Hydrodynamics Modes in Holography

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Contents

1 Introduction

In this dissertation, we are interested in building an axion model where the quasinormal modes can simulate the hydrodynamics with momentum dissipation. This idea is motivated by AdS/CFT correspondence [1], which indicates that a $d+1$ dimensional conformal field theory is dual to a $d+2$ supergravity theory at the low energy limit. The motivation that we are interested in the holographic model for hydrodynamics is that perturbation method is not valid for strongly coupled systems [2], and AdS/CFT correspondence is introduced as a method which can do the calculation. On the other hand, it is possible to design experiments in condensed matter to test the holographic principle, for example the optical lattice [3].

The AdS/CFT correspondence is the bridge between gravity theories on asymptotically Anti-de Sitter spacetimes to conformal field theories. The famous example is that $N = 4$ Super Yang-Mills theory in 3+1 dimensions is dual to IIB superstring theory on $AdS_5 \times S^5$. [4] For the purpose of this dissertation, gauge/-gravity duality has a very successful and important application in hydrodynamics. Heavy-ion experiments have shown that the quark-gluon plasma is best modeled by a strongly coupled relativistic fluid, which is hard to be computed by normal methods. [4] With the help of fluid/gravity correspondence, one can calculate transport coefficients systematically. It is worth noting that gauge/gravity duality successfully predict the universal result for the ratio of shear viscosity over entropy density:$\eta/s = 1/(4\pi)$ [4], which gives a better result than the traditional perturbative methods, which gives a much larger value in the range where perturbative methods are valid.

For the structure of this dissertation, in section 2, we introduce the basic knowledge in hydrodynamics, and derive the dispersion relations for both non-dissipative and dissipative hydrodynamic modes. On the gravity side, we derived the Einstein's field equation for the axion model from the perspective of the least action principle by the method of functional derivative in section 3. In the same section, we also introduce linearized gravity to derive the equations of motion for small perturbations around the black brane solution. Then we spend some pages on the detail of deriving the gauge invariant variables and their equations of motion, and finally we illustrate the numerical method to find the dispersion relation from the gravity side. In section 4, we present the numerical results of the dispersion relation for the choices of $\frac{m}{T} = \frac{1}{100}$ and $\frac{m}{T} = 100$, and compare it with the analytic dispersion relations directly derived by hydrodynamics.

2 Hydrodynamics

Hydrodynamics is an effective long-distance theory for a thermal system, whose length and time scales are much larger than the scale of corresponding microscopic constituents. In the context of this thesis, hydrodynamics constrained by Lorentz symmetry (e.g. relativistic hydrodynamics) is discussed. Relativistic hydrodynamics can not only be applied to describe the collective behavior of relativistic particles, but can also be generalized to describe other matter with Lorentz symmetry, for example, the quark-gluon plasma [5].

2.1 Degrees of Freedom for Hydrodynamics system

The degrees of freedom to describe hydrodynamics are the densities of conserved current of the corresponding microscopic field theory. According to the Noether theorem, the continuous symmetries of the microscopic theory are related to conserved currents obeying conservation equation. In particular, the continuous symmetries include Poincaré symmetry (i.e. translation, rotations and boosts in spacetime), and internal symmetry (e.g. $U(1)$ baryon number symmetry).

For example, in a translationally invariant theory with a $U(1)$ global symmetry, the $U(1)$ symmetry corresponds to a rank-1 current j^μ, and the translational symmetry in space-time corresponds to the rank-2 energy-momentum tensor $T^{\mu\nu}$, which can be defined by the variation of the action with respect to the metric [6].

According to Noether theorem, both j^μ and $T^{\mu\nu}$ are divergence-free,

$$\partial_\mu j^\mu = 0, \tag{1}$$

$$\partial_\mu T^{\mu\nu} = 0. \tag{2}$$

Separating the time and space index, these equations are equivalent to

$$\begin{aligned}\partial_0 j^0 + \partial_i j^i &= 0,\\ \partial_0 T^{00} + \partial_i T^{i0} &= 0,\\ \partial_0 T^{0i} + \partial_j T^{ji} &= 0.\end{aligned} \tag{3}$$

If we integrate these three equations above over the space, then applying the Gauss's theorem [7] and appropriate boundary conditions, we will get the expected conserved charges and the corresponding charge densities: j^0, T^{00}, T^{i0}. we further define

$$T^{00} \equiv \varepsilon, \tag{4}$$

$$T^{i0} \equiv \Pi^i, \tag{5}$$

where ϵ is the energy density and Π^i is the momentum density. Hence we can write rewrite equation (3) as

$$\partial_t j^0 = -\partial_i j^i, \tag{6}$$

$$\partial_t \varepsilon = -\partial_i \Pi^i, \tag{7}$$

$$\partial_t \Pi^i = -\partial_j T^{ji}. \tag{8}$$

These are the equations of motion of our hydrodynamic degrees of freedom j^0, ε and Π^i. They describe a hydrodynamic system with momentum conservation.

2.2 Constitutive relations

According to equations (1) and (2), the number of unknowns is larger than the number of equations. Therefore, constitutive relations the equations of state are needed to determine the system. The constitutive relations [8] are assumed to be able to express $T^{\mu\nu}$ and j^μ in terms of the hydrodynamic variables: a local temperature T, a local fluid velocity u^μ, and a local chemical potential μ. In other words, these relations express the spatial components j^i and T^{ij} in terms of the densities of conserved charges. In particular, the general constitutive relations for j^μ to the lowest order in derivatives can be written as:

$$j_i = -D\partial_i j^0, \tag{9}$$

where D is the diffusion constant. Then, taking the derivative with respect to spatial coordinates, equating with the conservation law (6), we get the diffusion equation for a conserved density j^0:

$$\partial_t j^0 - D\nabla^2 j^0 = 0, \tag{10}$$

where ∇^2 is the spatial Laplacian.

In turn, the stress-energy tensor can be decomposed as [8]

$$\begin{aligned} T^{\mu\nu} =& (\varepsilon + p) u^\mu u^\nu + p\eta^{\mu\nu} - \zeta\left(\eta^{\mu\nu} + u^\mu u^\nu\right)\partial_\alpha u^\alpha \\ & - \eta\left(\eta^{\mu\alpha} + u^\mu u^\alpha\right)\left(\eta^{\nu\beta} + u^\nu u^\beta\right)\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d}\eta_{\alpha\beta}\partial_\sigma u^\sigma\right) \end{aligned} \tag{11}$$

where u^μ is the local four-velocity of the fluid with normalization condition $u_\mu u^\mu = -1$; d is the spatial dimensions; $\eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, ..., +1)$ is the $(d+1)$ Minkowski metric,ϵ is the local energy density, p is the thermodynamic pressure in the local rest frame of the fluid, ζ is the bulk viscosity, η is the shear viscosity. [8]

To see the long-lived collective excitations predicted by hydrodynamics, we consider a field theory that is slightly perturbed away from equilibrium:

$$u_\alpha = (-1, v_i),\ i = 1, ..., d,\ |\mathbf{v}| \ll 1, \tag{12}$$

$$\varepsilon = \epsilon + \delta\varepsilon, \tag{13}$$

$$p = P + \delta p. \tag{14}$$

where $\epsilon =< \varepsilon >$, $P = \frac{1}{d} < T_{ii} >$ are the density and pressure of the fluid in thermodynamic equilibrium, and $\delta\varepsilon$, δp and v_i are the respective small perturbations. Substituting equations (12), (13) and (14) into equations (11), we get the perturbative expansion for the energy-momentum tensor. As a result, at zeroth order in derivatives, we have the constitutive relation

$$T^{\mu\nu} = (\epsilon + P)u^{\mu}u^{\nu} + P\eta^{\mu\nu}. \tag{15}$$

Equivalently,

$$T^{00} = \epsilon, \tag{16}$$

$$T^{ij} = P\delta^{ij}, \tag{17}$$

$$T^{0i} \equiv \Pi^{i} = (\epsilon + P)u^{0}u^{i} = (\epsilon + P)v^{i}. \tag{18}$$

Then with the substitution of v^i from equation (18), we have the constitutive relation at the first order in derivatives:

$$\delta T_{ij} = \delta_{ij}v_s^2\delta\varepsilon - \frac{\zeta}{\epsilon + P}\delta_{ij}\partial_k\Pi_k - \frac{\eta}{\epsilon + P}(\partial_i\Pi_j + \partial_j\Pi_i - \frac{2}{d}\delta_{ij}\partial_k\Pi_k) \tag{19}$$

where the free parameters: $v_s = \sqrt{\frac{\partial P}{\partial \epsilon}}$ is the speed of sound. Like in any effective field theory, these free parameters cannot be calculated directly by hydrodynamics but can be derived by microscopic kinetic theory, for example using the Boltzmann equation [9].

