Imperial College London

Department of Physics

Review of a General Relativistic Gravitating Source at Second-post-Newtonian Correction

Author:
Yikang Li

Supervisor:
Professor Kellogg Stelle

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Chapter 1

Introduction

1.1 Motivation and Structure of the Dissertation

In modern physics, General Relativity is a widely accepted theory that describes gravitation. Einstein's equation describes the relationship between the matter and the geometry of the space-time that encloses the matter source. This equation directly predicts the existence of gravitational waves. By measuring the properties of the emitting gravitational waves, we can retrieve the data of the gravitating sources. This information is crucial for us to understand the nature of massive astrophysical objects, e.g. neutron stars and black holes.
In recent decades, large gravitational wave detectors such as LIGO and VIRGO [10] [18] [16] are constructed to test the non-linear aspect of General Relativity and to measure the parameters of the sources [6], among which compact binary systems made of black holes or neutron stars are among the most promising ones. It is, therefore, crucial to theoretically derive not only the dynamics of the coalescing compact two objects but also the expressions of the gravitational radiation fields. The latter one will be used to compute the properties of the radiation generated by a coalescing compact binary system [1]. However, due to the highly non-linear aspect of Einstein's equation, it is hard to directly find a theoretical solution for a certain given gravitating source, e.g. a compact binary system. Therefore, it is natural to linearize Einstein's equation by expanding the equation to an appropriate order of c^{-1}. This process is called post-Newtonian expansion, which makes Einstein's equation theoretically solvable at certain post-Newtonian corrections. Although, nowadays, higher post-Newtonian corrections are needed to retrieve data with high accuracy [21] [11] [22], in this dissertation, we only consider the second post-Newtonian (2PN) corrections (corresponds to the order c^{-4}), which will demonstrate a similar approach and provide a reference when one calculates higher-order corrections.
This dissertation will be mainly divided into two chapters. In Chapter 3, we focus on the effective one-body approach that reduces the dynamics of com-

pact binary objects to the dynamics of an effective one object by a canonical transformation. In this chapter, we start from deriving the general two bodies' Lagrangian that is derived from-Einstein Infeld-Hoffmann equations. We then work in the center-of-mass frame and compute the Hamiltonian corresponding to the compact two-body system, from which we use the Hamilton-Jacobi approach to derive the real action. At the same time, we construct some effective metric, from which we derive the effective action from the Hamilton-Jacobi equation, with some undetermined coefficients, which can be evaluated by matching the real action to the effective action. Finally, we discuss the dynamics that arise from the effective metric and derive the canonical transformation that links the effective problem to the real problem.
In Chapter 4, we compute the gravitational radiation field corresponding to the 2PN correction. We start from introducing the irreducible Cartesian tensors, which are representations of the SO(3) rotation Lie group. We then find the solution for Einstein's vacuum equation in the exterior zone of the source by using the Multipolar-Post-Minkowskian expansion [7] [23]. We also compute the solution for Einstein's equation in the interior zone and express the solution in terms of the stress-energy tensor. By comparing the two solutions in their common zone of validity, we obtain a set of matching equations, which are used to deal with the cubic order of non-linearality of Einstein's equation at 2PN correction. Finally, from the matching equations, we derive the expressions for the source moments up to an appropriate order of correction. Previous work has been done in calculating the slowly moving isolated system at 1.5PN correction [14] [5].

1.2 Introduction to the Relevant Math

In Chapter 3, we will implement the Hamilton-Jacobi mechanism in dealing with the effective one-body problem. This mechanism is useful in finding the conserved quantities of a particular system. Once the Hamiltonian of the system is given, the Hamilton-Jacobi equation can be reduced to a set of first-order differential equations, each of which corresponds to a conserved quantity of the system. One can then express the resulting action in terms of these conserved quantities by solving the differential equations iteratively. The formal mathematical introduction of the Hamilton-Jacobi mechanism can be found in Section 2.3.
In Chapter 4, we compute Einstein's equation in the harmonic coordinate. We introduce some field $h^{\mu\nu}$ which is the deviation of the metric $g^{\mu\nu}$ from the Minkowski metric. Einstein's equation is then reduced to an equation of the field $h^{\mu\nu}$ together with the harmonic coordinate condition, i.e. $\partial_\mu h^{\mu\nu} = 0$. This new equation is linear at the order $O(h)$, but becomes nonlinear at higher orders. Details of deriving the equation can be found in Section 2.3. Another mathematical tool we will use in this Chapter is the symmetric trace-free (STF) tensors. Equivalent to the spherical harmonics Y_{lm}, the STF tensor with l indices generates a $2l+1$ dimensional vector space that corresponds to the vec-

tor space constructed by the irreducible representation of SO(3) rotation group [12]. Previous work by Campbell, Macek, and Morgan [9] has demonstrated the approach of solving electromagnetism and linearized gravity by spherical harmonics. Thorn [23] provided the mathematical construction that links the spherical harmonics and the STF tensors. In this dissertation, we will use the STF tensors to perform the multipole decomposition of the fields. Detail of the STF tensors can be found in Section 4.1 and the decomposition formula is proved in Appendix A.

Chapter 2

Preliminaries

Spacetime indices $(0, 1, 2, 3)$ are denoted by Greek indices while space indices $(1, 2, 3)$ are denoted by Latin indices. Our signature is $(-1, 1, 1, 1)$.

2.1 Motion in Schwarzschild Space-time

Let us first review the motion of a test particle in a Schwarzschild spacetime. In polar coordinate, the geometry can be described by the following metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{2.1}$$

This expression can describe a spherically symmetric space-time affinely parametrized by a parameter λ outside a star of mass M[19]. In Schwarzschild coordinate, $r = 2M$ is the coordinate singularity, while the curvature singularity is at $r = 0$. In the following calculation, we only consider the case where $r > 2M$. Since the metric $g_{\mu\nu}$ does not explicitly depend on t and ϕ, it admits two Killing vectors

$$k = \partial_t \tag{2.2a}$$

$$m = \partial_\phi \tag{2.2b}$$

From the Killing vectors, we can construct the conserved quantities

$$Q = V_\mu \frac{dx^\mu}{d\lambda} \tag{2.3}$$

where V_μ is the corresponding Killing vector and $dQ/d\lambda = 0$. Substituting Eq. (2.2) into Eq. (2.3), we obtain two conserved quantities which are identified as the energy and the angular momentum respectively

$$E = -g_{tt}k^t \frac{dt}{d\lambda} = (1 - \frac{2M}{r})\frac{dt}{d\lambda} \tag{2.4a}$$

$$J = g_{\phi\phi}m^\phi \frac{d\phi}{d\lambda} = r^2 sin^2\theta \frac{d\phi}{d\lambda} \tag{2.4b}$$

The minus sign in Eq. (2.4a) is to ensure the energy is positive.

Now, let us consider the motion of a test particle with rest mass m in the Schwarzschild spacetime. We assume that the motion is solely in equatorial plane, i.e. $\theta = \frac{\pi}{2}$ and $\frac{d\theta}{d\lambda} = 0$. We construct the four-momentum of the particle $P^\mu = (\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, \frac{d\theta}{d\lambda}, \frac{d\phi}{d\lambda})$, which, by using Eq. (2.4), can also be written as

$$P^\mu = (\frac{E}{1 - \frac{2M}{r}}, \frac{dr}{d\lambda}, 0, \frac{J}{r^2}) \tag{2.5}$$

The mass shell condition in General Relativity reads

$$g_{\mu\nu} P^\mu P^\nu + m^2 = 0 \tag{2.6}$$

Inserting Eq. (2.5) into Eq. (2.6) yields

$$-\frac{E^2}{1 - \frac{2M}{r}} + (\frac{dr}{d\lambda})^2 \frac{1}{1 - \frac{2M}{r}} + \frac{J^2}{r^2} + m^2 = 0 \tag{2.7}$$

Multiplying the equation by $1 - \frac{2M}{r}$ and rearranging the terms, we have

$$(\frac{dr}{d\lambda})^2 = E^2 - (1 - \frac{2M}{r})(\frac{J^2}{r^2} + m^2) \tag{2.8}$$

The test particle moves along a timelike geodesic, on which $m\lambda = \tau$. Therefore, it is natural to define the new variables $e \equiv \frac{E}{m}$ and $j \equiv \frac{J}{m}$. Eq. (2.8) then can be written as

$$(\frac{dr}{d\tau})^2 = e^2 - V_{eff}^2 \tag{2.9}$$

where we have defined the effective potential V_{eff}

$$V_{eff}^2 = (1 - \frac{2M}{r})(\frac{j^2}{r^2} + 1) \tag{2.10}$$

The test particle follows a circular motion, i.e. r is a constant, when V_{eff} reaches its minimum value. Expanding out the equation $\frac{dV_{eff}}{dr}\Big|_{r_0} = 0$ and defining a new variable $u = \frac{1}{r}$ yields

$$\frac{1}{2V}(2Mu^2(1 + j^2u^2) - 2j^2u^3(1 - 2Mu)) = 0 \tag{2.11}$$

The stable circular orbit corresponds to the solution

$$u_0 = \frac{1}{6M}(1 - \sqrt{1 - \frac{12M^2}{j^2}}) \tag{2.12}$$

The other root $u_1 = \frac{1}{6M}(1 + \sqrt{1 - \frac{12M^2}{j^2}})$ gives a smaller r, which represents an unstable orbit. Since r is a constant along the circular orbit, $\frac{dr}{d\tau}\Big|_{r_0} = 0$. Therefore, the energy per unit mass is equal to the effective potential

$$e = \sqrt{(1 - 2Mu_0)(j^2u_0^2 + 1)} \tag{2.13}$$

The angular velocity ω_0 is defined as

$$\omega_0 = \frac{d\phi}{dt}\Big|_{r_0} = (1 - 2Mu_0)u_0^2 \frac{j}{e} \tag{2.14}$$

Finally, simplifying both Eq. (2.13) and Eq. (2.14) using Eq. (2.12), we obtain

$$e = (1 - 2Mu_0)j\sqrt{\frac{u_0}{M}} \tag{2.15a}$$

$$\omega_0 = \sqrt{Mu_0^3} \tag{2.15b}$$

For consistency, we define the scaled variables: $\hat{u}_0 := Mu_0$, $\hat{j} := \frac{j}{M}$, and $\hat{\omega}_0 := M\omega_0$. In this notation, we conclude

$$\hat{u}_0 = \frac{1}{6}(1 - \sqrt{1 - \frac{12}{\hat{j}^2}}) \tag{2.16a}$$

$$e = (1 - 2\hat{u}_0)\hat{j}\sqrt{\hat{u}_0} \tag{2.16b}$$

$$\hat{\omega}_0 = \sqrt{\hat{u}_0^3} \tag{2.16c}$$

Now, let us consider again Eq. (2.10). It is worth discussing the monotonicity of the effective potential. For dV_{eff}/dr to be positive for all r, we require $d^2V_{eff}/dr^2 = 0$. We obtain the innermost stable circular orbit (ISCO) for the Schwarzschild metric, where $\hat{u}_0 = \frac{1}{6}$ and $\hat{j} = 2\sqrt{3}$. Therefore, to ensure the effective potential to have a local minimum, we require

$$\hat{j} \geq 2\sqrt{3} \tag{2.17}$$

For particles with angular momentum below this value, they will fall inside the event horizon.

2.2 Hamilton-Jacobi Equation and Its Solution

Let us now review the Hamilton-Jacobi description of a particle's motion [17]. Consider a Lagrangian $L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)$ and its corresponding action

$$S(\boldsymbol{x}, t) = \int_{t_i}^{t_f} L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)dt \tag{2.18}$$

where t_i and t_f are the initial and final time respectively. From the Lagrangian, we can construct the Hamiltonian $H(\boldsymbol{x}, \boldsymbol{\pi}, t)$

$$H(\boldsymbol{x}, \boldsymbol{\pi}, t) = \boldsymbol{\pi}\dot{\boldsymbol{x}} - L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) \tag{2.19}$$

where $\boldsymbol{\pi}$ is the conjugate momentum, i.e. $\boldsymbol{\pi} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}}$. From Eq. (2.18), we have

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = \frac{\partial S}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial t} + \frac{\partial S}{\partial t} \tag{2.20}$$

Therefore, we obtain the Hamilton-Jacobi equation (referred as HJE in the following context)

$$\frac{\partial S}{\partial t} = -(\frac{\partial S}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial t} - L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)) = -H(\boldsymbol{x}, \frac{\partial S}{\partial \boldsymbol{x}}, t) \tag{2.21}$$

A general HJE can be highly non-linear. However, if the Hamiltonian is explicitly independent on time, the resulting action can be separated into two parts

$$S(\boldsymbol{x}, t) = -Et + \hat{S}(\boldsymbol{x}) \tag{2.22}$$

where E is a constant of motion. Inserting this equation into the HJE gives

$$H(\boldsymbol{x}, \frac{\partial \hat{S}}{\partial \boldsymbol{x}}) = E \tag{2.23}$$

We can further partition the action if we assume the Hamiltonian can be written in the form

$$\begin{aligned} H(\boldsymbol{x}, \frac{\partial \hat{S}}{\partial \boldsymbol{x}}) =& H(\phi_i(x_i, \frac{\partial \hat{S}}{\partial x_i}), x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n, \\ & \frac{\partial \hat{S}}{\partial x_1}, \frac{\partial \hat{S}}{\partial x_2}, ..., \frac{\partial \hat{S}}{\partial x_{i-1}}, \frac{\partial \hat{S}}{\partial x_{i+1}}, ..., \frac{\partial \hat{S}}{\partial x_n}) \end{aligned} \tag{2.24}$$

where the dependence of the variable x_i is absorbed into a function ϕ_i. The action is then once more partitioned

$$S(\boldsymbol{x}, t) = -Et + \hat{S}_i(x_i) + \hat{S}^{(2)}(x_1, x_2, ...x_{i-1}, x_{i+1}, ..., x_n) \tag{2.25}$$

HJE Eq. (2.25) then reads

$$\begin{aligned} E =& H(\phi_i(x_i, \frac{d\hat{S}_i}{dx_i}), x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n, \\ & \frac{\partial \hat{S}^{(2)}}{\partial x_1}, \frac{\partial \hat{S}^{(2)}}{\partial x_2}, ..., \frac{\partial \hat{S}^{(2)}}{\partial x_{i-1}}, \frac{\partial \hat{S}^{(2)}}{\partial x_{i+1}}, ..., \frac{\partial \hat{S}^{(2)}}{\partial x_n}) \end{aligned} \tag{2.26}$$

Since the function ϕ_i only depends on x_i but not on other variables, we immediately see that it must be a constant, denoting as Φ_i. The partitioned action $\hat{S}_i$ then obeys a simple differential equation

$$\phi_i(x_i, \frac{d\hat{S}_i}{dx_i}) = \Phi_i \tag{2.27}$$

We similarly assume that x_j and $\frac{\partial \hat{S}^{(2)}}{\partial x_j}$ appear together as a function $\phi_j(x_j, \frac{\partial \hat{S}^{(2)}}{\partial x_j}, \phi_i)$ in the Hamiltonian. Applying the above discussion gives us a differential equation with respect to x_j

$$\phi_j(x_j, \frac{d\hat{S}_j^{(2)}}{dx_j}, \Phi_i) = \Phi_j \tag{2.28}$$

This process is iterated until we find the differential equations for all the variables. It is worth noticing that, through the Hamilton-Jacobi formalism, we can easily identify the constants of motion, i.e. Φ_i and E.
Now consider the Hamiltonian in Newtonian Gravity [19]. The particle's positions and momenta are denoted as $\boldsymbol{r}$ and $\boldsymbol{p}$ respectively. The Hamiltonian can be written as

$$H(\boldsymbol{r}, \boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2} + V(\boldsymbol{r}) \tag{2.29}$$

In polar coordinate,

$$\boldsymbol{p}^2 = p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 sin^2\theta} \tag{2.30}$$

$$V(\boldsymbol{r}) = -\frac{M}{r} \tag{2.31}$$

The HJE reads

$$-\frac{\partial S}{\partial t} = \frac{1}{2}((\frac{\partial S}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial S}{\partial \theta})^2 + \frac{1}{r^2 sin^2\theta}(\frac{\partial S}{\partial \phi}))^2 - \frac{M}{r} \tag{2.32}$$

Writing the action as

$$S = S_r(r) + S_\theta(\theta) + S_\phi(\phi) - Et \tag{2.33}$$

and inserting into Eq. (2.32) yields three separated differential equations

$$(\frac{dS_\phi}{d\phi})^2 = \Phi_\phi \tag{2.34}$$

$$(\frac{dS_\theta}{d\theta})^2 + \frac{1}{sin^2\theta}\Phi_\phi = \Phi_\theta \tag{2.35}$$

$$\frac{1}{2}(\frac{dS_r}{dr})^2 - \frac{M}{r} + \frac{1}{2r^2}\Phi_\theta = E \tag{2.36}$$

where Φ_ϕ, Φ_θ, and E are three constants of motion, which we can identify as the angular momentum in the ϕ direction, the angular momentum in the θ direction, and the total energy of the motion, respectively. Solving these equations, we can obtain the general expression for the action

$$S = \int \sqrt{2E + \frac{2M}{r} - \frac{1}{r^2}\Phi_\theta} dr + \int \sqrt{\Phi_\theta - \frac{1}{sin^2\theta}\Phi_\phi} d\theta + \sqrt{\Phi_\phi}\phi - Et + C \tag{2.37}$$

where C is the integrating constant.

2.3 Einstein's Equation in Harmonic Coordinate

In harmonic coordinate, we linearly expand the metric $g^{\mu\nu}$ around the Minkowski metric $\eta^{\mu\nu}$ [4]

$$h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu} \tag{2.38}$$

where g is the determinant of the metric $g_{\mu\nu}$, and $h^{\mu\nu}$ is assumed to be a small field deviation from the Minkowski metric, which, in our convention, is $diag(-1, 1, 1, 1)$. Since $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, the determinant of $g^{\mu\nu}$ is $\frac{1}{g}$. From Eq. (2.38), we deduce

$$det(\eta^{\mu\nu} + h^{\mu\nu}) = det(\sqrt{-g}g^{\mu\nu}) = g \tag{2.39}$$

Taylor-expanding Eq. (2.39) yields

$$g = -1 + \frac{\partial g}{\partial g^{\mu\nu}}\Big|_{g=\eta} h^{\mu\nu} + \frac{1}{2} h^{\mu\nu} h^{\rho\sigma} \frac{\partial^2 g}{\partial g^{\mu\nu} \partial g^{\rho\sigma}}\Big|_{g=\eta} + O(h^3) \tag{2.40}$$

Evaluating each term in Eq. (2.40), we have

$$-g = 1 + h + \frac{1}{2}h^2 - \frac{1}{2}h^{\mu\rho}h_{\mu\rho} + O(h^3) \tag{2.41}$$

where we have denoted $h = h^{\mu}_{\mu} = \eta_{\mu\nu}h^{\mu\nu}$.
From Eq. (2.38) and Eq. (2.41), up to the first order in h, we can derive

$$\eta^{\mu\nu} + h^{\mu\nu} = (1 + \frac{1}{2}h)g^{\mu\nu} + O(h^2) \tag{2.42}$$

Therefore, after rearranging, we obtain

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h + O(h^2) \tag{2.43a}$$

$$g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h + O(h^2) \tag{2.43b}$$

The deviation of $g_{\mu\nu}$ takes a relative minus sign because $g_{\mu\nu}$ is the inverse of $g^{\mu\nu}$. The Christoffel symbol is defined as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \tag{2.44}$$

Inserting Eq. (2.43) into Eq. (2.44) yields

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} = -\frac{1}{2}(&\partial_\mu h^{\rho}_{\nu} + \partial_\nu h^{\rho}_{\mu} - \partial^\rho h_{\mu\nu} \\ &- \frac{1}{2}\eta^{\rho\lambda}\eta_{\lambda\nu}\partial_\mu h - \frac{1}{2}\eta^{\rho\lambda}\eta_{\lambda\mu}\partial_\nu h + \frac{1}{2}\eta^{\rho\lambda}\eta_{\mu\nu}\partial_\lambda h) + O(h^2) \end{aligned} \tag{2.45}$$

The harmonic coordinate condition reads

$$g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = 0 \tag{2.46}$$

Now consider

$$\partial_\rho h^{\mu\nu} = \partial_\rho(\sqrt{-g}g^{\mu\nu}) \tag{2.47}$$

Recall that

$$\nabla_\rho(\sqrt{-g}) = \partial_\rho\sqrt{-g} - \sqrt{-g}\Gamma^{\alpha}_{\alpha\rho} = 0 \tag{2.48}$$

Therefore, one can rewrite Eq. (2.47) as

$$\partial_\rho h^{\mu\nu} = \nabla_\rho(\sqrt{-g}g^{\mu\nu}) + g^{\mu\nu}\sqrt{-g}\Gamma^{\alpha}_{\alpha\rho} - g^{\alpha\mu}\Gamma^{\nu}_{\alpha\rho}\sqrt{-g} - g^{\alpha\nu}\Gamma^{\mu}_{\alpha\rho}\sqrt{-g} \tag{2.49}$$

After contracting the indices μ and ρ, we easily notice that the first term and the third term on the right hand side of Eq. (2.49) become zero, while the second term cancels against the fourth term. Therefore, we deduce

$$\partial_\mu h^{\mu\nu} = 0 \tag{2.50}$$

under the harmonic coordinate condition. Now we can use Eq. (2.44) to calculate the Riemann curvature tensor, which, to the first order of h, can be denoted as

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\sigma\nu} - (\mu \longleftrightarrow \nu) + O(h^2) \tag{2.51}$$

where $(\mu \longleftrightarrow \nu)$ is the term with μ and ν indices exchanged compared to the first term on the right hand side of Eq. (2.51). Using Eq. (2.45), one can check

$$\begin{aligned} R^{\rho}_{\sigma\mu\nu} =& -\frac{1}{2}(\partial_\mu\partial_\sigma h^{\rho}_{\nu} - \partial_\mu\partial^\rho h_{\sigma\nu} - \partial_\nu\partial_\sigma h^{\rho}_{\mu} + \partial_\nu\partial^\rho h_{\sigma\mu}) \\ &+ \frac{1}{4}(\eta^{\rho\lambda}\eta_{\lambda\nu}\partial_\mu\partial_\sigma h - \eta^{\rho\lambda}\eta_{\lambda\mu}\partial_\nu\partial_\sigma h - \eta^{\rho\lambda}\eta_{\sigma\nu}\partial_\mu\partial_\lambda h + \eta^{\rho\lambda}\eta_{\sigma\mu}\partial_\nu\partial_\lambda h) + O(h^2) \end{aligned} \tag{2.52}$$

where terms that are symmetric in μ and ν cancel. Contracting indices ρ and μ, we find the Ricci tensor

$$R_{\sigma\nu} = \frac{1}{2}\partial_\rho\partial^\rho h_{\sigma\nu} - \frac{1}{4}\eta_{\sigma\nu}\partial_\rho\partial^\rho h + O(h^2) \tag{2.53}$$

Einstein tensor is denoted as

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}R_{\rho\sigma} \tag{2.54}$$

which obeys Einstein's field equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.55}$$

where we have neglected the cosmological constant. Inserting Eq. (2.53) into Eq. (2.55), to the first order of h, we have

$$\partial_\rho\partial^\rho h_{\mu\nu} = \frac{16\pi G}{c^4}T_{\mu\nu} \tag{2.56}$$

However, once considering the higher-order terms which we neglected in the previous calculations, one finds

$$\partial_\rho\partial^\rho h_{\mu\nu} = -\frac{16\pi G}{c^4}gT_{\mu\nu} + \Lambda_{\mu\nu}(h) \tag{2.57}$$

where $\Lambda_{\mu\nu}(h)$ is expressed by Eq. (1.4) to Eq. (1.6) in [4].

