GENERALIZATION OF NON-LINEAR DUAL SYMMETRIC ELECTROMAGNETIC LAGRANGIANS

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Abstract

This dissertation develops a new and relatively simple way of generating non-linear duality-symmetric Lagrangians for electromagnetism. We start with a general ansatz Lagrangian and use it to re-obtain some known and find new theories. The physical implications of the new theories are analyzed by considering their spherically symmetric electrostatic solutions. The resulting electrostatic fields are distinct, ranging from confinement-like effects, reshaped Coulomb's law, to particles having a finite radius.

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Contents

1 Introduction

Maxwell's electrodynamics is one of the most well-understood physical theories. Symmetries, from Lorentz invariance to charge conservation, played an important role in its development and analysis. From all of its symmetries, there is an interesting one that appears in fields absence of charge sources; the electric and magnetic fields can be transformed into one another by the so-called dual symmetry transformations [1, 2, 3]. This aspect of Maxwell's theory has inspired physicists to extend the theory of electromagnetism, for example, Dirac's magnetic monopoles [4], and the development of duality-symmetric formulations [5, 6, 7, 8]. More recently, there has been a renewed interest in non-linear electrodynamics theories that are duality-symmetric. Historically speaking, non-linear electrodynamics offered ways to solve classical field divergences of point charges [9], provided description of photon-matter coupling [10], and appeared in the context of string theory [11, 12]. In general, dual symmetric non-linear electrodynamics models have the potential to provide further insight into magnetic monopoles, string theory, and black holes.

One of the most well recognized non-linear dual symmetric theories would be that of Born and Infeld [9]. While it is widely celebrated within the community, it is not the only possible non-linear theory with duality symmetry.

The general Lagrangian for non-linear electrodynamics can be written in the form:

$$S = \int \mathrm{d}^4x \; \mathcal{L}(u,v) \,, \qquad u = s + \sqrt{s^2+p^2} \,, \quad v = -s + \sqrt{s^2+p^2} \,. \tag{1}$$

In this dissertation, the chosen convention is $u, v \geq 0$ and the variables s, p correspond to the usual Maxwell's Lagrangian and the so-called $\theta-$term respectively:

$$s = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \,, \quad p = \frac{1}{2} F_{\mu\nu} \star F^{\mu\nu} \,, \tag{2}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ being the Faraday tensor with its corresponding Hodge dual $\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}$. The electric-magnetic duality symmetry, that is, the symmetry with respect to the rotations of $F_{\mu\nu}$ and $G_{\mu\nu} = (\star\frac{\partial\mathcal{L}}{\partial F})_{\mu\nu}$:

$$F_{\mu\nu} \to \cos\alpha\, F_{\mu\nu} + \sin\alpha\, G_{\mu\nu}\,,$$
$$G_{\mu\nu} \to -\sin\alpha\, F_{\mu\nu} + \cos\alpha\, G_{\mu\nu}\,, \tag{3}$$

can be imposed by the Gaillard-Zumino-Gibbons-Rasheed (GZGR) duality symmetry condition [14, 15] on the Lagrangian (1):

$$\frac{\partial\mathcal{L}}{\partial u}\frac{\partial\mathcal{L}}{\partial v} = -1\,. \tag{4}$$

For simplicity, we have chosen u, v to be dimensionless; however, one may modify $u \to \frac{a}{T}u+b$ and $v \to \frac{1}{aT}v + c$ in such a way that a, b, c are dimensionless and act as scaling/shifting parameters, and T as a parameter of energy density (in the context of string theory it is regarded as tension of D-brane [12]). Then, the Lagrangian $\mathcal{L}'(u, v) = T\mathcal{L}(\frac{a}{T}u + b, \frac{1}{aT}v + c)$ satisfies the duality-symmetric condition (4) as long as $\mathcal{L}(u, v)$ does.

This dissertation aims to show a new and general method of constructing non-linear Lagrangians that are of the form (1), satisfying GZGR dual symmetry condition (4). In the proceeding section, we declare an ansatz Lagrangian that contains arbitrary functions $\rho = \rho(u, v)$ and $f(\rho)$, with their purpose being of providing a way to generate duality-symmetric theories. Furthermore, we derive the Lagrangian's equations of motion and Maxwell-like equations for the electric and magnetic fields. We analyze the physical implications of the theories by considering their spherically symmetric electrostatic solutions. Afterward, we reobtain the usual Maxwell in a vacuum and Born-Infeld Lagrangians and introduce new duality-symmetric theories. Later on, the dissertation suggests another construction that additionally maintains $u \leftrightarrow v$ exchange symmetry. Again, we show some more examples that have this additional symmetry.

2 Generalization of Non-Linear Dual Symmetric Lagrangians

2.1 General Form of Non-Linear Dual Symmetric Lagrangians

We start by introducing Bialynicki-Birula [17] electrodynamics, which is regarded as the $T \to 0$ limit of Born-Infeld theory:

$$\mathcal{L}_{BI} = 2\sqrt{(u+T)(-v+T)} - 2T\,. \tag{5}$$

The $T = 0$ limit is easy to derive:

$$\mathcal{L}_{BB} = 2\sqrt{-u\,v} = 2\sqrt{-p^2} = 2\,i\,|p|\,. \tag{6}$$

The latter expression is total derivative, and therefore the Lagrangian is equivalent to zero up to boundary terms[1]. We will, however, work with the non-trivial deformations of $\mathcal{L} \equiv \mathcal{L}_{BB} = 2\sqrt{-u\,v}$.

It is easy to see that $\mathcal{L}$ satisfies the dual symmetry condition (4). Due to its simple form, the Lagrangian acts as a useful starting point for constructing more general dual symmetric theories. For example, the model can be modified by shifting one of the parameters, say u, by some arbitrary function $\rho = \rho(u,v)$ and adding another $f(\rho)$ term. That is, we consider an ansatz Lagrangian

$$\mathcal{L} = 2\sqrt{v(\rho - u)} - f(\rho)\,, \tag{7}$$

where ρ and $f(\rho)$ will be chosen in such a way that dual symmetry of $\mathcal{L}$ is maintained. The condition on these functions can be found by looking at the $\mathcal{L}$ derivatives with respect to

[1] A recent discussion of Lagrangian formulation of BB theory can be found in [18]: the appropriate Lagrangian for BB theory is given by two constraints, $s = 0$ and $p = 0$, imposed by Lagrange multipliers.

u, v:

$$\frac{\partial \mathcal{L}}{\partial u} = -\sqrt{\frac{v}{\rho - u}} + \left[- f'(\rho) + \sqrt{\frac{v}{\rho - u}} \right] \frac{\partial \rho}{\partial u} \, ,$$
$$\frac{\partial \mathcal{L}}{\partial v} = \sqrt{\frac{\rho - u}{v}} + \left[- f'(\rho) + \sqrt{\frac{v}{\rho - u}} \right] \frac{\partial \rho}{\partial v} \, , \tag{8}$$

where $f'(\rho) = \partial f(\rho)/\partial \rho$. A simple and non-trivial way to ensure dual symmetry is to make the second terms vanish by choosing the condition

$$f'(\rho) = \sqrt{\frac{v}{\rho - u}} \, , \tag{9}$$

and in return giving us a direct relationship between ρ and $f'(\rho)$. Furthermore, the derivatives reduce to:

$$\frac{\partial \mathcal{L}}{\partial u} = -\sqrt{\frac{v}{\rho - u}} = -f'(\rho) \, ,$$
$$\frac{\partial \mathcal{L}}{\partial v} = \sqrt{\frac{\rho - u}{v}} = \frac{1}{f'(\rho)} \, , \tag{10}$$

which satisfies (4) and gives us two possible ways to express it: either having only ρ or the derivative of $f(\rho)$.