2.3 Dispersion relations of hydrodynamic modes

To find the dispersion relations of hydrodynamic modes, we need to take Fourier transform of hydrodynamic equation of motion and constitutive equations. In the case of the diffusion equation, we substitute for [8]

$$j^0(t, \mathbf{x}) = \int \frac{d^{(d+1)}q}{(2\pi)^{(d+1)}} e^{-i\omega t + i\mathbf{q}\cdot\mathbf{x}} \tilde{j}_0(\omega, \mathbf{q}) \tag{20}$$

into equation (10), to then get the diffusive mode

$$\omega = -i\mathrm{q}^2 D. \tag{21}$$

By introducing new variables

$$
\begin{aligned}
a &= v_s^2 \delta\varepsilon \\
b &= -\frac{\zeta}{\epsilon + P} \\
c &= -\frac{\eta}{\epsilon + P},
\end{aligned}
\tag{22}
$$

the constitutive relation (19) is simplified as

$$
\delta T_{ij} = a\delta_{ij} + b\partial_k \Pi_k + c(\partial_i \Pi_j + \partial_j \Pi_i - \frac{2}{d}\delta_{ij}\partial_k \Pi_k) \tag{23}
$$

Analog to the process for the diffusion equation, Fourier transform of the constitutive equation is given by

$$
\delta T_{ij} = a\delta_{ij} + biq_l \Pi_l + c(iq_i \Pi_j + iq_j \Pi_i - \frac{2}{d}\delta_{ij} iq_l \Pi_l). \tag{24}
$$

The Fourier transform of equation (8) is given by

$$
\begin{aligned}
\partial_j T^{ij} &= iq_j T^{ij} \\
&= iaq^i - bq^i(q\cdot\Pi) - c\left((q\cdot\Pi)q^i\left(1 - \frac{2}{d}\right) + q^2\Pi^i\right) \\
&= -\partial_t \Pi^i = i\omega\Pi^i
\end{aligned}
\tag{25}
$$

The Fourier transform of the small variation of equation (7) is given by

$$
-i\omega\delta\varepsilon = -iq_i\Pi^i \;\longrightarrow\; \omega\delta\varepsilon = q\cdot\Pi. \tag{26}
$$

From equations (25) and (26), we get:

$$
i\omega\Pi^i = (ia - b\omega\delta\varepsilon)q^i - c\left(\omega\delta\varepsilon q^i\left(1 - \frac{2}{d}\right) + q^2\Pi^i\right). \tag{27}
$$

It is natural to decompose the momentum in transverse and longitudinal components with respect to q_i [8]:

$$\Pi^i = \Pi^i_{\parallel} + \Pi^i_{\perp} \tag{28}$$

where $q_i \Pi^i_{\perp} = 0$. Multiplying equation (27) with $\Pi^i_{\perp}$, we get the shear mode:

$$\omega_{\perp} = -iq^2 \frac{\zeta}{\epsilon + P}. \tag{29}$$

Similarly, multiplying equation (27) with q_i and rearranging, we obtain:

$$\omega^2 \delta\varepsilon + i\omega\delta\varepsilon q^2(-b - c\frac{2d-2}{d}) - aq^2 = 0. \tag{30}$$

With the substitution for $a = v_s^2 \delta\varepsilon$, and $\delta\varepsilon$ canceling out, we get an implicit dispersion relation:

$$\omega^2 + i\omega q^2(-b - c\frac{2d-2}{d}) - v_s^2 q^2 = 0. \tag{31}$$

Solving this quadratic equation, we get:

$$\omega_{\pm} = \frac{iq^2(b + c(2d-2)/d) \pm \sqrt{q^2(4v_s^2 - (b + c(2d-2)/d)^2 q^2}}{2} \tag{32}$$

$$= \frac{iq^2(b + c(2d-2)/d) \pm 2v_s q + O(q^3)}{2}. \tag{33}$$

Replacing b and c, we find for the longitudinal mode:

$$\omega_{\parallel} = \pm v_s q - \frac{i}{2} q^2 (\zeta + \frac{2d-2}{d}\eta)/(\epsilon + P). \tag{34}$$

This is the standard dispersion relation of a sound mode propagating in viscous medium, where we observed that the real part of $\omega_{\parallel}$ is proportional to the wavenumber q, while its imaginary part is proportional to q^2. The imaginary part of $\omega_{\parallel}$ leads to the exponential decay of the hydrodynamic mode, hence corresponds to the damping of the mode.

2.4 Correlation Functions

The transport coefficients D, η, ζ of the hydrodynamics can be treated as parameters in the effective field theory which need to be determined by the microscopic theory. Different from the constitutive relations of first and second order hydrodynamics which depends on the choice of frame, the coefficients D, η, ζ are frame invariant. They can be matched to microscopic theory by the linear response theory, which allows to express the coefficients in terms of correlation functions via the so-called Kubo formulas. The retarded function [8] for momentum density is given by

$$G^R_{\Pi_I\Pi_j}(\omega,\mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)\frac{\eta\mathbf{k}^2}{i\omega - \gamma_\eta \mathbf{k}^2} + \frac{k_i k_j}{\mathbf{k}^2}\frac{w(\mathbf{k}^2 v_s^2 - i\omega\gamma_s\mathbf{k}^2)}{\omega^2 - \mathbf{k}^2 v_s^2 + i\omega\gamma_s\mathbf{k}^2} \tag{35}$$

$$G^R_{\epsilon\Pi_j}(\omega,\mathbf{k}) = G^R_{\Pi_j\epsilon}(\omega,\mathbf{k}) = \frac{w\omega k_i}{\omega^2 - \mathbf{k}^2 v_s^2 + i\omega\gamma_s\mathbf{k}^2} \tag{36}$$

$$G^R_{\epsilon\epsilon}(\omega,\mathbf{k} = \frac{w\mathbf{k}^2}{\omega^2 - \mathbf{k}^2 v_s^2 + i\omega\gamma_s\mathbf{k}^2} \tag{37}$$

where $w = \epsilon + P$ is the equilibrium enthalpy density, $v_s^2 = \partial p/\partial\epsilon$, $\gamma_\eta = \eta/(\epsilon + P)$, $\gamma_s = \left(\frac{2d-2}{d}\eta + \zeta\right)/(\epsilon + P)$, d is the number of spatial dimensions. Kubo formulas are found by evaluating the imaginary parts of the retarded functions [8]:

$$\eta = -\frac{\omega}{\mathbf{k}^2}\frac{1}{d-1}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) ImG^R_{\Pi_I\Pi_j}(\omega,\mathbf{k}\to 0), \tag{38}$$

$$\frac{2d-2}{d}\eta + \zeta = -\frac{\omega^3}{\mathbf{k}^4} ImG^R_{\epsilon\epsilon}(\omega,\mathbf{k}\to 0) \tag{39}$$

Evidently, the poles of the retarded functions give the dispersion relations.

2.5 Hydrodynamics with dissipation

To describe a hydrodynamics system with momentum dissipation, the translation symmetry in space is broken hence the momentum conservation equation (7) is modified to

$$\partial_t \Pi^i + \partial_j T^{ji} = -\Gamma\Pi^i \tag{40}$$

where the constant rate Γ is a parameter to ensure the momentum dissipates isotropically, while the constitutive relations (18) remain the same if Γ is sufficiently small [10]. In the case of relevance for this work, we consider a (2+1) dimensional system without bulk viscosity, hence the constitutive relations become

$$\delta T_{ij} = \delta_{ij} v_s^2 \delta\varepsilon - \frac{\eta}{\epsilon + P}(\partial_i \Pi_j + \partial_j \Pi_i - \delta_{ij}\partial_k \Pi_k) \tag{41}$$

Analog to the method used in section 2.3, from equations (7), (40) and (41), we can derive the modified momentum conservation equations for small perturbation around the equilibrium

$$\partial_t \delta\varepsilon + \partial_i \delta\Pi^i = 0, \tag{42}$$

$$\partial_t \delta\Pi^i + \Gamma\delta\Pi^i + \frac{\partial p}{\partial\epsilon}\partial_i \delta\varepsilon - \frac{\eta}{\epsilon + P}\partial_j \partial_j \delta\Pi^i = 0, \tag{43}$$

where if we Fourier transform these equations, we find the equations in momentum space $(\omega, \mathbf{k})$:

$$-i\omega\delta\varepsilon + i\mathbf{k}\cdot\delta\Pi = 0, \tag{44}$$

$$-i\omega\delta\Pi_i + \Gamma\delta\Pi_i + \frac{\partial p}{\partial\epsilon} i k_i \delta\epsilon - \frac{\eta}{\epsilon + P}(ik)^2 \delta\Pi^i = 0. \tag{45}$$

Cancelling $\mathbf{k}\cdot\Pi$, we get the dispersion relation

$$-i\omega^2 + (\Gamma + \frac{\eta}{\epsilon + P}k^2)\omega + \frac{\partial p}{\partial\epsilon} i k^2 = 0. \tag{46}$$

Solving for ω, we have

$$\omega_\pm = -\frac{i}{2}(\Gamma + \frac{\eta}{\epsilon + P}k^2) \pm k\sqrt{\frac{\partial p}{\partial\epsilon} - \frac{1}{4}\left(\Gamma k^{-1} + \frac{\eta}{\epsilon + P}\right)^2} \tag{47}$$

We compare this dispersion relation to the translational invariant case (34) for

$d = 2$ and $\zeta = 0$:

$$\omega_{\parallel} = -\frac{i}{2}\frac{\eta}{\epsilon + P}k^2 \pm \sqrt{\frac{\partial p}{\partial \epsilon}}k. \tag{48}$$

Supposing that Γ and η is small enough, in the range of k which is small enough so that the second terms of (47) is imaginary, the frequency ω becomes pure imaginary and hence the corresponding mode is heavily damped. In the complementary range of k, ω is a complex number and we observe that the momentum dissipation rate Γ makes the imaginary part of $omega$ increase, and the real part decrease; physically, the corresponding hydrodynamic modes damps more heavily and oscillates with less frequency due to the momentum dissipation analog to the damped simple harmonic oscillator.