Chapter 3

Effective One-Body

3.1 Lagrangian and Equations of Motion

Newtonian motions of multiple point-mass particles in a curved spacetime can be described by the Einstein-Infeld-Hoffmann equations [15] (denoted as the EIH equations in the following context). In our case, we only consider the motions of two particles in the ADM coordinate. The notation is the following: m_1 and m_2 are the two particles' masses; $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ their positions; $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ their velocities; $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ their accelerations. We consider the terms up to c^{-4} in the following calculation. The EIH equation for the first particle is

$$\begin{aligned}\frac{d^2\boldsymbol{x_1}}{dt^2} = &-\frac{Gm_2\boldsymbol{n}}{r^2} \\ &+\frac{1}{c^2}\frac{Gm_2\boldsymbol{n}}{r^2}[-\boldsymbol{v_1}^2-2\boldsymbol{v_2}^2+4(\boldsymbol{v_1}\cdot\boldsymbol{v_2})+\frac{3}{2}(\boldsymbol{n}\cdot\boldsymbol{v_2})^2+4\frac{Gm_2}{r}+\frac{Gm_1}{r}] \\ &+\frac{1}{c^2}\frac{Gm_2}{r^2}(4\boldsymbol{v_1}\cdot\boldsymbol{n}-3\boldsymbol{v_2}\cdot\boldsymbol{n})(\boldsymbol{v_1}-\boldsymbol{v_2}) \\ &+\frac{1}{2c^2}\frac{Gm_2}{r^2}\boldsymbol{n}[(\boldsymbol{x_1}-\boldsymbol{x_2})\boldsymbol{a_2}]+\frac{7}{2c^2}\frac{Gm_2}{r}\boldsymbol{a_2}+O(c^{-4})\end{aligned} \tag{3.1}$$

where we have defined

$$r = |\boldsymbol{x_1}-\boldsymbol{x_2}| \tag{3.2}$$

$$\boldsymbol{n} = \frac{\boldsymbol{x_1}-\boldsymbol{x_2}}{r} \tag{3.3}$$

In order to eliminate the acceleration dependence in Eq. (3.1), we apply the EIH equation for the second particle and expand $\boldsymbol{a_2}$ to the terms up to the order c^0

$$\boldsymbol{a_2} = \frac{Gm_1\boldsymbol{n}}{r^2}+O(c^{-2}) \tag{3.4}$$

Insertion of Eq. (3.4) into Eq. (3.1) gives

$$\begin{aligned}\frac{d^2\boldsymbol{x_1}}{dt^2} =& -\frac{Gm_2\boldsymbol{n}}{r^2} \\ &+ \frac{1}{c^2}\frac{Gm_2\boldsymbol{n}}{r^2}[-\boldsymbol{v_1}^2 - 2\boldsymbol{v_2}^2 + 4(\boldsymbol{v_1}\cdot\boldsymbol{v_2}) + \frac{3}{2}(\boldsymbol{n}\cdot\boldsymbol{v_2})^2 + 4\frac{Gm_2}{r} + 5\frac{Gm_1}{r}] \\ &+ \frac{1}{c^2}\frac{Gm_2}{r^2}(4\boldsymbol{v_1}\cdot\boldsymbol{n} - 3\boldsymbol{v_2}\cdot\boldsymbol{n})(\boldsymbol{v_1}-\boldsymbol{v_2}) + O(c^{-4})\end{aligned} \tag{3.5}$$

where we have used $\boldsymbol{n}\cdot\boldsymbol{n} = 1$.
A general Lagrangian for two particles relativistic motion is described in [13]

$$\begin{aligned}L(\boldsymbol{x_1},\boldsymbol{x_2},\boldsymbol{v_1},\boldsymbol{v_2}) =& \frac{1}{2}m_1\boldsymbol{v_1}^2 + \frac{1}{2}m_2\boldsymbol{v_2}^2 + \frac{Gm_1m_2}{r} + \frac{1}{c^2}\{\frac{1}{8}m_1\boldsymbol{v_1}^4 + \frac{1}{8}m_2\boldsymbol{v_2}^2 \\ &+ \frac{Gm_1m_2}{r}[\frac{3}{2}\boldsymbol{v_1}^2 + \frac{3}{2}\boldsymbol{v_2}^2 - \frac{7}{2}(\boldsymbol{v_1}\cdot\boldsymbol{v_2}) - \frac{1}{2}(\boldsymbol{n}\cdot\boldsymbol{v_1})(\boldsymbol{n}\cdot\boldsymbol{v_2}) \\ &- \frac{1}{2}\frac{Gm_1}{r} - \frac{1}{2}\frac{Gm_1}{r}]\} + O(c^{-4})\end{aligned} \tag{3.6}$$

Now let us prove that the equations of motion that are described by the Lagrangian in Eq. (3.6) coincide with the EIH equation in Eq. (3.5). Since the Lagrangian does not depend on accelerations nor higher derivative terms, the equation of motion is described by the Euler-Lagrange equations

$$\frac{\partial L}{\partial \boldsymbol{x_1}} - \frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{v_1}} = 0 \tag{3.7}$$

In order to write the equation of motion explicitly, it is useful to evaluate the following expressions

$$\frac{\partial}{\partial \boldsymbol{x_1}}\frac{1}{r} = -\frac{1}{r^2}\boldsymbol{n} \tag{3.8a}$$

$$\frac{\partial}{\partial \boldsymbol{x_1}}(\boldsymbol{n}\cdot\boldsymbol{u}) = \frac{1}{r}[\boldsymbol{u} - (\boldsymbol{n}\cdot\boldsymbol{u})\boldsymbol{n}] \tag{3.8b}$$

where $\boldsymbol{u}$ is an arbitrary vector and

$$\frac{d}{dt}\frac{1}{r} = \boldsymbol{v_1}\frac{\partial}{\partial \boldsymbol{x_1}}\frac{1}{r} + \boldsymbol{v_2}\frac{\partial}{\partial \boldsymbol{x_2}}\frac{1}{r} = -\frac{1}{r^2}(\boldsymbol{n}\cdot\boldsymbol{v_1} - \boldsymbol{n}\cdot\boldsymbol{v_2}) \tag{3.9a}$$

$$\frac{d}{dt}\boldsymbol{n} = \frac{\partial}{\partial \boldsymbol{x_1}}(\boldsymbol{n}\cdot\boldsymbol{v_1}) + \frac{\partial}{\partial \boldsymbol{x_2}}(\boldsymbol{n}\cdot\boldsymbol{v_2}) = \frac{1}{r}\{\boldsymbol{v_1} - \boldsymbol{v_2} - [(\boldsymbol{n}\cdot\boldsymbol{v_1}) - (\boldsymbol{n}\cdot\boldsymbol{v_2})]\boldsymbol{n}\} \tag{3.9b}$$

From Eq. (3.8), we find

$$
\begin{aligned}
\frac{\partial L}{\partial \boldsymbol{x_1}} =& -\frac{Gm_1m_2}{r^2}\boldsymbol{n} + \frac{1}{c^2}\frac{Gm_1m_2}{r^2}\boldsymbol{n}[-\frac{3}{2}\boldsymbol{v_1}^2 - \frac{3}{2}\boldsymbol{v_2}^2 + \frac{7}{2}(\boldsymbol{v_1}\cdot\boldsymbol{v_2}) \\
&+ \frac{1}{2}(\boldsymbol{n}\cdot\boldsymbol{v_1})(\boldsymbol{n}\cdot\boldsymbol{v_2}) + \frac{Gm_1}{r} + \frac{Gm_2}{r}] \\
&- \frac{1}{2c^2}\frac{Gm_1m_2}{r^2}\{[\boldsymbol{v_1} - (\boldsymbol{n}\cdot\boldsymbol{v_1})\boldsymbol{n}](\boldsymbol{n}\cdot\boldsymbol{v_2}) + [\boldsymbol{v_2} - (\boldsymbol{n}\cdot\boldsymbol{v_2})\boldsymbol{n}](\boldsymbol{n}\cdot\boldsymbol{v_1})\} \\
&+ O(c^{-4})
\end{aligned}
\tag{3.10}
$$

Similarly, from Eq. (3.9), we deduce

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{v_1}} =& m_1\boldsymbol{a_1} + \frac{1}{c^2}m_1[(\boldsymbol{v_1}\cdot\boldsymbol{a_1})\boldsymbol{v_1} + \frac{1}{2}\boldsymbol{v_1}^2\boldsymbol{a_1}] \\
&- \frac{1}{c^2}\frac{Gm_1m_2}{r^2}(\boldsymbol{n}\cdot\boldsymbol{v_1} - \boldsymbol{n}\cdot\boldsymbol{v_2})[3\boldsymbol{v_1} - \frac{7}{2}\boldsymbol{v_2} - \frac{1}{2}(\boldsymbol{n}\cdot\boldsymbol{v_2})\boldsymbol{n}] \\
&+ \frac{1}{c^2}\frac{Gm_1m_2}{r}\{3\boldsymbol{a_1} - \frac{7}{2}\boldsymbol{a_2} - \frac{1}{2}(\boldsymbol{n}\cdot\boldsymbol{a_2})\boldsymbol{n} \\
&- \frac{1}{2r}[(\boldsymbol{v_1}\cdot\boldsymbol{v_2})\boldsymbol{n} - \boldsymbol{v_2}^2\boldsymbol{n} + (\boldsymbol{n}\cdot\boldsymbol{v_2})\boldsymbol{v_1} - (\boldsymbol{n}\cdot\boldsymbol{v_2})\boldsymbol{v_2} \\
&- 2(\boldsymbol{n}\cdot\boldsymbol{v_1})(\boldsymbol{n}\cdot\boldsymbol{v_2})\boldsymbol{n} + 2(\boldsymbol{n}\cdot\boldsymbol{v_2})^2\boldsymbol{n}]\} \\
&+ O(c^{-4})
\end{aligned}
\tag{3.11}
$$

Again, Eq. (3.1) and Eq. (3.4) are used to eliminate the acceleration dependence in Eq. (3.11). Insertion of Eq. (3.10) and Eq. (3.11) into the Euler-Lagrange equation Eq. (3.7) gives

$$
\begin{aligned}
\frac{d^2\boldsymbol{x_1}}{dt^2} + \frac{Gm_2\boldsymbol{n}}{r^2} - \frac{1}{c^2}\frac{Gm_2}{r^2}[&-\boldsymbol{v_1}^2\boldsymbol{n} - 2\boldsymbol{v_2}^2\boldsymbol{n} + 4(\boldsymbol{v_1}\cdot\boldsymbol{v_2})\boldsymbol{n} + \frac{3}{2}(\boldsymbol{n}\cdot\boldsymbol{v_2})^2\boldsymbol{n} \\
&+ 4\frac{Gm_2\boldsymbol{n}}{r} + 5\frac{Gm_1\boldsymbol{n}}{r} + (4\boldsymbol{v_1}\cdot\boldsymbol{n} - 3\boldsymbol{v_2}\cdot\boldsymbol{n})(\boldsymbol{v_1} - \boldsymbol{v_2})] + O(c^{-4}) = 0
\end{aligned}
\tag{3.12}
$$

Eq. (3.12) is identical to Eq. (3.5). Therefore, the particles motion described by the EIH equations are encoded in a general Lagrangian expressed by Eq. (3.6). Higher-order terms in the general Lagrangian are expressed as Eq. (2.2c) in [13]. The Lagrangian for two particles relativistic motion, up to the order

c^{-4}, is

$$
\begin{aligned}
& L(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{v_1}, \boldsymbol{v_2}) = \frac{1}{2}m_1\boldsymbol{v_1^2} + \frac{1}{2}m_2\boldsymbol{v_2^2} + \frac{Gm_1m_2}{r} \\
& + \frac{1}{c^2}\{\frac{1}{8}m_1\boldsymbol{v_1^4} + \frac{1}{8}m_2\boldsymbol{v_2^2} + \frac{Gm_1m_2}{r}[\frac{3}{2}\boldsymbol{v_1^2} + \frac{3}{2}\boldsymbol{v_2^2} \\
& - \frac{7}{2}(\boldsymbol{v_1}\cdot\boldsymbol{v_2}) - \frac{1}{2}(\boldsymbol{n}\cdot\boldsymbol{v_1})(\boldsymbol{n}\cdot\boldsymbol{v_2}) - \frac{1}{2}\frac{Gm_1}{r} - \frac{1}{2}\frac{Gm_1}{r}]\} \\
& + \frac{1}{c^4}\{-\frac{1}{16}m_1\boldsymbol{v}_1^6 - \frac{1}{16}m_2\boldsymbol{v}_2^6 + \frac{Gm_1m_2}{r}[-\frac{5}{8}\boldsymbol{v}_1^4 - \frac{5}{8}\boldsymbol{v}_2^4 \\
& + \frac{11}{8}\boldsymbol{v}_1^2\boldsymbol{v}_2^2 + \frac{1}{4}(\boldsymbol{v}_1\cdot\boldsymbol{v}_2)^2 - \frac{5}{8}\boldsymbol{v}_1^2(\boldsymbol{n}\cdot\boldsymbol{v}_2)^2 - \frac{5}{8}\boldsymbol{v}_2^2(\boldsymbol{n}\cdot\boldsymbol{v}_1)^2 \\
& + \frac{3}{2}(\boldsymbol{v}_1\cdot\boldsymbol{v}_2)(\boldsymbol{n}\cdot\boldsymbol{v}_1)(\boldsymbol{n}\cdot\boldsymbol{v}_2) + \frac{3}{8}(\boldsymbol{n}\cdot\boldsymbol{v}_1)^2(\boldsymbol{n}\cdot\boldsymbol{v}_2)^2] \\
& + \frac{G^2m_1m_2}{r^2}[-\frac{1}{4}(19m_1 + 10m_2)\boldsymbol{v}_1^2 - \frac{1}{4}(10m_1 + 19m_2)\boldsymbol{v}_2^2 \\
& + \frac{1}{4}(m_1 + m_2)(27\boldsymbol{v}_1\cdot\boldsymbol{v}_2 + 6(\boldsymbol{n}\cdot\boldsymbol{v}_1)(\boldsymbol{n}\cdot\boldsymbol{v}_2))] \\
& + \frac{G^3m_1m_2}{4r^3}(m_1^2 + m_2^2 + 5m_1m_2)\} + O(c^{-6})
\end{aligned}
\tag{3.13}
$$

3.2 Hamiltonian and Action

From Eq. (3.13), we perform the Legendre transformation to find the Hamiltonian of the two-particles system, denoted as $\hat{H}$. The conjugate momentum of two particles are denoted as $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ respectively, i.e. $\boldsymbol{P_1} = \frac{\partial L}{\partial \boldsymbol{v}_1}$ and $\boldsymbol{P_2} = \frac{\partial L}{\partial \boldsymbol{v}_2}$. We will work in the center-of-mass frame where $\boldsymbol{P_1} = -\boldsymbol{P_2} = \boldsymbol{P}$. It is therefore convenient to define the following quantities

$$M := m_1 + m_2 \tag{3.14}$$

$$\mu := \frac{m_1m_2}{M} \tag{3.15}$$

$$\nu := \frac{m_1m_2}{M^2} \tag{3.16}$$

and the following reduced variables

$$\boldsymbol{p} := \frac{\boldsymbol{P}}{\mu} \tag{3.17}$$

$$H := \frac{\hat{H}}{\mu} \tag{3.18}$$

$$q := \frac{r}{GM} \tag{3.19}$$

The Hamiltonian can then be expressed as

$$
\begin{aligned}
H =&\frac{1}{2}\boldsymbol{p}^2 - \frac{1}{q} + \frac{1}{c^2}\{\frac{1}{8}(3\nu-1)\boldsymbol{p}^4 - \frac{1}{2q}[(3+\nu)\boldsymbol{p}^2 + \nu(\boldsymbol{n}\cdot\boldsymbol{p})^2] + \frac{1}{2q^2}\} \\
&+ \frac{1}{c^4}\{\frac{1}{16}(5\nu^2 - 5\nu + 1)\boldsymbol{p}^6 \\
&+ \frac{1}{8q}[(-3\nu^2 - 20\nu + 5)\boldsymbol{p}^4 - 2\nu^2\boldsymbol{p}^2(\boldsymbol{n}\cdot\boldsymbol{p})^2 - 3\nu^2(\boldsymbol{n}\cdot\boldsymbol{p})^4] \\
&+ \frac{1}{2q^2}[(8\nu+5)\boldsymbol{p}^2 + 3\nu(\boldsymbol{n}\cdot\boldsymbol{p})^2] - \frac{1}{4q^3}(3\nu+1)\} + O(c^{-6})
\end{aligned}
\tag{3.20}
$$

We now consider the Hamilton-Jacobi approach and assume the motion solely in the equatorial plane. The conjugate momentum $\boldsymbol{p}$ can be expressed in spherical coordinate

$$
\boldsymbol{p} = \boldsymbol{p}_q\hat{e}_r + \frac{1}{q^2}\boldsymbol{p}_\phi\hat{e}_\phi \tag{3.21}
$$

where we have used $\theta = \frac{\pi}{2}$. In spherical coordinate, the normal vector reduces to

$$
\boldsymbol{n} = \hat{e}_r \tag{3.22}
$$

As discussed in Section 2.2, we can partition the action

$$
S = S_q(q) + S_\phi(\phi) - E\hat{t} \tag{3.23}
$$

where $\hat{t}$ is the reduced time, i.e. $\hat{t} = \frac{t}{GM}$. By inserting Eq. (3.20) and Eq. (3.23) into Eq. (2.23), We then deduce

$$
\frac{dS_\phi(\phi)}{d\phi} = j \tag{3.24}
$$

where j is the conserved reduced angular momentum and

$$
(\frac{dS_q(q)}{dq})^2 = K(q, E, j) \tag{3.25}
$$

The function K obeys

$$
\begin{aligned}
E =&\frac{1}{2}(K + \frac{j^2}{q^2}) - \frac{1}{q} + \frac{1}{c^2}\{\frac{1}{8}(3\nu-1)(K + \frac{j^2}{q^2})^2 \\
&- \frac{1}{2q}[(3+\nu)(K + \frac{j^2}{q^2}) + \nu K] + \frac{1}{2q^2}\} + \frac{1}{c^4}\{\frac{1}{16}(5\nu^2 - 5\nu + 1)(K + \frac{j^2}{q^2})^3 \\
&+ \frac{1}{8q}[(-3\nu^2 - 20\nu + 5)(K + \frac{j^2}{q^2})^2 - 2\nu^2(K + \frac{j^2}{q^2})K - 3\nu^2K^2] \\
&+ \frac{1}{2q^2}[(8\nu+5)(K + \frac{j^2}{q^2}) + 3\nu K] - \frac{1}{4q^3}(3\nu+1)\} + O(c^{-6})
\end{aligned}
\tag{3.26}
$$

where we have used

$$\boldsymbol{n} \cdot \frac{dS}{d\boldsymbol{q}} = \sqrt{K} \tag{3.27a}$$

$$(\frac{dS}{d\boldsymbol{q}})^2 = K + \frac{j^2}{q^2} \tag{3.27b}$$

To explicitly find a solution, we expand the function K up to the order c^{-4}

$$K(q, E, j) = \sum_{i=0}^{5} \sum_{j=0}^{2} A_{ij} q^{-i} c^{-2j} + O(c^{-6}) \tag{3.28}$$

Elements of the matrix A_{ij} can be determined by substituting Eq. (3.28) into Eq. (3.26) and rearranging terms according to the order of q. For example, collection of terms with q^0 gives

$$\begin{aligned} E =& \frac{1}{2} A_{00} + \frac{1}{2c^2} A_{01} + \frac{1}{2c^4} A_{02} + \frac{1}{8c^2}(3\nu - 1)(A_{00}^2 + 2A_{00}A_{01}) \\ &+ \frac{1}{16c^4}(5\nu^2 - 5\nu + 1)A_{00}^3 + O(c^{-6}) \end{aligned} \tag{3.29}$$

By eliminating the terms of c^{-2} and c^{-4}, one can check

$$A_{00} = 2E \tag{3.30}$$

$$A_{01} = (1 - 3\nu)E^2 \tag{3.31}$$

$$A_{02} = (4\nu^2 - \nu)E^3 \tag{3.32}$$

Using the same procedure, we can explicitly determine all the elements of the matrix A_{ij}.

$$A_{ij} = \begin{pmatrix} 2E & (-3\nu+1)E^2 & (4\nu^2 - \nu)E^3 \\ 2 & 2(-\nu+4)E & 2(\nu^2 - 2\nu + 2)E^2 \\ -j^2 & \nu + 6 & 15E \\ 0 & -j^2\nu & (2 - Ej^2)\nu^2 + (\frac{5}{2} - Ej^2)\nu + \frac{17}{2} \\ 0 & 0 & -3j^2\nu^2 - j^2\nu \\ 0 & 0 & \frac{3}{4}j^4\nu^2 \end{pmatrix} \tag{3.33}$$

It is then useful to define

$$A_i = \sum_{j=0}^{2} A_{ij} c^{-2j} \tag{3.34}$$

Since $A_{i0} = 0$ for $i > 2$, it is clear that, when c^{-1} tends to 0, $A_i \neq 0$ only if $i \leq 2$. Under this condition, one can then check that, among 5 roots of $K(q, E, j)$, three of them tend to infinity while the other two remain non-zero and finite. We then denote these two roots as q_{min} and q_{max}. In this notation, we can solve Eq. (3.25)

$$S_q(q) = \int dq (A_0 + \frac{A_1}{q} + \frac{A_2}{q^2} + \frac{A_3}{q^3} + \frac{A_4}{q^4} + \frac{A_5}{q^5})^{\frac{1}{2}} \tag{3.35}$$

The reduced action is defined as

$$S_{red} = \frac{1}{\pi}\int_{q_{min}}^{q_{max}} dq(A_0 + \frac{A_1}{q} + \frac{A_2}{q^2} + \frac{A_3}{q^3} + \frac{A_4}{q^4} + \frac{A_5}{q^5})^{\frac{1}{2}} \tag{3.36}$$

This integral is explicitly evaluated in [13], such that

$$\begin{aligned} S_{red} = \frac{A_1}{2\sqrt{-A_0}} - \sqrt{-A_2}\{1 - \frac{1}{4}\frac{1}{A_2^2}[A_1A_3 + A_0A_4] \\ + \frac{3}{16}\frac{1}{A_2^3}[A_1^2A_4 + A_0A_3^2 + 2A_0A_1A_5] \\ - \frac{5}{64}\frac{1}{A_2^4}[3A_1^2A_3^2 + 2A_1^3A_5]\} + O(c^{-6}) \end{aligned} \tag{3.37}$$

Recall, for ϵ whose absolute value is small compared to an arbitrary finite value x, up to the order ϵ^2,

$$(x+\epsilon)^k = x^k[1 + k\frac{\epsilon}{x} + \frac{1}{2}k(k-1)(\frac{\epsilon}{x})^2] + O(\epsilon^3) \tag{3.38}$$

Use Eq. (3.33), Eq. (3.34), and Eq. (3.38), we can explicitly evaluate each term in Eq. (3.37) up to and including the order of c^{-4}

$$\frac{1}{2}A_1(-A_0)^{-\frac{1}{2}} = \frac{1}{\sqrt{-2E}}[1 + \frac{E}{c^2}(-\frac{1}{4}\nu + \frac{15}{4}) + \frac{E^2}{c^4}(\frac{3}{32}\nu^2 + \frac{15}{16}\nu + \frac{35}{32})] \tag{3.39a}$$

$$-\sqrt{-A_2} = -j + \frac{1}{c^2}\frac{\nu+6}{2j} + \frac{1}{c^4}[\frac{15E}{2j} + \frac{(\nu+6)^2}{8j^3}] \tag{3.39b}$$

$$\frac{1}{4}(-A_2)^{-\frac{3}{2}}[A_1A_3 + A_0A_4] = -\frac{1}{c^2}\frac{\nu}{2j} - \frac{1}{c^4}[\frac{3E}{j}(\frac{1}{2}\nu^2 + \nu) + \frac{1}{4j^3}(-\nu^2 + 13\nu - 17)] \tag{3.39c}$$

$$\frac{3}{16}(-A_2)^{-\frac{5}{2}}[A_1^2A_4 + A_0A_3^2 + 2A_0A_1A_5] = \frac{3}{16}\frac{1}{c^4}[-\frac{4}{j^3}(3\nu^2 + \nu) + \frac{8}{j}E\nu^2] \tag{3.39d}$$

$$\frac{5}{64}(-A_2)^{-\frac{7}{2}}[3A_1^2A_3^2 + 2A_1^3A_5] = \frac{15}{8}\frac{1}{j^3c^4}\nu^2 \tag{3.39e}$$

Inserting Eq. (3.39) into Eq. (3.37), we obtain

$$\begin{aligned} S_{red} = &-j + \frac{1}{\sqrt{-2E}} + \frac{1}{c^2}[\frac{3}{j} - \sqrt{-\frac{E}{2}}(\frac{15}{4} - \frac{\nu}{4})] \\ &+ \frac{1}{c^4}[\frac{E}{j}(\frac{15}{2} - 3\nu) + \frac{1}{j^3}(\frac{35}{4} - \frac{5}{2}\nu) + \sqrt{-\frac{E^3}{2}}(\frac{3}{32}\nu^2 + \frac{15}{16}\nu + \frac{35}{32})] \end{aligned} \tag{3.40}$$

3.3 Effective Metric

Eq. (3.40) is the reduced radial action obtained from the Newtonian expansion of a two-body Lagrangian. In this section, we aim to encode the motion described by Eq. (3.40) in a single effective metric. The effective variables are represented by adding ˜ to their representations in ADM coordinate. We assume that the coordinates, i.e. the positions and the conjugated momenta, in the effective problem are linked to the coordinates in the real problem by a canonical transformation. Detail of this canonical transformation can be found in Section 3.5. The metric $\tilde{g}_{\mu\nu}$ can be written as [8],

$$d\tilde{s}^2 = -M(\tilde{R})c^2 d\tilde{t}^2 + N(\tilde{R})d\tilde{R}^2 + L(\tilde{R})\tilde{R}^2(d\tilde{\theta}^2 + \tilde{\theta}^2 d\tilde{\phi}^2) \tag{3.41}$$

where we can expand the $\tilde{R}$-dependent functions M, N, and L

$$M(\tilde{R}) = 1 + \frac{GM_0 a_1}{c^2\tilde{R}} + \frac{G^2M_0^2 a_2}{c^4\tilde{R}^2} + \frac{G^3M_0^3 a_3}{c^6\tilde{R}^3} + O(c^{-8}) \tag{3.42a}$$

$$N(\tilde{R}) = 1 + \frac{GM_0 b_1}{c^2\tilde{R}} + \frac{G^2M_0^2 b_2}{c^4\tilde{R}^2} + O(c^{-6}) \tag{3.42b}$$

$$L(\tilde{R}) = 1 + \frac{GM_0 c_1}{c^2\tilde{R}} + \frac{G^2M_0^2 c_2}{c^4\tilde{R}^2} + O(c^{-6}) \tag{3.42c}$$

By analogy to the Schwarzschild metric describe in Eq. (2.1), we naturally define the mass scale M_0 such that

$$a_1 = -2 \tag{3.43}$$

We also define m_0 to be the mass of the effective particle. In these notations, the motion of the effective particle can be described as

$$\tilde{g}_{\mu\nu}\tilde{p}^\mu\tilde{p}^\nu + m_0^2c^2 = 0 \tag{3.44}$$

where $\tilde{p}_\mu$ is the conjugate momentum of the effective particle. Similarly, we assume the motion of the effective particle is purely in the equatorial plane, where $\tilde{\theta} = \frac{\pi}{2}$. We then apply the Hamilton-Jacobi approach and partition the effective action as

$$\tilde{S} = \tilde{S}_R + \tilde{S}_\phi - \tilde{E}_{tot}\tilde{t} \tag{3.45}$$

where $\tilde{E}_{tot}$ is the total energy of the effective particle

$$\tilde{E}_{tot} = \tilde{E}_0 + m_0c^2 \tag{3.46}$$

Applying the same method in Section 3.2, The Hamilton-Jacobi equations are

$$\frac{d\tilde{S}_\phi}{d\phi} = \tilde{J} \tag{3.47}$$

$$\frac{d\tilde{S}_R}{d\tilde{R}} = \sqrt{\tilde{K}(\tilde{R}, \tilde{E}_{tot}, \tilde{J})} \tag{3.48}$$

$$-\frac{\tilde{E}_{tot}^2}{Mc^2}+\frac{\tilde{J}^2}{L\tilde{R}^2}+\frac{1}{N}\tilde{K}+m_0^2c^2=0 \tag{3.49}$$

To simplify our calculation, we define the following scaled variables

$$\tilde{j}=\frac{\tilde{J}}{GM_0m_0} \tag{3.50}$$

$$\tilde{E}=\frac{\tilde{E}_0}{m_0} \tag{3.51}$$

$$\tilde{\mathcal{E}}=\frac{\tilde{E}_{tot}}{m_0} \tag{3.52}$$

$$\tilde{r}=\frac{\tilde{R}}{GM_0} \tag{3.53}$$

In the following calculations, we work in the Schwarzschild coordinate where $L(\tilde{r})=1$. We can then obtain an expression for $\tilde{K}$ by inserting Eq. (3.42) into Eq. (3.49)

$$\tilde{K}(\tilde{r},\tilde{E},\tilde{j})=\sum_{i=0}^{5}\sum_{j=0}^{2}\tilde{A}_{ij}\tilde{r}^{-i}c^{-2j} \tag{3.54}$$

where in the matrix notation

$$\tilde{A}_{ij}=m_0^2\begin{pmatrix} 2\tilde{E} & \tilde{E}^2 & 0 \\ -a_1 & 2\tilde{E}(b_1-a_1) & \tilde{E}^2(b_1-a_1) \\ -\tilde{j}^2 & (a_1^2-a_2-a_1b_1) & 2\tilde{E}(b_2+a_1^2-a_2-a_1b_1) \\ 0 & -b_1\tilde{j}^2 & 2a_1a_2-a_1^3-a_3+b_1a_1^2-b_1a_2-a_1b_2 \\ 0 & 0 & -b_2\tilde{j}^2 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.55}$$