Since, ρ depends on both u, v, there are many possible solutions for ρ and $f(\rho)$ that satisfy equation (9). This allows us to generate many possible non-linear dual symmetric theories: the challenge is to find functions $\rho(u, v)$ and $f(\rho)$, satisfying (9). There are two ways one can try to approach this problem. One method involves simply choosing a $f(\rho)$ function and solving the constraint to obtain ρ. This method is straightforward, and as shown later on in the examples section, can be used to obtain different non-linear theories. Another way to approach the problem would be by first considering desired equations of motion and finding suitable $\rho, f(\rho)$ solutions. Examples of this are also covered later on and are used to obtain already known theories. We do not know if our ansatz can cover all possible duality-symmetric theories. Nevertheless, it implicitly defines a large class of such theories.

2.2 Equations of Motion

We derive the equations of motion for the generalized duality-symmetric Lagrangian (7). Explicitly, we use Euler-Lagrange equations with respect to the field A_μ; thus,

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0\,. \tag{11}$$

Since the Lagrangian does not explicitly depend on the potential A_μ, the first term of the equation is equal to zero. Applying the chain rule for derivatives further expands it to

$$\partial_\mu \left[-\frac{\partial \mathcal{L}}{\partial u} \frac{\partial u}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial v} \frac{\partial v}{\partial(\partial_\mu A_\nu)} \right] = 0\,. \tag{12}$$

In the previous subsection, we chose $\mathcal{L}$ such, that its derivatives (10) would satisfy (4). We obtained two possible but equivalent ways to express them. In return, this gives equations of motion either in terms of ρ or $f'(\rho)$:

$$\partial_\mu \left[\sqrt{\frac{v}{\rho - u}} \frac{\partial u}{\partial(\partial_\mu A_\nu)} - \sqrt{\frac{\rho - u}{v}} \frac{\partial v}{\partial(\partial_\mu A_\nu)} \right] = \partial_\mu \left[f'(\rho) \frac{\partial u}{\partial(\partial_\mu A_\nu)} - \frac{1}{f'(\rho)} \frac{\partial v}{\partial(\partial_\mu A_\nu)} \right] = 0\,. \tag{13}$$

Next, we compute the derivatives of the u, v. Specifically, using the definitions of (1), in terms of s, p we obtain:

$$\begin{aligned} \frac{\partial u}{\partial(\partial_\mu A_\nu)} &= 2F^{\mu\nu} + \frac{1}{\sqrt{s^2 + p^2}} \left[2sF^{\mu\nu} + 2p \star F^{\mu\nu} \right], \\ \frac{\partial v}{\partial(\partial_\mu A_\nu)} &= -2F^{\mu\nu} + \frac{1}{\sqrt{s^2 + p^2}} \left[2sF^{\mu\nu} + 2p \star F^{\mu\nu} \right]. \end{aligned} \tag{14}$$

However, for our discussion, it is more useful to re-express the derivatives in terms of u, v. Using the fact that:

$$
\begin{aligned}
u + v &= 2\sqrt{s^2 + p^2}\,, \\
u - v &= 2\,s\,, \\
\text{sign}(p)\sqrt{uv} &= p\,,
\end{aligned} \tag{15}
$$

where sign(p) appears as s, p are arbitrary real numbers; we obtain the derivatives as:

$$
\begin{aligned}
\frac{\partial u}{\partial(\partial_\mu A_\nu)} &= 4\left[\frac{uF^{\mu\nu} + \text{sign}(p)\sqrt{uv} \star F^{\mu\nu}}{u+v}\right], \\
\frac{\partial v}{\partial(\partial_\mu A_\nu)} &= 4\left[\frac{-vF^{\mu\nu} + \text{sign}(p)\sqrt{uv} \star F^{\mu\nu}}{u+v}\right].
\end{aligned} \tag{16}
$$

Applying this to the equations of motion, gives us

$$
\begin{aligned}
&\partial_\mu\left[\left(u\sqrt{\frac{v}{\rho-u}} + v\sqrt{\frac{\rho-u}{v}}\right)\frac{F^{\mu\nu}}{u+v} + \left(\sqrt{\frac{v}{\rho-u}} - \sqrt{\frac{\rho-u}{v}}\right)\frac{\text{sign}(p)\sqrt{uv} \star F^{\mu\nu}}{u+v}\right] = \\
&\partial_\mu\left[\left(uf'(\rho) + \frac{v}{f'(\rho)}\right)\frac{F^{\mu\nu}}{u+v} + \left(f'(\rho) - \frac{1}{f'(\rho)}\right)\frac{\text{sign}(p)\sqrt{uv} \star F^{\mu\nu}}{u+v}\right] = 0\,.
\end{aligned} \tag{17}
$$

Since there are two ways to express the equations of motion, this gives us the convenience of choosing one or the other depending on the situation and how easy it is to compute these factors.

2.3 Electrostatic Solutions

We can describe the physical implications of a dual symmetric theory by reformulating the equations of motion into a form that resemble Maxwell's equations. Redefining the equations of motion in a more readable form

$$
\partial_\mu\left(\alpha \star F^{\mu\nu} - \beta\,F^{\mu\nu}\right) = 0\,, \tag{18}
$$

where $\alpha = \alpha(u,v)$ and $\beta = \beta(u,v)$ are the factors of equation (17):

$$\alpha(u,v) = \left(\sqrt{\frac{v}{\rho-u}} - \sqrt{\frac{\rho-u}{v}}\right)\frac{\text{sign}(p)\sqrt{uv}}{u+v} = \left(f'(\rho) - \frac{1}{f'(\rho)}\right)\frac{\text{sign}(p)\sqrt{uv}}{u+v}\,,$$

$$\beta(u,v) = -\left(u\sqrt{\frac{v}{\rho-u}} + v\sqrt{\frac{\rho-u}{v}}\right)\frac{1}{u+v} = -\left(uf'(\rho) + \frac{v}{f'(\rho)}\right)\frac{1}{u+v}\,, \tag{19}$$

we can expand equation (18) in a manner, similar to Maxwell equations. In natural units and metric $(+,-,-,-)$, the field tensor components correspond to electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields as: $E^i = F^{0i}$, $B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk}$, $B^i = -\star F^{0i}$, $E^i = \frac{1}{2}\epsilon^{ijk}\star F_{jk}$. Therefore, we obtain equations:

$$\begin{aligned} &\boldsymbol{\nabla}\cdot\big(\beta\boldsymbol{E} - \alpha\boldsymbol{B}\big) = 0\,, && \text{(Gauss's Law)} \\ &\boldsymbol{\nabla}\times\big(\beta\boldsymbol{B} + \alpha\boldsymbol{E}\ \big) + \frac{\partial}{\partial t}\big(\beta\boldsymbol{E} - \alpha\boldsymbol{B}\big) = 0\,. && \text{(Ampere's Law)} \end{aligned} \tag{20}$$

