R_{ab} g_{cd} + \frac{1}{16} R_{ad} g_{bc} - \frac{1}{8} R g_{ac} g_{bd} + (a \leftrightarrow b) + (c \leftrightarrow d) + (a \leftrightarrow b, c \leftrightarrow d). \end{aligned} \tag{2.65}$$

The path integral is then

$$Z = \exp(-\hat{I}[g_0]) \int D[\phi] \exp(-\hat{I}_2[g_0, \phi]) \tag{2.66}$$

Now one can decompose the metric perturbation in terms of the eigenfunctions

$$\phi^{ab} = \sum_n a_n \phi_n^{ab} \tag{2.67}$$

where ϕ_n is the eigenfunction. Now we could take $D(\phi) = \Pi_n \mu da_n / (32\pi G)^{1/2}$ and then get:

$$logZ = -\hat{I}[g_0] - \frac{1}{2} logdet(\mu^{-2} A) \tag{2.68}$$

$$detA = \prod_n \mu_n \tag{2.69}$$

Because the action is invariant under the following gauge transformation, A has many 0 eigenvalues.

$$\begin{aligned} x^a &\to x^a + \epsilon \xi^a \\ g_{ab} &\to g_{ab} + 2\epsilon \xi \end{aligned} \tag{2.70}$$

In order to eliminate this, one can choose to only integrate over in-equivalent ϕ^{ab}. One can add a term to achieve this:

$$\hat{I}_g = \frac{1}{32\pi G} \int \phi^{ab} B_{abcd} \phi^{cd} (g_0)^{1/2} d^4x \tag{2.71}$$

The gauge one could choose is harmonic gauge i.e.

$$\begin{aligned} B_{abcd} = &\frac{1}{4} g_{bd} \nabla_a \nabla_c - \frac{1}{8} g_{cd} \nabla_a \nabla_b - \frac{1}{8} g_{ab} \nabla_c \nabla_d + \frac{1}{16} g_{ab} g_{cd} \nabla_e \nabla^e \\ &(a \leftrightarrow b) + (c \leftrightarrow d) + (a \leftrightarrow b, c \leftrightarrow d). \end{aligned} \tag{2.72}$$

One should note that the eigenvalues of operator B is contained det(A+B). Furthermore, one can divide by det(B) in order to nullify them, where

$$det(B) = det(C)^2|_V. \tag{2.73}$$

. Here V is the space of the complete set of the vector fields that is 0 on the boundary. Also, here we have

$$C_{ab}\xi^b = -g_{0ad}(\xi^{d;b}_{;b} + R^d_b\xi^b. \tag{2.74}$$

Then one have

$$\log Z = -\hat{I}[g_0] - \frac{1}{2}\log\det(\mu^{-2}(A+B)) + \log\det(\mu^{-2}C) \tag{2.75}$$

One cannot just apply the zeta function technique here. One can see the following relationship:

$$A + B = -\frac{1}{16}\nabla_a\nabla^a + G \tag{2.76}$$

One can call the second term operator F. This operator is positive definite and in order for the integral to converge, one should take the contour of imaginary axis. As this operator only acts on trace-free part of the wavefunction, one can define another operator which acts on the trace part:

$$G_{abcd} = \frac{1}{8}(g_{ac}g_{bd} + g_{bc}g_{ad})\nabla_e\nabla^e + \frac{1}{4}(R_{acbd} + R_{adbc}) \tag{2.77}$$

This operator then can only integrate along the real axis. This however will introduce a factor of $(+i)^n$ into Z. Let P be the operator that projects on eigenfunctions with non-positive eigenvalues of G, One can define $\tilde{G} = G - P$ and zeta function:

$$\zeta(s, L) = \sum_n \lambda_n^{-s}(L). \tag{2.78}$$

Hence one can calculate the path integral:

$$\begin{aligned}\log Z = &-\hat{I}[g_0] + \frac{1}{2}\zeta'(0,F) + \frac{1}{2}\zeta'(0,\tilde{G}) \\ &+ \zeta'(0,C) - \frac{1}{2}\sum_i log\lambda_i \\ &+ \frac{1}{2}\log(\mu^2)(\zeta(0,F) + \zeta(0,\tilde{G}) + n - 2\zeta(0,C)) \\ &+ \frac{1}{2}in\pi\end{aligned} \tag{2.79}$$

Then one can use the result from Gilkey [13]:

$$\begin{aligned}\zeta(0,F) &= \frac{1}{2880\pi^2}\int R_{abcd}R^{abcd}(g_0^{1/2}d^4x \\ \zeta(0,\tilde{G}) + n &= \frac{21}{320\pi^2}\int R_{abcd}R^{abcd}(g_0)^{1/2}d^4x, \\ \zeta(0,C) &= \frac{-11}{2880\pi^2}\int R_{abcd}R^{abcd}(g_0)^{1/2}d^4x\end{aligned} \tag{2.80}$$

On can assume that the normalization factor is independent of the background metric.

$$\log \tilde{Z} = \log Z + (1-k)\hat{I} g_0 + \frac{1}{2} \log k(\zeta(0,F) + \zeta(0,\tilde{G}) + n - \zeta(0,C)). \quad (2.81)$$

This means that Z is very small for $k \ll 1$. If one define

$$T_{ab} = -2(g_0)^{-1/2} \frac{\delta \log Z}{\delta g_0^{ab}} \quad (2.82)$$

one have:

$$T_a^a = \frac{53}{720\pi^2} R_{abcd} R^{abcd} \quad (2.83)$$

One can also find the Euler number for the compact space of the solutions of the vacuum Einstein equation.

$$\chi = \frac{1}{32\pi^2} \int R_{abcd} R^{abcd} (g_0)^{1/2} d^4x \quad (2.84)$$

However, one should note that this method of fixing is only partly successful.

2.4 Minisuperspace Wave Functions

2.4.1 de Sitter space

There are several differences between the two approaches above. Let us consider action:

$$S = \int d^4x \sqrt{-g} (l_P^{-2} R - \rho_v) \quad (2.85)$$

where ρ_v is a constant vacuum energy, and we assume cosmological principle is valid:

$$ds^2 = \sigma^2 [N^2(t) dt^2 - a^2(t) d\Omega_3^2] \quad (2.86)$$

Thus the Lagrangian of this model is:

$$\mathscr{L} = \frac{1}{2} N[a(1 - \frac{\dot{a}^2}{N^2} - \Lambda a^3] \quad (2.87)$$

and the momentum conjugate to a is:

$$p_a = \frac{-a\dot{a}}{N} \quad (2.88)$$

Therefore, one can express the Lagrangian as

$$\mathscr{L} = p_a \dot{a} - N \mathscr{H} \quad (2.89)$$

where

$$\mathscr{H} = -\frac{1}{2} (\frac{p_a^2}{a} + a - \Lambda a^3) \quad (2.90)$$

This model has only one degree of freedom the scale factor a. The equation of motion is

$$\dot{a}^2 + 1 - \Lambda a^2 = 0 \tag{2.91}$$

Hence the solution is:

$$a(t) = H^{-1} cosh(Ht) \tag{2.92}$$

where

$$H = \frac{4}{3} G \rho_v^{1/2}$$

From here we can write out the Wheeler-DeWitt equation for $\psi(a)$:

$$(a^{-p} \frac{\partial}{\partial a} a^p \frac{\partial}{\partial a} - U(a))\psi = 0 \tag{2.93}$$

The solution for this equation are:

$$\psi_{\pm}^{(1)}(a) = exp(\pm i \int_{H^{-1}}^{a} p(a')da' \mp \frac{i\pi}{4}) \tag{2.94}$$

$$\psi_{\pm}^{(2)}(a) = exp(\pm \int_{a}^{H^{-1}} |p(a')|da') \tag{2.95}$$

Solution 2.94 is WKB solution i.e. solution with boundary condition that $U(a) \leqslant 0$. Solution 2.95 is the underbarrier solution where the boundary condition is $a \leqslant H^{-1}$.

We can write the tunnelling solution using these two solutions.

$$\psi_T(a > H^{-1}) = \psi_{-}^{(1)}(a) \tag{2.96}$$

$$\psi_T(a < H^{-1}) = \psi_{+}^{(2)}(a) - \frac{i}{2}\psi_{-}^{(2)}(a) \tag{2.97}$$

and for the special case where $a = H^{-1}$, we have:

$$\psi_T(H^{-1})/\psi_T(0) = exp(-\int_{0}^{H^{-1}} |p(a')|da' = exp(-\frac{3}{16G^2\rho_v}) \tag{2.98}$$

One should note that the sign of N in equation 2.88 is purely conventional. One could choose the opposite sign without significant change to the formulation other than a time reversal transformation $\psi- > \psi*$. Physically, this means that time coordinate is entirely arbitrary in General Relativity. However, once we choose one convention to follow, the wavefunction is unique.

Here one should note [16] one theory proposed by Strominger. The theory state that the boundary condition on ψ should be at small a instead of large a, as this process is at very small scale. And the large scale prediction can be solved without specifying the form of the boundary condition. Hence, one have the following solution for the theory:

$$\begin{aligned} \psi(a < H^{-1}) &= \tilde{\psi} - (a) \\ \psi(a > H^{-1}) &= \psi_{+}(a) + \psi_{-}(a) \end{aligned} \tag{2.99}$$

The argument is not perfect as there are many examples where small scale event are governed by large scale or even infinity-away boundary conditions.

By contrast, the Hartle-Hawking solution [17] are:

$$\psi_H(a < H^{-1}) = \psi_-^{(2)}(a) \tag{2.100}$$

and

$$\psi_H(a > H^{-1}) = \psi_+^{(1)}(a) + \psi_-^{(1)}(a) \tag{2.101}$$

Interestingly, as one can see these are essentially the same with quantum tunnelling approach with Strominger's proposal. Physically, this describes a expanding and then contracting universe and in region $a < H^{-1}$, ψ_H is exponentially suppressed.

Now, we shall discuss another method of obtaining wavefunction i.e. by analytical continuation.

One can consider this Lagrangian with $\rho_v < 0$. It is easy to see that the classical equation of motion has no solutions. However, quantum fluctuations still exist. Thus, the wave function will be peaked at very small scales.