Similarly, we define

$$\tilde{A}_i=\sum_{j=0}^{2}\tilde{A}_{ij}c^{-2j} \tag{3.56}$$

The scaled reduced effective action is therefore defined as

$$\tilde{S}_{red}=\frac{1}{m_0\pi}\int_{\tilde{r}_{min}}^{\tilde{r}_{max}}\sqrt{\tilde{K}(\tilde{r},\tilde{E}_{tot},\tilde{j})} \tag{3.57}$$

where, with the similar discussion in Section 3.2, $\tilde{r}_{min}$ and $\tilde{r}_{max}$ are two roots which are non-zero and finite when $\frac{1}{c}$ goes to zero. One can directly calculate this integral by using Eq. (3.37). Detail calculation gives the explicit expression for each term in Eq. (3.37)

$$\frac{1}{2}\tilde{A}_1(-\tilde{A}_0)^{-\frac{1}{2}}=\frac{m_0}{\sqrt{-2\tilde{E}}}[-\frac{1}{2}a_1+(b_1-\frac{7}{8}a_1)\frac{\tilde{E}}{c^2}+(\frac{1}{4}b_1-\frac{19}{64}a_1)\frac{\tilde{E}^2}{c^4}] \tag{3.58}$$

$$
\begin{aligned}
-\sqrt{-\tilde{A}_2} =& -m_0\tilde{j} + \frac{1}{c^2}\frac{m_0}{2\tilde{j}}(a_1^2 - a_2 - a_1 b_1) \\
&+ \frac{1}{c^4}[\frac{\tilde{E}m_0}{\tilde{j}}(b_2 + a_1^2 - a_2 - a_1 b_1) + \frac{1}{8}\frac{m_0}{\tilde{j}^3}(a_1^2 - a_2 - a_1 b_1)^2]
\end{aligned} \tag{3.59}
$$

$$
\begin{aligned}
\frac{1}{4}(-\tilde{A}_2)^{-\frac{3}{2}}[\tilde{A}_1\tilde{A}_3 + \tilde{A}_0\tilde{A}_4] =& \frac{m_0}{c^2\tilde{j}}\frac{1}{4}a_1 b_1 + \frac{\tilde{E}m_0}{c^4\tilde{j}}(\frac{1}{2}a_1 b_1 - \frac{1}{2}b_1^2 - \frac{1}{2}b_2) \\
&+ \frac{m_0}{c^4\tilde{j}^3}(\frac{1}{8}a_1^3 b_1 - \frac{1}{8}a_1 b_1 a_2 - \frac{3}{8}a_1^2 b_1^2 - \frac{1}{2}a_1^2 a_2 \\
&+ \frac{1}{4}a_1^4 + \frac{1}{4}a_1 a_3 + \frac{1}{4}a_1^2 b_2)
\end{aligned} \tag{3.60}
$$

$$
\frac{3}{16}(-\tilde{A}_2)^{-\frac{5}{2}}[\tilde{A}_1^2\tilde{A}_4 + \tilde{A}_0\tilde{A}_3^2 + 2\tilde{A}_0\tilde{A}_1\tilde{A}_5] = \frac{3}{8}\frac{\tilde{E}m_0}{\tilde{j}c^4}b_1^2 - \frac{3}{16}\frac{m_0}{\tilde{j}^3c^4}a_1^2 b_2 \tag{3.61}
$$

$$
\frac{5}{64}(-\tilde{A}_2)^{-\frac{7}{2}}[3\tilde{A}_1^2\tilde{A}_3^2 + 2\tilde{A}_1^3\tilde{A}_5] = \frac{15}{64}\frac{m_0}{\tilde{j}^3c^4}a_1^2 b_1^2 \tag{3.62}
$$

Therefore, we obtain

$$
\begin{aligned}
\tilde{S}_{red} =& -\tilde{j} - \frac{1}{2}a_1\frac{1}{\sqrt{-2\tilde{E}}} + \frac{1}{c^2}[\frac{1}{\tilde{j}}(\frac{1}{2}a_1^2 - \frac{1}{2}a_2 - \frac{1}{4}a_1 b_1) + \sqrt{-\frac{\tilde{E}}{2}}(\frac{7}{8}a_1 - b_1)] \\
&+ \frac{1}{c^4}[\sqrt{-\frac{\tilde{E}^3}{2}}(\frac{1}{4}b_1 - \frac{19}{64}a_1) + \frac{\tilde{E}}{\tilde{j}}(a_1^2 - \frac{1}{2}a_1 b_1 - a_2 - \frac{1}{8}b_1^2 + \frac{1}{2}b_2)] \\
&+ \frac{1}{\tilde{j}^3c^4}(\frac{3}{8}a_1^4 - \frac{3}{4}a_1^2 a_2 + \frac{1}{4}a_1 a_3 + \frac{1}{8}a_2^2 - \frac{1}{8}a_1^3 b_1 + \frac{1}{8}a_1 a_2 b_1 - \frac{1}{64}a_1^2 b_1^2 + \frac{1}{16}a_1^2 b_2)
\end{aligned} \tag{3.63}
$$

Canonical transformation ensures that the action of the effective problem is unchanged from the action of the real problem. Eq. (3.63) has the same form as Eq. (3.40). However, before we can determine the explicit values of coefficients a_i and b_i, we must know the relations between the effective quantities, i.e. M_0, m_0, $\tilde{j}$ and $\tilde{E}$, and the real quantities, i.e. M, μ, j and E. One can check that if we require all the effective quantities equal to their corresponding real ones, e.g. $M_0 = M$, $m_0 = \mu$, $\tilde{j} = j$, and $\tilde{E} = E$, by comparing Eq. (3.63) to Eq. (3.40), we obtain 5 independent equations with only 4 unknowns (recall $a_1 = 2$). No solution is found in this case. Therefore, there is at least one effective quantity that differs from its corresponding real quantity. There are several approaches which give different assumptions about their relations. However, in the following discussion, we will assume

$$M_0 = M \tag{3.64a}$$

$$m_0 = \mu \tag{3.64b}$$

$$\tilde{j} = j \tag{3.64c}$$

$$\tilde{E} = E(1 + c_1\frac{E}{c^2} + c_2\frac{E^2}{c^4}) + O(c^{-6}) \tag{3.64d}$$

where c_1 and c_2 are two unknown coefficients to be determined. Now, we insert Eq. (3.64) into Eq. (3.63) and obtain

$$
\begin{aligned}
\tilde{S}_{red} =& - j - \frac{1}{2}a_1\frac{1}{\sqrt{-2E}} \\
&+ \frac{1}{c^2}[\frac{1}{j}(\frac{1}{2}a_1^2 - \frac{1}{2}a_2 - \frac{1}{4}a_1b_1) + \sqrt{-\frac{E}{2}}(\frac{7}{8}a_1 - b_1 - \frac{1}{4}a_1c_1)] \\
&+ \frac{1}{c^4}\sqrt{-\frac{E^3}{2}}(-\frac{7}{16}a_1c_1 + \frac{1}{2}b_1c_1 - \frac{3}{16}a_1c_1^2 + \frac{1}{4}b_1 - \frac{19}{64}a_1 + \frac{1}{4}c_2) \\
&+ \frac{1}{c^4}\frac{E}{j}(a_1^2 - \frac{1}{2}a_1b_1 - a_2 - \frac{1}{8}b_1^2 + \frac{1}{2}b_2) \\
&+ \frac{1}{c^4j^3}(\frac{3}{8}a_1^4 - \frac{3}{4}a_1^2a_2 + \frac{1}{4}a_1a_3 + \frac{1}{8}a_2^2 - \frac{1}{8}a_1^3b_1 + \frac{1}{8}a_1a_2b_1 - \frac{1}{64}a_1^2b_1^2 + \frac{1}{16}a_1^2b_2)
\end{aligned} \tag{3.65}
$$

We aim to find the values of a_i, b_i, and c_i, such that the reduced effective radial action is equal to the real action. Comparing Eq. (3.65) to Eq. (3.40), we obtain a set of equations

$$\frac{1}{2}a_1^2 - \frac{1}{2}a_2 - \frac{1}{4}a_1b_1 = 3 \tag{3.66a}$$

$$\frac{7}{8}a_1 - b_1 - \frac{1}{4}a_1c_1 = -\frac{15}{4} + \frac{\nu}{4} \tag{3.66b}$$

$$-\frac{7}{16}a_1c_1 + \frac{1}{2}b_1c_1 - \frac{3}{16}a_1c_1^2 + \frac{1}{4}b_1 - \frac{19}{64}a_1 + \frac{1}{4}c_2 = \frac{3}{32}\nu^2 + \frac{15}{16}\nu + \frac{35}{32} \tag{3.66c}$$

$$a_1^2 - \frac{1}{2}a_1b_1 - a_2 - \frac{1}{8}b_1^2 + \frac{1}{2}b_2 = \frac{15}{2} - 3\nu \tag{3.66d}$$

$$\frac{3}{8}a_1^4 - \frac{3}{4}a_1^2a_2 + \frac{1}{4}a_1a_3 + \frac{1}{8}a_2^2 - \frac{1}{8}a_1^3b_1 + \frac{1}{8}a_1a_2b_1 - \frac{1}{64}a_1^2b_1^2 + \frac{1}{16}a_1^2b_2 = \frac{35}{4} - \frac{5}{2}\nu \tag{3.66e}$$

Now, since a_1 is known ($a_1 = 2$), we have 5 equations and 6 unknowns. We can therefore fix the value of b_1 without affecting the physics. By analogy to the Schwarzschild metric described in Eq. (2.1), b_1 is chosen to be 2. Eq. (3.66) then gives a unique solution

$$(a_1, a_2, a_3) = (-2, 0, 2\nu) \tag{3.67}$$

$$(b_1, b_2) = (2, 4 - 6\nu) \tag{3.68}$$

$$(c_1, c_2) = (\frac{\nu}{2}, 0) \tag{3.69}$$

We therefore conclude

$$
\begin{aligned}
d\tilde{s}^2 =& - (1 - \frac{2GM}{c^2\tilde{R}} + \frac{2G^3M^3\nu}{c^6\tilde{R}^3} + O(c^{-8}))c^2d\tilde{t}^2 \\
&+ (1 + \frac{2GM}{c^2\tilde{R}} + \frac{(4-6\nu)G^2M^2}{c^4\tilde{R}^2} + O(c^{-6}))d\tilde{R}^2 \\
&+ \tilde{R}^2(d\tilde{\theta}^2 + \tilde{\theta}^2d\tilde{\phi}^2)
\end{aligned} \tag{3.70}
$$

and

$$\tilde{E} = E(1 + \frac{\nu}{2}\frac{E}{c^2}) + O(c^{-6}) \tag{3.71}$$

Inserting Eq. (3.46) into Eq. (3.71), we have

$$\frac{\tilde{\mathcal{E}}}{c^2} = 1 + \frac{E}{c^2} + \frac{1}{c^4}\frac{\nu E^2}{2} + O(c^{-6}) \tag{3.72}$$

3.4 Dynamics

In order to discuss the dynamics of the effective one body system, by analogy to the Schwarzschild metric, we write the metric described in Eq. (3.70) in the following form

$$d\tilde{s}^2 = -M(\tilde{R})c^2 d\tilde{t}^2 + \frac{P(\tilde{R})}{M(\tilde{R})}d\tilde{R}^2 + \tilde{R}^2(d\tilde{\theta}^2 + \tilde{\theta}^2 d\tilde{\phi}^2) \tag{3.73}$$

where

$$\begin{aligned} P(\tilde{R}) &= M(\tilde{R})N(\tilde{R}) \\ &= 1 - \frac{6\nu G^2 M^2}{c^4 \tilde{R}^2} + O(c^{-6}) \end{aligned} \tag{3.74}$$

To simplify the calculation, we define

$$\tilde{u} := \frac{GM}{c^2\tilde{R}} \tag{3.75}$$

$$\tilde{h} := c\tilde{j} \tag{3.76}$$

The function $M(\tilde{R})$ reaches zero when

$$2\nu\tilde{u}^3 - 2\tilde{u} + 1 = 0 \tag{3.77}$$

From the definition of ν, we have

$$0 \leq \nu \leq \frac{1}{4} \tag{3.78}$$

If we require $\nu = 0$, Eq. (3.77) yields a single solution $\tilde{R} = \frac{2GM}{c^2}$, which is the event horizon of the Schwarzschild metric. However, if $\nu \neq 0$, Eq. (3.77) admits three real roots, among which two are positive. In the following discussion, we only consider the root $\tilde{R}_0$ that tends to $\frac{2GM}{c^2}$ when ν tends to zero. The other positive root is not physical, since it tends to infinity when ν tends to zero.
In order to obtain the radial effective potential, we recall, from Eq. (3.47)-(3.49), the Hamiltonian can be written as

$$\tilde{H}^2 = m_0^2 c^4[M(\tilde{R}) + \frac{M(\tilde{R})\tilde{J}^2}{m_0^2 c^2 \tilde{R}^2} + \frac{M(\tilde{R})}{m_0^2 c^2 N(\tilde{R})}(\frac{d\tilde{S}_r}{d\tilde{R}})^2] \tag{3.79}$$

The effective radial potential can be directly read as

$$\tilde{V}_{eff} = M(\tilde{R}) + \frac{M(\tilde{R})\tilde{J}^2}{m_0^2 c^2 \tilde{R}^2} \tag{3.80}$$

In terms of the notation we defined, Eq. (3.80) can be simplified

$$\tilde{V}_{eff} = 2\nu\tilde{h}^2\tilde{u}^5 + 2(\nu - \tilde{h}^2)\tilde{u}^3 + \tilde{h}^2\tilde{u}^2 - 2\tilde{u} + 1 \tag{3.81}$$

To calculate the ISCO of the effective one body metric, we require $\frac{d\tilde{V}_{eff}}{d\tilde{u}} = 0$ and $\frac{d^2\tilde{V}_{eff}}{d\tilde{u}^2} = 0$. This condition yields two equations

$$5\nu\tilde{h}^2\tilde{u}^4 + 3(\nu - \tilde{h}^2)\tilde{u}^2 + \tilde{h}^2\tilde{u} - 1 = 0 \tag{3.82a}$$

$$20\nu\tilde{h}^2\tilde{u}^3 + 6(\nu - \tilde{h}^2)\tilde{u} + \tilde{h}^2 = 0 \tag{3.82b}$$

which can be further simplified

$$30\nu^2\tilde{u}^5 - 20\nu\tilde{u}^3 - 3\nu\tilde{u}^3 + 6\tilde{u} - 1 = 0 \tag{3.83a}$$

$$\tilde{h} = \sqrt{\frac{-3\nu\tilde{u}^2 + 1}{5\nu\tilde{u}^4 - 3\tilde{u}^2 + \tilde{u}}} \tag{3.83b}$$

From Eq. (3.83), one can check that, if $\nu = 0$, $\tilde{u}|_{\nu=0} = \frac{1}{6}$ and $\tilde{h}\Big|_{\nu=0} = 2\sqrt{3}$. This is exactly the result we obtained from calculating the ISCO for the Schwarzschild metric. For general values of ν, one can numerically calculate the value of $\tilde{u}$ from Eq. (3.83a).

3.5 Canonical Transformation

In this section, we aim to map the coordinates and momenta in the real problem (ADM coordinate) to that in the effective problem. We denote the positions and momenta in the real problem as Q^i and P_i and the positions and momenta in the effective problem as $\tilde{Q}^i$ and $\tilde{P}_i$, where the index $i = 1, 2, 3$. In spherical coordinate, we have $\tilde{Q}^1 = \tilde{Q}\sin\tilde{\theta}\cos\tilde{\phi}$, $\tilde{Q}^2 = \tilde{Q}\sin\tilde{\theta}\sin\tilde{\phi}$, and $\tilde{Q}^3 = \tilde{Q}\cos\tilde{\theta}$. We can calculate the momentum in the spherical coordinate by using

$$\tilde{P}_i = \frac{\partial\tilde{S}}{\partial\tilde{Q}^i} = \tilde{P}_R\frac{\partial\tilde{Q}}{\partial\tilde{Q}^i} + \tilde{P}_\theta\frac{\partial\tilde{\theta}}{\partial\tilde{Q}^i} + \tilde{P}_\phi\frac{\partial\tilde{\phi}}{\partial\tilde{Q}^i} \tag{3.84}$$

When $\tilde{\theta} = \frac{\pi}{2}$, we obtain

$$\tilde{P}_1 = \tilde{P}_R\frac{\tilde{Q}^1}{\tilde{Q}} - \tilde{P}_\phi\frac{\tilde{Q}^2}{(\tilde{Q})^2} \tag{3.85}$$

$$\tilde{P}_2 = \tilde{P}_R\frac{\tilde{Q}^2}{\tilde{Q}} + \tilde{P}_\phi\frac{\tilde{Q}^1}{(\tilde{Q})^2} \tag{3.86}$$

$$\tilde{P}_3 = 0 \tag{3.87}$$

One can check that the Hamilton Jacobi equation in Eq. (3.49) can be written as

$$\tilde{H}_0 = m_0 c^2 \sqrt{M(\tilde{Q})(\frac{\tilde{P}_R^2}{N(\tilde{Q}) m_0^2 c^2} + \frac{\tilde{P}_\phi^2}{\tilde{Q}^2 m_0^2 c^2} + 1)} \tag{3.88}$$

Eq. (3.88) can be simplified

$$\tilde{H}_0 = m_0 c^2 \sqrt{M(\tilde{Q})(\frac{(\tilde{\boldsymbol{n}} \cdot \tilde{\boldsymbol{P}})^2}{N(\tilde{Q}) m_0^2 c^2} + \frac{\tilde{\boldsymbol{P}}^2 - (\tilde{\boldsymbol{n}} \cdot \tilde{\boldsymbol{P}})^2}{m_0^2 c^2} + 1)} \tag{3.89}$$

where we have used the following properties

$$\tilde{\boldsymbol{P}} \cdot \tilde{\boldsymbol{Q}} = \tilde{P}_R \tilde{Q} \tag{3.90a}$$

$$\tilde{\boldsymbol{P}}^2 - (\tilde{\boldsymbol{n}} \cdot \tilde{\boldsymbol{P}})^2 = \frac{\tilde{P}_\phi^2}{\tilde{Q}^2} \tag{3.90b}$$

$$\tilde{\boldsymbol{n}} = \frac{\tilde{\boldsymbol{Q}}}{\tilde{Q}} \tag{3.90c}$$

Again, to simply the following calculation, we define the following scaled variables

$$\tilde{\boldsymbol{q}} = \frac{\tilde{\boldsymbol{Q}}}{GM_0} \tag{3.91}$$

$$\tilde{\boldsymbol{p}} = \frac{\tilde{\boldsymbol{P}}}{m_0} \tag{3.92}$$

$$\tilde{H} = \frac{\tilde{H}_0}{m_0} \tag{3.93}$$

We can therefore express Eq. (3.89) with the scaled variables

$$\tilde{H} = c^2 \sqrt{M(\tilde{q})(\frac{(\tilde{\boldsymbol{n}} \cdot \tilde{\boldsymbol{p}})^2}{N(\tilde{q}) c^2} + \frac{\tilde{\boldsymbol{p}}^2 - (\tilde{\boldsymbol{n}} \cdot \tilde{\boldsymbol{p}})^2}{c^2} + 1)} \tag{3.94}$$

Since, the reduced action and the angular momentum are identical in the real and effective problem, e.g. $\tilde{S}_{red} = S_{red}$ and $\tilde{j} = j$, the Hamilton's equations are preserved. We can define a canonical transformation such that

$$p_i dq^i - \tilde{p}_i d\tilde{q}^i = dg(q, \tilde{q}) \tag{3.95}$$

where $g(q, \tilde{q})$ is the generating function. Introducing a new generating function $G(q, \tilde{p}) = \tilde{p}_i \tilde{q}^i + g(q, \tilde{q})$, we obtain

$$p_i dq^i + \tilde{q}^i d\tilde{p}_i = dG(q, \tilde{p}) \tag{3.96}$$

We can then expand the new generating function

$$G(q,\tilde{p}) = \tilde{p}_i q^i + G_{PN}(q,\tilde{p}) \tag{3.97}$$

where

$$G_{PN}(q,\tilde{p}) = \frac{1}{c^2}G_1(q,\tilde{p}) + \frac{1}{c^4}G_2(q,\tilde{p}) \tag{3.98}$$

One should be aware that $G_{PN}(q,\tilde{p})$ is of order c^{-2}. By dividing Eq. (3.96) by dq^i and $d\tilde{p}_i$ respectively, we yield

$$\tilde{p}_i = p_i - \frac{\partial G_{PN}(q,\tilde{p})}{\partial q^i} \tag{3.99a}$$

$$\tilde{q}^i = q^i + \frac{\partial G_{PN}(q,\tilde{p})}{\partial \tilde{p}_i} \tag{3.99b}$$

To the order of c^{-4}, we expand $K_{qi}(q,\tilde{p}) = \frac{\partial G_{PN}(q,\tilde{p})}{\partial q^i}$

$$K_{qi}(q,\tilde{p}) = K_{qi}(q,p) - K_{qj}(q,p)\frac{\partial K_{qi}(q,p)}{\partial p_j} + O(c^{-6}) \tag{3.100}$$

Inserting Eq. (3.100) into Eq. (3.99a) yields

$$\tilde{p}_i = p_i - \frac{\partial G_{PN}(q,p)}{\partial q^i} + \frac{\partial G_{PN}(q,p)}{\partial q^j}\frac{\partial^2 G_{PN}(q,p)}{\partial p_j \partial q^i} + O(c^{-6}) \tag{3.101}$$

Similarly, we expand $\tilde{K}_p^i(q,\tilde{p}) = \frac{\partial G_{PN}(q,\tilde{p})}{\partial \tilde{p}_i}$ and obtain

$$\tilde{q}^i = q^i + \frac{\partial G_{PN}(q,p)}{\partial p_i} - \frac{\partial G_{PN}(q,p)}{\partial q^j}\frac{\partial^2 G_{PN}(q,p)}{\partial p_i \partial p_j} + O(c^{-6}) \tag{3.102}$$

We can further reduce Eq. (3.101) and Eq. (3.102) using Eq. (33.98) and obtain

$$\tilde{p}_i = p_i - \frac{1}{c^2}\frac{\partial G_1(q,p)}{\partial q^i} + \frac{1}{c^4}(\frac{\partial G_1(q,p)}{\partial q^j}\frac{\partial^2 G_1(q,p)}{\partial p_j \partial q^i} - \frac{\partial G_2(q,p)}{\partial q^i}) + O(c^{-6}) \tag{3.103a}$$

$$\tilde{q}^i = q^i + \frac{1}{c^2}\frac{\partial G_1(q,p)}{\partial p_i} - \frac{1}{c^4}(\frac{\partial G_1(q,p)}{\partial q^j}\frac{\partial^2 G_1(q,p)}{\partial p_i \partial p_j} - \frac{\partial G_2(q,p)}{\partial p_i}) + O(c^{-6}) \tag{3.103b}$$

Let us recall that the reduced real Hamiltonian is related to the reduced effective Hamiltonian by Eq. (3.72)

$$\frac{\tilde{H}}{c^2} = 1 + \frac{1}{c^2}H + \frac{1}{c^4}\frac{\nu H^2}{2} + O(c^{-6}) \tag{3.104}$$

The reduced real Hamiltonian is explicitly expressed in Eq. (3.20). To deal with the square root in the reduced effective metric $\tilde{H}$, it is convenient to square both side of Eq. (3.104). We therefore have

$$M(\tilde{q})(\frac{(\tilde{\boldsymbol{n}}\cdot\tilde{\boldsymbol{p}})^2}{N(\tilde{q})c^2}+\frac{\tilde{\boldsymbol{p}}^2-(\tilde{\boldsymbol{n}}\cdot\tilde{\boldsymbol{p}})^2}{c^2}+1)=1+\frac{1}{c^2}R_2+\frac{1}{c^4}R_4+\frac{1}{c^6}R_6+O(c^{-8}) \tag{3.105}$$

where

$$R_2=\boldsymbol{p}^2-\frac{2}{q} \tag{3.106}$$

$$R_4=\nu\boldsymbol{p}^4-(4+2\nu)\frac{\boldsymbol{p}^2}{q}-\nu\frac{(\boldsymbol{n}\cdot\boldsymbol{p})^2}{q}+(2+\nu)\frac{1}{q^2} \tag{3.107}$$

$$\begin{aligned}R_6=&\frac{1}{8}\boldsymbol{p}^6(8\nu^2-2\nu)+\frac{1}{4}\frac{\boldsymbol{p}^4}{q}(-8\nu^2-33\nu)+\frac{1}{2}\frac{\boldsymbol{p}^2}{q}(\boldsymbol{n}\cdot\boldsymbol{p})^2(-2\nu^2-\nu)\\&+\frac{1}{4}\frac{(\boldsymbol{n}\cdot\boldsymbol{p})^4}{q}(-3\nu^2)+\frac{1}{2}\frac{\boldsymbol{p}^2}{q^2}(2\nu^2+28\nu+17)\\&+\frac{(\boldsymbol{n}\cdot\boldsymbol{p})^2}{q^2}(\nu^2+4\nu)+\frac{1}{2}\frac{1}{q^3}(-7\nu-3)\end{aligned} \tag{3.108}$$

In the following calculations, we will denote $G_1 := G_1(q,p)$ and $G_2 := G_2(q,p)$ for the sake of simplicity. Since the coordinates and momenta indices are all spatial, we are therefore allowed to define $u_iv_i = u^iv^i = u_iv^i = \boldsymbol{u}\cdot\boldsymbol{v}$ and $u_i^2 = u_iu_j\delta^{ij} = \boldsymbol{u}^2$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are some arbitrary functions of $\boldsymbol{q}$ and $\boldsymbol{p}$, i.e. $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{q},\boldsymbol{p})$ and $\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{q},\boldsymbol{p})$. To simplify the left hand side of Eq. (3.105), it is useful to check

$$\begin{aligned}\tilde{q}=&q\{1+\frac{1}{c^2}\frac{n^i}{q}\frac{\partial G_1}{\partial p_i}\\&+\frac{1}{2c^4}[\frac{2n^i}{q}\frac{\partial G_2}{\partial p_i}+\frac{1}{q^2}(\frac{\partial G_1}{\partial p_i})^2-\frac{2n^i}{q}\frac{\partial G_1}{\partial q^j}\frac{\partial^2 G_1}{\partial p_i\partial p_j}-\frac{n^in^j}{q^2}\frac{\partial G_1}{\partial p_i}\frac{\partial G_1}{\partial p_j}]\}\end{aligned} \tag{3.109}$$

$$\begin{aligned}M(\tilde{q})=&1-\frac{1}{c^2}\frac{2}{q}+\frac{1}{c^4}\frac{2n^i}{q^2}\frac{\partial G_1}{\partial p_i}\\&+\frac{1}{c^6}[\frac{2n^i}{q^2}\frac{\partial G_2}{\partial p_i}+\frac{1}{q^3}(\frac{\partial G_1}{\partial p_i})^2-\frac{2n^i}{q^2}\frac{\partial G_1}{\partial q^j}\frac{\partial^2 G_1}{\partial p_i\partial p_j}-3\frac{n^in^j}{q^3}\frac{\partial G_1}{\partial p_i}\frac{\partial G_1}{\partial p_j}+\frac{2\nu}{q^3}]\\&+O(c^{-8})\end{aligned} \tag{3.110}$$

$$\frac{1}{N(\tilde{q})}=1-\frac{2}{c^2q}+\frac{1}{c^4}(\frac{6\nu}{q^2}+\frac{2n^i}{q^2}\frac{\partial G_1}{\partial p_i})+O(c^{-6}) \tag{3.111}$$