Since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, two more equations can be derived from the Hodge dual condition $\partial_\mu \star F^{\mu\nu} = 0$. Explicitly,

$$\begin{aligned} &\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0\,, && \text{(Gauss's Law for Magnetisim)} \\ &\boldsymbol{\nabla}\times\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{B} = 0\,. && \text{(Maxwell-Faraday Equation)} \end{aligned} \tag{21}$$

The equations of (20) and (21) are the non-linear Maxwell equations for given duality-symmetric theories. In general, assuming that the gauge field A_μ is well-defined everywhere, (21) implies that these theories do not predict spherically symmetric magneto-static solutions: a feature, shared with the free Maxwell case. However, there appears an additional $\boldsymbol{E}$ and $\boldsymbol{B}$ mixing in both Gauss's and Ampere's laws, differently from the free Maxwell case. The additional α, β factors gives rise to more complicated equations. Specifically, they depend on s, p which can be expressed in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ as

$$s = \boldsymbol{B^2} - \boldsymbol{E^2}\,, \quad p = -2\boldsymbol{B}\cdot\boldsymbol{E}\,. \tag{22}$$

In this dissertation, we restrict our analysis to spherically symmetric electrostatic solutions. In the electrostatic limit we require the magnetic fields to vanish, $s \to -\boldsymbol{E}^2$ and $p \to 0$. As a consequence, the parameters u, v approach $u \to 0$ and $v \to 2\boldsymbol{E}^2$. In this limit, the Maxwell-like equations (20) and (21) reduce to:

$$\lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} \boldsymbol{\nabla}\cdot(\beta\boldsymbol{E}) = 0\,,$$

$$\lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} \boldsymbol{\nabla}\times(\alpha\boldsymbol{E}) = 0\,,$$

$$\lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} \boldsymbol{\nabla}\times\boldsymbol{E} = 0\,. \tag{23}$$

Since the electric field has no curl, we can solve the first of the upper equations to obtain spherically symmetric solutions. The parameter β in this limit will only dependent on $\boldsymbol{E}^2$, a scalar. Therefore, we can express the spherically symmetric solution as

$$\lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} \boldsymbol{\nabla}\cdot(\beta\boldsymbol{E}) = \lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} \frac{1}{r^2}\frac{\partial}{\partial r}(\beta E(r) r^2) = 0\,. \tag{24}$$

Following [9], we define the term inside the derivative, which has to be constant, electric charge Q,

$$\lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} \beta E(r) = \frac{Q}{r^2}\,. \tag{25}$$

As a result, the electrostatic solutions are determined by the β factor only and α plays no role. We can try to explicitly take the electrostatic limit of β. Let us assume that in the limit $\rho \neq 0$ or ∞. That is, we wish for β to be well defined in this limit. In this case,

$$\lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} \beta = \sqrt{\frac{\bar{\rho}}{2\boldsymbol{E}^2}}\,, \qquad \bar{\rho} = \lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} \rho\,. \tag{26}$$

Applying it to the electrostatic solution (25), we get a simple algebraic equation of the form

$$\bar{\rho} = \frac{2Q^2}{r^4}\,, \qquad \bar{\rho} = \lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} \rho \neq 0 \text{ or } \infty\,. \tag{27}$$

The resulting formula proves to be very useful in finding electrostatic solutions for a given theory. As long as $\bar{\rho}$ is non-zero or infinite, the formula lets us quickly work out the resulting $E(r)$ solution without the need of deriving the equations of motion from first principles. However, if $\bar{\rho}$ does not satisfy the upper condition, the limit of β must be explicitly evaluated as it might still be finite and give us electrostatic solutions.

3 Non-Linear Duality-Symmetric Theories

3.1 Examples

3.1.1 Maxwell's Electrodynamics

Let us start by applying the new developed method for the simplest dual symmetric theory, which would be Maxwell's electrodynamics. In order to obtain $\mathcal{L}$, we need to determine $\rho(u, v)$ and $f(\rho)$. By requiring the theory to satisfy Maxwell's equations in a vacuum

$$\partial_\mu F^{\mu\nu} = 0\,, \tag{28}$$

we compare it to the general equations of motion (18) obtained in the previous section. In this case, the factors are $\beta = -1$ and $\alpha = 0$. It is simple to obtain $f'(\rho)$ from the conditions:

$$\begin{aligned} 0 &= \left(f'(\rho) - \frac{1}{f'(\rho)}\right)\frac{\text{sign}(p)\sqrt{uv}}{u+v}\,,\\ 1 &= \left(uf'(\rho) + \frac{v}{f'(\rho)}\right)\frac{1}{u+v}\,. \end{aligned} \tag{29}$$

The solution gives us $f'(\rho) = 1$, which implies $f(\rho) = \rho$ (we neglect any constants that appear after integration). Next, we use the dual symmetric condition (9) that relates $f'(\rho)$

and $\rho(u,v)$. This in return gives us $\rho(u,v) = u + v$. Therefore, the Lagrangian becomes

$$\mathcal{L} = 2\sqrt{v(\rho - u)} - f(\rho) = v - u = -2s\,, \tag{30}$$

which is up to a factor Maxwell's electrodynamics in a vacuum. In the electrostatic limit, $\rho \to \bar{\rho} = 2\boldsymbol{E}^2$; hence, non-zero or infinite. Therefore, we can apply the formula of (27) to re-obtain Coulomb's law

$$E(r) = \frac{Q}{r^2}\,. \tag{31}$$

While Maxwell's theory is linear, we have obtained it from our general non-linear dual symmetric construction. This simple example highlights how one can use the developed method to construct more complicated theories, which will be shown in the following examples.

3.1.2 Born-Infeld Electrodynamics

Another test for our method would be to re-derive the Lagrangian which corresponds to Born-Infeld electrodynamics. We start by considering it's equations of motion [9]. Specifically,

$$\boldsymbol{\nabla} \cdot \left[\frac{\boldsymbol{E}^2 - p\boldsymbol{B}^2}{\sqrt{1 + s - p^2}}\right] = 0\,. \tag{32}$$

Where s, p can be converted back to u, v using (15). Comparing each term of Born-Infeld electrodynamics to our equations of motion, we have:

$$\begin{aligned} \alpha &= \frac{p}{\sqrt{1 + s - p^2}}\,, \\ \beta &= \frac{1}{\sqrt{1 + s - p^2}}\,. \end{aligned} \tag{33}$$

In other words, Born-Infeld's theory requires $p\beta = \text{sign}(p)\sqrt{uv}\beta = \alpha$. Thus, we obtain an equation that $f'(\rho)$ must satisfy,

$$-uf'(\rho) - \frac{v}{f'(\rho)} + f'(\rho) - \frac{1}{f'(\rho)} = 0\,. \tag{34}$$

When the equation is solved, the function is

$$f'(\rho) = \sqrt{\frac{1-v}{u+1}}\,. \tag{35}$$

However, we can not find $f(\rho)$ yet, as we do not know the form of $\rho(u,v)$. This can be obtained from (9), $\rho(u,v)$ is found to be equal to

$$\rho = \frac{u+v}{1-v}\,. \tag{36}$$

Therefore, we can now find $f(\rho)$. We can rearrange $\rho(u,v)$ as follows

$$\rho = \frac{u+v}{1-v} = \frac{u+1+v-1}{1-v} = \frac{u+1}{1-v} + \frac{v-1}{1-v} = \frac{1}{[f'(\rho)]^2} - 1\,. \tag{37}$$

