Imperial College London

MSc in Quantum Fields and Fundamental Forces

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ϕ^4 interaction in Causal Set Theory

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Abstract

This project focuses on Causal Set Theory as an approach to Quantum Gravity that replaces the idea of continuum spacetime with a discrete collection of events ordered by causality. As a consequence, the road to quantization is taken through a Sum-over-Histories framework, abandoning the canonical way.

After investigating some properties of the free decoherence functional, I deal with interactions, formulating an expression for the interacting k-point function in ϕ^4 theory on a fixed background causal set. Expanding perturbatively, I focus on the two-point function: I evaluate all the contributions for 2 points in the interaction region and up to third order for 3 interaction vertices, in terms of retarded and Feynman propagators. Then, I develop a diagrammatic method to represent the perturbation expansion for all orders and arbitrary number of interaction points, assigning a mathematical expression to each element of the pictorial representation. To conclude, I propose rules to construct the allowed diagrams, estimating all terms at any order without doing any calculation.

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It has been a pleasure to collaborate with Fay Dowker, inspiring me through each step of this project, Ian Jubb, pointing to always also look to a general framework, and Yasaman Kouchekzadeh Yazdi, suggesting new technical machinery. Thank you for the warm welcome in the causal set community! I would like to thank all my travel companions through life, especially those who have been beside me during my Kantian walks. ◈ æ

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Introduction

> But you have correctly grasped the drawback that the continuum brings. If the molecular view of matter is the correct (appropriate) one, i.e., if a part of the universe is to be represented by a finite number of moving points, then the continuum of the present theory contains too great a manifold of possibilities. I also believe that this too great is responsible for the fact that our present means of description miscarry with the quantum theory. The problem seems to me how one can formulate statements about a discontinuum without calling upon a continuum (space-time) as an aid; the latter should be banned from the theory as a supplementary construction not justified by the essence of the problem, which corresponds to nothing "real". But we still lack the mathematical structure unfortunately. How much have I already plagued myself in this way!
> - *Albert Einstein* [Ein16]

In 20^{th} century physics, the two established theories were general relativity and quantum mechanics, which had been supported by experimental evidences and observations. While general relativity accounts for gravity and deals with phenomena involving large scale objects such as orbiting planets and colliding galaxies, quantum mechanics describes particles and interactions between them, i.e. the microscopic behaviour of matter. However, the key features of the two theories seem to diverge from a common line: quantum theory demands for non-continuity, non causality and non-locality whereas relativity is based upon continuity, strict causality and locality. Then, the challenge arises to understand how these two pillars are related, leading theoretical physicists to work towards a still unknown unified theory, called quantum gravity.

1.1 Problems with the continuum

In our discussion, we can summarize the main inconsistencies of the current theories in physics, describing three infinities which point towards a discrete rather than a continuum spacetime [Sor97a]. The first one is expressed by the equation $Z = \infty$ in quantum field theory. This problematic contribution was approached via renormalization in the 1940 and 50s. This method suggests that the theoretical description deals with physical quantities, e.g. mass and charge, which are different from the ones measured in experiments. For instance, the discrepancy in the measured mass and charge is explained by the persistent particle self-interactions. Then, recasting the theory in terms of measured parameters gives finite results. However, the bare quantities become infinite as their values are renormalised in order to obtain the measured values. Arising due to spacetime continuity at very small length-scales, these divergences become problematic in many approaches to gravity quantization.
In classical general relativity, the gravitation equations of motion can break down, no longer providing a valid answer and leading to the second infinity. In physically realistic conditions, the spacetime geometry has a singularity where physical quantities diverge, such as $R_{abcd} = \infty$ at the singularities, meaning that the curvature blows up there. Although there are still doubts on whether the singularities

are a mathematical artifact or they actually happen, spacetime at a singularity can not be reliably described by general relativity.
The third infinity arises in quantum gravity, when evaluating the black hole entropy counting the degrees of freedom of the horizon, i.e. $S_{BH} = \infty$, [LBS86]. Consequently, Planck scale discreteness is required to recover the Area-Entropy law for black holes using statistical mechanics of a quantum theory.
In addition to physical arguments, conceptual problems with the continuum have been developed. Firstly, in a continuum spacetime, every region has the same number of points. Indeed, it is not possible to find the volume of a region of spacetime, since such counting leads to infinity. A continuum spacetime volume is defined through a volume measure, assigning a real number to a region to evaluate its volume. Despite mathematically well-defined, some measures result in physical absurdity. The Banach-Tarski paradox is a notorious example: a sphere is divided in pieces which are recombined obtaining twice the volume of the initial sphere.
Another conceptual complication relies on the non-uniqueness of mathematical definition of the continuum. For instance, the equivalence classes of Cauchy sequences of rational numbers construct the real numbers $\mathbb{R}$ from integers. However, a different equivalence relation would lead to, for example, non-standard analysis, allowing for infinitesimally small and infinitely great quantities. Then, the continuum we deal with seems the consequence of an historical accident. The physical appearance of continuity of space played an important role for the formulation of the continuum. On the other hand, the mathematical choices for its definition are not always physically motivated, leading to inconsistencies when describing the spacetime in our universe.

1.2 Discretness

There are numerous hints from our present theories of physics suggesting a discrete framework. Following the idea of quantum mechanics, where matter is treated as made of small building blocks, i.e. quanta, some approaches to quantum gravity take into account discrete spacetime [Ori06]. In this project, we deal with causal set theory, which is one of them.
The most common choice for the discreetness scale from the entropy of black holes [Sor97a] is evaluated starting from three important physical constants: G, Newton's constant, c, the speed of light in vacuum and $\hbar$ the (reduced) Planck's constant. The Planck scale is then defined as

$$l_P = \sqrt{\frac{8\pi G\hbar}{c^3}} = 1.6162 \cdot 10^{-35} \text{ m} \tag{1.1}$$

and the fundamental unit of volume $V_f = \nu l_P^4$ where ν still needs to be determined and is a number of order one. Supposing there is just one spacetime event per V_f then the number of events in 1 s m^3 is around $4.4 \cdot 10^{147}$.
Discretness has been considered fundamental with its twin principle of causality in various attempts to quantum gravity. In 1873 [Rie73], Bernhard Riemann arises the question on how geometry encapsulates both the notion of space and constructions in space. In particular, he focuses on the structure of space that allows to define measurable quantities, requiring a discrete formalism in contrast with the continuum framework. In 1959 [Coi59], Coish realizes that a new physical geometry is needed to account for the elementary particles, suggesting a finite geometry. This idea is later expanded by Shapiro [Sha60], focusing on weak interactions.
In 1978 [Myr78], Myrheim shows that the causal ordering of events and the four-dimensional volume element completely describe the structure of general relativistic spacetimes. Assuming spacetime to be discrete, coordinates and metric are derived as statistical concepts. His reformulation of Einstein's equations for the gravitational field in empty space allows for a statistical interpretation. However, his treatment is only valid for large enough regions so that a statistically significant number of points are contained by their volumes. The statistical framework is a key idea that causal set theory takes into account [LBS87], as the continuum spacetime emerges via a random Poisson sprinkling, see 2.4.

The year after, 't Hooft assumes spacetime to be discrete as a lattice, publishing [tH79]. Chronological ordering becomes manifestation of general invariance, giving structure to the lattice. Implicitly taking the ordering as locally finite, timelike and spacelike distances in the lattice are discussed. To conclude, a non-local action is formulated to describe the dynamics.
In 1988, Hemion [Hem88] focuses on the theory of action at a distance to model classical electrodynamics on a discrete spacetime. It is central in his discussion the notion of position in a partially ordered set.
Starting from these developments toward discreteness, causal set theory has been formulated [LBS87].

1.3 Overview

This project approaches quantum gravity through causal set theory, developing an interacting formalism. After setting the mathematical background in Chapter 2, we choose the sum over histories formalism over canonical quantisation to deal with free quantum field theory on a fixed causal set background. In Chapter 3, preparing the ground with some preliminary calculations, we introduce the interaction terms. Then, we evaluate the two point function in Chapter 4, leading to propose rules associated with our diagrammatic representation of the interactions in Chapter 5. The aim of this project is to present a set of modified Feynman rules to implement a diagrammatic expansion for the history based formalism of interacting causal set theory, in order to merge the useful characteristics of this framework with the efficiency of the graphical representation.

1.4 A taste of phenomenology

Before diving in, I want to emphasise the most important phenomenological result from causal set theory: the successful prediction of the cosmological constant Λ [Sor91]. From the classical unimodular theory, Λ enters into the action for general relativity in the term $-\Lambda V$ so that the 4-volume is conjugate to the cosmological constant. In causal set, the uncertainty on the volume is related to the number N of elements contained into V, undergoing a Poisson distribution as illustrated by section 2.4. Accordingly, Λ should fluctuate about its "target value", which is zero.
The smallness of Λ is regarded as the statistical significance due to the large number of spacetime elements constituting the universe, following the relation

$$\Lambda \sim \frac{1}{\sqrt{N}}, \tag{1.2}$$

where N is the number of ancestors at a given cosmological epoch. As the number of spacetime elements in the observable universe today is 10^{240}, Λ has fluctuations of order 10^{-120} in the present epoch. This prediction has been subsequently confirmed by observation.
It would be interesting to expand the discussion about the cosmological constant in a complete "quantum caual set dynamics". Moreover, given its initial success, the next step would be to develop a model taking into account not only the instantaneous magnitude of the fluctuations but also their relations throughout their evolution.

Chapter 2

Causal set theory: background and definitions

> The noun 'dividation' means an unrestricted and generalized totality of acts of seeing things as separate. As has been indicated earlier, di-vidation implies a division in the attention-calling function of the word, in the sense that di-vidation is seen to be different from vidation. [...] Rather, their very forms indicate that dividation is a kind of vidation, indeed a special case of the latter. So ultimately, wholeness is primary, in the sense that these meanings and functions pass into each other to merge and interpenetrate. Division is thus seen to be a convenient means of giving a more articulated and detailed description to this whole, rather than a fragmentation of 'what is'. The movement from division to oneness of perception is through the action of ordering. For example, a ruler may be divided into inches, but this set of divisions is introduced into our thinking only as a convenient means of expressing a simple sequential order, which we can communicate and understand something that has bearing on some whole object, which is measured with the aid of such a ruler. [...] But, of course, more complex orders are possible, and these have to be expressed in terms of more subtle divisions and categories of thought, which are significant for more subtle forms of movement. [...] Beyond all these orders is that of the movement of attention. This movement has to have an order that fits the order in that which is to be observed, or else we will miss seeing what is to be seen.
> \- *David Bohm* [Boh80]

2.1 Causality and spacetime

In 1905, bringing a paradigm change, Albert Einstein merged the separate concepts of space and time into spacetime, a new arena where phenomena take place. The theory of special relativity replaced Newtonian mechanics with relativistic mechanics, abolishing the notion of absolute time and space. Following with general relativity, the curvature of spacetime has been related to gravitational effects. Henceforth, the model of spacetime has not changed, becoming the background framework for the elaboration of new theories, such as quantum mechanics. Those developments brought light on unexpected relations between quantum theory and gravity. Therefore, it is interesting to review how spacetime is modelled by general relativity, in particular focusing on the concept of relativistic causality.
In general relativity, a four-dimensional Lorentzian manifold (M, g) defines spacetime, where g is a symmetric non-degenerate tensor on M of signature $(-,+,+,+)$. A collection of spacetime events is represented by points in M in the limit of the event happening in smaller and smaller regions of spacetime.
The tangent vectors $X \in T_pM$, $\forall\ p \in M$ follow the classification in timelike, null or spacelike, given $g(X, X)$ is positive, zero or negative respectively. Furthermore, given timelike $X, Y \in T_pM$, we arbitrarily label Y as future directed if $g(X, Y) > 0$ and past-directed if $g(X, Y) < 0$, choosing $-X$

instead of X would have reversed the definition. Then, if everywhere in the manifold the choice of future-directed and past-directed timelike or null vectors is continuous and consistent, the Lorentzian manifold is *time-orientable*.
Accordingly, smooth curves in M are categorized from the properties of their tangent vectors as:

- *Chronological* (or timelike) : the tangent vector is always timelike,
- *Null* : the tangent vector is always null,
- *Spacelike* : the tangent vector is always spacelike,
- *Causal* (or non-spacelike) : the tangent vector is always timelike or null.

