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Topological Magnetic Monopoles and Electroweak Sphalerons

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Abstract

Topological solitons (also solitons) are solutions of the field equations with finite energy and non-trivial topology sutures. Unlike the elementary particles which own the wave-like property, they can be treated classically with non-linear behaviours. In this dissertation, the 't Hooft-Polyakov monopoles in the Georgi-Glashow $SU(2)$ model and sphalerons in Weinberg-Salam electroweak theory are examined carefully as two solitons. Their topological structures and field configurations are illustrated and enable us to use the spherical ansatz to solve their equations of motion, where the numerical methods are applied instead of analytical ones. To check the validity of the numerical results, their energies are calculated and comparisons are made with the existing analytical results. The lattice discretization method is also employed to find monopole energies, in which way the magnetic Coulomb effect of virtual monopole-antimonopole pairs can be accounted for. With the finite-size effect inspected closely, we conclude that it is important to treat the lattice spacing carefully and suitable values should always be applied to get the right energies.

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Contents

1 Introduction

1.1 What are monopoles?

A magnetic monopole is a hypothetical particle that behaves like an isolated magnetic pole (north/south) without its counterpart (south/north). Magnets are widely used in our world and we are very familiar with it that we know from childhood that magnets have two poles, north and south. But why is it the case and why no magnet simply exists as a sole magnetic pole (monopole)? The answer to it is still not known yet and no theory gives proof for its non-existence.

In topological interpretation, magnetic monopoles are solitons [1] with non-trivial topological structures, which means they are homotopically distinct from the vacuum. While the elementary particles of quantum field theory are the ones with no topological structure, which are known as the quantised form of wave-like excitation of the fields, such as photons and leptons, with linear field equations. The one-to-one correspondence between particles and solutions of fields also applies to solitons, and it can be interpreted as the solutions of the classical fields equations, with some nonlinear character. The energy density of soliton is localised in a finite region in the smooth form, and this field configuration gives the uniqueness of the soliton.

The topological nature of the particles are usually classified by the topological charge, or the topological degree N, which is an integer and it is also the generalised winding number. Some examples of solitons in different dimensions are kinks of ϕ^4 and sine-Gordon model in one-dimensional space [2, 3], sigma model lumps [4] and the (Baby) Skyrmions [5] in 2D space, while Skyrmions [6, 7], 't Hooft-Polyakov monopoles [8, 9], and electroweak sphalerons [10, 11] are solitons in $\mathbb{R}^3$, and there are instantons in 4-dimensional Euclidean spacetime [12] . Detailed information about other solitons except for monopoles and electroweak sphalerons will not be examined in this dissertation, and the book [13] contains comprehensive information for beginners is recommended for interested readers.

1.2 History and motivation of finding monopoles

Magnetism is of continuous inspection since the discovery of lodestone, people found it can attract iron but no other metals [14]. also, the lodestone always aligns itself to a certain direction, i.e. north and south, when it is suspended by a string. William Gilbert drew the conclusion that the origin of the force which aligns the magnetized needle in a certain way is the earth, publishing in his 1600 book *De Magnete* [15].

In the 19th-century, Ampere found the magnetism is produced by electric current, and Faraday confirmed the non-existence of magnetic fluids. In 1864, Maxwell made the great breakthrough establishing electromagnetism [16] through Maxwell's equations, where the magnetic charges are assumed to not exist in Maxwell's theory. But on the other side, the duality symmetry of electric and magnetic charge in Maxwell equations seems to provide a hint for the existence of monopole. And if the magnetic charge and electric charge can be treated equally, the magnetic charges and fields will behave in the the same way as its electric counterpart, that a universe with magnetic charges replacing electric charges will be the same as ours.

With the development of quantum mechanics at the beginning of the 20th century, magnetic charges also seemed to be inconsistent with it. But later in 1931, Paul Dirac proposed that certain magnetic charges can exist in quantum mechanics, where they need to be quantised [17]. Moreover, the existence could give an explanation of the quantised electric charge. After the standard model of $SU(3)_C \times SU(2)_L \times U(1)_Y$ is established, the electroweak phenomenons are successfully explained. Then different genres of the Grand Unified Theories (GUTs) in a very high energy regime were brought out by physicists, such as the (minimal) $SU(5)$ Georgi-Glashow model and $Spin(10)$ grand unification theories, which aims to unify the strong nuclear force and the electroweak force. Though $SU(5)$ model has been ruled out by experiment showed the longer lifetime of proton decay.

In 1974, Gerard 't Hooft [8] and Alexander Polyakov [9] showed it is possible that magnetic monopoles exist in a model similar to the electroweak theory within the standard model but with 3-real-component adjoint Higgs field instead of 2-complex components. But since the fact that the standard model predicts our world with elementary particles in a very precise way, which is up to 10 significant figures using electrodynamics, the possibility for 't Hooft and Polyakov monopole existing the standard model is ruled out. However, the standard model is not a theory of everything. It cannot explain many fundamental problems, such as dark matter, cosmological inflation and the gravitational force is missed from the theory. In contrast, monopoles should exist in all the GUTs, and the forefront of modern theories e.g. superstring theory also predicts the magnetic charges [18]. If monopoles were discovered, it would be a great step forward in the high energy physics area. More attractively, due to the strong interaction between monopoles and electromagnetic fields, experiments based on monopoles could be designed to test grand unified theories and even superstring theories.

1.3 Experimental status of searching monopoles

The existence of magnetic monopole has not been discovered by experiment yet, but there is no reason to against it from the view of theoretical completeness. The GUT monopole was predicted to exist in all grand unified theories has a mass of about 10^{17} GeV, but the Large Hadron Collider (LHC) at CERN can only provide up to maximum 13 TeV energy using a proton-proton collision, which is the largest energy human can achieve using collider currently. So there is no way to produce or test GUT monopoles using modern accelerators. However, the elementary monopoles produced from electroweak symmetry breaking have a much lower mass (intermediate-mass monopole), about 10^4 GeV [19–22], which is comparable to the energy scale that LHC can achieve. But unfortunately semi-classical calculation shows that the production rate of 't Hooft-Polyakov monopoles (intermediate-mass/composite monopoles) during collisions of point-like particles will be suppressed by a huge factor of $\geq 10^{30}$, which makes it not feasible to produce composite monopoles in practice [23]. Some recent numerical lattice field theory simulations are also applied to account for the effect of virtual monopole-antimonopole pairs and important progress has been made in terms of the interaction and mass form factors [24–26].

On the other hand, if monopoles are the elementary particles that have no relation with grand unified theories. It would be possible to have them with mass below 10^4 GeV (elementary monopoles), and hence can be produced in LHC. The MoEDAL project is designed especially for tracking possible elementary monopoles produced through collisions in LHCb experiments. Some similar detectors have been set before, such as Tevatron, LEP and HERA. But none of them found monopoles successfully, which basically ruled out the probability for finding monopole with mass below 10^4 GeV [27].

Monopoles are considered to be stable once they are produce, which only annihilate with antimonopoles. It is proposed the (anti) monopoles together with other elementary particles and intermediate-mass monopoles are produced through our early universe in a stage of symmetry breaking cosmological phase transitions (GUT phase transition) at high temperature. After which, monopoles and antimonopoles efficiently annihilate with each other as the temperature falls.

In the traditional Big Bang Theory, where there is no inflation, it is required the mass density produced by the monopoles in the early universe is lower than the limit of mass density derived from cosmic observation. However, for the GUT monopoles $\sim 10^{17}$ GeV/c^2, a much larger mass density would be produced than the limit, but for a GUT monopole with a mass lower than 10^{10} GeV/c^2, the limit imposed on mass density would be satisfied [28], and this is what we generally accepted. This mass discrepancy leads to the so-called monopole problem, which made the traditional Big Bang theory defective, as it does not allow a grand unified situation to happen.

To solve this problem, it is proposed that our universe went through an accelerating inflation stage after the GUT phase transition [29], in which process the monopole density got diluted to an extremely low value. After which, the universe cannot reach a temperature as high as the phase transitions stage though reheats exist. The present-day measurements based on cosmic microwave background radiation [30] and other experiments' results suggest the temperature after the acceleration do have an upper limit around 10^{16}GeV/k_B, supported this the cosmological inflation argument. However, there are also other models, suggesting the inflation happened before (or at the beginning of) GUT phase transition [31].

Due to the stability of monopoles, we should expect to find the relic monopoles today, most likely in the cosmic rays or trapped in some long-lasting olden materials, like moonstone or seawater. But research on the latter did not manage to get hints for the monopoles [32]. Meanwhile, the former method gives an important bound for possible expected monopole flux. If there are monopoles in the cosmic rays, they will get accelerated by magnetic field and cause the field lose energy. The higher magnetic field strength is, the lower the flux of monopole we would expect. The astronomic observation of a magnetic field $\sim$3 μG, setting the upper limit of the flux,

$$F = \begin{cases} 10^{-15}\ \mathrm{cm}^{-2}\ \mathrm{sec}^{-1}\ \mathrm{sr}^{-1}, & M_m \leq 10^{17}\mathrm{GeV}, \\ 10^{-15}(\frac{M_m}{10^{17}\mathrm{GeV}})\ \mathrm{cm}^{-2}\ \mathrm{sec}^{-1}\ \mathrm{sr}^{-1}, & M_m \geq 10^{17}\mathrm{GeV}, \end{cases} \tag{1.1}$$

known as the Parker bound [33, 34], where M_m is the mass of monopole. Similarly, other experiments carried out all over the world also made contributions to the upper limit of magnetic flux. These experiment include MARCO [35] in Italy; AMANDA [36], Bailkal [37] and ANTARES based on neutrino telescope; RICE [38], ANITA and IceCube at the South Pole. Detailed limits are plotted and compared in [39, 40].

1.4 Dissertation aim and outline

This dissertation aims to review two topological solitons, monopoles and sphalerons. It begins with the general topological concept of solitons and specifies the topological nature of our research objects. Then the history and current searching status of monopoles are elaborated, as well as their classifications. The direct search includes particle colliders aiming to produce intermediate-mass monopoles and experiments like MoEDAL searching the hints of monopoles. The indirect search is from astronomic observations, such as MARCO, together with Parker bound, also provide upper limits of monopole flux in the cosmology.

In the main part of the essay, the Dirac monopole is introduced as a starter, then our main focus will be on the 't Hooft-Polyakov monopole, which is static solution existing in Georgi-Glashow $SU(2)$ theory of $\mathbb{R}^3$. The finite energy configuration and non-trivial topology structure are the main reasons to account for its existence, and the Derrick argument [41] indicates that the stationary point of energy can possibly be a soliton solution. The spherical ansatz is used to solve the monopole solution, which reduces the dimensions and turns two partial differential equations (PDEs) into ordinary differential equations (ODEs). Particularly, solving the ODEs will be categorized to boundary value problem and can be solved by numerical approach, where shooting method is applied using *Wolfram Mathematica 12.3*. Regarding the monopole energy (or equivalently mass), numerical integration and lattice discretization method are conducted, where the lattice calculations are performed on Imperial College's HPC (High Performance Computing), and the finite-size effect of lattice simulation is demonstrated. The properties of 't Hooft-Polyakov monopoles and the energy lower bound imposed on Bogomolny-Prasad-Sommerfield (BPS) limit [42, 43] are discussed afterwards. The asymptotic magnetic field outside the core region of a monopole/antimonopole can be treated abelianized. This leads to a well-defined total magnetic charge assigned similarly to the electric charge. Hence the magnetic Coulomb force exists between monopoles (antimonopoles) and it is expected to behave in the same way as the electric Coulomb force. A generalization of monopole with electric charge, known as dyon, together with its properties are then presented.

The remaining part of the dissertation gives a basic introduction of sphalerons in Weinberg-Salam electroweak theory [11], which is a saddle point solution over the energy function manifold for all field configurations. The energies of sphalerons are also calculated using the numerical shooting method the same as monopoles, and comparisons are made between the numerical method and analytical method [10].

2 Background

2.1 General theory of electromagnetism

The Maxwell's theory of electromagnetism assumes the absence of magnetic monopoles. Maxwell's equations which describe electromagnetic fields without electric charges or magnetic charges are,

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = 0, \tag{2.1a}$$

$$\overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}, \tag{2.1b}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0, \tag{2.1c}$$

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \frac{\partial \overrightarrow{E}}{\partial t}. \tag{2.1d}$$

However, if we assume the existence of magnetic charge in a similar way as the electric charge and embed both of them into Maxwell' equations, we would have,

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = \rho_e, \tag{2.2a}$$

$$\overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} - \overrightarrow{J}_m, \tag{2.2b}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = \rho_m, \tag{2.2c}$$

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \frac{\partial \overrightarrow{E}}{\partial t} + \overrightarrow{J}_e. \tag{2.2d}$$

It can be seen when replacing $\overrightarrow{E}$ with $\overrightarrow{B}$, and$\overrightarrow{B}$ to $-\overrightarrow{E}$, the equations stay unchanged, which means electric and magnetic fields behave the same, and this implies the duality symmetry between electric $\overrightarrow{E}$ field and magnetic $\overrightarrow{B}$.

If we apply the definition of magnetic charge the same as the electric charge, that for a point magnetic charge of strength q_m sitting at the origin, will generate the magnetic field $\overrightarrow{B}$,

$$\overrightarrow{B} = \frac{q_m}{4\pi r^2}\hat{x}, \tag{2.3}$$

with the static field direction points outwards along the radius. The magnetic flux out of a closed surface S, containing the origin, will be

$$\int_S \vec{B} \cdot d\vec{S} = q_m. \tag{2.4}$$

When including the scalar ϕ and vector A potentials of the gauge field,

$$\vec{B} = -\vec{\nabla} \times \vec{A}, \tag{2.5}$$

However, if one applies the equation (2.5) to (2.2c) and takes the divergence, it yields,

$$-\vec{\nabla}\cdot(\vec{\nabla} \times \vec{A}) = \rho_m = 0, \tag{2.6}$$

because the left-hand side vanishes naturally from vector calculus, which implies the non-existence of magnetic charge. Despite the failure of interpolating magnetic charge in the classical electromagnetic theory, Dirac demonstrated the existence possibility of quantised magnetic charge in quantum mechanics in his 1931 paper [17]. It is proposed that the magnetic charge could exist provided that the charge satisfies certain quantisation condition which we shall discuss in more detail in the following.

2.2 Dirac monopoles

Dirac illustrated the possibility of existence in his configuration, where the north and south monopoles are connected by an invisible Dirac string (can be imagined as an infinitesimally thin and long solenoid), which makes sure the continuity and confinement of the magnetic field lines between two monopoles. To realise this setup, the strength of the monopoles, i.e. the magnetic charges have to satisfy the *Dirac quantisation condition*,

$$q_m q_e = -2\pi\hbar N, \qquad N \in \mathbb{Z}. \tag{2.7}$$

If there is some particle with electric charge q_e, it has to be quantised by the condition(2.7), which we shall give proof later. And indeed, it is true that all the electric charges are quantised to be integer multiples of electron charge e. Though the Dirac monopole can be implemented in a quantisation theory, it can not be placed inside quantum electrodynamics yet as the magnetic charges are not quantised intrinsically. Meanwhile, another issue appears when introducing the Dirac string. The vector potential for the monopoles are

$$\overrightarrow{A}(\overrightarrow{r}) = \frac{q_m}{4\pi|\overrightarrow{r}|}\frac{\overrightarrow{r} \times \hat{k}}{|\overrightarrow{r}| - \overrightarrow{r} \cdot \hat{k}}, \tag{2.8}$$

where $\hat{k}$ is a unit vector in the direction along solenoid from the monopole. According to (2.8), there are singularities along the string where $|\overrightarrow{r}| = 0$. But Dirac pointed out that the vector potential A can be defined only in $\mathbb{R}^3 - \{0\}$, with the origin $|\overrightarrow{r}| = 0$ removed. In this case, A is actually a connection but not globally defined quantity. Two patches are enough to cover the area of $\mathbb{R}^3 - \{0\}$, and each of them is topologically trivial as they are both contractible.