Hence, we have $f'(\rho) = 1/\sqrt{\rho+1}$ which implies that

$$f(\rho) = 2\sqrt{\rho+1}\,. \tag{38}$$

Therefore, we have all of the information needed to re-obtain the Lagrangian which described Born-Infeld electrodynamics

$$\mathcal{L} = 2\sqrt{v(\rho - v)} - f(\rho) = -2\sqrt{(1-v)(1+u)}\,. \tag{39}$$

As for the electrostatic solution we have that in the electrostatic limit $\rho \to \bar{\rho}$ is not equal to zero or infinity. Explicitly,

$$\bar{\rho} = \frac{2\boldsymbol{E}^2}{1-2\boldsymbol{E}^2}\,. \tag{40}$$

Applying the result to formula (27), the electrostatic field is equal to

$$E(r) = \frac{Q}{\sqrt{r^4 + 2Q^2}}\,. \tag{41}$$

The field has the characteristic of being finite at $r = 0$ with a field value $E(r = 0) = \text{sign}(Q)/\sqrt{2}$. Since we are working with dimensionless u, v, the proper way to restore the dimensions of fields would be by reintroducing $u \to u/T$, $v \to v/T$ and $\mathcal{L} \to T\mathcal{L}$ into our formulation, as mentioned in the Introduction.

3.1.3 Function $\ln \rho$

This is going to be the first example of a new type of non-linear dual symmetric theory. We will show that the electrostatic solutions predict a confinement-like effect. With this example, we directly assume that the Lagrangian has $f(\rho) = \ln \rho$. Directly calculating $\rho(u, v)$ from the (9) constraint

$$f'(\rho) = \frac{1}{\rho} = \sqrt{\frac{v}{\rho - u}}\,, \tag{42}$$

returns us a polynomial $v\rho^2 - \rho + u = 0$. When solved, it has two possible $\rho_\pm(u, v)$ solutions

$$\rho_\pm(u, v) = \frac{1 \pm \sqrt{1 - 4uv}}{2v}\,, \qquad\qquad 0 \le uv \le \frac{1}{4}\,. \tag{43}$$

Since we obtain two separate solutions, this implies that there are two possible $\mathcal{L}_+$ and $\mathcal{L}_-$ dual symmetric theories with the same $f(\rho)$ function.

$$\mathcal{L}_\pm = \sqrt{2\big[1 - 2uv \pm \sqrt{1 - 4uv}\,\big]} - \ln\left[\frac{1 \pm \sqrt{1 - 4uv}}{2v}\right]. \tag{44}$$

Next, we derive their electrostatic solutions. Let us start with $\mathcal{L}_+$ with its corresponding $\rho_+(u, v)$. Clearly, in the electrostatic limit it becomes equal to

$$\bar{\rho}_+ = \frac{1}{2\boldsymbol{E}_+^2}\,. \tag{45}$$

Where we denote $\boldsymbol{E}_+$ as the electrostatic field for the $\mathcal{L}_+$ Lagrangian. Therefore, we can simply apply formula (27) to obtain the static field as

$$E_+(r) = \frac{r^2}{2Q} \,. \tag{46}$$

As for the second case, $\mathcal{L}_-$, we need to directly evaluate the β limit in (25), since $\bar{\rho}_- = 0$. Explicitly, the limit is

$$\begin{aligned}
\lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} \beta &= \lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} -\left(uf'(\rho) + \frac{v}{f'(\rho)}\right)\frac{1}{u+v} \\
&= \lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} -\left(\frac{u}{\rho_-} + v\rho_-\right)\frac{1}{u+v} \\
&= \lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} -\left(\frac{2uv}{1-\sqrt{1-4uv}} + \frac{1-\sqrt{1-4uv}}{2}\right)\frac{1}{u+v} \\
&= \lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} -\left(\frac{2uv}{1-(1-2uv+O(u^2))} + \frac{1-\sqrt{1-4uv}}{2}\right)\frac{1}{u+v} \\
&= \lim_{v\to 2\boldsymbol{E}^2} \frac{-1}{v} \\
&= \frac{-1}{2\boldsymbol{E}_-^2} \,.
\end{aligned} \tag{47}$$

Therefore, solving (25) gives us the same electrostatic solution as it was with ρ_+ but with a sign change,

$$E_-(r) = -\frac{r^2}{2Q} \,. \tag{48}$$

However, the minus sign can be reabsorbed into the charge if one assumes that the sign of the charge is arbitrary. Thus, the two different Lagrangians give rise to the same electrostatic solution.

It is interesting to point out the form of the electrostatic solution. In both Lagrangian $\mathcal{L}_+$ and $\mathcal{L}_-$, we get the same electrostatic field, which increases from the origin of the charge where the field strength is zero. Another aspect of the theory is that $E_\pm(r)$ becomes dependent on the charge Q inversely compared to Coulomb. In general, this might hint a possibility of a confinement-like effect. For example, two opposite charge particles could experience a force on each other that does not let them escape each other. It would be interesting to investigate the possibility of stable multi-particle configurations with finite energy.

3.1.4 Function arcsinh(ρ)

In this example, we will obtain two separate theories that have the same function $f(\rho)$ (as in previous example) but different electrostatic solutions. Both of these theories have electrostatic solutions that are only defined in certain regions of space, that is, the elementary particle in this theory has to have a non-zero radius r_0. One Lagrangian provides a confinement-like effect as discussed in the previous example and the other one shows a Coulomb-like behavior when far away from the charge source of finite radius.
In this scenario, we choose a function $f(\rho) = \text{arcsinh}\ \rho$. For $f'(\rho) = (1+\rho^2)^{-1/2}$ the constraint

$$\frac{1}{\sqrt{1+\rho^2}} = \sqrt{\frac{v}{\rho - u}} \tag{49}$$

implies $\rho(u,v)$ has two distinct solutions

$$\rho_\pm(u,v) = \frac{1 \pm \sqrt{1-4uv-4v^2}}{2v}\,, \qquad\qquad 0 \leq 4v(v+u) \leq 1\,. \tag{50}$$

The chosen $f(\rho)$ produces two possible Lagrangians $\mathcal{L}_+$ and $\mathcal{L}_-$

$$\mathcal{L}_\pm = \sqrt{2\big[1 - 2uv \pm \sqrt{1-4uv-4v^2}\,\big]} - \text{arcsinh}\left[\frac{1 \pm \sqrt{1-4uv-4v^2}}{2v}\right]. \tag{51}$$

In the electrostatic limit, both $\rho_\pm(u,v)$ are finite

$$\bar{\rho}_\pm = \frac{1 \pm \sqrt{1-16\boldsymbol{E}_\pm^4}}{4\boldsymbol{E}_\pm^2}\,, \qquad\qquad \boldsymbol{E}_\pm^2 \leq \frac{1}{4}\,, \tag{52}$$

and using equation (27) we get

$$\frac{1 \pm \sqrt{1 - 16\boldsymbol{E}_{\pm}^4}}{4\boldsymbol{E}_{\pm}^2} = \frac{2Q^2}{r^4} \,.$$

$$\frac{8Q^2}{r^4}\boldsymbol{E}_{\pm}^2 - 1 = \pm\sqrt{1 - 16\boldsymbol{E}_{\pm}^4} \,. \tag{53}$$

This is the point where $\mathcal{L}_{\pm}$ theories start to deviate.