Now, we are ready to build the causal structure on a manifold M, starting from its smooth curves. Given two points $u, v \in M$, if and only if there exists a future-directed timelike curve from u to v, then u chronologically precedes v, i.e $u \ll v$, while if and only if there exists a future-directed causal curve from u to v, then u causally precedes v, i.e. $u \preceq v$. A Lorentzian manifold is causal if $\nexists\, u, v \in M$ such that $u \preceq v$ and $v \preceq u$, meaning that there are no closed causal curves.
The chronological relation on a causal Lorentzian manifold has the following properties:

- $\forall\, u \in M, u \not\ll u$, irreflexive,
- $\forall\, u, v, t \in M, u \ll v \ll t \Rightarrow u \ll t$, transitive,

while the causal relation is:

- $\forall\, u \in M, u \preceq u$, reflexive,
- $\forall\, u, v \in M, u \preceq v \preceq u \Rightarrow u = v$, antisymmetric,
- $\forall\, u, v, t \in M, u \preceq v \preceq t \Rightarrow u \preceq t$, transitive.

Hence, we introduce two key theorems involving the causal structures of different Lorentzian manifolds. Firstly, let us define *a chronological isomorphism* for two Lorentzian manifolds (M, g) and (M', g') as a bijection $f : M \to M'$, such that $\forall\, u, v \in M$, $u \ll v \iff f(u) \ll f(v)$, preserving the chronological structure. Analogously, if $\forall\, u, v \in M$, $u \preceq v \iff f(u) \preceq f(v)$, then the bijection in *a causal isomorphism*, as it preserves the causal relations.
Malament's Theorem Suppose that (M, g) and (M', g') are two distinguishing Lorentzian manifolds and $f : M \to M'$ is a chronological isomorphism, then f is a smooth conformal isometry. This means f is a smooth map and that $f_* g = \Omega^2 g'$ where Ω is a conformal factor: $M' \to R$ [Mal77].
Levichev's Theorem Suppose that (M, g) and (M', g') are two distinguishing Lorentzian manifolds and $f : M \to M'$ is a causal isomorphism then f is a smooth conformal isometry [Lev87].
To conclude, we notice that the causal structure of a Lorentzian manifold determines its conformal geometry. For instance, one of the 10 independent components of the metric of a four-dimensional Lorentzian manifold is fixed by the conformal factor, determing $det(g)$, while the other 9 are given by the causal structure.
A more detailed discussion on relativity and causality is illustrated by Penrose [Pen72], Hawking and Ellis [SH73].

2.2 Causal set theory

Causal set theory establishes a discrete framework with an order relation that reflects the causal structure of a continuum spacetime. A causal set (or causet) is defined as a locally finite partially ordered set (poset). Precisely, it is described by a pair $(\mathcal{C}, \preceq)$ where $\mathcal{C}$ is a set with a partial order relation $\preceq$ which satisfies:

- Reflexivity $\forall\, a \in \mathcal{C},\;\; a \preceq a$
- Acyclicity $\forall\, a, b \in \mathcal{C},\;\; a \preceq b \preceq a \Rightarrow a = b$
- Transitivity $\forall\, a, b, c \in \mathcal{C},\;\; a \preceq b \preceq c \Rightarrow a \preceq c$
- Locally finiteness $\forall\, a, c \in \mathcal{C}, \mid [a, c] \mid < \infty$, where the set $[a, c] := \{b \in C \mid\; a \preceq b \preceq c\}$ is a causal interval and |X| is the cardinality of a set X .

Then, the set of spacetime events is represented by the set $\mathcal{C}$ and the causal dependence between pairs of events is given by the partial order $\preceq$. For instance, for a $\preceq$ b, "a precedes b" means that a is in the causal past of b.
The first three requirements are in agreement with the conditions of causal relations on causal Lorentzian spacetimes. However, spacetime discretness enters in the the fourth requirement, since in a continuum spacetime any causal interval contains an uncountably infinite number of spacetime events. Hence, it is possible to define the notion of volume for a subset of $\mathcal{C}$, as the number of spacetime events in the spacetime region, giving always a finite result. Due to local finiteness, there is no distinction between volume and number: "Number = Volume", in contrast with the continuum case where the volume measure is derived from a metric, requiring extra structure. Then, a causal set has an ordering relation and an implicit notion of volume, motivating the claim that the large-scale structure of spacetime can be specified by just these two complementary information : "Order + Number = Geometry". The most naive starting point is, to consider as spacetime, a single large causal set: it approximates a Lorentzian manifold thanks to its "manifold-like" causal structure and the high number of events it contains. This initial claim could be a motivation for why Lorentzian manifolds frameworks have been found to work reasonably well in describing physics on large scales. With more insight, a classical collection of causets or a large quantum superposition of many different of them could be regarded as spacetimes.
My work focuses on spacetime as a single fixed causal set, elaborating how quantum field interactions work on it.

2.3 Embeddings

The definition of embedding is useful to relate causal sets to continuum Lorentzian manifolds. An embedding of a causal set $(\mathcal{C}, \preceq)$ into a Lorentzian manifold (M, g) is a map $f :\; \mathcal{C} \to M$, such that if

$$u \preceq v \in \mathcal{C} \Longleftrightarrow f(u) \preceq f(v) \in M,$$

preserving the causal structure.
An embedding of a causal set $(\mathcal{C}, \preceq)$ into a Lorentzian manifold (M, g) is *faithful* if the images of elements in the causal set follow a uniform distribution in M in agreement with the volume measure on M. Moreover, we restrict to the case where the scale over which the variation of the manifold geometry happens is larger than the embedding scale.

Conjecture 2.3.1. Hauptvermutung *If a causal set can be faithfully embedded into two Lorentzian manifolds* (M, g) *and* (M', g')*, then the two manifolds are* similar *on large scales, where large means* $> 10^{-15}$ *cm.*

It is important to notice that we did not use the term "identical" since the structure of the manifold at scales smaller than the discreteness scale can not be described by a faithful embedding. The limiting case of the conjecture where the density of causet points in a unit volume tends to infinity is proven by [BM89], however more steps are needed for a general proof.

2.4 Sprinkling

We posed the question about the relation between a given causal set and a spacetime, and vice versa. However, it is difficult to determine if a given Lorentzian manifold is a faithful embedding of a causal set. It is possible to construct causal sets which automatically faithfully embed into Lorentzian manifolds through *sprinkling*. This method consists in a random process for generating a causal set selecting points from a causal Lorentzian manifold. Placing points according to a Poisson distribution, the probability that a randomly chosen $\mathcal{C}$ has n points in a volume V is

$$P(|\mathcal{C} \cap V| = n) = \frac{(\rho V)^n e^{-\rho V}}{n!}, \tag{2.1}$$

where ρ is the sprinkling density which estimate the number of points placed in a unit volume. Then, in a volume V, the expected number of points is evaluated by ρV. Hence, a causal set is generated:

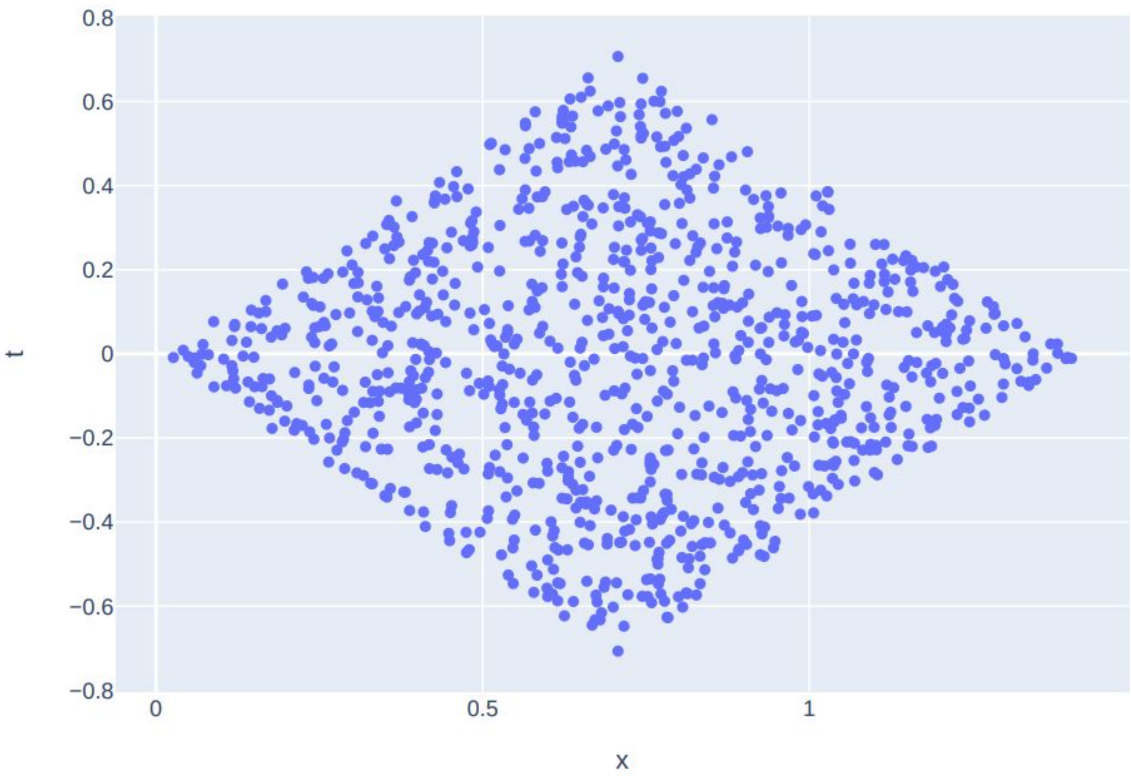


Figure 2.1: Given a 2-dimensional Minkowski spacetime, this is a possible distribution of 1000 points sprinkled into a causal interval. [Ric21]

the elements are chosen through sprinkling and the causal relation is induced by the order relations among the selected elements within the continuum spacetime. An example of a resulting distribution in 1+1 dimensional spacetime is shown in 2.1 for 1000 points. To avoid that causal relations violate antisymmetry, we exclude manifolds with closed causal curves, restricting to causal Lorentzian manifolds. To summarize, if a causal set $\mathcal{C}$ is originated by sprinkling into a spacetime M, then it can be faithfully embedded into the causal Lorentzian manifold.

2.4.1 Motivation - Lorentz invariance

The justification of the choice of a Poisson distribution is related to local Lorentz invariance. Infact, we want to avoid generating a causal set which selects a particular direction in spacetime. For instance, performing a Lorentz transformation on all the sprinkled points in a fixed frame will change their coordinates but not their distribution. They will be statistically identical, still following a Poisson distribution. The reason is that the sprinkling process depends only on spacetime volumes which are Lorentz-invariant quantities. Specifically, before and after performing the Lorentz transformation, the expected number of points in a region of volume V is given by ρV. On the other hand, if we perform a Lorentz transformation on points arranged on a regular lattice, the resulting distribution of points will be statistically different from the original one. Applying a boost will give irregularly distributed points, breaking Lorentz invariance. Given such a discrete structure as fundamental, highly boosted frames would not be described as manifolds. Therefore, our choice of sprinkling through a random Poisson process guarantees "statistical Lorentz invariance" so that in all frames the statistical distribution of points stays unchanged. Among the possible approaches to discretisations of Lorentzian manifolds, one of the main features which distinguish causal set theory is its local Lorentz invariance. However, the problem of sprinklings into infinite Minkowski spacetime is that the number of "nearest neighbours" of each causal set element is infinite, in contrast with a hyper-cubic lattice, for which it is finite. This first infinity is one of the criticisms against causal set theory [Moo88], indicating that the theory supports the idea that every point is affected by an infinity of neighbours arbitrarily far back in time. However, this obstacle has no relevance in this work since we are considering only finite causal sets.

