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Quantum Cosmology-An introduction

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Abstract

Quantum cosmology is based on the idea that quantum physics should apply to anything in nature, including the whole universe. If quantum mechanics is truly a fundamental theory, one would expect that we can apply it to the Universe as a whole. Simply we treat the universe as a quantum system and this approach attempts to answer open questions of classical physical cosmology. For example some questions concerning the first phases of the universe. However, there is no unique version and no completely well-defined theory yet to describe the universe as a quantum system.

The main aim of this dissertation is to give an introduction to this interesting field of Physics which constantly gives new results compatible with observations.

Science cannot solve the ultimate mystery of nature. And that is because, in the last analysis, we ourselves are a part of the mystery that we are trying to solve.

Max Planck

Acknowledgments

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Finally, I cannot neglect thanking my family and friends for their spiritual support and encouragement, not only throughout the writing of this thesis, but for the whole period of my studies.

Contents

Chapter 1

Introduction

The Cosmological Principle proposed by Milne in 1933 states that no observer occupies a special place in the universe (Copernican Principle). It is based on two principles of spatial invariance. The universe is homogeneous and isotropic on large distances. This means that the universe looks the same at each point and in all directions.

Figure 1.1: The oldest galaxy in the Universe, which is located at the boundary of the observable Universe.

A subject of interest in quantum cosmology is the description of a closed universe by the wave function:

$$\psi[h_{ij}(x), \phi(x), B]. \tag{1.1}$$

This functional was first introduced by DeWitt in 1967 [1] and describes the probability amplitude that the Universe contains a three-surface B where $\phi(x)$ is the matter configuration, h_{ij} is the three-metric. This object should describe the past, present and future of a closed Universe. Unlike the familiar particle wave functions of quantum mechanics, the wave function of the Universe is not defined on spacetime but rather, actually being a functional, is defined on an infinite dimensional manifold known as superspace.

The governing equation of such a function, and the central equation to quantum cosmology is known as the Wheeler-DeWitt equation. This equation, which the wave function must satisfy, takes steps towards a theory of quantum gravity as it blends ideas from both quantum mechanics and general relativity by Dirac quantizing the Hamiltonian constraint of a gravity plus matter system. The equation takes the form of a second order hyperbolic functional differential equation and permits infinitely many solutions. The explicit form of this equation is rather complicated due to the fact that it is defined on an infinite dimensional manifold. In an attempt to understand properties of its solutions, we usually restrict ourselves to a finite dimensional manifold known as minisuperspace. Upon doing so, the Wheeler-DeWitt equation is reduced to a wave equation which can be solved by techniques that are described in [2].

In conventional cosmology there is no a way to assign the initial conditions for the evolution of the universe (for example, for inflation) and relatively, one faces a landscape problem [3]. Next, we turn to quantum cosmology to construct a measure that could potentially provide answers in terms of probabilities. However, this requires certain boundary conditions to determine it. These cannot arise from the theory itself, and once applied, new physical laws are effectively made. Observations in the universe led to the idea that the function ψ is a no boundary quantum state. This idea was first introduced by J. B. Hartle and S. W. Hawking in [4]. This proposal was of great importance for subsequent research in this field, and reference will be made to it in a subsequent chapter.

The structure of this thesis is as follows:

- Chapter 1 gives a brief introduction about Quantum Cosmology which is the main object of review in this dissertation. Some basic principles are described as well as some points of interest in Quantum Cosmology.

- Chapter 2 describes the general formalism that is needed to determine the equation of the universe as the solution of the Wheeler-DeWitt equation. More precisely, the second chapter describes the Hamiltonian Formulation of General Relativity, the Canonical Quantization procedure and the path integral approach. Finally, a description is given for the spaces in which the wave function of the Universe is defined.

- Chapter 3 focuses on the two most studied and comprehensive boundary condition proposals. These boundary conditions are important to ensure the uniqueness of the solution that describes the Universe. The first one is Vilenkin's and Linde's tunnelling proposal and the second one is the no-boundary proposal of Hartle and Hawking. Using these boundary conditions, the wave function of the Universe was determined for the classically allowed and for the classically forbidden region. Finally, some information about the inflation of the Universe is given.

- In Chapter 4, Picard-Lefschetz technique was introduced so as to identify the Lorentzian path integral for quantum gravity during a semiclassical expansion. This technique describes an alternative version of Hartle-Hawking proposal that gives new results.

- In conclusion, a brief summary of what follows is given. Also, some recent work concerning Quantum Cosmology is mentioned.

Chapter 2

General Formalism

2.1 Hamiltonian Formulation of General Relativity

Describing general relativity as a field theory is based on the Lagrangian formulation. This formalism leads to the general formalism of quantum cosmology. It has been suggested by Roger Penrose that a physically appropriate spacetime must be globally hyperbolic [5]. Any manifold of this form admits a smooth time function t, such that the set of points satisfying $t = \mathrm{constant}$ form a spacelike Cauchy hypersurface Σ. We may split a manifold $\mathcal{M}$, of this type in terms of the orthogonal product [6]:

$$\mathcal{M} = \mathrm{R} \times \Sigma \tag{2.1}$$

One considers a compact of three surface (since we are considering only closed universes) which is embedded in a four-manifold on which the four-metric is $g_{\mu\nu}$. The relevant metric is h_{ij}, with some matter field configuration. The embedding of three surface is described by the Standard (3 + 1) form of the four metrics:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(N^2 - N_i N^i\right)dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j \tag{2.2}$$

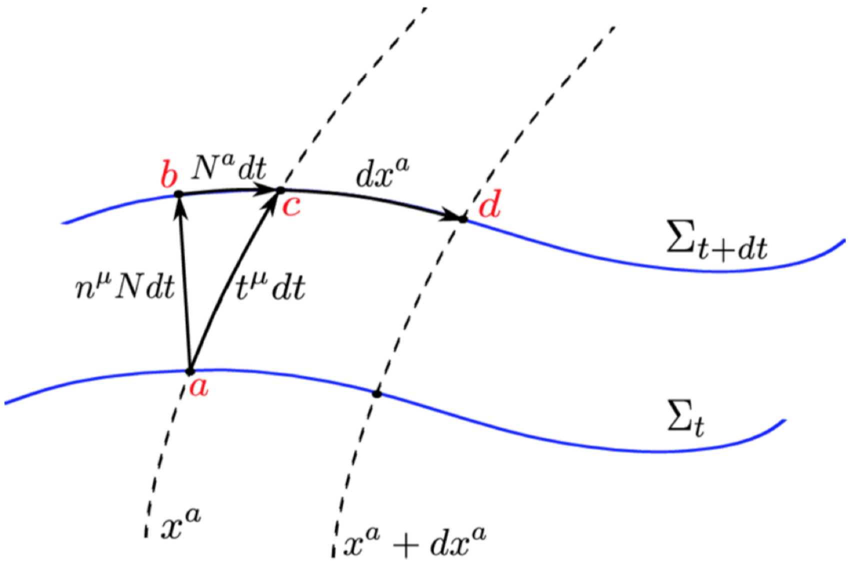

Figure 2.1: Graphical illustration of the lapse function N and the shift vector N^a (taken from [7]).

where $\mu, \nu = 0, 1, 2, 3$ and $i, j = 1, 2, 3$. N is known as the lapse function and is fully determined by the difference between the elapsed coordinate time t, and proper time τ on curves normal to the hypersurfaces. The defining equation of this function takes the form $d\tau + Ndt$. The shift vector N_i describes how the hypersurface Σ_t differs from the neighbouring hypersurface Σ_{t+dt}. Consider a point in Σ_t. For the special case in which $N_i = 0$, the spatial coordinates are said to be "comoving". N and N_i are arbitrary as they are related to choice of coordinates.