Let us first consider the $\mathcal{L}_{+}$ case. We require that the left side of the upper equation remains positive. Hence, the solution is

$$E_{+}(r) = \frac{r^2 Q}{\sqrt{r^8 + 4Q^4}} \,, \qquad \frac{r^2}{2\sqrt{2}Q} \leq E(r)_{+} \leq \frac{1}{2} \,. \tag{54}$$

We wish to determine the range of r for which the solution holds true. First, we consider $E \leq 1/2$. The inequality gives

$$\frac{r^2 Q}{\sqrt{r^8 + 4Q^4}} \leq \frac{1}{2}$$

$$r^8 - 4Q^2 r^4 + 4 \geq 0$$

$$(r^4 - 2Q^2)^2 \geq 0 \,. \tag{55}$$

Which is a trivial condition and it is always satisfied. At the point $r_0 = (2Q^2)^{1/4}$ the $E(r)$ field reaches its maximum value. This point is also the radius of the charge r_0. In order to see this, we consider $r^2/(2\sqrt{2}Q) \leq E(r)_{+}$. This gives the inequality

$$\frac{r^2}{2\sqrt{2}Q} \leq \frac{r^2 Q}{\sqrt{r^8 + 4Q^4}}$$

$$\sqrt{r^8 + 4Q^4} \leq 2\sqrt{2}Q^2$$

$$r \leq (2Q^2)^{1/4} = r_0 \,. \tag{56}$$

In conclusion, the $\mathcal{L}_+$ theory gives an electrostatic solution defined inside the particle's radius

$$E_+(r) = \frac{r^2 Q}{\sqrt{r^8 + r_0^8}}\,, \qquad\qquad 0 \leq r \leq r_0 = (2Q^2)^{1/4}\,. \tag{57}$$

and the field value at the radius is $E(r_0)_+ = Q/(\sqrt{2}r_0^2)$. Considering the field near the origin $r = 0$ by expanding the electrostatic solution, gives us

$$E_+(r) \approx \frac{r^2}{2Q}\,. \tag{58}$$

We see that the field gives a confinement effect. However, there is an additional constraint for where the electrostatic field exists. The field is defined inside the radius r_0. It is important to point out the at the radius point r_0, the field has a finite value. This means it is physically ambiguous as to what happens outside the radius.

Next, we analyze the $\mathcal{L}_-$ case. In this situation, we require

$$\frac{8Q^2}{r^4}\boldsymbol{E}_\pm^2 - 1 \leq 0\,, \tag{59}$$

that is, we need the left side to be strictly negative. Again, we obtain the electrostatic solution as

$$E_-(r) = \frac{r^2 Q}{\sqrt{r^8 + 4Q^4}}\,, \qquad\qquad E(r)_- \leq \frac{r^2}{2\sqrt{2}Q}\,, \quad E(r)_- \leq \frac{1}{2}\,. \tag{60}$$

However, we already established $r_0 = (2Q^2)^{1/4}$ and $E(r)_- \leq 1/2$ is always satisfied. Therefore, we consider the inequality

$$\begin{aligned} \frac{r^2 Q}{\sqrt{r^8 + r_0^8}} &\leq \frac{r^2}{2r_0^2} \\ \sqrt{2}r_0^4 &\leq \sqrt{r^8 + r_0^8} \\ r_0 &\leq r\,. \end{aligned} \tag{61}$$

Hence, we get an electrostatic field defined outside the radius r_0,

$$E_-(r) = \frac{r^2 Q}{\sqrt{r^8 + r_0^8}}\,, \qquad\qquad r \geq r_0 = (2Q^2)^{1/4}\,. \tag{62}$$

Again, we see that the field value at the radius $E(r_0)_+ = Q/(\sqrt{2}r_0^2)$ is finite. There is no field defined inside the radius. If we consider distances $r \gg r_0$, we observe Coulomb like electrostatics

$$E_-(r) \approx \frac{Q}{r^2}\,. \tag{63}$$

It is interesting to note that both of the Lagrangians gave the same field $E(r)$ function but defined at different regions around the radius point r_0. However, there is a bit of physical ambiguity as to what happens to the field when one considers $\mathcal{L}_\pm$ inside/outside the radius respectively.

In order to fix the vanishing of the field inside the radius for the $\mathcal{L}_-$ theory, let us modify the Lagrangian by adding a current term $\mathcal{L}_- \to \mathcal{L}_- + A_\mu J^\mu$. We add this modification to obtain field solutions inside the region of the charge Q by assuming that it has a constant charge density ρ_0 within it's radius $r < r_0$. In this case, we can modify the electrostatic solution of equation (24) to include this constant distribution

$$\lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} \frac{1}{r^2}\frac{\partial}{\partial r}(\beta E(r) r^2) = 4\pi\rho_0(r)\,, \quad 0 \leq r \leq r_0\,. \tag{64}$$

If we compute this differential equation by assuming that at point $r = 0$ the field is finite, then

$$\lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}} \beta\boldsymbol{E}(r) r^2 = \frac{4\pi}{3}\rho_0(r) r^3 \quad 0 \leq r \leq r_0\,. \tag{65}$$

Taking the charge density $\rho_0(r) = \frac{Q}{\frac{4}{3}\pi r_0^3}$, and performing the limiting process of β, we get an equation of similar form to (27) as

$$\bar{\rho} = 2Q^2 \left(\frac{r}{r_0^3} \right)^2 , \quad 0 \leq r \leq r_0 \,. \tag{66}$$

That is, we can redefine the $r \to \sqrt{r_0^3/r}$ in the $E_-(r)$ solutions from before. Explicitly, we get a field defined inside the radius as

$$E_-(r) = \frac{rQ}{r_0\sqrt{r^4 + r_0^4}} \qquad 0 \leq r \leq r_0 \,. \tag{67}$$

This is consistent with the requirement $r \geq r_0$ in the previous case. When we redefine $r \to \sqrt{r_0^3/r}$, we get that $r \leq r_0$. This means we get an electric field defined everywhere in space. In summary, the Lagrangian $\mathcal{L}_-$ with a constant charge density within radius $r_0 = (2Q^2)^{1/4}$ gives electrostatic solutions

$$E_-(r) = \begin{cases} \dfrac{r^2 Q}{\sqrt{r^8 + r_0^8}} \,, & r \geq r_0 \\ \dfrac{rQ}{r_0\sqrt{r^4 + r_0^4}} \,, & 0 \leq r \leq r_0 \end{cases} \tag{68}$$

As for the $\mathcal{L}_+$ case, where there is a field defined inside the radius, we can reason that there is also a constant charge density ρ_0 defined outside the charge. Again, we use equation (64) and leave ρ_0 as some kind of arbitrary constant. This time, we solve the differential equation by assuming that the terms $E(r)r^2 \to 0$ at $r \to \infty$. Then preforming the limit process, we obtain

$$\bar{\rho} = 2Q^2 \left[\frac{4\pi\rho_0 r}{3Q} \right]^2 , \quad r \geq r_0 \,. \tag{69}$$