To perform an analytic continuation, one need an exact solution. Setting $\gamma = 1$ and boundary condition to be

$$\psi(a \to \infty) = 0 \tag{2.102}$$

one obtain the solution as:

$$\psi(a) = Ai(z), \tag{2.103}$$

where

$$z = (-2\Lambda)^{-2/3}(1 - \Lambda a^2) \tag{2.104}$$

Using asymptotic approximation, one can show that at large a :

$$\psi(a) \propto a^{-1/2} \exp[-(-\Lambda)^{1/2} a^3/3] \tag{2.105}$$

In order to analytically continue the solution to positive Λ, one has to substitute $(-2\Lambda)^{-2/3}$ with $(-2\Lambda)^{-2/3}\exp(\mp 2\pi i/3)$. With the relation:

$$2e^{\pm\pi i/3} Ai(ze^{\mp 2\pi i/3}) = Ai(z) \pm iBi(z) \tag{2.106}$$

one obtain:

$$\psi(A) = Ai(\tilde{z}) + Bi(\tilde{z}), \tag{2.107}$$

with

$$\tilde{z} = (a\Lambda)^{-2/3}(1 - \Lambda a^2) \tag{2.108}$$

2.4.2 Scalar Field

Scalar field theory is more realistic than de Sitter space. The Lagrangian of scalar field theory is

$$S = d^4x\sqrt{-g}[l_P^{-1} R + \frac{1}{2}(\partial_\mu \tilde{\phi})^2 - V(\tilde{\phi})] \tag{2.109}$$

The scalar field is homogeneous and isotropic. For simplicity reasons, dimensionless quantities are introduced:

$$\phi = (4\pi G/3)^{1/2}\widetilde{\phi},$$
$$V = (4G/3)^2\widetilde{V}$$

We subsequently can write the Wheeler-DeWitt equation:

$$(\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial \phi^2} - U(a,\phi))\psi = 0 \tag{2.110}$$

where

$$U(a,\phi) = a^2[1 - a^2 V(\phi)] \tag{2.111}$$

The minisuperspace is $0 < a < \infty, -\infty < \phi < \infty$. The nonsingular boundary is the line a=0 with $—\phi| < \infty$ and the singular boundary is boundary with at least one variable being ∞.

Similar to the de Sitter space, we can write out Hamilton-Jacobi equation 2.33 in the form:

$$\left(\frac{\partial S_n}{\alpha}\right)^2 - \left(\frac{\partial S_n}{\phi}\right)^2 + U = 0 \tag{2.112}$$

where new variable $\alpha = lna$ is introduced.

The potential $U(a,\phi)$ goes to zero if a is small. andψ takes form of:

$$\psi = \sum_k \psi_k = \sum_k e^{ik(a\mp\phi)} \tag{2.113}$$

ψ_k with $k > 0$ gives the model of universe which collapsing to a singularity and those with $k < 0$ describes universe expanding out of singularity. The tunnelling boundary condition means that ψ_k can only exist if $k > 0$. Note if a approaches to 0, the wave function will approach a constant. Only regions near these points i.e. early stage of universe have significant quantum effect. Other part of mini-superspace can be represented by classical model.

Now let us consider the solution of equation 2.110. We assume the following conditions for $V(\phi)$:

$$|V^{-1}dV/d\phi| \ll 1 \tag{2.114}$$

and

$$|V| \ll 1 \tag{2.115}$$

When condition 2.115 is violated, the effect of quantum gravity becomes significant and we cannot use semiclassical solution to represent the physical effect.

As ψ is also a slow varying function of ϕ. This means we can rewrite equation 2.110 in the form

$$(\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - U(a,\phi))\psi = 0 \tag{2.116}$$

This makes the problem identical to one dimensional minisuperspace model studied in de-Sitter case. There are two part of minisuperspace (a,ϕ), $U <$

$0 and U > 0$, which are corresponding to classically forbidden region and classically allowed region. In classically allowed region, ψ is oscillatory and in classically forbidden region ψ is exponential. Note that p does not affect the semi-classical probability. Hence, with the choice of p=-1 we can solve equation 2.116 exactly. Introducing new variable $z = -(2V)^{-2/3}(1 - a^2V$ and we have

$$\left(\frac{\partial^2}{\partial z^2} + z\right)\psi = 0 \tag{2.117}$$

As only an outgoing wave can exist in the classically allowed region, the tunnelling wave function can be solve from condition 2.114:

$$\psi_T = \frac{Ai(-z) + iBi(-z)}{Ai(-z_0) + iBi(-z_0)} \tag{2.118}$$

where $z_0 = z(a = 0) = -(2V)^{-2/3}$. For $V(\phi) < 0$, z and z_0 are complex. In order that the function can be continuous at V(ϕ)=0, we set:

$$\begin{aligned} V(\phi) &= e^{-i\pi}|V(\phi)|, \\ -z &= e^{2\pi i/3}|z| \\ -z_0 &= e^{2\pi/3}|z_0| \end{aligned} \tag{2.119}$$

We can re-write the solution to

$$\psi_T = \frac{Ai(|z|)}{Ai(|z_0|)} \tag{2.120}$$

for $V(\phi) < 0$ as the Airy function has the relation:

$$Ai(e^{2\pi i/3}z) + iBi(e^{2\pi i/3}z) = 2e^{\pi i/3}Ai(z) \tag{2.121}$$

Note that this wave function is real.

In the range $a^2V > 1$, z and z_0 are both large, but z is positive and z_0 is negative. Then we can write the solution in classical allowed region as

$$\psi_T = e^{i\pi/4}(a^2V - 1)^{-1/4}exp\left(-\frac{1 + i(a^2V - 1)^{3/2}}{3V}\right)(a^2V > 1) \tag{2.122}$$

and in classical forbidden region as

$$\psi_T = (1 - a^2V)^{-1/4}exp\left(\frac{(1 - a^2V)^{3/2} - 1}{3V}\right)(a^2V < 1) \tag{2.123}$$

For Hartle-Hawking wave function, we find:

$$\psi_H = \frac{Ai(-z)}{Ai(-z_0} \tag{2.124}$$

using the requirement of it being exponentially growing function if a in classically forbidden range. We can obtain WKB approximations for ψ_H in classically allowed range:

$$\psi_H = 2(a^2V-1)^{-1/4}exp(\frac{1}{3V})cos(\frac{(a^2V-1)^{3/2})}{3V} - \frac{\pi}{4}) \tag{2.125}$$

and in classically forbidden range

$$\psi_H = (1-a^2V)^{-1/4}exp(\frac{1-(1-a^2V)^{3/2}}{3V}) \tag{2.126}$$

Using relation 2.121, and change a to $e^{i\pi/2}a$, we have:

$$\psi_H = \psi_T(V->e^{i\pi/2}V, a->e^{i\pi/2}a) \tag{2.127}$$

This relation shows the possibility that two wavefunction is related by an analytic continuation.

2.5 Physical Predictions

The purpose of introducing quantum effect into cosmology is trying to determine the initial state of the universe as the latter evolution of the universe can be determined using classical model.

For the tunnelling approach, we can write out the conserved current:

$$\begin{aligned} j^a &= \frac{i}{2}a^2(\psi^*\partial_a\psi - \psi\partial_a\psi^*) \\ j^\phi &= -\frac{i}{2}a^{p-2}(\psi^*\partial_\phi\psi - \psi\partial_\phi\psi^*) \end{aligned} \tag{2.128}$$

Here, j^a is the probability density for ϕ at a specific value of a. The classical model can be expressed as:

$$a \approx V^{-1/2}cosh(V^{1/2}t), \phi \approx const \tag{2.129}$$

We can then define the probability density $\rho(a,\phi)$. The corresponding density of ψ_T is

$$\rho_T(a,\phi) = C_T exp(-\frac{2}{3V(\phi)} \tag{2.130}$$

We can then determine the normalization constant to be

$$C_T^{-1} = \int_{[V(\phi)>0]} d\phi exp(-\frac{2}{3V(\phi)}) \tag{2.131}$$

The integral will not be normalizable if none of the following three conditions are satisfied:

$$i) V(\phi) < 0 \ as \ \phi \to 0$$

$$ii) V(\phi) \ goes \ to \ 0 \ faster \ than \ \frac{2}{3}ln(\phi)$$

$$iii) \phi \ is \ cyclic \ variable \ with \ finite \ range \ 0 < \phi < \phi_0 \ and$$
$$both \ 0 \ and \ \phi_0 \ are \ identified$$

If we use the no boundary approach, [18] the current will be zero and the wave-function will be real. The wavefunction can be written as:

$$\psi_H = (a^2V-1)^{-1/4}\exp(1/3V)\exp(-\frac{i(a^2V-1)^{3/2}}{3V}+\frac{i\pi}{4})+c.c. \tag{2.132}$$

The second term is not very useful in this case as it represents time-reversal of universes that are contracting. Then we can find out the probability distribution using the first term:

$$\rho_H(a,\phi) = C_H\exp(\frac{2}{3V}) \tag{2.133}$$

with the normalization factor:

$$C_H^{-1} = \int_{[V(\phi)>0]} d\phi\exp(\frac{2}{3V(\phi)} \tag{2.134}$$

It is easy to see that if we want this integral to be well-defined, V_ϕ cannot be 0 at any point. This can be achieved in two cases. Either ϕ does not have an infinite range or $V(\phi)$ is positive-defined.

The model where Hartle-Hawking wave function gives the prediction of inflating universe is the one with $V(\phi)$ unbounded below.

It should be noted that these theories only gives probability distribution of the initial state of universes. However, the only experimental data we can acquire is our universe. Hence, our best assumption is that our universe is a typical universe i.e. is near the maximum of the probability distribution. The other possible situation is that the most probable universe is not suitable for life. Then our best guess is that our universe is the most probable universe that can generate life. Hence, one can argue that inflation is somehow essential to provide isotropy and homogeneity which are necessary for life. However, this does not provide the full explanation as it will predict a much smaller inflation than what we observe today.