We are considering the standard Einstein-Hilbert action coupled to matter:

$$S = \frac{m_p^2}{16\pi}\left[\int_M d^4x(-g)^{1/2}(R - 2\Lambda) + 2\int_{\partial M} d^3x h^{1/2} K\right] + S_{matter} \tag{2.3}$$

where K_{ij} is the extrinsic curvature at the boundary ∂M of the four-manifold M. $K = tr(K_{ij})$. Components K_{ij} satisfy the relation:

$$K_{ij} = \frac{1}{2N}\left[-\frac{\partial h_{ij}}{\partial t} + 2D_{(i}N_{j)}\right] \tag{2.4}$$

where the symbol D_i refers to the covariant derivative in this three-surface. For a scalar field Φ, and in terms of the (3+1) variables, the action is:

$$S = \frac{m_p^2}{16\pi}\int d^3x dt N h^{1/2}\left[K_{ij}K^{ij} - K^2 + {}^3R - 2\Lambda\right] - \frac{1}{2}\int d^4x(-g)^{1/2}[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V(\Phi)]. \tag{2.5}$$

The action takes the following Hamiltonian form:

$$S = \int d^3x dt\left[\dot{h}_{ij}\pi^{ij} + \dot{\Phi}\pi_\Phi - N\mathcal{H} - N^i\mathcal{H}_i\right]. \tag{2.6}$$

The symbols π^{ij} and π_Φ denote the momenta conjugate to h_{ij} and Φ respectively:

$$\pi^{ij} = \frac{\delta\mathcal{L}}{\delta\dot{h}_{ij}} = -\frac{m_p^2}{16\pi}h^{1/2}(K^{ij} - h^{ij}K) \tag{2.7}$$

$$\pi_\Phi = \frac{\delta\mathcal{L}}{\delta\dot{\Phi}} = \frac{1}{N}h^{1/2}(\dot{\Phi} - N^i\partial_i\Phi) \tag{2.8}$$

The Hamiltonian is derived from a sum of constraints, where the lapse N and shift N^i are actually Lagrange multipliers. The momentum constraint is given by:

$$\mathcal{H}_i = -2D_j\pi_i^j + \mathcal{H}_i^{matter} = 0 \tag{2.9}$$

and the Hamiltonian constraint:

$$\mathcal{H} = \frac{16\pi}{m_p^2}G_{ijkl}\pi^{ij}\pi^{kl} - \frac{m_p^2}{16\pi}h^{1/2}({}^3R - 2\Lambda) + \mathcal{H}^{matter} = 0 \tag{2.10}$$

$$G_{ijkl} = \frac{1}{2}h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}) \tag{2.11}$$

The above constrains are useful in the canonical quantization procedure and equivalent to components of Einstein's classical equations. In the absence of matter these constrains can be described geometrically, using the Gauss-Codazzi equations [8]. In the expression 2.10 G_{ijkl} is the Wheeler-DeWitt metric which has signature (-+++++) at every point x^μ on the hypersurface.

2.2 Canonical Quantization

A wave functional $\Psi[h_{ij}, \Phi]$, which is a functional on superspace, represents the quantum state of a system. When our constraints are taken on-shell, the integrand and thus the Hamiltonian vanish. This is rather suspicious as in quantum mechanics, the Hamiltonian is responsible for generating translations in time which we interpret as being the flow of time. A vanishing Hamiltonian therefore implies that there is no such flow. The issue that has arisen is one of the main difficulties encountered when attempting to construct theories of quantum gravity [9]. At its core, this is a conflict due to the fact that quantum mechanics and general relativity each have independent and incompatible notions of time. In quantum mechanics time is treated as a background parameter with an absolute and rigid flow that is external to the system itself. On the other hand, in general relativity time is a coordinate that can flow in a malleable way depending on the motion and position of the system under consideration. Finding a resolution to this conflict is vital if we are to ever replace quantum theory and general relativity with a unified framework to treat situations where the effects of both are important i.e. during the early Universe or inside of a black hole.

We use the usual substitutions for momenta:

$$\pi^{ij} \longrightarrow -i\frac{\delta}{\delta h_{ij}} \tag{2.12}$$

$$\pi^{\Phi} \longrightarrow -i\frac{\delta}{\delta \Phi} \tag{2.13}$$

According to Dirac quantization procedure, the wave function is described by the operator versions of the classical constraints. Some appropriate replacements and operating on the wave function with the quantized Hamiltonian constraint, gives the

Wheeler - DeWitt equation:

$$\mathcal{H}\Psi = \left[-G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} - h^{1/2}(^3R - 2\Lambda) + \mathcal{H}^{matter} \right]\Psi = 0. \tag{2.14}$$

In the above expression we have ignored operator ordering problems. Make no mistake though, the Wheeler-DeWitt equation suffers from issues of operator ordering as is often the case in quantum theory. Although solutions to this equation will clearly depend on how we choose to resolve these issues, it will not be too big of a concern to us as predictions in quantum cosmology can only be trusted to leading semiclassical order. That is to say, the operator ordering will only affect the prefactor and not the exponential contribution to the wave function in which we are interested. The significance of the momentum constraint was first realised and proven by Peter Higgs in 1958 [10].

The 2.9 indicate that the wave function remains unchanged for configurations $(h_{ij}(x), \Phi(x))$. These configurations are associated with coordinate transformations in the three-surface. To illustrate this, a diffeomorphism in the three-surface can be used (in the absence of matter):

$$x^i \longrightarrow x^i - \xi^i \tag{2.15}$$

Under this change in coordinates, the wave function transforms as:

$$\Psi[h_{ij} + D_{(i}\xi_{j)}] = \Psi[h_{ij}] + \int d^3x D_{(i}\xi_{j)}\frac{\delta\Psi}{\delta h_{ij}} \tag{2.16}$$

We are interested in compact geometries and as a result of this, the boundary term (that arises from integration by parts of the last term) can be ignored. The change in Ψ then is the following:

$$\delta\Psi = -\int d^3x\xi_j D_i\left(\frac{\delta\Psi}{\delta h_{ij}}\right) = \frac{1}{2i}\int d^3x\xi_i\mathcal{H}^i\Psi \tag{2.17}$$

Now we have:

$$\Psi[h_{ij} + D_{(i}\xi_{j)}] = \Psi[h_{ij}] - \int d^3x \xi_j D_i \left(\frac{\delta\Psi}{\delta h_{ij}}\right) = \Psi[h_{ij}] + \frac{1}{2i}\int d^3x \xi_i \mathcal{H}^i \Psi \tag{2.18}$$

If the momentum constraint is satisfied, the second term on the RHS of the above equation vanishes, and we have:

$$\Psi[h_{ij} + D_{(i}\xi_{j)}] = \Psi[h_{ij}] \tag{2.19}$$

Similarly to the wave functions from non-relativistic particle mechanics, there is a probability interpretation associated with our wave function. Because the Wheeler-DeWitt equation has a Klein-Gordon form, there exist a corresponding conserved current:

$$J = \frac{i}{2}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) \tag{2.20}$$

Although tempting to naively interpret J as a probability flux, it is not positive definite. For this reason, many physicists reject probabilities constructed from the conserved current J. Instead, many adopt Hawking's proposal that $|\Psi[\tilde{h}_{ij}, \tilde{\Phi}, \sum]|^2$ is to be interpreted because it is proportional to the probability of the Universe containing a three-surface $\sum$ with metric $\tilde{h}_{ij}$ and matter field $\tilde{\Phi}$. Explicitly, the probability of finding that the Universe is in a configuration contained within a region $\mathcal{V}$ of our superspace [11] is then:

$$P(\mathcal{V}) \propto \int_{\mathcal{V}} |\Psi|^2 * 1 \tag{2.21}$$

with $*1$ being the volume element.