We then have

$$M(\tilde{q})(\frac{(\tilde{\boldsymbol{n}}\cdot\tilde{\boldsymbol{p}})^2}{N(\tilde{q})c^2}+\frac{\tilde{\boldsymbol{p}}^2-(\tilde{\boldsymbol{n}}\cdot\tilde{\boldsymbol{p}})^2}{c^2}+1)=1+\frac{1}{c^2}L_2+\frac{1}{c^4}L_4+\frac{1}{c^6}L_6+O(c^{-8}) \tag{3.112}$$

where

$$L_2 = \boldsymbol{p}^2 - \frac{2}{q} \tag{3.113}$$

$$L_4 = \frac{2n^i}{q^2}\frac{\partial G_1}{\partial p_i} - 2p_i\frac{\partial G_1}{\partial q^i} - \frac{2}{q}\boldsymbol{p}^2 - \frac{2(\boldsymbol{p}\cdot\boldsymbol{n})^2}{q} \tag{3.114}$$

$$\begin{aligned} L_6 =& \frac{2n^i}{q^2}\frac{\partial G_2}{\partial p_i} - 2p_i\frac{\partial G_2}{\partial q^i} - \frac{2n^i}{q^2}\frac{\partial G_1}{\partial q^j}\frac{\partial^2 G_1}{\partial p_i \partial p_j} + 2p_i\frac{\partial G_1}{\partial q^j}\frac{\partial^2 G_1}{\partial p_j \partial q^i} \\ &+ \frac{1}{q^3}(\frac{\partial G_1}{\partial p_i})^2 - 3\frac{n^i n^j}{q^3}\frac{\partial G_1}{\partial p_i}\frac{\partial G_1}{\partial p_j} + (\frac{\partial G_1}{\partial q^i})^2 \\ &+ (-\frac{4(\boldsymbol{n}\cdot\boldsymbol{p})p_i}{q^2} + \frac{6(\boldsymbol{n}\cdot\boldsymbol{p})^2 n^i}{q^2} + \frac{2\boldsymbol{p}^2 n^i}{q^2})\frac{\partial G_1}{\partial p_i} \\ &+ (\frac{4(\boldsymbol{n}\cdot\boldsymbol{p})n^i}{q} + \frac{4p_i}{q})\frac{\partial G_1}{\partial q^i} + \frac{2\nu}{q^3} + \frac{6\nu+4}{q^2}(\boldsymbol{n}\cdot\boldsymbol{p})^2 \end{aligned} \tag{3.115}$$

We choose the ansatz such that

$$G_1 = (\alpha_1\boldsymbol{p}^2 + \alpha_2\frac{1}{q})(\boldsymbol{p}\cdot\boldsymbol{q}) \tag{3.116}$$

we therefore have

$$\frac{\partial G_1}{\partial p_i} = \alpha_1 q^i \boldsymbol{p}^2 + 2\alpha_1 p^i(\boldsymbol{p}\cdot\boldsymbol{q}) + \alpha_2 n^i \tag{3.117a}$$

$$\frac{\partial G_1}{\partial q^i} = \alpha_1 p_i \boldsymbol{p}^2 + \alpha_2\frac{p_i}{q} - \alpha_2\frac{n_i}{q}(\boldsymbol{n}\cdot\boldsymbol{p}) \tag{3.117b}$$

and

$$\frac{\partial^2 G_1}{\partial p_i \partial p_j} = 2\alpha_1(p^i q^j + p^j q^i + \delta^{ij}(\boldsymbol{p}\cdot\boldsymbol{q})) \tag{3.118a}$$

$$\frac{\partial^2 G_1}{\partial q^i \partial p_j} = \alpha_1(\delta_i^j\boldsymbol{p}^2 + 2p_i p^j) + \frac{\alpha_2}{q}(\delta_i^j - n_i n^j) \tag{3.118b}$$

Inserting Eq. (3.117) into (3.114) and equating $L_4 = R_4$, we have

$$\alpha_1 - \alpha_2 = -1 - \nu \tag{3.119a}$$

$$2\alpha_1 + \alpha_2 = 1 - \frac{\nu}{2} \tag{3.119b}$$

$$-2\alpha_1 = \nu \tag{3.119c}$$

$$2\alpha_2 = 2 + \nu \tag{3.119d}$$

Eq. (3.119) yields a single solution

$$\alpha_1 = -\frac{\nu}{2} \tag{3.120a}$$

$$\alpha_2 = 1 + \frac{\nu}{2} \tag{3.120b}$$

Once we deduce the explicit expression for G_1, we can insert Eq. (3.117) and Eq. (3.118) into Eq. (3.115)

$$\begin{aligned}L_6 =& \frac{2n^i}{q^2}\frac{\partial G_2}{\partial p_i} - 2p_i\frac{\partial G_2}{\partial q^i} \\ &+ \frac{7\nu^2}{4}\boldsymbol{p}^6 + (-4\nu^2 - 8\nu)\frac{\boldsymbol{p}^4}{q} + (-\frac{1}{2}\nu^2 + 4\nu)\frac{\boldsymbol{p}^2}{q}(\boldsymbol{n}\cdot\boldsymbol{p})^2 + (-3\nu^2 - 6\nu)\frac{(\boldsymbol{n}\cdot\boldsymbol{p})^4}{q} \\ &+ (\frac{11}{4}\nu^2 + 10\nu + 9)\frac{\boldsymbol{p}^2}{q^2} + (\frac{1}{4}\nu^2 + 4\nu - 1)\frac{(\boldsymbol{n}\cdot\boldsymbol{p})^2}{q^2} + (-\frac{1}{2}\nu^2 - 2)\frac{1}{q^3}\end{aligned} \tag{3.121}$$

Similarly, we choose the ansatz for G_2

$$G_2 = (\beta_1\boldsymbol{p}^4 + \beta_2\frac{\boldsymbol{p}^2}{q} + \beta_3\frac{(\boldsymbol{n}\cdot\boldsymbol{p})^2}{q} + \beta_4\frac{1}{q^2})(\boldsymbol{p}\cdot\boldsymbol{q}) \tag{3.122}$$

and evaluate the following

$$\frac{\partial G_2}{\partial p_i} = \beta_1(\boldsymbol{p}^4 q^i + 4p^i\boldsymbol{p}^2(\boldsymbol{p}\cdot\boldsymbol{q})) + \beta_2(\boldsymbol{p}^2 n^i + 2(\boldsymbol{n}\cdot\boldsymbol{p})p^i) + 3\beta_3(\boldsymbol{n}\cdot\boldsymbol{p})^2 n^i + \beta_4\frac{n^i}{q} \tag{3.123}$$

$$\begin{aligned}\frac{\partial G_2}{\partial q^i} =& \beta_1\boldsymbol{p}^4 p_i + \beta_2(\frac{\boldsymbol{p}^2}{q}p_i - \frac{\boldsymbol{p}^2(\boldsymbol{n}\cdot\boldsymbol{p})}{q}n_i) \\ &+ \beta_3(3\frac{(\boldsymbol{n}\cdot\boldsymbol{p})^2}{q}p_i - 3\frac{(\boldsymbol{n}\cdot\boldsymbol{p})^3}{q}n_i) + \beta_4(\frac{p_i}{q^2} - 2\frac{(\boldsymbol{n}\cdot\boldsymbol{p})}{q^2}n_i)\end{aligned} \tag{3.124}$$

Inserting Eq. (3.123) and Eq. (3.124) into Eq. (3.121) and equating $L_6 = R_6$, we have

$$-\beta_1 = -\frac{\nu}{8}(3\nu + 1) \tag{3.125a}$$

$$\beta_1 - \beta_2 = \frac{\nu}{8}(8\nu - 1) \tag{3.125b}$$

$$3\beta_3 = \frac{3\nu}{8}(3\nu + 8) \tag{3.125c}$$

$$\beta_4 = \frac{1}{4}(\nu^2 - 7\nu + 1) \tag{3.125d}$$

$$4\beta_1 + \beta_2 - 3\beta_3 = -\frac{\nu}{4}(\nu + 9) \tag{3.125e}$$

$$\beta_2 - \beta_4 = \frac{1}{8}(-7\nu^2 + 16\nu - 2) \tag{3.125f}$$

$$2\beta_2 + 3\beta_3 + 2\beta_4 = \frac{1}{8}(3\nu^2 + 4) \tag{3.125g}$$

From Eq. (3.125), we have a unique solution

$$\beta_1 = \frac{\nu}{8}(3\nu + 1) \tag{3.126a}$$

$$\beta_2 = \frac{\nu}{8}(-5\nu + 2) \tag{3.126b}$$

$$\beta_3 = \frac{\nu}{8}(3\nu + 8) \tag{3.126c}$$

$$\beta_4 = \frac{1}{4}(\nu^2 - 7\nu + 1) \tag{3.126d}$$

Once the explicit expressions of G_1 and G_2 are found, one can easily deduce the canonical transformation between the effective coordinates and the real coordinates using Eq. (3.103)

$$\begin{aligned}
\tilde{p}_i &= p_i - \frac{1}{c^2}(\alpha_1 p_i \boldsymbol{p}^2 + \alpha_2 \frac{p_i}{q} - \alpha_2 \frac{n_i}{q}(\boldsymbol{n}\cdot\boldsymbol{p})) + \frac{1}{c^4}[(3\alpha_1^2 - \beta_1)p_i\boldsymbol{p}^4 \\
&+ (4\alpha_1\alpha_2 - \beta_2)\frac{\boldsymbol{p}^2 p_i}{q} + (-2\alpha_1\alpha_2 + \beta_2)\frac{\boldsymbol{p}^2(\boldsymbol{n}\cdot\boldsymbol{p})n_i}{q} + (-2\alpha_1\alpha_2 - 3\beta_3)\frac{(\boldsymbol{n}\cdot\boldsymbol{p})^2 p_i}{q} \\
&+ 3\beta_3\frac{(\boldsymbol{n}\cdot\boldsymbol{p})^3 n_i}{q} + (\alpha_2^2 - \beta_4)\frac{p_i}{q^2} + (-\alpha_2^2 + 2\beta_4)\frac{(\boldsymbol{n}\cdot\boldsymbol{p})n_i}{q^2}] + O(c^{-6})
\end{aligned} \tag{3.127}$$

$$\begin{aligned}
\tilde{q}^i &= q^i + \frac{1}{c^2}(\alpha_1 q^i \boldsymbol{p}^2 + 2\alpha_1 p^i(\boldsymbol{p}\cdot\boldsymbol{q}) + \alpha_2 n^i) - \frac{1}{c^4}[(2\alpha_1^2 - \beta_1)\boldsymbol{p}^4 q^i \\
&+ (4\alpha_1^2 - 4\beta_1)(\boldsymbol{p}\cdot\boldsymbol{q})\boldsymbol{p}^2 p^i + (2\alpha_1\alpha_2 - \beta_2)\boldsymbol{p}^2 n^i + (2\alpha_1\alpha_2 - 2\beta_2)(\boldsymbol{n}\cdot\boldsymbol{p})p^i \\
&+ (-4\alpha_1\alpha_2 - 3\beta_3)(\boldsymbol{n}\cdot\boldsymbol{p})^2 n^i - \beta_4\frac{n^i}{q}] + O(c^{-6})
\end{aligned} \tag{3.128}$$

where α_i and β_i are expressed in Eq. (3.120) and Eq. (3.126) respectively.

Chapter 4

2PN Approximation of an Isolated Gravitating Source

4.1 Symmetric Trace-Free Tensors

In this section, we introduce the notation used by Blanchet and Damour [2]. Latin letters denote the spatial indices, while Greek letters denote the spacetime indices. The Minkowski metric is denoted by $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ and the Cartesian metric is denoted by $\delta_{ij} = diag(1, 1, 1)$. The Levi-Civita tensor is ϵ_{ijk}. A tensor which includes a Cartesian multi-index is denoted by an upper-case Latin index, i.e. $U_L = U_{i_1 i_2 \ldots i_l}$, $U_{iL} = U_{i i_1 i_2 \ldots i_l}$. We also introduce $x^L = \prod_{n=1}^{l} x^{i_n}$ and $\partial_L = \partial_{i_1}\partial_{i_2}\ldots\partial_{i_l}$. For simplicity, we introduce $U_i V^i = U_i V_i = \sum_{i=1}^{3} U_i V_i$ and $U_{ii} = U_i^j \delta_{ij} = U_{11} + U_{22} + U_{33}$. For a positive integer k, we denote $k! = \prod_{n=1}^{k} n$ and $k!! = k(k-2)\ldots 2 \; or \; 1$. We shall also denote $0! = 0!! = 1$. The symmetric part of a Cartesian tensor is denoted by $U_{(L)} := \frac{1}{l!}\sum_a U_{i_{a_1} i_{a_2} \ldots i_{a_l}}$. The symmetric trace-free (STF) part of a Cartesian tensor U_L is denoted by $\hat{U}_L \equiv U_{\langle L\rangle} \equiv \mathrm{STF}_L(U_L)$, which can be explicitly expressed by [20],

$$U_{\langle L\rangle} = \sum_{n=0}^{[\frac{1}{2}l]} a_n^l \delta_{(i_1 i_2} \ldots \delta_{i_{2n-1} i_{2n}} S_{i_{2n+1} \ldots i_l) p_1 p_1 \ldots p_n p_n} \tag{4.1}$$

where $[\frac{1}{2}l]$ denotes the integer part of $\frac{1}{2}l$, i.e. $[\frac{5}{2}] = 2$ and

$$S_L = U_{(L)} \tag{4.2}$$

$$a_n^l = \frac{(-1)^n l!(2l-2n-1)!!}{(l-2n)!(2l-1)!!(2n)!!} \tag{4.3}$$

We also need to introduce the notation to exclude certain indices from the action of taking STF part of a tensor, i.e. $U_{\langle i_1 \ldots i_m | pq | i_{m+1} \ldots i_l\rangle} = \mathrm{STF}_L\{U_{i_1 \ldots i_m pq i_{m+1} \ldots i_l}\}$.

Equivalent to the spherical harmonics, each STF tensor represents an irreducible representation of the rotation Lie group SO(3). From Eq. (4.1), we can decompose the product of a rank 1 tensor and a STF tensor of rank l [2],

$$U_i \hat{V}_L = \hat{A}^{(+1)}_{iL} + \frac{l}{l+1}\epsilon_{ai\langle i_l}\hat{A}^{(0)}_{L-1\rangle a} + \frac{2l-1}{2l+1}\delta_{i\langle i_l}\hat{A}^{(-1)}_{L-1\rangle} \tag{4.4}$$

where

$$\hat{A}^{(+1)}_{iL} = U_{\langle i_l}\hat{V}_{iL-1\rangle} \tag{4.5a}$$

$$\hat{A}^{(0)}_{L-1a} = U_p\hat{V}_{q\langle L-1}\epsilon_{a\rangle pq} \tag{4.5b}$$

$$\hat{A}^{(-1)}_{L-1} = U_p\hat{V}_{pL-1} \tag{4.5c}$$

Detail proof of Eq. (4.4) is given in Appendix A. Similarly, we can also decompose the product of a rank 2 symmetric tensor and a STF tensor of rank l [12],

$$\begin{aligned} U_{ij}\hat{V}_L = B^{(+2)}_{ijL} &+ \mathrm{STF}_{ij}\mathrm{STF}_L(\epsilon_{aii_l}B^{(+1)}_{L-1ja} \\ &+ \delta_{ii_l}B^{(0)}_{L-1j} + \epsilon_{aii_{l-1}}\delta_{ji_l}B^{(-1)}_{L-2a} + \delta_{ii_{l-1}}\delta_{ji_l}B^{(-2)}_{L-2}) + \delta_{ij}C_L \end{aligned} \tag{4.6}$$

where

$$B^{(+2)}_{ijL} = \hat{U}_{\langle ij}\hat{V}_{L\rangle} \tag{4.7a}$$

$$B^{(+1)}_{L-1ja} = \frac{2l}{l+2}\hat{U}_{p\langle j}\hat{V}_{|q|L-1}\epsilon_{a\rangle pq} \tag{4.7b}$$

$$B^{(0)}_{L-1j} = \frac{6l(2l-1)}{(l+1)(2l+3)}\hat{U}_{p\langle j}\hat{V}_{L-1\rangle p} \tag{4.7c}$$

$$B^{(-1)}_{L-2a} = \frac{2(l-1)(2l-1)}{(l+1)(2l+1)}\hat{U}_{rp}\hat{V}_{qr\langle L-2}\epsilon_{a\rangle pq} \tag{4.7d}$$

$$B^{(-2)}_{L-2} = \frac{2l-3}{2l+1}\hat{U}_{pq}\hat{V}_{pqL-1} \tag{4.7e}$$

$$C^{(0)}_L = \frac{1}{3}U_{pp}\hat{V}_L \tag{4.7f}$$

Eq. (4.4) and Eq. (4.6) can be interpreted by a well known result that, given any two irreducible representations of SO(3) of weights p and q respectively $R^{(p)}$ and $R^{(q)}$, their tensor product can be decomposed into irreducible representations of SO(3) such that

$$R^{(p)} \otimes R^{(q)} = \bigoplus_{i=|p-q|}^{p+q} R^{(i)} \tag{4.8}$$

4.2 Multipole Expansion and Linearized Gravity

Let us consider a compacted supported source $J(\boldsymbol{x}, t)$ which is defined inside the region $|x| < r_0$, such that $J(\boldsymbol{x}, t) = 0$ for $|\boldsymbol{x}| > r_0$. We can therefore define the interior zone [4], $D_i = (\boldsymbol{x}, t)/|\boldsymbol{x}| < r_i$, where r_i satisfies $r_0 < r_i \ll \infty$. Similarly, we define the exterior zone $D_e = (\boldsymbol{x}, t)/|\boldsymbol{x}| > r_e$, where $r_0 < r_e < r_i$. We assume the weak gravitation in the exterior zone such that $T_{\mu\nu}$ is approximately zero. The retarded potential which satisfies the equation $\Box V = -4\pi J$ can be written as [3]

$$V(\boldsymbol{x}, t) = \int \frac{d^3\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|} J(\boldsymbol{y}, t - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c}) \tag{4.9}$$

In the region $|x| > r_0$, by denoting $r \equiv |\boldsymbol{x}|$, this potential admits a multipole expansion

$$V^{(M)}(\boldsymbol{x}, t) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L [\frac{1}{r} V_L(t - \frac{r}{c})] \tag{4.10}$$

where the subscript (M) denotes the multipole expansion and

$$V_L(t) = \int d^3\boldsymbol{y} \hat{y}_L \int_{-1}^{1} da \delta_l(a) J(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.11}$$

$$\delta_l(a) = \frac{(2l+1)!!}{2^{l+1} l!} (1-a^2)^l \tag{4.12}$$

Now, let us consider the deviation of the metric $h^{\mu\nu}$ which satisfies Eq. (2.38). From Section 2.3, we obtain

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4} g T_{\mu\nu} + \Lambda_{\mu\nu}(h) \tag{4.13}$$

We can expand $\Lambda_{\mu\nu}$ [4]

$$\Lambda_{\mu\nu} = N_{\mu\nu}(h^2) + M_{\mu\nu}(h^3) + O(h^4) \tag{4.14}$$

where

$$\begin{aligned} N_{\mu\nu}(h^2) = &-h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} - \frac{1}{4}\partial_\mu h \partial_\nu h - 2\partial_{(\mu} h_{|\alpha\beta|}\partial^\alpha h^\beta_{\nu)} \\ &+ \partial_\beta h^\alpha_\mu(\partial^\beta h_{\nu\alpha} + \partial_\alpha h^\beta_\nu) + \eta_{\mu\nu}(-\frac{1}{4}\partial_\rho h_{\alpha\beta}\partial^\rho h^{\alpha\beta} + \frac{1}{8}\partial_\alpha h \partial^\alpha h + \frac{1}{2}\partial_\alpha h_{\beta\rho}\partial^\beta h^{\alpha\rho}) \end{aligned} \tag{4.15}$$

$$\begin{aligned} M_{\mu\nu}(h^3) = &-h^{\alpha\beta}(\partial_\mu h_{\alpha\rho}\partial_\nu h^\rho_\beta + \partial_\rho h_{\alpha\mu}\partial^{rho} h_{\beta\nu} - \partial_\alpha h_{\rho\mu}\partial_\beta h^\rho_\nu) \\ &+ h_{\mu\nu}(-\frac{1}{4}\partial_\rho h_{\alpha\beta}\partial^\rho h^{\alpha\beta} + \frac{1}{8}\partial_\alpha h \partial^\alpha h + \frac{1}{2}\partial_\alpha h_{\beta\rho}\partial^\beta h^{\alpha\rho}) \\ &+ \frac{1}{2}h^{\alpha\beta}\partial_{(\mu} h_{|\alpha\beta|}\partial_{\nu)} h + 2h^{\alpha\beta}\partial_\rho h_{\alpha(\mu}\partial_{\nu)} h^\rho_\beta + h^\alpha_{(\mu}\partial_{\nu)} h_{\beta\rho}\partial_\alpha h^{\beta\rho} \\ &- 2h^\alpha_{(\mu}\partial_{|\beta|} h_{\nu)\rho}\partial_\alpha h^{\beta\rho} - \frac{1}{2}h^\alpha_{(\mu}\partial_{\nu)} h \partial_\alpha h \\ &+ \eta_{\mu\nu}(\frac{1}{8}h^{\alpha\beta}\partial_\alpha h \partial_\beta h - \frac{1}{4}h^{\alpha\beta}\partial_\rho h_{\alpha\beta}\partial^\rho h - \frac{1}{4}h^{\rho\sigma}\partial_\rho h_{\alpha\beta}\partial_\sigma h^{\alpha\beta} \\ &- \frac{1}{2}h^{\rho\sigma}\partial_\alpha h_{\rho\beta}\partial^\beta h^\alpha_\sigma + \frac{1}{2}h^{\rho\sigma}\partial_\alpha h^\beta_\rho \partial^\alpha h_{\sigma\beta}) \end{aligned} \tag{4.16}$$

We now consider Einstein's equation in the exterior zone. We can further expand the field in the external zone $h_{\mu\nu}$, such that

$$h^{\mu\nu} = \sum_{k=1}^{\infty} G^k h^{\mu\nu}_{(k)} \tag{4.17}$$

This expansion is called Multipolar-post-Minkowskian (MPM) expansion [2]. The first coefficient $h^{\mu\nu}_{(1)}$ is called 'linearized field'. Inserting Eq. (4.17) into Eq. (4.13) gives

$$\Box h^{\mu\nu}_{(1)} = \frac{16\pi G}{c^4} T^{\mu\nu} \tag{4.18}$$

Since in the exterior zone, $T^{\mu\nu}$ is assumed to be negligible. We therefore obtain

$$\Box h^{\mu\nu}_{(1)} = 0 \tag{4.19}$$

Inserting Eq. (4.17) into Eq. (2.50) yields the harmonic condition for $h^{\mu\nu}_{(1)}$

$$\partial_\mu h^{\mu\nu}_{(1)} = 0 \tag{4.20}$$

Now, we need to impose three other constraints on the deviation of the metric for completeness. First, we assume that each term of the MPM expansion admits a finite multipolar expansion

$$h^{\mu\nu}_{(i)} = \sum_{l=0}^{n} h^{\mu\nu}_{(i)L} \hat{n}^L \tag{4.21}$$

Expansion in Eq. (4.20) is explained in [23]. Second, we assume that the deviation of the metric is constant in some past such that

$$\partial_t h^{\mu\nu}_{(1)} = 0 \tag{4.22}$$

for all $t < -T$, where $-T$ is some time in the past. Finally, we assume that the metric $g^{\mu\nu}$ is asymptotically Minkowskian at infinity for all time $t < -T$, i.e.

$$\lim_{r\to\infty} h^{\mu\nu}_{(1)} = 0 \tag{4.23}$$

In the following context, we shall solve Eq. (4.19) and Eq. (4.20) under the three constraints Eq. (4.21), Eq. (4.22), and Eq. (4.23), which follows the process described in [2]. We prove some useful formulas in Appendix B. Recall that $\Box f(r) = \frac{1}{r^2}\partial_r(r^2\partial_r f(r)) - \frac{1}{c^2}\partial_t^2$. Inserting Eq. (4.21) into Eq. (4.19) and using Eq. (B.2), we obtain

$$(-\frac{1}{c^2}\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2}) h^{\mu\nu}_{(1)L} = 0 \tag{4.24}$$

Now we change the variables, i.e.

$$u = t + \frac{r}{c} \tag{4.25a}$$

$$v = t - \frac{r}{c} \tag{4.25b}$$

such that

$$\partial_t^2 = \partial_u^2 + 2\partial_u\partial_v + \partial_v^2 \tag{4.26a}$$

$$\partial_r^2 = \frac{1}{c^2}(\partial_u^2 - 2\partial_u\partial_v + \partial_v^2) \tag{4.26b}$$

Inserting Eq. (4.26) into Eq. (4.24), we obtain

$$(-\partial_u\partial_v + \frac{\partial_u - \partial_v}{u - v} - \frac{l(l+1)}{(u-v)^2})h^{\mu\nu}_{(1)L} = 0 \tag{4.27}$$

Now, we define a new function

$$j^{\mu\nu}_{(1)L} := \frac{1}{(u-v)^l}h^{\mu\nu}_{(1)L} \tag{4.28}$$

One can easily see that

$$(\partial_u - \partial_v)h^{\mu\nu}_{(1)L} = -2l(v-u)^{l-1}j^{\mu\nu}_{(1)L} + (v-u)^l(\partial_u - \partial_v)j^{\mu\nu}_{(1)L} \tag{4.29a}$$

$$\partial_u\partial_v h^{\mu\nu}_{(1)L} = (v-u)^l\partial_u\partial_v j^{\mu\nu}_{(1)L} + l(v-u)^{l-1}(\partial_u - \partial_v)j^{\mu\nu}_{(1)L} - l(l-1)(v-u)^{l-2}j^{\mu\nu}_{(1)L} \tag{4.29b}$$

Inserting Eq. (4.29) into Eq. (4.27) gives

$$[(u-v)\partial_u\partial_v + (l+1)(\partial_v - \partial_u)]j^{\mu\nu}_{(1)L} = 0 \tag{4.30}$$

Eq. (4.30) is a particular case of Euler-Poisson-Darboux equation. Let's define the operator $E_{m,n} := (u-v)\partial_u\partial_v + m\partial_v - n\partial_u$. Eq. (4.30) can be rewritten as

$$E_{l+1,l+1}(j^{\mu\nu}_{(1)L}) = 0 \tag{4.31}$$

We now differentiate Eq. (4.29) by u and v, respectively, and obtain

$$\partial_u E_{l,l}(j^{\mu\nu}_{(1)L}) = E_{l+1,l}(\partial_u j^{\mu\nu}_{(1)L}) \tag{4.32a}$$

$$\partial_v E_{l,l}(j^{\mu\nu}_{(1)L}) = E_{l,l+1}(\partial_v j^{\mu\nu}_{(1)L}) \tag{4.32b}$$

From Eq. (4.32), we notice that, if $j^{\mu\nu}_{(1)L}$ is a solution for $E_{l,l}(j^{\mu\nu}_{(1)L}) = 0$, then $\frac{\partial^{2n}}{\partial_u^n\partial_v^n}j^{\mu\nu}_{(1)L}$ is a solution for $E_{l+n,l+n}(j^{\mu\nu}_{(1)L}) = 0$. Therefore, to find a solution for Eq. (4.31), it is sufficient to find a solution for

$$E_{1,1}(j^{\mu\nu}_{(1)L(1,1)}) = 0 \tag{4.33}$$

A general solution for Eq. (4.33) can be described as

$$j^{\mu\nu}_{(1)L(1,1)} = a\frac{P^{\mu\nu}_L(u) + Q^{\mu\nu}_L(v)}{u - v} \tag{4.34}$$

where a is an arbitrary constant and $P_L^{\mu\nu}(u)$ and $Q_L^{\mu\nu}(u)$ are two arbitrary functions. The solution for Eq. (4.31) is

$$j^{\mu\nu}_{(1)L(l+1,l+1)} = a\frac{\partial^{2l}}{\partial^l u \partial^l v}\frac{P_L^{\mu\nu}(u) + Q_L^{\mu\nu}(v)}{u-v} \tag{4.35}$$