Writing the gauge fields as differential forms, A is a differential 1-form.

$$A = A_i dx^i = A_1 dx^1 + A_2 dx^2 + \cdots + A_m dx^m, \tag{2.9}$$

where m is 3 in three dimensions, and the field strength F is a locally closed 2-form as A is not globally well-defined, which can be expressed as a summation of all the field strength tensor components,

$$F = dA = \sum_{i<j}(\partial_i A_j - \partial_j A_i) dx^i \wedge dx^j. \tag{2.10}$$

Putting the differential forms in spherical coordinate using (r, θ, φ) and a suitable solution of A^1, A^2 can be found as,

$$A^1 = \frac{q_m}{4\pi}(-1+\cos\theta)d\varphi, \tag{2.11}$$

$$A^2 = \frac{q_m}{4\pi}(1+\cos\theta)d\varphi, \tag{2.12}$$

$$d\alpha^{21} = -\frac{q_m}{2\pi}d\varphi, \tag{2.13}$$

where $\alpha^{21} = -\frac{q_m \phi}{2\pi}$. The vector potentials A^1, A^2 and the scalar potentials ϕ^1, ϕ^2 defined on the two covers only differ by a gauge transformation,

$$\overrightarrow{A}^2 = \overrightarrow{A}^1 - \overrightarrow{\nabla}\alpha^{21}, \tag{2.14}$$

$$\phi^2(x) = e^{-i\alpha^{21}}(x)\phi^1(x), \tag{2.15}$$

where $e^{-i\alpha^{21}} \in U(1)$ group.

To maintain the single value of $e^{-i\alpha^{21}}$ when $\phi = 2\pi$ and $\phi = 0$, a condition of $q_m = 2\pi N$, with N an integer, must be included. Considering field with smallest electric charge e, then $q_m e = 2\pi n$ should be the case. If the field related particle has an electric charge of $q_e = -e\hbar$, it yields the *Dirac quantisation condition* $q_m q_e = -2\pi N\hbar$ (2.7).
Expressing the field strength tensor F in a differential 2-forms,

$$F = dA^1 = dA^2 = -\frac{q_m}{4\pi}\sin\theta d\theta \wedge d\varphi. \tag{2.16}$$

It can be seen that the field strength is exactly $-q_m$ times the normalised volume form of 2-sphere S^2, thus the Dirac monopole is very likely to be spherically symmetric, which has also been confirmed by more rigorous mathematical consideration regarding the connection

1-form A. In conclusion, the Dirac monopole is defined symmetrically on the space of 2-sphere, with singularity on the origin removed, which is equivalent $\mathbb{R}^3 - \{0\}$.

More generally, the *Dirac quantisation conditions* is implied by topology analysis. The first Chern form for abelian gauge filed is defined as $C_1 = \frac{1}{2\pi}F$ and the first Chern number c_1 is the integral of the first Chern form. It has been shown [13] that for a compact Riemann surface X with no boundary,

$$c_1 = \frac{1}{2\pi}\int_X F = N, \tag{2.17}$$

with N an integer (monopole number), when locally defined gauge and scalar fields are considered instead of globally defined, whereas in the latter case c_1 will be zero implied by Stokes theorem,

$$c_1 = \frac{1}{2\pi}\int_X F = \frac{1}{2\pi}\int_{\partial X} A = 0. \tag{2.18}$$

The integration $\int_X F$ on a closed S^2 surface which contains the monopole is equal to $-q_m$, achieved by substituting the 2-form F in (2.16) to (2.17), so this yields to

$$q_m = -2\pi N, \tag{2.19}$$

which satisfies the *Dirac quantisation conditions* (2.7).

However, Dirac monopole is not an inevitable product that leads from quantum field theory, because the perturbation method no longer works for the Dirac monopole with a much larger fine-structure constant $\alpha_e = e^2/4\pi\hbar c\epsilon_0 \approx 1/137$ comparing with $\alpha_m = q_m^2\mu_0/4\pi\hbar c \approx 34$. Moreover, the Dirac monopole contains an invisible Dirac string between two poles, which makes it difficult to be compatible with Lorentz invariance and gauge invariance.

2.3 't Hooft and Polyakov monopoles

2.3.1 Non-abelian gauge theory and field configuration

In 1974, 't Hooft and Polyakov found monopoles can exist in non-abelian Yang-Mills gauge theory with singularities smoothed out [8, 9]. Though the pure Yang-Mills theory does not allow topological solitons in $\mathbb{R}^3$, but it is possible when Higgs fields get involved.

In the Georgi-Glashow theory, the $SU(2)$ gauge group spontaneously broke into $U(1)$ by Higgs Mechanism, and the monopole solutions can exist in this theory. The theory of $SU(2)$ broken into $U(1)$ had been a promising candidate for electroweak theory for a while but later it was confirmed to be failed by the experiment finding of neutral current mediated by Z bosons. What replaced it is the $SU(2) \times U(1)$ Glashow-Weinberg-Salam electroweak theory broken into $U(1)_{EM}$ with a complex doublet Higgs field. It successfully predicts and the existence of elementary particles with their masses, which are massless photons, massive $W^{\pm}$ bosons and Z bosons. What should be pointed out is that monopole solutions cannot exist in the electroweak theory, but it does allow sphaleron solutions which we shall discuss later.

The monopole solution of Georgi-Glashow $SU(2)$ model with the gauge potential A_μ coupled to an adjoint Higgs field Φ is considered here. Or equivalently, the model is the same with $SO(3)$ gauge group broken into $U(1)$, with a three-component Higgs field in its fundamental representation. The basis of Lie algebra $su(2)$ are chosen to be $\{t^a = \frac{1}{2}\sigma^a, a = 1, 2, 3\}$ (hermitian and traceless), where σ^a are the Pauli matrices, to write them out explicitly,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.20}$$

The commutation relation they obey is $[t^a, t^b] = if_{abc}t^c$, where $f_{abc} = \epsilon_{abc}$ is structure constant of $SU(2)$ group. The trace of generators, $Tr(t^at^b) = C\delta_{ab} = \frac{1}{2}\delta_{ab}$ (the convention followed here is $C = \frac{1}{2}$ for $SU(N)$ groups and C=2 for $SO(N)$ groups). Both the Higgs field and gauge field can be expressed in the $su(2)$ algebra, and written in their components forms are,

$$A_\mu = A_\mu^a t^a, \qquad \Phi = \Phi^a t^a. \tag{2.21}$$

where $\mu, \nu, \cdots$, Greek letters are Lorentz indices, and $a, b, \cdots$, Roman letters are regarding the indices of individual component, whereas $i, j, \cdots \in \{1, 2, 3\}$ represent the three dimensional indices. In this notation, the Lagrangian density will be,

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \mathrm{Tr}(D_\mu \Phi D^\mu \Phi) - \lambda(v^2 - \mathrm{Tr}(\Phi)^2)^2, \tag{2.22}$$

where the covariant derivative and trace of matrices are introduced to insure the Lorentz invariance of the Lagrangian. The potential $U(\Phi)$ is the last term,

$$U(\Phi) = \lambda(v^2 - \mathrm{Tr}(\Phi)^2)^2. \tag{2.23}$$

It has the minimum value $U_{min} = 0$, when $\mathrm{Tr}(\Phi^2) = v^2$. The expectation value (vev) of the Higgs scalar field is attained by setting $\mathrm{Tr}(\Phi^2) = \frac{1}{2}|\Phi|^2 = \frac{1}{2}\Phi^a\Phi^a = v^2$. Hence the vev $|\Phi| = \sqrt{2}v$ sets the scale of the Lagrangian density. The covariant derivative of the Higgs field and gauge field strength tensor are,

$$D_\mu \Phi = \partial_\mu \Phi + ig[A_\mu, \Phi], \tag{2.24}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \tag{2.25}$$

where g is the gauge coupling constant. To write out the components explicitly,

$$D_\mu \Phi^a = \partial_\mu \Phi^a - g\epsilon_{abc} A_\mu^b \Phi^c, \tag{2.26}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g\epsilon_{abc} A^b_\mu A^c_\mu. \tag{2.27}$$

From the knowledge of electromagnetism, the electric field and magnetic field can also be written as components of field strength tensor. They are respectively,

$$E_i = F_{0i}, \tag{2.28}$$

$$B_i = -\frac{1}{2}\epsilon_{ijk} F_{jk}. \tag{2.29}$$

The energy (mass) is obtained by integrating the spatial components of the Lagrangian density (2.22) over $\mathbb{R}^3$ as only the static field are considered here.

$$\begin{aligned} E &= -L = -(K - V) = V \\ &= -\int \mathcal{L}\, d^3x = \int \Big(\frac{1}{2}\mathrm{Tr}(F_{ij}F_{ij}) + \mathrm{Tr}(D_i\Phi D_i\Phi) + \lambda(v^2 - \mathrm{Tr}(\Phi)^2)^2\Big)\, d^3x. \end{aligned} \tag{2.30}$$

Where the kinetic energy K is zero for the static field, and thus only the potential energy V should be considered. Writing (2.30) in the component form, yields,

$$E = \int \Big(\frac{1}{4} F^a_{ij} F^a_{ij} + \frac{1}{2} D_i\Phi^a D_i\Phi^a + \lambda(v^2 - \frac{1}{2}\Phi^a\Phi^a)^2\Big)\, d^3x. \tag{2.31}$$

2.3.2 The reason for expecting monopole solutions in $\mathbb{R}^3$

Assuming the configuration of the field Φ is $\mathcal{V}$ when the energy density is minimum. For topological solitons, a necessary requirement is that the field energy density has to be zero at infinity with boundary conditions imposed, i.e. $\mathcal{V}$ has to be the vacuum manifold. Moreover, the vacuum manifold must be a constant map when the gradient term in (2.30) exists. The field at infinity Φ^∞ is essentially an asymptotic map taking the real space sphere $S_\infty^{(d-1)}$ in $\mathbb{R}^3$ to the vacuum manifold $\mathcal{V}$, where now $d = 3$.

$$\Phi^\infty : S_\infty^2 \longmapsto \mathcal{V}. \tag{2.32}$$

In other words, the map Φ^∞ belongs to the homotopy class $\pi_2(\mathcal{V})$, which manifests the topological features of the field configuration $\Phi(x)$. The most important impact of introducing a gauge field is that the gradient energy term in the integral (2.30) will be finite rather than diverging because the covariant derivative term vanish rapidly when $|\vec{x}| \to \infty$ and thus a finite energy configuration of soliton is possible. The vacuum manifold $\mathcal{V}$ can be non-trivial in general, it could be a point or a sphere under $SO(n)$ internal symmetry or it could be more complicated, $\mathbb{CP}^2$ for example. For the $SO(3)$ gauge field and 3-real-component Higgs field, $\mathcal{V} = S^2$. From homotopy theory, $\pi_2(S^2) = S^2 \to S^2 = \mathbb{Z}$, where $\mathbb{Z}$ is the topological charge and here it is known as the monopole number.

To conclude, the finite energy density and the topological structure make it possible to have topological solitons in the three-dimensional gauge theory. Strictly speaking, the Dirac monopole within abelian $U(1)$ gauge is not a topological soliton as its energy is not finite at the origin, but when the singularity being removed, it has finite mass. While 't Hooft-Polyakov monopole is a topological soliton within non-abelian $SU(2)$ gauge theory in three dimension space.

Another reason accounting for the possibility of the existence of monopole solutions in $\mathbb{R}^3$ is the *non-existence* theorem by Derrick [41] for flat space; which argued that for an arbitrary field with finite energy, if there is no stationary point of the rescaled energy function, then the only static solution with finite energy is the vacuum so that no topological solitons can exist in these theories. On the other hand, Derrick's argument suggests that if a theory allows the existence of stationary points of the energy function, they would possibly be the soliton solutions.

To be more specific, let us consider a spatial rescaling applied to a gauge theory in d dimensional space,

$$x \to \mu(x), \qquad\qquad 0 < \mu < \infty. \tag{2.33}$$

And as defined in Ref. [13] for non-abelian field, the general scalar field ϕ and 1-form gauge field A can be rescaled as,

$$\phi(x) \to \phi^{(\mu)}(x) = \phi(\mu x), \tag{2.34}$$

$$A(x) \to A^{(\mu)}(x) = \mu A(\mu x). \tag{2.35}$$

Consequently, the derivative and the covariant derivative with respect to ϕ and the field strength tensor F are rescaled to be,

$$\nabla\phi(x) \to \nabla\phi^{(\mu)}(x) = \mu\phi(\mu x), \tag{2.36}$$

$$D^{A(x)}\phi(x) \to D^{A^{\mu}(x)}\phi^{(\mu)}(x) = \mu D^{A}\phi(\mu x), \tag{2.37}$$

$$F(x) \to F^{\mu}(x) = \mu^2 F(\mu x). \tag{2.38}$$

Writing the energy function in a less unwieldy form,

$$E = \int \left(|F|^2 + |D\Phi|^2 + U(\Phi)\right) d^d x \equiv E_4 + E_2 + E_0, \tag{2.39}$$

where the coefficient of each terms is ignored . After rescaling, the scalar and vector fields Φ and A turn to Φ^{μ}, A^{μ}, and the energy function is now $e(\mu)$,

$$e(\mu) = \mu^{4-d} E_4 + \mu^{2-d} E_2 + \mu^{-d} E_0. \tag{2.40}$$

Stationary points are obtained when $\frac{de}{d\mu} = 0$. When d=3,

$$\frac{de}{d\mu} = E_4 - \frac{1}{\mu^2} E_2 - \frac{3}{\mu^4} E_0. \tag{2.41}$$

Setting it equals to zero, there are solutions for μ exist. However, in the four dimensional space,

$$e(\mu) = E_4 + \frac{1}{\mu^2} E_2 + \mu^4 E_0. \tag{2.42}$$

where $e(\mu)$ is a monotonic function of μ, so there is no stationary point in $\mathbb{R}^4$ (4-dimensional Euclidean spacetime), and thus no soliton.

2.3.3 Methods of finding monopole solutions

It has been shown by theorists that the solution of a field configuration that optimizes energy has a maximum symmetry structure. So it is reasonable to assume it has rotation or reflection symmetries, while translation symmetry is not possible for solitons due to its localised energy density. To obtain such a solution, one method is to take an "ansatz" of the field, which assumes an internal rotational symmetry under which the field is invariant, then substitute the ansatz into the energy function and the integral will depend only on the radial part in the form of some unknown radial functions and their derivatives, which are introduced from the ansatz. In such a way the dimensions get reduced efficiently from d dimensional space, and applying the Euler-Lagrangian equitation afterwards will take us to the equations of motion.