In quantum field theory, solutions to the Klein-Gordon equation are quantized and turned into field operators, thinking in a similar manner about the Wheeler-DeWitt equation has led to proposals that Ψ should be "third quantized" and turned into an operator $\hat{\Psi}$. This operator creates and annihilates Universes in the same way that the familiar ladder operators associated with the Klein-Gordon field create and annihilate particles. Difficulties with this method arise due to the fact that we clearly cannot make measurements on a statistical ensemble of Universes the same way we can for particles. Therefore, it is not known how using this method could lead to measurable probabilities [12], [13], [14].

2.3 Superspace

Superspace is the space on which our wave function is defined. To construct this space, we start by considering the configuration space that contains the Riemannian three-metrics h_{ij} and matter field configurations $\Phi(x)$ on a spatial hypersurface Σ.

$$\mathrm{Riem}(\Sigma) = \{h_{ij}(x), \Phi(x) | x \in \Sigma\} \tag{2.22}$$

If we can find a diffeomorphism relating a set of configurations, those configurations must have the same intrinsic geometry and we consider them to be equivalent. We now proceed by partitioning this space into equivalence classes such that if two configurations are related by a diffeomorphism, they belong to the same equivalence class. Superspace is defined as the following quotient:

$$\mathrm{Sup}(\Sigma) = \mathrm{Riem}(\Sigma)/\mathrm{Diff}_0(\Sigma) \tag{2.23}$$

where the subscript zero specify that only diffeomorphisms that are connected to the identity are considered. The metric on the infinite dimensional superspace is the Wheeler-DeWitt metric G_{ijkl}.

2.4 Minisuperspace

Instead of speaking about superspace, which is infinite dimensional so it's not trivial to make exact statements about it, we can focus to a finite number of degrees of freedom $x^\alpha(t)$. By doing this we can create minisuperspace which is finite dimensional and which can be approached with an isotropic and homogeneous three-surface [15]. It is not sure with certainty that minisuperspace is a part of a systematic approximation of the full theory. Some arguments about the validity of this finite dimensional minisuperpace is given in [16]. In this space, the shift is $N^i = 0$ and the lapse is homogeneous and thus $N = N(t)$. Then, the space element in this space is given by:

$$ds^2 = -N^2(t)dt^2 + h_{ij}(\boldsymbol{x}, t)dx^i dx^j. \tag{2.24}$$

The matter in this approximation is homogeneous as well. According to [17], the inverse of the DeWitt metric is the following:

$$G^{ijkl} = \frac{1}{2}h^{1/2}(h^{ik}h^{jl} + h^{il}h^{jk} - 2h^{ij}h^{kl}). \tag{2.25}$$

Also it can be proved that:

$$G^{ijkl}\dot{h}^{ij}\dot{h}^{kl} = 4h^{1/2}N^2(K_{ij}K^{ij} - K^2). \tag{2.26}$$

Now, one can rewrite the action 2.5 as:

$$S = \int dtL = \int dtN\left[\frac{1}{2N^2}G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta - U(x)\right] \tag{2.27}$$

where:

$$U(x) = \int d^3x h^{1/2}\left[\frac{m_p^2}{16\pi}(-^3R + 2\Lambda) + V(\phi)\right] \tag{2.28}$$

$$G_{\alpha\beta}dx^\alpha dx^\beta = \int d^3x\left[\frac{m_p^2}{32\pi}G^{ijkl}\dot{h}_{ij}\dot{h}_{kl} + h^{1/2}\delta\phi\delta\phi\right]. \tag{2.29}$$

The equation 2.29 defines the metric in minisuperspace.

Now, the canonical momenta is given by:

$$\pi_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha} = G_{\alpha\beta}\frac{\dot{x}^\beta}{N}, \tag{2.30}$$

and therefore, the Hamiltonian is:

$$H = \pi_\alpha \dot{x}^\alpha - L = N\left[\frac{1}{2}G^{\alpha\beta}\pi_\alpha\pi_\beta + U(x)\right]. \tag{2.31}$$

The above expression arises from the Hamiltonian constraint of the full theory, integrated over the spatial hypersurfaces. Having the precise Hamiltonian, one can proceed with the quantization procedure.

2.5 Path Integral Approach

A path integral is called the multiple integral, in which integration is done by adding a functional $G[f(x)]$ to a continuous range of functions $f(x)$, as opposed to adding a function $f(x)$ to a continuous range of values of the variable x, as in the normal integral. That is, instead of the usual integral $\int dx f(x)$, the functional integral is defined as follows:

$$\int \mathcal{D}f G[f] = \int \prod_x df(x) G[f(x)] \tag{2.32}$$

The implementation of such an integral is useful for representation of correlation functions (such as the propagator of fields) in systems of many continuous degrees of freedom $q(t)$, as it turns out to be simpler than the process of canonical quantization. Another reason the path integral approach is preferred, is because while functional integral is based on the basic principles of quantum mechanics, it does not display the formalism of operators, that is, it does not require the conversion of fields into operators. Fields are treated as functions thus making it easier to manage them. Also, unlike canonical quantization, in gauge theories, the control of the independence of physical quantities from gauge selection is more direct using a functional integral. In addition, a very close analogy with statistical mechanics occurs when using functional integrals, which allows the use of common tricks of statistical mechanics and Quantum Field Theory. Adding to this, the functional quantization of fields is also used for non-perturbative calculations in theories of interacting fields.

Finally, the path integral approach is important for the formulation of the no-boundary proposal. The boundary conditions of this proposal can be expressed in a more natural way when using the path integral approach. In analogy with particle mechanics, we can now write the wave function of the Universe as a path integral [18,19,20]:

$$\Psi[\tilde{h}, \tilde{\Phi}, \Sigma] = \sum_{\mathcal{M}} \int_{\mathcal{C}} \mathcal{D}g \mathcal{D}\Phi e^{iS[g,\phi]} \tag{2.33}$$

In the above expression, we sum over compact three-manifolds $\mathcal{M}$ with boundary conditions containing Σ, a three-surface with three-metric $\tilde{h}$. $\tilde{\Phi}$ is the field matter configuration.

Hartle and Hawking in their paper [4] argue that a closed universe can be described by a wave function satisfying the Wheeler-DeWitt second order functional differential equation. They propose that the state of minimum excitation of the Universe should be the cosmological analog of a quantum mechanical ground state. However in quantum cosmology the ground state is not the state with the lowest energy and the energy of the Universe is not well-defined as there is no natural definition for the energy.

Chapter 3

Boundary conditions

3.1 Introduction

According to DeWitt's paper in 1967, the Wheeler-DeWitt equation gives one and only solution. However, is seems that when differential equations are combined with dynamical theories, like Wheeler-DeWitt equation, there is no just a unique solution. In order to get just a unique solution that describes the Universe, we must apply a set of boundary conditions that lead to our observable universe. These boundary conditions are not universally agreed upon. In this chapter we will focus on some boundary condition proposals, the two most studied and comprehensive. The first one is Vilenkin's and Linde's tunnelling proposal and the no-boundary proposal of Hartle and Hawking.

3.2 Classical Solutions

As mentioned earlier, the wave function that describes the universe is a solution of the Wheeler-DeWitt equation and using the path integral approach is important for the formulation of the no-boundary proposal. Boundary conditions that will be discussed in this chapter, ensure the uniqueness of the solution for the wave function.

One possible question is what kind of solutions do we expect to find. If the solution is the correct one, then the wave function should give the classical spacetime at the limit where the universe is large.