2.5 Coarse graining

Given a spacetime (M, g), it is possible to define the causal set $(\mathcal{C}, \preceq)$ for different ρ. Starting with a causal set $(\mathcal{C}, \preceq)$, which may or may not faithfully embed into (M, g) with ρ_c, a smaller subcausal set $\mathcal{C}' \subset \mathcal{C}$ which faithfully embeds into (M, g) at $\rho'_c < \rho_c$ can be obtained following a process called *coarse-graining* [Sor02]. The most common example is a random selection of n' points in $\mathcal{C}$ such that $n' = (\rho'_c \backslash \rho_c)\ n$ for every n elements of $\mathcal{C}$.
Hence, removing some points from the causal set $\mathcal{C}$, we look for a faithfully embeddable subset. For instance, analysing quantum regimes, the causets representing the microscopic states can not faithfully embed into a manifold on the large scales. However, we expect them to have a common coarse graining from which a manifold can be recovered. Being itself a random process, causet theory allows to define coarse graining in agreement with the inverse procedures of continuum approximation and discretisation.
Futhermore, this method guarantees the notion of scale dependent topology to be realised in causal set theory. In many quantum gravity approaches, the topology of spacetime at scales close to the Planck scale is described by wormholes or other nontrivial excitations [Ori06]. However, it is difficult to find a continuum correspondence of such a structure. Instead, in causal set theory, coarse graining may generate various causets resulting in continuum approximations with different topologies.

2.6 Definitions

It is useful to state some definitions which will be frequently recalled in this project.

- Given $a \preceq b$ and $a \neq b$, we write $a \prec b$, where $a \prec b$ is an irreflexive relation known as *strict* causal relation. If $a \npreceq b$ and $b \npreceq a$, a, b are *unrelated*, writing $a||b$.
- Given the pair $(Q, \preceq')$ and $(P, \preceq)$, if Q is a subset of P and the order relation $\preceq'$ is equivalent to $\preceq$ when imposed to just elements in Q, then $(Q, \preceq')$ is a *sub-poset* of a poset $(P, \preceq)$.
- An *Alexandrov set* is defined as the set $[a, c] := \{b \in \mathcal{C}, a \preceq b \preceq c\}$ given $a \prec c$. This definition includes the end points of the interval, then $a, c \in [a, c]$.
- An integer is assigned to the elements of $\mathcal{C}$ by *labelling*, such as a_x for $x = 1, \ldots, |\mathcal{C}|$. The *natural* labelling is defined as $a_x \preceq a_y \Rightarrow x \leq y$

- A partial order in which there are no unrelated elements is called *total order*. A subset of a causal set $(\mathcal{C}, \preceq)$ which is totally ordered when considered as a sub-poset of $(\mathcal{C}, \preceq)$ is called a *chain*; a chain with repeated elements is a *multichain*. A set with elements which are mutually unrelated is an *antichain*

- A relation $a \prec c$ so that $\nexists\ b \in \mathcal{C}$ with $a \prec b \prec c$, is a *link*, expressed as $a \prec *b$, where a and b are called nearest neighbours.

- A subset $P \subset \mathcal{C}$ is a *path* if it is a maximal (or saturated) chain, i.e. $\nexists\ b \in \mathcal{C} - P$ such that $a \prec b \prec c$ for some $a, c \in P$

- A total order $(\mathcal{C}, \leq)$ which is consistent with the partial order of a causal set $(\mathcal{C}, \preceq)$ is called *linear extension*, i.e. $a \preceq b \Rightarrow a \leq b\ \forall\ a, b \in \mathcal{C}$.

2.6.1 Dimensions

Assuming the relation between causal set and causal Lorentzian manifold holds, it is interesting to study how geometrical information can be extracted from an order relation. We focus on the most fundamental feature of a manifold's topology: its dimension. The effective continuum dimensions of a causal set can be evaluated following different approaches. We present three of them [Sor02], which assign a dimension to an interval I in a causet $\mathcal{C}$, where I $\approx$ L and L is a double light cone interval in Minkowski space $\mathbb{M}^d$.
Myrheim-Meyer dimension [Mey88] We label with N the number of elements in I and R the number of relations in I, evaluating the number of pairs x, y such that x $\prec$ y. Let us define $f(d) = \frac{3}{2}\binom{3d\backslash 2}{d}^{-1}$. Then, $f^{-1}(R/\binom{N}{2})$ gives an estimate of d for $N \gg (27\backslash 16)^d$
Midpoint scaling dimension Given $I = \text{interval}(a, b)$, $m \in I$ is called the "midpoint" if it maximizes $N' = min\{|\text{interval}(a, m)|, |\text{interval}(m, b)|\}$. Then, we can calculate the dimension as $d = log_2(N\backslash N')$
A third method The dimensions are determined by $d = \frac{lnN}{ln\ lnK}$ where K is the total number of chains in I. A good accuracy is assured for exponentially large N due to the logarithmic relation.

2.7 Causal Sets representation

There are multiple ways of representing causal sets. Let us introduce an example to illustrate the most common ones:

$$\mathcal{C} = \{a_1, a_2, a_3, a_4, a_5\}$$

$$a_1 \preceq a_1, a_1 \preceq a_2, a_1 \preceq a_3, a_1 \preceq a_4, a_1 \preceq a_5,$$

$$a_2 \preceq a_2, a_2 \preceq a_4, a_2 \preceq a_5,$$

$$a_3 \preceq a_3, a_3 \preceq a_5,$$

$$a_4 \preceq a_4, a_4 \preceq a_5,$$

$$a_5 \preceq a_5,$$

It is clear that $(\mathcal{C}, \preceq)$ satisfies the four requirements for a causal set.

2.7.1 Hesse Diagrams

The elements of the causal set are represented as dots and if $a_x \prec * a_y$ so that they are linked, then a_x is drawn lower than a_y connected by a line, as shown by 2.2

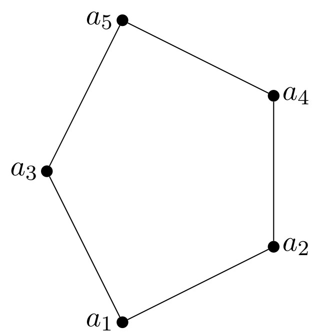


Figure 2.2: The Hesse Diagram of the example causal set

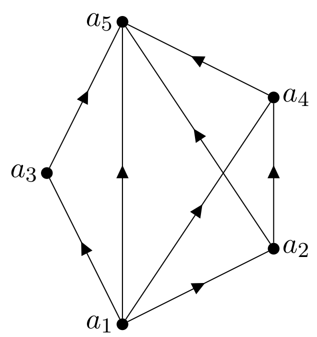


Figure 2.3: The directed graph of the example causal set

2.7.2 Directed Graphs

The elements are again given by points but here all relations are highlighted with lines connecting the involved elements, as represented in 2.3

2.7.3 Adjacency matrices

Given a causal set $\mathcal{C}$ with p elements labelled as $a_1, \ldots, a_p$, $\mathcal{C}$ can be represented with two different p $\times$ p matrices.
The *causal matrix* given by

$$C_{ij} := \begin{cases} 1 & \text{if } a_i \prec a_j \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

and the *link matrix* defined as

$$L_{ij} := \begin{cases} 1 & \text{if } a_i \prec * \, a_j \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

We can notice that both matrices are zero on the main diagonal and if natural labelling is chosen, L and C are strictly upper triangular matrices. In the example, the resulting matrices are

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad L = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.4}$$

2.8 Free Scalar Field theory on a causal set in operator form

At a fundamental level quantum field theory is the best theory to describe matter. Combining quantum mechanics with special relativity, the most successful result is the Standard Model which accounts for the electromagnetic, weak and strong forces. However, the theory is not mathematically well-defined due to the presence of divergences typically related to integrals. In 1940, renormalization has been introduced to deal with such complications, leading to admit our ignorance about the behavior at very small length scales. Alternatively, considering discrete spacetime, a model for matter is formulated to try to avoid the divergences from the beginning.

2.8.1 Propagators in the continuum

Firstly, we start with a brief review of the continuum case. Consider a *gaussian* free real bosonic scalar field ϕ in flat spacetime $\mathbb{M}^d$. The theory consists of

1. Equations of motion for all the operators $\phi(x)$, ie. Klein-Gordon equation $(\Box + m^2)\phi(x) = 0$
2. Equal time canonical commutation relations, such as $[\phi(x), \partial\phi(y)/\partial t] = i\delta^{d-1}(x-y)$ where $x = (x^0, x^i)$ with $i = 1, \dots, d-1$, m is the mass of the field and $\Box = \partial^2_{x_i} - \partial^2_{x_0}$ is the d'Alembertian.
3. Creation and annihilation operators $(a^\dagger_x, a_x)$ which act on a Fock space F, including a Poincarè invariant vacuum state $|0\rangle$ for which $a_x|0\rangle = 0$.

As the commutation relations are invariant under the time-evolution generated by the field equations, the fundamental consistency between the two expressions is ensured in our theory. Therefore, we decide to express the commutation relation in Peierls form from which Lorentz invariance is evident [Pei52]. Let $G^R(x, y)$ to be the retarded Green function, which vanishes unless y is in the past light cone of x, i.e. $y \prec x$, satisfying $(\Box + m^2)G^R(x, y) = \delta^d(x - y)$. Then, the difference between the retarded and advanced Green functions defines the *Pauli-Jordan function* as

$$\Delta(x, y) = G_R(x, y) - G_R(y, x) = G_R(x, y) - G_A(x, y). \tag{2.5}$$

This expression holds because the advanced Green function can be evaluated as the "transpose" of the retarded one in any globally hyperbolic spacetime. Hence, the commutation relations in Peierls form are given by

$$[\phi(x), \phi(y)] = i\Delta(x, y), \tag{2.6}$$

vanishing if the wave operator $(\Box + m^2)$ acts on them. In addition, the Feynman propagator is evaluated as

$$G_F(x, y) = i\langle 0|\mathcal{T}\phi(x)\phi(y)|0\rangle, \tag{2.7}$$

representing the vacuum expectation value of the time-ordered product of two field operators. The time ordering sets the time increasing from right to left. From this last expression without time ordering, the two-point function (or *Wightman function*) is defined as

$$W(x, y) = \langle 0|\phi(x)\phi(y)|0\rangle. \tag{2.8}$$

To summarize, the steps to define a real scalar free theory in continuum spacetime are:
wave operator $\rightarrow$ $G_R \rightarrow [\phi, \phi]$.

2.8.2 Propagators on a causal set

Now, as we deal with causal set, we immediately realize that there is no such thing as an equation of motion for the field operator. Therefore, the approach starts from the retarded Green function. Take a finite causal set $(\mathcal{C}, \preceq)$ with p elements $c_1, \ldots, c_p$ which is obtained by a sprinkling with density ρ into a finite interval of $\mathbb{M}^d$. To define the propagator in this framework, we need to choose the appropriate trajectories to which assign an amplitude. Then, the obvious options are all chains or paths between two elements. Due to local finiteness, there are only a finite number of such relations. As shown in [Joh08], the retarded propagator K_R is given by a p $\times$ p matrix of the form
if d=2

$$K_R^{(2)} := aC(I - abC)^{-1}, \tag{2.9}$$

where $a = \frac{1}{2}$, $b = -\frac{m^2}{\rho}$, I is the identity matrix, C is the causal matrix 2.2 and m is the mass of the field,
or if d=4

$$K_R^{(4)} := aL(I - abL)^{-1}, \tag{2.10}$$

where $a = \frac{\sqrt{\rho}}{2\pi\sqrt{6}}$, $b = -\frac{m^2}{\rho}$ and L is the link matrix 2.3. Then, deriving the advanced propagator as $K_A = K_R^T$, it is possible to define the analogue of the *Pauli Jordan function* as the real matrix

$$\Delta := K_R - K_A. \tag{2.11}$$

The matrix $i\Delta$ results as skew-symmetric ($i\Delta^T = -i\Delta$) and Hermitian ($i\Delta^\dagger = i\Delta$). These two features guarantee this matrix to have even rank, namely 2s, and real positive and negative pairs of non-zero eigenvalues, as shown in [Per58]. We can identify its eigenvalues and normalised eigenvectors as

$$i\Delta u_i = \lambda_i u_i \qquad i\Delta v_i = -\lambda_i v_i \qquad i\Delta w_k = 0, \tag{2.12}$$

where $\lambda_i > 0$, i = 1,. . . , s ; k = 1,. . . , p − 2s.
Hence, it is possible to write the Pauli-Jordan function as

$$i\Delta = \sum_{i=1}^{s} \lambda_i u_i u_i^\dagger - \sum_{i=1}^{s} \lambda_i v_i v_i^\dagger, \tag{2.13}$$

where we can distinguish its Hermitian, positive semi-definite part as

$$W = \sum_{i=1}^{s} \lambda_i u_i u_i^\dagger \ \geq 0, \tag{2.14}$$

so that

$$i\Delta = W - W^T = W - W^*. \tag{2.15}$$

These operators have orthogonal support so that

$$W\ W^* = W^*\ W = 0. \tag{2.16}$$

In addition, we express

$$W = \frac{1}{2}(\sqrt{(i\Delta)^2} + i\Delta), \tag{2.17}$$

leading to recognise its real and imaginary part as

$$\mathfrak{R}[W] = \frac{\sqrt{(i\Delta)^2}}{2} \quad \mathfrak{I}[W] = \frac{\Delta}{2} \tag{2.18}$$