Another method is to calculate the equations of motion first by working with the energy density function directly, which involve multiple variables and they are harder to solve being partial differential equations (PDEs). But once substitute the ansatz into PDEs, it transforms to ordinary differential equations (ODEs), which are easier to solve compared to the PDEs. Both of the ways lead to the same equations of motion and hence the same solutions of monopoles in $\mathbb{R}^3$ when the same boundary conditions applied. For the $SU(2)$ gauge field potential A_μ and adjoint Higgs field Φ (2.21), if we take the second method to obtain the equations of motion by directly applying the Euler-Lagrangian equation to (2.30), this yields,

$$D_i D_i \Phi = -\lambda(2v^2 - (\Phi^a)^2)\Phi, \tag{2.43}$$

$$D_i F_{ij} = -ig[D_j\Phi, \Phi], \tag{2.44}$$

which corresponds to the stationary point of the field energy by the minimal action principle. The ansatz is assumed to be of rotation and reflection symmetries, that the fields are [13],

$$\Phi = h(r)\frac{x^a}{r}t^a, \tag{2.45}$$

$$A_i = -\epsilon_{aij}\frac{x^j}{gr^2}\big(1 - k(r)\big)t^a. \tag{2.46}$$

Where r is the distance from the origin, and $h(r), k(r)$ are two unknown functions of r. These are known as the "hedgehog" configuration (Figure 1), where the directions of the scalar field ϕ are different at each different position (x^1, x^2, x^3). Specifically, the direction of each point is along the line away from the origin. The hedgehog solution is topologically stable as it cannot continuously deform to the vacuum, as the field is continuous so it cannot be a vacuum state in the origin. On the contrary, it is expected to have the monopole in the origin which has the localised energy density.

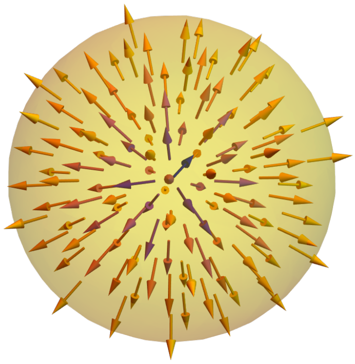


Figure 1. Hedgehog configuration

From (2.30), the minimum energy is attained when $\mathrm{Tr}(\Phi^2) = v^2$, $D_i\Phi = 0$, and $F_{ij} = 0$, which is called the pure gauge condition. By some gauge transformations, A_i can be made vanished and Φ proportional to t^3 generator. Then the $SU(2)$ internal symmetry is spontaneously broken to $U(1)$, with the residual symmetry group generated by t^3, and $M(x)$ is an element of the $U(1)$ group.

$$M(x)t^3M(x)^{-1} = t^3, \qquad\qquad M(x) \in e^{i\alpha x}, \alpha \in \mathbb{R}. \tag{2.47}$$

The boundary conditions when $x^3 \to \infty$ should be determined by the vacuum expectation values of $\Phi(0, 0, x^3) \to \sqrt{2}vt^3$, and correspondingly $M(0, 0, x^3) \to 1$ constrained by (2.47). Applying the boundary conditions to (2.45), (2.46) and nominate values for $h(r = 0)$ and $k(r = 0)$ to avoid the singularity at the origin. Thus the boundary conditions for $h(r), k(r)$ are,

$$\begin{aligned} h(0) &= 0, \qquad & k(0) &= 1, \\ h(\infty) &= \sqrt{2}v, \qquad & k(\infty) &= 0. \end{aligned} \tag{2.48}$$

By expanding the Higgs field Φ around the vev, the Higgs particle mass $M_H = 2\sqrt{\lambda}v$ is obtained. In the symmetry broken pattern $SU(2) \to U(1)$, the unbroken U(1) group corresponds to the massless photon, and the broken gauge group $SU(2)/U(1)$ corresponds to the $W^{\pm}$ bosons, with mass $M_W = \sqrt{2}gv$.

3 Numerical approaches for the 't Hooft-Polyakov monopole

3.1 Numerical 2D shooting method

While through analytical approach, the asymptotic forms of $h(r)$, $k(r)$ and the monopole mass have been studied in details [19] [44], and the existence of solutions has been proofed mathematically. The aim here is to use the numerical method to find the solutions when λ are of different values, and then calculate their energies respectively. When substitute the ansatz into the energy described by (2.31), each term reads,

$$\frac{1}{4}F^a_{ij}F^a_{ij} = \frac{1}{g^2}\Big[\frac{1}{2r^4}(1-k^2)^2 + \frac{1}{r^2}(\frac{dk}{dr})^2\Big], \tag{3.1}$$

$$\frac{1}{2}D_i\Phi^a D_i\Phi^a = \Big[\frac{1}{2}(\frac{dh}{dr})^2 + (\frac{kh}{r})^2\Big], \tag{3.2}$$

$$\lambda(v^2 - \frac{1}{2}(\Phi^a)^2)^2 = \frac{\lambda}{4}(2v^2 - h^2)^2, \tag{3.3}$$

Putting the three terms together into the integral in (2.31) and replacing the integral sign $\int_{-\infty}^{+\infty} d^3x$ with $\int_0^{\infty} 4\pi r^2\, dr$, it will gives us the energy (mass) of monopole,

$$E = 4\pi\int_0^{\infty} dr\ \Big[\frac{1}{g^2}\Big(\frac{1}{2r^2}(1-k^2)^2 + (\frac{dk}{dr})^2\Big) + \frac{1}{2}(\frac{dh}{dr})^2 r^2 + k^2h^2 + \frac{\lambda}{4}(2v^2-h^2)^2 r^2\Big]. \tag{3.4}$$

Following what haven been mentioned early in the begin of subsection 2.3.3, the first method to find the solution of monopole will be applied here. After taking the ansatz, the energy function E (3.4) is an expression depends on $h(r)$, $k(r)$, $h'(r)$, and $k'(r)$. Applying Euler-Lagrange function respect to h and k, the equation of motions are obtained,

$$\frac{d^2h}{dr^2} + \frac{2}{r}\frac{dh}{dr} = \frac{2}{r^2}k^2h + \lambda(h^2 - 2v^2)h, \tag{3.5}$$

$$\frac{d^2k}{dr^2} = \frac{1}{r^2}(k^2 - 1)k + g^2h^2k, \tag{3.6}$$

which is the same as using the second method to apply the ansatz (2.45) and (2.46) into equations of motion (2.43), (2.44). To solve equation (3.5), (3.6), realising that they are second order non-linear ODEs. The method used here is to split the domains of the ODEs into two parts respectively. In the first part $r \to 0$, they can be linearized by using the approximation that

$$h(r) = \delta h(r), \tag{3.7}$$

$$k(r) = 1 + \delta k(r). \tag{3.8}$$

Substituting (3.7) and (3.8) into the ansatz (2.45), (2.46), it gives two linear ODEs.

$$r^2\delta h'' + 2r\delta h' - 2\delta h = 0, \tag{3.9}$$

$$r^2\delta k'' - 2\delta k = 0. \tag{3.10}$$

And these can be solved, they yield,

$$\delta h(r) = C_h r, \tag{3.11}$$

$$\delta k(r) = C_k r^2, \tag{3.12}$$

where C_h, C_k are two undetermined constants. In the second range of domain, they are not linear but we have already known the initial values and their derivatives, which are determined by the end values of the first linear part. For instance, setting the initial value of r to be $r_{ini} = 10^{-5}$ to avoid singularities in the origin, and the terminal of the first domain is set to be $r_1 = 0.01$, then when $r_{ini} < r < r_1$,

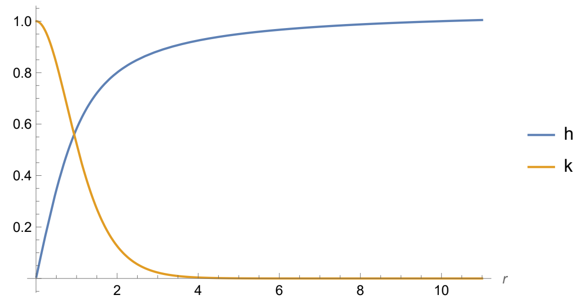
$$\begin{aligned} h(r_1) &= C_h r_1\,, \\ h'(r_1) &= C_h\,, \\ k(r_1) &= 1 + C_k r_1^2\,, \\ k'(r_1) &= 2C_k r_1\,. \end{aligned} \tag{3.13}$$

The 2D shooting and bisection approaches are used when different values of C_h and C_k are tried. When they are adjusted to be different values, either h or k can be larger (overshoot) or smaller (undershoot) than the values we expect, which are $h(\infty) = 1$, $k(\infty) = 0$. Thus the bisection method is used to approach a correct value. For example, the constant of $C_h = \frac{C_{h1}+C_{h2}}{2}$ was taken when one of C_{h1} and C_{h2} are overshoot and the other is undershoot. By continuously changing C_h and C_k, the results which reach the boundary values at limit r_{bc} are shown in Table 1 and Figure 2. When λ is not large, i.e. 0, 0.01, 0.1 and 1, the r_{bc} are set to be 10; but when $\lambda = 10, 100$, the r_{bc} has to be set smaller as the functions are more sensitive to minimal variations. Because the Higgs mass $M_H = 2\sqrt{\lambda}v$ set a length scale for the monopole, consequently a larger error is expected.

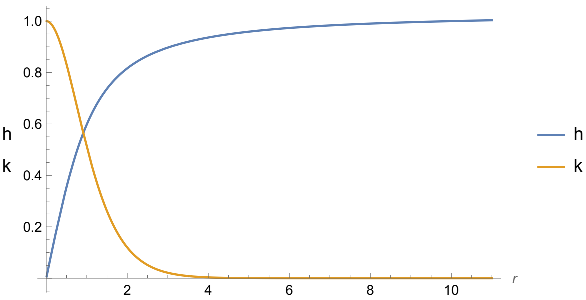
Note in the shooting method, the gauge coupling constant g is chosen to be 2, and the constant v which defining the value of vacuum is fixed to be $1/\sqrt{2}$ by convention.

λ	C_h	C_k	r_{bc}
0	0.734972658	-0.733150961284	10
0.01	0.7606734375	-0.758246985703	10
0.1	0.88762566	-0.87996433252999	10
1	1.2580204858	-1.2036522975561399	10
10	2.2927593341894	-1.88125150743252	5
100	5.5391853320003	-3.007353214277487253	2.5

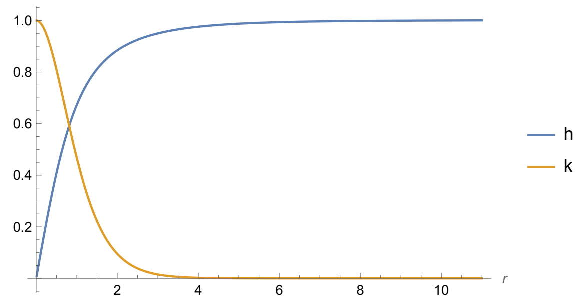
Table 1. The values of C_h, C_k corresponding to different λ values.



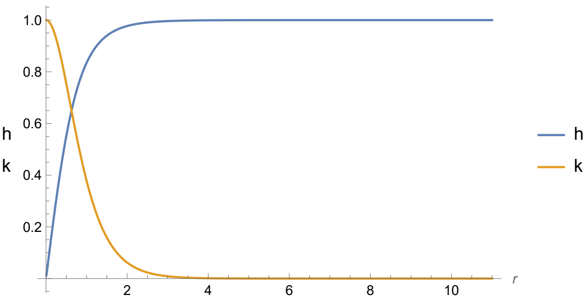
(a) $\lambda = 0$.



(b) $\lambda = 0.01$.



(c) $\lambda = 0.1$.



(d) $\lambda = 1$.

Figure 2. Functions $h(r)$, $k(r)$ of λ =0, 0.01, 0.1 and 1, $r_{bc} = 10$.

The energies of the monopole solutions are also calculated. In the first range where $r_{ini} < r < r_1$, the linear forms of $h(r)$ and $k(r)$ are used, and the energy is denoted by E_1,

$$E_1 = 4\pi \int_{r_{ini}}^{r_1} dr \Bigg[\left(\frac{1}{8r^2} \left(1 - (1 + C_k r^2)^2\right)^2 + C_k^2 r^2 \right) + \left(\frac{1}{2} C_h^2 r^2 + (1 + C_k r^2)^2 C_h^2 r^2 \right) + \left(\frac{\lambda}{4} (1 - C_h^2 r^2)^2 r^2 \right) \Bigg] \quad (3.14)$$

The second range should be $r_1 < r < \infty$, but as the limit reached here is up to $r = r_{bc}$, and singularities will appear somewhere after the limit, so we shall separate the second range into $r_1 < r < r_{bc}$ and $r > r_{bc}$.

The integral of the second part E_2 are calculated by the equation (3.4), using Numerical Integral in Mathematical.

$$E_2 \; = \; 4\pi \int_{r_1}^{r_{bc}} dr \left[\frac{1}{8r^2}(1-k^2)^2 + \frac{1}{4}(\frac{dk}{dr})^2 + \frac{r^2}{2}(\frac{dh}{dr})^2 + k^2h^2 + \frac{\lambda}{4}(1-h^2)^2 r^2 \right] \tag{3.15}$$

Then the last part is simply a constant after integration, reads,

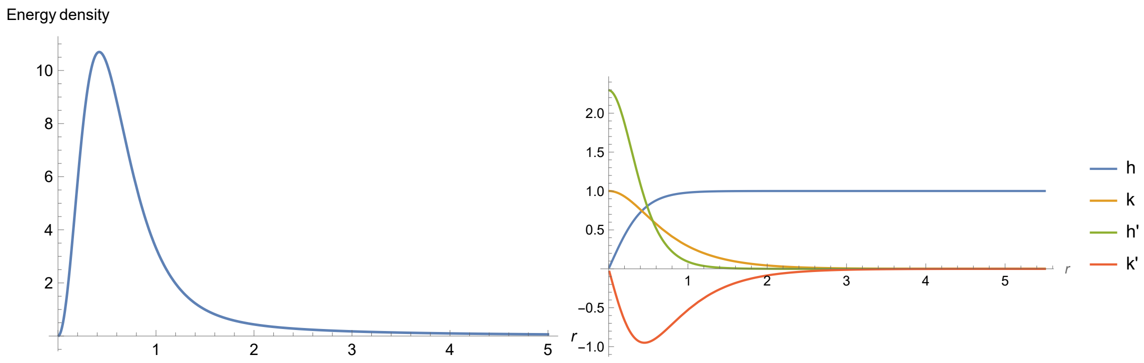
$$E_3 = \int_{r_{bc}}^{\infty} dr \; \frac{\pi}{2r^2} = \Big[-\frac{\pi}{2r} \Big]_{r_{bc}}^{\infty} = \frac{\pi}{2r_{bc}}. \tag{3.16}$$

The total energy subjected to shooting method E_{sho}=$E_1 + E_2 + E_3$ are shown in Table 2.

λ	E_1	E_2	E_3	E_{sho}
0	6.78×10^{-6}	6.28321	$\pi/20$	6.44030
0.01	7.27×10^{-6}	6.34629	$\pi/20$	6.50338
0.1	9.95×10^{-6}	6.63398	$\pi/20$	6.79107
1	2.04×10^{-5}	7.32008	$\pi/20$	7.47718
10	5.89×10^{-5}	8.29869	$\pi/10$	8.61291
100	$2.95\ \times10^{-4}$	8.80562	$\pi/5$	9.43424

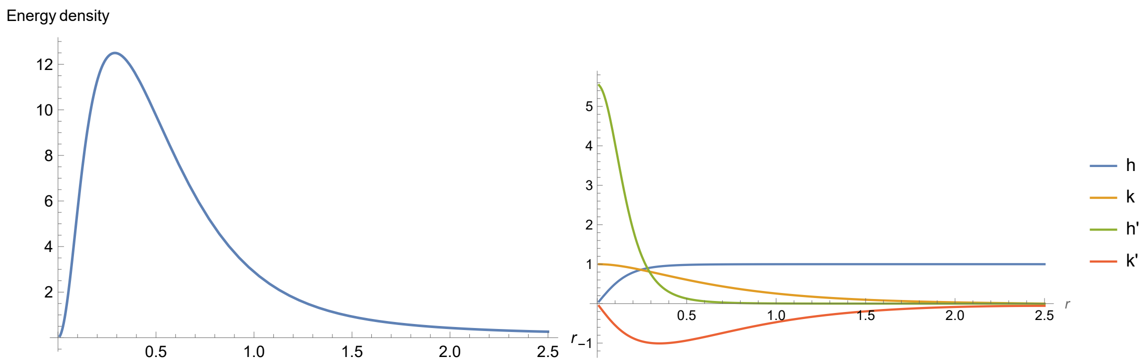
Table 2. Energies of monopole corresponds to different λ values.