3 Axion Model

The microscopic theory for the dissipative hydrodynamics is a field theory for a specific, strongly coupled thermal state. This theory is dual to an axion model. On the gravity side, the presence of axion fields breaks the translation symmetry which is what we expected to be happened on the field theory side. In practice, we are interested in using the perturbation abound the black brane solution for the axion model to calculate the numerical results of dispersion relation for the dual hydrodynamics with dissipation.

3.1 Einstein Field Equations

The Einstein's field equations [11] describe the dynamics of the spacetime, which can be expressed in a general form:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{49}$$

where $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, Λ is the cosmological constant and the constant κ can be expressed in terms of the Newton's gravitational constant G by $\kappa^2 = 8\pi G$. These equations relate the geometric properties of spacetime to the matter, given by the stress-energy tensor $T_{\mu\nu}$. Moreover, in the weak field limit, Einstein's field equations boil down to the Newtonian gravity. Einstein equations are encoded in the action of the gravitational system, and derived by the least action principle. The generic action can be written as

$$S[g_{\mu\nu}, \phi] = S_{\mathrm{EH}}[g_{\mu\nu}] + S_{\mathrm{matter}}[g_{\mu\nu}, \phi], \tag{50}$$

where S_{matter} is the action of any possible field ϕ living in the spacetime, which can be any type of matter fields, for instance, scalar, vector and spinor field, and $S_{EH}[g_{\mu\nu}]$ is the Einstein-Hilbert action for the spacetime.

In particular, the Einstein-Hilbert action with cosmological constant in $(d+1)$

spacetime dimensions is given by

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}(R+2\Lambda). \tag{51}$$

Varying the action S with respect to the metric $g_{\mu\nu}$, we will get:

$$\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = \frac{\sqrt{-g}}{2\kappa^2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right). \tag{52}$$

To match to the Einstein equations (49), we define the generic form of the energy-momentum tensor $T_{\mu\nu}$ in terms of the functional derivative of the matter action as:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{53}$$

These are the basic equations governing the spacetime dynamics, where the specific expression of the right hand side of the equations changes with respect to different matter content. Note that here and throughout we are using the mostly plus signature.

3.2 Maximally symmetric Anti-de Sitter spacetime

Here we briefly review the concepts of symmetry and killing vector in the context of general relativity. Although any system in the real world has somewhat symmetry breaking, the symmetric metric can be regarded as the background in the perturbation theory. The spacetime has a symmetry if the geometry is invariant under a certain coordinate transformation. In other words, the corresponding metric is invariant under such transformation [11]. In particular, the symmetry of metric is also called an isometry and isometries are related to Killing vector fields K which satisfy the Killing's equation [11]:

$$\mathscr{L}_K g_{\mu\nu} = \nabla_{(\mu}K_{\nu)} = 0, \tag{54}$$

where $\mathscr{L}_K g_{\mu\nu}$ is the Lie derivative of the metric $g_{\mu\nu}$ and ∇_μ is the normal covariant derivative with respect to $g_{\mu\nu}$ and K is also the generator of corresponding isometries.

n-dimensional spacetime either Euclidean or not contains at most $\frac{1}{2}n(n+1)$ Killing vectors [11], and hence it is obvious that an n-dimensional spacetime with $\frac{1}{2}n(n+1)$ Killing vectors is the so-called maximally symmetric space. It has remarkable properties: first the curvature is a constant everywhere and in every direction. In a normal coordinate system, the Riemann tensor $R_{\rho\sigma\mu\nu}$ with maximal symmetry should be proportional to $g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}$, since we require the Riemann tensor to be invariant under local Lorentz transformation and its symmetries under the permutations of the indices [11]. Finally, by contracting the tensors twice, we can confirm the coefficient as:

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}). \tag{55}$$

In other words, the metric of maximally symmetric spacetime should have the same Riemann tensor $R_{\rho\sigma\mu\nu}$ as equation (55).

From the point view of general relativity, Anti-de Sitter (AdS) spacetime is the maximally symmetric solution of source-free Einstein field equation with negative cosmological constant [4]. AdS space time has negative constant curvature [4], compared to Minkowski and de Sitter spacetimes which have zero and positive curvature respectively.

The $(d+1)$-dimensional Anti-de Sitter spacetime can be defined as a spacetime embedded into a $(d+2)$-dimensional Minkowski spacetime with metric $\eta^{\mu\nu}$ = diag(-, +, +, ... , +,-). Suppose the Minkowski spacetime has coordinates (X^0,X^1, ...,X^d,X^{d+1}) $\in R^{d,2}$, with the line element,

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + \cdots + (dX^d)^2 - (dX^{d+1})^2 = \eta_{\mu\nu}dX^\mu dX^\nu, \tag{56}$$

where the indices $\mu,\nu \in (0,\ldots,d+1)$. The AdS_{n+1} in this spacetime is a hypersurface, satisfying the relation:

$$-(X^0)^2 + (X^1)^2 + \cdots + (X^d)^2 - (X^{d+1})^2 = \eta_{\mu\nu}X^\mu X^\nu = -L^2, \tag{57}$$

where L is the radius of curvature of this spacetime.

From the point view of an intrinsic definition of a manifold, the AdS_{d+1} spacetime can be described by the following metric:

$$ds^2 = \frac{r^2}{R^2}(-d\tau^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2 \tag{58}$$

where $-d\tau^2 + d\vec{x}^2$ is the metric of a d-dimensional Minkowski spacetime, r is a coordinate ranging from 0 to infinity, L is a parameter of this spacetime called the AdS curvature radius. It is worth noting that as expected AdS_{d+1} spacetime has a maximally symmetric Riemann tensor:

$$R_{\mu\nu\alpha\beta} = -L^2(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) \tag{59}$$

and hence the corresponding Ricci scalar is constant:

$$R = -d(d+1)L^{-2}. \tag{60}$$

There is a convenient way to rewrite equation (59) by a coordinate transformation $z = L^2/r$:

$$ds^2 = \frac{L^2}{z^2}(-d\tau^2 + d\vec{x}^2 + dz^2). \tag{61}$$

In this work, we are interested in spacetime where the scale invariance is broken by a non-zero temperature.

3.3 The Axion model and the Equation of Motions

We are interested in the axion model which is dual to the hydrodynamics with energy and momentum dissipation. This model has an action including scalar fields:

$$S = \int dx^4 \sqrt{-g}\left(\frac{1}{2}M_{Pl}^2\left(R+\frac{6}{L^2}\right)-\frac{1}{2}\partial^\mu\phi^I\partial_\mu\phi_I\right), \tag{62}$$

where the index $I,J = 1,2$, M_{Pl} is the Planck mass, R is the Riemann scalar, L is the Anti-de Sitter radius, ϕ_I is a doublet of scalar fields. Hence we can identify S_{EH} and S_{matter} as

$$S_{EH} = \int dx^4 \frac{1}{2}M_{Pl}^2\sqrt{-g}\left(R+\frac{6}{L^2}\right), \tag{63}$$

$$S_{\text{matter}} = \int dx^4 \left(-\frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial^\mu\phi^I\partial_\mu\phi_I\right). \tag{64}$$

To get the Einstein field equation for this model, we apply the variational method as shown in section (2.1): first varying the S_{EH} with respect to the metric $g^{\mu\nu}$

$$\delta S_{\text{gravity}} = \int dx^4 \frac{M_{Pl}^2}{2}\left(\delta(\sqrt{-g})(R+\frac{6}{L^2})+\sqrt{-g}\delta R\right) \tag{65}$$

$$= \int dx^4 \frac{M_{Pl}^2}{2}\delta g^{\mu\nu}\sqrt{-g}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}-\frac{3}{L^2}g_{\mu\nu}\right)+\sqrt{-g}\delta R, \tag{66}$$

where we use the following identities

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{67}$$

and the Palatini equation:

$$\delta R_{\mu\nu} = \nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu})-\nabla_\nu(\delta\Gamma^\alpha_{\alpha\nu}). \tag{68}$$

Moreover, the boundary term $\sqrt{-g}\delta R$ is cancelled by introducing the Gibbson-Hawking-York boundary term [12].