A quantum system is classical when it satisfy some requirements. Firstly the solution must strongly correlate the canonical variables with one or more classical configurations and also the interference between these configurations must decohere. It can be proved that the solutions are single wave packets strongly peaked about a single classical trajectory. In Hartle's lectures (Hartle, 1990), path integral methods are described when there are predictions (peaks) from a theory of initial conditions.

3.3 WKB Approximation

The WKB approximation is used to focus in the oscillatory region of the solutions of Wheeler-DeWitt equation:

$$\left[-\frac{1}{2m_p^2}\nabla^2 + m_p^2 U(q)\right]\Psi(q) = 0 \tag{3.1}$$

where m_p is the Planck mass. In the presence of a cosmological constant Λ, one can take Λm_p^{-4} to be a small parameter to control the dimensionless WKB expansion.

Using the WKB approximation, we are interested in solutions whose behaviour is strictly exponential or oscillatory, (i.e. solutions of the form e^{-I} or e^{iS}). Solutions of 3.1 have the following form in the WKB approximation:

$$\Psi(q) = C(q)e^{-m_p^2 I(q)} + O(m_p^{-2}), \tag{3.2}$$

where I and C are complex. Doing this, one can found that peaked wave functions in the oscillatory region determine a set of solutions to the classical field equations. Also for every solution of the above form, there is an associated current

$J_n = -|C_n|^2 \nabla S_n.$

3.4 The tunnelling proposal

Alexander Vilenkin in 1982 [21] introduce the tunnelling proposal as a possible approach to quantum cosmology that involves the spontaneously nucleating of the Universe from the vacuum (no matter, time and space) into de Sitter spacetime and after that the entrance into an inflationary period. This proposal description is given through a path integral [22,23]. This proposal regards the transition amplitude between two three-geometries $h^1_{i,j}$ and $h^2_{i,j}$. The path integral is the following:

$$\sum_{\mathcal{M}} \int_{h^1_{ij},\Phi^1}^{h^2_{ij},\Phi^2} \mathcal{D}g_{\mu\nu}\mathcal{D}\Phi e^{iS[g_{\mu\nu},\Phi]}, \tag{3.3}$$

where Φ^1 and Φ^2 are the corresponding matter fields.

Now, one can compute the wave function of the Universe by shrinking the three geometry h^1_{ij} to a single point. Then:

$$\sum_{\mathcal{M}} \int_{0}^{\tilde{h}^2_{ij},\tilde{\Phi}^2} \mathcal{D}g_{\mu\nu}\mathcal{D}\Phi e^{iS[g_{\mu\nu},\Phi]}, \tag{3.4}$$

where the above expression is the transition amplitude between the configuration of the Universe $(\tilde{h}_{ij}, \tilde{\Phi})$ and the three geometry. The integration over events of the past $(\tilde{h}_{ij}, \tilde{\Phi})$ and as a result of this, the Universe exhale from a vanishing three-geometry. In tunnelling proposal, the integration is over Lorentzian metrics with no restrictions to compact Euclidean geometries. In the Hartle-Hawking proposal we face this restriction. In this proposal, there is the issue with singularity which follows from the fact that any Lorentzian geometry that connects the vanishing three-geometry to $\tilde{h}_{ij}$ is inherently singular. Focusing on spacetime manifolds at smaller scales than the Planck length can conquer this issue [24]. Also this issue can be avoided by a sufficient quantum gravity theory.

This proposal was remodelled some years after, in terms of boundary conditions in hyperspace [24]. Even though there is no obvious equivalence with the original proposal, the condition is that at singular boundaries of superspace, the wave function including modes that carry flux out of superspace (outgoing modes). Vilenkin tried to associate the solutions of the Klein-Gordon equation (which are positive and negative frequency) with the solutions of the Wheeler-DeWitt equations and attempted to classify these solutions as ingoing or outside at the boundary. However these solutions are not that well defined outside of the semiclassical case. Now, as a regularity condition, the wave function should be bounded everywhere. The boundary of superspace will in general incorporate singular configurations because of points or regions (like Φ, $\partial_i\Phi$) which may be infinite.

The boundary of the superspace can be split into two regions. The first region is the non-singular boundary that consists of three-geometries where singularities are credited to slicing of regular four-geometries. The second region is the singular boundary consisting of everything not in the first region.

The Hamiltonian - Jacobi equation on superspace is given by:

$$(1/2)(\nabla S_n)^2 + U = 0 \tag{3.5}$$

Every semi - classical wave function can be written as the following sum:

$$\Psi = \sum_n C_n e^{iS_n} \tag{3.6}$$

where S_n are the solutions of the Hamiltonian - Jacobi equation. As we mentioned before, in the WKB approximation, for every exponential or oscillatory (i.e. solutions of the form $e^{-I}\ e^{iS}$), there exist an associated current:

$$J_n = -|C_n|^2 \nabla S_n \tag{3.7}$$

The term $-\nabla S_n$ is defined to point out of superspace at the boundary.

Now, one can consider the FRW metric:

$$ds^2 = -dt^2 + a(t)^2\big(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)\big) \tag{3.8}$$

where $0 \leq \chi \leq \pi$, $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. The above metric can describe a closed universe with the following action:

$$S = \int d^4x(-g)^{1/2}\left[\frac{3a^{-2}}{8\pi}(1 + \dot{a}^2 + a\ddot{a}) + \frac{1}{2}g^{\mu\nu}\partial\mu\Phi\partial\nu\Phi - V(\Phi)\right] \tag{3.9}$$

The homogeneous and isotropic minisuperspace will be defined by the scale factor $a(t) \geq 0$ and $\Phi(t)$. Now, we can rewrite the action as:

$$S = \int dt\bigg(\frac{3\pi a}{4}(1 + \dot{a}^2 + a\ddot{a}) + \pi^2 a^3\dot{\Phi}^2 - 2\pi^2 a^3 V(\Phi)\bigg) \tag{3.10}$$

where in the above expression we used that:

$$\int d^3x(-g)^{1/2} = 2\pi^2 a^3 \tag{3.11}$$

Using integration by parts and setting the boundary term equal to zero, we simplify:

$$S = \int dt\bigg(\frac{3\pi}{4}(1 - \dot{a}^2)a + \pi^2 a^3\dot{\Phi}^2 - 2\pi^2 a^3 V(\Phi)\bigg) = \int dtL \tag{3.12}$$

where:

$$L = \frac{3\pi}{4}(1 - \dot{a}^2)a + \pi^2 a^3\dot{\Phi}^2 - 2\pi^2 a^3 V(\Phi) \tag{3.13}$$

We can now calculate the canonical momenta:

$$\pi_a = \frac{\partial L}{\partial \dot{a}} = -\frac{3\pi}{2}a\dot{a} \tag{3.14}$$

$$\pi_\Phi = \frac{\partial L}{\partial \dot{\Phi}} = 2\pi^2 a^3 \dot{\Phi} \tag{3.15}$$

Then, the Hamiltonian is given by:

$$H = -\frac{1}{3\pi a}\pi_a^2 + \frac{1}{4\pi^2 a^3}\pi_\Phi - \frac{3\pi}{4}a\left(1 - \frac{8\pi}{3}a^2 V(\Phi)\right) \tag{3.16}$$