To verify the feasibility of taking a shorter r_{bc} when $\lambda = 10$ and 100, the energy density inside the integral (3.4) is plotted in the range of $r_1 < r < r_{bc}$. If it tends to zero at r_{bc}, then the second part of energy E_2 can be obtained acceptably by the numerical integration (3.4) from $r_1 < r < r_{bc}$. And the results presented in Figure 3, 4 do show the vanishing energy density at r_{bc} which ensure the validity of doing so.



(a) $\lambda = 10,\ r_{bc} = 5$ (b) $\lambda = 10,\ r_{bc} = 5$

Figure 3. Energy density and functions of $\lambda = 10,\ r_{bc} = 5$.



(a) $\lambda = 100,\ r_{bc} = 2.5$ (b) $\lambda = 100,\ r_{bc} = 2.5$

Figure 4. Energy density and functions of $\lambda = 100,\ r_{bc} = 2.5$.

To express in a more conventional way for monopole energy, a rescaling $u = \sqrt{2}gvr,\ h = \sqrt{2}vH$ are applied to (3.4). Then functions $H,\ k$ will become functions of new variable u, which yields $H(u) = h(\sqrt{2}gvr)$, and $k(u) = k(\sqrt{2}gvr)$. It can be seen clearly that after rescaling, $H(u)$ should satisfy the boundary condition $H(u) \to 1$ when $x_3 \to \infty$. As (2.48) implies $h(\infty) = \sqrt{2}v$, and $H = h/\sqrt{2}v$. And the new boundary conditions are,

$$H(0) = 0, \qquad k(0) = 1,$$

$$H(\infty) = 1, \qquad k(\infty) = 0.$$

After rescaling, the equation (3.4) becomes an expression only depends on the quantity $\frac{\lambda}{g^2}$, which is

$$E=\frac{4\pi\sqrt{2}v}{g}\int_0^{\infty}du\left[\frac{1}{2u^2}(1-k^2)^2+(\frac{dk}{du})^2+\frac{1}{2}(\frac{dH}{du})^2u^2+k^2H^2+\frac{\lambda}{4g^2}(H^2-1)^2u^2\right]. \quad (3.17)$$

Defining a dimensionless quantity $\beta = M_H/M_W = \sqrt{2\lambda/g^2}$ and substituting the mass of the vector bosons $M_W = \sqrt{2}gv$ into E, yields,

$$\begin{aligned}E&=\frac{4\pi M_W}{g^2}\int_0^{\infty}du\left[\frac{1}{2u^2}(1-k^2)^2+(\frac{dk}{du})^2+\frac{1}{2}(\frac{dH}{du})^2u^2+k^2H^2+\frac{\beta^2}{8}(H^2-1)^2u^2\right]\\&=\frac{4\pi M_W}{g^2}\tilde{E}(\beta).\end{aligned} \quad (3.18)$$

Though the coefficient in front of the integral $\tilde{E}(\beta)$ can be different due to different conventions, the $\tilde{E}(\beta)$ function will be the same and the value of which is close to 1 for all β [45]. So it is calculated and will be used to make comparisons with data obtained by lattice methods later. If we choose to set $g = 2$, $v = \frac{1}{\sqrt{2}}$, and $\beta = \sqrt{\frac{\lambda}{2}}$, then

$$\tilde{E}(\beta)=\frac{E}{2\pi}, \quad (3.19)$$

according to which, the numerical values of $\tilde{E}(\beta)_{sho}$ are shown in Table 3.

λ	β	$\tilde{E}(\beta)_{sho}$
0	0	1.02501
0.01	0.07071	1.03504
0.1	0.22361	1.08083
1	0.70711	1.19003
10	2.23607	1.37079
100	7.07107	1.50151

Table 3. The energies $\tilde{E}(\beta)_{sho}$ of monopole correspond to different β (λ) values.

The analytical expressions of energy for both small and large λ are known, When λ is small, [44, 46]

$$E = 2\pi(1 + \frac{1}{2}\beta + \frac{1}{2}\beta^2 log\beta + c_3\beta^2 + \cdots) = 2\pi\tilde{E}(\beta). \tag{3.20}$$

and it is proved that the values of $\tilde{E}(\beta)$ are always close to 1 no matter what β is [45]. The value of $c_3 = 0.7071$ is originally proposed by asymptotic analysis [44]. But in the later article, it is reexamined using numerical approach and a constant $c_3 = 0.44297 \pm 1.8 \times 10^{-4}$ is brought forward instead [45]. While when λ is large, the analytical result for the monopole mass is [19],

$$E = 2\pi(1.787 - \frac{2.228}{\beta} + \cdots) = 2\pi\tilde{E}. \tag{3.21}$$

According to (3.20), the energy $\tilde{E}(\beta)$ for $\beta = 0$ should be strictly equal to 1, while we see the $\tilde{E}(\beta)_{sho}$ for $\beta = 0$ case shows a comparable result of 1.02501.

3.2 Lattice Monte-Carlo simulation method

Another numerical method for calculating monopole energy which is going to be examined in the essay is lattice discretization method [47]. The action is discretized on the lattice, that each point on the lattice position is represented by $\vec{x} = \{n_x, n_y, n_z\}a$, where a is the lattice spacing. The adjoint Higgs field $\Phi(\vec{x})$ is defined on position $\vec{x}$, and the gauge field is defined via the link $U_i(\vec{x}) \in SU(2)$ matrix group, i.e. $U_i = exp(igA_\mu)$, between lattice sites $\vec{x}$ and $\vec{x}+\hat{i}$, where $\vec{x} \in \{0, \cdots, (N-1)a\}^3$, N is the total number of lattice sites in either x, y, or z direction and U_{ij} is the standard Wilson plaqutte [25], that

$$U_{ij} = U_i(\vec{x})U_j(\vec{x}+\hat{i})U_i^\dagger(\vec{x}+\hat{j})U_j^\dagger(\vec{x}).$$

The Lagrangian density is given by [24],

$$\begin{aligned}\mathcal{L}_{lat} = &\frac{4}{ag^2}\sum_{i<j}(1-\frac{1}{2}Re\mathrm{Tr}U_{ij}(\vec{x})) \\ &+\sum_i 2a\big[\mathrm{Tr}\Phi^2(\vec{x}) - \mathrm{Tr}\Phi(\vec{x})U_i(\vec{x})\Phi(\vec{x}+\hat{i})U_i^\dagger(\vec{x})\big] \\ &+ m^2a^3\mathrm{Tr}\Phi^2(\vec{x}) + a^3\lambda\mathrm{Tr}(\Phi^2(\vec{x}))^2,\end{aligned} \tag{3.22}$$

where the last line of the equation is the potential energy U. Note the Wick rotation and discretization are performed in order to get the Lagrangian density. The standard gradient descent optimisation is used to find the static solution which minimises the energy $E_{lat} = \sum_{\vec{x}}\mathcal{L}_{lat}$. The code used here is part of the TFMONOPOLES PYTHON package [48] and the lattice is set to be 1 in the code.

The classical monopole mass M_m has been calculated as (3.18) shows,

$$M_m = \frac{4\pi M_W}{g^2}\tilde{E}(\beta). \tag{3.23}$$

Substituting $M_H = 2\sqrt{\lambda}v, M_W = \sqrt{2}gv$ in the equation (3.23), reads,

$$M_m = \frac{4\sqrt{2}\pi v}{g}\tilde{E}(\beta). \tag{3.24}$$

3.2.1 Lattice size and lattice spacing effect

It is noticed that though the lattice method can be use to calculate energy, it still has deviation caused by finite-size effects and discretisation errors [25] that depends on the lattice size (total number of discretized lattice) and the boundary conditions used. As the code fixes lattice spacing a to be 1, we could only change it by changing the unit $M_W = \sqrt{2}gv$ and the Higgs mass $M_H = 2\sqrt{\lambda}v$, When M_W or M_H get larger, the lattice space will become larger.

If the lattice spacing is too large, the monopole will fall into the lattice spacing and we cannot capture it using the lattice discretization method. On the contrary, if the spacing is too small that the monopole takes up the whole lattice size, then the energy cannot be integrated out and tend to infinity. To examine the lattice size effect, the energies of monopoles are calculated by using the collection of many separate computers cluster (HPC) of Imperial College. The parameters being fixed here are $g = 2$, $\lambda = 1$, which yields $\beta = \sqrt{\frac{2\lambda}{g^2}} = 0.70711$. The bosons mass M_W, or in other words, the vev v will be consequently only variable.

The monopole mass M_m of 32^3 and 16^3 lattice size are shown in Figure 5. It is observed that in the range of $0 < 1/v < 2$, the monopole masses stay almost the same ~ 4.6 for 32^3 lattice and ~ 4.5 for 16^3 lattice respectively. Between $2 < 1/v < 10$, the masses are monotonically decreasing for both two lattice sizes, and it can be easily noticed that a discontinuity appears at $1/v = 8$ for 16^3 lattice. From (3.24), we know that if $\tilde{E}(\beta)$ a constant, the mass M_m will be inversely proportional to $1/v$, and the curves indicate a similar argument that $\tilde{E}(\beta)$ could possibly be close to some constant.

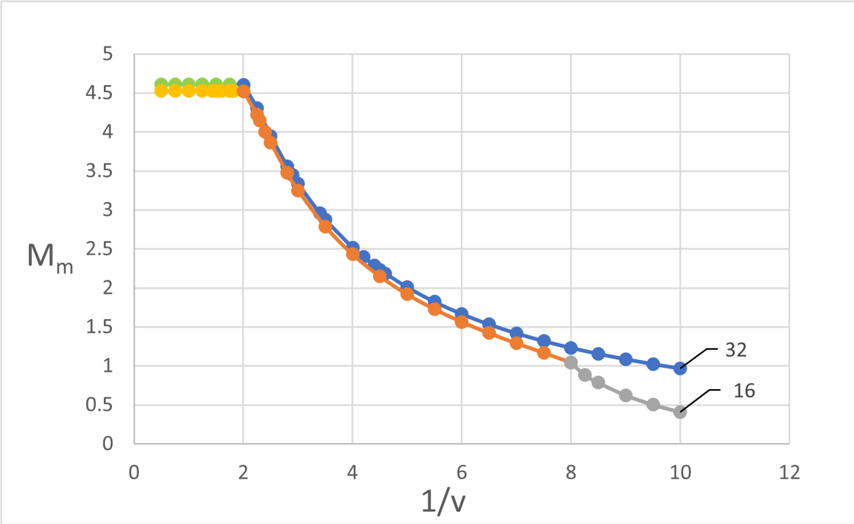


Figure 5. Monopole mass attained from 32^3 and 16^3 lattice through HPC, $\beta = 0.70711$.

To compare $\tilde{E}(\beta)$ attained by lattice simulation and shooting method, where $\tilde{E}(\beta)_{sho} = 1.19003$ (Table 3, $\beta = 0.70711$), their plots are presented in Figure 6. It can be seen that between $0 < 1/v < 2$, the energies $\tilde{E}$ of two lattice sizes nearly overlap with each other, and they connected smoothly at $v = 0.5$ with their second pieces. In the middle regions of $2 < 1/v < 10$ and $2 < 1/v < 8$ for 32^3 and 16^3 lattice size respectively, the monopole energy ~ 1 and hence implying their suitable lattice spacing.

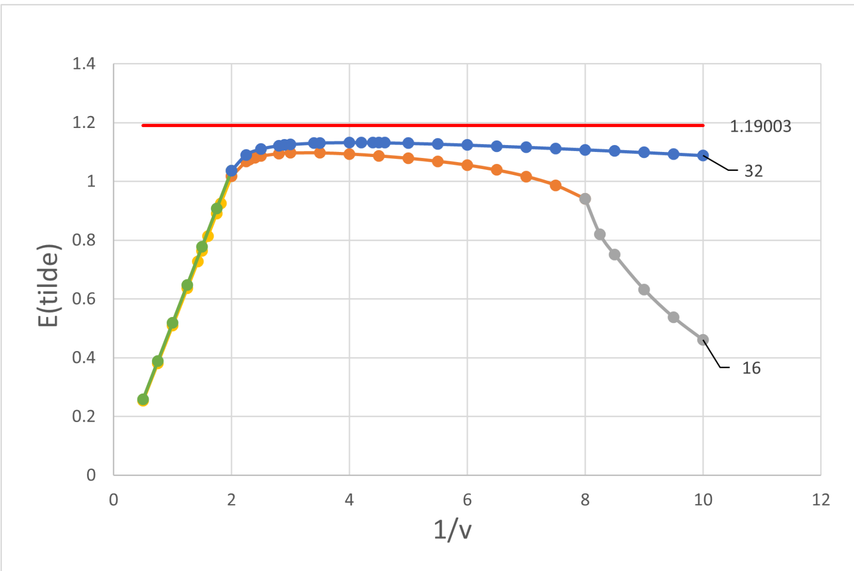


Figure 6. Monopole energy attained from 32^3 and 16^3 lattice, $\beta = \sqrt{\frac{2\lambda}{g^2}} = 0.70711$.

To compare the lattice size effect, we could first turn to the energies of 16^3 lattice size. When v=1/8, the lattice spacing is too short because the vev (scale) is smaller than what is needed. Thus the monopole, which is the solution of local minimum of energy, falls into one unit cube of lattice, hence the gradient descent algorithm cannot capture the monopole effectively. We will see this problem mitigates when the lattice size becomes larger, in which way the monopole gets larger and correspondingly the risk of being not captured is reduced. On the other side, the $v = 1/2$ corresponds to an opposite condition, that the length scale is too large that the whole lattice volume cannot accommodate the monopole. The energy should be obtained by integrating the whole space occupied by the monopole, but in this condition, only part of the space is counted, so the energies are lower than the actual value.

For the lattice size of 32^3, the suitable range of lattice spacing is larger, with the lower limit of vev being 1/10 compared with 1/8 for 16^3 lattice, and this is what we expected. With the larger lattice size applied, the monopole will occupy more lattices. To remain the same length of the monopole, a smaller length scale, hence a shorter lattice spacing would be appropriate, which allows it to have a larger lower bound of $1/v$.

Another feature of the two plots is the energies obtained from 32^3 lattice is more close to the numerical value 1.19003. We should expect a more accurate result when the number of lattice sites is increased, and when it tends to infinity in the ideal case, the results will reach the continuum limit.

Apart from this, the existence of the magnetic Coulomb effect (will continue to discuss later) of monopole-antimonopole pair also predicts lower energy than a sole monopole, and this can be illustrated more clearly when a large lattice size is applied. However, the algorithm using a large number of lattice sites requires intensive computational resources, which might be conducted on supercomputers.