Finally, we get the functional derivative:

$$\frac{\delta S_{\text{gravity}}}{\delta g^{\mu\nu}} = \frac{M_{Pl}^2}{2}\sqrt{-g}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R-\frac{3}{L^2}\right). \tag{69}$$

Similarly, varying S_{matter} with respect to $g^{\mu\nu}$, we get:

$$\delta S_{\text{matter}} = \int d^4x(-\frac{1}{2}\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\partial_\alpha\phi^I\partial^\alpha\phi_I + \partial_\mu\phi^I\partial_\nu\phi_I\right)\delta g^{\mu\nu} \tag{70}$$

and the corresponding functional derivative:

$$\frac{\delta S_{matter}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\partial_\alpha\phi^I\partial^\alpha\phi_I + \partial_\mu\phi^I\partial_\nu\phi_I\right). \tag{71}$$

Then according to the least action principle, we get the equation of motion for the metric:

$$\begin{aligned}
&\frac{\delta S_{\text{gravity}}}{\delta g^{\mu\nu}} + \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} = 0 \ \rightarrow \\
&\frac{M_{Pl}^2}{2}\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{3}{L^2}\right) - \frac{1}{2}\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\partial_\alpha\phi^I\partial^\alpha\phi_I + \partial_\mu\phi^I\partial_\nu\phi_I\right) = 0.
\end{aligned} \tag{72}$$

Rearrange it:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{3}{L^2}g_{\mu\nu} + 8\pi G T_{\mu\nu}, \tag{73}$$

where

$$T_{\mu\nu} = \left(-\frac{1}{2}g_{\mu\nu}\partial_\alpha\phi^I\partial^\alpha\phi_I + \partial_\mu\phi^I\partial_\nu\phi_I\right), \tag{74}$$

and the cosmological constant can be expressed in terms of the Anti de Sitter radius as:

$$\Lambda = \frac{3}{L^2}. \tag{75}$$

Varying S_{matter} with respect to the scalar field, we get:

$$\delta S_{\text{matter}} = -\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\delta\phi_I \tag{76}$$
$$= -\int d^4x\sqrt{-g}g^{\mu\nu}\nabla_\mu\phi^I\nabla_\nu\delta\phi_I \tag{77}$$
$$= \int d^4x\nabla_\nu(\sqrt{-g}g^{\mu\nu}\nabla_\mu\phi^I)\delta\phi_I, \tag{78}$$

where we have used expansion of Gauss's theorem to general relativity and assumed that the boundary terms vanish. And then, equation of motion for the scalar fields is given by

$$\nabla_\nu(\sqrt{-g}g^{\mu\nu}\nabla_\mu\phi^I) = 0. \tag{79}$$

In my calculation, I use the conventions of [10], and set $M_{Pl}^2 = 2$ so that the action becomes:

$$S = \int dx^4\sqrt{-g}\left(\left(R + \frac{6}{L^2}\right) - \frac{1}{2}\partial^\mu\phi^I\partial_\mu\phi_I\right), \tag{80}$$

and the stress-energy tensor becomes:

$$T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}\partial_\alpha\phi^I\partial^\alpha\phi_I + \partial_\mu\phi^I\partial_\nu\phi_. \tag{81}$$

The equations of motion become:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu} + \frac{1}{2}T_{\mu\nu} \tag{82}$$
$$\nabla_\nu(\sqrt{-g}g^{\mu\nu}\nabla_\mu\phi^I) = 0 \tag{83}$$

where the cosmological constant is $\Lambda = \frac{3}{L^2}$. In the following we use the units $L \equiv 1$.

The theory with the action in eq.(80) has a planar Schwarzschild-AdS$_4$ black brane solution [10]:

$$\begin{aligned}
&ds^2 = -r^2 f dt^2 + r^2(dx^2+dy^2) + \frac{dr^2}{r^2 f},\\
&\phi_1 = mx,\ \phi_2 = my,\\
&f(r) = 1 - \frac{m^2}{2r^2} - \frac{r_0^3}{r^3}\left(1 - \frac{m^2}{2r_0^2}\right),
\end{aligned} \tag{84}$$

where the radial coordinate r ranges from r_0 (the horizon radius of the black brane) to infinity (the asymptotically AdS boundary), and the coordinates t, x, y is the direction of the dual field theory.

We use the thermodynamic relations of the field theory state computed via the standard AdS/CFT dictionary [10], [13], [14]:

$$T = \frac{r_0}{4\pi}\left(3 - \frac{m^2}{2r_0^2}\right), \tag{85}$$

$$\epsilon = 2r_0^3\left(1 - \frac{m^2}{2r_0^2}\right), \tag{86}$$

$$P = r_0^3\left(1 + \frac{m^2}{2r_0^2}\right), \tag{87}$$

$$s = 4\pi r_0^2, \tag{88}$$

where T is the temperature, ϵ is the equilibrium energy density, P is the equilibrium pressure and s is the entropy density of the state and r_0 is the position of the black brane horizon. The momentum dissipation rate for sufficiently small values of m

$$\Gamma = \frac{s}{4\pi(\epsilon + P)} m^2 = \frac{m^2}{4\pi T}, \tag{89}$$

which after substituting in the dispersion relation in equation (47) gives [10]

$$\omega_\pm = \pm k\sqrt{\sqrt{\frac{1}{4} + \frac{3m^2}{32\pi^2 T^2} - \frac{(m^2+k^2)^2}{64\pi^2 k^2 T^2}} - \frac{i}{8\pi T}(m^2+k^2) + \ldots..} \tag{90}$$

For the purpose of comparison to the numerical results in section (4), we can expand this equation for small $\frac{m}{T}$ and small k, which gives

$$\begin{aligned}\omega(k)_{\pm} = &\left(\pm\frac{k}{\sqrt{2}} - \frac{ik^2}{8\pi T} \mp \frac{k^3}{64(\sqrt{2}\pi^2 T^2)} + O[k]^4\right) + \\ &\left(-\frac{iT}{8\pi} + \frac{(-1+3\pi)k}{32\sqrt{2}\pi^2} + \frac{(-1+3\pi)k^3}{2048\sqrt{2}\pi^4 T^2}\right)\left(\frac{m}{T}\right)^2 + O[\frac{m}{T}]^4. \end{aligned} \tag{91}$$

It is clear that the dominant terms for small k and small $\frac{m}{T}$ are

$$\omega(k) = \pm\frac{k}{\sqrt{2}} - \frac{ik^2}{8\pi T}, \tag{92}$$

where we see that the hydrodynamic modes at small k and small $\frac{m}{T}$ behave like standard sound modes.

By comparison, we are also interested in the behaviour of the hydrodynamics modes for small k but large $\frac{m}{T}$. In this region, equivalently, we expand equation (90) for small k and small $\frac{T}{m}$, which is given by

$$\begin{aligned}\omega(k) = &e\left(\mp i\sqrt{\frac{3\pi}{2}}\frac{k^2}{T} + O\left(k^4\right)\right) + \\ &e^3\left(\mp i\frac{2\sqrt{2/3}\pi^{3/2}}{T}k^2 + O\left(k^4\right)\right) + O\left(e^4\right), \end{aligned} \tag{93}$$

where $\frac{1}{e} \equiv \frac{m}{T}$, and we can see the dominant terms are purely imaginary

$$\omega(k)_{\pm} = \pm i(-\frac{T}{m} - \frac{T^2}{m^3}2\sqrt{\frac{2}{3}}\pi^{3/2})k^2. \tag{94}$$

We shall verify the behaviour (92) and (94) in section 4.

3.4 Linearized Gravity and the equations of motion for the Axion model

In this dissertation, we are interested in small perturbations around the black brane solution (84) so that we can ignore any terms of order higher than the first order :

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu},\ |h_{\mu\nu}| \ll 1, \tag{95}$$

$$\tilde{\phi}_I = \phi^{(0)}_I + \delta\phi_I,\ |\delta\phi_I| \ll 1, \tag{96}$$

where $g^{(0)}_{\mu\nu}$ and $\phi^{(0)}_I$ correspond to the black brane solution, $h_{\mu\nu}$ and $\delta\phi_I$ are the perturbation fields. In particular, we focus on the longitudinal sector [10] (h_{tt}, h_{xx}, h_{yy}, h_{rr}, $h_{xt}, h_{xr}, h_{tr}, \delta\phi_1$) at non-zero frequency ω and wavenumber $\mathbf{k}$ pointing in the direction of x. In other words, we are interested in the Fourier modes

$$h_{tt}(t,r,x) = h_{tt}(r)e^{ikx-it\omega},\ h_{tr}(t,r,x) = h_{tr}(r)e^{ikx-it\omega},\ etc. \tag{97}$$

Substituting these fields with the perturbation in longitudinal sector back into the equations of motion (82) and (83), we get eight independent equations of motion (The symmetric Einstein equations gives $3 \times (3+1)/2 + 1 = 7$ equations and the equation of motion for the scalar fields contributes one equation).

We thus find that the longitudinal perturbations obey eight coupled equations of motions:

$$\begin{aligned}
&\frac{1}{4}f(2h^r{}_r(2r^3f' + 6r^2f + k^2) - 2h^t{}_t(2r^3f' + 6r^2f + m^2 - 6r^2) + 2ikr^4h^x{}_r f' - r^4f'h^x{}_x{}' - \\
&r^4f'h^y{}_y{}' + 4r^3fh^r{}_r{}' + 4ikr^4fh^x{}_r{}' + 16ikr^3fh^x{}_r - 8r^3fh^x{}_x{}' - 2r^4fh^x{}_x{}'' - 8r^3fh^y{}_y{}' - \\
&2r^4fh^y{}_y{}'' + m^2h^x{}_x + 2k^2h^y{}_y + m^2h^y{}_y - 2ikm\delta\phi_1) = 0,
\end{aligned} \tag{98}$$

$$\begin{aligned}
&\frac{1}{4r^4f}\{2h^r{}_t(2r^3f' + k^2 + m^2 - 6r^2) - ir^4f'(2kh^x{}_t + \omega(h^x{}_x + h^y{}_y)) + \\
&2r^2f[-2ir\omega h^r{}_r + 6h^r{}_t + r^2(k\omega h^x{}_r + i(kh^x{}_t{}' + \omega(h^x{}_x{}' + h^y{}_y{}')))]\} = 0,
\end{aligned} \tag{99}$$