Now, we define a free real bosonic scalar field on a causal set by considering an algebra of operators $\phi_x = \phi_x^\dagger$ such that [Joh08]

1. $[\phi_x, \phi_y] = i\Delta_{xy}$, analogue of 2.6 in the continuum
2. $i\Delta w = 0 \Rightarrow \sum_{x'=1}^{p} w_{x'}\phi_{x'} = 0$, meaning that any linear combination of field operators that commutes with all the field operators vanish. This is equivalent to imposing the wave equation on operators as shown in [Joh10].
3. $\phi_x = \sum_{i=1}^{s} \sqrt{\lambda_i}(u_i)_x a_i + \sqrt{\lambda_i}(v_i)_x a_i^\dagger$ where a_i and $a_i^\dagger$ are annihilation and creation operators acting on a Fock space in which a *Fock vacuum* state vector $|0\rangle$ satisfies the conditions $a_i|0\rangle = 0 \ \forall \ i = 1, \dots, s$

From the last point, it is straightforward to derive the *Wightman function* as

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_{i=1}^{s}\sum_{j=1}^{s} \sqrt{\lambda_i\lambda_j}(u_i)_x(v_j)y\langle 0|a_i a_j^\dagger|0\rangle = \sum_{i=1}^{s}\sum_{j=1}^{s} \sqrt{\lambda_i\lambda_j}(u_i)_x(v_j)_y\delta_{ij} = W_{xy}. \quad (2.19)$$

Again, equivalently to the continuum, we can estimate the Feynman propagator, after fixing a linear extension $(\mathcal{C}, \leq)$ with causal matrix $\bar{A}$ as

$$G_F^{xy} = i\langle 0|\mathcal{T}\phi(x)\phi(y)|0\rangle := \begin{cases} iW_{yx} & \text{if } c_x \leq c_y \\ iW_{xy} & \text{if } c_y \leq c_x \end{cases} = i(\bar{A}_{xy}W_{yx} + \bar{A}_{yx}W_{xy} + \delta_{xy}W_{xy}), \quad (2.20)$$

where δ_{xy} is the Kronecker delta. Generally, an arbitrary order can be assigned to pairs of unrelated elements by different linear extensions without ambiguity on G_F since the field operators commute for pairs of unrelated elements. Noticing that $\bar{A}_{xy}(i\Delta_{xy}) = (iK_R)_{xy}$ and using 2.15, we get $\bar{A}_{xy}W_{yx} = \bar{A}_{xy}(W_{xy} - i\Delta_{xy}) = \bar{A}_{xy}W_{xy} - i(K_R)_{xy}$. As this result is substituted in the second expression for G_F in 2.20 and using $\bar{A}_{xy} + \bar{A}_{yx} + \delta_{xy} = 1 \ \forall \ x, y = 1, \dots, p$, we get

$$G_F = K_R + iW, \quad (2.21)$$

which clearly shows the real and imaginary part to be

$$\Re[G_F] = K_R - \frac{\Delta}{2} = \frac{K_R + K_A}{2} \quad \Im[G_F] = \Re[W]. \quad (2.22)$$

Therefore, the new path has been traced as $K_R \to \Delta \to W$ to find propagators in causal set, avoiding to consider the wave equation.
It is important to notice that 2.14, 2.15 and 2.16 can be considered as "ground-state condition" imposed on W beyond what follows from its definition as the two-point function of a selfadjoint operator ϕ_x. Then, the *Sorkin-Johnston (SJ) vacuum* is defined through its Wightman function satisfying the three conditions above [NA12]. Then, from now we consider the expectation values to be taken with respect to Ψ, denoting the SJ vacuum state.
As in the classical analogue, we are dealing with a *gaussian* theory, however, it is necessary to specify its meaning in the causal set context. Since there is no Schrödinger representation, any vacuum wave function will not have an exponential form that satisfies the definition of a gaussian. On the contrary, given the SJ vacuum, Wick's theorem can be applied as usual since the Wightman function is known and the ϕ_xs are linear combinations of raising and lowering operators. Then, we characterize the possibility of applying Wick's pattern as a manifest feature of gaussian distributions. For instance, considering Φ as a superposition of ϕ_x, we can show that [Sor11]

$$\langle exp\{\Phi\}\rangle = \sum_n \frac{\langle\Phi^n\rangle}{n!} = \sum_n \frac{\langle\Phi^{2n}\rangle}{2n!} = \sum_n \frac{(2n-1)!!\langle\Phi\Phi\rangle^n}{2n!} = \sum_n \frac{1}{n!}\frac{\langle\Phi\Phi\rangle^n}{2^n} = exp\{\langle\frac{\Phi\Phi}{2}\rangle\}, \quad (2.23)$$

which is a consequence of Wick's theorem we will require, showing explicitly a gaussian form.

2.9 A history-based framework

In general relativity, the notions of "event" and "4-D spacetime history" are fundamental while the idea of a "3-D state" of the whole world frozen in time has no relevance. In quantum theory, the path integral gives an approach to quantum foundations that takes into account the concepts of event and history that are primary in general relativity [FD20].
The canonical approach is the most standard route to quantisation, which starts with the phase space of a classical system and then construct Hilbert spaces and self adjoint operators. The spacetime has a split as $M = \Sigma \times \mathbb{R}$, where Σ is a Cauchy hypersurface, which is necessary to define the canonical phase space variables capturing the intrinsic and extrinsic geometry of Σ. However, *a Cauchy hypersurface cannot be meaningfully defined in the continuum approximation of causal set.* To show our last statement, we recall some terminology. An antichain is a set of unrelated elements in $\mathcal{C}$, and an *inextendible antichain* is an antichain $P \subset \mathcal{C}$ so that every element $u \in \mathcal{C}\backslash P$ is related to an element of P. Therefore, the discrete counterpart of a Cauchy hypersurface is an inextendible antichain P, dividing $\mathcal{C}$ into its future and past such that $\mathcal{C} = Fut(P) \sqcup Past(P) \sqcup P$, where $\sqcup$ denotes the disjoint union. However, it is possible to trace a link from an element in $Past(P)$ to an element in $Fut(P)$, "ignoring" P. Since P is not a summary of its past, canonical quantization through the definition of Cauchy hypersurfaces can not be applied [Sur19].
Hence, dynamics can not be based on the idea of Hamiltonian evolution since a continuous time variable is not present in causal set theory, abandoning the canonical approach and demanding a histories-based formalism for quantization. A histories-based formulation of a field theory works directly with field configurations ϕ, or "histories", avoiding to introduce field operators or state-vectors, except as technical tools [Sor07b]. Then, the path-integral is recasted from its role as a mere mathematical device to compute propagators or S-matrix elements; it becomes the fundamental dynamical object of the theory [Sor10]. It is important to specify that with this approach we are not dealing with the quantum theory of causal sets themselves but the quantum theory of a field on a fixed spacetime background, whether continuous (Lorentzian manifold) or discrete (causal set), with no back reaction.
The starting point is a sample space Ω of possible histories which is related to a physical, quantum system. Ω is the space over which the integration of the path integral takes place. In a scalar field theory, a history ξ is associated to a real or complex function on spacetime and gives a complete classical description of physical reality within the theory.
While in the continuum the histories are field configurations on a fixed Lorentzian manifold of general relativity, in causal set scalar QFT, the background is a fixed causal set or locally finite partial order to which the manifold is only an approximation.
Following the sum-over-histories formalism, we assign an amplitude to each process which is simply the sum of the amplitudes for all the ways the process can occur, computing physically meaningful probabilities instead of a transition amplitude. In the continuum this sum diverges, while if it was evaluated on a causal set, we would expect a finite summation because the number of diagrams to represent propagation that could be squeezed onto a finite causal set is finite [Joh10].
The amplitudes are assigned to pair of histories in contrast with the familiar path-integral formulation which considers individual histories. Such a pair of histories is known as "Schwinger history" from the Schwinger-Keldysh treatment of the path integral, or *in-in formalism*.

2.9.1 Event algebra

Given a sample space, any subset of Ω which gives a proposition about physical reality is called event. A non-empty collection, $\mathfrak{U}$, of subsets of Ω such that

1. $\forall\ \alpha \in \mathfrak{U},\ \Omega\backslash\alpha \in \mathfrak{U}$;
2. $\forall\ \alpha, \beta \in \mathfrak{U},\ \alpha \cup \beta \in \mathfrak{U}$

defines an *event algebra* on a sample space Ω. Given this algebra of sets, it is straightforward that $\emptyset \in \mathfrak{U}$, $\Omega \in \mathfrak{U}$ where $\emptyset$ is the empty set and $\mathfrak{U}$ satisfies the closure property under finite unions and intersections. In addition, the event algebra is a *Boolean algebra* under union (logical 'or'), intersection ('and') and complement ('not') with unit element Ω and zero element $\emptyset$. It is also a *unital ring* with

1. identity element Ω
2. $\alpha \cdot \beta := \alpha \cap \beta$ multiplication as intersection,
3. $\alpha + \beta := (\alpha \backslash \beta) \cup (\beta \backslash \alpha)$ addition as symmetric difference

This ring is Boolean since $\alpha \cdot \alpha = \alpha$ and is also an algebra over $\mathbb{Z}_2$. More details on the event algebra are elucidated in [Sor07a].

2.9.2 Decoherence functional

A decoherence functional on an event algebra $\mathfrak{U}$ is a map $D : \mathfrak{U} \times \mathfrak{U} \to \mathbb{C}$ satisfying the following properties [FD10]:

1. Hermiticity, $\forall\ \alpha, \beta \in \mathfrak{U}, D(\alpha, \beta) = D(\beta, \alpha)^*$;
2. Linearity, $\forall\ \alpha, \beta, \gamma \in \mathfrak{U}$ with $\beta \cap \gamma = \emptyset,\ D(\alpha, \beta \cup \gamma) = D(\alpha, \beta) + D(\alpha, \gamma)$;
3. Normalization, $D(\Omega, \Omega) = 1$;
4. Strong positivity, the N $\times$ N matrix $D(\alpha_i, \alpha_j)$ is positive semidefinite where $\alpha_i \in \mathfrak{U}$ is any finite collection of events with i = 1, . . . , N .

We also define a *quantal measure* on an event algebra $\mathfrak{U}$ as a map $\mu : \mathfrak{U} \to \mathbb{R}$, fulfilling the requirements of:

1. positivity, $\forall\ \alpha \in \mathfrak{U}, \mu(\alpha) \geq 0$;
2. quantal sum rule, for mutually disjoint $\alpha, \beta, \gamma \in \mathfrak{U},\ \mu(\alpha \cup \beta \cup \gamma) - \mu(\alpha \cup \beta) - \mu(\beta \cup \gamma) - \mu(\alpha \cup \gamma) + \mu(\alpha) + \mu(\beta) + \mu(\gamma) = 0$;
3. normalization, $\mu(\Omega) = 1$.

The second condition captures the "quadratic nature" of μ, which is a feature mathematically distinguishing quantum physics from classical physics.[1] Let the decoherence functional be $D : \mathfrak{U} \times \mathfrak{U} \to \mathbb{C}$, then a quantal measure is given by the map $\mu : \mathfrak{U} \to \mathbb{R}$ such that $\mu(\alpha) := D(\alpha, \alpha)$.
A *quantum measure system* is represented by (Ω, $\mathfrak{U}$, D): the sample space expresses the kinematics, while the dynamics is defined by the event algebra and decoherence functional. It is important to highlight the lack of the bridge between this formal structure and its physical meaning, although many attemps have been made [Sor07b][Sor97b].