Using the canonical quantization procedure:

$$\pi_a \rightarrow -i\frac{\partial}{\partial a} \tag{3.17}$$

$$\pi_\Phi \rightarrow -i\frac{\partial}{\partial \Phi}, \tag{3.18}$$

we can determine the Wheeler-DeWitt equation for the minisuperspace model. In this equation there will be some operator ordering issues because of terms like $1/a$ and ∂a. According to Hawking and Page [25], if one replace the differential operator with the Laplacian one in the metric on superspace, this issue can be ignored. Then:

$$\left(a\frac{\partial}{\partial a}\left(a\frac{\partial}{\partial a}\right) - \frac{3}{4\pi}\frac{\partial^2}{\partial \Phi^2} - \frac{9\pi^2}{4}a^4\left(1 - \frac{8\pi}{3}a^2 V(\Phi)\right)\right)\Psi = 0 \tag{3.19}$$

Vilenkin in his paper introduced a parameter p to group the operator orderings:

$$\left(\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - \frac{3}{4\pi a^2}\frac{\partial^2}{\partial \Phi^2} - \frac{9\pi^2}{4}a^2\left(1 - \frac{8\pi}{3}a^2 V(\Phi)\right)\right)\Psi = 0 \tag{3.20}$$

Setting $p = 1$ to the above expression leads to Hawking and Page argument. Vilenkin introduced a suitable scaling for Φ and $V(\Phi)$ to simplify the above equation:

$$\left(\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial \Phi^2} - U\right)\Psi = 0 \tag{3.21}$$

where:

$$U = a^2(1 - a^2 V(\Phi)) \tag{3.22}$$

is the superpotential. The Euclidean region is described by $U > 0$ and the Lorentzian region by $U < 0$. $U = 0 \Rightarrow V(\Phi) = 1/a^2$ is the boundary [26]. Assuming that the potential is far from the boundary, we get the following condition [27]:

$$|dV(\Phi)/d\Phi| \ll \max\left[|V(\Phi)|, a^{-2}\right]. \tag{3.23}$$

Due to the above assumption, we can ignore terms $\partial/\partial\Phi$ and rewrite the equation 3.21 as:

$$\left(\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - U\right)\Psi = 0 \tag{3.24}$$

If one set the value of p to be equal to -1, the above equation can be solved exactly.

The tunneling wave function for the classically allowed region (oscillatory region) is described by the approximation:

$$a^2 V(\Phi) > 1, \tag{3.25}$$

and it is of the form:

$$\Psi_T \propto \exp\left(-\frac{1 + i(a^2V(\Phi) - 1)^{3/2}}{3V(\Phi)} + \frac{i\pi}{4}\right). \tag{3.26}$$

For the classically forbidden region (exponential region) the approximation is:

$$a^2 V(\Phi) < 1. \tag{3.27}$$

The tunnelling wave function for this region has the following form:

$$\Psi_T \propto \exp\left(-\frac{1}{3V(\Phi)}\left[1 - \left(1 - a^2V(\Phi)\right)^{3/2}\right]\right) \tag{3.28}$$

The two components of the conserved current J according to WKB approximation are given by:

$$J^a = \frac{i}{2} a^p (\Psi \partial_a \Psi^* - \Phi^* \partial_a \Psi) \tag{3.29}$$

$$J^\Phi = -\frac{i}{2} a^{p-2} (\Psi \partial_\Phi \Psi^* - \Phi^* \partial_\Phi \Psi). \tag{3.30}$$

According to Vilenkin [24], the above solution for the wave function is not arbitrary and consist to an expanding de Sitter minisuperspace:

$$a \approx V^{-1/2} \cosh \left(V^{1/2} t \right). \tag{3.31}$$

where the above scale factor can play the role of the time variable, giving a current $J > 0$ for expanding Universes.

3.5 No-Boundary Proposal

Hartle and Hawking in their paper propose that the amplitude of the state of minimum excitation for a three-geometry is given by a path integral over all compact positive and definite four - geometries where the three - geometry is the boundary. Also according to Hawking's assumption, the geometry of the Universe was a regular Euclidean geometry with four spatial dimensions that after a quantum transition became a Lorentzian geometry with one time dimension and three spacial.

In the previous chapter we mentioned that the path integral approach is important for the formulation of the no-boundary proposal. The boundary conditions of this proposal can be expressed in a more natural way when using the path integral approach. According to this, the solution of the Wheeler-DeWitt equation that describes the Universe is expressed as a Euclidean path integral. This Euclidean geometry is compact, having a compact three - surface Σ as the only boundary.

Using the gauge where $\dot{N}$ vanishes, we express the wave function of the Universe as a Euclidean path integral:

$$\Psi_{NB} = \int dN \int \mathcal{D}\Phi \mathcal{D}a \exp[-I(a(\tau), \Phi(\tau), N)] \tag{3.32}$$

The subscript NB denotes the no-boundary wave function for the closed homogeneous Universe.

We can now parametrize the three - surface Σ as follows:

$$h_{ij}(\boldsymbol{x}, 1) = \tilde{h}_{ij}(\boldsymbol{x}) \tag{3.33}$$

$$\Phi(\boldsymbol{x}, 1) = \tilde{\Phi}(\boldsymbol{x}) \tag{3.34}$$

$$a(1) = \tilde{a} \tag{3.35}$$

where we set $\tau = 1$ in the above expressions. The Euclidean action that describes this model is given by:

$$I = \frac{1}{2}\int_0^1 d\tau N\bigg[-\frac{a}{N^2}\left(\frac{da}{d\tau}\right)^2 + \frac{a^3}{N^2}\left(\frac{d\Phi}{d\tau}\right)^2 - a + a^3V(\Phi)\bigg]. \tag{3.36}$$

In the above expression, we chose the initial point $\tau = 0$. Now, by variation of the action with respect to a and Φ, we obtain the following field equations:

$$\frac{1}{N^2}\frac{d^2\Phi}{d\tau^2} + \frac{3}{Na}\frac{da}{d\tau}\frac{d\Phi}{d\tau} - \frac{1}{2}\frac{dV(\Phi)}{d\Phi} = 0 \tag{3.37}$$

$$\frac{1}{N^2a}\frac{d^2a}{d\tau^2} + \frac{2}{N^2}\left(\frac{d\Phi}{d\tau}\right)^2 + V(\Phi) = 0 \tag{3.38}$$

The saddle-point condition $(\partial I/\partial N) = 0$ gives:

$$\frac{1}{N^2}\left(\frac{da}{d\tau}\right)^2 - \frac{a^2}{N^2}\left(\frac{d\Phi}{d\tau}\right)^2 - 1 + a^2V(\Phi) = 0 \tag{3.39}$$

Let's now consider the Euclidean four-geometry (FRW metric) where the line element is given by:

$$ds^2 = N^2 dt^2 + a^2(\tau) d\Omega_3^2 \tag{3.40}$$

In Hartle and Hawking no-boundary proposal, we require $a(\tau) \approx N\tau$ at the limit $\tau \to 0$ and thus $a(0) = 0$ That's because we want the four-geometry to close off in a regular way. In this case, the metric 3.40 approaches the flat space (in spherical coordinates), which has the following metric:

$$ds^2 = dt^2 + r^2 \Omega_3^2 \tag{3.41}$$

The above requirement is not enough because it leads to singularities of the fields equations. However if we require:

$$\left.\frac{d\Phi}{d\tau}\right|_{\tau=0} = 0 \tag{3.42}$$

for the scalar field Φ, then the solution will be regular.