To check the validity of results attained from the shooting method and lattice discretization method, their values are compared with the analytical predictions. The $\tilde{E}(\beta)$ are plotted regarding different values of β in Figure 7, with all other parameters variable. The red line (small) and the blue line (large) are the analytical results of expressions (3.20) and (3.21) plotting in ranges $0.001 < \beta < 0.1$ and $10 < \beta < 1000$ respectively, where

$$\tilde{E}(\beta) = 1 + \frac{1}{2}\beta + \frac{1}{2}\beta^2 log\beta + 0.7071\beta^2 + \cdots, \tag{3.25}$$

for small value of β. When β is large, we have,

$$\tilde{E}(\beta) = 1.787 - \frac{2.228}{\beta} + \cdots. \tag{3.26}$$

Whereas the E_{num} (black dot with line) is the energy obtained by numerical shooting method, which presented in Table 3. The energies of 32^3 and 64^3 lattice size are attained from lattice discretization through the HPC, note their parameters, v, λ, and g are different.

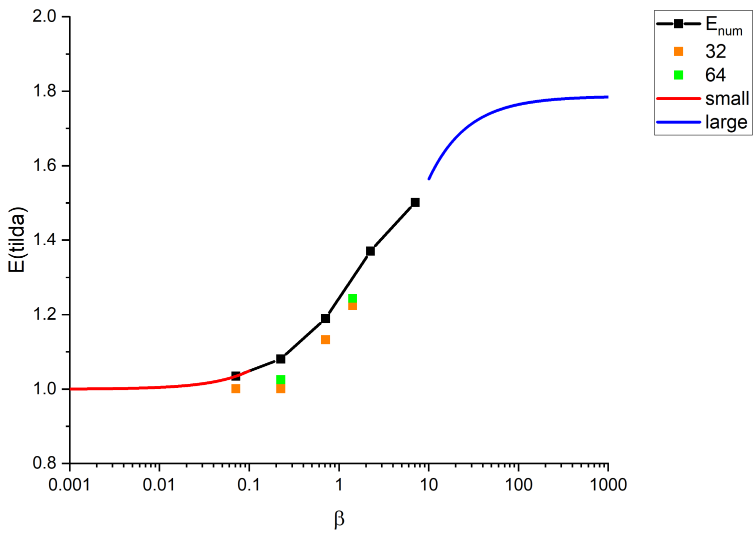


Figure 7. Monopole energy $\tilde{E}(\beta)$ (log plot).

It can be seen that in the media region of $0.1 < \beta < 10$, the numerical plot shows a trend of connecting both the small and large β parts, which is in line with our expectation. For the lattice results in this region, we can see the highest $\tilde{E}$ are close to numerical results.

Another aspect affecting the energies is that due to the existence of virtual monopole-antimonopole pairs, the magnetic Coulomb effect will lower the total energy when a pair gets closer. But this also presents the significance of the lattice method, while the 't Hooft-Polyakov monopole-antimonopole pairs cannot be produced via LHC practically as the production rate is greatly suppressed [23], thus it is important to verify the possibility of having the pairs in the quantum theory.

To conclude, a larger lattice size could yield a more precise result but the lattice spacing has to be suitable. Considering the finite-size effect, if one pursues a more precise result, the lattice size could be set larger, however, the vev need also to be shrunk to an appropriate range. In addition, a larger lattice size is also better to allow us to study the effect of monopole-antimonopole pair, but at the same time, higher computational resources are demanded.

4 Some properties and generalization of 't Hooft-Polyakov monopoles

4.1 Magnetic charge

In this finite energy configuration, one requirement is that the covariant derivative $D_i\Phi$ has to decrease faster than $r^{-\frac{3}{2}}$. Once it is satisfied, we can assume out of some region R, the $SU(2)$ Yang-Mils field is abelianized, that f_{ij} and b_i will be used following to represent the abelian fields, and we have

$$\frac{1}{8\pi}\epsilon_{ijk}\int_{S^2_\infty} dS^i \left[\hat{\Phi}\cdot D_j\hat{\Phi}\times D_k\hat{\Phi}\right] = 0, \tag{4.1}$$

where $h(r)\hat{\Phi} = \Phi$, with $|\Phi|^2 = 2\mathrm{Tr}(\Phi^2) = 1$. Expanding the above equation, yields,

$$
\begin{aligned}
&\frac{1}{8\pi}\epsilon_{ijk}\int_{S^2_\infty} dS^i\left[\hat{\Phi}\cdot D_j\hat{\Phi}\times D_k\hat{\Phi}\right] \\
&=\frac{1}{8\pi}\epsilon_{ijk}\int_{S^2_\infty} dS^i\left[\hat{\Phi}\cdot\partial_j\hat{\Phi}\times\partial_k\hat{\Phi}+g\partial_j\hat{\Phi}\cdot A_k-g\partial_k\hat{\Phi}\cdot A_j+g^2A_j\times A_k\cdot\hat{\Phi}\right] \\
&=\frac{1}{8\pi}\epsilon_{ijk}\int_{S^2_\infty} dS^i\Big[\hat{\Phi}\cdot\partial_j\hat{\Phi}\times\partial_k\hat{\Phi}-g\hat{\Phi}\cdot f_{jk}\Big]=0.
\end{aligned}
\tag{4.2}
$$

Where integrate by part is used in the last step. The f_{ij} here satisfies the abelian Bianchi Identity $\epsilon^{\mu\nu\lambda\rho}\partial_i f_{\lambda\rho}=0$, and this implies $D_iB_i=0$, where $\epsilon^{\mu\nu\lambda\rho}$ is the 4-dimensional fully antisymmetric Levi-Civita tensor. By (4.2), reads,

$$
\frac{1}{8\pi}\epsilon_{ijk}\int_{S^2_\infty} dS^i\left[\hat{\Phi}\cdot\partial_j\hat{\Phi}\times\partial_k\hat{\Phi}\right]=\frac{1}{8\pi}\epsilon_{ijk}\int_{S^2_\infty} dS^i\left[g\hat{\Phi}\cdot f_{jk}\right].
\tag{4.3}
$$

If one defines,

$$
b_i=-\frac{1}{2}\epsilon_{ijk}f_{jk},
\tag{4.4}
$$

then$\overrightarrow{\nabla}\cdot\overrightarrow{b}$ =0, $\overrightarrow{\nabla}\times\overrightarrow{b}$ =0 are satisfied, and the topological charge

$$
\begin{aligned}
N&=\frac{1}{8\pi}\epsilon_{ijk}\int_{S^2_\infty} dS^i\Big[\,\hat{\Phi}\cdot\partial_j\hat{\Phi}\times\partial_k\hat{\Phi}\Big] \\
&=-\frac{g}{4\pi}\int_{S^2_\infty} dS^i\left[\hat{\Phi}\cdot b_i\right] \\
&=-\frac{gq_m}{4\pi}.
\end{aligned}
\tag{4.5}
$$

Noting that we replace the gauge coupling constant g to $-e$ afterwards as they are the same in $SU(2)$ George-Glashow theory, so $eq_m=4\pi N$, which is the Schwinger's condition, is obtained instead of the *Dirac quantization condition* $eq_m=2\pi N$ in (2.7) for $q_e=-e\hbar$. The basic monopole with monopole number $N=1$ has a magnetic charge $q_m=\frac{4\pi}{e}$, which is twice the value for that of elementary particles with electric charges $\pm e$ as shown in the

Dirac quantization condition . But if the elementary particles can be quantised to have their electric charges $\pm\frac{e}{2}$, the *Dirac quantization condition* will automatically change to the Schwinger condition. It is the case when another $SU(2)$ doublet field is coupled with our existing $SU(2)$ field, and now the new gauge field can be interpreted as the fermionic field.

It is confirmed by numerical analysis that the solution for 't Hooft and Polyakov monopole demonstrated here is indeed stable with respect to spherical and non-spherical perturbations [49]. Thus it is a stable point of the field energy, and also being a local minimum. Another property of monopole is that it is spin 0 particle as the spherical symmetry form prohibit it to have any rotational angular momentum. The properties of particles in the theory where an adjoint Higgs field coupled to a $SU(2)$ gauge field are listed below in Table 4.

	mass	q_e	q_m	spin
photon	0	0	0	0
Higgs boson	$2\sqrt{\lambda}v$	0	0	1
$W^{\pm}$ bosons	$\sqrt{2}ev$	$\pm$ e	0	1
monopole	$\frac{4\pi\sqrt{2}v}{e}\tilde{E}(\beta)$	0	$\frac{4\pi}{e}$	0

Table 4. Particles' properties in George-Glashow model (note $\tilde{E}(\beta) \sim 1$).

We could similarly get the antimonopole solution when x in the ansatz (2.45) of Φ is substituted with $-x$. So the difference between monopole and antimonopole is their magnetic charge is opposite, whereas of the same mass. When two monopoles of charge $q_m = \frac{4\pi}{e}$ are put close to each other separated by a distance of r, the repulsive Coulomb force is expected, that,

$$F = \frac{(\frac{4\pi}{e})^2}{r^2} = \frac{4\pi}{e^2 r^2}. \tag{4.6}$$

Similarly, when a monopole and an antimonopole are at a distance r close to each other, the attractive Coulomb force will take place with the same magnitude. Thus in this adjoint Higgs field coupled to $SU(2)$ gauge field configuration, not only a solution of the monopole is feasible, but also the Coulomb force is predicted at the same time, which gives monopoles the concept of particles to some extent with their properties as Table 4 demonstrated.

Outside of the region R, $D_i\Phi$ tends to zero exponentially fast and $h(r)$, $k(r)$ tend to their asymptotic values (2.48) exponentially fast, [50]

$$k(r) \sim e^{-\sqrt{2}evr} = e^{-M_W r}, \tag{4.7}$$

and

$$h(r) = \begin{cases} \sqrt{2}v - e^{-2\sqrt{\lambda}vr} = \sqrt{2}v - e^{-M_H r}, & \dfrac{\lambda}{e^2} < 2, \\ \sqrt{2}v - \dfrac{e^{-2\sqrt{2}ev}}{r^2} = \sqrt{2}v - \dfrac{e^{-2M_W}}{r^2}, & \dfrac{\lambda}{e^2} > 2, \end{cases} \tag{4.8}$$

where $\lambda/e^2 = \beta^2/2 > 2$ is simply $\beta > 2$. Outside the core region R, abelianized fields are expected as mentioned above. The radial magnetic field b can be calculated using the ansatz (2.45), (2.46) and ignore the exponential small terms,

$$b_i = \frac{1}{2}\epsilon_{ijk}F_{jk} = \frac{\hat{r}^i}{er^2} = \frac{x^i}{er^3}, \tag{4.9}$$

which is the magnetic field generated by a monopole of charge $q_m = \frac{4\pi}{e}$ (2.3), and it is in line with the Coulomb force we expected as (4.6). Considering the contribution of the magnetic field to the field energy when $r > R$, we have,

$$\begin{aligned} E_{Coulomb} &= 4\pi \int_R^\infty r^2 dr \big[\frac{1}{2}B^2\big] \\ &= \frac{2\pi}{e^2}\int_R^\infty \frac{1}{r^2}dr \\ &= \frac{2\pi}{e^2 R}. \end{aligned} \tag{4.10}$$

In the core region, it is suggested the energy density is a constant ρ, then $\rho \sim 4e^2v^4$. If we take λ/e^2 to be of unit order, then $\rho \sim 4\lambda v^4$. The total energy E_{tot} will be a sum,

$$E_{tot} \approx E_{core} + E_{Coulomb} \approx \frac{4\pi R^3}{3}\rho + \frac{2\pi}{e^2 R}. \tag{4.11}$$

In the monopole's field configuration, the total energy should be a local minimum, so taking the derivative and set it to be 0,

$$\begin{aligned} \frac{dE_{tot}}{dR} &= 0 \\ 4\pi R^2 \rho &= \frac{2\pi}{e^2 R^2} \\ R &= (\frac{1}{2e^2\rho})^{1/4} \\ &= \frac{1}{2^{1/4}\sqrt{2}ev} \sim \frac{1}{\sqrt{2}ev} = \frac{1}{M_W}, \end{aligned} \tag{4.12}$$

which confirmed the assumption that $h(r)$ and $k(r)$ tend to their asymptotic values exponentially fast with the exponential coefficient being about M_W as in (4.7), (4.8). And the energy (mass) can be calculated when substituting (4.12) into (4.11), $E = \frac{10\sqrt{2}\pi v}{3e}$ is obtained.

4.2 Monopoles in BPS limit and duality

The Bogomolny-Prasad-Sommerfield (BPS) limit is reached when the self-coupling $\lambda = 0$ and correspondingly the expectation value for scalar Higgs field is zero. The BPS limit was first proposed by Bogomolny[42], and Prasad and Sommerfield [43]. In the BPS limit, some interesting and deeper relations have been scrutinised and there are further research on duality and supersymmetry developed based on the BPS limit. In 1977, C. Montonen and D. Olive [51] proposed that there is an underlying duality symmetry of 't Hooft-Polyakov monopoles when they are quantised in the $SU(2)$ George-Glashow model in the BPS limit. They proposed that if replace the electric charge of W bosons and the magnetic charge of monopole (replacing e with $\frac{4\pi}{e}$ in Table 4), the particle spectrum will stay the same.

Let us consider what happen in limit of $\lambda = 0$ first, where the equations of motion (3.5) and (3.6) will take the forms,

$$\frac{d^2h}{dr^2} + \frac{2}{r}\frac{dh}{dr} = \frac{2}{r^2}k^2h, \tag{4.13}$$

$$\frac{d^2k}{dr^2} = \frac{1}{r^2}(k^2-1)k + g^2h^2k. \tag{4.14}$$

They are solved to be,

$$h(r) = \sqrt{2}v\ \coth(\sqrt{2}ver) - \frac{1}{er}, \tag{4.15}$$

$$k(r) = \frac{\sqrt{2}ver}{\sinh(\sqrt{2}ver)}, \tag{4.16}$$

with the monopole mass $4\pi\sqrt{2}v/e$ and $\tilde{E}(\beta) = 1$ (3.24). Bogomolny realised the energy for the monopole of a static field has more interesting behaviours. It is noted [42] that the total energy,

$$\begin{aligned} E &= \int \Big(\frac{1}{2}\mathrm{Tr}(F_{ij}F_{ij}) + \mathrm{Tr}(D_i\Phi D_i\Phi) + U\Big)\, d^3x \\ &= \int \Big(\mathrm{Tr}(B_iB_i) + \mathrm{Tr}(D_i\Phi D_i\Phi) + U\Big)\, d^3x, \end{aligned} \tag{4.17}$$

where B_i is of the form (2.29), and E_i vanishes because the monopoles carry no electric charge but we shall see $E_i \neq 0$ in dyon's case. With the third term U being always non-negative, the sum of the first two terms can be written as,

$$
\begin{aligned}
E_{ftt} &= \int d^3x\Big(\mathrm{Tr}(B_i + D_i\Phi)(B_i + D_i\Phi) - 2\mathrm{Tr}(B_i D_i \Phi)\Big) \\
&= \int d^3x\Big(\mathrm{Tr}(B_i + D_i\Phi)(B_i + D_i\Phi) - 2\partial_i \mathrm{Tr}(B_i \Phi)\Big),
\end{aligned}
\tag{4.18}
$$

as the Bianchi Identity $\epsilon^{\mu\nu\lambda\rho} D_\mu F_{\lambda\rho} = 0$ implies $D_i B_i = 0$, where $\epsilon^{\mu\nu\lambda\rho}$ is the fully antisymmetric Levi-Civita tensor. Then we could derive what used above,

$$
2\partial_i \mathrm{Tr}(B_i\Phi) = 2\mathrm{Tr}(B_i D_i \Phi + (D_i B_i)\Phi) = 2\mathrm{Tr}(B_i D_i \Phi). \tag{4.19}
$$