2.9.3 Free Decoherence functional in causal set

Given a causet $(\mathcal{C}, \preceq)$ of cardinality N, the decoherence functional is expressed by

$$D(\xi, \bar{\xi}|\Psi) = \langle \delta(\phi^1 - \bar{\xi}^1)\delta(\phi^2 - \bar{\xi}^2)...\delta(\phi^N - \bar{\xi}^N)\delta(\phi^N - \xi^N)...\delta(\phi^2 - \xi^2)\delta(\phi^1 - \xi^1)\rangle_\Psi, \tag{2.24}$$

where $\phi^i = \phi(x_i), \xi^i = \xi(x_i)$, and i = 1 ... N is any *natural labeling* of all the elements $x \in \mathcal{C}$, i.e $x_j \prec x_k \Rightarrow j < k$. Furthermore, all the ϕs are field operators and ξ and $\bar{\xi}$ are two independent histories.

[1]The squares of sums of amplitudes are the probabilities, alternatively to just sums of elementary probabilities. This corresponds physically to the phenomenon of interference.

Each history is a list of real numbers, one for each element of the causet.
Expressed differently with the δ-functions as integrals, 2.9.3 becomes

$$D(\xi, \bar{\xi}|\Psi) = \int_{-\infty}^{\infty} \frac{d\lambda_1}{2\pi} \frac{d\lambda_2}{2\pi} \cdots \frac{d\bar{\lambda}_1}{2\pi} \frac{d\bar{\lambda}_2}{2\pi} \langle e^{-i\bar{\lambda}_1\bar{\xi}^1 - i\bar{\lambda}_2\bar{\xi}^2 \cdots - i\lambda_2\xi^2 - i\lambda_1\xi^1} \langle e^{i\bar{\lambda}_1\phi^1} e^{i\bar{\lambda}_2\phi^2} \cdots e^{i\lambda_2\phi^2} e^{i\lambda_1\phi^1} \rangle_\Psi, \quad (2.25)$$

where the real numbers λ_i and $\bar{\lambda}_i$ are independent parameters of the Fourier transform. This form is known as the "non-commutative characteristic function", playing the same role as its homonym in ordinary probability theory. It is important we recall that we are assuming the expectation value is taken in some fixed Gaussian state Ψ i.e. we are assuming we know $W^{xy} := \langle \phi^x \phi^y \rangle_\Psi$ (= $\langle \Psi | \phi^x \phi^y | \Psi \rangle$ if the state is pure). The subscript Ψ denotes the SJ state defined through the conditions in 2.8.2, which is the closest analogue on the causal set to the Minkowski vacuum in the continuum [NA12].

Chapter 3

Discussion

3.1 Preliminary calculations

Before proceeding to deal with interactions, let us work through the free theory. Firstly, we want to make sure that the normalization condition is satisfied, i.e.

$$\begin{aligned} \int d\xi^1 \delta(\phi^1 - \xi^1) = \mathbf{1} \, \text{etc. and so} \\ \int d^N\bar{\xi}\, \delta(\phi^1 - \bar{\xi}^1)\delta(\phi^2 - \bar{\xi}^2)\dots\delta(\phi^N - \bar{\xi}^N) = \mathbf{1}, \\ \int d^N\xi\, \delta(\phi^N - \xi^N)\dots\delta(\phi^2 - \xi^2)\delta(\phi^1 - \xi^1) = \mathbf{1}\,, \\ \text{leading to} \int d^N\bar{\xi} \int d^N\xi\ D(\xi,\bar{\xi}|\Psi) = \mathbf{1}, \end{aligned} \tag{3.1}$$

where $\mathbf{1}$ is the identity operator. Then, we integrate all the $\bar{\xi}$ and ξ variables except for one ξ^x so that we evaluate the marginal probability density for ϕ^x. With the aid of 2.23, we get

$$\int \frac{d\alpha}{2\pi} exp\{-i\alpha\xi^x\}\langle exp\{i\alpha\phi^x\}\rangle = \int \frac{d\alpha}{2\pi} exp\{\frac{-w\alpha^2}{2} - i\alpha\xi^x\} = \frac{1}{\sqrt{2\pi w}} exp\{\frac{-(\xi^x)^2}{2w}\}, \tag{3.2}$$

where $w = \langle\phi^x\phi^x\rangle$ giving a gaussian, as expected for a free field.
We introduce the following notation to highlight the opposite order of the causet elements in the two halves of the Schwinger history:

$$\overrightarrow{W}^{xy} := \begin{cases} W^{xy} & \text{if x} \leq \text{y} \\ W^{yx} & \text{if y} \leq \text{x} \end{cases} \quad \overleftarrow{W}^{xy} := \begin{cases} W^{xy} & \text{if y} \leq \text{x} \\ W^{yx} & \text{if x} \leq \text{y} \end{cases} = -iG_F^{xy}, \tag{3.3}$$

where G_F^{xy} is the Feynman Green function and noting that $\overleftarrow{W}$ and $\overrightarrow{W}$ are complex conjugates of each other. In addition, define $\Box_F := i\overleftarrow{W}^{-1}$.
Let us say $x \prec y$. Then

$$\begin{aligned} \int d^N\bar{\xi}\ D(\xi,\bar{\xi}|\Psi)\xi^x\xi^y &= \langle\delta(\phi^N - \xi^N)\dots\xi^y\delta(\phi^y - \xi^y)\dots\xi^x\delta(\phi^x - \xi^x)\dots\delta(\phi^1 - \xi^1)\rangle_\Psi \\ &= \langle\delta(\phi^N - \xi^N)\dots\phi^y\delta(\phi^y - \xi^y)\dots\phi^x\delta(\phi^x - \xi^x)\dots\delta(\phi^1 - \xi^1)\rangle_\Psi \end{aligned} \tag{3.4}$$

and so

$$\begin{aligned} \int d^N\xi \int d^N\bar{\xi}\ D(\xi,\bar{\xi}|\Psi)\xi^x\xi^y &= \langle\phi^y\phi^x\rangle_\Psi \\ &= \langle\mathcal{T}[\phi^x\phi^y]\rangle_\Psi\,. \end{aligned} \tag{3.5}$$

where $\mathcal{T}$ is time ordered product, ordering the factors from later to earlier from left to right. Let $y \prec x$, then

$$\begin{aligned}\int d^N\bar{\xi}\ D(\xi,\bar{\xi}|\Psi)\xi^x\xi^y &= \langle\delta(\phi^N-\xi^N)\dots\xi^x\delta(\phi^x-\xi^x)\dots\xi^y\delta(\phi^y-\xi^y)\dots\delta(\phi^1-\xi^1)\rangle_\Psi \\ &= \langle\delta(\phi^N-\xi^N)\dots\phi^x\delta(\phi^x-\xi^x)\dots\phi^y\delta(\phi^y-\xi^y)\dots\delta(\phi^1-\xi^1)\rangle_\Psi \end{aligned} \tag{3.6}$$

and so

$$\begin{aligned}\int d^N\xi\int d^N\bar{\xi}\ D(\xi,\bar{\xi}|\Psi)\xi^x\xi^y &= \langle\phi^x\phi^y\rangle_\Psi \\ &= \langle\mathcal{T}[\phi^x\phi^y]\rangle_\Psi \\ &= \overleftarrow{W}^{xy}\,. \end{aligned} \tag{3.7}$$

This is an operator argument which can be confirmed from firstly integrating all but two of the field values in the decoherence functional. For $y \prec x$, it results as

$$\begin{aligned}&\int\frac{d\alpha}{2\pi}\frac{d\beta}{2\pi}exp\{-i\alpha\xi^x-i\beta\xi^y\}\langle exp\{i\alpha\phi^x\}exp\{i\beta\phi^y\}\rangle \\ &= \frac{1}{4\pi^2}\int d^2\alpha\, exp\{-i\vec{\alpha}\cdot\vec{\xi}\}\, exp\{-\frac{1}{2}\alpha^T\overleftarrow{W}\vec{\alpha}\} \\ &= \frac{1}{4\pi^2}\int d^2\alpha\, exp\{i\vec{\alpha}\cdot(-\vec{\xi})\}\, exp\{\frac{i}{2}\alpha^T(i\overleftarrow{W})\vec{\alpha}\} \\ &= \frac{1}{4\pi^2}\sqrt{\frac{4\pi^2}{det\overleftarrow{W}}}\ exp\{\frac{i}{2}\xi^T(i\overleftarrow{W}^{-1})\vec{\xi}\} \\ &= \frac{1}{2\pi}\sqrt{\frac{1}{det\overleftarrow{W}}}\ exp\{\frac{i}{2}\xi^T\Box_F\vec{\xi}\}, \end{aligned} \tag{3.8}$$

where $\vec{\xi}=\begin{pmatrix}\xi^x\\ \xi^y\end{pmatrix}$ and $\vec{\alpha}=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. The following identity was used in the integration:

$$\int d^n v\ \ exp\{\frac{i}{2}v^TA\vec{v}+i\vec{J}\cdot\vec{v}\} = \sqrt{\frac{(2\pi i)^n}{detA}}\,exp\{-\frac{i}{2}J^TA^{-1}\vec{J}\}. \tag{3.9}$$

If we integrate 3.8 over ξ^x and ξ^y, we get $\sqrt{\frac{1}{det\overleftarrow{W}}}\sqrt{\frac{-1}{det(\Box_F)}}$, giving the identity as expected. Now let us evaluate the following correlation

$$\begin{aligned}
\int d^N\xi \int d^N\bar{\xi}\; D(\xi,\bar{\xi}|\Psi)\xi^x\xi^y &= \frac{1}{4\pi^2}\sqrt{\frac{4\pi^2}{det\overleftarrow{W}}}\int d^2\xi\,\xi^x\xi^y\,exp\{\frac{i}{2}\xi^T\Box_F\vec{\xi}\} \\
&= \frac{\partial}{i\partial\Box_F^{xy}}\frac{1}{4\pi^2}\sqrt{\frac{4\pi^2}{det\overleftarrow{W}}}\int d^2\xi\,exp\{\frac{i}{2}\xi^T\Box_F\vec{\xi}\} \\
&= \sqrt{\frac{4\pi^2}{det\overleftarrow{W}}}\frac{\partial}{i\partial\Box_F^{xy}}\frac{1}{4\pi^2}\sqrt{\frac{(2\pi i)^2}{det\Box_F}} \\
&= \frac{1}{\sqrt{det\overleftarrow{W}}}\frac{\partial}{\partial\Box_F^{xy}}\frac{1}{\sqrt{det\Box_F}} \\
&= \frac{1}{\sqrt{det\overleftarrow{W}}}\frac{\partial}{\partial\Box_F^{xy}}\frac{1}{\sqrt{\Box_F^{xx}\Box_F^{yy}-(\Box_F^{xy})^2}} \\
&= \frac{1}{\sqrt{det\overleftarrow{W}}}\frac{\Box_F^{xy}}{(\Box_F^{xx}\Box_F^{yy}-(\Box_F^{xy})^2)^{3/2}} \\
&= \frac{1}{\sqrt{det\overleftarrow{W}}}\frac{i(\overleftarrow{W}^{-1})^{xy}}{(det(i\overleftarrow{W}^{-1}))^{3/2}} \\
&= -det\overleftarrow{W}\cdot(\overleftarrow{W}^{-1})^{xy} = \overleftarrow{W}^{xy},
\end{aligned} \tag{3.10}$$

where we used that $W^{-1} = \frac{1}{det\overleftarrow{W}}\begin{pmatrix} W^{yy} & -W^{xy} \\ -W^{yx} & W^{xx}\end{pmatrix}$, verifying 3.6.