$$
\begin{aligned}
&\frac{1}{2}(6r^3{h^x}_t f' + r^4 {h^x}_t f'' + r^2 f(r(-ir\omega {h^x}_r{}' - 4{h^x}_t{}' - r{h^x}_t{}'') - \\
&4ir\omega {h^x}_r + 6{h^x}_t) - k\omega {h^r}_r + ik{h^r}_t{}' + m^2{h^x}_t - 6r^2{h^x}_t - k\omega {h^y}_y + \\
&im\omega\delta\phi_1) = 0, \qquad (100)\\
&\frac{1}{4r^5 f^2}(rf(-2ikr^4{h^x}_r f' + r^4 f'{h^x}_x{}' + r^4 f'{h^y}_y{}' + 2{h^r}_r(m^2 - 6r^2) - \\
&2k^2{h^t}_t - m^2{h^x}_x - 2k^2{h^y}_y - m^2{h^y}_y + 2ikm\delta\phi_1) + 4r^4 f^2({h^t}_t{}' - \\
&2ik{h^x}_r + {h^x}_x{}' + {h^y}_y{}') + 2\omega(-4i{h^r}_t + 2kr{h^x}_t + r\omega({h^x}_x + {h^y}_y))) = 0, \qquad (101)\\
&\frac{1}{4}(\frac{i(k{h^r}_r f' - k{h^t}_t f' + 2i\omega^2{h^x}_r + 2\omega{h^x}_t{}')}{f} + \\
&2{h^x}_r(6r^3 f' + r^4 f'' + m^2 - 6r^2) - \frac{2k\omega{h^r}_t}{r^4 f^2} + 12r^2 f{h^x}_r + \frac{4ik{h^r}_r}{r} - \\
&2i(k{h^t}_t{}' + k{h^y}_y{}' - im\delta\phi_1{}')) = 0, \qquad (102)\\
&\frac{1}{4f^2}(f^2(r^4(-f'){h^r}_r{}' - 12r^3{h^r}_r f' - 2r^4{h^r}_r f'' + 3r^4 f'{h^t}_t{}' + \\
&{h^x}_x(12r^3 f' + 2r^4 f'' + m^2 - 12r^2) + 2r^4 f'{h^y}_y{}' - m^2{h^y}_y - 2ikm\delta\phi_1) + \\
&2i\omega{h^r}_t f' + 2r^2 f^3(r(-2{h^r}_r{}' + 4{h^t}_t{}' + r{h^t}_t{}'' + 4{h^y}_y{}' + r{h^y}_y{}'') - \\
&6{h^r}_r + 6{h^x}_x) + 2\omega f(\omega{h^r}_r - 2i{h^r}_t{}' + \omega{h^y}_y)) = 0, \qquad (103)\\
&\frac{1}{4f^2}(-f^2(r^4 f'{h^r}_r{}' + 2{h^r}_r(6r^3 f' + r^4 f'' + k^2) - 3r^4 f'{h^t}_t{}' + \\
&4ikr^4{h^x}_r f' - 2r^4 f'{h^x}_x{}' - 12r^3{h^y}_y f' - 2r^4{h^y}_y f'' + 2k^2{h^t}_t + \\
&m^2{h^x}_x - m^2{h^y}_y + 12r^2{h^y}_y - 2ikm\delta\phi_1) + 2i\omega{h^r}_t f' + \\
&2r^2 f^3(-2r{h^r}_r{}' - 6{h^r}_r + r^2{h^t}_t{}'' + 4r{h^t}_t{}' - 2ikr^2{h^x}_r{}' - 8ikr{h^x}_r + \\
&r^2{h^x}_x{}'' + 4r{h^x}_x{}' + 6{h^y}_y) + 2\omega f(\omega{h^r}_r - 2i{h^r}_t{}' + 2k{h^x}_t + \omega{h^x}_x)) = 0, \qquad (104)\\
&-2mr^4{h^x}_r f' + 2r^4 f'\delta\phi_1{}' + 2r^3 f\left(-mr{h^x}_r{}' - 4m{h^x}_r + 4\delta\phi_1{}' + r\delta\phi_1{}''\right) + \\
&\frac{2\omega(\omega\delta\phi_1 - im{h^x}_t)}{f} + ikm{h^r}_r + ikm{h^t}_t - ikm{h^x}_x + ikm{h^y}_y - 2k^2\delta\phi_1 = 0, \qquad (105)
\end{aligned}
$$

where the index are raised by the background metric, and are given in the orders of tt, tr, tx, rr, rx, xx, yy components of the equations of motion for the metric perturbations, followed by the equation of motion for the scalar field perturbation.

For convenience, we refer the reader to the equations from (A1) to (A8) in the appendix A of [10]. We have checked by explicit calculations that the linearised Einstein and scalar field equations of motion with equations (A.1), (A.2), ...,(A.8):

$$\mathrm{Eq.}(A.1) \propto \mathrm{EOM\ for}\ \phi_I, \tag{106}$$
$$\mathrm{Eq.}(A.2) \propto \mathrm{EOM}_{xx} - \mathrm{EOM}_{yy}, \tag{107}$$
$$\mathrm{Eq.}(A.3) \propto \mathrm{EOM}_{tx}, \tag{108}$$
$$\mathrm{Eq.}(A.4) \propto \mathrm{EOM}_{tr}, \tag{109}$$
$$\mathrm{Eq.}(A.5) \propto \mathrm{EOM}_{rx}, \tag{110}$$
$$\mathrm{Eq.}(A.6) \propto \mathrm{EOM}_{rr}, \tag{111}$$
$$\mathrm{Eq.}(A.7) \propto \mathrm{EOM}_{tt}, \tag{112}$$
$$\mathrm{Eq.}(A.8) \propto \mathrm{EOM}_{xx} + \mathrm{EOM}_{yy}. \tag{113}$$

In the context of linearized gravity, the perturbation fields behave like gauge fields under the coordinates transformation $x^\mu \longrightarrow x^\mu + \xi^\mu$:

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{114}$$
$$\delta\phi_1 \longrightarrow \delta\phi_1 + m\xi^i. \tag{115}$$

In the following we shall use this freedom of coordinates transformations to eliminate the non-dynamic metric components from the linearised field equations.

3.5 Deriving the gauge invariant fields and their EOMs

There are only two independent degrees of freedom in the longitudinal sector and the redundancy of the field equations can be eliminated by using the combinations of the fields which are gauge invariant [10]. In this project, I calculated the gauge transformation of the perturbation fields in longitudinal sector explicitly as

$$h^t{}_t \longrightarrow h^t{}_t + \frac{2\xi^r}{r} - 2i\omega\xi^t + \frac{\xi^r f'}{f}, \tag{116}$$

$$h^r{}_r \longrightarrow h^r{}_r - \frac{2\xi^r}{r} - \frac{\xi^r f'}{f} + 2\xi^{r\prime}, \tag{117}$$

$$h^x{}_x \longrightarrow h^x{}_x + \frac{2\xi^r}{r} + 2ik\xi^x, \tag{118}$$

$$h^y{}_y \longrightarrow h^y{}_y + \frac{2\xi^r}{r}, \tag{119}$$

$$h^x{}_t \longrightarrow h^x{}_t - ikf\xi^t - i\omega\xi^x, \tag{120}$$

$$h^x{}_r \longrightarrow h^x{}_r + \frac{ik\xi^r}{r^4 f} + \xi^{x\prime}, \tag{121}$$

$$h^r{}_t \longrightarrow h^r{}_t - i\omega\xi^r - r^4 f^2 \xi^{t\prime}, \tag{122}$$

$$\delta\phi_1 \longrightarrow \delta\phi_1 + m\xi^x. \tag{123}$$

By eliminating the terms ξ^r, ξ^t, ξ^x, f and their derivatives, we obtain the gauge invariant fields:

$$h_1 = h^t{}_t - \frac{2\omega h^x{}_t}{kf} - h^y{}_y - \frac{2i\omega^2\delta\phi_1}{kmf} - \frac{rh^y{}_y f'}{2f}, \tag{124}$$

$$h_2 = h^r{}_r + (rh^y{}_y f')/(2f) - rh^y{}_y{}', \tag{125}$$

$$h_3 = h^x{}_x - h^y{}_y - (2ik\delta\phi_1)/m, \tag{126}$$

$$h_4 = h^x{}_r - (ikh^y{}_y)/(2r^3 f) - \delta\phi_1'/m, \tag{127}$$

$$\begin{aligned} h_5 = {} & h^r{}_t + 1/2ir\omega h^y{}_y - (ir^4 f h^y{}_y f')/(4\omega) + (ir^5 h^y{}_y f'^2)/(4\omega) + (ir^4 f^2 h^{t\prime}{}_t)/(2\omega), \\ & -(ir^4 f^2 h^{y\prime}{}_y)/(2\omega) - (ir^5 f f' h^{y\prime}{}_y)/(4\omega) - (ir^5 f h^y{}_y f'')/(4\omega). \end{aligned} \tag{128}$$