When $r \to \infty$, the field Φ approaches its boundary value $\sqrt{2}vt^3$ at S^2_∞ sphere and $h(r)$ goes to the vev $\sqrt{2}v$, whereas the external fields are all abelianized. We could rewrite (4.18) as,

$$
\begin{aligned}
E_{ftt} &= \int_{\mathbb{R}^3} d^3x\Big(\mathrm{Tr}(B_i + D_i\Phi)(B_i + D_i\Phi)\Big) - 2\sqrt{2}v \int_{S^2_\infty} dS\Big(\mathrm{Tr}(b_i\hat{\Phi})\Big) \\
&= \int_{\mathbb{R}^3} d^3x\Big(\mathrm{Tr}(B_i + D_i\Phi)(B_i + D_i\Phi)\Big) - \sqrt{2}v \int_{S^2_\infty} dS^i \; b_{3i} \\
&= \int_{\mathbb{R}^3} d^3x\Big(\mathrm{Tr}(B_i + D_i\Phi)(B_i + D_i\Phi)\Big) - \sqrt{2}v q_m,
\end{aligned}
\tag{4.20}
$$

for that b_i and $\hat{\Phi}$ are in the same direction of $su(2)$ algebra, i.e. $b_i = b_{3i}t^3$. Then, recall that the integral in last term is simply q_m (2.4) which equals $-2\pi N$ (2.19). It is finally,

$$
E_{ftt} = \int_{\mathbb{R}^3} d^3x\Big(\mathrm{Tr}(B_i + D_i\Phi)(B_i + D_i\Phi)\Big) + \sqrt{2}v(2\pi N). \tag{4.21}
$$

Therefore, the total energy of (4.17) has a lower bound,

$$
E \geqslant \sqrt{2}v(2\pi N), \tag{4.22}
$$

with the equality achieved when

$$B_i = -D_i\Phi, \tag{4.23}$$

where (4.21) and (4.23) are known as the Bogomolny energy bound and the Bogomolny equation. If we rescale the Higgs field, replacing $\Phi \to \Phi/\sqrt{2}v$, the (4.22) will be simplified as a common form,

$$E \geqslant 2\pi N. \tag{4.24}$$

For antimonopoles with a negative N value, the Bogomolny equation will be the same, but with the energy bound,

$$E \geqslant 2\pi |N|. \tag{4.25}$$

We know that from the second order differential equations of motion (2.43) (2.44), the monopole field configuration is a stable point of the field energy and also the local minimum. But the Bogomolny equation for $\lambda = 0$ is a first order one, which describes this field configuration as being a global minimum. So it is meaningful to check whether the global minimum condition also enables the field to be a stable point. Form the Bogomolny equation(4.23), we have

$$\begin{aligned} D_iF_{ij} &= -\epsilon_{ijk}D_iB_k = \epsilon_{ijk}D_iD_k\Phi \\ &= \frac{1}{2}\epsilon_{ijk}[D_i, D_k]\Phi \\ &= \frac{ig}{2}\epsilon_{ijk}F_{ik}\Phi = \frac{ig}{2}\epsilon_{ijk}[F_{ik}, \Phi] \\ &= ig[B_j, \Phi] \\ &= -ig[D_j\Phi, \Phi], \end{aligned} \tag{4.26}$$

which is of the same form as (2.44), hence the energy of the field that satisfies the Bogomolny equation is also a stable point. In other words, the global minimum is a stable point of energy.

The 't Hooft-Polyakov monopoles have monopole number $N = 1$, while $N > 1$ stands for multi-monopole case, though spherical symmetry may not exist anymore. It is suggested that in the BPS limit since the Higgs field becomes massless, there will be a scalar attractive force between static monopoles or antimonopoles at the same magnitude as the magnetic Coulomb forces [13]. Hence the acceleration between a pair of monopole and antimonopole will be twice as much as that with a massive Higgs field, and which will vanish between two monopoles or two antimonopoles. Thus the static solutions are possible with multi charged monopoles of any number [52].

As the Montonen–Olive duality implies the duality symmetry between monopoles and W bosons, the force between two static W bosons with the same electric charge should be zero as well. However, considering the monopoles are spinless, whereas the W bosons are of spin 1, a supersymmetric theory with fermions being added might be needed to solve this problem. The supersymmetric theory will not be discussed in this dissertation, but the content is available in reference [50].

4.3 Dyons

The concept of dyons was brought by Julia and Zee in 1975 [53], that Dyons are generalization of monopoles with electric charges. A spherical symmetric ansatz is still applied in this case, where the ansatz in (2.45) (2.46) are the same but with an additional equation with respect to A_0,

$$A_0 = j(r)\frac{x^a}{r}t^a. \tag{4.27}$$

and the equations of motion of static field are modified, that we now have,

$$\frac{d^2h}{dr^2} + \frac{2}{r}\frac{dh}{dr} = \frac{2}{r^2}k^2h + \lambda(h^2 - 2v^2)h, \tag{4.28}$$

$$\frac{d^2k}{dr^2} = \frac{1}{r^2}(k^2 - 1)k + g^2k(h^2 - j^2), \tag{4.29}$$

$$\frac{d^2j}{dr^2} = \frac{2k^2j}{r^2} - \frac{2}{r}\frac{dj}{dr} \tag{4.30}$$

The boundary conditions remain unchanged as (2.48) for $h(r)$ and $k(r)$, but the new boundary conditions for $j(r)$ are,

$$j(0) = 0, \qquad\qquad j(\infty) = \frac{\omega}{g}, \tag{4.31}$$

where ω is introduced through a phase factor of $e^{-i\omega t}$.

The energy of the field can be expressed in the same form of (4.17), but now with a non-vanishing E field.

$$\begin{aligned} E &= \int \Big(\frac{1}{2}\mathrm{Tr}(F_{ij}F_{ij}) + \mathrm{Tr}(D_i\Phi D_i\Phi) + U\Big)\, d^3x \\ &= \int \Big(\mathrm{Tr}(E_iE_i) + \mathrm{Tr}(B_iB_i) + \mathrm{Tr}(D_i\Phi D_i\Phi) + U\Big)\, d^3x. \end{aligned} \tag{4.32}$$

Here we shall consider the first three terms only as the last term U is always non-negative.

$$\begin{aligned} E &= \int \Big(\mathrm{Tr}(E_iE_i) + \mathrm{Tr}(B_iB_i) + \mathrm{Tr}(D_i\Phi D_i\Phi) + V\Big)\, d^3x \\ &= \int \Big[\mathrm{Tr}((E_i + \sin\mu D_i\Phi)^2) + \mathrm{Tr}((B_i + \cos\mu D_i\Phi)^2) \\ &\quad - 2\sin\mu \mathrm{Tr}(E_iD_i\Phi) - 2\cos\mu \mathrm{Tr}(B_iD_i\Phi)\Big]\, d^3x, \end{aligned} \tag{4.33}$$

where μ is an arbitrary real constant of angle. Writing the equation of motion for the gauge field and include the time component when a $SU(2)$ gauge field coupled with an adjoint Higgs field,

$$D_\mu F_{\mu\nu} = -ig[D_\nu \Phi, \Phi]. \tag{4.34}$$

When $\nu = 0$, it gives the Gauss law,

$$D_i F_{0i} = D_i E_i = ig[D_0 \Phi, \Phi], \tag{4.35}$$

which implies $D_i E_i = 0$ when $D_0 \Phi = 0$. Hence, by a similar argument of (4.19), the term of $\mathrm{Tr}(E_i D_i \phi)$ can be replaced by a total derivative,

$$2\partial_i \mathrm{Tr}(E_i \Phi) = 2\mathrm{Tr}(E_i D_i \Phi + (D_i E_i)\Phi) = 2\mathrm{Tr}(E_i D_i \Phi). \tag{4.36}$$

And the equation (4.33) can be rewritten as,

$$\begin{aligned} E = &\int d^3x \Big[\mathrm{Tr}((E_i + \sin\mu D_i \Phi)^2) + \mathrm{Tr}((B_i + \cos\mu D_i \Phi)^2)\Big] \\ &- \sin\mu\sqrt{2}v \int_{S^2_\infty} e_i dS^i - \cos\mu\sqrt{2}v \int_{S^2_\infty} b_i dS^i \\ = &\int d^3x \Big[\mathrm{Tr}((E_i + \sin\mu D_i \Phi)^2) + \mathrm{Tr}((B_i + \cos\mu D_i \Phi)^2)\Big] \\ &+ \sin\mu\sqrt{2}v q_e + \cos\mu\sqrt{2}v q_m, \end{aligned} \tag{4.37}$$

where the lower characters for b_i and e_i stand for the abelian field outside the core region as well. The energy of the field also has a lower limit, reads,

$$E \geqslant \sqrt{2}v(\sin\mu|q_e| + \cos\mu|q_m|), \tag{4.38}$$

with the equality achieved when

$$E_i = -\sin\mu D_i \Phi, \tag{4.39}$$

$$B_i = -\cos\mu D_i \Phi, \tag{4.40}$$

which should correspond to the dyon solution, that is the local stable point of the field configuration as well as the global minimum of energy. Solving (4.39) and (4.40), gives the solutions,

$$h(r) = a\coth(gar\cos\mu) - \frac{1}{gr\cos\mu}, \tag{4.41}$$

$$k(r) = \frac{gar\cos\mu}{\sinh(gar\cos\mu)}, \tag{4.42}$$

$$j(r) = \frac{\tan\mu}{gr} - a\sin\mu\coth(gr\cos\mu). \tag{4.43}$$

Where we replace $\sqrt{2}v$ with a for simplicity. One could check that the solution actually solves the equation of motion (3.5), (3.6) and (4.13).

It is also worth mention that the energy of dyon field (4.38) depends on both q_e and q_m. When (4.39), (4.40) and the prerequisite that the fields are of their asymptotic (abelian) forms $b_i = \frac{q_m}{4\pi r^2}\hat{r}^i$, $e_i = \frac{q_e}{4\pi r^2}\hat{r}^i$ are satisfied, as well as $D_i\Phi \to 0$. It cam be derived that,

$$\frac{q_e}{q_m} = \tan\mu. \tag{4.44}$$

Then the energy of the dyon as shown in (4.38) will have a lower bound of

$$E \geqslant \sqrt{2}v\sqrt{q_e^2 + q_m^2}. \tag{4.45}$$

5 Electroweak sphaleron

5.1 Saddle point and field realisation

The electroweak sphaleron [10, 11] is an unstable, finite-energy solution of static field existing in the Weinberg-Salam electroweak theory. As its name suggested, "$\sigma\phi\alpha\lambda\epsilon\rho os$" ("sphaleron") is a classical Greek adjective means "ready to fall" [10]. Actually, the sphaleron solution is a saddle point of its field energy function.

We shall elaborate the idea of a saddle point using Figure 8. The surface is constructed by a function that represents the surface's height, and there are two points x_1 and x_2 are two local minimum points on the surface X. One could take infinite paths s_1, s_2, $\cdots$, s_c, $\cdots$, s_n to connect these two points. Along all the paths, there is always one point which has the maximum height V_i on every path (sit on the green line), $i \in \{1, 2, \cdots, n\}$; and there is one point x_c being the maximum along the path s_c, which is also the infimum among all points V_i. The x_c here is a saddle point or a minimax point of the surface. It is a stationary point but is not a local extremum of the function, where the first order derivative is always zero, and its second order variation operator, i.e. Hessian, has a finite number of both positive and negative eigenvalues.

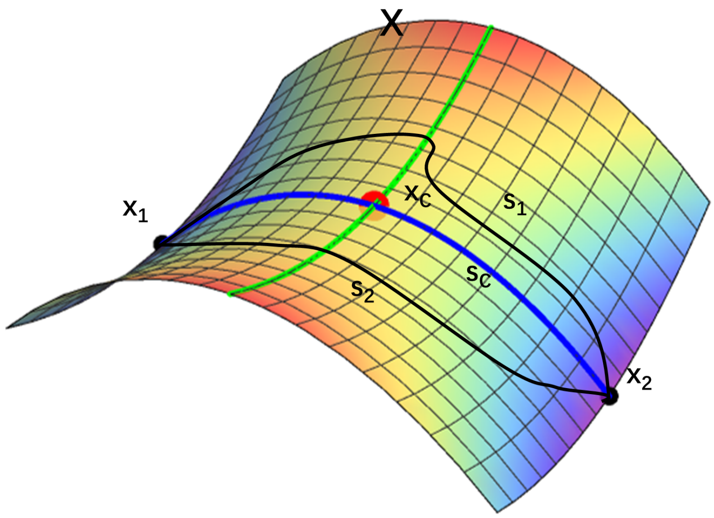


Figure 8. Saddle point x_c (red) on a surface

Ljusternik and Šnirelman first proved that there are at least two saddle points on a torus [54] by topological requirement, and later Morse Theory [55] pointed out more general relation between the saddle points of a function and a compact smooth manifold subjected to it. However, the compactness requirement is not so strict. If one extends this idea to field theory, the manifold will be a function space of finite energy for all field configurations, and potentially one could find the existence of a saddle point. Taubes in 1982 employed this view to the $SU(2)$ Yang-Mills-Higgs field theory, where the Higgs field is in the adjoint representation as we discussed in monopole case, and rigorously proved the existence of non-minimal (saddle point) solution in the BPS limit [56].

However, as Manton suggested [11], there are three main difficulties to apply Ljusternik and Šnirelman theory to search saddle point solution in field theory. The first one is the infinite dimensions of the manifold (configuration space); the second is regarding the non-compactness of the field configuration space; the third one is how to keep gauge invariance in this theory. One problem posed by the non-compact manifold is the existence of stationary point. We can rescale the fields as (2.33) - (2.38), and stationary points are not always allowed to exist in different-dimensional space. For example, we have seen the magnetic monopoles are allowed to exist in $\mathbb{R}^3$ but not in $\mathbb{R}^4$ Euclidean space. On the other hand, pure Yang-Mill theory in $\mathbb{R}^4$ space do allow stationary points (instantons) while in $\mathbb{R}^3$ it is not allowed. In the pure gauge theory, only the first field strength tensor term will appear in the rescaled energy function (2.40), where,

$$e(\mu) = \mu^{4-d} E_4 = \begin{cases} E_4, & \dfrac{de(\mu)}{d\mu} = 0, \\ \mu E_4, & \dfrac{de(\mu)}{d\mu} = E_4 \neq 0. \end{cases} \tag{5.1}$$

The gauge invariance can be dealt with gauge orbits [57]. Following Manton's [13] notation, the infinite linear space of all field configurations is $\mathcal{A}$, and the space $\mathcal{G}$ is a map from position x to the Lie group G. The quotient space $\mathcal{C} = \mathcal{A}/\mathcal{G}$, is the orbits in the real configuration space. Particularly, the gauge potential and scalar potential $\{A(x), \phi(x)\}$ and $\{A'(x), \phi'(x)\}$ are gauge equivalent on a same gauge orbit. It is discovered that there are non-contractible loops which connect the vacuum can exist in the classical Weinberg-Salam electroweak theory, which implies the existence of stationary point of energy on these loops, being the unstable solution of the finite energy field (sphaleron) [11].

5.2 Weinberg-Salam electroweak theory

Before we illustrate sphaleron, we may give a brief review of the Weinberg-Salam electroweak theory established between 1967-1968 [58, 59] and familiar readers with the notations. The gauge group considered here is $U(2)$ or equivalently $SU(2) \times U(1)$, which breaks into $U(1)_{EM}$. The Higgs scalar is a complex doublet, $\phi = \frac{v}{\sqrt{2}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$.