2.6 Perturbative theory

We shall now consider perturbative theory for quantum tunnelling approach. Let us assume the potential is bounded from above. Hence ψ is at its maximum when $V(\phi)$ is at its maximum. Now, we shall consider the theory near its maximum. We can write:

$$V(\phi) = H^2 - \mu^2\phi^2 + O(\phi^3) \tag{2.135}$$

We can write the small perturbation around $\phi = 0$ in spherical harmonic form:

$$\phi(x) = (2\pi^2)^{1/2}\sum_{n,l,m} f_{nlm}(t)Q^n_{lm}(x^i) \tag{2.136}$$

where n=1,2,3... ;l=0,1,2,3...n-1; m=-l.....,l; just like the atomic quantum number.

Then, the Wheeler-DeWitt equation can be written in form of:

$$\left[a^2\frac{\partial^2}{\partial a^2} + pa\frac{\partial}{\partial a} - a^4(1-H^2a^2) - \sum_n\left(\frac{\partial^2}{\partial f_n^2} - (n^2-1)a^4 f_n^2 + \mu^2 a^6 f_n^2\right)\right]\psi = 0 \tag{2.137}$$

First, we write the wave-function in the form of

$$\psi = e^{iS} \tag{2.138}$$

with

$$S = S_0 + \frac{1}{2}\sum_n S_n(a) f_n^2 + O(f_n^3) \tag{2.139}$$

Hence, we can write the Wheeler-Dewitt equation as:

$$S_0'^2 + a^2(1 - H^2 a^[2]) = 0 \tag{2.140}$$

$$a^2 S_0' S_n' - S_n' - (n^2-1)a^4 + \mu^2 a^6 = 0 \tag{2.141}$$

where $S' = \frac{\partial S}{\partial a}$. One should note that equation 2.140 is just the equation for one-dimensional semi-classical de-Sitter space. One solution for $a > H^{-1}$ is:

$$S_0(a) = -\frac{1}{3H^2}(H^2a^2-1)^{3/2} \tag{2.142}$$

Now we substitute $V(\phi) = H^2 - \mu^2 f_1^2$ into equation 2.123.

$$S_1(a) = \frac{2i\mu^2}{3H^4} + \frac{\mu^2}{3H^4}(H^2a^2-1)^{1/2}(H^2a^2+2) \tag{2.143}$$

We now note the relation:

$$S_0' = -\frac{a\dot{a}}{N(a)} \tag{2.144}$$

which gives the conformal time:

$$a = (Hcost)^{-1} \tag{2.145}$$

where we choose N(a)=a. Now we shall discuss about equation 2.141. One can linearize this equation with

$$S_n(t) = a^2\dot{v}_n/v_n \tag{2.146}$$

. This gives

$$\ddot{v_n} + 2(\dot{a}/a)\dot{v_n} + (n^2-1)v_n = 0 \tag{2.147}$$

The general solution is a superposition of two modes

$$v_n^{(1)}(y) = (y-1)^{(n-1)/2}(y+1)^{-(n+1)/2}(1+y/n) \tag{2.148}$$

and

$$v_n^{(2)}(y) = (y+1)^{(n-1)/2}(y-1)^{-(n+1)/2}(1-y/n) \tag{2.149}$$

The first mode represents a de-Sitter invariant vacuum state for graviton. The only case to have a de Sitter-invariant state is a massive scalar field. Hence, the prediction made by quantum cosmology is as good as a de Sitter-invariant vacuum.

In classically-forbidden range, we have:

$$S_0(a) = \pm\frac{i}{3}(1 - H^2a^2)^{3/2} \tag{2.150}$$

For Hartle-Hawking wave function [8], we first need to perturb the Friedmann model. We write the three metric in Friedmann model as

$$h_{ij} = a^2(\omega_{ij} + \epsilon_{ij}) \tag{2.151}$$

where ϵ_{ij} is the perturbation. We can write the Hamiltonian as:

$$H = N_0(H_{|0} + \sum_n H^n_{|2} + \sum_n g_n H^n_{|1}) + \sum_n (k^S_n H^n_{\_1} + j^V_n H^n_{\_1}) \tag{2.152}$$

where the number denotes the orders of perturbation. Together with zero energy Schrodinger equation, we have an infinite-dimensional second-order differential equation:

$$\left(H_{|0} + \sum_n ({}^S H^n_{|2} +^V H^n_{|2} +^T H^n_{|2})\right)\psi = 0 \tag{2.153}$$

The solution has the form:

$$\Psi = Re(\Psi_0(a,\phi)\prod_n \Psi^{(n)}) = Re(Ce^{iS} \tag{2.154}$$

where S is a rapidly varying function. Substituting equation 2.154 into equation 2.153, we obtain:

$$\begin{aligned} &-\frac{\nabla_2^2\Psi_0}{2\Psi_0} - \sum_n \frac{\nabla_2^2\Psi^{(n)}}{2\Psi^{(n)}} - \sum_{n,m} \frac{(\nabla_2^2\Psi^{(n)})(\nabla_2^2\Psi^{(m)})}{2\Psi^{(n)}\Psi^{(m)}} \\ &-\frac{\nabla_2\Psi_0}{\Psi_0}\cdot(\sum_n \frac{\nabla_2\Psi^{(n)}}{\Psi^{(n)}}) + \sum_n \frac{H^n_{|2}\Psi}{\Psi} + e^{-3\alpha}V(\alpha,\phi) = 0 \end{aligned} \tag{2.155}$$

In order for the ansatz solution to be valid, we must have

$$\frac{\nabla_2\Psi}{\Psi}\cdot\nabla_2\Psi^{(n)} + \frac{1}{2}\nabla_2^2\Psi^{(n)} = \frac{H^n_{|2}\Psi}{\Psi}\Psi^{(n)} \tag{2.156}$$

and

$$(-\frac{1}{2}\nabla_2^2 + e^{-3\alpha}V + \frac{1}{2}(\sum_n \frac{\nabla_2^2\Psi^{(n)}}{2\Psi^{(n)}})\cdot(\sum_n \frac{\nabla_2^2\Psi^{(n)}}{2\Psi^{(n)}})\psi_0 = 0 \tag{2.157}$$

The first two terms of equation 2.157 can be seen as the Wheeler-Dewitt equation and the last term as the perturbations.

Chapter 3

Further Development

3.1 Beyond Minisuperspace

Now consider Wheeler-DeWitt equation in form 2.6. One should note that ∇^2 is the superspace Laplacian, G_{ijkl} is the superspace metric and U is the superpotential. The metric can be written as:

$$h_{ij} = e^{2\alpha}\tilde{h_{ij}} \tag{3.1}$$

Then one have:

$$\int d^3xNh^{1/2}[-R^{(3)} + \frac{1}{2}h^{ij}\phi_{,i}\phi_{,j} \propto \exp(\alpha) \tag{3.2}$$

and

$$\int d^3xNh^{1/2}V(\phi) \propto \exp(3\alpha) \tag{3.3}$$

Now one can represent the scalar field as:

$$\phi(x) = (2\pi^2)^{1/2}\sum_n f_n Q_n(x) \tag{3.4}$$

Then the superspace Laplacian is :

$$\nabla^2 = e^{-3\alpha}(\frac{\partial^2}{\partial\alpha^2} - \sum_n \frac{\partial^2}{\partial f_n^2} \tag{3.5}$$

Hence the solution is :

$$\psi(\alpha, f_n) = \exp(ik_\alpha\alpha + i\sum_n k_n f_n) \tag{3.6}$$

One should note that:

$$k_\alpha^2 - \sum_n k_n^2 = 0 \tag{3.7}$$

Boundary condition is the tunnelling wave function includes only out-going modes.

Not all metric and matter fields can be included in the superspace. The limitation of these fields is essentially the boundary of superspace. If all three terms in the superspace metric can integratable, the metric-fields configuration should be included. The superpotential is then divergent yet finite everywhere. Writing the wavefunction as:

$$\psi(c,q) = \sum_N e^{iS_N(c)} \chi_N(c,q). \tag{3.8}$$

where S_N is the Hamilton Jacobi function. The classical path is $p_i = -\frac{\partial S}{\partial c_i}$. The superspace here involves many configurations, including some matter fields that are not differentiable. Examples are scalar fields with discontinuous derivatives. This agrees with the path integral approach where differentiability are not required for the path integral.

Now we shall discuss the topology change since the very natural of creating universe is change of topology. The first thing we should note is that the Wheeler-Dewitt equation is defined on a fixed topology $R \times \Sigma$. The superspace hence only include one topology. One can expand the superspace into including all possible topologies. And one can divide the superspace into different sectors. The topology change is then the transition from one topology sector to another.

The creation of the universe is then the transition from null topology to the sector with a universe of S_3. Similar to the instanton solution of path integral, One can expect the topology change to be represented by a smooth Euclidean manifold interpolating between the initial and the final sector. [39][40]

Now let us define the following mathematical concept: a smooth function $f(x)$ on a manifold is called a Morse function if

i)$f(x) \in [0,1]$

ii)$f(x) = 0 \ iff x \in \Sigma_1$

iii)$f(x) = 1 \ iff x \in \Sigma_2$

iv) all point x_0 where $\partial_\mu f(x_0)$, obeys $\det[\partial_\mu \partial_\nu f(x_0)]$ and are in the interior of the manifold. Morse function are well-defined in any situation.

Now one can slice the manifold into surfaces of constant f. The level-plane will have a smooth geometry barring the slice with critical points. The function near the critical point can be written as:

$$f(x) = \sum_{i=1}^{d} a_i x_i^2 \tag{3.9}$$

For $d \geqslant 3$ one has a singularity of the form

$$R \propto r^{-2} \tag{3.10}$$

where:

$$r^2 = \sum x_i^2 \tag{3.11}$$

one should note the topology that is the creation of the universe is actually very special, as there is no initial configuration. The boundary of the superspace are either regular or singular. The regular boundary refers to those which can be obtained by slice of manifold. These boundaries correspond to the transition between different topological sectors. The idea of this is the probability flux should be conserved between sector transition. However, the out-going wave boundary condition are corresponds to the singular part of the boundary. The whole picture is the probability flux is injected by the creation of the universe and then flow between different topological sector.