Using Eq. (4.28), we, therefore, find a general solution for Eq. (4.27)

$$h^{\mu\nu}_{(1)L} = a(u-v)^l \frac{\partial^{2l}}{\partial^l u \partial^l v}\frac{P_L^{\mu\nu}(u) + Q_L^{\mu\nu}(v)}{u-v} \tag{4.36}$$

Recall that Leibniz formula, i.e. $\frac{\partial^n}{\partial^n x}(f(x)g(x)) = \sum\limits_{i=0}^{n} \frac{n!}{i!(n-i)!}\frac{\partial^i}{\partial^i x}f(x)\frac{\partial^{n-i}}{\partial^{n-i}x}g(x)$. We obtain

$$\frac{\partial^{2l}}{\partial^l u \partial^l v}(P_L^{\mu\nu}(u)(u-v)^{-1}) = \sum_{i=0}^{l}\frac{l!(2l-i)!}{i!(l-i)!}(-1)^{l-i}(u-v)^{-(2l-i+1)}\frac{\partial^i}{\partial^i u}P_L^{\mu\nu}(u) \tag{4.37a}$$

$$\frac{\partial^{2l}}{\partial^l u \partial^l v}(Q_L^{\mu\nu}(v)(u-v)^{-1}) = \sum_{i=0}^{l}\frac{l!(2l-i)!}{i!(l-i)!}(-1)^{l}(u-v)^{-(2l-i+1)}\frac{\partial^i}{\partial^i v}Q_L^{\mu\nu}(v) \tag{4.37b}$$

Inserting Eq. (4.37) into Eq. (4.36) gives

$$h^{\mu\nu}_{(1)L} = a(-1)^l \sum_{i=0}^{l}\frac{l!(2l-i)!}{i!(l-i)!}\frac{(-1)^i\frac{\partial^i}{\partial^i u}P_L^{\mu\nu}(u) + \frac{\partial^i}{\partial^i v}Q_L^{\mu\nu}(v)}{r^{l-i+1}}(\frac{c}{2})^{l-i+1} \tag{4.38}$$

Since $P_L^{\mu\nu}$ and $Q_L^{\mu\nu}$ are arbitrary function, we can fix a value for a without changing the physics. After careful comparison between Eq. (4.38) and Eq. (B.4), one can check that, if we choose $a = \frac{2}{l! c^{l+1}}$, Eq. (4.38) admits the same form as Eq. (B.4). We can therefore rewrite Eq. (4.37)

$$\hat{n}^L h^{\mu\nu}_{(1)L} = \hat{\partial}_L[\frac{P_L^{\mu\nu}(t+\frac{r}{c}) + Q_L^{\mu\nu}(t-\frac{r}{c})}{r}] \tag{4.39}$$

We now prove that we are free to change $P_L^{\mu\nu}(t+\frac{r}{c})$ to $P_L^{\mu\nu}(t-\frac{r}{c})$ in Eq. (4.38). Applying the constraint Eq. (4.22) to Eq. (4.39), we obtain

$$\hat{\partial}_L[\frac{\partial_t P_L^{\mu\nu}(t+\frac{r}{c}) + \partial_t Q_L^{\mu\nu}(t-\frac{r}{c})}{r}] = 0 \tag{4.40}$$

for any time t before $-T$. From Eq. (B.1), we notice that $\hat{\partial}_L$ is proportional to $\frac{\partial^l}{\partial(r^2)^l}$. Therefore, the solution for $\hat{\partial}_L F(r) = 0$ must have the form

$$F(r) = \sum_{i=0}^{l-1} a_i r^{2i+1} \tag{4.41}$$

where a_i is some arbitrary constants. We can then write

$$\partial_t P_L^{\mu\nu}(t+\frac{r}{c}) + \partial_t Q_L^{\mu\nu}(t-\frac{r}{c}) = \sum_{i=1}^{2l} a_i[(t+\frac{r}{c})^i - (t-\frac{r}{c})^i] \tag{4.42}$$

We can integrate Eq. (4.42) with respect to $t+\frac{r}{c}$ and $t-\frac{r}{c}$ and obtain

$$P_L^{\mu\nu}(t+\frac{r}{c}) = \sum_{i=1}^{2l+1} a_i(t+\frac{r}{c})^i + c_p \tag{4.43a}$$

$$Q_L^{\mu\nu}(t-\frac{r}{c}) = -\sum_{i=1}^{2l+1} a_i(t-\frac{r}{c})^i + c_q \tag{4.43b}$$

where we have redefined a_i, such that $a_i := \frac{a_{i-1}}{i}$. One can easily check that

$$\hat{\partial}_L r^k = 0 \tag{4.44}$$

for positive even integer k less than $2l$. Using Eq. (4.44), we can insert Eq. (4.43) into Eq. (4.39) and obtain

$$\hat{n}^L h_{(1)L}^{\mu\nu} = \hat{\partial}_L[-\frac{2}{c^{2l+1}} a_{2l+1} r^{2l} + \frac{c_p + c_q}{r}] \tag{4.45}$$

We then apply constraint Eq. (4.23) to Eq. (4.45) and easily find

$$a_{2l+1} = 0 \tag{4.46}$$

From Eq. (4.44), one can easily check

$$\hat{\partial}_L(\frac{(t+\frac{r}{c})^i}{r}) = \hat{\partial}_L(\frac{(t-\frac{r}{c})^i}{r}) \tag{4.47}$$

for positive integer i less and equal to $2l$. Therefore, since Eq. (4.46), we have

$$\hat{\partial}_L P_L^{\mu\nu}(t+\frac{r}{c}) = \hat{\partial}_L P_L^{\mu\nu}(t-\frac{r}{c}) \tag{4.48}$$

We can then naturally define $U_L^{\mu\nu}(t-\frac{r}{c}) := P_L^{\mu\nu}(t-\frac{r}{c}) + Q_L^{\mu\nu}(t-\frac{r}{c})$ and rewrite Eq. (4.21) by applying Eq. (4.39)

$$h_{(1)}^{\mu\nu} = \sum_l \hat{\partial}_L(\frac{U_L^{\mu\nu}(t-\frac{r}{c})}{r}) \tag{4.49}$$

Since $\hat{\partial}_L$ is STF in L, we can replace $U_L^{\mu\nu}$ in Eq. (4.49) by $U_{\langle L\rangle}^{\mu\nu}$ and replace $\hat{\partial}_L$ by ∂_L. By using Eq. (4.4), we can decompose $U_{\langle L\rangle}^{0i}$, such that

$$U_{i\langle L\rangle}^{0} = \hat{A}_{iL}^{(+1)} + \frac{l}{l+1}\epsilon_{ai\langle i_l}\hat{A}_{L-1\rangle a}^{(0)} + \frac{2l-1}{2l+1}\delta_{i\langle i_l}\hat{A}_{L-1\rangle}^{(-1)} \tag{4.50}$$

where

$$\hat{A}^{(+1)}_{iL} = U^0_{\langle iL\rangle} \tag{4.51a}$$

$$\hat{A}^{(0)}_{L-1a} = U^0_{pq\langle L-1}\epsilon_{a\rangle pq} \tag{4.51b}$$

$$\hat{A}^{(-1)}_{L-1} = U^0_{ppL-1} \tag{4.51c}$$

Inserting Eq. (4.50) into Eq. (4.49), we have

$$\begin{aligned} h^{0i}_{(1)} &= \sum_{l\geq 1}\partial_{L-1}(\frac{1}{r}\hat{A}^{(+1)}_{iL-1}) + \sum_{l\geq 1}\partial_L(\frac{l}{l+1}\epsilon_{ai(i_l}\hat{A}^{(0)}_{L-1)a} + \frac{1}{l+1}\epsilon_{aib}\hat{A}^{(0)}_{ab(L-2}\delta_{i_{l-1}i_l)}) \\ &+ \sum_{l\geq 1}\partial_L(\frac{2l-1}{2l+1}\delta_{i(i_l}\hat{A}^{(-1)}_{L-1)}) + \sum_{l\geq 2}\partial_L(\frac{2l-1}{2l+1}\frac{2a^l_1}{l}\hat{A}^{(-1)}_{i(L-2}\delta_{i_{l-1}i_l)}) \end{aligned} \tag{4.52}$$

From Eq. (4.52), one should notice that $\epsilon_{aib}\hat{A}^{(0)}_{abL-2}$ is zero, since $\hat{A}^{(0)}_L$ is totally symmetric with respect to all its indices. Also, since ∂_L is totally symmetric with respect to its indices, we can ignore the symmetrizing operations in the terms, i.e. $\partial_L U_{(L)} = \partial_L U_L$. Using Eq. (4.3) to compute a^l_1, we obtain

$$\begin{aligned} h^{0i}_{(1)} &= \sum_{l\geq 0}\partial_{iL}(\frac{2l+1}{2l+3}\frac{1}{r}\hat{A}^{(-1)}_L) + \sum_{l\geq 1}\partial_{L-1}(\frac{1}{r}\hat{A}^{(+1)}_{iL-1} - \frac{l}{2l+3}\triangle(\frac{1}{r}\hat{A}^{(-1)}_{iL-1})) \\ &+ \sum_{l\geq 1}\epsilon_{iba}\partial_{bL-1}(\frac{l}{l+1}\frac{1}{r}\hat{A}^{(0)}_{L-1a}) \end{aligned} \tag{4.53}$$

One can then check $\triangle(\frac{1}{r}\hat{A}^{(-1)}_{iL-1}) = \frac{1}{c^2}\frac{1}{r}\ddot{\hat{A}}^{(-1)}_{iL-1}$, where $\ddot{f}(x) := \partial^2_x f(x)$. We can thus further simplify Eq. (4.53)

$$h^{0i}_{(1)} = \sum_{l\geq 0}\partial_{iL}(r^{-1}B_L) + \sum_{l\geq 1}\partial_{L-1}(r^{-1}C_{iL-1}) + \sum_{l\geq 1}\epsilon_{iba}\partial_{bL-1}(r^{-1}D_{L-1a}) \tag{4.54}$$

where

$$B_L = \frac{2l+1}{2l+3}\hat{A}^{(-1)}_L \tag{4.55a}$$

$$C_{iL-1} = \hat{A}^{(+1)}_{iL-1} - \frac{l}{(2l+3)c^2}\ddot{\hat{A}}^{(-1)}_{iL-1} \tag{4.55b}$$

$$D_{L-1a} = \frac{l}{l+1}\hat{A}^{(0)}_{L-1a} \tag{4.55c}$$

Similarly, one can insert Eq. (4.6) into Eq. (4.49) and obtain

$$\begin{aligned} h^{ij}_{(1)} &= \sum_{l\geq 0}[\partial_{ijL}(r^{-1}E_L) + \delta_{ij}\partial_L(r^{-1}F_L)] \\ &+ \sum_{l\geq 1}[\partial_{L-1(i}(r^{-1}G_{j)L-1}) + \epsilon_{ab(i}\partial_{j)aL-1}(r^{-1}H_{bL-1})] \\ &+ \sum_{l\geq 2}[\partial_{L-2}(r^{-1}I_{ijL-2}) + \partial_{aL-2}(r^{-1}\epsilon_{ab(i}J_{j)bL-2})] \end{aligned} \tag{4.56}$$

where E, F, G, H, I and J are functions of $B^{(+2)}, B^{(+1)}, B^{(0)}, B^{(-1)}, B^{(-2)}$ and $C_L^{(0)}$. For completeness, we define

$$A_L := U_L^{00} \tag{4.57}$$

such that

$$h_{(1)}^{00} = \sum_{l\geq 0} \partial_L (r^{-1} A_L) \tag{4.58}$$

Now, we apply the harmonic condition Eq. (4.20) to Eq. (4.54), Eq. (4.56), and Eq. (4.58), which will impose constraints on functions A to J. For example, the time component of Eq. (4.20) reads

$$\frac{1}{c}\partial_0 h_{(1)}^{00} + \partial_i h_{(1)}^{0i} = 0 \tag{4.59}$$

Inserting Eq. (4.54) and Eq. (4.58) into Eq. (4.59), we obtain

$$\sum_{l\geq 0} \partial_L (r^{-1}\dot{A}_L) + \sum_{l\geq 0} \partial_L \triangle (r^{-1} B_L) + \sum_{l\geq 1} \partial_{iL-1}(r^{-1} C_{iL-1}) = 0 \tag{4.60}$$

where we have used $\partial_i \partial_a \epsilon_{iba} = 0$. Using the identity of the Laplace operator, one can easily check

$$C_L + \frac{1}{c}\dot{A}_L + \frac{1}{c^2}\ddot{B}_L = 0 \tag{4.61}$$

We now define $A_L^{(n)} := \frac{1}{c^n}\frac{\partial^n}{\partial t^n} A_L$. One should be careful that indices (n) which represent higher derivatives is distinct from indices $(\pm n)$ in Eq. (4.50) which label different irreducible representations of the rotation Lie group. Therefore, we can rewrite Eq. (4.61)

$$C_L + A_L^{(1)} + B_L^{(2)} = 0 \tag{4.62}$$

Similarly, using the spatial component of Eq. (4.20),

$$\frac{1}{c}\partial_0 h_{(1)}^{j0} + \partial_i h_{(1)}^{ji} = 0 \tag{4.63}$$

We obtain the following constraints

$$G_L + 2B_L^{(1)} + 2E_L^{(2)} + 2F_L = 0 \tag{4.64a}$$

$$I_L - A_L^{(2)} - 2B_L^{(3)} - E_L^{(4)} - F_L^{(2)} = 0 \tag{4.64b}$$

$$J_L + 2D_L^{(1)} + H_L^{(2)} = 0 \tag{4.64c}$$

For future convenience, we now redefine the functions, such that

$$M_L := A_L + 2B_L^{(1)} + E_L^{(2)} + F_L := \frac{(-1)^{l+1}4}{c^2 l!}\tilde{M}_L \tag{4.65a}$$

$$S_L := -D_L - \frac{1}{2}H_L^{(1)} := \frac{(-1)^{l+1}4l}{c^3(l+1)!}\tilde{S}_L \tag{4.65b}$$

and

$$\Gamma_L := B_L + \frac{1}{2}E_L^{(1)} \tag{4.66a}$$

$$\Theta_L := \frac{1}{2}E_L \tag{4.66b}$$

$$\Phi_L := -B_L^{(1)} - E_L^{(2)} - F_L \tag{4.66c}$$

$$\Psi_L := \frac{1}{2}H_L \tag{4.66d}$$

In terms of the new functions, from Eq. (4.58), Eq. (4.54), and Eq. (4.56), we obtain

$$h^{00}_{(1)} = \frac{-4}{c^2}\sum_{l\geq 0}\frac{(-1)^l}{l!}\partial_L(r^{-1}\tilde{M}_L) + \sum_{l\geq 0}\partial_L(r^{-1}(-\Gamma_L^{(1)} + \Theta_L^{(2)} + \Phi_L)) \tag{4.67}$$

$$\begin{aligned} h^{0i}_{(1)} = {} & \frac{4}{c^3}\sum_{l\geq 1}\frac{(-1)^l}{l!}\partial_{L-1}(r^{-1}\tilde{M}^{(1)}_{iL-1}) + \frac{4}{c^3}\sum_{l\geq 1}\frac{(-1)^l l}{(l+1)!}\epsilon_{iab}\partial_{aL-1}(r^{-1}\tilde{S}_{bL-1}) \\ & + \sum_{l\geq 0}\partial_{iL}(r^{-1}(\Gamma_L - \Theta_L^{(1)})) - \sum_{l\geq 1}[\partial_{L-1}(r^{-1}\Phi^{(1)}_{iL-1}) + \epsilon_{iab}\partial_{aL-1}(r^{-1}\Psi^{(1)}_{bL-1})] \end{aligned} \tag{4.68}$$

$$\begin{aligned} h^{ij}_{(1)} = {} & \frac{-4}{c^4}\sum_{l\geq 2}\frac{(-1)^l}{l!}\partial_{L-2}(r^{-1}\tilde{M}^{(2)}_{ijL-2}) - \frac{8}{c^4}\sum_{l\geq 2}\frac{(-1)^l l}{(l+1)!}\partial_{aL-2}(\epsilon_{ab(i}\tilde{S}^{(1)}_{j)bL-2}) \\ & + \sum_{l\geq 0}[2\partial_{ijL}(r^{-1}\Theta_L) - \delta_{ij}\partial_L(r^{-1}(\Gamma_L^{(1)} + \Theta_L^{(2)} + \Phi_L))] \\ & + 2\sum_{l\geq 1}[\partial_{L-1(i}(r^{-1}\Phi_{j)L-1}) + \epsilon_{ab(i}\partial_{j)aL-1}(r^{-1}\Psi_{bL-1})] \end{aligned} \tag{4.69}$$

where we have used the inverse of Eq. (4.65) and Eq. (4.66) and also the constraints Eq. (4.62) and Eq. (4.64). Now, one can check that Eq. (4.67) to Eq. (4.69) can be rewritten as

$$h^{\mu\nu}_{(1)} = h^{\mu\nu}_{(1)can}(\tilde{M}, \tilde{S}) + K^{\mu\nu}(\Gamma, \Theta, \Phi, \Psi) \tag{4.70}$$

Using $r^{-1}\Theta_L^{(2)} = \triangle(r^{-1}\Theta_L)$, we can rewrite $K^{\mu\nu}(\Gamma, \Theta, \Phi, \Psi)$

$$K^{00} = -\frac{1}{c}\partial_0\sum_{l\geq 0}\partial_L(r^{-1}\Gamma_L) + \partial_i[\sum_{l\geq 0}\partial_{iL}(r^{-1}\Theta_L) + \sum_{l\geq 1}\partial_{L-1}(r^{-1}\Phi_{iL-1})] \tag{4.71}$$

$$\begin{aligned} K^{0i} = {} & -\frac{1}{c}\partial_0[\sum_{l\geq 0}\partial_{iL}(r^{-1}\Theta_L) + \sum_{l\geq 1}(\partial_{L-1}(r^{-1})\Phi_{iL-1} + \epsilon_{iab}\partial_{aL-1}(r^{-1}\Psi_{bL-1}))] \\ & + \partial_i\sum_{l\geq 0}\partial_L(r^{-1}\Gamma_L) \end{aligned} \tag{4.72}$$

$$K^{ij} = -\delta^{ij}\frac{1}{c}\partial_0\sum_{l\geq 0}\partial_L(r^{-1}\Gamma_L) - \delta^{ij}\partial_k(\sum_{l\geq 0}\partial_{kL}(r^{-1}\Theta_L) + \sum_{l\geq 1}\partial_{L-1}\Phi_{L-1k})$$
$$+ \partial_{(i}[\sum_{l\geq 0}\partial_{j)L}(r^{-1}\Theta_L) + \sum_{l\geq 1}(\partial_{L-1}(r^{-1}\Phi_{j)L-1}) + \epsilon_{j)ab}\partial_{aL-1}(r^{-1}\Psi_{bL-1}))] \tag{4.73}$$

We can therefore write $K^{\mu\nu}$ in the form

$$K^{\mu\nu} = \partial^{(\mu}k^{\nu)} - \eta^{\mu\nu}\partial_\rho k^\rho \tag{4.74}$$

where

$$k^0 = \sum_{l\geq 0}\partial_L(r^{-1}\Gamma_L) \tag{4.75}$$

$$k^i = \sum_{l\geq 0}\partial_{iL}(r^{-1}\Theta_L) + \sum_{l\geq 1}(\partial_{L-1}(r^{-1})\Phi_{iL-1} + \epsilon_{iab}\partial_{aL-1}(r^{-1}\Psi_{bL-1})) \tag{4.76}$$

Therefore, Eq. (4.70) is canonical transformation of $h^{\mu\nu}_{(1)}$ used in [23]. Now, we obtain an expression for the canonical linearized field $h^{\mu\nu}_{(1)}$ in the exterior zone D_e.

$$h^{00}_{(1)can} = \frac{-4}{c^2}\sum_{l\geq 0}\frac{(-1)^l}{l!}\partial_L(r^{-1}\tilde{M}_L) \tag{4.77}$$

$$h^{0i}_{(1)can} = \frac{4}{c^3}\sum_{l\geq 1}\frac{(-1)^l}{l!}\partial_{L-1}(r^{-1}\tilde{M}^{(1)}_{iL-1}) + \frac{4}{c^3}\sum_{l\geq 1}\frac{(-1)^l l}{(l+1)!}\epsilon_{iab}\partial_{aL-1}(r^{-1}\tilde{S}_{bL-1}) \tag{4.78}$$

$$h^{ij}_{(1)can} = \frac{-4}{c^4}\sum_{l\geq 2}\frac{(-1)^l}{l!}\partial_{L-2}(r^{-1}\tilde{M}^{(2)}_{ijL-2}) - \frac{8}{c^4}\sum_{l\geq 2}\frac{(-1)^l l}{(l+1)!}\partial_{aL-2}(\epsilon_{ab(i}\tilde{S}^{(1)}_{j)bL-2}) \tag{4.79}$$

where $\tilde{M}_L$ and $\tilde{S}_L$ are sets of totally arbitrary functions. We identify the functions $\tilde{M}_L$ as the mass multipole moments and the functions $\tilde{S}_L$ the current multipole moments [23]. In the following discussion, we will drop the tilde on both functions for convenience.

4.3 Solution of Einstein's Equation in the Interior Zone

In this section, we solve Einstein's equation in the interior zone D_i. We introduce a notation for small order terms in the post-Newtonian expansion [4]. If a totally symmetric tensor $T^{\mu_1\mu_2\cdots\mu_l}$ obeys

$$T^{\mu_1\mu_2\cdots\mu_l} = O(p_0, p_1, \cdots, p_l) \tag{4.80}$$

we mean the following

$$T^{i_1\cdots i_m 0\cdots 0} = O(c^{-m}) \tag{4.81}$$

For example, $G^{\mu\nu} = O(a,b,c)$ is equivalent to, $G^{00} = O(c^{-a})$, $G^{0i} = O(c^{-b})$, and $G^{ij} = O(c^{-c})$. Now we define the following quantities from the stress-energy tensor $T^{\mu\nu}$

$$\sigma = \frac{T^{00} + T^{ii}}{c^2} \tag{4.82a}$$

$$\sigma_i = \frac{T^{0i}}{c} \tag{4.82b}$$

$$\sigma_{ij} = T^{ij} \tag{4.82c}$$

We also define the related retarded potential, such that

$$\Box V = -4\pi G\sigma \tag{4.83a}$$

$$\Box V_i = -4\pi G\sigma_i \tag{4.83b}$$

$$\Box W_i j = -4\pi G\sigma_{ij} + (\frac{1}{2}\delta_{ij}\partial_k V\partial_k V - \partial_i V\partial_j V) \tag{4.83c}$$

Using Eq. (4.9), we obtain

$$V(\boldsymbol{x},t) = G\int d^3\boldsymbol{y}\frac{\sigma(\boldsymbol{y},t-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c})}{|\boldsymbol{x}-\boldsymbol{y}|} \tag{4.84a}$$

$$V_i(\boldsymbol{x},t) = G\int d^3\boldsymbol{y}\frac{\sigma_i(\boldsymbol{y},t-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c})}{|\boldsymbol{x}-\boldsymbol{y}|} \tag{4.84b}$$

$$W_{ij}(\boldsymbol{x},t) = G\int d^3\boldsymbol{y}\frac{(\sigma_{ij} + \frac{1}{4\pi G}(\partial_i V\partial_j V - \frac{1}{2}\delta_{ij}\partial_k V\partial_k V))(\boldsymbol{y},t-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c})}{|\boldsymbol{x}-\boldsymbol{y}|} \tag{4.84c}$$

In the following context, we assume that σ, σ_i, and σ_{ij} are of order c^0. From Eq. (4.13), one can easily check that $h^{\mu\nu}$ is of order $O(2,3,4)$. Recall that the stress-energy tensor obeys the conservation law

$$\nabla_\mu T^{\mu\nu} = 0 \tag{4.85}$$

where ∇ is the covariant derivative operator, such that,

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + T^{\rho\nu}\Gamma^\mu_{\rho\mu} + T^{\mu\rho}\Gamma^\nu_{\rho\mu} \tag{4.86}$$

First, we consider the '0' component of Eq. (4.85), e.g. $\nu = 0$. We, therefore, obtain

$$\partial_\mu T^{\mu 0} + T^{\rho 0}\Gamma^\mu_{\rho\mu} + T^{\mu\rho}\Gamma^0_{\rho\mu} = 0 \tag{4.87}$$

Rewriting Eq. (4.87), we have

$$\partial_t\sigma + \partial_i\sigma_i + \frac{1}{c}(T^{00}\Gamma^\mu_{0\mu} + T^{i0}\Gamma^\mu_{i\mu} + T^{00}\Gamma^0_{00} + T^{0i}\Gamma^0_{0i} + T^{ij}\Gamma^0_{ij}) = 0 \tag{4.88}$$

One should notice that, from Eq. (2.45), $\Gamma^\rho_{\mu\nu}$ is of order $O(2)$. Therefore, terms with T^{i0} and T^{ij} must have order smaller or equal to $O(2)$. We can then compute $\Gamma^{\mu}_{0\mu}$ by using Eq. (2.45)

$$\Gamma^{\mu}_{0\mu} = \frac{1}{2c}\partial_t h \tag{4.89}$$

which is of order $O(3)$. Thus, inserting Eq. (4.89) into Eq. (4.88), we obtain

$$\partial_t \sigma + \partial_i \sigma_i = O(2) \tag{4.90}$$

Second, we consider the 'i' component of Eq. (4.85) and obtain

$$\partial_i \sigma + \partial_j \sigma_{ij} + T^{00}\Gamma^i_{00} + T^{0j}\Gamma^i_{0j} = O(2) \tag{4.91}$$

where we have ignored all the terms whose order is smaller or equal to $O(2)$ by the same reason as before. From Eq. (2.45), we have

$$\Gamma^i_{00} = \frac{1}{4}\partial_i(h_{00} + \sum_j h_{jj}) \tag{4.92a}$$

$$\Gamma^i_{j0} = \frac{1}{2}(\partial_i h_{j0} - \partial_j h_{i0}) \tag{4.92b}$$

From Eq. (4.92b), we immediately see that the term $T^{0j}\Gamma^i_{0j}$ is of order $O(2)$. Inserting Eq. (4.92a) into Eq. (4.13), we obtain, at lowest order,

$$\Box(h_{00} + \sum_j h_{jj}) = \frac{16\pi G}{c^2}\sigma \tag{4.93}$$

Comparing Eq. (4.93) with Eq. (4.83a), we obtain, at lowest order,

$$\Gamma^i_{00} = -\frac{1}{c^2}\partial_i V \tag{4.94}$$

Inserting Eq. (4.94) into Eq. (4.91) gives

$$\partial_i \sigma + \partial_j \sigma_{ij} = \sigma \partial_i V + O(2) \tag{4.95}$$

In order to replace σ, σ_i, and σ_{ij} in Eq. (4.90) and Eq. (4.95) by corresponding retarded potentials, one can check

$$\partial_j W_{ij} = G\int d^3\boldsymbol{y}\frac{(\partial_j \sigma_{ij} + \frac{1}{4\pi G}(\partial_i V \triangle V))(\boldsymbol{y}, t - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c})}{|\boldsymbol{x}-\boldsymbol{y}|} \tag{4.96}$$