We can now find a solution of 3.38 that meets the above requirements:

$$a(\tau) \approx \frac{\tilde{a} \sin\left(V^{1/2} N \tau\right)}{\sin\left(V^{1/2} N\right)} \tag{3.43}$$

Substituting this solution to 3.39, we get:

$$\frac{V \tilde{a}^2 \cos^2\left(V^{1/2} N \tau\right)}{\sin^2\left(V^{1/2} N\right)} - 1 + a^2 V = 0 \tag{3.44}$$

For $\tau = 1$, we get:

$$\tilde{a}^2 V \left(1 + \cot^2(V^{1/2} N)\right) = 1 \tag{3.45}$$

and thus:

$$\sin^2(V^{1/2}N) = \tilde{a}^2 N \tag{3.46}$$

For the above expression we consider real values for the potential and the scale factor a and for this reason:

$$\tilde{a}^2 V < 1 \tag{3.47}$$

Now, the solutions are solutions are parameterized by $n \in \mathbb{Z}$:

$$N_n^{\pm} = \frac{1}{V^{1/2}}\left[\left(n + \frac{1}{2}\right)\pi \pm \cos^{-1}(\tilde{a}V^{1/2})\right]. \tag{3.48}$$

In the above expression, $\cos^{-1}(\tilde{a}V^{1/2})$ lies in the principle range $\left(0, \frac{\pi}{2}\right)$. For $n = 0$ we have [27]:

$$a(\tau) \approx \frac{1}{V^{1/2}} \sin\left[\left(\frac{\pi}{2} \pm \cos^{-1}(\tilde{a}V^{1/2})\right)\tau\right]. \tag{3.49}$$

Now, we can calculate the action, where:

$$I_{\pm} = -\frac{1}{3V(\tilde{\Phi})}\left[1 \pm \left(1 - \tilde{a}^2 V(\tilde{\Phi})\right)^{3/2}\right] \tag{3.50}$$

As we can see, there are two possible solutions:

- (-) describes the three-sphere which is closed off by less than half of a four-sphere.
- (+) describes the three-sphere which is closed off by more than half of a four-sphere.

Finding these two possible solutions is not enough as we still need to specify the contour that we will perform the integration over it.

According to Halliwell et Louko [28,29,30], different convergent contours are dominated by different saddle-points leading to different wave function. Hartle and

Hawking proposal doesn't offer guidance in making a choice. However, there are no preferred contours but only some better than others.

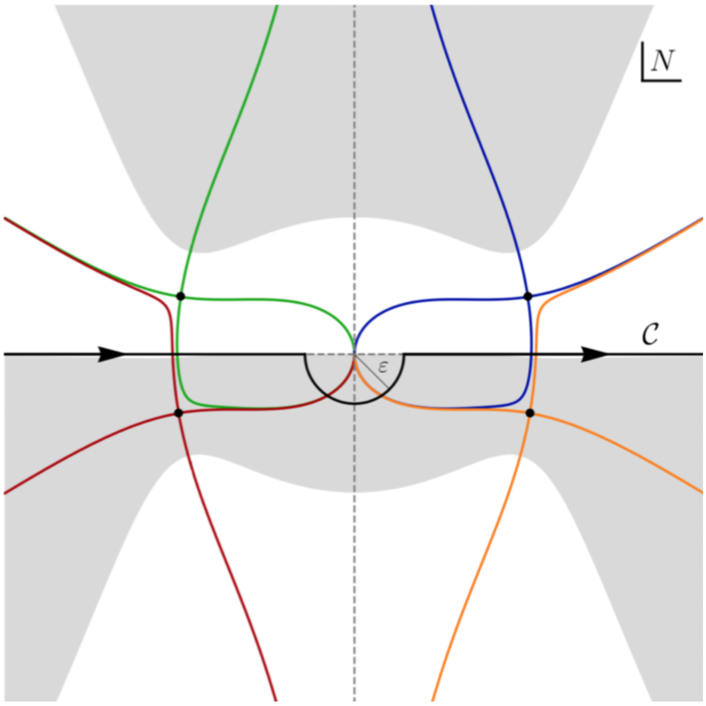

Figure 3.1: Steepest descent, ascent and integration contours for Hartle and Hawking proposal (taken from [31] where more details are provided).

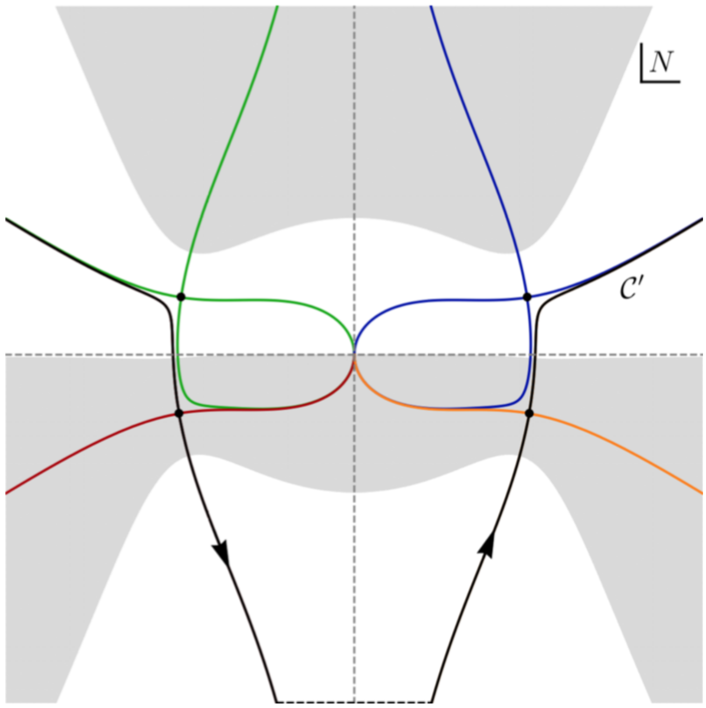

Figure 3.2: Modification of the contour (taken from [31] where more details are provided).

In [32], the following conditions, that any sensible contour should satisfy, are presented:

- Classical spacetime should be predicted in the case of a large Universe.
- Convergence of the integral that defines the wave function.
- Compatibility between wave function and diffeomorphims invariance implemented by momentum constraint.
- The correct field theory in curved spacetime should be reproduced in this spacetime.
- To the extent that wormholes make the cosmological constant dependent on initial conditions the wave function should predict its vanishing.

Halliwell and Hartle in the same paper showed that contours dominated by saddle-points corresponding to negative lapse functions lead to difficulties in recovering quantum field theory in a curved space time. While this provides us with a good reason for excluding contours for which the dominating contribution is from a saddle-point with $n < 0$ there does not appear to be any good reason for preferring a contour for which the dominating contribution comes from a saddle-point associated with any particular $n \geq 0$.

Hartle and Hawking in (1983) derived to the semi-classical no-boundary wave function. As mentioned before, the condition for the classically allowed region is:

$$a^2 V(\Phi) > 1 \tag{3.51}$$

and for the classically forbidden region is:

$$a^2 V(\Phi) < 1 \tag{3.52}$$

The no-boundary wave function for the classically allowed region is:

$$\Psi_{NB} \approx \exp\left(\frac{1}{3V(\Phi)}\right)\cos\left[\frac{1}{3V(\Phi)}\left(a^2V(\Phi)-1\right)^{3/2}-\frac{\pi}{4}\right] \tag{3.53}$$

and the one that corresponds to the classically forbidden region is:

$$\Psi_{NB} \approx \exp\left[\frac{1}{3V(\Phi)}\left(1-\left(1-a^2V(\Phi)\right)^{3/2}\right)\right] \tag{3.54}$$

Hawking in 1985, published a paper [33] on the arrow of time in cosmology, in which he made what he considered his ”biggest mistake” [34]. Previously, he had proposed that the thermodynamic arrow of time and the cosmological arrow of time must always point in the same direction [35]. With this suggestion in mind, Hawking concluded that the thermodynamic arrow would reverse at the moment of maximum expansion and entropy and coincide with the cosmological arrow when the universe eventually re-collapsed. Since the psychological arrow of time is presumably a consequence of the thermodynamic arrow, this would bizarrely mean that a conscious observer would then remember the future but not the past. A crucial factor leading him to this conclusion was the fact that his unbounded quantum state was CPT invariant. The CPT theorem is a recognised property of all fundamental physical laws and states that these laws are invariant under the combination of charge conjugation (Charge), space inversion (Parity), and time reversal (Time). Therefore, the fact that the wave function without limits exhibits this property is an encouraging sign.