However, since we wish the field to be continuous after crossing the radius r_0, we must equate $r_0 = \sqrt{\frac{3Q}{4\pi\rho_0 r_0}}$, which helps us get rid of the factors and in return give us the same equation as before

$$\bar{\rho} = 2Q^2 \left(\frac{r}{r_0^3} \right)^2 , \quad r \geq r_0 . \tag{70}$$

only the region is now defined outside the radius. Again, we can substitute $r \to \sqrt{r_0^3/r}$ in $E_+(r)$ and obtain the same curve

$$E_+(r) = \frac{rQ}{r_0 \sqrt{r^4 + r_0^4}} , \quad r \geq r_0 . \tag{71}$$

Again, this is consistent with the requirement $r \leq r_0$ as in the previous case. When we redefine $r \to \sqrt{r_0^3/r}$, we get that $r \geq r_0$. Thus, we get the electrostatics of $\mathcal{L}_+$ described by

$$E_+(r) = \begin{cases} \dfrac{r^2 Q}{\sqrt{r^8 + r_0^8}} , & 0 \leq r \leq r_0 \\ \dfrac{rQ}{r_0 \sqrt{r^4 + r_0^4}} , & r \geq r_0 \end{cases} \tag{72}$$

Therefore, by assuming a constant density of charge, we managed to obtain electrostatic fields defined in all points in space.

3.1.5 Function $-\ln(1-\rho)$

We will construct a theory which gives an electrostatic solution with a charge radius r_0. In this theory, there is no field defined inside the radius, and the field for $r \gg r_0$ behaves Coulomb-like. At the radius point, the field strength is infinite; thus suggesting an interpretation of the theory having finite sized particle.

Explicitly, we consider $f(\rho) = -\ln(1-\rho)$, with ρ function satisfying $\rho < 1$. The corresponding $\rho(u,v)$ function satisfies the polynomial equation $v\rho^2 - (1+2v)\rho + (u+v) = 0$, giving two possible solutions:

$$\rho_\pm(u,v) = \frac{1 + 2v \pm \sqrt{1 - 4uv + 4v}}{2v} . \tag{73}$$

However, applying the constraint $\rho < 1$ we obtain an inequality

$$\begin{aligned} \rho_{\pm} &< 1 \\ \pm\sqrt{1-4uv+4v} &< -1\,. \end{aligned} \tag{74}$$

Clearly, for $u, v \geq 0$, ρ_+ does not satisfy this condition, so the only physical solution is ρ_-. However, we still need to check the range of u, v for which the equality holds true. Since $u, v \geq 0$, meaning all terms are strictly positive, then

$$\begin{aligned} -\sqrt{1-4uv+4v} &< -1 \\ \sqrt{1-4uv+4v} &> 1 \\ 1-4uv+4v &> 1 \\ v(1-u) &> 0\,. \end{aligned} \tag{75}$$

Thus, constraining $0 \leq u < 1$ with $0 < v$. In the electrostatic limit, u vanishes and the ρ_- function becomes equal to

$$\bar{\rho}_- = \frac{1+4\boldsymbol{E}^2 - \sqrt{1+8\boldsymbol{E}^2}}{4\boldsymbol{E}^2} = \frac{2Q^2}{r^4}\,. \tag{76}$$

Therefore, the theory produces one Lagrangian

$$\mathcal{L} = \sqrt{2\big[1+2v-\sqrt{1-4uv+4v}\,\big]} + \ln\left[\frac{\sqrt{1-4uv+4v}-1}{2v}\right]. \tag{77}$$

Solving it for the electrostatic field yields us

$$E(r) = \frac{Qr^2}{|r^4-2Q^2|}\,, \qquad\qquad E(r)^2 \leq \frac{r^4}{4(2Q^2-r^4)}\,. \tag{78}$$

Again, we wish to obtain the range of r for which the field is defined. There clearly is a singularity at point $r_0 = (2Q^2)^{1/4}$ and we define it as the radius of the charge. The inequality requires

$$\frac{Q^2 r^4}{(r^4 - 2Q^2)^2} \leq \frac{r^4}{4(2Q^2 - r^4)}$$

$$(2Q^2 - r^4)^2 - 4Q^2(Q^2 - r^4) \geq 0$$

$$-(2Q^2 - r^4)(2Q^2 + r^4) \geq 0\,. \tag{79}$$

Which implies that $r \geq (2Q^2)^{1/4} = r_0$, meaning the field is defined outside the radius. In conclusion, the model gives us an electrostatic field

$$E(r) = \frac{Qr^2}{r^4 - r_0^4}\,, \qquad\qquad r > r_0 = (2Q^2)^{1/4}\,. \tag{80}$$

This model offers a non-linear theory where charges have a finite radius r_0. The field is finite everywhere except inside the radius and on the radius point (surface of the particle). For $r > r_0$ the field strength decreases with distance. Taking $r \gg r_0$ we have Coulomb's law

$$E(r \gg r_0) \approx \frac{Q}{r^2}\,. \tag{81}$$

Thus, providing an alternative formulation to the usual electrostatic solutions but with the inclusion of finite-size particles.

Again, we can use the same reasoning as in the previous example. We assume that the charge density ρ_0 inside the charge is constant. Using equation (24) we obtain that

$$\bar{\rho} = 2Q^2 \left(\frac{r}{r_0^3}\right)^2, \quad 0 \leq r \leq r_0\,. \tag{82}$$

We can redefine the $r \to \sqrt{r_0^3/r}$ in our $E(r)$ solution to obtain

$$E(r) = \frac{Qr}{r_0|r_0^2 - r^2|}\,, \qquad\qquad 0 \leq r < r_0,. \tag{83}$$

Which gives a field value $E(r=0)=0$. Therefore, the electric field in all regions would be equal to

$$E(r) = \begin{cases} \dfrac{Qr}{r_0(r_0^2 - r^2)}\,, & 0 \le r < r_0 \\ \dfrac{Qr^2}{r^4 - r_0^4}\,, & r > r_0 \end{cases} \tag{84}$$

Comparing it to the function $f(\rho) = \text{arcsinh}\,\rho$, we obtain a field singularity on the radius r_0.

3.1.6 Function $\frac{2}{3}\rho^{3/2}$

As a final example, we obtain a dual symmetric theory that in the electrostatic limit resembles a reshaped $E(r)$ of Coulomb's law.
In this case, $f(\rho) = \frac{2}{3}\rho^{3/2}$. Solving the constraint equation gives us the polynomial $\rho^2 - u\rho - v = 0$. When solved, it returns two solutions

$$\rho_\pm(u,v) = \frac{u \pm \sqrt{u^2+4v}}{2}\,. \tag{85}$$

Again, we get two possible Lagrangians $\mathcal{L}_+$ and $\mathcal{L}_-$ for the same function $f(\rho)$, that is

$$\mathcal{L}_\pm = \sqrt{2v\big[\pm\sqrt{u^2+4v} - u\big]} - \frac{2}{3}\left[\frac{u \pm \sqrt{u^2+4v}}{2}\right]^{3/2} \tag{86}$$

In the electrostatic limit the functions approach

$$\bar{\rho}_\pm = \pm\sqrt{2\boldsymbol{E}^2}\,. \tag{87}$$

Therefore, the electrostatic solution looks like

$$|E_\pm(r)| = \pm\frac{Q^2}{\sqrt{2}r^4}\,. \tag{88}$$

Since $Q^2 \ge 0$, this implies that E_- can be only equal to zero. Meaning, that $\mathcal{L}_-$ produces only zero spherically symmetric electrostatic fields. On the other hand, $\mathcal{L}_+$ is non-trivial with solution

$$E_+(r) = \frac{Q^2}{\sqrt{2}r^4}\,. \tag{89}$$

Comparing the result to Coulomb's law, the fields are the same at point $\sqrt{2}r_0^2 = Q$. However, as $r > r_0$, this theory produces weaker fields and for $r < r_0$ stronger fields compared to Coulomb's law. In any case, both derived theories suggest singularities at $r = 0$. However, both sign charges produce electric fields of the same direction, since $Q^2 \geq 0$.