As two gauge groups are implemented in this theory, the overall gauge transformations is,

$$\phi \to e^{i\theta^a t^a + i\eta \mathbb{1}} \phi, \tag{5.2}$$

where θ^a and η are real numbers, and the generators $\{t^a = \frac{1}{2}\sigma^a, a = 1, 2, 3\}$, where σ^a are the Pauli matrices as shown in (2.20). The Lagrangian is,

$$\mathcal{L} = -\frac{1}{2} Tr(F_{\mu\nu} F^{\mu\nu}) + \frac{1}{4} Y_{\mu\nu} Y_{\mu\nu} + (D_\mu \phi)^\dagger D^\mu \phi - U(\phi). \tag{5.3}$$

writing the components explicitly,

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\epsilon_{abc} W^b_\mu W^c_\nu, \tag{5.4}$$

$$Y_{\mu\nu} = \partial_\mu Y_\nu - \partial_\nu Y_\mu, \tag{5.5}$$

$$D_\mu \phi = \partial_\mu \phi + \frac{1}{2} i g W^a_\mu \sigma^a \phi + \frac{1}{2} i g' Y_\mu \mathbb{1} \phi, \tag{5.6}$$

$$U(\phi) = \lambda(\frac{v^2}{2} - \phi^\dagger \phi)^2. \tag{5.7}$$

Where $W_\mu = W_\mu^a t^a$ is the $SU(2)$ non-abelian gauge field, $a \in \{1, 2, 3\}$, as the dimensionality of the group is $2^2 - 1 = 3$. While Y_μ is abelian $U(1)$ hypercharge gauge group with dimensionality 1. The gauge coupling constant is different for the two gauge groups, where g is for $SU(2)$, and g' is for $U(1)$.

As we only consider static field here, the time components of each term in (5.3) vanish and what left is,

$$E = -\int \mathcal{L}\, d^3x = \int d^3x \Big[\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{4} Y_{ij} Y_{ij} + (D_i\phi)^\dagger D_i\phi + \lambda(\frac{v^2}{2} - \phi^\dagger\phi)^2\Big]. \tag{5.8}$$

By varying the energy function or applying the Euler-Lagrange equation on it, the equations of motion are obtained,

$$D_i D_i \phi = 2\lambda(\phi^\dagger\phi - \frac{1}{2}v^2)\phi. \tag{5.9}$$

$$(D_j F_{ij})^a = \frac{1}{2} ig\Big(\phi^\dagger \sigma^a D_i\phi - (D_i\phi)^\dagger \sigma^a \phi\Big), \tag{5.10}$$

$$\partial_j Y_{ij} = \frac{1}{2} ig'\Big(\phi^\dagger D_i\phi - (D_i\phi)^\dagger \phi\Big). \tag{5.11}$$

By looking into the symmetry breaking pattern and symmetry breaking matrix, one can obtain the masses of W and Higgs bosons are $m_W = \frac{1}{2} g v$ and $m_H = \sqrt{2\lambda} v$. The *Weinberg angle* (weak mixing angle) $\tan\theta_W$ is defined to be

$$\tan\theta_W = \frac{g'}{g}. \tag{5.12}$$

The physical masses of W bosons and Higgs bosons are known [59],

$$m_W \approx 80.4GeV, \tag{5.13}$$

$$m_H \approx 125.2GeV, \tag{5.14}$$

and

$$\sin^2\theta_W \approx 0.223. \tag{5.15}$$

With the classical electroweak theory established, we can now build the setup for sphaleron in Yang-Mills-Higgs field following Manton's method [11].

Suppose a polar gauge condition is implemented on the field, that the radial part of the gauge potential $W_r = 0$. For any field configuration with $W_r \neq 0$, a gauge transformation can be applied and make it zero. The gauge potential in the spherical coordinate is related with the Cartesian coordinate with,

$$W_x dx = W_r dr + W_\theta d\theta + W_\varphi d\varphi = W_\theta d\theta + W_\varphi d\varphi, \tag{5.16}$$

as the first term vanishes in polar gauge condition. The Higgs complex doublet scalar can be equivalently expressed as a real 4-component scalar, where,

$$\phi_{Re} = \frac{v}{\sqrt{2}} \begin{pmatrix} \mathrm{Re}\phi_1 \\ \mathrm{Im}\phi_1 \\ \mathrm{Re}\phi_2 \\ \mathrm{Im}\phi_2 \end{pmatrix} \tag{5.17}$$

At $r \to \infty$, the Higgs field should approach its asymptotic value, i.e. the vev.

$$\lim_{r\to\infty} \phi(r,\theta,\varphi) = \phi^{\infty}(\theta,\varphi), \tag{5.18}$$

whose magnitude is,

$$\begin{aligned} |\phi^{\infty}(\theta,\varphi)| &= \sqrt{(\phi^{\infty})^{\dagger}\phi^{\infty}} \\ &= \frac{v}{\sqrt{2}}\sqrt{(\phi_1^*\phi_1 + \phi_2^*\phi_2)} \\ &= \frac{v}{\sqrt{2}}\sqrt{(\mathrm{Re}\phi_1)^2 + (\mathrm{Im}\phi_1)^2 + (\mathrm{Re}\phi_2)^2 + (\mathrm{Im}\phi_1)^2} \\ &= \frac{v}{\sqrt{2}}. \end{aligned} \tag{5.19}$$

Implying the Higgs field at infinity is the S^3 sphere vacuum manifold, denoted by $S^3_{\infty(Higgs)}$. So that a map ϕ^{∞}_{Re} is set up now,

$$\phi^{\infty}_{Re} : S^2_{\infty} \to S^3_{\infty(Higgs)}, \tag{5.20}$$

which brings the real space S^2_{∞} sphere to the vacuum Higgs field manifold $S^3_{\infty(Higgs)}$. From the homotopy theory, this map belongs to the homotopy group (homotopy class of maps) $\pi_2(S^3) = I$, where I is the identity element and hence this is a trivial representation. Since S^3 is connected and in this case, it is simply connected, so every map in the homotopy class along with its image in S^3 constitute a contractible loop, homotopic to I. We are free to fix the global gauge, setting the ϕ^∞ to be a constant, and still leaves an unbroken global $U(1)$ group,

$$\phi^\infty(\theta = 0, \varphi) = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.21}$$

Or equivalently,

$$\phi^\infty_{Re}(\theta = 0, \varphi) = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{5.22}$$

is a constant map as well, any maps in the space $\mathcal{W} = \pi_0(Maps(S^2 \to S^3)) = \pi_2(S^3) = I$, can be deformed into it.

Since the space $\mathcal{W}$ and the field configuration space $\mathcal{C}$ is topologically equivalent, the topological degree is consequently zero, which means solitons (monopoles) cannot exist in this electroweak field theory. However, $\mathcal{C}$ can be non-trivial and contains non-contractible loops by regarding $\mathcal{W} = \pi_1(Maps(S^2 \to S^3)) = \pi_3(S^3) = \mathbb{Z}$. A new parameter $\mu \in [0, \pi]$ shall be introduced, which parameterize the map in (5.29) to be $\phi^\infty_{Re}(\mu)$ defined on the non-contractible loop, with

$$\phi^\infty_{Re}(\mu = 0, \theta, \varphi) = \phi^\infty_{Re}(\mu = \pi, \theta, \varphi) = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \forall\ \theta, \varphi \in [0, 2\pi]. \tag{5.23}$$

And for all μ values when $\theta = 0$, we have

$$\phi^{\infty}_{Re}(\mu, \theta = 0, \varphi) = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{5.24}$$

constrained by the gauge (5.22) A map $\Psi(p) : S^3_{\infty(dom)} \to S^3_{\infty(Higgs)}$ can be directly constructed, with each point on the S^3 denoted by p,

$$\begin{aligned} p(\mu, \theta, \varphi) = (& \sin\mu \sin\theta \cos\varphi, \sin\mu \sin\theta \ \sin\varphi, \\ & \sin^2\mu \ \cos\theta + \cos^2\mu, \sin\mu \ \cos\mu(\cos\theta - 1)). \end{aligned} \tag{5.25}$$

Which is a 4-component vector with unit length $|p| = 1$. To get an intuitive view, the relation between maps are,

$$\begin{tikzcd} S^2_{\infty}(\mu) \arrow[rr, dash, "\phi^{\infty}_{Re}"] \arrow[dr, dash] & & S^3_{\infty(Higgs)} \arrow[dl, dash, "\Psi(p)"] \\ & S^3_{\infty(dom)} & \end{tikzcd}$$

$$p(\mu, \theta, \varphi)$$

with the following properties should be satisfied,

$$p((\mu = 0, \theta, \varphi) = p(\mu = \pi, \theta, \varphi) = \frac{v}{\sqrt{2}}(0, 0, 1, 0), \tag{5.26}$$

$$p(\mu, \theta = 0, \varphi) = \frac{v}{\sqrt{2}}(0, 0, 1, 0), \tag{5.27}$$

and

$$p((\mu, \theta, \varphi) = p((\mu, \theta + 2\pi, \varphi). \tag{5.28}$$

The map ϕ^{∞}_{Re} from $S^2_{\infty} \to S^3_{\infty(Higgs)}$ can be expressed equally as,

$$\phi^{\infty}_{Re}(\mu(p), \theta(p), \varphi(p)) = \Psi(p(\mu, \theta = 0, \varphi)). \tag{5.29}$$

Another requirement imposed on every $\mu(p)$ is, if the corresponding point p in $S^3_{\infty(Higgs)}$ is not the vev $\frac{v}{\sqrt{2}}(0, 0, 1, 0)$, then every $\mu(p)$ is unique.

The geometric interpretation of (5.25) is a two-sphere being the intersection between a unit three-sphere and a hyperplane denoted by $p_3 \cos\mu - p_4 \sin\mu = \cos\mu$. When the parameter μ continuously changes from 0 to π, the S^2 sphere continuously covers the whole S^3.

To represent the map $\Psi(p) : S^3_{\infty(dom)} \to S^3_{\infty(Higgs)}$, we could first consider the defining (fundamental) representation, and this has topology degree 1. A suitable expression for the asymptotic Higgs field is,

$$\phi^{\infty}_{Re}(\mu, \theta, \varphi) = \frac{v}{\sqrt{2}} \begin{pmatrix} \sin\mu \sin\theta \cos\varphi \\ \sin\mu \sin\theta \sin\varphi \\ \sin^2\mu \cos\theta + \cos^2\mu \\ \sin\mu \cos\mu(\cos\theta - 1) \end{pmatrix}. \tag{5.30}$$

Or we could use its form of complex doublet,

$$\phi^{\infty}(\mu, \theta, \varphi) = \frac{v}{\sqrt{2}} \begin{pmatrix} e^{i\varphi} \sin\mu \sin\theta \\ e^{-i\mu}(\cos\mu + i \sin\mu \cos\theta) \end{pmatrix}. \tag{5.31}$$

As mentioned above, this map with its image is a non-contractible loop in $S^3_{\infty(Higgs)}$ manifold. We shall discuss the case when $g' = 0$ for simplicity, where $SU(2)$, $U(1)$ fields decouple. The field will be spherical symmetric in $g' = 0$ limit, otherwise it will not own this symmetry. The covariant derivative of ϕ (5.6) now turns to,

$$D_\mu\phi = \partial_\mu\phi + \frac{1}{2}igW^a_\mu\sigma^a\phi. \tag{5.32}$$

An ansatz of asymptotic gauge potential can be chosen as,

$$W^\infty_\theta = \frac{i}{g}\partial_\theta M^\infty (M^\infty)^{-1}, \tag{5.33}$$

$$W^\infty_\varphi = \frac{i}{g}\partial_\varphi M^\infty (M^\infty)^{-1}. \tag{5.34}$$

With $M^\infty \in U(2)$, being

$$\begin{aligned} M^\infty &= \frac{v}{\sqrt{2}}\begin{pmatrix} \phi_2^{\infty *} & \phi_1^\infty \\ \phi_2^{-\infty *} & \phi_2^\infty \end{pmatrix} \\ &= \frac{v}{\sqrt{2}}\begin{pmatrix} e^{i\mu}(\cos\mu - i\sin\mu\cos\theta) & e^{i\varphi}\sin\mu\sin\theta \\ -e^{-i\varphi}\sin\mu\sin\theta & e^{-i\mu}(\cos\mu + i\sin\mu\cos\theta) \end{pmatrix}. \end{aligned} \tag{5.35}$$

It is easy to notice that

$$\phi^\infty = \begin{pmatrix} 0 \\ 1 \end{pmatrix} M^\infty, \tag{5.36}$$

such that the covariant derivative $D_\theta\phi$ and $D_\varphi\phi$ vanishes when the Higgs field reaches its asymptotic value, as

$$D_\theta\phi = \partial_\theta M^\infty + ig\frac{i}{g}\partial_\theta M^\infty (M^\infty)^{-1} M^\infty = \partial_\theta M^\infty - \partial_\theta M^\infty = 0, \tag{5.37}$$

and similar for $D_\varphi\phi$. Therefore the field will have finite energy as we required, which is of the same property as we mentioned for monopoles. Then a suitable ansatz for the field configuration is,

$$\phi(\mu,\theta,\varphi,r) = \frac{v}{\sqrt{2}}\left[\Big(1-h(r)\Big)\begin{pmatrix} 0 \\ e^{-i\mu}\cos\mu \end{pmatrix} + h(r)M^\infty\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right], \tag{5.38}$$

$$W_\theta(\mu,\theta,\varphi,r) = f(r)W_\theta^\infty(\mu,\theta,\varphi) = \frac{i}{g}f(r)\partial_\theta M^\infty(M^\infty)^{-1}, \tag{5.39}$$

$$W_\varphi(\mu,\theta,\varphi,r) = f(r)W_\varphi^\infty(\mu,\theta,\varphi) = \frac{i}{g}f(r)\partial_\varphi M^\infty(M^\infty)^{-1}, \tag{5.40}$$

$$W_r(\mu,\theta,\varphi,r) = 0. \tag{5.41}$$

Substituting the ansatz into the energy (5.8), it yields,

$$\begin{aligned} E = 4\pi\int dr\Bigg[&\frac{4}{g^2}f'^2\sin^2\mu + \frac{8}{g^2r^2}\big(f(1-f)\big)^2\sin^4\mu \\ &+\frac{v^2}{2}\Big(h'^2r^2\sin^2\mu + 2v^2\big(h(1-f)\big)^2\sin^2\mu + \Big[\big(f(1-h)\big)^2 - 2fh(1-f)(1-h)\Big]\cos^2\mu\sin^2\mu\Big) \\ &+\frac{\lambda v^4}{4}(h^2-1)^2r^2\sin^4\mu\Bigg]. \end{aligned} \tag{5.42}$$

Note here that f, h are functions of radial distance r. It can be checked that maximum energy is obtained when $\mu = \frac{\pi}{2}$, where we have,

$$
\begin{aligned}
E = 4\pi \int dr \Big[&\frac{4}{g^2} f'^2 + \frac{8}{g^2 r^2}\big(f(1-f)\big)^2 + \frac{v^2}{2}\Big(h'^2 r^2 + 2v^2\big(h(1-f)\big)^2\Big) \\
&+ \frac{\lambda v^4}{4}(h^2-1)^2 r^2 \Big].
\end{aligned} \tag{5.43}
$$

Using a similar rescale argument $\xi = gvr$, $f(r) \to f(\xi)$, $h(r) \to h(\xi)$. The energy (5.43) can be expressed by the new dimensionless variable ξ, that,

$$
\begin{aligned}
E = \frac{4\pi v}{g} \int d\xi \Big[&4f'^2 + \frac{8}{\xi^2}\big(f(1-f)\big)^2 + \frac{\xi^2}{2} h'^2 + \big(h(1-f)\big)^2 \\
&+ \frac{1}{4}\frac{\lambda}{g^2}\xi^2 (h^2-1)^2 \Big].
\end{aligned} \tag{5.44}
$$

The quantities with a prime here stands for the first derivative with respect to ξ, and double primes means the second derivative, e.g. $h' = \frac{dh}{d\xi}$, $h'' = \frac{d^2 h}{d\xi}$.