Similarly, we have that

$$\begin{aligned}
&\int \frac{d\alpha}{2\pi}\frac{d\beta}{2\pi}exp\{-i\alpha\xi^x - i\beta\bar{\xi}^y\}\langle exp\{i\alpha\phi^x\}exp\{i\beta\phi^y\}\rangle \\
&= \int \frac{d\alpha}{2\pi}\frac{d\beta}{2\pi}exp\{-i(\beta\;\alpha)\begin{pmatrix}\bar{\xi}^y\\ \xi^x\end{pmatrix}\}\langle exp\{-\frac{1}{2}(\beta\,\alpha)\begin{pmatrix}W^{yy} & W^{yx}\\ W^{xy} & W^{xx}\end{pmatrix}\begin{pmatrix}\beta\\ \alpha\end{pmatrix}\}\rangle \\
&= \int \frac{d\alpha}{2\pi}\frac{d\beta}{2\pi}exp\{-i(\alpha\;\beta)\begin{pmatrix}\bar{\xi}^x\\ \xi^y\end{pmatrix}\}\langle exp\{-\frac{1}{2}(\alpha\,\beta)\begin{pmatrix}W^{xx} & W^{xy}\\ W^{yx} & W^{xx}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\}\rangle \\
&= \frac{1}{2\pi}\sqrt{\frac{1}{detW}}\;exp\{\frac{i}{2}\eta^T\Box\vec{\eta}\},
\end{aligned} \tag{3.11}$$

where $\eta = \begin{pmatrix}\bar{\xi}^x\\ \xi^y\end{pmatrix}$ and $\Box \equiv iW^{-1}$.

Carrying out the η integral we would get $\int d\eta\, exp\{\frac{i}{2}\eta^T\Box\vec{\eta}\} = \sqrt{\frac{(2\pi i)^2}{det\Box}}$.
Now we estimate the correlation function as

$$\begin{aligned}
\int d^N\xi \int d^N\bar{\xi}\; D(\xi,\bar{\xi}|\Psi)\bar{\xi}^x\xi^y &= \frac{1}{2\pi}\sqrt{\frac{1}{detW}}\int d\eta\,\bar{\xi}^x\xi^y\,exp\{\frac{i}{2}\eta\Box\eta\} \\
&= \frac{1}{2\pi}\sqrt{\frac{1}{detW}}\frac{\partial}{i\partial\Box^{xy}}\int d\eta\,exp\{\frac{i}{2}\eta\Box\eta\} \\
&\;\;\vdots \\
&= W^{xy}.
\end{aligned} \tag{3.12}$$

Finally, we have that

$$\int d^N\xi \int d^N\bar{\xi} D(\xi,\bar{\xi}|\Psi)\bar{\xi}^x\bar{\xi}^y = \frac{1}{2\pi}\sqrt{\frac{1}{det\overrightarrow{W}}}\int d^2\bar{\xi}\,\bar{\xi}^x\bar{\xi}^y\, exp\{\frac{i}{2}\bar{\xi}\Box\bar{\xi}\} = \overrightarrow{W}^{xy} \tag{3.13}$$

Without doing any further work we see that

$$\int d^N\xi \int d^N\bar{\xi}\; D(\xi,\bar{\xi}|\Psi)\xi^x\xi^x = \langle\phi^x\phi^x\rangle_\Psi =: W^{xx}\,. \tag{3.14}$$

Also for

$$\int d^N\xi \int d^N\bar{\xi}\; D(\xi,\bar{\xi}|\Psi)(\xi^x)^m = \langle(\phi^x)^m\rangle_\Psi \tag{3.15}$$

we can use Wick's theorem, e.g. for m= 4, we have that

$$\int d^N\xi \int d^N\bar{\xi}\; D(\xi,\bar{\xi}|\Psi)(\xi^x)^4 = \langle(\phi^x)^4\rangle_\Psi = 3W^{xx}\,. \tag{3.16}$$

Hence, we can state the general expression

$$\int d^N\xi \int d^N\bar{\xi}\; D(\xi,\bar{\xi}|\Psi)\xi^{x_1}\xi^{x_2}\ldots\xi^{x_k} = \langle\mathcal{T}[\phi^{x_1}\phi^{x_2}\ldots\phi^{x_k}]\rangle_\Psi \tag{3.17}$$

$$=: -iG_{0F}(x_1,x_2,\ldots,x_k), \tag{3.18}$$

where we define $-iG_{0F}$ as the free "Feynman k-point function", where the term "Feynman" is used to highlight that this is an expectation value of a time ordered product of field operators in the SJ state Ψ. Then, in the case of a gaussian state, this can be easily evaluated because knowing the Wightman function W, all the higher free k-point functions can be calculated by Wick's formula. The path integral version of Wick's theorem states that for any gaussian integral

$$G_{0F}(x_1,x_2,\ldots,x_k) = i\sum W^{x_{j_1}x_{j'_1}}\ldots W^{x_{j_{N/2}}x_{j'_{N/2}}} \tag{3.19}$$

where k is even and the sum is over all the possible ways of arranging the set {1, 2, . . . , k} into pairs. Now, we are ready to approach the interacting case where we replace $D(\xi,\bar{\xi}|\Psi)$ in 3.17 by the interacting decoherence functional.

3.2 Interacting Decoherence functional for ϕ^4 theory

Given a causet $(\mathcal{C},\preceq)$ of cardinality N, we propose the interacting decoherence functional for ϕ^4 theory:

$$D(\xi,\bar{\xi}|\Psi,\underline{\Lambda}) := \mathcal{N}\,\langle\delta(\phi^1-\bar{\xi}^1)\delta(\phi^2-\bar{\xi}^2)\ldots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\ldots \tag{3.20}$$

$$\delta(\phi^1-\xi^1)\rangle_\Psi\; e^{i(\underline{\xi}^4-\underline{\bar{\xi}}^4)\cdot\underline{\Lambda}} \tag{3.21}$$

where

$$\underline{\Lambda} = (\lambda_1,\lambda_2,\lambda_3,\ldots\lambda_N) \tag{3.22}$$

is a vector of N coupling constants, one for each causet element and

$$\underline{\xi}^4\cdot\underline{\Lambda} := \sum_{x=1}^{N}(\xi^i)^4\Lambda_i \quad etc.\,. \tag{3.23}$$

and $\mathcal{N}$ is a normalisation factor that needs to be worked out perturbatively.
The interacting decoherence functional $D(\xi, \bar{\xi}|\Psi, \underline{\Lambda})$ satisfies the "bi-additive property" of decoherence functionals, following from its definition 2.9.2. In addition, it is possible to show that $D(\xi, \bar{\xi}|\Psi, \underline{\Lambda})$ is strongly positive, requiring that there is the same $\underline{\Lambda}$ of coupling constants on both sides of the "Schwinger history" for related ξ and $\bar{\xi}$. As proven in [FD20], we can think to the free decoherence functional as the set of all inner products $\langle\Psi|O^\dagger O|\Psi\rangle$ where $O = \delta(\phi^N - \xi^N)\ldots\delta(\phi^1 - \xi^1)$ and Ψ a pure state, defining a strongly positive matrix. The interaction terms add a phase (a complex number) to the expression, preserving strong positivity.
In the first instance, we choose all the $\Lambda_i's$ to be zero except for one at point z, $\Lambda_z = \lambda$: $\underline{\Lambda} = (0, 0, \ldots, 0, \lambda, 0, \ldots, 0, 0)$. Later, we will set more of the $\Lambda_i's$ to equal λ. However, we are also free to choose different values of the coupling at different points. For example, the $\Lambda_i's$ could be zero except in some interaction region and constant in the middle of that region and tend "smoothly" to zero in some boundary region of the interaction region. This would describe an interaction which is turned on smoothly.
Referring to [Car10], we deviate from a treatment that emphasises the spacetime approach and spacetime Feynman diagrams using Euclidean techniques and "Wick rotation" to get Lorentzian path integral expressions. We work in the Lorentzian ("real time") domain, meaning:

- we require that if two points are spacelike, the field operators at those points commute (microcausality condition),
- we introduce the factor $e^{i\lambda\phi^4}$ for interaction instead of $e^{-\lambda\phi^4}$ as in the Euclidean case.

Then, we aim to recast the results in [Car10] in terms of the causal set decoherence functional in Lorentzian domain.

3.3 Hermiticity of $D(\xi, \bar{\xi}|\Psi, \underline{\Lambda})$

Since $\phi^\dagger = \phi$ for any point in the causal set, then

$$\delta(\phi - \xi) = \int_{\mathbb{R}} \frac{dt}{2\pi} e^{-it\xi} e^{it\phi} \,, \tag{3.24}$$

and so

$$\begin{aligned} \delta(\phi - \xi)^\dagger &= \int_{\mathbb{R}} \frac{dt}{2\pi} e^{it\xi} e^{-it\phi} \\ &= \int_{\mathbb{R}} \frac{dt'}{2\pi} e^{-it'\xi} e^{it'\phi} \\ &= \delta(\phi - \xi) \,, \end{aligned} \tag{3.25}$$

by changing variables to $t' = -t$, proving $\delta(\phi - \xi)^\dagger = \delta(\phi - \xi)$. Then, we have that

$$\begin{aligned} D(\xi, \bar{\xi})^* &= \langle\delta(\phi^1 - \bar{\xi}^1)\ldots\delta(\phi^1 - \xi^1)\rangle^* \\ &= \langle\delta(\phi^1 - \xi^1)^\dagger\ldots\delta(\phi^1 - \bar{\xi}^1)^\dagger\rangle \\ &= \langle\delta(\phi^1 - \xi^1)\ldots\delta(\phi^1 - \bar{\xi}^1)\rangle \\ &= D(\bar{\xi}, \xi) \,. \end{aligned} \tag{3.26}$$

3.4 Normalization

The normalisation condition is

$$1 = \mathcal{N}(\underline{\Lambda})\mathcal{M}(\underline{\Lambda}) \,. \tag{3.27}$$

where

$$\mathcal{M}(\underline{\Lambda}) = \int d^N\xi \int d^N\overline{\xi}\, D(\xi,\overline{\xi})\; e^{i(\underline{\xi}^4-\underline{\overline{\xi}}^4)\cdot\underline{\Lambda}}\;, \tag{3.28}$$

is real, as

$$\begin{aligned}\mathcal{M}(\underline{\Lambda})^* &= \int d^N\xi \int d^N\overline{\xi}\, D(\xi,\overline{\xi})^*\; e^{i(\underline{\xi}^4-\underline{\overline{\xi}}^4)\cdot\underline{\Lambda}} \\ &= \int d^N\overline{\xi} \int d^N\xi\, D(\overline{\xi},\xi)\; e^{i(\underline{\xi}^4-\underline{\overline{\xi}}^4)\cdot\underline{\Lambda}} \\ &= \mathcal{M}(\underline{\Lambda})\;,\end{aligned} \tag{3.29}$$

by relabelling ξ and $\overline{\xi}$. More generally, any double integral over a function $f(\xi,\overline{\xi})$, satisfying $f(\xi,\overline{\xi})^* = f(\overline{\xi},\xi)$, is real, given we can always relabel ξ and $\overline{\xi}$. Note that since $\mathcal{M}(\underline{\Lambda})$ is real, $\mathcal{N}(\underline{\Lambda})$ is as well.
To begin with, we need to work out the normalization factor perturbately. Let us consider *one interaction point* and expand the exponential at different order in the coupling constant λ
Order λ^0 $\mathcal{N}$=1
Order λ^1

$$\begin{aligned}\int d^N\xi \int d^N\bar{\xi}\; \mathcal{N}\langle\delta(\phi^1-\bar{\xi}^1)\delta(\phi^2-\bar{\xi}^2)\ldots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\ldots\delta(\phi^2-\xi^2)\delta(\phi^1-\xi^1)\rangle_\Psi \\ (i\lambda(\xi^z)^4 - i\lambda(\bar{\xi}^z)^4) \\ = i\lambda\mathcal{N}(\langle(\phi^z)^4\rangle - \langle(\phi^z)^4\rangle) = \mathcal{N}i\lambda\; 3(W^{zz})^2 - 3(W^{zz})^2 = 0\end{aligned} \tag{3.30}$$

Hence, $\mathcal{N}$=1
Order λ^2

$$\begin{aligned}\int d^N\xi \int d^N\bar{\xi}\; \mathcal{N}\langle\delta(\phi^1-\bar{\xi}^1)\delta(\phi^2-\bar{\xi}^2)\ldots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\ldots\delta(\phi^2-\xi^2)\delta(\phi^1-\xi^1)\rangle_\Psi \\ (i\lambda)^2((\xi^z)^4-(\bar{\xi}^z)^4)^2 = \\ \mathcal{N}(i\lambda)^2\int d^N\xi \int d^N\bar{\xi}\langle\delta(\phi^1-\bar{\xi}^1)\ldots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\ldots\delta(\phi^1-\xi^1)\rangle_\Psi \\ (i\lambda)^2((\xi^z)^8+(\bar{\xi^z})^8-2(\xi^z)^4(\bar{\xi}^z)^4) = \\ \mathcal{N}(i\lambda)^2(\langle(\phi^z)^8\rangle_\Psi + \langle(\phi^z)^8\rangle_\Psi - 2\langle(\phi^z)^8\rangle_\Psi) = \mathcal{N}\;(i\lambda)^2 105(W^{zz})^4 + 105(W^{zz})^4 - 210(W^{zz})^4 = 0\end{aligned} \tag{3.31}$$