By using $\triangle V = \Box V + O(2)$, one can check that Eq. (4.95) can be written as

$$\partial_i V_i + \partial_j W_{ij} = O(2) \tag{4.97}$$

For consistency, we also rewrite Eq, (4.90) in terms of the retarded potentials

$$\partial_t V + \partial_i V_i = O(2) \tag{4.98}$$

Now, let us consider Einstein's equation (Eq. (4.13) to Eq. (4.16)) in the interior zone. From Eq. (4.13) and Eq. (4.93), we have $h^{00} = -\frac{4V}{c^2} + O(2)$, $h^{0i} = O(3)$, and $h^{ij} = O(4)$. Inserting these relations into Eq. (4.13), we can obtain Einstein's equation at order $O(6, 5, 6)$. For example, the '00' component of Eq. (4.13) reads

$$\Box h^{00} = \frac{16\pi G}{c^4}(1+h)T^{00} + N^{00}(h^2) + O(6) \tag{4.99}$$

Using the fact that $h^{\mu\nu} = O(2, 3, 4)$, we can simply Eq. (4.15), such that

$$N^{00}(h^2) = \partial_i h^{00}\partial^i h^0_0 + \frac{1}{4}\partial_i h_{00}\partial^i h^{00} - \frac{1}{8}\partial_i h \partial^i h + O(6) \tag{4.100}$$

One can then explicitly compute $N^{00}(h^2)$ using Eq. (4.100).

$$N^{00}(h^2) = -\frac{14}{c^4}\partial_i V \partial_i V \tag{4.101}$$

Inserting Eq. (4.101) into Eq. (4.99), we yield

$$\Box h^{00} = \frac{16\pi G}{c^4}(1 + \frac{4V}{c^2})T^{00} - \frac{14}{c^4}\partial_i V \partial_i V + O(6) \tag{4.102}$$

Using the same method, we obtain

$$\Box h^{i0} = \frac{16\pi G}{c^4}T^{i0} + O(5) \tag{4.103}$$

$$\Box h^{ij} = \frac{16\pi G}{c^4}T^{ij} + \frac{4}{c^4}(\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V) + O(6) \tag{4.104}$$

From Eq. (4.83b) and Eq. (4.83c), we can easily see

$$h^{i0} = -\frac{4}{c^3}V_i + O(5) \tag{4.105}$$

$$h^{ij} = -\frac{4}{c^4}W_{ij} + O(6) \tag{4.106}$$

However, to solve Eq. (4.102), one can check

$$\Box(W - 2V^2) = -4\pi G T^{ii} - \frac{7}{2}\partial_i V \partial_i V + \frac{16\pi G}{c^2}V T^{00} + O(2) \tag{4.107}$$

where we have used $\Box V^2 = 2\partial_i V \partial_i V + 2V\Box V + O(2)$ and $W \equiv W_{ii}$. We can solve Eq. (4.102)

$$h^{00} = -\frac{4}{c^2}V + \frac{4}{c^4}(W - 2V^2) + O(6) \tag{4.108}$$

Using Eq. (4.105), Eq. (4.106), and Eq. (4.108), we can iterate the above process again to find Einstein's equation at order $O(8, 7, 8)$. Using $h = \eta_{\mu\nu}h^{\mu\nu}$, we obtain

$$h = \frac{4}{c^2}V - \frac{8}{c^4}(W - V^2) \tag{4.109}$$

From Eq. (2.41), we find

$$-g = 1 + \frac{4}{c^2}V - \frac{8}{c^4}(W - V^2) + O(6) \qquad (4.110)$$

One should notice that the term $\frac{1}{2}h^2$ cancels with the term $\frac{1}{2}h_{\mu\rho}h^{\mu\rho}$ at order $O(4)$. Inserting Eq. (4.105), Eq. (4.106), Eq. (4.108), and Eq. (4.109) into Eq. (4.14), we have

$$\begin{aligned}\Lambda^{00} =& -\frac{14}{c^4}\partial_i V\partial_i V + \frac{2}{c^6}[-8V\partial_t^2 V - 16V_i\partial_t\partial_i V + 5(\partial_t V)^2 + 4\partial_i V_j\partial_i V_j \\ &+ 12\partial_i V_j\partial_j V_i + 8\partial_t V_i\partial_i V - 28V\partial_i V\partial_i V - 8W_{ij}\partial_i\partial_j V + 16\partial_i V\partial_i W] + O(8)\end{aligned} \qquad (4.111)$$

$$\Lambda^{0i} = \frac{4}{c^5}(4\partial_j V\partial_i V_j - 4\partial_j V\partial_j V_i + 3\partial_i V\partial_t V) + O(7) \qquad (4.112)$$

$$\begin{aligned}\Lambda^{ij} =& \frac{2}{c^4}(2\partial_i V\partial_j V - \delta_{ij}\partial_k V\partial_k V) \\ &+ \frac{2}{c^6}[-8(\partial_i V_k\partial_j V_k + \partial_k V_i\partial_k V_j) + 16\partial_{(i}V\partial_t V_{j)} + 16\partial_{(i}V_k\partial_k V_{j)} \\ &- \delta_{ij}(3(\partial_t V)^2 + 8\partial_k V\partial_t V_k - 4\partial_k V_l(\partial_k V_l - \partial_l V_k))] + O(8)\end{aligned} \qquad (4.113)$$

Therefore, we obtain Einstein's equation at order $O(8, 7, 8)$

$$\Box h^{\mu\nu} = -\frac{16\pi G}{c^4}\bar{g}T^{\mu\nu} + \bar{\Lambda}^{\mu\nu} + O(8, 7, 8) \qquad (4.114)$$

where $\bar{g}$ and $\bar{\lambda}^{\mu\nu}$ are functions of the retarded potentials V, V_i, and W_{ij} and obey

$$g = \bar{g} + O(6) \qquad (4.115a)$$

$$\Lambda^{\mu\nu} = \bar{\Lambda}^{\mu\nu} + O(8, 7, 8) \qquad (4.115b)$$

Solution of Eq. (4.114) can be written as

$$h^{\mu\nu} = -G\int d^3y \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}(\frac{4}{c^4}\bar{g}T^{\mu\nu} + \frac{1}{4\pi G}\bar{\Lambda}^{\mu\nu})(\boldsymbol{y}, t - \frac{|\boldsymbol{x} - \boldsymbol{y}|}{c}) + O(8, 7, 8) \qquad (4.116)$$

In the following context, for convenience, we rewrite the solution as

$$h^{\mu\nu} = \Box_R^{-1}(-\frac{16\pi G}{c^4}\bar{g}T^{\mu\nu} + \bar{\Lambda}^{\mu\nu}) + O(8, 7, 8) \qquad (4.117)$$

where the subscript R represents the name 'retarded'.

4.4 Solution of Einstein's Equation in the Exterior Zone

In this section, we extend the discussion in Section 4.2 and compute the canonical external field $h^{\mu\nu}_{can}(x)$ up to the order $O(7, 7, 7)$ in the exterior zone D_e.

In Section 4.2, one can see that the canonical linearized field $h^{\mu\nu}_{(1)can}(x)$ can be written in an expression that only depends on two sets of functions M_L and S_L. Now, we introduce the potentials in the exterior zone $\tilde{V}$, $\tilde{V}_i$, and $\tilde{V}_{ij}$, such that

$$\tilde{V} = G\sum_{l\geq 0}\frac{(-1)^l}{l!}\partial_L(r^{-1}M_L) \tag{4.118a}$$

$$\tilde{V}_i = -G[\sum_{l\geq 1}\frac{(-1)^l}{l!}\partial_{L-1}(r^{-1}M^{(1)}_{iL-1}) + \sum_{l\geq 1}\frac{(-1)^l l}{(l+1)!}\epsilon_{iab}\partial_{aL-1}(r^{-1}S_{bL-1})] \tag{4.118b}$$

$$\tilde{V}_{ij} = G[\sum_{l\geq 2}\frac{(-1)^l}{l!}\partial_{L-2}(r^{-1}M^{(2)}_{ijL-2}) + \sum_{l\geq 2}\frac{(-1)^l l}{(l+1)!}\partial_{aL-2}(\epsilon_{ab(i}S^{(1)}_{j)bL-2})] \tag{4.118c}$$

From Eq. (4.77), Eq. (4.78), and Eq. (4.79), one can easily check $h^{00}_{(1)can} = -\frac{4}{Gc^2}\tilde{V}$, $h^{0i}_{(1)can} = -\frac{4}{Gc^3}\tilde{V}_i$, and $h^{ij}_{(1)can} = -\frac{4}{Gc^4}\tilde{V}_{ij}$. Using Einstein's vacuum equation (Eq. (4.13) with $T^{\mu\nu} = 0$), we can compute the higher-order coefficient in the MPM expansion Eq. (4.17) by

$$\Box h^{\mu\nu}_{(k)} = \Lambda^{\mu\nu}_{(k)}(h) \tag{4.119}$$

where the non-linear term $\Lambda^{\mu\nu}_{(k)}$ obeys

$$\Lambda^{\mu\nu}_{(k)} = \sum_{i+j=k} N^{\mu\nu}(h_{(i)can}, h_{(j)can}) + \sum_{a+b+c=k} M^{\mu\nu}(h_{(a)can}, h_{(b)can}, h_{(c)can}) \tag{4.120}$$

To obtain a solution for Eq. (4.119), we introduce a notation in [4], such that

$$h^{\mu\nu}_{(k)} = \mathrm{FP}_{z=0}\Box^{-1}_R[r^z\Lambda^{\mu\nu}_{(k)}(h)] + c^{\mu\nu}_{(k)} \tag{4.121}$$

where z is a complex number and $c^{\mu\nu}_{(k)}$ is an arbitrary function that obeys

$$\Box c^{\mu\nu}_{(k)} = 0 \tag{4.122}$$

For a function f, the function $g(z) = \Box^{-1}_R(r^z f)$ admits a Laurent expansion, i.e. $g(z) = \sum a_l z^l$. Then we define

$$\mathrm{FP}_{z=0}\Box^{-1}_R(r^z f) = a_0 \tag{4.123}$$

Formal definition and detail discussion of this notation can be found in [2]. We will see the convenience of introducing this notation in the following computation. Now, we will assume that $c^{\mu\nu}_{(2)} = O(7,7,7)$ and $c^{\mu\nu}_{(3)} = O(8,7,8)$, which are proved in detail in Appendix A of [4]. Using the iteration equation Eq. (4.119) and Eq. (4.120) and recalling $h^{\mu\nu}_{(1)} = O(2,3,4)$, one can check that the coefficient $h^{\mu\nu}_{(4)}$ will be smaller or equal to the order $O(8,8,8)$. Therefore, the canonical external field $h^{\mu\nu}_{can}$ can be written as

$$h^{\mu\nu}_{can}(x_{can}) = Gh^{\mu\nu}_{(1)can}(x_{can}) + \mathrm{FP}_{z=0}\Box^{-1}_R[r^z(G^2\Lambda^{\mu\nu}_{(2)}(h) + G^3\Lambda^{\mu\nu}_{(3)}(h))] + O(7,7,7) \tag{4.124}$$

where the term $O(7,7,7)$ comes from the term $c^{\mu\nu}_{(2)}$ in Eq. (4.121). We can now explicitly compute the coefficient $h^{\mu\nu}_{(2)can}$ in terms of the external potentials by inserting the expression for the linearized field $h^{\mu\nu}_{(1)can}$ into the non-linear term $\Lambda^{\mu\nu}_{(2)can}$. This is exactly the identical calculation in Section 4.3. We obtain

$$h^{00}_{can} = -\frac{4}{c^2}\tilde{V} + \frac{4}{c^2}(\tilde{W} - 2\tilde{V}^2) + O(6) \tag{4.125a}$$

$$h^{0i}_{can} = -\frac{4}{c^3}\tilde{V}_i + O(5) \tag{4.125b}$$

$$h^{ij}_{can} = -\frac{4}{c^4}\tilde{W}_{ij} + O(6) \tag{4.125c}$$

where

$$\tilde{W}_{ij} = \tilde{V}_{ij} - \mathrm{FP}_{z=0}\Box^{-1}_R[r^z(\partial_i\tilde{V}\partial_j\tilde{V} - \frac{1}{2}\delta_{ij}\partial_k\tilde{V}\partial_k\tilde{V})] \tag{4.126}$$

Eq. (4.125) admits the identical form to Eq. (4.105), Eq. (4.106), and Eq. (4.108). Therefore, we can directly apply the calculation in Section 4.2 to the case of exterior zone. By analogy to Eq. (4.117), we obtain the solution for the canonical external field

$$h^{\mu\nu}_{can}(x_{can}) = Gh^{\mu\nu}_{(1)can}(x_{can}) + \mathrm{FP}_{z=0}\Box^{-1}_R[r^z\bar{\Lambda}^{\mu\nu}(\tilde{V}, \tilde{V}_i, \tilde{W}_{ij})] + O(7,7,7) \tag{4.127}$$

where $\bar{\Lambda}^{\mu\nu}(\tilde{V}, \tilde{V}_i, \tilde{W}_{ij})$ admits the exact same form as $\bar{\Lambda}^{\mu\nu}(V, V_i, W_{ij})$

4.5 The Matching Equations

In previous sections, we introduced the solutions to Einstein's equation in the exterior zone D_e and in the interior zone D_i. In this section, we aim to find the matching equations that link between the two solutions in the overlapping zone $D_m = \{(\boldsymbol{x}, t) | r_e < |\boldsymbol{x}| < r_i\}$.
Now, let us consider a canonical transformation, i.e.

$$x^\mu_{can} = x^\mu + \phi^\mu \tag{4.128}$$

We assume that ϕ^μ is of order $O(3,4)$. From the transformation law of the metric, we can write

$$\det(\frac{\partial x^\alpha_{can}}{\partial x^\beta})(\eta^{\mu\nu} + h^{\mu\nu}_{can}(x_{can})) = \frac{\partial x^\mu_{can}}{\partial x^\rho}\frac{\partial x^\nu_{can}}{\partial x^\sigma}(\eta^{\rho\sigma} + h^{\rho\sigma}(x)) \tag{4.129}$$

We can then expand the determinant

$$\det(\frac{\partial x^\alpha_{can}}{\partial x^\beta}) = 1 + \partial_\alpha\phi^\alpha + O(\phi^2) \tag{4.130}$$

Taking the derivative of x^μ on both sides of Eq. (4.129) gives

$$\partial_\mu[(1+\partial_\alpha\phi^\alpha)(\eta^{\mu\nu}+h^{\mu\nu}_{can}(x)+\phi^\rho\partial_\rho h^{\mu\nu}_{can}(x))] = \partial_\mu[(\delta^\mu_\rho+\partial_\rho\phi^\mu)(\delta^\nu_\sigma+\partial_\sigma\phi^\nu)(\eta^{\rho\sigma}+h^{\rho\sigma}(x))] \tag{4.131}$$

where we have used $h^{\mu\nu}_{can}(x_{can}) = h^{\mu\nu}_{can}(x) + \phi^\rho\partial_\rho h^{\mu\nu}_{can}(x)$. Now, expanding both sides of Eq. (4.131), we obtain

$$\begin{aligned}\eta^{\mu\nu} + h^{\mu\nu}_{can}(x) + \phi^\rho\partial_\rho h^{\mu\nu}_{can}(x) + \eta^{\mu\nu}\partial_\rho\phi^\rho + h^{\mu\nu}_{can}(x)\partial_\rho\phi^\rho \\ = \eta^{\mu\nu} + h^{\mu\nu}(x) + \partial^\mu\phi^\nu + \partial^\nu\phi^\mu + h^{\rho\nu}(x)\partial_\rho\phi^\mu + h^{\rho\mu}(x)\partial_\rho\phi^\nu \\ + \eta^{\rho\sigma}\partial_\rho\phi^\mu\partial_\sigma\phi^\nu + O(8,7,8)\end{aligned} \tag{4.132}$$

After some rearrangement, one can check

$$\begin{aligned}h^{\mu\nu}_{can}(x) = h^{\mu\nu}(x) + \partial^\mu\phi^\nu + \partial^\nu\phi^\mu - \eta^{\mu\nu}\partial_\rho\phi^\rho \\ + h^{\rho\nu}(x)\partial_\rho\phi^\mu + h^{\rho\mu}(x)\partial_\rho\phi^\nu - \phi^\rho\partial_\rho h^{\mu\nu}_{can}(x) - h^{\mu\nu}_{can}(x)\partial_\rho\phi^\rho + \eta^{\rho\sigma}\partial_\rho\phi^\mu\partial_\sigma\phi^\nu \\ + O(8,7,8)\end{aligned} \tag{4.133}$$

One should, however, be aware that terms which contain both h and ϕ in Eq, (4.133) are smaller or equal to $O(6)$. This is simply because the '0' component of the derivative ∂_ρ contains another factor of $\frac{1}{c}$. Therefore, the transformation law Eq. (4.129) reduces to

$$h^{\mu\nu}_{can}(x) = h^{\mu\nu}(x) + \partial^\mu\phi^\nu + \partial^\nu\phi^\mu - \eta^{\mu\nu}\partial_\rho\phi^\rho + O(6,7,8) \tag{4.134}$$

Recall $h^{\mu\nu}(x)$ obeys the harmonic condition, i.e. $\partial_\mu h^{\mu\nu}(x) = 0$. Inserting Eq. (4.134) into Eq. (4.131), we obtain

$$\Box\phi^\rho + h^{\mu\nu}(x)\partial_\mu\partial_\nu\phi^\rho = 0 \tag{4.135}$$

It is also useful to compute the '00' component of the transformation law Eq. (4.119) to the next order $O(8)$. Inserting Eq. (4.124) into Eq. (4.123), we obtain

$$\begin{aligned}h^{00}_{can}(x) = h^{00}(x) + 2\partial^0\phi^0 + \partial_\rho\phi^\rho + 2h^{\rho 0}(x)\partial_\rho\phi^0 \\ - \phi^\rho\partial_\rho h^{00}(x) - h^{00}(x)\partial_\rho\phi^\rho + \partial_i\phi^0\partial_i\phi^0 + O(8)\end{aligned} \tag{4.136}$$

We therefore write Eq. (4.134) as

$$h^{\mu\nu}_{can}(x) = h^{\mu\nu}(x) + \Phi^{\mu\nu} + \Sigma^{\mu\nu} + O(8,7,8) \tag{4.137}$$

where

$$\Phi^{\mu\nu} = \partial^\mu\phi^\nu + \partial^\nu\phi^\mu - \eta^{\mu\nu}\partial_\rho\phi^\rho \tag{4.138}$$

and

$$\Sigma^{00} = 2h^{\rho 0}(x)\partial_\rho \phi^0 - \phi^\rho \partial_\rho h^{00}(x) - h^{00}(x)\partial_\rho \phi^\rho + \partial_i \phi^0 \partial_i \phi^0 \tag{4.139a}$$

$$\Sigma^{0i} = 0 \tag{4.139b}$$

$$\Sigma^{ij} = 0 \tag{4.139c}$$

Now we can match the exterior potentials to the interior potentials by inserting Eq. (4.105), Eq. (4.106), Eq. (4.108), and Eq. (4.125) into Eq. (4.137). We therefore obtain a set of matching equations

$$-\tilde{V} + \frac{1}{c^2}(\tilde{W} - 2\tilde{V}^2) = -V + \frac{1}{c^2}(W - 2V^2) - \frac{c}{2}\partial_t \phi^0 + \frac{c^2}{4}\partial_\rho \phi^\rho + O(4) \tag{4.140a}$$

$$-\tilde{V}_i = -V_i - \frac{c^2}{4}\partial_t \phi^i + \frac{c^3}{4}\partial_i \phi^0 + O(2) \tag{4.140b}$$

$$-\tilde{W}_{ij} = -W_{ij} + \frac{c^4}{4}(\partial_i \phi_j + \partial_j \phi_i - \delta_{ij}\partial_\rho \phi^\rho) + O(2) \tag{4.140c}$$

From Eq. (4.140c), one can easily check

$$\tilde{W} = W + \frac{c^4}{4}\partial_i \phi_i + \frac{3c^3}{4}\partial_t \phi^0 \tag{4.141}$$

Inserting Eq. (4.141) into Eq. (4.140a), we obtain

$$\tilde{V} = V + c\partial_t \phi^0 + O(4) \tag{4.142}$$

From Eq. (4.140b), we obtain

$$\tilde{V}_i = V_i - \frac{c^3}{4}\partial_i \phi^0 + O(2) \tag{4.143}$$

where we have used $\phi^i = O(4)$. We can then use Eq. (4.10) to expand interior potential V and V_i. This procedure is to ensure both external potentials and interior potentials to have the same mathematical expression. Therefore, we write

$$V^{(M)} = G\sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\partial_L[\frac{1}{r}V_L(t - \frac{r}{c})] \tag{4.144a}$$

$$V_i^{(M)} = G\sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\partial_L[\frac{1}{r}V_{iL}(t - \frac{r}{c})] \tag{4.144b}$$

where

$$V_L(t) = \int d^3\boldsymbol{y}\hat{y}_L \int_{-1}^{1} da\delta_l(a)\sigma(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.145a}$$

$$V_{iL}(t) = \int d^3\boldsymbol{y}\hat{y}_L \int_{-1}^{1} da\delta_l(a)\sigma_i(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.145b}$$

We have matched the exterior potentials $\tilde{V}$ and $\tilde{V}_i$ to the corresponding interior potentials V and V_i in the previous discussion. Now we need to match the exterior potential $\tilde{V}_{ij}$ to the interior potential W_{ij}. Inserting Eq. (4.126) into Eq. (4.140c), we obtain

$$\tilde{V}_{ij} = W_{ij} + \mathrm{FP}_{z=0}\Box_R^{-1}[r^z(\partial_i\tilde{V}\partial_j\tilde{V} - \frac{1}{2}\delta_{ij}\partial_k\tilde{V}\partial_k\tilde{V})] - \frac{c^4}{4}(\partial_i\phi_j + \partial_j\phi_i - \delta_{ij}\partial_\rho\phi^\rho) + O(2) \tag{4.146}$$

Since $\Box W_{ij} = -4\pi G\sigma_{ij} - \partial_i V\partial_j V + \frac{1}{2}\delta_{ij}\partial_k V\partial_k V$, we can write

$$\begin{aligned}\tilde{V}_{ij} &= \Box_R^{-1}(-4\pi G\sigma_{ij} - \partial_i V\partial_j V + \frac{1}{2}\delta_{ij}\partial_k V\partial_k V) \\ &\quad + \mathrm{FP}_{z=0}\Box_R^{-1}[r^z(\partial_i\tilde{V}\partial_j\tilde{V} - \frac{1}{2}\delta_{ij}\partial_k\tilde{V}\partial_k\tilde{V})] - \frac{c^4}{4}(\partial_i\phi_j + \partial_j\phi_i - \delta_{ij}\partial_\rho\phi^\rho) + O(2)\end{aligned} \tag{4.147}$$

Since $\tilde{V} = V^{(M)} + O(2)$, we can replace the exterior potential $\tilde{V}$ by the multipole expansion of its corresponding interior potential V. Thus, we can rewrite Eq. (4.147)

$$\tilde{V}_{ij} = \Box_R^{-1}(-4\pi G\sigma_{ij}) - (Z_{ij} - \frac{1}{2}\delta_{ij}Z_{kk}) - \frac{c^4}{4}(\partial_i\phi_j + \partial_j\phi_i - \delta_{ij}\partial_\rho\phi^\rho) + O(2) \tag{4.148}$$

where we have defined

$$Z_{ij} := \Box_R^{-1}(\partial_i V\partial_j V) - \mathrm{FP}_{z=0}\Box_R^{-1}[r^z(\partial_i V^{(M)}\partial_j V^{(M)})] \tag{4.149}$$

Since the first term in Eq. (4.149) is regular at $r = 0$, we can write Eq. (4.149) in the following form

$$Z_{ij} = \mathrm{FP}_{z=0}\Box_R^{-1}[r^z(\partial_i V\partial_j V - \partial_i V^{(M)}\partial_j V^{(M)})] \tag{4.150}$$

The convenience of writing Z_{ij} in this form is that now we can multipole expand Z_{ij}, such that

$$Z_{ij}^{(M)} = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L[\frac{1}{r}Z_{ijL}(t - \frac{r}{c})] \tag{4.151}$$

where

$$Z_{ijL} = -\frac{1}{4\pi}\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\hat{y}_L\int_{-1}^{1}da\delta_l(a)(\partial_i V\partial_j V - \partial_i V^{(M)}\partial_j V^{(M)})(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.152}$$

From Eq. (4.144a) and Eq. (4.145a), one can compute that the multipole expansion of V has the form

$$V^{(M)} = \sum \hat{n}_K r^{-m} f(t + (a-1)\frac{|\boldsymbol{y}|}{c}) \tag{4.153}$$

where K is some multi-index, m is an integer, and f is some function. One can then insert Eq. (4.153) into the integral Eq. (4.152) and obtain

$$\begin{aligned} \mathrm{FP}_{z=0} \int d^3\boldsymbol{y}|\boldsymbol{y}|^z \hat{y}_L \int_{-1}^{1} da\delta_l(a)(\partial_i V^{(M)}\partial_j V^{(M)})(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \\ = \sum \hat{n}_{K'} g(t)\mathrm{FP}_{z=0} \int_0^\infty d|\boldsymbol{y}||\boldsymbol{y}|^{z+n} \end{aligned} \tag{4.154}$$

Let us define

$$I_1 = \int_X^\infty d|\boldsymbol{y}||\boldsymbol{y}|^{z+n} \tag{4.155a}$$

$$I_2 = \int_0^X d|\boldsymbol{y}||\boldsymbol{y}|^{z+n} \tag{4.155b}$$

For I_1, we choose the complex number z, such that $Re(z)+n+1<0$. One can check that

$$I_1 = \frac{1}{z+n+1}|X|^{z+n+1} \tag{4.156}$$

Similarly, for I_2, we choose the complex number z, such that $Re(z)+n+1>0$. We then obtain

$$I_2 = -\frac{1}{z+n+1}|X|^{z+n+1} \tag{4.157}$$

Therefore, at $z=0$, by analytic continuation, we obtain

$$\mathrm{FP}_{z=0} \int_0^\infty d|\boldsymbol{y}||\boldsymbol{y}|^{z+n} = 0 \tag{4.158}$$

From Eq. (4.152), we then have

$$Z_{ijL} = \mathrm{FP}_{z=0} \int d^3\boldsymbol{y}|\boldsymbol{y}|^z \hat{y}_L \int_{-1}^{1} da\delta_l(a)(\partial_i V\partial_j V)(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.159}$$

Inserting Eq. (4.151) into Eq. (4.148), we obtain

$$\tilde{V}_{ij} = G\sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\partial_L[\frac{1}{r}W_{ijL}(t-\frac{r}{c})] - \frac{c^4}{4}(\partial_i\phi_j + \partial_j\phi_i - \delta_{ij}\partial_\rho\phi^\rho) + O(2) \tag{4.160}$$

where

$$W_{ijL}(t) = \mathrm{FP}_{z=0} \int d^3\boldsymbol{y}|\boldsymbol{y}|^z \hat{y}_L \int_{-1}^{1} da\delta_l(a)[\sigma_{ij} + \frac{1}{4\pi G}(\partial_i V\partial_j V - \frac{1}{2}\partial_k V\partial_k V)](\boldsymbol{y}, t+a\frac{|\boldsymbol{y}|}{c}) \tag{4.161}$$

and

$$\tilde{W}_{ij} = W_{ij}^{(M)} - \frac{c^4}{4}(\partial_i\phi_j + \partial_j\phi_i - \delta_{ij}\partial_\rho\phi^\rho) + O(2) \tag{4.162}$$

where

$$
\begin{aligned}
W_{ij}^{(M)} = G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L[\frac{1}{r}W_{ijL}(t-\frac{r}{c})] \\
- \mathrm{FP}_{z=0}\Box_R^{-1}[r^z(\partial_i V^{(M)}\partial_j V^{(M)} - \frac{1}{2}\delta_{ij}\partial_k V^{(M)}\partial_k V^{(M)})]
\end{aligned}
\tag{4.163}
$$