Vilenkin state that he do not believe that picture is the general case, as the topology transition are not required to exist between configurations at the boundary of topological sectors. The start point or the end point can be in the interior of the sectors. This implies that the Wheeler-Dewitt equation has to be modified to accommodate the topological change. One possible modification is adding an operator $\tilde{\delta}$ into the Hamitonian

$$S = \int [dh]\psi^* H\psi + \int [dh][dh']\psi^*(h)\tilde{\delta}(h,h')\psi(h') \tag{3.12}$$

Varying :

$$H\psi_N(h) + \sum \int [dh']\tilde{\delta}_{NN'}(h,h')\psi_{N'}(h') = 0 \tag{3.13}$$

one can obtain the Wheeler-Dewitt equation. The singular part of the boundary contains configuration with superpotenial goes to infinity and the null part of boundary where α goes to negative infinity. The outgoing wave would carry probability flux:

$$J_N = i(\psi_N^* \nabla\psi_N - \psi_N \nabla\psi_N^*) \tag{3.14}$$

The flux flowing into and out-of the regular boundary is the transitions between different sectors. In this case the wavefunction is equivalent to the wavefunction defined by the path integral formulation.

Another approach to quantum cosmology is third quantization. One can promote the ψ to be a quantum operator. Topology change that we have discussed above is then the self-interacting ψ. This view is good for one dimensional universe. However, things get more complicated for higher dimensional situation. Therefore, it is still unclear whether this approach provides any additional insight comparing the approach that we have just discussed.

3.2 Equivalence of two formulations

The Chern-Simons state 3.15, also known as the Kodama state, is not purely imaginary and hence criticized for its non-normalizability.

$$\psi(A) = \mathscr{N}\exp(-\frac{3}{2l_p^2\Lambda}Y_{CS}) \tag{3.15}$$

We can derive this state using the Einstein-Cartan formulation. The action is given by 3.16.

$$S = 3\kappa V_c \int dt(2a^2\dot{b} + 2Na(b^2 + k - \frac{\Lambda}{3}a^2)) \tag{3.16}$$

Here $\kappa = \frac{1}{16\pi G_N}$ and a is the expansion factor $V_c$is the co-moving volume of the region under study.

Using the knowledge from them classical physics, we can write out the Poisson bracket from this action. Hence, we can quantize it and write out the commutation relationship. 3.17

$$[\hat{b}, \hat{a}^2] = \frac{il_P^2}{2V_c} \tag{3.17}$$

Hence we can give the form of the two operator explicitly.

$$\hat{a}^2 = -\frac{il_P^2}{2V_c}\frac{d}{db} \tag{3.18}$$

$$\hat{b} = \frac{il_P^2}{2V_c}\frac{d}{da^2} \tag{3.19}$$

Hence we can write the two version of Hamiltonian constraint equation. The solution to those equations are just the Chern-Simons state and Hartle and Hawking and the Vilenkin or tunnelling wave functions depending on the boundary conditions. The Chern-Simons state can be seen as the Fourier transform of the Hartle-Hawking and Vilenkin wave function 3.20

$$\psi_{a^2}(a^2) = \frac{3V_c}{l_p^2}\int \frac{db}{\sqrt{2\pi}} e^{-i\frac{3V_c}{l_p^2}a^2 b}\psi_b(b) \tag{3.20}$$

As previous mentioned, Hartle-Hawking and Vilenkin wave function just differ by the boundary conditions, this means they integrate over different path of b if we treat them as the Fourier transform of the Chern-Simons state.

One can write the Vilenkin solution as:

$$\psi_v \propto Ai(-z) + iBi(-z) \tag{3.21}$$

and Hartle-Hawking as:

$$\psi_H \propto Ai(-z) \tag{3.22}$$

Airy function can represent as an integration:

$$\phi(z) = \frac{1}{2\pi}\int e^{i(\frac{t^3}{3}+zt)}dt \tag{3.23}$$

The integration contour must start and finish as complex infinity within the following 3 regions:

$$\begin{aligned} &S1 : 1 < arg(t) < \frac{\pi}{3}; \\ &S2 : \frac{2\pi}{3} < arg(t) < \pi; \\ &S3 : \frac{5\pi}{3} < arg(t) < \frac{7\pi}{3}; \end{aligned} \tag{3.24}$$

The Hartle-Hawking wave function start at S2 and ends at S1 and the Vilenkin wave function start at S3 and ends at S1. [6]

If one can accept an extended sense of convergence, it is possible to let the regions defined in equation 3.24 to be non-strict. Then one need to include delta function when try to normalize CS state.

We need to note some special case. First, if b is real, the only possible wavefunction dual to CS state are the Hartle-Hawking wavefunction. In this case the contour of integration in equation 3.23 is taken at the real line. However, if we decide to use the strict convergence condition i.e. exclude the delta-function, we can shift the contour:

$$b \to b + i\eta \tag{3.25}$$

and take the limit where η goes to 0.

The Vilenkin wavefunction is different, as it requires the contour over real b not over the negative part of the axis. This is because the Vilenkin wavefunction can only have the outgoing mode. However, the contour that starts at 0 and then goes over the positive part of the axis is also not acceptable. This contour will give out a wavefunction that solves:

$$\psi'' + z\psi = \frac{1}{\pi} \tag{3.26}$$

instead of the Airy/WdW equation.

Hence, the Vilenkin wavefunction requires the imaginary part of b. Alternative way to achieve the same result may be found. If one use the stationary approximation to equation 3.23, the WKB approximation can be recovered. We can write the integral as e^{iS} and write:

$$\psi_{a^2}(a^2) \propto \int \frac{db}{2\pi} \exp[\frac{9iV_c}{\Lambda l_P^2}(\frac{b^3}{3} + kb - \frac{\Lambda b a^2}{3})] \tag{3.27}$$

so that:

$$\frac{\partial S}{\partial t} \propto \frac{\partial S}{\partial b} \propto H = b^2 + k - \frac{\Lambda a^2}{3} \tag{3.28}$$

Hence the stationary points of S are the solutions to Hamiltonian constraint:

$$b_{\pm} = \pm\sqrt{\frac{\Lambda a^2}{3} - k} \tag{3.29}$$

Using Taylor expansion, one have:

$$S_{\pm} = -\frac{2}{3}t_{\pm}^3 + t_{\pm}(t - t_{\pm})^2 \tag{3.30}$$

This is the WKB solutions. More specifically, this is the HH wavefunction if we include both S_+ and S_-, and this is the Vilenkin wavefunction if only S_+ is included.

Now we need to discuss the implication of this mathematical equivalence. First, one must note the Chern-Simons wavefunction defined for $b \in D_1 = \mathbb{R}$

and for $b \in D_2 = (-i\infty, 0) \cup (0, \infty)$, as the function itself does not specify a quantum state. We must specify a contour in order to do so. Secondly, one must also note that the Cherin-Simons state and Hartle-Hawkings and Vilenkin state are just same quantum state in different representation. One can easily write these in the quantum form:

$$\psi_V(a^2) = \langle a^2|\psi_V\rangle \tag{3.31}$$

$$\psi_{CS}(b; b \in D_2) = \langle b|\psi_V\rangle \tag{3.32}$$

In Ashtekar formulation, we require E^a_I to be real or hermitian and the anti-self dual to be complex conjugate to the self dual connection. This then implies that a^2 and b must be real. Thus it rejects the Vilenkin function. However, if we change the interpretation of the condition, the conclusion may be changed. It is worth mention here that both Hartle-Hawking and Vilenkin wavefunction can live in classically forbidden region.

Reversely, this requirement only says that a^2 has to be real. It does not require it to be positive.

The Chern-Simons theory has been criticised for its non-normalizability. But has we have already established, this will not be the case for Euclidean formulation. It is possible to imitate the MSS treatment. First, we propose a modified state:

$$\psi_{CS} = N' \exp(-\frac{3i}{l_p^2\Lambda}\Im Y_{CS}) \tag{3.33}$$

The state is normalised as a plane wave across all space. That is to say a delta function normalisation with real b.

The prediction one can make is that b will distribute uniformly over the real line. One can use the momentum cut-off method here. This is the same as the distribution of a^2 implied by Hartle-Hawing wavefunction. In the a^2 representation, we can write:

$$\int_{-\infty}^{\infty} dz \psi^*_{HH}(z + x)\psi_{HH}(z + y) = \delta(x - y) \tag{3.34}$$

We may think that the Vilenkin state in b space predicts that b will be distributed uniformly over the real line. The prediction is:

$$P_V(\Im b) = \frac{1}{2\pi} \exp[\frac{18V_c}{\Lambda l_p^2}(\frac{\Im b^3}{3} - k\Im b)] \tag{3.35}$$

This means that P is peak at b=-i and the decrease exponentially. However, one would find several problems if one tries to map this into a^2 space.

Tunnelling state is suggested to be non-zero for $a > 0$. In this case the state will solve a modified WdW equation. Vilenkin state then will be a different wavefunction:

$$\psi_{V_1} = (a^2) = \langle a^2|\psi_{V_1}\rangle = \psi_V(a^2)\theta(a^2) \tag{3.36}$$

One should note this wavefunction's dual is no longer a Chern-Simons wavefunction. In order for the relation to hold, one must modify the tunnelling state again:

$$\psi_{V_2}(b) = \langle a^2|\psi_{V_2}\rangle = \psi_V(b)\theta(b) \tag{3.37}$$

and then one can get the tunnelling wavefunction in a^2 space:

$$\psi_{V_2}(a^2) = \langle a^2|\psi_{V2}\rangle \propto Ai) - z) + iGi(-z) \tag{3.38}$$

This state only have outgoing mode, just like Vilenkin proposal. The wavefunction is defined over a contour that obeys the reality condition. The integral over the imaginary axis has form:

$$\phi_{IM}(z) = \frac{i}{\pi}\int_{-\infty}^{0} e^{(\frac{t^3}{3}-zt)}dt = iHi(-z) \tag{3.39}$$

3.3 Application to Other Cases

Until now we have assumed a homogeneous universe, which is the case at large scale. However, the homogeneity is broken at small scale. [33]Classically, we attribute this to a small perturbation $\delta\rho/\rho \approx 10^{-4}$.However, this is not explained in the big bang model. Physicist used to contribute this to the boundary conditions. However, a new solution is possible, i.e. this is due to the pre-inflationary quantum fluctuation and then amplified by the inflation. One should note that the quantum fluctuation calculated depends on the vacuum state one choose. However, there is no clear choice here. Hence, we need to discuss this in superspace model.