Hence, $\mathcal{N}$=1
Order λ^3

$$\begin{aligned}\int d^N\xi \int d^N\bar{\xi}\; \mathcal{N}\langle\delta(\phi^1-\bar{\xi}^1)\delta(\phi^2-\bar{\xi}^2)\ldots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\ldots\delta(\phi^2-\xi^2)\delta(\phi^1-\xi^1)\rangle_\Psi \\ (i\lambda)^3((\xi^z)^4-(\bar{\xi}^z)^4)^3 = \\ \int d^N\xi \int d^N\bar{\xi}\mathcal{N}\langle\delta(\phi^1-\bar{\xi}^1)\ldots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\ldots\delta(\phi^1-\xi^1)\rangle_\Psi \\ (i\lambda)^3((\xi^z)^{12}-(\bar{\xi^z})^{12}-3(\xi^z)^8(\bar{\xi}^z)^4+3(\xi^z)^4(\bar{\xi}^z)^8 = \\ \mathcal{N}(i\lambda)^2(\langle(\phi^z)^8\rangle_\Psi - \langle(\phi^z)^8\rangle_\Psi - 3\langle(\phi^z)^{12}\rangle_\Psi + 3\langle(\phi^z)^{12}\rangle_\Psi) = 0\end{aligned} \tag{3.32}$$

Hence, $\mathcal{N}$=1
Let us now consider *two interaction points z and x* with the same coupling constant λ and expand.

For now we choose $x \prec z$
Order λ^0 $\mathcal{N}$=1
Order λ^1

$$\begin{aligned}\int d^N\xi \int d^N\bar{\xi}\ \mathcal{N}\langle\delta(\phi^1-\bar{\xi}^1)\delta(\phi^2-\bar{\xi}^2)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots\delta(\phi^2-\xi^2)\delta(\phi^1-\xi^1)\rangle_\Psi\\ (i\lambda(\xi^z)^4-i\lambda(\bar{\xi}^z)^4+i\lambda(\xi^x)^4-i\lambda(\bar{\xi}^x)^4)=\\ \mathcal{N}i\lambda(\langle\delta(\phi^1-\bar{\xi}^1)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots(\phi^z)^4\delta(\phi^z-\xi^z)\delta(\phi^1-\xi^1)\rangle_\Psi-\\ -\langle\delta(\phi^1-\bar{\xi}^1)(\phi^x)^4\delta(\phi^z-\xi^z)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots\delta(\phi^1-\xi^1)\rangle_\Psi+\\ +\langle\delta(\phi^1-\bar{\xi}^1)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots(\phi^x)^4\delta(\phi^x-\xi^x)\delta(\phi^1-\xi^1)\rangle_\Psi-\\ -\langle\delta(\phi^1-\bar{\xi}^1)(\phi^x)^4\delta(\phi^x-\xi^x)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots\delta(\phi^1-\xi^1)\rangle_\Psi=\\ \mathcal{N}i\lambda\ 3(W^{zz})^2-3(W^{zz})^2+3(W^{xx})^2-3(W^{xx})^2=0\end{aligned} \tag{3.33}$$

Hence, $\mathcal{N}$=1
Order λ^2

$$\begin{aligned}\mathcal{N}(i\lambda)^2\int d^N\xi\int d^N\bar{\xi}\ \langle\delta(\phi^1-\bar{\xi}^1)\delta(\phi^2-\bar{\xi}^2)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots\delta(\phi^2-\xi^2)\delta(\phi^1-\xi^1)\rangle_\Psi\\ ((\xi^z)^4-(\bar{\xi}^z)^4+(\xi^x)^4-(\bar{\xi}^x)^4)^2=\\ \mathcal{N}(i\lambda)^2\int d^N\xi\int d^N\bar{\xi}\langle\delta(\phi^1-\bar{\xi}^1)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots\delta(\phi^1-\xi^1)\rangle_\Psi\\ ((\xi^z)^8+(\bar{\xi}^z)^8+(\xi^x)^8+(\bar{\xi}^x)^8+2(\xi^z)^4(\xi^x)^4+2(\bar{\xi}^z)^4(\bar{\xi}^x)^4-2(\xi^z)^4(\bar{\xi}^x)^4-2(\xi^x)^4(\bar{\xi}^z)^4)=\\ \mathcal{N}(i\lambda)^2(\langle\delta(\phi^1-\bar{\xi}^1)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots(\phi^z)^8\delta(\phi^z-\xi^z)\delta(\phi^1-\xi^1)\rangle_\Psi+\\ +\langle\delta(\phi^1-\bar{\xi}^1)(\phi^z)^8\delta(\phi^z-\bar{\xi}^z)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots\delta(\phi^1-\xi^1)\rangle_\Psi+\\ +\langle\delta(\phi^1-\bar{\xi}^1)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots(\phi^x)^8\delta(\phi^x-\xi^x)\delta(\phi^1-\xi^1)\rangle_\Psi+\\ +\langle\delta(\phi^1-\bar{\xi}^1)(\phi^x)^8\delta(\phi^x-\bar{\xi}^x)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots\delta(\phi^1-\xi^1)\rangle_\Psi+\\ +2\langle\delta(\phi^1-\bar{\xi}^1)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)(\phi^z)^4\delta(\phi^z-\xi^z)\dots(\phi^x)^4\delta(\phi^x-\xi^x)\delta(\phi^1-\xi^1)\rangle_\Psi+\\ +2\langle\delta(\phi^1-\bar{\xi}^1)(\phi^x)^4\delta(\phi^x-\bar{\xi}^x)(\phi^z)^4\delta(\phi^z-\bar{\xi}^z)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots\delta(\phi^1-\xi^1)\rangle_\Psi-\\ -2\langle\delta(\phi^1-\bar{\xi}^1)(\phi^x)^4\delta(\phi^x-\bar{\xi}^x)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots(\phi^z)^4\delta(\phi^z-\xi^z)\delta(\phi^1-\xi^1)\rangle_\Psi\\ -2\langle\delta(\phi^1-\bar{\xi}^1)(\phi^z)^4\delta(\phi^z-\bar{\xi}^z)\dots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\dots(\phi^x)^4\delta(\phi^x-\xi^x)\delta(\phi^1-\xi^1)\rangle_\Psi=\\ \mathcal{N}\ (i\lambda)^2 105(W^{zz})^4+105(W^{zz})^4+105(W^{xx})^4+105(W^{xx})^4+2[9(W^{zz})^2(W^{xx})^2+24(W^{zx})^2]+\\ +2[9(W^{zz})^2(W^{xx})^2+24(W^{xz})^2]-2[9(W^{zz})^2(W^{xx})^2+24(W^{xz})^2]-2[9(W^{zz})^2(W^{xx})^2+24(W^{zx})^2]=0\end{aligned} \tag{3.34}$$

Hence, $\mathcal{N}$=1
Order λ^3

$$\mathcal{N}(i\lambda)^3\int d^N\xi\int d^N\bar{\xi}\langle\delta(\phi^1-\bar{\xi}^1)\delta(\phi^2-\bar{\xi}^2)\ldots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\ldots\delta(\phi^2-\xi^2)\delta(\phi^1-\xi^1)\rangle_\Psi$$
$$((\xi^z)^4-(\bar{\xi}^z)^4+(\xi^x)^4-(\bar{\xi}^x)^4)^3=$$
$$\mathcal{N}(i\lambda)^3\int d^N\xi\int d^N\bar{\xi}\langle\delta(\phi^1-\bar{\xi}^1)\ldots\delta(\phi^N-\bar{\xi}^N)\delta(\phi^N-\xi^N)\ldots\delta(\phi^1-\xi^1)\rangle_\Psi$$
$$((\xi^x)^{12}+(\xi^z)^8(\xi^x)^4+(\bar{\xi}^x)^8(\xi^x)^4+(\bar{\xi}^z)^8(\xi^x)^4+2(\xi^x)^8(\xi^z)^4-2(\xi^x)^8(\bar{\xi}^z)^4$$
$$-2(\xi^x)^8(\bar{\xi}^x)^4-2(\xi^z)^4(\bar{\xi}^x)^4(\xi^x)^4-2(\xi^z)^4(\bar{\xi}^z)^4(\xi^x)^4+2(\xi^x)^4(\bar{\xi}^x)^4(\bar{\xi}^z)^4)+$$
$$+((\xi^z)^{12}+(\xi^x)^8(\xi^z)^4+(\bar{\xi}^x)^8(\xi^z)^4+(\bar{\xi}^z)^8(\xi^z)^4+2(\xi^z)^8(\xi^x)^4-2(\xi^x)^4(\bar{\xi}^z)^4(\xi^z)^4$$
$$-2(\xi^x)^4(\xi^z)^4(\bar{\xi}^x)^4-2(\xi^z)^8(\bar{\xi}^x)^4-2(\xi^z)^8(\bar{\xi}^z)^4+2(\xi^z)^4(\bar{\xi}^x)^4(\bar{\xi}^z)^4)+$$
$$(+(\xi^x)^{12}+(\xi^z)^8(\xi^x)^4+(\bar{\xi}^x)^8(\xi^x)^4+(\bar{\xi}^z)^8(\xi^x)^4+2(\xi^x)^8(\xi^z)^4-2(\xi^x)^8(\bar{\xi}^z)^4$$
$$-2(\xi^x)^8(\bar{\xi}^x)^4-2(\xi^z)^4(\bar{\xi}^x)^4(\xi^x)^4-2(\xi^z)^4(\bar{\xi}^z)^4(\xi^x)^4+2(\xi^x)^4(\bar{\xi}^x)^4(\bar{\xi}^z)^4)-$$
$$-((\xi^x)^8(\bar{\xi}^x)^4+(\xi^z)^8(\bar{\xi}^x)^4+(\bar{\xi}^x)^{12}+(\bar{\xi}^z)^8(\bar{\xi}^x)^4+2(\xi^z)^4(\xi^x)^4(\bar{\xi}^x)^4-2(\xi^x)^4(\bar{\xi}^z)^4(\bar{\xi}^x)^4$$
$$-2(\xi^x)^4(\bar{\xi}^x)^8-2(\xi^z)^4(\bar{\xi}^x)^8-2(\xi^z)^4(\bar{\xi}^z)^4(\bar{\xi}^x)^4+2(\bar{\xi}^x)^8(\bar{\xi}^z)^4)$$
$$-((\xi^x)^8(\bar{\xi}^z)^4+(\xi^z)^8(\bar{\xi}^z)^4+(\bar{\xi}^x)^8(\bar{\xi}^z)^4+(\bar{\xi}^z)^{12}+2(\xi^z)^4(\xi^x)^4(\bar{\xi}^z)^4-2(\xi^x)^4(\bar{\xi}^z)^8$$
$$-2(\xi^x)^4(\bar{\xi}^x)^4(\bar{\xi}^z)^4-2(\xi^z)^4(\bar{\xi}^x)^4(\bar{\xi}^z)^4-2(\xi^z)^4(\bar{\xi}^z)^4(\bar{\xi}^z)^4+2(\bar{\xi}^z)^8(\bar{\xi}^x)^4)=$$
$$\mathcal{N}(i\lambda)^3(\langle(\phi^x)^{12}\rangle_\Psi+\langle(\phi^z)^8(\phi^x)^4\rangle_\Psi+\langle(\phi^x)^{12}\rangle_\Psi+\langle(\phi^z)^8(\phi^x)^4\rangle_\Psi+2\langle(\phi^z)^4(\phi^x)^8\rangle_\Psi-2\langle(\phi^z)^4(\phi^x)^8\rangle_\Psi-$$
$$-2\langle(\phi^x)^{12}\rangle_\Psi-2\langle(\phi^x)^4(\phi^z)^4(\phi^x)^4\rangle_\Psi-2\langle(\phi^z)^8(\phi^x)^4\rangle_\Psi+2\langle(\phi^x)^4(\phi^z)^4(\phi^x)^4\rangle_\Psi+$$
$$\langle(\phi^z)^{12}\rangle_\Psi+\langle(\phi^z)^4(\phi^x)^8\rangle_\Psi+\langle(\phi^x)^8(\phi^z)^4\rangle_\Psi+\langle(\phi^z)^{12}\rangle_\Psi+2\langle(\phi^z)^8(\phi^x)^4\rangle_\Psi-2\langle(\phi^z)^8(\phi^x)^4\rangle_\Psi-$$
$$-2\langle(\phi^x)^4(\phi^z)^4(\phi^x)^4\rangle_\Psi-2\langle(\phi^x)^4(\phi^z)^8\rangle_\Psi-2\langle(\phi^z)^{12}\rangle_\Psi+2\langle(\phi^x)^4(\phi^z)^8\rangle_\Psi-$$
$$\langle(\phi^x)^{12}\rangle_\Psi+\langle(\phi^x)^4(\phi^z)^8\rangle_\Psi+\langle(\phi^x)^{12}\rangle_\Psi+\langle(\phi^x)^4(\phi^z)^8\rangle_\Psi+2\langle(\phi^z)^4(\phi^x)^4(\phi^z)^4\rangle_\Psi-2\langle(\phi^z)^4(\phi^x)^4(\phi^z)^4\rangle_\Psi-$$
$$-2\langle(\phi^x)^{12}\rangle_\Psi-2\langle(\phi^x)^8(\phi^z)^4\rangle_\Psi-2\langle(\phi^x)^4(\phi^z)^8\rangle_\Psi+2\langle(\phi^x)^8(\phi^z)^4\rangle_\Psi-$$
$$-\langle(\phi^z)^4(\phi^x)^8\rangle_\Psi+\langle(\phi^z)^{12}\rangle_\Psi+\langle(\phi^x)^8(\phi^z)^4\rangle_\Psi+\langle(\phi^z)^{12}\rangle_\Psi+2\langle(\phi^z)^8(\phi^x)^4\rangle_\Psi-2\langle(\phi^z)^8(\phi^x)^4\rangle_\Psi-$$
$$-2\langle(\phi^x)^4(\phi^z)^4(\phi^x)^4\rangle_\Psi-2\langle(\phi^x)^4(\phi^z)^8\rangle_\Psi-2\langle(\phi^z)^{12}\rangle_\Psi+2\langle(\phi^x)^4(\phi^z)^8\rangle_\Psi=0 \tag{3.35}$$