We have related all the exterior potentials, i.e. $\tilde{V}$, $\tilde{V}_i$, and $\tilde{W}_{ij}$, to the multipole expansion of the interior potentials, i.e. $V^{(M)}$, $V_i^{(M)}$, and $W_{ij}^{(M)}$ in Eq. (4.142), Eq. (4.143), and Eq. (4.162), respectively. We have also obtained the solution for Einstein's equation in the interior zone D_i (Eq. (4.117)) and in the exterior zone D_e (Eq. (4.127)). We have also explicitly computed the form of the non-linear term $\bar{\Lambda}^{\mu\nu}$ in Eq. (4.111), Eq. (4.112), and Eq. (4.113). Inserting Eq. (4.142), Eq. (4.143), and Eq. (4.162) into Eq. (4.111), we obtain

$$
\begin{aligned}
\bar{\Lambda}^{00}(\tilde{V},\tilde{V}_i,\tilde{W}_{ij}) = \bar{\Lambda}^{00}(V^{(M)},V_i^{(M)},W_{ij}^{(M)}) - \frac{28}{c^3}\partial_i\partial_t\phi^0\partial_i V^{(M)} \\
+ \frac{2}{c^6}[4c^3\partial_i\phi^0\partial_t\partial_i V^{(M)} + 12(-\frac{c^3}{2}\partial_i\partial_j\phi^0\partial_j V_i^{(M)} + \frac{c^6}{16}\partial_i\partial_j\phi^0\partial_i\partial_j\phi^0) \\
+ 4(-\frac{c^3}{2}\partial_i\partial_j\phi^0\partial_i V_j^{(M)} + \frac{c^6}{16}\partial_i\partial_j\phi^0\partial_i\partial_j\phi^0) - 2c^3\partial_i V^{(M)}\partial_i\partial_t\phi^0 \\
+ 2c^4(\partial_i\phi_j + \partial_j\phi_i - \delta_{ij}(\frac{1}{c}\partial_t\phi^0 + \partial_k\phi_k))\partial_i\partial_j V^{(M)} \\
- 4c^4\partial_i V^{(M)}(-\partial_i\partial_k\phi_k - \frac{3}{c}\partial_i\partial_t\phi^0)] + O(8)
\end{aligned}
\tag{4.164}
$$

Rearranging the terms in Eq. (4.164) and using $\Box V^{(M)} = \triangle V^{(M)} + O(2)$, we have

$$
\begin{aligned}
\bar{\Lambda}^{00}(\tilde{V},\tilde{V}_i,\tilde{W}_{ij}) = \bar{\Lambda}^{00}(V^{(M)},V_i^{(M)},W_{ij}^{(M)}) - \frac{8}{c^3}\partial_i\partial_t\phi^0\partial_i V^{(M)} \\
+ \frac{8}{c^3}\partial_i\phi^0\partial_i\partial_t V^{(M)} - \frac{16}{c^3}\partial_i V_j^{(M)}\partial_i\partial_j\phi^0 \\
+ 2\partial_i\partial_j\phi^0\partial_i\partial_j\phi^0 + \frac{8}{c^2}\partial_i\phi_j\partial_i\partial_j V^{(M)} \\
+ \frac{8}{c^2}\partial_i V^{(M)}\partial_i\partial_j\phi_j + O(8)
\end{aligned}
\tag{4.165}
$$

Similarly, one can simply check that

$$
\bar{\Lambda}^{0i}(\tilde{V},\tilde{V}_i,\tilde{W}_{ij}) = \bar{\Lambda}^{0i}(V^{(M)},V_i^{(M)},W_{ij}^{(M)}) + O(7) \tag{4.166a}
$$

$$
\bar{\Lambda}^{ij}(\tilde{V},\tilde{V}_i,\tilde{W}_{ij}) = \bar{\Lambda}^{ij}(V^{(M)},V_i^{(M)},W_{ij}^{(M)}) + O(8) \tag{4.166b}
$$

Now we claim, using Eq. (4.165) and Eq. (4.166),

$$
\bar{\Lambda}^{\mu\nu}(\tilde{V},\tilde{V}_i,\tilde{W}_{ij}) = \bar{\Lambda}^{\mu\nu}(V^{(M)},V_i^{(M)},W_{ij}^{(M)}) + \Box\Sigma'^{\mu\nu} + O(8,7,8) \tag{4.167}
$$

where

$$\Sigma'^{00} = -\frac{4}{c^3}(V^{(M)}\partial_t\phi^0 - \partial_t V^{(M)}\phi^0 - c\partial_i V^{(M)}\phi_i - cV^{(M)}\partial_i\phi_i) - \frac{8}{c^3}V_i^{(M)}\partial_i\phi^0 + \partial_i\phi^0\partial_i\phi^0 \tag{4.168a}$$

$$\Sigma'^{0i} = 0 \tag{4.168b}$$

$$\Sigma'^{ij} = 0 \tag{4.168c}$$

To prove Eq. (4.167), one can easily check, for two functions f and g,

$$\Box(fg) = (\Box f)g + f(\Box g) + 2\partial_j f\partial_j g + O(2) \tag{4.169}$$

However, one should notice that $\Box V^{(M)} = 0$ and $\Box V_i^{(M)} = 0$, since the multipole expansion is evaluated outside the source. From Eq. (4.135), one can easily see that $\Box\phi^\mu = O(7,8)$. Therefore, one can check that $\Box\Sigma^{\mu\nu}$ gives the exact expression in Eq. (4.165). One can also check that $\Sigma'^{\mu\nu}$ defined in Eq. (4.168) is the same as $\Sigma^{\mu\nu}$ that we have defined in Eq. (4.139). In the overlapping zone D_m, we have the canonical field $h^{\mu\nu}_{can}(x)$ relates to the internal field $h^{\mu\nu}(x)$ by Eq. (4.137). Inserting Eq. (4.167) into Eq. (4.127) and comparing with Eq. (4.117), we have

$$\begin{aligned} Gh^{\mu\nu}_{(1)can}(x) = {} & \Box_R^{-1}(-\frac{16\pi G}{c^4}\bar{g}T^{\mu\nu} + \bar{\Lambda}^{\mu\nu}(V, V_i, W_{ij})) \\ & - \mathrm{FP}_{z=0}\Box_R^{-1}[r^z\bar{\Lambda}^{\mu\nu}(V^{(M)}, V_i^{(M)}, W_{ij}^{(M)})] + \Phi^{\mu\nu} + O(7,7,7) \end{aligned} \tag{4.170}$$

where we have defined $\Phi^{\mu\nu}$ in Eq. (4.138). Since the first term on the right hand side of Eq. (4.170) is regular at $r = 0$, we can rewrite Eq. (4.170) as

$$\begin{aligned} Gh^{\mu\nu}_{(1)can}(x) = {} & \mathrm{FP}_{z=0}\Box_R^{-1}[r^z(-\frac{16\pi G}{c^4}\bar{g}T^{\mu\nu} + \bar{\Lambda}^{\mu\nu}(V, V_i, W_{ij}) \\ & - \bar{\Lambda}^{\mu\nu}(V^{(M)}, V_i^{(M)}, W_{ij}^{(M)}))] + \Phi^{\mu\nu} + O(7,7,7) \end{aligned} \tag{4.171}$$

One should recall that, as we discussed in Section 4.2, the linearized field $h^{\mu\nu}_{(1)can}$ is a function of two set of functions M_L and S_L. We can multipole expand Eq. (4.171), such that,

$$Gh^{\mu\nu}_{(1)can}[M,S] = -\frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L[\frac{1}{r}P_L^{\mu\nu}(t-\frac{r}{c})] + \Phi^{\mu\nu} + O(7,7,7) \tag{4.172}$$

where

$$\begin{aligned} P_L^{\mu\nu}(t) = {} & \mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\hat{y}_L\delta_l(a)(p^{\mu\nu} \\ & - \frac{c^4}{16\pi G}\bar{\Lambda}^{\mu\nu}(V^{(M)}, V_i^{(M)}, W_{ij}^{(M)})))(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \end{aligned} \tag{4.173}$$

The new source $p^{\mu\nu}$ is defined as

$$p^{\mu\nu}(V, V_i, W_{ij}) = -\bar{g}(V,W)T^{\mu\nu} + \frac{c^4}{16\pi G}\bar{\Lambda}^{\mu\nu}(V, V_i, W_{ij}) \tag{4.174}$$

As we have discussed in Eq. (4.152), the integral of $\bar{\Lambda}^{\mu\nu}(V^{(M)}, V_i^{(M)}, W_{ij}^{(M)})$ is proportional to $\sum \hat{n}_K g(t)\mathrm{FP}_{z=0}\int_0^\infty d|\boldsymbol{y}||\boldsymbol{y}|^{z+n}$, and, therefore, vanishes due to analytic continuation. We then have

$$P_L^{\mu\nu}(t) = \mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\hat{y}_L\delta_l(a)p^{\mu\nu}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.175}$$

Eq. (4.172) is an equation that relates the two set of functions M_L and S_L to the internal potentials V, V_i, and W_{ij}. One should be aware that

$$\partial_\mu p^{\mu\nu} = O(3,4) \tag{4.176}$$

4.6 Relation Between Source Moments and Interior Potentials

In this section, we will explicitly write the multipole moments of the source in terms of the interior potentials. The process of solving Eq. (4.175) is very similar to the process of finding Einstein's equation in the exterior zone, which we discussed in detail in Section 4.2. It is also thoroughly discussed in [12]. First, we define

$$\mathcal{P}^{\mu\nu} = \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L[\frac{1}{r}P_L^{\mu\nu}(t - \frac{r}{c})] \tag{4.177}$$

and

$$Q_L(t) = \mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\hat{y}_L\delta_l(a)p^{00}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.178a}$$

$$K_{iL}(t) = \mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\hat{y}_L\delta_l(a)p^{0i}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.178b}$$

$$L_{ijL}(t) = \mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\hat{y}_L\delta_l(a)p^{ij}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.178c}$$

We can then decompose K_{iL} and L_{ijL} into irreducible representations of the rotation Lie group. This process is done in Section 4.2. We write the result as below [12]. Using Eq. (4.4) and Eq. (4.5), we have

$$K_{iL} = K_{iL}^{(+1)} + \frac{l}{l+1}\epsilon_{ai\langle i_l}K_{L-1\rangle a}^{(0)} + \frac{2l-1}{2l+1}\delta_{i\langle i_l}K_{L-1\rangle}^{(-1)} \tag{4.179}$$

where

$$K_{L+1}^{(+1)}(t) = \mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\delta_l(a)\hat{y}^{\langle L}p^{i_{l+1}\rangle 0}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.180a}$$

$$K_L^{(0)}(t) = \mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\delta_l(a)\epsilon_{ab\langle i_l}\hat{y}_{L-1\rangle b}p^{0a}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.180b}$$

$$K_{L-1}^{(-1)}(t) = \mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\delta_l(a)\hat{y}_{L-1a}p^{0a}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.180c}$$

Similarly, from Eq. (4.6) and Eq. (4.7), we have

$$\begin{aligned}L_{ijL} = L^{(+2)}_{ijL} &+ \text{STF}_{ij}\text{STF}_L(\frac{2l}{l+2}\epsilon_{aii_l}L^{(+1)}_{L-1ja} \\ &+ \frac{6l(2l-1)}{(l+1)(2l+3)}\delta_{ii_l}L^{(0)}_{L-1j} + \frac{2(l-1)(2l-1)}{(l+1)(2l+1)}\epsilon_{aii_{l-1}}\delta_{ji_l}L^{(-1)}_{L-2a} \\ &+ \frac{2l-3}{2l+1}\delta_{ii_{l-1}}\delta_{ji_l}L^{(-2)}_{L-2}) + \delta_{ij}\bar{L}_L\end{aligned} \tag{4.181}$$

where

$$L^{(+2)}_{L+2}(t) = \text{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\delta_l(a)\hat{y}_{\langle L}\hat{p}_{i_{l+1}i_{l+2}\rangle}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.182a}$$

$$L^{(+1)}_{L+1}(t) = \text{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\delta_l(a)\epsilon_{ab\langle i_{l+1}}\hat{y}_{L-1|b|}\hat{p}_{i_l\rangle a}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.182b}$$

$$L^{(0)}_{L}(t) = \text{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\delta_l(a)\hat{y}_{a\langle L-1}\hat{p}_{i_l\rangle a}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.182c}$$

$$L^{(-1)}_{L-1}(t) = \text{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\delta_l(a)\epsilon_{ab\langle i_{l-1}}\hat{y}_{L-2\rangle bc}\hat{p}_{ca}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.182d}$$

$$L^{(-2)}_{L-2}(t) = \text{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\delta_l(a)\hat{y}_{abL-2}\hat{p}_{ab}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.182e}$$

$$\bar{L}_L(t) = \frac{1}{3}\text{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\delta_l(a)\hat{y}_L p_{aa}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c}) \tag{4.182f}$$

One should recall that we have defined $\hat{p}_{ij} = p_{\langle ij\rangle}$. With the decomposition of $K_{iL}(t)$ and L_{ijL} in Eq. (4.179) and Eq. (4.181) respectively, one can decompose $\mathcal{P}^{\mu\nu}$ and write Eq. (4.177) in the following form

$$\mathcal{P}^{00} = \sum_{l\geq 0}\partial_L(r^{-1}A_L) \tag{4.183}$$

$$\mathcal{P}^{0i} = \sum_{l\geq 0}\partial_{iL}(r^{-1}B_L) + \sum_{l\geq 1}\partial_{L-1}(r^{-1}C_{iL-1}) + \sum_{l\geq 1}\epsilon_{iba}\partial_{bL-1}(r^{-1}D_{L-1a}) \tag{4.184}$$

$$\begin{aligned}\mathcal{P}^{ij} &= \sum_{l\geq 0}[\partial_{ijL}(r^{-1}E_L) + \delta_{ij}\partial_L(r^{-1}F_L)] \\ &+ \sum_{l\geq 1}[\partial_{L-1(i}(r^{-1}G_{j)L-1}) + \epsilon_{ab(i}\partial_{j)aL-1}(r^{-1}H_{bL-1})] \\ &+ \sum_{l\geq 2}[\partial_{L-2}(r^{-1}I_{ijL-2}) + \partial_{aL-2}(r^{-1}\epsilon_{ab(i}J_{j)bL-2})]\end{aligned} \tag{4.185}$$

One can immediately see that

$$A_L = \frac{4G}{c^4}\frac{(-1)^l}{l!}Q_L \tag{4.186}$$

We have calculated the functions B_L, C_L, and D_L in Eq. (4.55). However, due to the factor $\frac{(-1)^l}{l!}$ in Eq. (4.177), one should be careful about the coefficient in front of each term. We have

$$B_L = -\frac{4G}{c^4}\frac{(-1)^l}{l!}\frac{2l+1}{(l+1)(2l+3)}K_L^{(-1)} \tag{4.187a}$$

$$C_L = \frac{4G}{c^4}\frac{(-1)^l}{(l-1)!}[-K_L^{(+1)} + \frac{1}{(l+1)(2l+3)c^2}\ddot{K}_L^{(-1)}] \tag{4.187b}$$

$$D_L = \frac{4G}{c^4}\frac{(-1)^l l}{(l+1)!}K_L^{(0)} \tag{4.187c}$$

Using the same method, we can compute the terms in $\mathcal{P}^{ij}$ [12]

$$E_L = \frac{4G}{c^4}\frac{(-1)^l}{(l+2)!}\frac{2l+1}{2l+5}L_L^{(-2)} \tag{4.188a}$$

$$F_L = \frac{4G}{c^4}\frac{(-1)^l}{l!}[\bar{L}_L - \frac{2l(2l-1)}{(l+1)(2l+3)}L_L^{(0)} - \frac{2l+1}{(l+1)(l+2)(2l+3)(2l+5)c^2}\ddot{L}_L^{(-2)}] \tag{4.188b}$$

$$G_L = \frac{4G}{c^4}\frac{(-1)^l}{l!}[\frac{6l(2l-1)}{(l+1)(2l+3)}L_L^{(0)} - \frac{2l(2l+1)}{(l+1)(l+2)(2l+3)(2l+5)c^2}\ddot{L}_L^{(-2)}] \tag{4.188c}$$

$$H_L = -\frac{4G}{c^4}\frac{(-1)^l}{l!}\frac{2l(2l+1)}{(l+1)^2(l+2)(2l+3)}L_L^{(-1)} \tag{4.188d}$$

$$\begin{aligned} I_L =& \frac{4G}{c^4}\frac{(-1)^l}{l!}[l(l-1)L_L^{(+2)} - \frac{6l(l-1)}{(l+1)(2l+3)c^2}L_L^{(0)} \\ &+ \frac{l(l-1)}{(l+1)(l+2)(2l+3)(2l+5)c^4}\frac{d^4}{dt^4}L_L^{(-2)}] \end{aligned} \tag{4.188e}$$

$$J_L = \frac{4G}{c^4}\frac{(-1)^l}{l!}[-\frac{2l(l-1)}{l+1}L_L^{(+1)} + \frac{2l(l-1)(2l+1)}{(l+1)(l+2)(2l+1)(2l+3)c^2}\ddot{L}_L^{(-1)}] \tag{4.188f}$$

Following the process in Section 4.2, we have to determine the constraints on functions A_L to J_L using the conservation law of the stress-energy tensor. One should recall the new source $p^{\mu\nu}$ obeys Eq. (4.176), i.e. $\partial_\mu p^{\mu\nu} = O(3,4)$. Let's compute

$$\frac{d}{cdt}P_L^{0\mu}(t) = \frac{d}{cdt}[\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1}da\hat{y}_L\delta_l(a)p^{0\mu}(\boldsymbol{y}, t + a\frac{|\boldsymbol{y}|}{c})] \tag{4.189}$$

Inserting Eq. (4.176) into Eq. (4.189), we have

$$\frac{d}{cdt}P_L^{0\mu}(t) = -\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\hat{y}_L\delta_l(a)\partial_i p^{i\mu}(\boldsymbol{y}, t+a\frac{|\boldsymbol{y}|}{c}) + O(3,4) \tag{4.190}$$

One should be careful that

$$\frac{d}{dy^i}p^{i\mu} = \partial_i p^{i\mu} + \frac{a}{c}n_i\frac{d}{dt}p^{i\mu} \tag{4.191}$$

where we have used $\frac{d|\boldsymbol{y}|}{dy^i} = n_i$. Inserting Eq. (4.191) into Eq. (4.190), we obtain

$$\begin{aligned}\frac{d}{cdt}P_L^{0\mu}(t) =& -\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} da\hat{y}_L\delta_l(a)\frac{d}{dy^i}p^{i\mu}(\boldsymbol{y}, t+a\frac{|\boldsymbol{y}|}{c})\\ &+\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} dan_i\hat{y}_L\delta_l(a)a\frac{d}{dt}p^{i\mu}(\boldsymbol{y}, t+a\frac{|\boldsymbol{y}|}{c}) + O(3,4)\end{aligned} \tag{4.192}$$

Now, we notice that

$$\frac{d}{da}\delta_{a+1}(a) = -(2l+3)\delta_l(a)a \tag{4.193}$$

Therefore, for the first term on the right hand side of Eq. (4.192), we can integrate by parts with regard to y^i and, for the second term on the right hand side of Eq. (4.192), we can integrate by parts with respect to a. We, thus, yield

$$\begin{aligned}\frac{d}{cdt}P_L^{0\mu}(t) =& \mathrm{FP}_{z=0}\int d^3\boldsymbol{y}\frac{d}{dy^i}(|\boldsymbol{y}|^z\hat{y}_L)\int_{-1}^{1} da\delta_l(a)p^{i\mu}(\boldsymbol{y}, t+a\frac{|\boldsymbol{y}|}{c})\\ &+\frac{1}{2l+3}\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1} dan_i\hat{y}_L\delta_{l+1}(a)\frac{d}{da}[\frac{d}{cdt}p^{i\mu}(\boldsymbol{y}, t+a\frac{|\boldsymbol{y}|}{c})]\\ &+O(3,4)\end{aligned} \tag{4.194}$$

We can explicitly compute the derivatives inside the integral. We have

$$\frac{d}{dy^i}(|\boldsymbol{y}|\hat{y}_L) = z\hat{y}_L|\boldsymbol{y}|^{z-1}n_i + |\boldsymbol{y}|^z\sum_{m=1}^{l}\delta_{i\langle i_m}\hat{y}_{L/m\rangle} \tag{4.195}$$

and

$$\frac{d}{da}\frac{d}{cdt}p^{i\mu}(\boldsymbol{y}, t+a\frac{|\boldsymbol{y}|}{c}) = |\boldsymbol{y}|(\frac{d}{cdt})^2p^{i\mu}(\boldsymbol{y}, t+a\frac{|\boldsymbol{y}|}{c}) \tag{4.196}$$

where we have used $\frac{c}{|\boldsymbol{y}|}\frac{d}{da}p^{i\mu}(\boldsymbol{y},t+a\frac{|\boldsymbol{y}|}{c}) = \frac{d}{dt}p^{i\mu}(\boldsymbol{y},t+a\frac{|\boldsymbol{y}|}{c})$ in Eq. (4.196). Inserting Eq. (4.195) and Eq. (4.194) into Eq. (4.194) gives

$$\begin{aligned}\frac{d}{cdt}P_L^{0\mu}(t) = \mathrm{FP}_{z=0}[z\int d^3\boldsymbol{y}\hat{y}_L y_i|\boldsymbol{y}|^{z-2}\int_{-1}^{1}da\delta_l(a)p^{i\mu}(\boldsymbol{y},t+a\frac{|\boldsymbol{y}|}{c})]\\ + l\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1}da\delta_l(a)\hat{y}_{\langle L-1}p^{\mu}_{i_l\rangle}(\boldsymbol{y},t+a\frac{|\boldsymbol{y}|}{c})\\ + \frac{1}{2l+3}\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1}day_i\hat{y}_L\delta_{l+1}(a)(\frac{d}{cdt})^2p^{i\mu}(\boldsymbol{y},t+a\frac{|\boldsymbol{y}|}{c})\\ + O(3,4)\end{aligned} \tag{4.197}$$

Using the definition of $P_L^{\mu\nu}$ in Eq. (4.175), we can rewrite Eq. (4.197)

$$\begin{aligned}\frac{d}{cdt}P_L^{0\mu}(t) = l\mathrm{STF}_L P^{\mu i_l}_{L-1} + \frac{1}{2l+3}(\frac{d}{cdt})^2P^{i\mu}_{iL}\\ + \mathrm{FP}_{z=0}[z\int d^3\boldsymbol{y}\hat{y}_L y_i|\boldsymbol{y}|^{z-2}\int_{-1}^{1}da\delta_l(a)p^{i\mu}(\boldsymbol{y},t+a\frac{|\boldsymbol{y}|}{c})] + O(3,4)\end{aligned} \tag{4.198}$$

Now we will assume

$$\mathrm{FP}_{z=0}[z\int d^3\boldsymbol{y}\hat{y}_L y_i|\boldsymbol{y}|^{z-2}\int_{-1}^{1}da\delta_l(a)p^{i\mu}(\boldsymbol{y},t+a\frac{|\boldsymbol{y}|}{c})] = 0 \tag{4.199}$$

This arises from the fact that [4]

$$\mathrm{FP}_{z=0}[\int d^3\boldsymbol{y}\hat{y}_L|\boldsymbol{y}|^{z+a_0}(|\boldsymbol{y}|-|\boldsymbol{x}_1|)^{a_1}\cdots(|\boldsymbol{y}|-|\boldsymbol{x}_n|)^{a_n}] = 0 \tag{4.200}$$

for any integers a_0, $\cdots$, a_n that obey $\sum_{i=0}^{n}a_i$ is even. Full proof of this lemma can be found in [4]. Therefore, Eq. (4.198) reduces to

$$\frac{d}{cdt}P_L^{0\mu}(t) = l\mathrm{STF}_L P^{\mu i_l}_{L-1} + \frac{1}{2l+3}(\frac{d}{cdt})^2P^{i\mu}_{iL} + O(3,4) \tag{4.201}$$

Let us recalculate the constraints arise from Eq. (4.62) and Eq. (4.64). In this case

$$C_L + \frac{1}{c}\dot{Q}_L + \frac{1}{c^2}\ddot{B}_L = \frac{4G}{c^4}\frac{(-1)^l}{l!}(-lK_L^{(+1)} - \frac{1}{(2l+3)c^2}\ddot{K}_L^{(-1)} + \frac{1}{c}\dot{Q}_L) \tag{4.202}$$

Inserting Eq. (4.178a), Eq. (4.180a), and Eq. (4.180c) into Eq. (4.202) gives

$$\begin{aligned}C_L + \frac{1}{c}\dot{Q}_L + \frac{1}{c^2}\ddot{B}_L = \frac{4G}{c^4}\frac{(-1)^l}{l!}[\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\hat{y}_L\int_{-1}^{1}da\delta_l(a)\frac{d}{cdt}p^{00}(\boldsymbol{y},t+a\frac{|\boldsymbol{y}|}{c})\\ - l\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\hat{y}_L\int_{-1}^{1}da\delta_l(a)\hat{y}_{\langle L-1}p^0_{i_l\rangle}(\boldsymbol{y},t+a\frac{|\boldsymbol{y}|}{c})\\ - \frac{1}{2l+3}\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1}da\hat{y}_L y_a\delta_{l+1}(a)(\frac{d}{cdt})^2p^{0a}(\boldsymbol{y},t+a\frac{|\boldsymbol{y}|}{c})]\end{aligned} \tag{4.203}$$

One can easily check that Eq. (4.203) is exactly the '0' component of Eq. (4.201). Therefore, we have

$$C_L + Q_L^{(1)} + \ddot{B}_L^{(2)} = O(7) \tag{4.204}$$

Using the 'i' component of Eq. (4.201), we obtain

$$G_L + 2B_L^{(1)} + 2E_L^{(2)} + 2F_L = O(8) \tag{4.205a}$$

$$I_L - A_L^{(2)} - 2B_L^{(3)} - E_L^{(4)} - F_L^{(2)} = O(8) \tag{4.205b}$$

$$J_L + 2D_L^{(1)} + H_L^{(2)} = O(8) \tag{4.205c}$$

Similar to what we have done in Section 4.2, we define the following new variables

$$\mathcal{I}_L(t) := \frac{(-1)^l c^2 l!}{4}(A_L + 2B_L^{(1)} + E_L^{(2)} + F_L) \tag{4.206a}$$

$$\mathcal{J}_L(t) := \frac{(-1)^l c^3 (l+1)!}{4l}(-D_L - \frac{1}{2}H_L^{(1)}) \tag{4.206b}$$

and

$$\gamma_L(t) := B_L + \frac{1}{2}E_L^{(1)} \tag{4.207a}$$

$$\theta_L(t) := \frac{1}{2}E_L \tag{4.207b}$$

$$\phi_L(t) := -B_L^{(1)} - E_L^{(2)} - F_L \tag{4.207c}$$

$$\psi_L(t) := \frac{1}{2}H_L \tag{4.207d}$$

Therefore, we obtain (see Eq. (4.70) to Eq. (4.79))

$$\mathcal{P}^{\mu\nu} = \mathcal{P}^{\mu\nu}_{can} + \mathcal{K}^{\mu\nu}(\gamma, \theta, \phi, \psi) \tag{4.208}$$

where $\mathcal{P}^{\mu\nu}_{can}$ has the same form as Eq. (4.77) to Eq. (4.79) with $\tilde{M}_L$ and $\tilde{S}_L$ replaced by $\mathcal{I}_L$ and $\mathcal{J}_L$ respectively and $\mathcal{K}^{\mu\nu}$ has the same form as Eq. (4.74)-Eq. (4.76). One should also note that the $O(7)$ and $O(8)$ terms in Eq. (4.204) and Eq. (4.205) can be absorbed into the $O(7,7,7)$ term in Eq. (4.172). Now, we choose the gauge, such that $\Phi^{\mu\nu} = \mathcal{K}^{\mu\nu} + O(0,7,8)$, we can rewrite Eq. (4.172)

$$Gh^{\mu\nu}_{(1)can}[M,S] = h^{\mu\nu}_{(1)can}[\mathcal{I},\mathcal{J}] + O(7,7,7) \tag{4.209}$$

One can immediately see

$$GM_L(t) = \mathcal{I}_L(t) + O(4) \tag{4.210a}$$

$$GS_L(t) = \mathcal{J}_L(t) + O(4) \tag{4.210b}$$

From the definition of $\mathcal{I}_L(t)$ and $\mathcal{J}_L(t)$ in Eq. (4.206), we can write

$$\begin{aligned}\mathcal{I}_L(t) = &\frac{1}{c^2}\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1}da[\delta_l(a)\hat{y}_L(p^{00}+p^{ii})\\&-\frac{4(2l+1)}{(l+1)(2l+3)}\delta_{l+1}(a)\hat{y}_{iL}\frac{\partial}{c\partial t}p^{0i}\\&+\frac{2(2l+1)}{(l+1)(l+2)(2l+5)}\delta_{l+2}(a)\hat{y}_{ijL}(\frac{\partial}{c\partial t})^2p^{ij}](\boldsymbol{y},t+a\frac{|\boldsymbol{y}|}{c})\end{aligned} \tag{4.211a}$$

$$\begin{aligned}\mathcal{J}_L(t) = &\frac{1}{c}\mathrm{FP}_{z=0}\int d^3\boldsymbol{y}|\boldsymbol{y}|^z\int_{-1}^{1}da[\delta_l(a)\epsilon_{ab\langle i_l}\hat{y}_{L-1\rangle}p^{0b}\\&-\frac{2l+1}{(l+2)(2l+3)}\delta_{l+1}(a)\epsilon_{ab\langle i_l}\hat{y}_{L-1\rangle ac}\frac{\partial}{c\partial t}p^{bc}](\boldsymbol{y},t+a\frac{|\boldsymbol{y}|}{c})\end{aligned} \tag{4.211b}$$

Chapter 5

Conclusion

This dissertation provides a general review of the dynamics of a compact two-body system and derives explicit expressions of the multipole moments in terms of the stress-energy tensor of a gravitating source at 2PN correction. In discussing the dynamics, we have transferred the motion of a two-body system into the motion of an effective one-body system, by the means of canonical transformation. By matching the effective action $\tilde{S}$ to the real action S, we have obtained an effective metric $d\tilde{s}^2$, which is a deformation of the Schwarzschild metric with mass $M = m_1 + m_2$. From this metric, we have derived expressions of the radius and the angular momentum of the effective particle moving in the ISCO of this metric. Finally, for completeness, we have computed the explicit canonical transformation that matches the real problem to the effective problem at 2PN correction. This effective one-body approach illustrates a way to reduce the complicated relativistic two-body problem to a rather simple effective one-body problem at 2PN correction. However, further work can be done to extend this approach to higher-order correction. Furthermore, this approach may also be extended by adding the electromagnetic interaction and the spin to the two-body system.