Before actually going into the cosmology case, we can first revisit the quantum field theory in curved space time. Let us consider the model of which :

$$S_m = -\frac{1}{2}\int d^4x\sqrt{-g}[(\partial\Psi)^2 + m^2\Psi^2] \tag{3.40}$$

One can quantise this by using:

$$(\Box - m^2)u_k(x,t) = 0 \tag{3.41}$$

where u_k is the mode functions. The field operator is then:

$$\tilde{\Psi}(x,t) = \sum_k(\hat{a}_k u_k + \hat{a}_k^\dagger u_k^*) \tag{3.42}$$

The vacuum state is then given by:

$$\hat{a}_k|0\rangle = 0 \tag{3.43}$$

Unlike in Minkowski space, there is no unique choice of vacuum state. A functional Schrodinger picture can be used. First one need to break the action down to a (3+1) form.

$$S_m = \frac{1}{2}\int d^3xdtNh^{\frac{1}{2}}[\frac{\dot{\Psi}^2}{N^2} - h^{ij}\partial_i\Psi\partial_j\Psi + m^2\Psi^2] \tag{3.44}$$

The Hamiltonian is then:

$$H_m = \frac{1}{2}\int d^3xNh^{\frac{1}{2}}[h^{-1}\pi_\Psi^2 + h^{ij}\partial_i\Psi\partial_j\Psi + m^2\Psi^2] \tag{3.45}$$

The quantum state is represent by a wave functional instead of a function. The functional is governed by the Schrodinger equation:

$$i\frac{\partial\Psi_m}{\partial t} = H_m\Psi_m \tag{3.46}$$

where the momentum operator is:

$$\pi_\Phi(x) \to -i\frac{\delta}{\delta\Phi(x)} \tag{3.47}$$

The Heinsenburgh and Shcrodinger picture stated are related as they are in normal quantum field theory, namely:

$$|\Psi_S(t)\rangle = \exp(-i\int^t dt' H_m(t'))|\Psi_H\rangle \tag{3.48}$$

The wavefuntional is defined as:

$$|\Psi_S\rangle = \int D\Phi(x)|\Phi\rangle\Psi_S[\Phi(x)] \tag{3.49}$$

Then the choice of vacuum state before becomes the choice of solution to the functional equation.

It is then can be shown the solution is:

$$\Psi(q,\Phi) = C(q)e^{im_p^2 S_0(q)}\tilde{\psi}(q,\Phi) \tag{3.50}$$

Hence the Wheeler-DeWitt equation reduce to the normal quantum field theory.

Now we will examine how to pick out the vacuum state for the quantum cosmology. As physicist are interested in inflation, vacuum state in de-Sitter space is of particular importance. The vacuum state here is known as the Euclidean vacuum.

There is one vacuum that is invariant under the Poincare group and thus holds the same for all observers in Minkowski space. And such vacuum is unique up to a trivial Bogoliubov transformation. The de-Sitter version of Poincare group is known as the de Sitter group.

The symmetric two-point function is defined as:

$$G_\lambda(x,y) = \langle\lambda|(\Phi(x)\Phi(y) + \Phi(y)\Phi(x)|\lambda\rangle. \tag{3.51}$$

A de-Sitter invariant state is such a state that the two-point function depends on the geodesic distance between x and y and is independent of x and y otherwise. Hence, the following holds:

$$G_\lambda(x,y) = f_\lambda(\mu) \tag{3.52}$$

Unlike the situation in Minkowski space, there are more than one de-Sitter invariant vacuum. However, one can identify a one-parameter family of de-Sitter vacuum.

Let $\bar{x}$ be the point that is antipodal to the point x at the de-Sitter space. Then one can find two poles in f_λ i.e. the one where x is at the light-cone of y and the one where y is on the light-cone of $\bar{x}$.

Another equivalent way to obtain the Euclidean vacuum is through a particular choice of mode functions. First one can write the field operator in forms of mode functions:

$$\hat{\Phi}(x,t) = \sum_{nlm} (u_{nlm}(x,t)\hat{a}_{nlm} + u*_{nlm}(x,t)\hat{a}^\dagger_{nlm} \tag{3.53}$$

Then the vacuum state is defined by:

$$\hat{a}_{nlm}|0\rangle = 0 \tag{3.54}$$

Now onw need to choose the mode functions:

$$u_{nlm}(x,t) = y_n(t)Q^n_{lm}(x) \tag{3.55}$$

where Q is the harmonics and y obeys:

$$\ddot{y}_n + 3\frac{\dot{a}}{a}\dot{y}_n + (\frac{n^2-1}{a^2} + m^2)y_n = 0 \tag{3.56}$$

The normalization is done through the Wroskian condition:

$$y_n\dot{y}*_n - y*_n\dot{y}_n = \frac{i}{a^3} \tag{3.57}$$

The vacuum is then defined as the $y_n(t)$ being regular on the Euclidean section. Euclidean section is obtained by changing $t \rightarrow -i(\tau - \frac{\pi}{2H}$.

Now one should regard the mode to be perturbations on a homogeneous and isotropic background. The Hartle-Hawking wave-function is given by:

$$\Psi_{NB}(\tilde{a},\tilde{\Psi}) = \int Dg_{\mu\nu}D\Psi \exp(-I_g[g_{\mu\nu}] - I_m[g_{\mu\nu},\Phi]) \tag{3.58}$$

Around saddle-point one have:

$$\Psi_{NB}(\tilde{a},\tilde{\Phi}) \approx \exp(-I_g[\tilde{g}_{\mu\nu}])\int D\Phi \exp(-I_m[\bar{g}_{\mu\nu},\Phi]) \tag{3.59}$$

where $\bar{g}$ is the metric near the saddle point. Whether g is real or complex depends on whether aH is smaller or larger than 1. If $aH < 1$, the metric gives a four sphere with a three sphere of radius a.

The matter wave functional is given by the following formula:

$$\psi[\tilde{a},\tilde{\Phi}] = \int D\Phi \exp(-I_m[\bar{g}_{\mu]nu},\Phi]) \tag{3.60}$$

One can expand the scalar field using three-sphere harmonics:

$$\Phi(x,\tau) = \sum_{nlm} f_{nlm}(\tau) Q^n_{lm}(x) \tag{3.61}$$

The Euclidean action is then written as:

$$I_m[a(\tau),\Phi] = \frac{1}{2}\sum_{nlm}\int_0^1 d\tau N a^3 [\frac{1}{N^2}(\frac{df_{nlm}}{d\tau}^2 + (\frac{n^2-1}{a^2}+m^2) f^2_{nlm}] \tag{3.62}$$

The field equation is then:

$$\frac{d^2 f_{nlm}}{d\tau^2} + \frac{3}{a}\frac{da}{d\tau}\frac{df_{nlm}}{d\tau} - N^2(\frac{n^2-1}{a^2}+m^2) f_{nlm} = 0 \tag{3.63}$$

There two constrains for the background a(0)=0 and a(1)=$\tilde{a}$. More explicitly one can write:

$$a(\tau) - \frac{1}{H}\sin(NH\tau),\ N = \frac{1}{H}(\frac{\pi}{2} - \cos^{-1}(\tilde{a}H)) \tag{3.64}$$

We can write the solution in terms of hypergeometric functions. The solutions are regular except for region near $\tau = 0$. In the region where $\tau = 0$, one have:

$$a(\tau) \sim N\tau \tag{3.65}$$

and it can be shown that the solutions will be like τ^{-n-1} or τ^{n-1}. However, one can still pick out a regular case in this region by imposing:

$$f_{nlm}(0) = 0, for\ n = 2,3,4... \tag{3.66}$$

and

$$\frac{df_{nlm}}{d\tau}(0_{=}0\ for\ n = 1 \tag{3.67}$$

Another condition will also be satisfied:

$$f_{nlm}(1) = \tilde{f}_{nlm} \tag{3.68}$$

One may write:

$$\psi[\tilde{a},\tilde{\Phi}(x)] = \prod_{nlm}\psi_{nlm}(\tilde{a},\tilde{f}_{nlm}) \tag{3.69}$$

Then one can write:

$$\psi(\tilde{a},\tilde{f}_{nlm}) = \int D f_{nlm} e^{-I_{nlm}} \tag{3.70}$$

As I_{nlm} is quadratic here, equation 3.70 yield an expression:

$$\psi(\tilde{a},\tilde{f}_{nlm}) = A_{nlm}(\tilde{a})\exp(-\bar{I}_{nlm}(\tilde{a},\tilde{f}_{nlm})) \tag{3.71}$$

where $\tilde{I}$ is the action to the field equation with boundary condition 3.66, 3.67 and 3.68. One can denote this solution as g_n and then have:

$$I_{nlm}(\tilde{a},\tilde{f}_{nlm}) = \frac{1}{2}[a^3(\tau) g_n(\tau)\frac{dg_n(\tau)}{d\tau}]^1_0 = \frac{1}{2}\tilde{a}^3 \tilde{f}^2_{nlm}[\frac{1}{g_n}\frac{dg_n}{d\tau}]_{\tau=1} \tag{3.72}$$

The matter wavefunction in Hartle-Hawking theory is therefore:

$$\psi_{nlm}(\tilde{a}, \tilde{f}_{nlm}) = A_{nlm}(\tilde{a}) \exp(\frac{1}{2}\tilde{a}^3 \tilde{f}^2_{nlm}[\frac{1}{g_n}\frac{dg_n}{d\tau}]_{\tau=1}) \tag{3.73}$$

We should note that it involves $\frac{\dot{g}_n}{g_n}$. In order to prove that this wave functional is a Euclidean vacuum state, one need to determine what the vacuum state looks like in this picture. First we promote f_{nlm} to be an operator:

$$\hat{f}_{nlm}(t) = y_n(t)\hat{a}_{nlm} + y*_{nlm}(t)\hat{a}^\dagger_{nlm} \tag{3.74}$$

The momentum operator is then:

$$\hat{\pi}_{nlm}(t) = a^3\dot{\hat{g}}_{nlm} = a^3\dot{y_n}(t)\hat{a}_{nlm} + a^3\dot{y}^*_n(t)\hat{a}^\dagger_{nlm} \tag{3.75}$$

One can write these in reverse form:

$$\hat{a}^\dagger_{nlm} = -iy^*_n(a^3\frac{\dot{y*}_n}{y*_n}\hat{f}_{nlm} - \hat{\pi}_{nlm}) \tag{3.76}$$

Thus the vacuum state would obey:

$$(a^3\frac{\dot{y*}_n}{y*_n}\hat{f}_{nlm} + i\frac{\partial}{\partial f_{nlm}})\psi_{nlm}(f_{nlm}) = 0 \tag{3.77}$$

We can solve this equation and obtain:

$$\psi_{nlm} = \exp(\frac{i}{2}a^3\frac{\dot{y*}_n}{y*_n}f^2_{nlm}) \tag{3.78}$$

This is thus the vacuum in the functional Schrodinger picture. We can therefor write:

$$\psi_{nlm} = \exp(-\frac{1}{2}a^3\frac{1}{y*_n}\frac{dy*_n}{d\tau}f^2_{nlm}) \tag{3.79}$$

Hence, we have proven that the solution is indeed a vacuum state. And y_n and $y*_n$ is regular one the Euclidean section.