3.2 Addition of $u \leftrightarrow v$ Symmetry

3.2.1 General Form of Non-Linear Dual Symmetric Lagrangians with $u \leftrightarrow v$ Symmetry

In the first part of the dissertation, we introduced a new method for constructing non-linear dual symmetric Lagrangian and showed some examples of the method. However, The previous model did not necessarily impose a discrete $u \leftrightarrow v$ symmetry. That is, if we consider the electric and magnetic fields can be interchanged as $E^i \to B^i$ and $B^i \to -E^i$, then the field tensors transform as $F \to \star F$ and $\star F \to -F$. This implies:

$$\begin{aligned} s = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} &\to \frac{1}{2}\star F_{\mu\nu}\star F^{\mu\nu} \\ &\qquad = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\epsilon^{\mu\nu\alpha\beta}\frac{F_{\alpha\beta}F^{\rho\lambda}}{4} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = -s\,, \\ p = \frac{1}{2}F_{\mu\nu}\star F^{\mu\nu} &\to -\frac{1}{2}\star F_{\mu\nu}F^{\mu\nu} = -p\,. \end{aligned} \tag{90}$$

Therefore, $u \leftrightarrow v$ are interchangeable in a dual symmetric Lagrangian.

We can incorporate this discrete symmetry by extending (7). For example, let us consider the Lagrangian

$$\mathcal{L}(u,v) = 2\sqrt{(u-\rho)(\rho-v)} + f(\rho)\,. \tag{91}$$

As $\mathcal{L}(u,v) = \mathcal{L}(v,u)$ (that is, $u \leftrightarrow v$ invariant), we require $\rho(u,v) = \rho(v,u)$ symmetry to make sure the Lagrangian is preserved under this discrete transformation. Following the same line of thought as in the first construction, we find a way to impose the dual symmetry condition on $f(\rho)$ and $\rho(u,v)$ by considering the derivatives of the Lagrangian with respect to u, v:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial u} &= \sqrt{\frac{\rho - v}{u - \rho}} + \left[f(\rho)' - \sqrt{\frac{\rho - v}{u - \rho}} + \sqrt{\frac{u - \rho}{\rho - v}} \right] \frac{\partial \rho}{\partial u}\,, \\ \frac{\partial \mathcal{L}}{\partial v} &= -\sqrt{\frac{u - \rho}{\rho - v}} + \left[f(\rho)' - \sqrt{\frac{\rho - v}{u - \rho}} + \sqrt{\frac{u - \rho}{\rho - v}} \right] \frac{\partial \rho}{\partial v}\,. \end{aligned} \tag{92}$$

As previously, we constrain

$$f(\rho)' - \sqrt{\frac{\rho - v}{u - \rho}} + \sqrt{\frac{u - \rho}{\rho - v}} = 0\,, \tag{93}$$

with the derivatives becoming

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial u} &= \sqrt{\frac{\rho - v}{u - \rho}}\,, \\ \frac{\partial \mathcal{L}}{\partial v} &= -\sqrt{\frac{u - \rho}{\rho - v}}\,, \end{aligned} \tag{94}$$

and giving us Lagrangians that satisfy the dual symmetry condition.

Thus, we obtain a another general dual symmetric Lagrangian, with the addition of $u \leftrightarrow v$ symmetry. Comparing this construction to the previous one, we see that the $f'(\rho)$ and $\rho(u,v)$ relating equation (93) becomes more complicated. When constructing new models of $f(\rho)$ such that it is a polynomial in ρ, it might be useful to consider

$$\begin{aligned}\left[f'(\rho)\right]^2 &= \left[\sqrt{\frac{\rho - v}{u - \rho}} - \sqrt{\frac{u - \rho}{\rho - v}} \right]^2 \\ &= \frac{\left[2\rho - (v + u)\right]^2}{(u - \rho)(\rho - v)}\,, \end{aligned} \tag{95}$$

and it's polynomial form

$$\left(\left[f'(\rho)\right]^2+4\right)\left(\rho^2-(u+v)\rho\right)+uv\left[f'(\rho)\right]^2+(v+u)^2=0\,. \tag{96}$$

In general, a degree n polynomial in ρ of $f(\rho)$ gives back an $2n$ polynomial equation for ρ which satisfies the (93) constraint. Hence, making it a challenge to find analytic solutions.

3.2.2 Equations of Motion and Electrostatic Solution

The equations of motion can be derived using the same process as before. We only need to exchange α, β in equation (18) with the terms in the modified Lagrangian:

$$\begin{aligned}
\alpha &= \left(\sqrt{\frac{\rho-v}{u-\rho}}-\sqrt{\frac{u-\rho}{\rho-v}}\right)\frac{\text{sign}(p)\sqrt{uv}}{u+v}\,, \\
\beta &= -\left(u\sqrt{\frac{\rho-v}{u-\rho}}+v\sqrt{\frac{u-\rho}{\rho-v}}\right)\frac{1}{u+v}\,.
\end{aligned} \tag{97}$$

In return, we obtain the same Maxwell like equations in (20) and (21). Again, we are going to focus on the radial electrostatic solutions

$$\lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}}\beta E(r)=\frac{Q}{r^2}\,,\qquad \lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}}\,,\beta=\lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}}\left(u\sqrt{\frac{\rho-v}{u-\rho}}+v\sqrt{\frac{u-\rho}{\rho-v}}\right)\frac{1}{u+v}\,. \tag{98}$$

Assuming that in the electrostatic limit $\rho\to\bar{\rho}\neq 0$ or $\neq\infty$, then the upper equation reduces to

$$\frac{\bar{\rho}E^2}{2E^2-\bar{\rho}}=\frac{Q^2}{r^4}\qquad\qquad \bar{\rho}=\lim_{\substack{u\to 0\\ v\to 2\boldsymbol{E}^2}}\rho\neq 0\text{ or }\neq\infty\,, \tag{99}$$

and thus providing a quick way to compute electrostatic solutions.

3.3 Examples with $u\leftrightarrow v$ Symmetry

3.3.1 Function Equal to a Constant

We start by considering the simplest case, which would be an always positive defined Maxwell Lagrangian. For this we choose $f(\rho)$ = constant. From the polynomial equation (96) we get that

$$\rho^2 - (u+v)\rho + \frac{(u+v)^2}{4} = \left(\rho - \frac{u+v}{2}\right)^2 = 0 \tag{100}$$

with ρ causing the Lagrangian to be dual symmetric and $u \leftrightarrow v$ invariant as

$$\rho = \frac{u+v}{2}\,. \tag{101}$$

Therefore, we obtain a Lagrangian of the form (neglecting the constant that comes from $f(\rho)$)

$$\mathcal{L} = 2\sqrt{(u-\rho)(\rho-v)} + f(\rho) = |u-v| = 2|s|\,. \tag{102}$$

That is, we get Maxwell's electrodynamics with the difference being that the Lagrangian is always positive defined. Using the fact that in the electrostatic limit $\bar{\rho} = \boldsymbol{E}^2$ and using equation (99) we obtain the electrostatic solution

$$E(r) = \frac{Q}{r^2}\,. \tag{103}$$

That is, we get the expected Coulomb's law.