We have then discussed the fields and the related potentials of a gravitating source. We have defined the field $h^{\mu\nu}$ and have computed Einstein's equation in the harmonic coordinate. Together with the harmonic condition, we have derived the solution for Einstein's equation in both interior and exterior zones. The solutions are represented by the multipole expansion and are decomposed into the irreducible representations of the SO(3) rotation group. We have identified the two sets of functions M_L and S_L to be the mass multipole moments and the current multipole moments respectively. By matching the solutions of Einstein's equation in the overlapping zone, we have expressed the multipole moments in terms of the stress-energy tensor of the gravitating source. These expressions are mathematically well defined. They are significant in deducing the explicit gravitational waveform of a given source and computing the energy associated. Further work has been done in applying these expressions to the inspiralling compact binaries [1].

Appendix A

Decomposition of the Product of Representations

In this appendix, we give a proof of Eq. (4.4), copied as below

$$U_i \hat{V}_L = \hat{A}^{(+1)}_{iL} + \frac{l}{l+1} \epsilon_{ai\langle i_l} \hat{A}^{(0)}_{L-1\rangle a} + \frac{2l-1}{2l+1} \delta_{i\langle i_l} \hat{A}^{(-1)}_{L-1\rangle} \tag{A.1}$$

where

$$\hat{A}^{(+1)}_{iL} = U_{\langle i_l} \hat{V}_{iL-1\rangle} \tag{A.2a}$$

$$\hat{A}^{(0)}_{L-1a} = U_p \hat{V}_{q\langle L-1} \epsilon_{a\rangle pq} \tag{A.2b}$$

$$\hat{A}^{(-1)}_{L-1} = U_p \hat{V}_{pL-1} \tag{A.2c}$$

In order to write the following derivation conveniently, we introduce a notation for indices, where, for a positive integer $m \leq l$, $U_{L/i_m = U_{i_1 \cdots i_{m-1} i_{m+1} \cdots i_l}}$ and, for distinct positive integers $m, n \leq l$,

$$U_{L/(i_m i_n)} = \begin{cases} U_{i_1 \ldots i_{m-1} i_{m+1} \ldots i_{n-1} i_{n+1} \ldots i_l} & \text{if m < n} \\ U_{i_1 \ldots i_{n-1} i_{n+1} \ldots i_{m-1} i_{m+1} \ldots i_l} & \text{if m > n} \end{cases} \tag{A.3}$$

For a STF tensor $\hat{V}_L$, we notice that if the tensor has repeated indices, it must be zero, i.e.

$$\hat{V}_{bbL-2} = 0 \tag{A.4}$$

This is simply because we have removed all the traces of the tensor, therefore, any trace of a STF tensor must be zero.
We can then evaluate each of the term in Eq. (A.1) by Eq. (4.1). For $\hat{A}^{(+1)}_{iL}$, we have

$$\hat{A}^{(+1)}_{iL} = U_{(i_l} \hat{V}_{iL-1)} + a_1^{l+1} \delta_{(i_1 i} K^{(+1)}_{i_2 \ldots i_l)bb} \tag{A.5}$$

where

$$K^{(+1)}_{i_2 \ldots i_l bb} = U_{(b} \hat{V}_{bi_2 \ldots i_l)} := \sum_{\textit{all permutations}} U_{i_{a_1}} \hat{V}_{i_{a_2} \ldots i_{a_{l+1}}} \tag{A.6}$$

However, from Eq. (A.4), we notice that among all $(l+1)!$ permutations, only $2(l-1)!$ ones are non-zero. We also notice that, since $\hat{V}_L$ is symmetric, $\hat{V}_L = \hat{V}_{i_{a_1}\dots i_{a_l}}$. Therefore, we obtain

$$K^{(+1)}_{i_2\dots i_l bb} = \frac{2}{l+1} U_b \hat{V}_{bi_2\dots i_l} \tag{A.7}$$

Now we consider the last term in Eq. (A.5). One should be careful with the position of the i index. Since $\hat{V}_L$ is totally symmetric, once the indices in Kronecker Delta are chosen, there will be $2(l-1)!$ numerically identical expressions of $K^{(+1)}_{i_{a_1}\dots i_{a_{l-1}} bb}$. Therefore, we can write the last term in Eq. (A.5) as

$$\begin{aligned} a_1^{l+1}\delta_{(i_1 i}K^{(+1)}_{i_2\dots i_l)bb} = &\sum_{m=1}^{l} \frac{2}{l+1}\frac{2(l-1)!}{(l+1)!} a_1^{l+1}\delta_{ii_m} U_b \hat{V}_{bL/i_m} \\ &+ \sum_{c=1}^{l}\sum_{d\neq c} \frac{2}{l+1}\frac{(l-1)!}{(l+1)!} a_1^{l+1}\delta_{i_c i_d} U_b \hat{V}_{biL/(i_c i_d)} \end{aligned} \tag{A.8}$$

There is an extra factor of 2 in the first expression on the RHS of Eq. (A.8) because $\delta_{ii_m} = \delta_{i_m i}$. However, in the second term, we already include both $\delta_{i_c i_d}$ and $\delta_{i_d i_c}$ through the double sum. Similar discussion applies for the first term on the RHS of Eq. (A.5). We then have

$$U_{(i_l}\hat{V}_{iL-1)} = \frac{1}{l+1}U_i\hat{V}_L + \sum_{m=1}^{l}\frac{1}{l+1}U_{i_m}\hat{V}_{iL/i_m} \tag{A.9}$$

For the second term on the right hand side of Eq. (A.1), we firstly consider $\hat{A}^{(0)}_{L-1a}$. From Eq. (4.1), we have

$$\hat{A}^{(0)}_{L-1a} = U_p\hat{V}_{q(L-1}\epsilon_{a)pq} + a_1^l U_p \delta_{(ai_{l-1}}K^{(0)}_{L-2)bbp} \tag{A.10}$$

where

$$K^{(0)}_{L-2bbp} = \hat{V}_{q(L-2b}\epsilon_{b)pq} = \frac{2}{l}\hat{V}_{L-2bq}\epsilon_{bpq} \tag{A.11}$$

The extra factor of $\frac{2}{l}$ in the last term of Eq. (A.11) comes from $2(l-2)!$ non-zero permutations among $l!$ permutations, where we have applied the same discussion as what we do in Eq. (A.7). Inserting Eq. (A.11) into Eq. (A.10) yields

$$\begin{aligned} \hat{A}^{(0)}_{L-1a} =& \frac{1}{l}U_p\hat{V}_{qL-1}\epsilon_{apq} + \frac{1}{l}\sum_{m=1}^{l-1} U_p\epsilon_{i_m pq}\hat{V}_{aqL-1/i_m} \\ &+ \frac{4}{l^2(l-1)}a_1^l U_p \sum_{m=1}^{l-1}\delta_{ai_m}\hat{V}_{L-1/i_m bq}\epsilon_{bqp} \\ &+ \frac{4}{l^2(l-1)}a_1^l U_p \sum_{c=1}^{l-1}\sum_{d\neq c}\delta_{i_c i_d}\hat{V}_{L-1/(i_c i_d)abq}\epsilon_{bpq} \end{aligned} \tag{A.12}$$

One can easily check that the last two terms in Eq. (A.12) is zero, because ϵ_{bpq} is totally anti-symmetric but $\hat{V}_{L-1/i_m bq}$ and $\hat{V}_{L-1/(i_c i_d)abq}$ is symmetric with respect to indices b and q. Inserting Eq. (A.12) into the second term on the RHS of Eq. (A.1) yields

$$\frac{l}{l+1}\epsilon_{ai\langle i_l}\hat{A}^{(0)}_{L-1\rangle a} = I_1 + I_2 \tag{A.13}$$

where

$$I_1 = \frac{1}{l+1}\epsilon_{ai\langle i_l}\hat{V}_{L-1\rangle q}U_p\epsilon_{apq} \tag{A.14a}$$

$$I_2 = \frac{1}{l+1}\sum_{m=1}^{l-1}\epsilon_{ai\langle i_l}\epsilon_{i_m|pq|}\hat{V}_{L-1/i_m\rangle aq}U_p \tag{A.14b}$$

Using Eq. (4.1) and the discussion used in Eq. (A.7), we obtain

$$I_1 = \frac{1}{l+1}\epsilon_{ai(i_l}\hat{V}_{L-1)q}U_p\epsilon_{apq} + \frac{1}{l+1}\frac{2}{l}a_1^l\epsilon_{aib}\epsilon_{apq}\delta_{(i_l i_{l-1}}\hat{V}_{L-2)qb}U_p \tag{A.15}$$

When expanding I_2, we will obtain a term which contains $\epsilon_{ai(b}\epsilon_{b|pq|}\hat{V}_{L-2)qa}$. One can easily notice that either $\epsilon_{aib}\epsilon_{i_m|pq|}\hat{V}_{L-2/i_m bqa}$ or $\epsilon_{aii_m}\epsilon_{b|pq|}\hat{V}_{L-2/i_m bqa}$ are zero due to the properties of Levi-Civita tensor. Therefore, we obtain

$$\epsilon_{ai(b}\epsilon_{b|pq|}\hat{V}_{L-2)qa} = \frac{2}{l(l-1)}\epsilon_{aib}\epsilon_{b|pq|}\hat{V}_{L-2qa} \tag{A.16}$$

From Eq. (A.16), we have

$$\begin{aligned} I_2 = &\sum_{m=1}^{l-1}\frac{1}{l+1}\epsilon_{ai(i_l}\epsilon_{i_{l-1}|pq|}\hat{V}_{L-2)aq}U_p \\ &+ \sum_{m=1}^{l-1}\frac{1}{l+1}\frac{2}{l(l-1)}a_1^l\epsilon_{aib}\epsilon_{bpq}\delta_{(i_l i_{l-1}}\hat{V}_{L-2)qa}U_p \end{aligned} \tag{A.17}$$

We notice that the dummy index m disappears in Eq. (A.17), therefore, we can simplify Eq. (A.17)

$$I_2 = \frac{l-1}{l+1}\epsilon_{ai(i_l}\epsilon_{i_{l-1}|pq|}\hat{V}_{L-2)aq}U_p + \frac{1}{l+1}\frac{2}{l}a_1^l\epsilon_{aib}\epsilon_{bpq}\delta_{(i_l i_{l-1}}\hat{V}_{L-2)qa}U_p \tag{A.18}$$

By switching the indices a and b in the last term in Eq. (A.15), one can easily see that the last term in Eq. (A.15) cancels with the last term in Eq. (A.18). We can now simplify Eq, (A.13)

$$\frac{l}{l+1}\epsilon_{ai\langle i_l}\hat{A}^{(0)}_{L-1\rangle a} = \frac{1}{l+1}\epsilon_{ai(i_l}V_{L-1)q}U_p\epsilon_{apq} + \frac{l-1}{l+1}\epsilon_{ai(i_l}\epsilon_{i_{l-1}|pq|}\hat{V}_{L-2)aq}U_p \tag{A.19}$$

We can further expand Eq. (A.19)

$$\frac{l}{l+1}\epsilon_{ai\langle i_l}\hat{A}^{(0)}_{L-1\rangle a} = \frac{1}{(l+1)l}\sum_{m=1}^{l}\epsilon_{apq}\epsilon_{aii_m}\hat{V}_{L/i_m q}U_p + \frac{l-1}{l+1}\frac{1}{l(l-1)}\sum_{c=1}^{l}\sum_{d\neq c}\epsilon_{aii_c}\epsilon_{pqi_d}\hat{V}_{L/(i_c i_d)aq}U_p \tag{A.20}$$

Recall the epsilon-delta identities, where $\epsilon_{aij}\epsilon_{apq} = 2!\delta^i_{[p}\delta^j_{q]}$ and $\epsilon_{aij}\epsilon_{bpq} = 3!\delta^a_{[b}\delta^i_p\delta^j_{q]}$. Therefore, we obtain

$$\begin{aligned}\frac{l}{l+1}\epsilon_{ai\langle i_l}\hat{A}^{(0)}_{L-1\rangle a} =& \frac{1}{l+1}U_i\hat{V}_L - \frac{1}{(l+1)l}\sum_{m=1}^{l}U_{i_m}\hat{V}_{iL/i_m} \\ &+ \frac{1}{(l+1)l}\sum_{c=1}^{l}\sum_{d\neq c}[U_i\hat{V}_L - U_{i_c}\hat{V}_{iL/i_c} \\ &+ \delta_{i_c i_d}\hat{V}_{ipL/(i_c i_d)}U_p - \delta_{ii_d}\hat{V}_{L/i_d}U_p]\end{aligned} \tag{A.21}$$

Further simplifying Eq. (A.21) gives

$$\begin{aligned}\frac{l}{l+1}\epsilon_{ai\langle i_l}\hat{A}^{(0)}_{L-1\rangle a} =& \frac{l}{l+1}U_i\hat{V}_L - \frac{1}{l+1}\sum_{m=1}^{l}U_{i_m}\hat{V}_{iL/i_m} \\ &+ \frac{1}{(l+1)l}\sum_{c=1}^{l}\sum_{d\neq c}\delta_{i_c i_d}\hat{V}_{ipL/(i_c i_d)}U_p - \frac{l-1}{(l+1)l}\sum_{c=1}^{l}\delta_{ii_c}\hat{V}_{L/i_c}U_p\end{aligned} \tag{A.22}$$

Now let us consider the last term in Eq. (A.1). From Eq. (A.2), it can be written as

$$\frac{2l-1}{2l+1}\delta_{i\langle i_l}\hat{A}^{(-1)}_{L-1\rangle} = \frac{2l-1}{2l+1}\delta_{i\langle i_l}\hat{V}_{L-1\rangle p}U_p \tag{A.23}$$

Using Eq. (4.1), we obtain

$$\frac{2l-1}{2l+1}\delta_{i\langle i_l}\hat{A}^{(-1)}_{L-1\rangle} = \frac{2l-1}{2l+1}U_p\delta_{i(i_l}\hat{V}_{L-1)p} + \frac{2l-1}{2l+1}\frac{2}{l}a^l_1\delta_{ib}\delta_{(i_l i_{l-1}}\hat{V}_{L-2)ap}U_p \tag{A.24}$$

Further expanding Eq. (A.24) gives

$$\frac{2l-1}{2l+1}\delta_{i\langle i_l}\hat{A}^{(-1)}_{L-1\rangle} = \frac{2l-1}{2l+1}\frac{1}{l}\sum_{m=1}^{l}U_p\delta_{ii_m}\hat{V}_{L/i_m p} + \frac{2(2l-1)}{(2l+1)l}\frac{1}{l(l-1)}a^l_1\sum_{c=1}^{l}\sum_{d\neq c}\delta_{i_c i_d}\hat{V}_{L/(i_c i_d)p}U_p \tag{A.25}$$

We then insert Eq. (A.8), Eq. (A.9), Eq. (A.22), and Eq. (A.25) into Eq. (A.1) and obtain

$$\begin{aligned}U_i\hat{V}_L =& U_i\hat{V}_L \\ &+ [\frac{4}{(l+1)^2 l}a_1^{l+1} - \frac{l-1}{(l+1)l} + \frac{2l-1}{(2l+1)l}]\sum_{m=1}^{l} U_p \delta_{i i_m} \hat{V}_{L/i_m p} \\ &+ [\frac{2}{(l+1)^2 l}a_1^{l+1} + \frac{1}{(l+1)l} + \frac{2(2l-1)}{(2l+1)l^2(l-1)}a_1^l]\sum_{c=1}^{l}\sum_{d\neq c}\delta_{i_c i_d}\hat{V}_{L/(i_c i_d)p}U_p\end{aligned} \tag{A.26}$$

From Eq. (4.3), for any integer $p > 1$, we can evaluate

$$a_1^p = \frac{-p(p-1)}{2(2p-1)} \tag{A.27}$$

Inserting Eq. (A.27) into Eq. (A.26), one can easily check

$$\frac{4}{(l+1)^2 l}a_1^{l+1} - \frac{l-1}{(l+1)l} + \frac{2l-1}{(2l+1)l} = 0 \tag{A.28a}$$

$$\frac{2}{(l+1)^2 l}a_1^{l+1} + \frac{1}{(l+1)l} + \frac{2(2l-1)}{(2l+1)l^2(l-1)}a_1^l = 0 \tag{A.28b}$$

Therefore, we have proved Eq. (A.1).

Appendix B

Some Useful Equations and Their Proofs

We here introduce a set of equations that are useful in finding the general multipole expansion solution for the vacuum Einstein's equation.

$$\hat{\partial}_L g(r) = \hat{n}_L (2r)^l \frac{\partial^l}{\partial (r^2)^l} g(r) \tag{B.1}$$

$$\triangle \hat{n}_L = -l(l+1) r^{-2} \hat{n}_L \tag{B.2}$$

$$\hat{\partial}_L r^{-k} = (-1)^l \frac{(k+2l-2)!!}{(k-2)!!} (r^{-(k+l)} \hat{n}_L) \tag{B.3}$$

$$\hat{\partial}_L (\frac{G(t-r)}{r}) = (-1)^l \hat{n}_L \sum_{n=0}^{l} \frac{(l+n)!}{2^n n!(l-n)!} \frac{G^{(l-n)}(t-r)}{r^{n+1}} \tag{B.4}$$

where, in this appendix, $G^{(n)}(x) = \frac{d^n}{dx^n} G(x)$, We now give detail proofs to Eq. (B.1) to Eq. (B.4).

To prove Eq. (B.1), one should notice that the function $g(r)$ solely depends on r and, therefore, use $\partial_{i_m} g(r) = \partial_{i_m} r \partial_r g(r)$. The left hand side of Eq. (B.1) then becomes

$$\hat{\partial}_L g(r) = \mathrm{STF}_L [(\prod_{m=1}^{l} n_{i_m} \partial_r) g(r)] \tag{B.5}$$

where we have used $\partial_{i_m} r = n_{i_m}$. Now, one can easily check that Eq. (B.5) can be further simplified

$$\hat{\partial}_L g(r) = \hat{x}_L (\frac{1}{r} \partial_r)^l g(r) \tag{B.6}$$

which is identical to Eq. (B.1).
In order to prove Eq. (B.2), one should notice that

$$\partial_s n_{i_m} = \frac{1}{r}(\delta_{i_m s} - n_{i_m} n_s) \tag{B.7a}$$

$$\partial_s n_s = \frac{2}{r} \tag{B.7b}$$

We then rewrite the Laplace operator as $\triangle = \partial_s \partial_s$. Using Eq. (B.7a), one can expand the left hand side of Eq. (B.2)

$$\triangle \hat{n}_L = \mathrm{STF}_L[\partial_s(\sum_{m=0}^{l} \frac{1}{r}\delta_{i_m s} n_{L/m} - \frac{l}{r} n_s n_L)] \tag{B.8}$$

which can be further expanded

$$\begin{aligned}\triangle \hat{n}_L = \mathrm{STF}_L[&\sum_{m=0}^{l}(\partial_s \frac{1}{r}) n_{L/m}\delta_{i_m s} + \sum_{m=0}^{l}\sum_{n\neq m}\frac{1}{r}\delta_{i_m s}(\partial_s n_{i_n}) n_{L/(mn)} \\ &- (\partial_s \frac{1}{r}) l n_s n_L - \frac{2l}{r^2} n_L - \frac{l}{r} n_s(\partial_s n_L)]\end{aligned} \tag{B.9}$$

One should note that the second term on the right hand side of Eq. (B.9) gives a term of $\frac{1}{r^2}\sum_{m=0}^{l}\sum_{n\neq m}\delta_{i_m s}\delta_{i_n s} n_{L/(mn)}$, which is clearly zero since m and n are distinct. Simplifying Eq. (B.9) using Eq. (B.7) gives exactly the same equation as expressed in Eq. (B.2).
To prove Eq. (B.3), one can use Eq. (B.6) and let $g(r) = r^{-k}$. Therefore, one can directly find

$$\hat{\partial}_L r^{-k} = r^l \hat{n}_L (\frac{1}{r}\partial_r)^l r^{-k} \tag{B.10}$$

We therefore obtain

$$\hat{\partial}_L r^{-k} = (-1)^l \frac{(k+2l-2)!!}{(k-2)!!}(r^{-(k+l)}\hat{n}_L) \tag{B.11}$$

Finally, we use induction to prove Eq. (B.4). Using Eq. (B.6), one can find

$$\hat{\partial}_L(\frac{G(t-r)}{r}) = \hat{n}_L r^l (\frac{1}{r}\partial_r)^l (\frac{G(t-r)}{r}) \tag{B.12}$$

Therefore, proving Eq. (B.4) is equivalent to proving

$$(\frac{1}{r}\partial_r)^l(\frac{G(t-r)}{r}) = (-1)^l \sum_{n=0}^{l}\frac{(l+n)!}{2^n n!(l-n)!}\frac{G^{(l-n)}(t-r)}{r^{n+l+1}} \tag{B.13}$$

Now, we use induction to prove Eq. (B.13). One can easily see

$$(\frac{1}{r}\partial_r)(\frac{G(t-r)}{r}) = -\frac{1}{r}(\frac{G^{(1)}(t-r)}{r} + \frac{G(t-r)}{r^2}) \tag{B.14}$$

Now, assuming Eq. (B.13) to be correct, we need to prove

$$(\frac{1}{r}\partial_r)^{(l+1)}(\frac{G(t-r)}{r}) = (-1)^{(l+1)}\sum_{n=0}^{l+1}\frac{(l+n+1)!}{2^n n!(l-n+1)!}\frac{G^{(l-n+1)}(t-r)}{r^{n+l+2}} \quad \text{(B.15)}$$

The left hand side of Eq. (B.15) can be rewritten as

$$(\frac{1}{r}\partial_r)^{(l+1)}(\frac{G(t-r)}{r}) = -(\frac{1}{r}\partial_r)^l[\frac{1}{r}K(t-r)] \quad \text{(B.16)}$$

where we have defined $K(t-r) := -\partial_r(\frac{G(t-r)}{r})$. One can now apply Eq. (B.4) to Eq. (B.16) and obtain

$$(\frac{1}{r}\partial_r)^{(l+1)}(\frac{G(t-r)}{r}) = (-1)^{(l+1)}\sum_{n=0}^{l}\frac{(l+n)!}{2^n n!(l-n)!}\frac{K^{(l-n)}(t-r)}{r^{n+l+1}} \quad \text{(B.17)}$$

Recall that Leibniz formula gives $(UV)^{(j)} = \sum_{i=0}^{j}\frac{j!}{i!(j-i)!}U^{(i)}V^{(j-i)}$. Applying Leibniz formula to $K^{(l-n)}(t-r)$, we obtain

$$K^{(l-n)}(t-r) = \sum_{i=0}^{l-n+1}\frac{(l-n+1)!}{i!}G^{(i)}(t-r)r^{n+i-l-2} \quad \text{(B.18)}$$

Inserting Eq. (B.18) into Eq. (B.17) gives

$$(\frac{1}{r}\partial_r)^{(l+1)}(\frac{G(t-r)}{r}) = (-1)^{l+1}\sum_{n=0}^{l}\sum_{i=0}^{l-n+1}\frac{(l+n)!(l-n+1)}{i!2^n n!}G^{(i)}(t-r)r^{-2l+i-3} \quad \text{(B.19)}$$

By carefully analyzing Eq. (B.19), one can find that the coefficient for term $G^{(l-n+1)}(t-r)r^{-l-n-2}$ is $(-1)^l\frac{1}{(l-n+1)!}\sum_{k=0}^{n}\frac{(l+k)!(l-k+1)}{2^k k!}$. Recall that

$$\sum_{k=0}^{n}(l+k)!(l-k+1)2^{n-k}\frac{n!}{k!} = (l+n+1)! \quad \text{(B.20)}$$

Therefore, the coefficient for $G^{(l-n+1)}(t-r)r^{-l-n-2}$ is $(-1)^{l+1}\frac{(l+n+1)!}{2^n n!(l-n+1)!}$, which agrees with Eq. (B.4).

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