By expressing $h^t{}_t$, $h^r{}_r$, $h^x{}_x$, $h^x{}_r$, $h^r{}_t$ in terms of the gauge invariant variables, and substitute for them into the equations of motion, we have the equations of motion for the gauge invariant fields

$$4(k^2-m^2+6r^2)h_2+2m^2h_3+(-2ikm^2r+12ikr^3+20ikr^3f)h_4+$$
$$(8r^3fh_2{}'+m^2r-6r^3-10r^3f)h_3{}'+8ikr^4fh_4{}'-4r^4fh_3{}''=0, \quad (129)$$
$$\omega(r(-m^2+6r^2)\omega h_3+4ik^2h_5)+2k^2r^4f^2h_1{}'+$$
$$2r^3\omega^2f(4h_2-3h_3+2ikrh_4-2rh_3{}')=0, \quad (130)$$
$$-2k\omega^2h_2+f(-8ir^3\omega^2h_4+kr(-m^2+6r^2)h_1{}'-2ir^4\omega^2h_4{}')+2ik\omega h_5{}'+$$
$$kr^3f^2(-2h_1{}'+rh_1{}'')=0, \quad (131)$$
$$4\omega(r\omega h_3-4ih_5)+2r^4f^2(-2ikh_4+h_3{}')-rf(4k^2h_1-4h_2\left(m^2-6r^2\right)+m^2rh_3{}'-$$
$$6r^3h_3{}'+2m^2h_3-2ikm^2rh_4+12ikr^3h_4+m^2rh_3{}'-6r^3h_3{}')=0, \quad (132)$$
$$2r^3f^2(-krh_1{}'+3kh_1+kh_2-2im^2rh_4)+rf(kh_1(m^2-6r^2)-kh_2(m^2-6r^2)+$$
$$4ir^3\omega^2h_4)+4ik\omega h_5=0, \quad (133)$$
$$-2i(m^2-6r^2)\omega h_5+r^3f^2(-4k^2h_1-4(k^2+6r^2)h_2-2m^2h_3+4ikm^2rh_4-24ikr^3h_4$$
$$+18m^2h^y{}_y-108r^2h^y{}_y+36r^3h^y{}_yf'+m^2rh_2{}'-6r^3h_2{}'-2m^2rh_3{}'+12r^3h_3{}')$$
$$+4r^2\omega f(r\omega h_2+r\omega h_3-i(3h_5+2rh_5{}'))$$
$$+2r^5f^3(-4ikrh_4+54h^y{}_y+r(-h_2{}'+2h_3{}'-4ikrh_4{}'+2rh_3{}''))=0, \quad (134)$$
$$ikmh_1+ikmh_2-ikmh_3-8mr^3fh_4-2mr^4f'h_4-2mr^4fh_4{}'=0. \quad (135)$$

We observed that h_1 and h_3 are dynamical fields (i.e. the second derivative of h_1 and h_4 appear in the equations of motion (129), (131), and (134). By canceling the non-dynamical fields h_2, h_4, h_5 and their derivatives, we have two dynamical equations for h_1 and h_3. The coupled dynamical equation is rather lengthy and will not be reproduced here, while the decoupled one for h_3 is simple:

$$-(k^2+m^2)h_3+\omega^2h_3/f+(-(m^2r)/2+3r^3+r^3f)h_3'+r^4fh_3{}''=0 \quad (136)$$

One find that it is convenient to use the combination of the gauge invariant variables h_1, h_2, h_3, h_4, h_5, which is given by [10]:

$$\psi_{||} = \frac{r}{m^2+k^2}\left[m(h^x{}_x - h^y{}_y) - 2ik\delta\phi_1\right] - \frac{2r^4 f}{m(k^2+r^3 f')}\left[h^{x\prime}{}_x + h^{y\prime}{}_y - \frac{2}{r}h^r{}_r - 2ikh^x{}_r - \frac{k^2+r^3f'}{r^3 f}h^y{}_y\right]. \tag{137}$$

By substituting the fields h^x_x, h^y_y, h^r_r, h^x_r in terms of h_1, h_2, h_3, h_4, h_5 into the expression for $\psi_{||}$, we find that

$$\psi_{||} = \frac{mrh_3}{k^2+m^2} + \frac{2r^3 f(2h_2 + 2ikrh_4 - rh_3')}{m(k^2+r^3 f')}. \tag{138}$$

It is clear that the linear combination of gauge invariant fields $\psi_{||}$ is also gauge invariant. We have checked that the field $\psi_{||}$ obey the equation of motion [10]:

$$\frac{d}{dr}[r^2 f\psi_{||}'] + \left(\frac{\omega^2 - k^2 f}{r^2 f} + V(r)\right)\psi_{||} = 0 \tag{139}$$

where

$$V(r) = -\frac{3r_0(m^2-2r_0^2)}{2r^3[2k^2r + m^2(2r-3r_0) + 6r_0^3]^2}[4k^4r^2 + m^4(-4r^2+6rr_0-3r_0^2) + 12m^2r_0(r^3 - rr_0^2 + r_0^3) - 12r_0^3(2r^3+r_0^3)]. \tag{140}$$

Another convenient choice for the gauge invariant field in [15] is considered, and is given by

$$\psi_1 = \frac{r^2 f}{(k^2+r^3f')^2}\frac{d}{dr}[r^4 f(h^x{}_x{}' + h^y{}_y{}' - 2ikh^x{}_r - 2\frac{h^r{}_r}{r} - \frac{k^2+r^3f'}{r^3f}h^y{}_y) - \frac{mr(k^2+r^3f')}{2(k^2+m^2)(m(h^x{}_x - h^y{}_y) - 2ik\phi_1)}]. \tag{141}$$

ψ_1 obeys a simpler equation of motion compared to $\psi_{||}$:

$$\frac{d}{dr}\left[\frac{r^2 f(k^2+r^3f')^3}{\omega^2(k^2+r^3f') - k^2(k^2+m^2)f}\psi_1{}'\right] + \frac{(k^2+r^3f')^2}{r^2 f}\psi_1 = 0. \tag{142}$$

We observe that ψ_1 can be expressed in terms of $\psi_{||}$:

$$\psi_1 = \frac{r^2 f}{k^2 + r^3 f'} \frac{d}{dr} \left(-\frac{m}{2} (k^2 + r^3 f') \psi_{||} \right). \tag{143}$$

In the next section we will discuss how to derive the decoupled equation of motion for ψ_1 and $\psi_{||}$.

3.6 Method of deriving the equations of motion

The derivation is based on the dynamical equations for the gauge invariant fields h_1 h_2, h_3, h_4 and h_5. By eliminating additional variables of h_2, h_4, h_5, we can get one dynamical equation for h_3 only and a coupled dynamical equation for h_1 and h_3, equation given in (136).

We find that we can express the gauge invariant variables $\psi_{||}$ and ψ_1 in terms of h_2, h_4 and h_3, as shown in equation (137). We constructed a new variable in terms of the ψ_1 called $d\psi_1$:

$$d\psi_1 \equiv \frac{d}{dr} \left(\frac{(k^2 + r^3 f')^2 (k^2 + m^2)}{k r^2 f} \psi_1 \right) \tag{144}$$

and then we find it include h_2' and h_4' explicitly. Hence we would like to find the h_2' and h_4' in terms of h_2 and h_4 by massaging equations (A1), (A2) and (A7), then substitute h_2' and h_4' back into the above new variable. Finally, we get expressions for $\psi_{||}$, ψ_1 and $d\psi_1$ in terms of h_2 and h_4 without their derivatives. Hence we get 3 variables $\psi_{||}$, ψ_1 and $d\psi_1$, which include two unwanted h_2 and h_4. We can eliminate h_2 and h_4 by three equations including h_2 and h_4, $\psi_{||}$, ψ_1 and $d\psi_1$, finally we get a decoupled differential equation for $\psi_{||}$:

$$C_{\psi_{||}} \psi_{||} + C_{\psi_{||}'} \psi_{||}' + C_{\psi_{||}''} \psi_{||}'' = 0, \tag{145}$$

where the coefficients are

$$
\begin{aligned}
C_{\psi_{||}} =&2r^2f(r^3f'(k^2(6r^2-5m^2)-6k^4-108m^2r^2+7m^4+12r^2\omega^2+396r^4)+\\
&2r^6f'^2(k^2+4m^2-48r^2)+12r^9f'^3+4k^2(-15m^2r^2+m^4+3r^2\omega^2+54r^4)+\\
&k^4(66r^2-7m^2)+4k^6-6r^2(m^2-6r^2)^2)-(2k^2-m^2+6r^2)(r^3f'+k^2)\times\\
&(2r^3(m^2-6r^2)f'-12m^2r^2+m^4+4r^2(9r^2+\omega^2))+\\
&12r^4f^2(-r^3f'(k^2-6m^2+96r^2)+16r^6f'^2+k^2(m^2-18r^2)-5k^4-\\
&18m^2r^2+m^4+120r^4)+72r^6f^3(8r^3f'+m^2-22r^2)+576r^8f^4, \qquad (146)\\
C_{\psi_{||}'} =&-2r^3f(24r^4f^2(4r^3f'-k^2+m^2-12r^2)-2r^2f(r^3f'(8k^2-7m^2+66r^2)-\\
&6r^6f'^2-k^2(m^2-6r^2)+2k^4+12r^2(m^2-6r^2))+\\
&(m^2-6r^2)(-2k^2+m^2-6r^2)(r^3f'+k^2)+144r^6f^3), \qquad (147)\\
C_{\psi_{||}''} =&4r^6f^2(6r^2f-2k^2+m^2-6r^2)(r^3f'+k^2), \qquad (148)
\end{aligned}
$$

which coincides with the dynamical equation for $\psi_{||}$ in the appendix A of [10]. Written in equations (139) and (140) above with $r_0 = 1$.

To get the equation of motion for ψ_1, analog to what we have done for ψ_1, we construct a new variable called $d^2\psi_1$:

$$
d^2\psi_1 = \frac{d}{dr}\left[r^3f^2\frac{d}{dr}\left(\frac{(k^2+m^2)(k^2+r^3f')^2}{kr^2f}\psi_1\right)\right] \qquad (149)
$$

We can express ψ_1, $d\psi_1$, $d^2\psi_1$ in terms of h_2, and h_4 only, and then eliminate h_2 and h_4 to get the equation of motion for ψ_1 which coincide with equation (142).