I can be shown that the Vilenkin approach can also pick out the same vacuum state.

Chapter 4

Cosmology with Torsion

4.1 Classical theory with Torsion

4.1.1 Early Universe

As stated before, the normal general relativity are formulated as a torsion-free theory. However, as we have fermionic particles in our universe and we have not yet include in the standard theory of general relativity, it seems that one should consider the effect of spin to the manifold. Thus, it seems plausible to introduce torsion into the theory and explain some of the phenomenon that we observe.[41][42][43][44][45][46] One example would be using to torsion to explain the cosmic inflation [20].

Cosmic inflation are the reason why today's universe is homogeneous and isotropic. However, there is no explanation why such rapid expansion exist. One proposal is to introduce intrinsic angular momentum of matter and explain the homogeneity without the inflation. Such a theory is called Einstein-Cartan-Kibble-Sciama theory of gravity.

In this theory the restriction of non-torsion is removed. Torsion tensor is added to the theory and is treated like a dynamical variable. We define the energy-momentum tensor as:

$$\sigma_i^i - \Theta_i^j/\sqrt{-det g_{mn}}. \tag{4.1}$$

and the spin tensor of matter as:

$$s_{ij}^k = \sigma_{ij}^k/\sqrt{-det g_{nm}} \tag{4.2}$$

And we can vary the total action with metric and obtain:

$$-\frac{1}{2}Rg_{ik} + R_{ik} = \kappa\sigma_{ki} \tag{4.3}$$

Also, if we vary the action with respect to torsion, we can obtain:

$$S_{jik} - S_{il}^l g_{jk} + S_{kl}^l g_{ji} = -\frac{1}{2}\kappa s_{ijk} \tag{4.4}$$

The equation 4.3 gives the relationship between the curvature and tensor σ and the equation 4.4 gives relationship between the torsion and spin tensor.

One can define a symmetric tensor:

$$T_{ik} = \sigma_{ik} - \frac{1}{2}(\nabla_j - 2S^l_{jl})(s^j_{ik} - s^j_{k\ i} + s^j_{ik} \tag{4.5}$$

Combine above relationships together, one have:

$$\begin{aligned} G^{ik} &= \kappa T^{ik} + \frac{1}{2}\kappa^2(s^{ij}_j s^{kl}_l - s^{ij}_l s^{kl}_j - s^{ijl}s^k_{jl} + \frac{1}{2}s^{jli}s^k_{jl} \\ &+ \frac{1}{4}g^{ik}(2s^l_{jm}s^{jm}_l - 2s^l_{jl}s^{jm}_m + s^{jlm}s_{jlm})) \end{aligned} \tag{4.6}$$

One should note if spin vanishes, this equation will reduce to normal Einstein equation.

Now we need to consider the quarks and leptons. At macroscopic scale, these particles can be described by spin fluid model. The canonical energy momentum tensor is:

$$\sigma_{ij} = c\Pi_i u_j - p(g_{ij} - u_i u_j) \tag{4.7}$$

The spin tensor is:

$$s^k_{ij} = s_{ij}u^k,\ s_{ij}u^j = 0 \tag{4.8}$$

The theory including spin fluid gives:

$$\begin{aligned} G^{ij} &= \kappa(\epsilon - \frac{1}{4}\kappa s^2)u^i u^j - \kappa(p - \frac{1}{4}\kappa s^2)(g^{ij} - u^i u^j) \\ &- \frac{1}{2}\kappa(\delta^l_k + u_k u^l)\nabla_l(s^{ki}u^j + s^{kj}u^i) \end{aligned} \tag{4.9}$$

As we know the universe we observe can be described by Friedman-Lemaitre-Robertson-Walker metric:

$$ds^2 = c^2dt^2 - \frac{a^2(t)}{(1+kr^2/4)^2}(dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2) \tag{4.10}$$

We can combine this with equation 4.9 and obtain:

$$\dot{a}^2 + 1 = \frac{1}{3}\kappa(\epsilon - \frac{1}{4}\kappa s^2)a^2 \tag{4.11}$$

$$\dot{a}^2 + 2a\ddot{a} + 1 = -\kappa(p - \frac{1}{4}\kappa s^2)a^2 \tag{4.12}$$

and the conservation law:

$$\frac{d}{dt}((\epsilon - \kappa s^2/4)a^3) + (p - \kappa s^2/4)\frac{d}{dt}(a^3) = 0 \tag{4.13}$$

The conservation law is the equivalent of second Friedman equation.

The average density of particle is given by :

$$\frac{dn}{n} = \frac{d\epsilon}{\epsilon + p} \tag{4.14}$$

where ϵ is the energy density.

Let us assume that the fluid obeys the barotropic equation:

$$p = \omega\epsilon \tag{4.15}$$

Then the spin density for the fluid will be:

$$s^2 = \frac{1}{8}(\hbar cn)^2 \tag{4.16}$$

and subsequently:

$$\epsilon \propto a^{-3(1+w)} \tag{4.17}$$

The total energy density caused by the spin density is given by:

$$\epsilon_S = -\frac{1}{4}\kappa s^2 \propto a^{-6} \tag{4.18}$$

It is apparently independent of w. Thus, ϵ_S decouples from ϵ.

These particles has energy larger than rest energy at the very early universe. Thus, the relativistic effect must be considered. As we know, the mist common particles at the time was background photons and neutrinos. Thus, we have $\epsilon \approx \epsilon_R \approx \epsilon_\gamma + \epsilon_\nu$. Hence, we can write the first Friedman equation as:

$$H^2 + \frac{c^2}{a^2} = \frac{1}{3}\kappa(\epsilon + \epsilon_S)c^2 \tag{4.19}$$

The total density parameter can be written as:

$$\Omega(\hat{a}) = \frac{\kappa c^2}{3H^2}(\epsilon + \epsilon_S) \tag{4.20}$$

and it will obey:

$$a|H|\sqrt{\Omega(\hat{a} - 1} = c \tag{4.21}$$

If we combine these equations together, we have:

$$|H| = H_0(\Omega_R \hat{a}^{-4} + \Omega_S \hat{a}^{-6})^{\frac{1}{2}} \tag{4.22}$$

where:

$$\Omega_R = \frac{\epsilon_r}{\epsilon_S},\ \Omega_S = \epsilon_S/\epsilon_c \tag{4.23}$$

Hence we can write the total density parameter as a function of $\hat{a}$:

$$\Omega(\hat{a}) = 1 + \frac{(\Omega - 1)\hat{a}^4}{\Omega_R \hat{a}^2 + \Omega_S} \tag{4.24}$$

Now we can show the torsion can prevent the singularity. When the expansion started, we have $\hat{a} = \hat{a}_m$ where:

$$\hat{a}_m = \sqrt{-\frac{\Omega_S}{\Omega_R}} = 3.1 \times 10^{-33} \tag{4.25}$$

When the universe contract with $H < 0$, we have:

$$-\frac{\Omega_R^{3/2}}{\Omega_S} t = f(x) = \frac{x}{2}\sqrt{x^2 - 1} + \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| \tag{4.26}$$

When $x \gg 1$, we have the usual radiation-dominated universe. In usual general relativity, we set $\Omega_S = 0$ and will have flatness problem. This is solved by introduce cosmic inflation. In the ECKS gravity, we have a $\Omega_S < 0$. Hence, $\Omega(\hat{a})$ will be infinite at $\hat{a}_m$ and be at its minimum at $\sqrt{2}\hat{a}_m$:

$$\Omega(\sqrt{2}\hat{a}_m) = 1 - \frac{4\Omega_S(\Omega - 1)}{\Omega_R^2} = 1 + 8.9 \times 10^{-64} \tag{4.27}$$

This way the universe seems to expand and rapidly reduce Ω to the level we observe today. The time taken is :

$$t = -\frac{\Omega_S}{\Omega_R^{3/2} H_0} f(\sqrt{2}) = 5.3 \times 10^{-46} s \tag{4.28}$$

In this model, the universe only expand $\sqrt{2}$ of its size, and it is naturally caused by a small negative torsion. In cosmic inflationary model, we have to introduce a 20^{26} scale inflation.

We also have the relation:

$$\dot{a} = \frac{1}{\sqrt{\Omega(\hat{a}) - 1}}. \tag{4.29}$$

We can calculate the velocity of the point that is antipodal to the origin:

$$v_a = \pi c \dot{a} \tag{4.30}$$

and find its maximum which is $1.1 \times 10^{32} c$. The velocity increase from 0 to this value just when the universe expands to $\sqrt{2}\hat{a}_m$. As the universe expand further, the velocity will decrease. This has important implication i.e. if the universe is casually connected before the expansion, it will remain connected afterwards. Thus, the horizon problem is solved by introduce a small negative torsion.