To finish the discussion about calculation of the no-boundary wave function, we construct a probability measure on a set of paths $J \cdot d\Sigma$:

$$dP_{NB} = J \cdot d\Sigma \propto \exp\left(\frac{2}{3V(\Phi)}\right)d\Phi \tag{3.55}$$

where Σ is a surface where a is constant.

3.6 Inflation

Obviously the universe has reached a sufficient level of inflation for large-scale structures and even observers to emerge. At least in the case of a minisuperspace model, the two boundary conditions mentioned lead to a wave function that predicts an inflationary period. The amount of inflation a universe experiences is determined by the initial value Φ_0 of our scalar field. It is the potential energy of this field that allows the universe to expand exponentially for a short period of time. If this value is too small, the model will predict that the universe will expand and collapse again in a time period too short for a large-scale structure to emerge. Therefore, we will make the same restriction as Hawking and Page did in an earlier paper [25] and not deal with values of Φ_0 that are below a certain small value, which we will call Φ_{min}. We would like our model to predict a sufficient amount of inflation, say 60 e-folds [36, 37], to provide a satisfactory explanation for the flatness and horizon problems. This motivates us to define a value Φ_{sugg} for which the universe experiences sufficient inflation for $\Phi_0 > \Phi_{\mathrm{sugg}}$. Now we can write:

$$N_e = 6\int_{\Phi_e}^{\Phi_0} d\Phi \frac{V(\Phi)}{V'(\Phi)} \tag{3.56}$$

where N_e is the number of e-folds, Φ_e is the value of the scalar field at the end of inflationary period. Considering a chaotic potential of the form $V(\Phi) = m^2\Phi^2$, we get:

$$N_e = \frac{3}{2}(\Phi_0^2 - \Phi_e^2) \tag{3.57}$$

It can be proved that a sufficient amount of inflation occurs at $\Phi_{\mathrm{suff}} \approx 6.3$. Vilenkin in his paper [28] pointed out that the potential $V(\Phi)$ is likely to far exceed the Planck energy density for large values of Φ, unless it has a very special form. Since the derivation of our probability density is based on a semiclassical approximation, it would not be wise to trust the predictions of our mini-space model in this domain.

We are therefore again motivated to define another value Φ_{max}, which is the largest value of Φ_0 for which we trust our model. Since the initial value for the scalar field is somewhere within the range $\Phi_{\text{min}} < \Phi_0 < \Phi_{\text{max}}$, we would like to compute the probability that it is greater than Φ_{stuff}. That is, we want to evaluate the conditional probability:

$$P(\Phi_0 > \Phi_{\text{suff}} | \Phi_{\text{min}} < \Phi_0 < \Phi_{\text{max}}) = 1 - \frac{\int_{\Phi_{\text{min}}}^{\Phi_{\text{suff}}} d\Phi \exp\left(\pm \frac{2}{3V\Phi} \right)}{\int_{\Phi_{\text{min}}}^{\Phi_{\text{max}}} d\Phi \exp\left(\pm \frac{2}{3V\Phi} \right)} \tag{3.58}$$

We can think of this result graphically:

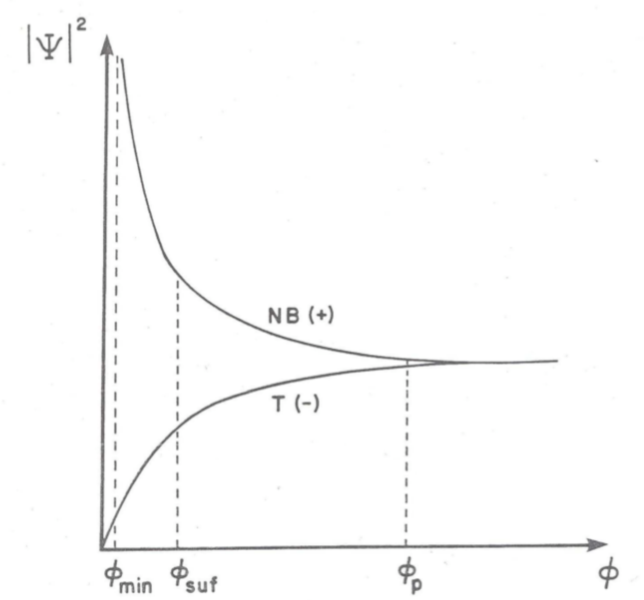

Figure 3.3: Probability distributions for the tunnelling and no-boundary wave function (taken from [26] where more details are provided).

According to the figure for the tunnelling wave function we see that for values $\Phi \to \Phi_{\text{min}}$:

$$\int_{\Phi_{\text{min}}}^{\Phi_{\text{max}}} d\Phi \exp\left(\frac{2}{3V(\Phi)} \right) \gg \int_{\Phi_{\text{min}}}^{\Phi_{\text{suff}}} d\Phi \exp\left(\frac{2}{3V(\Phi)} \right) \tag{3.59}$$

and thus:

$$P(\Phi_0 > \Phi_{\text{suff}} | \Phi_{\text{min}} < \Phi_0 < \Phi_{\text{max}}) \approx 1. \tag{3.60}$$

On the other hand for the no-boundary wave function we see that:

$$\int_{\Phi_{\text{min}}}^{\Phi_{\text{max}}} d\Phi \exp\left(-\frac{2}{3V(\Phi)}\right) \approx \int_{\Phi_{\text{min}}}^{\Phi_{\text{suff}}} d\Phi \exp\left(-\frac{2}{3V(\Phi)}\right), \tag{3.61}$$

and:

$$P(\Phi_0 > \Phi_{\text{suff}} | \Phi_{\text{min}} < \Phi_0 < \Phi_{\text{max}}) \ll 1. \tag{3.62}$$

Chapter 4

Alternative approach of the no-boundary proposal

4.1 Introduction

In this Chapter, a mathematical technique called Picard-Lefschetz theory is introduced. Turok et al. in [38] used this technique to implement an alternative version of the no-boundary proposal which gives different results to the traditional implementation.

4.2 Picard-Lefschetz Theory

One can use this technique to approximate integrals of the following form:

$$\int_D dx e^{iS[x]/\hbar} \tag{4.1}$$

In the above path integral $\hbar$ is a real small parameter, $S[x]$ a real function, and D a real domain. This technique was first introduced by Arnol'd et al. [39]. To calculate the above integral, we need a generalisation of the methods of stationary phase or steepest descent. As in these cases, and assuming that Cauchy's theorem is applicable, one then allows x to be complex and deforms the domain of integration to a contour of steepest descent that is bounded by critical points of S.

At these points, the equations of Cauchy-Riemann, imply that Re(iS) (which controls the decay of the integrand) has a saddle point. We then take the steepest descent contour through a to be the path along which Re(iS) decreases most rapidly. These contours generally lead to a convergent integral and in this case are called Lefschetz's thimbles $\mathcal{J}_\sigma$.