3.7 Numerical Method for finding the dispersion relation on the boundary

In this section, we will illustrate how to find the dispersion relation using the numerical method. First, we apply the conventional coordinates transformation for the equation of motion (142) multiplied by a factor $F(r)$,where

$$F(r) \equiv \frac{f(r)\left(r^3 f'(r)+k^2\right)^2\left(k^2 f(r)\left(k^2+m^2\right)-\omega^2\left(r^3 f'(r)+k^2\right)\right)^2}{r^2}. \tag{150}$$

and the coordinates transformation are

$$r=\frac{1}{u},\; r\in(r_0,\infty)\to u\in(0,\frac{1}{r_0}) \tag{151}$$

We then get the new equation of motion with respect to the new coordinate u:

$$C_{\psi_1(u)}\psi_1(u)+C_{\psi_1{}'(u)}\psi_1{}'(u)+C_{\psi_1{}''(u)}\psi_1{}''(u)=0, \tag{152}$$

where

$$\begin{aligned}
C_{\psi_1(u)} =&\frac{1}{4}(k^2\left(k^2+m^2\right)\left(m^2u^2(r_0u-1)-2r_0^3u^3+2\right)-\\
&\omega^2\left(2k^2+m^2(2-3r_0u)+6r_0^3u\right))^2 && (153)\\
C_{\psi_1(u)'} =&\frac{1}{8}(u\omega^2(-\frac{1}{2}m^2u^2(r_0u-1)+r_0^3u^3-1)(m^2(2-3r_0u)+6r_0^3u)\times\\
&(2k^2+m^2(2-3r_0u)+6r_0^3u)^2-6r_0\omega^2(m^2-2r_0^2)(-\frac{1}{2}m^2u^2(r_0u-1)+\\
&r_0^3u^3-1)(m^2(u^2-r_0u^3)+2r_0^3u^3-2)(2k^2+m^2(2-3r_0u)+6r_0^3u)-\\
&9k^2r_0(k^2+m^2)(m^2-2r_0^2)(-\frac{1}{2}m^2u^2(r_0u-1)+r_0^3u^3-1)(m^2u^2(r_0u-1)\\
&-2r_0^3u^3+2)^2) && (154)\\
C_{\psi_1{}''(u)} =&\frac{1}{8}(\omega^2(-\frac{1}{2}m^2u^2(r_0u-1)+r_0^3u^3-1)(m^2(u^2-r_0u^3)+2r_0^3u^3-2)(2k^2+\\
&m^2(2-3r_0u)+6r_0^3u)^2+k^2(k^2+m^2)(-\frac{1}{2}m^2u^2(r_0u-1)+r_0^3u^3-1)\times\\
&(m^2u^2(r_0u-1)-2r_0^3u^3+2)^2(2k^2+m^2(2-3r_0u)+6r_0^3u)) && (155)
\end{aligned}$$

We have checked that this equation coincides with equation (142). We then substitute for the ansatz

$$\psi_1(u)=(u-1)^{-\frac{i\omega}{4\pi T(m)}}\psi_{hor}(u), \tag{156}$$

satisfying the ingoing boundary conditions near the horizon into the above equation of motion (152), where $\psi_{hor}(u)$ is a field dependent of u, $T(m) = r_0(3 - \frac{m^2}{2{r_0}^2})/(4\pi)$ is the thermal temperature of the field theory [13][14]. It is worth noting that the reason for exclude the outgoing wave boundary conditions near the horizon is that classically the horizon does not radiate. [16] And then we get the equation of motion for $\psi_{hor}(u)$. It is convenient to set $r_0 = 1$ so that the equation of motion for ψ_{hor} is given by

$$C_{\psi_{hor}}\psi_{hor} + C_{\psi_{hor}'}{\psi_{hor}}' + C_{\psi_{hor}''}{\psi_{hor}}'' = 0 \tag{157}$$

where the coefficients are

$$\begin{aligned}
C_{\psi_{hor}} =& -i\omega^3(m^2(u-1)u^2 - 2u^3 + 2)^2(2k^2 + m^2(2-3u) + 6u)(2k^2(m^2 - 2i\omega - 6) + \\
& m^2(6iu(\omega + 4i) - 4i\omega + 36) + m^4(3u-4) + 12(-6 + u(3 - i\omega))) + \\
& i(m^2-6)(u-1)u\omega^3(m^2(3u-2) - 6u)(m^2(u-1)u^2 - 2u^3 + 2)\times \\
& (2k^2 + m^2(2-3u) + 6u)^2 + 2(m^2-6)^2(u-1)^2(k^2(k^2+m^2)\times \\
& (m^2(u-1)u^2 - 2u^3 + 2) - \omega^2(2k^2 + m^2(2-3u) + 6u))^2 + \\
& ik^2\omega(k^2+m^2)(m^2(u-1)u^2 - 2u^3 + 2)^3(2k^2(m^2 - 2i\omega - 6) + \\
& m^2(6iu(\omega + 8i) - 4i\omega + 60) + m^4(6u-7) + 12(-9 + u(6 - i\omega))),
\end{aligned} \tag{158}$$

$$\begin{aligned}
C_{\psi_{hor}'} =& 4(3 - \frac{m^2}{2})(u-1)(-\frac{1}{2}m^2(u-1)u^2 + u^3 - 1)(-2\omega^2(-m^2(u-1)u^2 + 2u^3 - 2)(2k^2 + \\
& m^2(2-3u) + 6u)(-\frac{3}{2}(m^2-6)(m^2-2)(u-1) + i\omega(2k^2 + m^2(2-3u) + 6u)) - 2k^2\times \\
& (k^2+m^2)(m^2(u-1)u^2 - 2u^3 + 2)^2(-\frac{9}{4}(m^2-6)(m^2-2)(u-1) + i\omega(2k^2 + m^2(2- \\
& 3u) + 6u)) + (3 - \frac{m^2}{2})(u-1)u\omega^2(m^2(2-3u) + 6u)(2k^2 + m^2(2-3u) + 6u)^2),
\end{aligned} \tag{159}$$

$$\begin{aligned}
C_{\psi_{hor}''} =& (m^2-6)^2(u-1)^2(-\frac{1}{2}m^2(u-1)u^2 + u^3 - 1)(-2k^2 + m^2(3u-2) - 6u)(-\omega^2(-m^2\times \\
& (u-1)u^2 + 2u^3 - 2)(2k^2 + m^2(2-3u) + 6u) - k^2(k^2+m^2)(m^2(u-1)u^2 - 2u^3 + 2)^2).
\end{aligned} \tag{160}$$

We can have a near horizon expansion of ψ_{hor}:

$$\psi_{hor} = C\psi_0 + (u-1)C\psi_1 + (u-1)^2 C\psi_2 + \mathcal{O}((u-1)^2), \tag{161}$$

where $C\psi_0$, $C\psi_1$ and $C\psi_2$ are expansion factors. To find the boundary condition, we first substitute this near horizon expansion into its equation of motion (157). We can normalise this equation by setting $C\psi_0 = 1$. To make the expansion solve the equation, we expect that the coefficient for coefficient at any order of $(u-1)$ must vanish, and finally we get the expression for $C\psi_1$:

$$\begin{aligned} C\psi_1 = &-\frac{2i}{(m^2-6)^2\omega(2k^2-m^2+6)(m^2+4i\omega-6)}(k^2(4m^4(\omega^2-6i\omega+27)+ \\ &4im^2(4\omega^3+9i\omega^2+18\omega+54i)+2im^6(\omega+9i)+m^8+24\omega^2(3-2i\omega))+ \\ &k^4(m^2-6)^2(m^2+2i\omega-6)-2(m^2-6)\omega^2(m^2(-33+4i\omega)+4m^4+6(9-2i\omega))) \end{aligned} \tag{162}$$

In the process of calculating $C\psi_1$, we find that it is enough to expand ψ_{hor} up to the first order of $(r-1)$ to find the coefficient $C\psi_1$. If more accurate expansion approximation is expected, we need to expand ψ_{hor} up to more higher order terms.

3.7.1 Boundary conditions

We expect that the ψ_{hor} behaves the same as the near horizon expansion on the boundary. First we define the boundary field as

$$\psi_{bdy}(u) = C_{\psi_0} + (u-1)C_{\psi_1}, \tag{163}$$

where C_{ψ_0} and C_{ψ_1} are coefficients independent of u. In practice, we then set the boundary conditions

$$\psi_{bdy}(u=1-\epsilon) - \psi_{hor}(u=1-\epsilon) = 0, \tag{164}$$

$$\psi_{bdy}{}'(u=1-\epsilon) - \psi_{hor}{}'(u=1-\epsilon) = 0, \tag{165}$$

where ϵ is an infinitely small number. Practically, we set $\epsilon = 5 \times 10^{-3}$.

Given equation of motion (157), and the boundary conditions (164) and (165), we can solve for the field $\psi_{hor}(u)$ with parameters ω and k numerically using the build-in function NDsovle command in Mathematica. The dispersion relation can be found numerically by imposing the Dirichlet boundary conditions [16]:

$$\psi_{hor}(0)(\omega,k) = 0. \tag{166}$$

It was observed in [17] and [18] that for both (2+1) and higher dimensional black holes, to make the quasinormal frequencies match with the singularities of the retarded Green function in a field theory dual to corresponding gravity theory, we need to impose the Dirichlet boundary conditions.