Now we can discuss another recently developed model involving torsion. [22] [19] Again we will consider the Einstein equation but with torsion:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \tag{4.31}$$

We can write the energy momentum tensor in terms of torsion by using Cartan field equation:

$$S_{\alpha\nu\mu} = -\frac{1}{4}\kappa(2s_{\mu\nu\alpha} + g_{\nu\alpha}s_\mu - g_{\alpha\mu}s_\nu). \tag{4.32}$$

The Friedmann equation is:

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} - 4\phi^2 - 4(\frac{\dot{a}}{a})\phi \tag{4.33}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) - 2\dot{\phi} - 2(\frac{\dot{a}}{a})\phi \tag{4.34}$$

For the barotropic matter, we have:

$$\dot{\rho} + 3(1+\omega)H\rho + 2(1+3\omega)\phi\rho = 0 \tag{4.35}$$

We can define a deceleration parameter to help us understand the effect of torsion:

$$q = \frac{4\pi G}{3H^2}(\rho + 3o) + 2\frac{\dot{\phi}}{H^2} + 2\frac{\phi}{H} \tag{4.36}$$

Now we have several choices for the torsion function. But the choice with the best property is:

$$\phi(t) = -\alpha H(t)(\frac{\rho_m(t)}{\rho_{0c}})^n \tag{4.37}$$

This case means the torsion is from the spin of ordinary matter. In this case we have a clear solution to equation 4.35:

$$\rho_m(a) = \rho_{0c}\frac{3^{1/n}}{2\alpha + 3C_1(a/a_0)^{3n})^{1/n})} \tag{4.38}$$

In this case, the Friedmann equation will explicitly be:

$$H^2 = \frac{8\pi G}{3}\rho_m - \frac{k}{a^2} + 4\alpha H^2(\frac{\rho_m}{\rho_{0c}})^n - 4\alpha^2 H^2(\frac{\rho_m}{\rho_{0c}})^{2n} \tag{4.39}$$

The most general and realistic case is however:

$$\phi(t) = -\alpha H_0(\frac{H_0}{H(t)})^m(\frac{\rho_m(t)}{\rho_{0c}})^n \tag{4.40}$$

Now we choose a special case where k=0 and write the Friedmann equation as:

$$H^2 = \frac{8\pi G}{3}\rho_m + 4\alpha H_0^{m+1}H^{-m+1}(\frac{\rho_m}{\rho_{0c}})^n - 4\alpha^2 H_0^{m+1}H^{-m+1}(\frac{\rho_m}{\rho_{0c}})^{2n} \tag{4.41}$$

We can derive the following equation from the Friedmann equation:

$$\Omega_m = (1 + \frac{2\phi_0}{H_0})^2 = (1 - 2\alpha\Omega_m^n)^2 \tag{4.42}$$

Now let us assume that $m = n \approx 1$ and $k = 0$. Therefore, we obtain:

$$H(t)^2 = \frac{1}{3}\kappa\rho(t) - 4\phi^2 - 4H(t)\phi(t) \tag{4.43}$$

and

$$\phi(t) = -\alpha H_0(\frac{H_0}{H(t)})(\frac{\rho_m(t)}{\rho_{0c}}) \tag{4.44}$$

One can see that $\phi(t)$ is very attached to constants. Combining equation 4.43 and 4.44, we have:

$$\frac{H_0^2}{\rho_0} = [\frac{\kappa}{3(1-2\alpha)^2}] \tag{4.45}$$

Hence we can define a more general form of ϕ:

$$\phi(t) = -\alpha\frac{\kappa}{3(1-2\alpha)^2}\frac{\rho(t)}{H(t)} \tag{4.46}$$

The energy of the torsion will dominate the early universe as the dark matter are not yet fully formed. Just as mentioned before, this will cause a inflationary phase. Assuming $\dot{\rho} \sim 0.$, we have:

$$\rho \approx \rho_0[\frac{3}{2}\frac{(1+\omega}{\alpha(1+3\omega)}\frac{H^2}{H_0^2}] \tag{4.47}$$

We can assume that the Hubble radius is constant before and after the inflation and find the relation: ;

$$\alpha = \frac{3}{2}\frac{(1+\omega)}{(1+3\omega)} \tag{4.48}$$

This is the constraint on α and ω. We can now use the condition for the initial matter density:

$$\rho_0 = \frac{2\alpha H_0^2}{\kappa}\frac{(1+3\omega)}{(1+\omega)}(1-2\alpha)^2 \tag{4.49}$$

The solution for a(t) at the inflationary era is then:

$$a(t) = a_i e^{\beta t} \tag{4.50}$$

where:

$$\beta = (\frac{\kappa}{3}\frac{\rho_0}{(1-2\alpha)^2})^{1/2} \tag{4.51}$$

Now we can consider the classical result of the scale factor. Using equation 4.46 and the continuity equation, we can have:

$$\dot{\rho} + 3(1+\omega)\gamma\rho^{3/2} - \frac{2\alpha}{\gamma\rho_i}(1+3\omega)\rho^{3/2} = 0 \tag{4.52}$$

The solution will be:

$$\rho = \frac{4\rho_i}{(\beta_i(t-t_i)\sqrt{\rho_i}+2)^2} \tag{4.53}$$

We know that $H = \gamma\sqrt{\rho}$. Hence, the solution for scale factor is:

$$a(t) = \frac{a_i}{2}(2+\beta_i\sqrt{\rho_i}(t-t_i))^{\frac{2}{3(1+\omega)-2\alpha(1+3\omega)}} \tag{4.54}$$

The solution to matter density is:

$$\rho(t) = \rho_i (\frac{a(t)}{a_i}^{3(1+\omega)-2\alpha(1+3\omega)} \quad (4.55)$$

Now we will try to work out the linearized Einstein equation in Poisson gauge. The perturbed metric we will use is:

$$ds^2 = -a^2(\eta)(1-2\Phi)d\eta + a^2(\eta)[(1+2\Psi)\delta_{ij} + 2t_{ij}]dx^i dx^j \quad (4.56)$$

where Φ Ψ are the Bardeen potentials.

From this metric one can show that if one set $H = \Phi = 0$, one can obtain the normal Newtonian limit.If we exclude torsion's contribution to the tensor perturbations, we have:

$$t''_{ij} + 2Ht'_{ij} - \Delta t_{ij} = -16\pi G a^2 \delta T^i_{j(T)} \quad (4.57)$$

Using the linearized Einstein equation, we have the equation:

$$\Psi'' - c_s^2 \Delta\Psi + 3(1-c_s^2)\beta\Psi' + \Psi(\beta^2(1+3c_s^2) + 2\beta') = 4\pi G a^2 \delta S \quad (4.58)$$

If we assume $\Psi = u(\eta, x^i) f(\eta)$, we can write:

$$u'' - c_s^2 \Delta u - u(\frac{\theta''}{\theta} + G) = S \quad (4.59)$$

A special case is the adiabatic perturbation where we have:

$$u'' - c_s^2 \Delta u - u(\frac{\theta''}{\theta} + G) = 0 \quad (4.60)$$

There are two solutions to this equation, one at the long-wavelength limit and another at the short wavelength limit.

In the long-wavelength limit, equation 4.60 will take the form:

$$u'' - u(\frac{\theta''}{\theta} + G) = 0 \quad (4.61)$$

Here G is the contribution from the torsion.

Let u_{can} be a solution of canonical Mukhanov-Sasaki equation. We can assume the solution to the equation 4.61 is $u = u_{can}h$. Hence, we have:

$$h'' + 2(\frac{u''_{can}}{u_{can}})h' - Gh = 0 \quad (4.62)$$

Using equation 4.46, we can see that G is a small term. Thus, we have:

$$h = 1 + \int \frac{1}{u_{can}}[\int G u_{can}^2 d\eta] d\eta \quad (4.63)$$

Therefore one can find the equation for ψ is

$$\Psi = \Psi_{can} h e^{2\int \phi d\eta} \quad (4.64)$$

Using this equation, one can write out the power spectrum:

$$P_s \propto k^{n_s-1} \tag{4.65}$$

One can also find equation to determine the index:

$$n_s - 1 = \frac{d\ln P_s}{d\ln k} \tag{4.66}$$

One can solve this equation and obtain:

$$n_s = 1 - 3(1+\frac{p}{\rho})_{can} - \frac{8\alpha^2}{3-6\alpha} \tag{4.67}$$

Now we can fix the value of α between -0.157 and 0.12 and get:

$$0.92 \leqslant n_s \leqslant 0.97 \tag{4.68}$$

In short wavelength limit, similar method is used. First we can write out equation 4.60 in this limit:

$$u'' + k^2 u = 0 \tag{4.69}$$

The solution is:

$$\Psi = \Psi_{can} e^{-2\int \phi d\eta} \tag{4.70}$$

Then we can find the equation to determine the spectral index:

$$n_s = 1 - 3(1+\frac{p}{\rho})_{can} - \frac{8\alpha^2}{3-6\alpha} \tag{4.71}$$

It is clear that in both case the contribution of torsion to the spectral index is the same.

Another data worth noting is the tensor to scalar ratio. This remains unchanged due to the small torsion contribution:

$$r = 2\frac{p_T}{R_s} < 0.11 \tag{4.72}$$

It is worth mention that this model has no reheating era after the inflation. Therefore, the transition between the inflation and matter dominant era need more careful analysis. One way to do this is introduced in paper [50].

4.1.2 Late Universe

We have now discussed how torsion model behave in the early universe. Torsion model also has impact to late cosmology and can be used to explain dark energy[23]. From the Friedmann equation with torsion, we can define a new variable known as the deceleration parameter:

$$q = -1 - \frac{\dot{H}}{H^2} = \frac{1}{2}\{\frac{(1+3\omega)\rho + 12(\dot{\phi} + \phi H}{\rho - 12\phi(\phi + H)}\} \tag{4.73}$$

Again here we assumed barotropic equation of state $p = \omega\rho$. The deceleration parameter will be smaller or equal to 1 as long as:

$$\omega \leqslant -1 - \frac{4}{\rho}[\dot{\phi} - \phi(2\phi + H)]. \tag{4.74}$$

The universe with q=-1 is the universe with a constant cosmological constant and universes with $q < -1$ is known as phantom cosmology.