However, the requirement for the absolute value has some interesting consequences. Since the Lagrangian is of the form $\mathcal{L} = |u-v|$, then the equations of motion are not defined for when $u = v$, since the derivative of $\mathcal{L}$ is not defined at that point:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u} &= \frac{u-v}{|u-v|}\,, \\ \frac{\partial \mathcal{L}}{\partial v} &= \frac{v-u}{|v-u|}\,. \end{aligned} \tag{104}$$

This means that the equations of motion are undefined when $\boldsymbol{B}^2 - \boldsymbol{E}^2 = 0$. For example, in regular Maxwell's theory, a plane electromagnetic wave has the fields related by $|\boldsymbol{E}| = |\boldsymbol{B}|$. In this modified theory, such waves could not exist. This might raise a question as to why this happens, since Maxwell's equations in vacuum incorporate $\boldsymbol{E} \to \boldsymbol{B}$ and

$\boldsymbol{B} \to -\boldsymbol{E}$ transformations. The problem comes from the fact that Maxwell's Lagrangian does not exhibit $u \leftrightarrow v$ symmetry itself (explicitly, $\mathcal{L}(u,v) \to -\mathcal{L}(v,u)$), while the modified Lagrangian does. This means, that constraining the Lagrangian to maintain this discrete symmetry could result in very different theories from the original.

3.3.2 Function ρ

In this example, we consider a simple function $f(\rho) = \rho$. Looking at equation (96), we see that we obtain a polynomial equation of degree two, meaning it is easily solved. This gives us an interesting result, an electrostatic field similar to Coulomb's law, only re-scaled by the golden ratio φ_+.

For our chosen $f(\rho)$, we obtain a polynomial $5\rho^2 - 5(u+v)\rho + (u+v)^2 + uv = 0$. When solved, we get two possible theories

$$\rho_\pm(u,v) = \frac{u+v}{2} \pm \frac{1}{2}\sqrt{\frac{(u-v)^2}{5}}, . \tag{105}$$

That is, for the same function $f(\rho) = \rho$, there is a corresponding $\mathcal{L}_+$ and $\mathcal{L}_-$ Lagrangian

$$\mathcal{L}_\pm = \left[\frac{4 \pm 1}{2\sqrt{5}}\right]|u - v| + \frac{u+v}{2} . \tag{106}$$

Now the interesting part comes from the electrostatic solutions. In the electrostatic limit, we get

$$\bar{\rho}_\pm = \left[1 \pm \frac{1}{\sqrt{5}}\right]\boldsymbol{E}^2 \tag{107}$$

Therefore the electrostatic solution is

$$E_\pm(r) = \sqrt{\frac{1 \pm \sqrt{5}}{-1 \pm \sqrt{5}}}\frac{Q}{r^2} . \tag{108}$$

However, we can observe the golden ratio φ_+ (and it's corresponding second root to the golden ratio polynomial $\varphi^2 - \varphi - 1 = 0$, φ_-), which is defined as

$$\varphi_\pm = \frac{1 \pm \sqrt{5}}{2}, \qquad \varphi_- = -\varphi_+^{-1}. \tag{109}$$

Hence, reducing electrostatic solution to

$$E_\pm(r) = (\varphi_+)^{\pm 1} \frac{Q}{r^2}, \tag{110}$$

which is Coulomb's law re-scaled with the golden ratio φ_+ for the $\mathcal{L}_+$, and $1/\varphi_+$ for $\mathcal{L}_-$ theories. Of course, φ_+ can be reabsorbed into Q and thus giving back the regular Coulomb's law.

3.3.3 Function $2\rho\,\sinh\gamma$

In this example, we obtain a theory that corresponds to constant derivatives of the Lagrangian and a more generalized electrostatic solution to the previous example. This can be done by choosing:

$$\begin{aligned} e^{\gamma} &= \sqrt{\frac{\rho - v}{u - \rho}} = \frac{\partial \mathcal{L}}{\partial u}, \\ -e^{-\gamma} &= \sqrt{\frac{u - \rho}{\rho - v}} = \frac{\partial \mathcal{L}}{\partial v}, \end{aligned} \tag{111}$$

where γ is some constant. Using the constraint, we get that $f'(\rho) = e^{\gamma} - e^{-\gamma} = 2\,\sinh\gamma$. In return, we obtain the function

$$f(\rho) = 2\rho\,\sinh\gamma, \tag{112}$$

That is, we get a more generalized function of the previous example. Solving for ρ yields us

$$\rho = \frac{e^{2\gamma} u + v}{1 + e^{2\gamma}}. \tag{113}$$

We observe, that under $u \to v$ and $v \to u$ the factor in the exponent transforms as $\gamma \to -\gamma$; thus, making $\rho(u,v)$ symmetric under $u \leftrightarrow v$. The corresponding Lagrangian is therefore

$$\mathcal{L} = \frac{|u-v|}{\cosh\gamma} + (e^{\gamma}u + e^{-\gamma}v)\tanh\gamma\,. \tag{114}$$

The electrostatic solution gives ρ that is non zero or infinite; explicitly,

$$\bar{\rho} = \frac{2\boldsymbol{E}^2}{1+e^{2\gamma}}\,. \tag{115}$$

This means we can use the simple electrostatic formula (99) to obtain

$$E(r) = e^{\gamma}\frac{Q}{r^2}\,. \tag{116}$$

Which is re-scaled Coulomb's law. However, compared to the last part, the $f(\rho)$ function does not split the Lagrangian into two parts $\mathcal{L}_{\pm}$.

4 Conclusions and Future Research

In this dissertation, we have obtained a new and relatively simple way to construct non-linear dual symmetric theories using the Lagrangian of the form (7) with the constraint on $\rho, f(\rho)$ being (9). Using the method discussed in this dissertation, we re-obtain Maxwell's and Born-Infeld electrodynamics. Furthermore, we generated more non-linear dual symmetric Lagrangians. These new theories have different physical implications when one considers spherically symmetric electrostatic solutions. The predicted electrostatic fields range from confinement-like behavior to reshaped Coulomb's law and particles having a finite radius. Therefore, we conclude there is a wide range of non-linear dual symmetric electromagnetic theories with different properties of their electrostatic solutions.

Since we did not exhaust all possible non-linear dual symmetric theories, one can try to generate more Lagrangians or maybe even try to classify them. We restricted our work to 4d Minkowski spacetime, while generalisations to higher dimensions and higher forms can be considered [18, 19, 20], in particular for the cases relevant for string theory. Another research path could involve considering a way to introduce magnetic monopoles, quantum and non-abelian theories, and more. The manifestly democratic approach of [19, 20, 21] can be more instrumental there.

One more interesting application of this work would be generating gravitational solution via double copy of the spherically symmetric solutions discussed here, following [13]. In particular, those solutions that exhibit horizon already for electromagnetism, could generate non-trivial black hole solutions for gravity.

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