In practice, we do this by using the FindRoot command in Mathematica, by evaluating the value of ψ_{hor} at $u \to 0$, that is,

$$\psi_{hor}(0.0001)(\omega,k) = 0. \tag{167}$$

4 Results and Discussion

In this section, we represent the numerical results for the choice of small $\frac{m}{T}$, in particular at $\frac{m}{T} = \frac{1}{100}$, where the result is shown in Fig. 1 and compared with the analytic relation (90) and its approximation 92. We also present the numerical results for the choice of large $\frac{m}{T}$, in particular at $\frac{m}{T} = 100$, where the result is shown in Fig. (2).

The upper diagram of Fig. 1 shows that the imaginary part of ω behaves like quadratic function with respect to k for small k for the choice of $\frac{m}{T} = \frac{1}{100}$. The numerical results from the gravity side is well-fitted to the analytic dispersion relation (90) and its approximation dispersion relation (92) for the k range from 0 to 0.5. By comparison, the lower diagram of Fig. 1 shows that the real part of ω is proportional to k for small k range from 0 to 0.5. It is also well-fitted to the analytic dispersion relation (90) and its approximation dispersion relation (92) in the same range of k. This is what we expect that hydrodynamic modes behaves like a sound modes in the limit of small k and small $\frac{m}{T}$. We also observed that beyond the range of small k, the difference between the numerical results and the analytical results increase rapidly as the k increases.

The upper diagram of fig. 2 shows that the imaginary part of ω behaves like quadratic function with respect to k for small k for the choice of $\frac{m}{T} = 100$, and the lower diagram of fig. 1 shows that the real part of ω vanished at the range of k from 0 to 0.7. These numrical result from the gravity side is well-fitted to what we expect in equation (94), that is, for the large m/T and small k approximation, the quasinormal frequency is purely imaginary and is proportional to k^2. This means that such quasinormal modes are heavily damped. Furthermore, for the large k, we observe a bifurcation point both in the upper and lower diagrams. This implies that hydrodynamics breaks down at this point and is no longer a valid approximation beyond the bifurcation point.

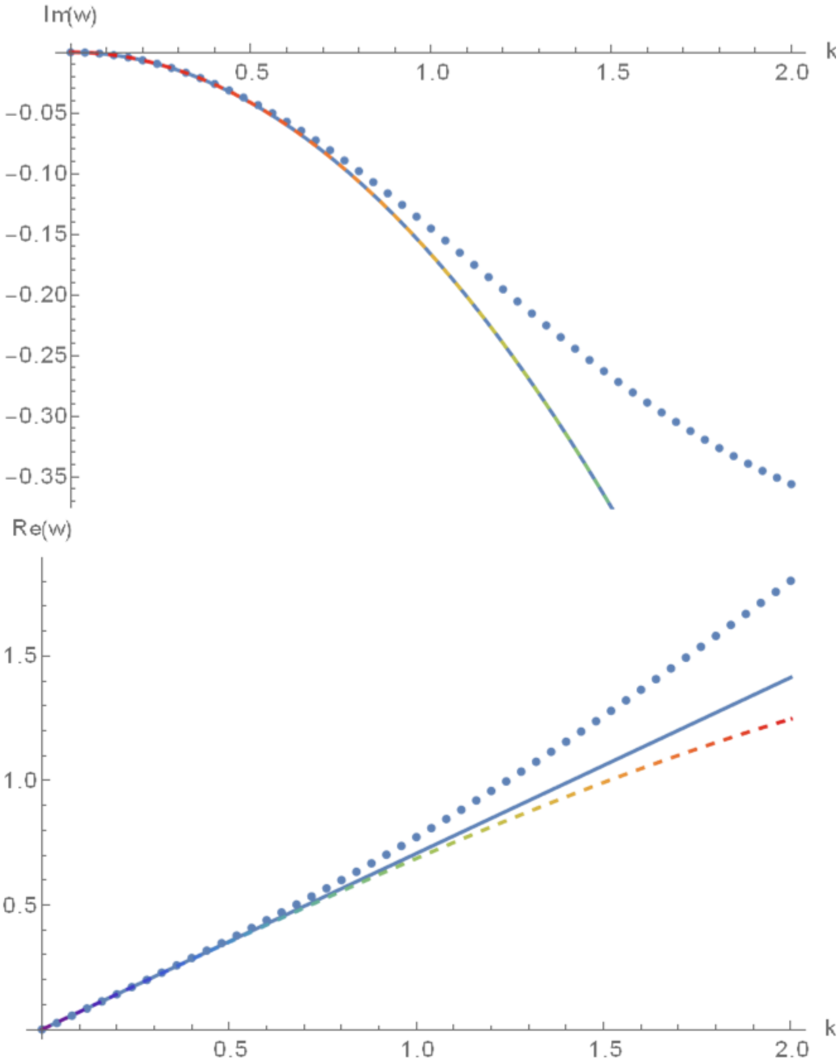

Figure 1: These plots show the dispersion relation $\omega(k)$ of the hydrodynamic modes for $m/T = 0.01$. The upper panel is for the imaginary part of ω. The lower panel is for the real part of ω. The blues dots are the numerical result. The dashed rainbow line are hydrodynamic approximation (90). The solid blue line are hydrodynamic approximation (92) for small k and small m/T.

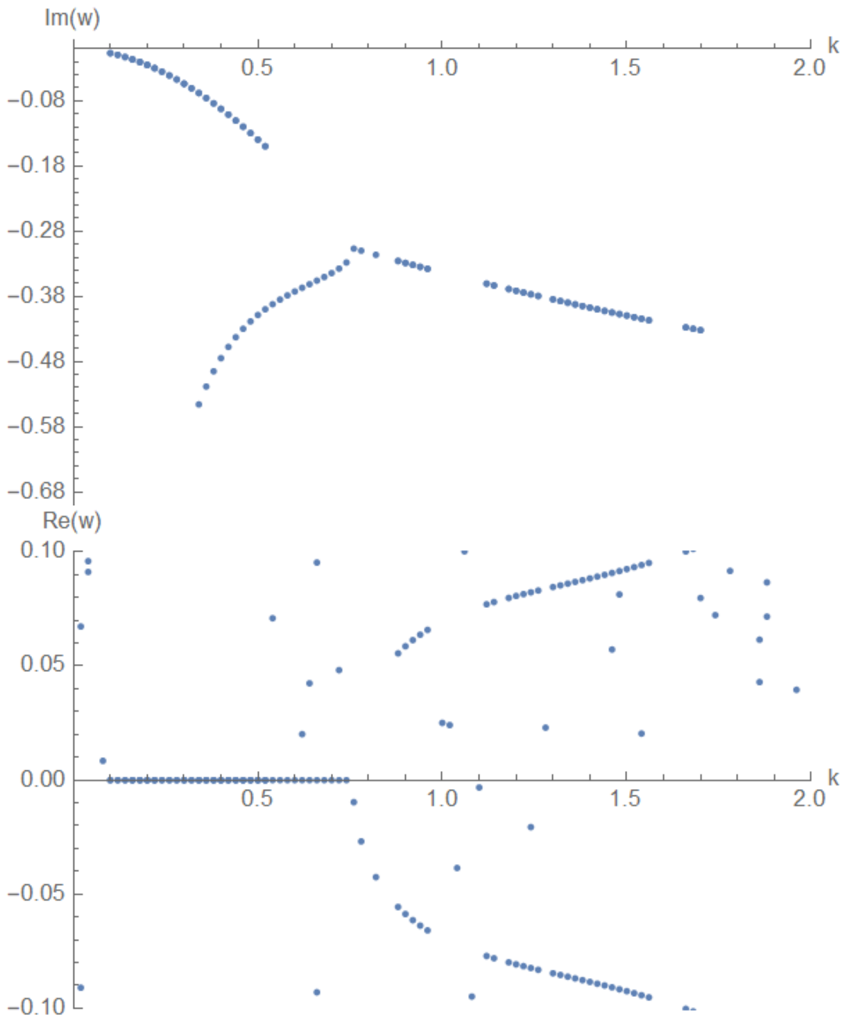

Figure 2: These plots show the dispersion relation $\omega(k)$ of the hydrodynamic modes for $m/T = 100$. The upper panel is for the imaginary part of ω. The lower panel is for the real part of ω. The blues dots are the numerical result. Some points missing in the upper panel and some strange points in the lower panel are the fake root picked by FindRoot command in Mathematica.

5 Conclusion

In this project, we review the background of relativistic hydrodynamics and the quasinormal modes for linearized hydrodynamics, deriving the dispersion relation of non-dissipative and dissipative hydrodynamics. On the gravity side, we review the basic knowledge of axion model, deriving the corresponding Einstein equation in the Lagrangian formalism, then introduce linearized gravity to derive the dispersion relation of quasinormal modes of the axion model. We spend some effort in the derivation of gauge invariant fields and their equations of motion. We also illustrate how to find the dispersion relation on the gravity side numerically. At last, we show the numerical results of dispersion relation for the choice of $\frac{m}{T} = 1/100$ and $\frac{m}{T} = 100$. We observe that in the case of $\frac{m}{T} = \frac{1}{100}$, the quasinormal modes are sound-like for small k, whereas in the case of $\frac{m}{T} = 100$, the quasinormal modes are heavily damped. Such observations agree with the analysis on the hydrodynamics side.

Because of lack of time, the numerical results are not complete. For the further work, we can first take time to optimize the Mathematica program used to find the numerical results so that we can discard the strange points and find the missing points in and Fig. 2. Then, we can do machinery calculations to check the dispersion relation for the different choice of $\frac{m}{T}$.

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