Now we can introduce a normalised Hubble parameter:

$$E(z) = \\ (1+z)^{3(1+\omega)/2}\sqrt{\Omega_\rho(0)\exp[6(\frac{1}{3}+\omega)\int_0^z \frac{\phi(z)}{(1+z)H_0E(z)}dz]} - 2\frac{\phi(z)}{H_0} \tag{4.75}$$

The first torsion field we will consider is $\phi = \lambda H$, where λ can be obtained from the observational data. The Hubble parameter in this scenario is:

$$E(z) = \frac{\sqrt{\Omega_\rho(0)}}{(1+2\lambda}(1+z)^{\frac{1}{2}[3(1+\omega)+6\lambda(\frac{1}{3}+\omega)]} \tag{4.76}$$

and the scale factor in this universe is:

$$a(t) = a_0[\frac{(1+2\lambda)^2}{\Omega_\rho(0)H_0^2}]^{-1/\Delta}(t_s - t)^{2/\Delta} \tag{4.77}$$

where $\Delta := 3(1+\omega) + 6\lambda(1/3+\omega)$ and

$$t_s = t_0 - \frac{2}{\Delta}\sqrt{\frac{(1+2\lambda)^2}{\Omega_\rho(0)H_0^2}} \tag{4.78}$$

One need to note that scale factor will diverge if $\Delta < 0$ and $t = t_s$. One should also note the normalized Hubble parameter will be divergent in this case. No singularity will occur in the case where $\Delta > 0$.

The second torsion field is given by the following equation:

$$\frac{\phi(z)}{H(z)} = -\alpha(\frac{\rho(Z)}{3H_0^2})^n \tag{4.79}$$

We can now define an effective parameter:

$$\omega_{eff} = \frac{2}{3}\frac{\phi}{H} + \omega(1 + \frac{2\phi}{H}) \tag{4.80}$$

Hence the conservation equation can be written as:

$$\dot{\rho} + 3H\rho(1 + \omega_{eff}) = 0 \tag{4.81}$$

Let us compare this model with ΛCDM model. The effective interval will lie between -0.5 and 0 if we set ω to be 0. Hence, the torsion term cannot simulate the ΛCDM model. Also, this model can be used to explain dark energy. We can write:

$$\omega_{eff} = -\frac{1}{3}[1 + \sqrt{\Omega_\rho(0)}] \tag{4.82}$$

Using the data from the density parameter, we have $-0.522 < \omega_{eff} < -0.518$. In this case, the torsion will behave like dark energy.

Thermodynamic behaviour of the universe in this model is also an interesting topic. If we differentiate both sides of the first law of thermodynamics with respect to time, we have:

$$\frac{T}{V}\frac{dS}{dt} = 4\phi\rho \tag{4.83}$$

Here we used the relation:

$$\frac{dV}{v} = 3H(1 + \frac{2\phi}{H})dt \tag{4.84}$$

and the continuity equation.

One should note the relation $S = constant$ does not hold even in adiabatic case as we have an extra torsion term here. The Gibbs equation can be written as:

$$nTdS = -(\rho + p)\frac{dn}{n} + d\rho \tag{4.85}$$

If one take the time derivative of this equation, one can obtain:

$$nT\frac{dS}{dt} = -(\rho + p)\frac{\dot{n}}{n} + \rho \dot{=} 4\phi\rho \tag{4.86}$$

From the equation 4.85, we can define the temperature:

$$\dot{T} = \frac{\partial T}{\partial n}\dot{n} + \frac{\partial T}{\partial \rho}\dot{\rho} \tag{4.87}$$

A more general form of temperature can be written as:

$$\begin{aligned} T(z) =& T_0 \exp[-4\int \frac{\rho\phi}{H(z)}\frac{\partial T}{\partial \rho}\frac{dz}{(1+z)} \\ &+ 3\omega\int(1 + \frac{2\phi}{H(z)})\frac{dz}{(1+z)}] \end{aligned} \tag{4.88}$$

In the case where $\phi = \lambda H$, this can be simplified:

$$T(z) = T_0(1+z)^\alpha \exp[-4\lambda\int \rho(\frac{\partial T}{\partial \rho}\frac{dz}{(1+Z)}] \tag{4.89}$$

where $\alpha = 3\omega(1 + 2\lambda)$.

One can see that this temperature will always be positive. Hence, we have:

$$TS = (1 + \omega)\rho V \tag{4.90}$$

In the phantom regime we have $TS < 0$ thus implies a negative entropy. Just like normal cosmology, this problem cam be solved by introduce chemical potential:

$$\mu = \frac{4\phi\rho}{nv}(\frac{N-1}{N}) \tag{4.91}$$

In order for the second law of thermodynamics to hold, one must have:

$$\frac{4\phi}{\nu}\frac{N-1}{N} > -|1+\omega| \tag{4.92}$$

Let us go back to the scenario defined by $\phi = \lambda H$. The internal energy here can be found using Misner-Sharper term $U = \rho V$. However, this expression of energy is not compatible with the thermodynamics described before. Therefore, a more general energy term needs to be found. A simple way to do this is Komar energy which is defined by:

$$U_k = (3p + \rho)V \tag{4.93}$$

With the definition of pressure, we can write out the following equation:

$$\frac{dp}{d\rho} + \frac{2}{3}\frac{p}{\rho} + \frac{1}{2} = 0 \tag{4.94}$$

The solution to this equation is:

$$p(\rho) = \rho(\frac{c_1}{\rho^{5/3}} - \frac{3}{10}) \tag{4.95}$$

where c_1 is an arbitrary constant.

Combined with continuity equation, one can obtain the Hubble parameter:

$$H^2(a) = \frac{\rho(a)}{3(1+2\lambda)^2} \tag{4.96}$$

One can also study the entropy in this case. The equation to determine the entropy is:

$$\frac{T}{V}\frac{dS}{dt} = 4\phi\rho + 3[\dot{p} + 3Hp(1 + \frac{2\phi}{H})] \tag{4.97}$$

It is worth mentioning there are also some development on the experiment front.[48][49]

4.2 Quantum Theory with Torsion

Now that we have seen how torsion can be useful in the classical cosmology, we can step into the quantum regime. First we need to determine the potential caused by torsion. [21] Let us a four vector:

$$n_\mu = (\frac{1}{\sqrt{2}}(\psi_1^* \ \psi_2^*)\sigma_\mu \begin{pmatrix} \psi_1 \\ \psi 2 \end{pmatrix} \tag{4.98}$$

where:

$$\psi_1 = (cos\theta/2)e^{i\phi/2} \tag{4.99}$$

and

$$\psi_2 = (sin\theta/2)e^{-i\phi/2} \tag{4.100}$$

One can construct the topological current as:

$$J_\mu = (\frac{1}{24\pi^2})\epsilon_{\mu\nu\lambda\sigma}Tr[(g^{-1}\partial^\nu g)(g^{-1}\partial^\lambda g)(g^{-1}\partial^\sigma g)] \tag{4.101}$$

Therefore one can find out the invariant charge:

$$Q_P = \frac{1}{16\pi^2}\int d^4x\partial_\mu J^\mu \tag{4.102}$$

One can prove the action of torsion can be written as:

$$S_T = \frac{M_p^2}{2}\int J_\mu^2 J_\mu^2 d^4x \tag{4.103}$$

Using the relation:

$$j^{\mu(2)} = \epsilon^{\mu\nu\lambda\sigma}\partial_\nu f_{\lambda\sigma} \tag{4.104}$$

one can rewrite the action as:

$$S_T = \int d^4x\sqrt{-g_{(4)}}\frac{m^2}{2}\phi^2 \tag{4.105}$$

By inspection one can find the potential is:

$$V_T(\phi) = -\frac{m^2}{2}\phi^2 \tag{4.106}$$

Following the step of the Vilenkin's paper, one can find the WKB solution of this theory are:

$$\psi_T = (\frac{1-a^2V-c^2}{1-c^2})^{-1/4}\exp(\frac{(1-c^2-a^2V)^{3/2}-(1-c^2)^{3/2}}{3V}) \tag{4.107}$$

and

$$\psi_T = e^{i\pi/4}(\frac{-1+a^2V+c^2}{1-c^2})^{-1/4}\exp(-\frac{(1-c^2)^{3/2}+i(a^2V-1+c^2)^{3/2}}{3V}) \tag{4.108}$$

for classical allowed and classical forbidden region respectively where c is the torsion field. Further study is needed for quantum cosmology with torsion.

Chapter 5

Conclusion

In this dissertation we have explored the subject of quantum cosmology. In the first chapter, Wheeler-DeWitt equation is introduced and two ways of quantising the cosmology is discussed. In the second chapter, we have explored the basic formulation of quantum cosmology and discuss the problem of boundary condition. Two important approaches has been discussed in detail: quantum tunnelling approach first introduced by Vilenkin and 'no-boundary' approach first developed by Hartle and Hawking. We have compared them in two different models and discussed their physical predictions and perturbative theory.

In the third chapter, we have discussed some further development of quantum cosmology. Theory in superspace is discussed. We have seen that the two approaches to the boundary problem can be seen as one. This is recently discussed by Magueijo.

In forth chapter, we have discussed the implication of torsion in cosmology. Some recent studies have been introduced in both classical and quantum cosmology. In the classical regime, we have discussed the implication of torsion to the early universe. Specifically, We have used the torsion to model the cosmic inflation. Two different cases were discussed. Case one is $\phi(t) = -\alpha H(t)(\frac{\rho_m}{\rho})^n$ and case two is $\phi(t) = H_0(\frac{H_0}{H(t)})^m(\frac{\rho_m(t)}{\rho_{0c}})^n$. Case two was studies in more details. Both long-wavelength and short-wavelength limit were studies. The fact that there is no reheating phase is mentioned. We also discussed late cosmology model with torsion and see how it can be used to explain dark energy. The thermodynamics of cosmology with torsion is discussed, and we have discussed the need for an extra torsion term. On the other hand, certain unusual behaviour caused by torsion term is introduced. The concept of energy was extended to Kormar energy. In quantum regime, we have derived the potential caused by torsion and gives the WKB solution for tunnelling approach. However, this is still an active area of research and there remain many questions of quantum cosmology with torsion yet to be discussed and studied.

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