Now, we can write:

$$iS[x]/\hbar \equiv h + iH \tag{4.2}$$

$$x \equiv u^1 + iu^2 \tag{4.3}$$

Using λ as a parameter for the path, we can define a downward flow that satisfies the following equation:

$$\frac{du^i}{d\lambda} = -g_{ij}\frac{\partial h}{\partial u^j} \tag{4.4}$$

Use of the chain rule gives:

$$\frac{dh}{d\lambda} = \sum_i \frac{\partial h}{\partial u^i}\frac{du^i}{d\lambda} = -\sum_i \left(\frac{\partial h}{\partial u^i}\right) < 0 \tag{4.5}$$

Therefore, we have the real part of the exponent h decreases on such a flow away from its critical point. In addition, the decrease is the fastest because the gradient has the largest possible amplitude and thus we use the direction of the fastest descent $\mathcal{J}_\sigma$ to identify this flow. Adding to this, one can similarly define the upward flow or steepest ascent path $\mathcal{K}_\sigma$.

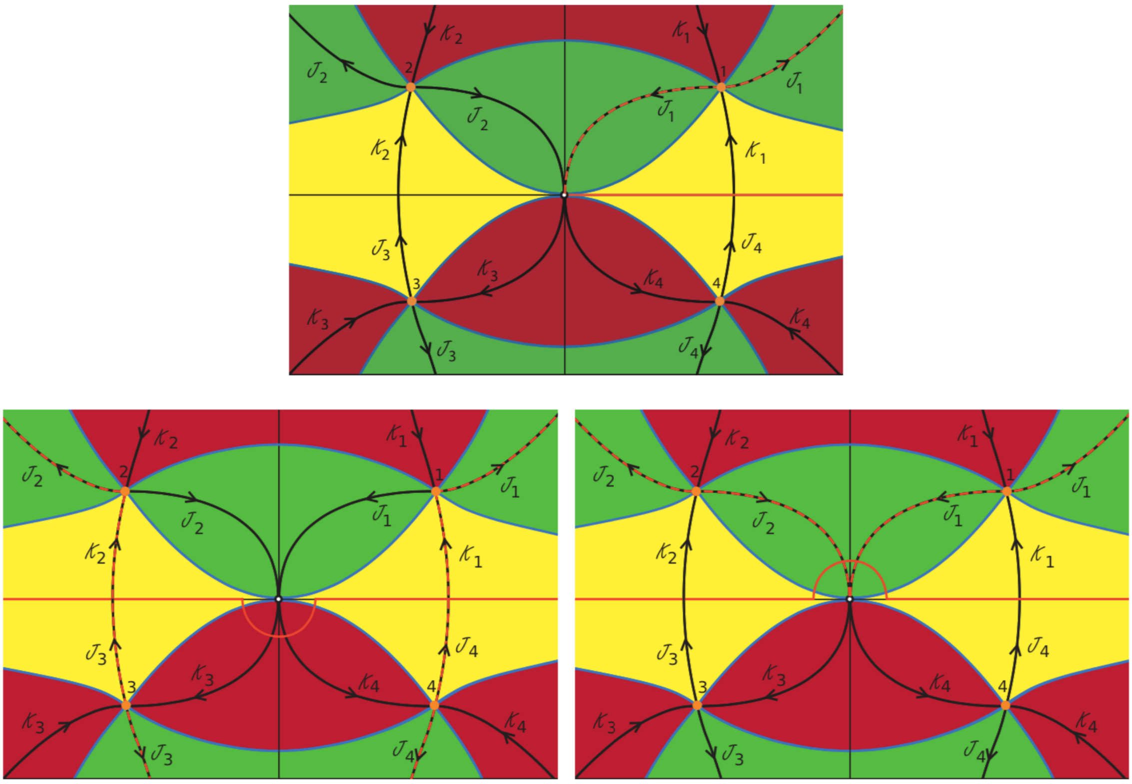

Figure 4.1: The integrand, in the complex N-plane, for a closed, homogeneous and isotropic Universe. (Taken from [38])

For the flat Riemannian metric, we have:

$$\frac{dx}{d\lambda} = -\frac{\partial \bar{I}}{\partial \bar{x}} \tag{4.6}$$

Also, the imaginary part of the exponent is conserved:

$$\frac{dH}{d\lambda} = \frac{1}{2i}\frac{d(I-\bar{I})}{d\lambda} = \frac{1}{2i}\left(\frac{\partial I}{\partial x}\frac{dx}{d\lambda} - \frac{\partial \bar{I}}{\partial \bar{x}}\frac{d\bar{x}}{d\lambda}\right) = 0 \tag{4.7}$$

According to the above results, we see that the integrand does not oscillate along the downward flow, but decreases monotonically, so that the integral converges most rapidly. This is not entirely true, however, since it may be that a steepest downward path through one saddle point coincides with a steepest upward path through another saddle point. This degeneracy is usually caused by symmetries of action and is solved by introducing perturbations. Once this is resolved, a one-to-one relationship can be established between the saddle points and the relatively steepest descent and ascent contours. The general situation is that every steepest descent contour terminates on a singularity where $h \to -\infty$. Therefore, we can consistently modify the integration domain D to a new contour:

$$\mathcal{C} = \sum_{\sigma} n_{\sigma}\mathcal{J}_{\sigma} \tag{4.8}$$

where the coefficients n_{σ} (intersection number) takes values 0 or ± 1. For more details, see [40]. Critical points that make very large contributions to the integral,are described by setting $n_{\sigma} = 0$ [41].

Finally, one can write:

$$\int_D dx e^{iS[x]/\hbar} = \int_{\mathcal{C}} dx e^{iS[x]/\hbar} = \sum_{\sigma} n_{\sigma} \int_{\mathcal{J}_{\sigma}} dx e^{iS[x]/\hbar} \tag{4.9}$$

The above integral converges when:

$$\int_{\mathcal{J}_\sigma} |dx| e^{h(x)} < \infty \tag{4.10}$$

$\forall$ Lefschetz trimbles $\mathcal{J}_\sigma$.

Using the semiclassical expansion of the integral of interest in powers of $\hbar$, we get:

$$\int_D dx e^{iS[x]/\hbar} = \sum_\sigma n_\sigma e^{iH(p_\sigma)} \int_{\mathcal{J}_\sigma} e^h dx \approx \sum_\sigma n_\sigma e^{iS(p_\sigma)/\hbar}[A_\sigma + \mathcal{O}(\hbar)], \tag{4.11}$$

where S_σ is the value of the action at the critical point and A_σ is the value of the leading order Gaussian integral about the same saddle point.

Chapter 5

Conclusion

Quantum cosmology is based on the idea that quantum physics should apply to anything in nature, including the whole universe. Universe is treated as a quantum system and this approach attempts to answer open questions of classical physical cosmology. In this dissertation, a description of general relativity as a field theory (that is based on the Lagrangian formulation) is given. This formalism leads to the general formalism of quantum cosmology. Also some issues related to this formalism are briefly described. Universe is described by the central equation known as the Wheeler-DeWitt equation. Describing this equation requires the approximation of minisuperspace which is the space on which the wave function is defined. Some boundary conditions ensure the uniqueness of the wave function that describes the Universe. This dissertation gives a brief review of the two most studied and comprehensive boundary condition proposals. Finally, Picard-Lefschetz technique was introduced so as to identify the Lorentzian path integral for quantum gravity during a semiclassical expansion. This technique describes the formulation of Hartle-Hawking proposal, as a sum over real Lorentzian four-geometries.

Some recent work about Quantum Cosmology is also given in [42]. Turok at al. in this paper made calculations concerning the no-boundary amplitude for a closed Universe. They found the opposite semiclassical exponent from what was originally proposed by Hartle and Hawking. As an extension of the work that Diaz Dorronsoro et al. did [43], they showed that the integration contour gives an additional non-perturbative contribution (correction) which continue render the perturbations un-suppressed. Adding to this, they prove that there is no an integration contour that solves this problem.

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