Regarding equations of motion (5.9), (5.10), they can be expressed as,

$$
(\xi^2 h')' = 2h(1-f)^2 + \frac{\lambda}{g^2}\xi^2(h^2-1)h, \tag{5.45}
$$

$$
\xi^2 f'' = 2f(1-f)(1-2f) - \frac{1}{4}\xi^2 h^2 (1-f). \tag{5.46}
$$

The boundary conditions required are,

$$
\begin{aligned}
h(0) &= 0, \qquad f(0) = 0, \\
h(\infty) &= 1, \qquad f(\infty) = 1.
\end{aligned} \tag{5.47}
$$

To find the sphaleron solutions, we could similarly do the same thing as monopole discussed in subsection 3.1. Linearizing the second order ODEs when $\xi \to 0$ by using approximations,

$$h(\xi) = \delta h(\xi), \tag{5.48}$$

$$f(\xi) = \delta f(\xi), \tag{5.49}$$

and substituting them into the equations of motion (5.45), (5.46), which can be reduced to,

$$2\xi\delta h'(\xi) + \xi^2\delta h''(\xi) = 2\delta h(\xi), \tag{5.50}$$

$$\xi^2\delta f''(\xi) = 2\delta f(\xi). \tag{5.51}$$

Solving these equations, reads,

$$\delta h(\xi) = \alpha\xi, \tag{5.52}$$

$$\delta f(\xi) = \beta\xi^2. \tag{5.53}$$

A similar approach as (3.13) is adapted here, the whole range from $\xi = 0$ to $\xi = \infty$ is separated to three pieces, $10^{-5} = \xi_{ini} < \xi < \xi_1 = 0.01$, $\xi_1 < \xi < \xi_{bc}$, and $\xi_{bc} < \xi < \infty$. The ξ_{bc} is the value of the dimensionless distance when $h(\xi)$, $f(\xi)$ reach their boundary conditions, and we should expect it to become smaller as λ/g^2 increases due to a similar argument that it is a scale of the length. The second part can be solved numerically by setting boundary conditions of r_1 properly. To get the two constants α and β which determine the shapes of the functions $h(\xi)$, $f(\xi)$ in the second range, the 2D shooting method is used. The appropriate constants of α, β which ensure the stability of functions up to ξ_{bc} with different values of λ/g^2 are shown in Table 5.

λ/g^2	α	β	ξ_{bc}
0	0.2979475	0.10604056	10
0	0.23565820300011	0.08373055652146128	60
0.01	0.3152031	0.1113821	10
0.1	0.4155135	0.1407652	10
1	0.7426141	0.2111319229	10
10	1.764247878	0.3165541388588	6

Table 5. The values of coefficients α, β corresponding to different λ/g^2 with restrictions of ξ_{bc}.

In equation (5.44), the coefficient is the same in Weinberg-Salam theory, that $\frac{4\pi v}{g} \sim$ 5.0 TeV, obtained by experiment [10]. Thus what we interested in is the integral, which can be calculated by summing the energies of the three pieces. When $\xi_{ini} < \xi < \xi_1$, the functions $h(\xi)$ and $f(\xi)$ are linear and the energy E_1 is,

$$E_1 = \int_{\xi_{ini}}^{\xi_1} d\xi \Big[16\beta^2\xi^2 + 8\beta^2\xi^2 \big(1-\beta\xi^2\big)^2 + \frac{\alpha^2\xi^2}{2} + \alpha^2\xi^2\big(1-\beta\xi^2\big)^2 + \frac{1}{4}\frac{\lambda}{g^2}\xi^2(\alpha^2\xi^2-1)^2 \Big]. \tag{5.54}$$

The second piece is $\xi_1 < \xi < \xi_{bc}$, in which the integral E_2 are obtained using Numerical Integral in Mathematical,

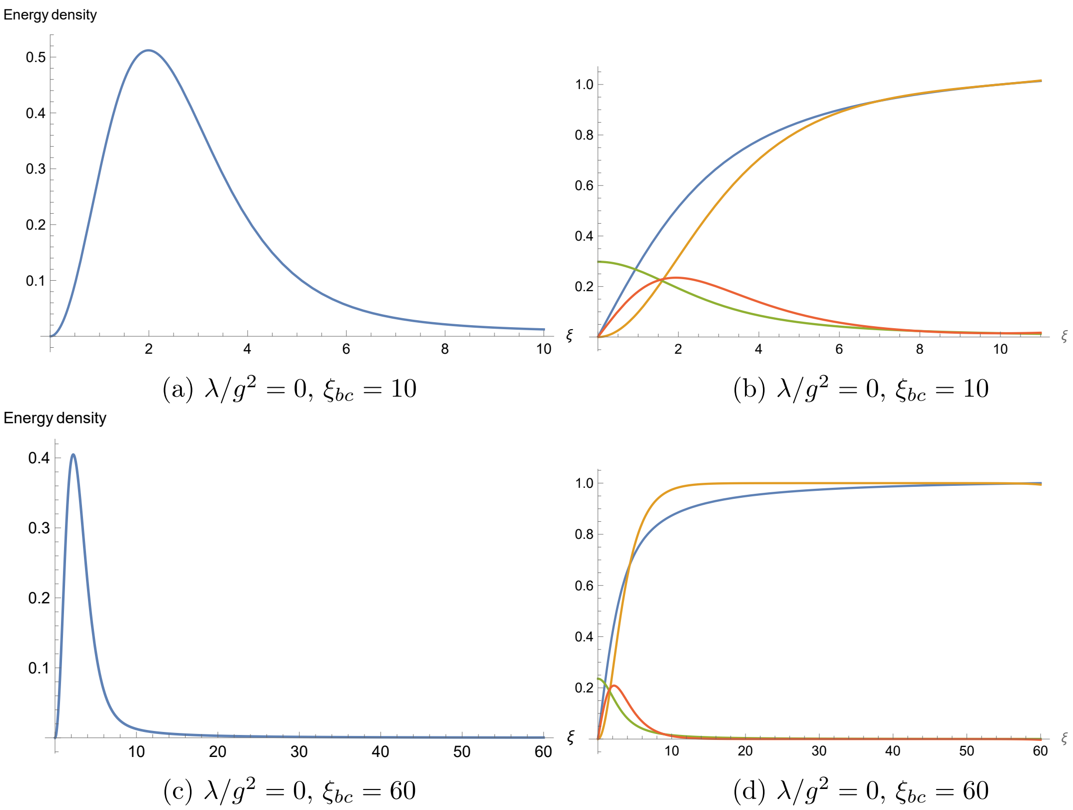
$$E_2 = \int_{\xi_1}^{\xi_{bc}} d\xi \Big[4f'^2 + \frac{8}{\xi^2}\big(f(1-f)\big)^2 + \frac{\xi^2}{2}h'^2 + \big(h(1-f)\big)^2 + \frac{1}{4}\frac{\lambda}{g^2}\xi^2(h^2-1)^2 \Big]. \tag{5.55}$$

If the boundary values of $h(\xi_{bc}) = 1$ and $f(\xi_{bc}) = 1$ are strictly reached at ξ_{bc}, then the last part energy E_3 will vanish, which we can infer from (5.55). Due to the differences are taken to be minimal during the calculation, we shall assume $E_3 = 0$. Then the total energy of sphaleron using numerical method is E_{sph-n}=$E_1 + E_2$, which is presented in Table 6, along with the data E_{sph-a} extracted from Ref. [10] using analytical method (ansatz b).

λ/g^2	E_1	E_2	E_{sph-n}	E_{sph-a}	ξ_{bc}
0	1.34×10^{-7}	1.63667	1.63667	1.566	10
0	$8.38\ \times10^{-8}$	1.53958	1.53958	1.566	60
0.01	1.45×10^{-7}	1.66528	1.66528	1.67	10
0.1	2.53×10^{-7}	1.80014	1.80014	1.83	10
1	7.16×10^{-7}	2.06621	2.06621	2.1	10
10	3.19×10^{-6}	2.37562	2.37562	2.41	6

Table 6. Energies of sphalerons corresponding to different λ/g^2 values from numerical and analytical methods.

The validity of taking different values of the boundary ξ_{bc} can be verified similarly as the monopole. If the energy density inside the integral vanishes at ξ_{bc} then the limit is acceptable. When $\lambda/g^2 = 0$ (see Figure 9), since the energy density function inside the integral (5.44) does not vanish completely at $\xi_{bc} = 10$, the limit can be set higher to $\xi_{bc} = 60$.



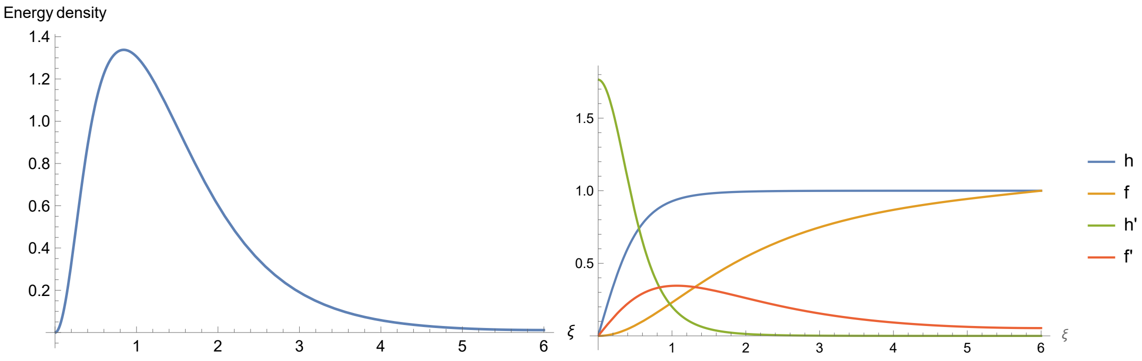
(a) $\lambda/g^2 = 0$, $\xi_{bc} = 10$ (b) $\lambda/g^2 = 0$, $\xi_{bc} = 10$

(c) $\lambda/g^2 = 0$, $\xi_{bc} = 60$ (d) $\lambda/g^2 = 0$, $\xi_{bc} = 60$

Figure 9. Energy density and functions of $\lambda/g^2 = 0$, $\xi_{bc} = 10$, 60.

For λ/g^2= 0.01, 0.1 and 1, the similar argument holds, and it is acceptable to take

$\xi_{bc} = 10$. However, when $\lambda/g^2 = 10$ the functions $h(\xi)$ and $f(\xi)$ are more sensitive to minimal fluctuation, and $\xi_{bc} = 6$ is set, thus larger errors are unavoidable, see Figure 10 for $\lambda/g^2 = 10$.



(a) $\lambda/g^2 = 10$, $\xi_{bc} = 6$ (b) $\lambda/g^2 = 10$, $\xi_{bc} = 6$

Figure 10. Energy density and functions of $\lambda/g^2 = 10$,, $\xi_{bc} = 6$.

A more clear illustration of the sphaleron energies differences is presented in Figure 11. It can be seen that our numerical results (red line with dot) E_{sph-n} fit well with the analytical results E_{sph-a}, and generally, the E_{sph-n} are sightly lower than E_{sph-a}.

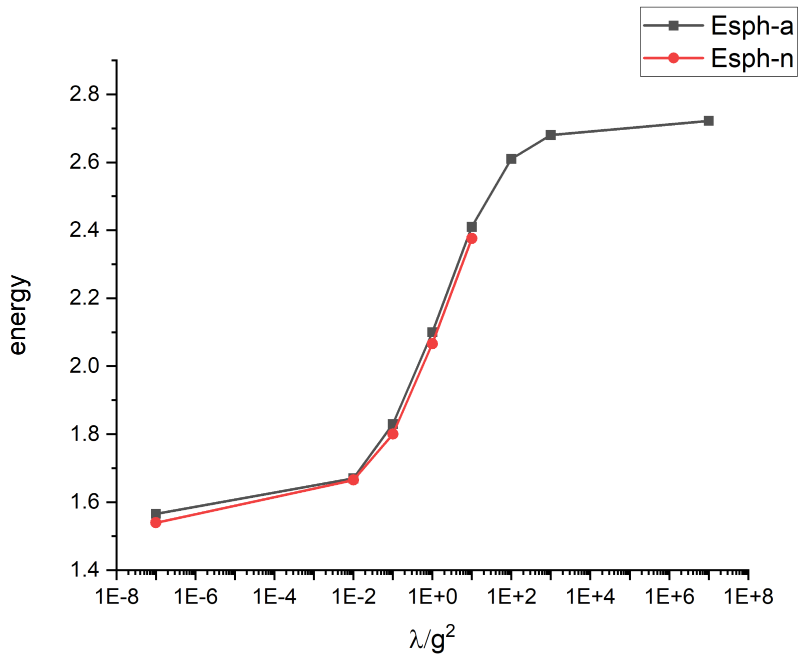


Figure 11. Sphaleron energy, unit=5.0 TeV, $\lambda/g^2 = 10^{-7}$ and 10^7 are taken to represent 0 and ∞ respectively for presentation in the log plot.

6 Conclusion and Outlooks

The 't Hooft-Polyakov monopoles and electroweak sphalerons have been demonstrated from the topological perspectives, where the former is defined in Georgi-Glashow $SU(2)$ model with adjoint Higgs field and the sphalerons are defined in Weinberg-Salam electroweak theory with a $SU(2)$ Higgs doublet. While they are both defined in $\mathbb{R}^3$ and their existences are predicted by homotopy theorem. The properties of monopole are scrutinised and its generalization dyon with electric charge is introduced, that they can be treated as point-like particles and their energy bounds are manifested.

We have demonstrated the spherical ansatz for both monopole and sphaleron, and the numerical shooting method is used to calculate their energies. For monopole, the lattice Monte-Carlo simulation is also performed via HPC. The results of which indicate that the finite-size effects cannot be underestimated, and more tests are required to specify the suitable range of lattice spacing for different lattice sizes. To inspect the reliability of those numerical methods, their results are compared with the analytical one, and we see a promising outcome of our methods, though more data will be necessary to verify their accuracy. It is appreciable that the lattice method can take the Coulomb effect into account, while other techniques lack the ability to conduct the calculation effectively, hence it gives a way for us to explore the underlying physics between monopole-antimonopole pairs. For sphalerons, the energy attained from the numerical shooting method fits the analytical results markedly, and overall slightly lower energies are presented compared with the analytical one, but we should note the results obtained by numerical approach are in fact closer to the true values.

Though the experiments and cosmic observations see no hint for monopoles yet, whether of high energy or intermediate energy regions, they could still exist to a high possibility and intriguing new physics can be derived from the search of them, and whether a theory allows the monopole to exist has become a guideline for their reliability. As more research continues, hopefully superstring [60] and supersymmetry [61, 62] theories could cast more fresh light on monopoles, as the BPS limit has implied, and we expect to see the discovery of monopoles in the long future.

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