Hence, $\mathcal{N}$=1.

Lemma 3.4.1. *The interacting decoherence functional is already normalized for all orders in $\underline{\Lambda}$, i.e $\mathcal{N}$=1.*

Proof. Let us consider the expression of the decoherence functional as

$$\int d^N\xi\int d^N\bar{\xi}\;D(\xi,\bar{\xi}|\Psi,\underline{\Lambda})=\langle O^\dagger O\rangle_\Psi, \tag{3.36}$$

where O is the time ordered operator

$$O = \int d^N\xi \; \delta(\phi^N - \xi^N)\dots\delta(\phi^2 - \xi^2)\delta(\phi^1 - \xi^1)\; e^{i\underline{\Lambda}\cdot\underline{\xi}^4}, \tag{3.37}$$

while $O^\dagger$ is the anti-time ordered operator

$$O^\dagger = \int d^N\bar{\xi} \; \delta(\phi^1 - \bar{\xi}^1)\delta(\phi^2 - \bar{\xi}^2)\dots\delta(\phi^N - \bar{\xi}^N)\; e^{-i\underline{\Lambda}\cdot\underline{\bar{\xi}}^4}. \tag{3.38}$$

In order to prove our conjecture, we need to show that as well as being the adjoint of each other, they are also inverse of each other. Applying the delta function, let us rewrite O as

$$\begin{aligned} O = \mathcal{T}e^{i(\lambda_N\phi_N^4+\dots+\lambda_1\phi_1^4)} = \mathcal{T}\lim_{m\to+\infty} e^{i(\lambda_N\phi_N^{4/m}+\dots+\lambda_1\phi_1^{4/m})^m} = \\ = \lim_{m\to+\infty}\mathcal{T}e^{i\lambda_N\phi_N^{4/m}}\dots e^{i\lambda_1\phi_1^{4/m}} \overset{\text{m-2 times}}{\dots} e^{i\lambda_N\phi_N^{4/m}}\dots e^{i\lambda_1\phi_1^{4/m}} \\ = \lim_{m\to+\infty} e^{i\lambda_N\phi_N^4}\dots e^{i\lambda_1\phi_1^4}, \end{aligned} \tag{3.39}$$

using that the time order and the limit commute. In the same way, working with the anti-time ordered operator we get

$$O^\dagger = \mathcal{T}^{-1}e^{-i(\lambda_1\phi_1^4+\dots+\lambda_N\phi_N^4)} = e^{-i\lambda_1\phi_1^4}\dots e^{-i\lambda_N\phi_N^4}. \tag{3.40}$$

Hence, the resulting expression is

$$\langle O^\dagger O\rangle_\Psi = \langle e^{-i\lambda_1\phi_1^4}\dots e^{-i\lambda_N\phi_N^4}e^{i\lambda_N\phi_N^4}\dots e^{i\lambda_1\phi_1^4}\rangle_\Psi = \mathbf{1}, \tag{3.41}$$

where **1** is the identity operator, highlighting that the interacting decoherence functional is already normalized. □

3.5 Interacting k-point functions

In the interacting case, we define the interacting Feynman k-point function replacing $D(\xi,\bar{\xi}|\Psi)$ in 3.17 by the interacting $D(\xi,\bar{\xi}|\Psi,\underline{\Lambda})$:

$$-iG_{\Lambda F}(x_1,x_2,\dots,x_k) := \int d^N\xi\int d^N\bar{\xi}\; D(\xi,\bar{\xi}|\Psi,\underline{\Lambda})\;\xi^{x_1}\xi^{x_2}\dots\xi^{x_k} \tag{3.42}$$

Let us find an expression for ϕ_x^{int}, interacting operators, proceeding in the following way. Let $P(x) = \{y\in\mathcal{C}\,|\,y\prec x\}$ be the (noninclusive) past of x. $P(x)$ is a subcauset of $\mathcal{C}$ that is a *past set* or *down set*. i.e. $P(x)$ is a *stem* in $\mathcal{C}$; a stem is a finite subcausal set that contains its own past. Let R be the cardinality of $P(x)$ and choose a linear extension of $P(x)$: $y_1,y_2,y_3,\dots y_R$. Then, define

$$\phi_x^{int} := e^{-i\lambda_{y_1}(\phi_{y_1})^4}e^{-i\lambda_{y_2}(\phi_{y_2})^4}\dots e^{-i\lambda_{y_R}(\phi_{y_R})^4}\phi_x e^{i\lambda_{y_R}(\phi_{y_R})^4}\dots e^{i\lambda_{y_2}(\phi_{y_2})^4}e^{i\lambda_{y_1}(\phi_{y_1})^4}, \tag{3.43}$$

Lemma 3.5.1.

$$G_{\Lambda F}(x_1,x_2,\dots,x_k) = i\langle\mathcal{T}[\phi_{x_1}^{int}\phi_{x_2}^{int}\dots\phi_{x_k}^{int}]\rangle_\Psi, \tag{3.44}$$

where $G_{\Lambda F}$ is defined by 3.42 and ϕ_{int}^x by 3.43. The time ordering is on all operators in expression.

Proof. Consider a causet $(\mathcal{C}, \preceq)$ of cardinality N and some labelling $i = 1, \ldots, N$ consistent with the causal order, i.e. natural labelling. Then, define the interacting k-point function with both ξ and $\bar{\xi}$ as

$$G_{\Lambda F}(n_1, \ldots, n_N; m_1 \ldots m_N) := i \int d^N\xi \int d^N\bar{\xi} D(\xi, \bar{\xi}|\Psi, \underline{\Lambda})(\xi^1)^{n_1} \ldots (\xi^N)^{n_N} (\bar{\xi}^1)^{m_1} \ldots (\bar{\xi}^N)^{m_N}$$

$$= i \int d^N\xi \int d^N\bar{\xi} \ \langle \delta(\phi^1 - \bar{\xi}^1) \ldots \delta(\phi^N - \bar{\xi}^N) \delta(\phi^N - \xi^N) \ldots \delta(\phi^1 - \xi^1) \rangle_\Psi \ e^{(i\underline{\Lambda} \cdot \underline{\xi}^4 - i\underline{\Lambda} \cdot \underline{\bar{\xi}}^4)}$$
$$(\xi^1)^{n_1} \ldots (\xi^N)^{n_N} (\bar{\xi}^1)^{m_1} \ldots (\bar{\xi}^N)^{m_N} =$$
$$= i \int d^N\xi \int d^N\bar{\xi} \ \langle e^{-i\lambda_1(\bar{\xi}^1)^4} (\bar{\xi}^1)^{m_1} \delta(\phi^1 - \bar{\xi}^1) \ldots e^{-i\lambda_N(\bar{\xi}^N)^4} (\bar{\xi}^N)^{m_N} \delta(\phi^N - \bar{\xi}^N)$$
$$e^{i\lambda_N(\xi^N)^4} (\xi^N)^{n_N} \delta(\phi^N - \xi^N) \ldots e^{i\lambda_1(\xi^1)^4} (\xi^1)^{n_1} \delta(\phi^1 - \xi^1) \rangle_\Psi =$$
$$= i \langle e^{-i\lambda_1(\phi_1)^4} (\phi_1)^{m_1} \ldots e^{-i\lambda_N(\phi_N)^4} (\phi_N)^{m_N} e^{i\lambda_N(\phi_N)^4} (\phi_N)^{n_N} \ldots e^{i\lambda_1(\phi_1)^4} (\phi_1)^{n_1} \rangle_\Psi, \tag{3.45}$$

where the ϕ_x are free, "Heisenberg" operators.

Let $\phi_x^{int} := e^{-i\lambda_1(\phi_1)^4} e^{-i\lambda_2(\phi_2)^4} \ldots e^{-i\lambda_N(\phi_N)^4} \phi_x e^{i\lambda_N(\phi_N)^4} \ldots e^{i\lambda_2(\phi_2)^4} e^{i\lambda_1(\phi_1)^4}$.
We can notice that

$$(\phi_x^{int})^n = e^{-i\lambda_1(\phi_1)^4} \ldots e^{-i\lambda_N(\phi_N)^4} \phi_x e^{i\lambda_N(\phi_N)^4} \ldots e^{i\lambda_1(\phi_1)^4} \overset{\text{n-2 times}}{\ldots} e^{-i\lambda_1(\phi_1)^4} \ldots e^{-i\lambda_N(\phi_N)^4} \phi_x e^{i\lambda_N(\phi_N)^4} \ldots e^{i\lambda_1(\phi_1)^4} =$$
$$= e^{-i\lambda_1(\phi_1)^4} \ldots e^{-i\lambda_N(\phi_N)^4} (\phi_x)^n e^{i\lambda_N(\phi_N)^4} \ldots e^{i\lambda_1(\phi_1)^4}$$

Then,

$$\langle \mathcal{T}[(\phi_1^{int})^{n_1} (\phi_2^{int})^{n_2} \ldots (\phi_N^{int})^{n_N}] \rangle_\Psi =$$
$$= \langle \mathcal{T}[e^{-i\lambda_1(\phi_1)^4} \ldots e^{-i\lambda_N(\phi_N)^4} (\phi_1)^{n_1} e^{i\lambda_N(\phi_N)^4} \ldots e^{i\lambda_1(\phi_1)^4} \ldots$$
$$e^{-i\lambda_1(\phi_1)^4} \ldots e^{-i\lambda_N(\phi_N)^4} (\phi_N)^{n_N} e^{i\lambda_N(\phi_N)^4} \ldots e^{i\lambda_1(\phi_1)^4}] \rangle_\Psi =$$
$$= \langle \mathcal{T}[e^{-i\underline{\Lambda} \cdot \underline{\bar{\xi}}^4} (\phi_1)^{n_1} \ldots (\phi_N)^{n_N} e^{i\underline{\Lambda} \cdot \underline{\xi}^4}] \rangle_\Psi =$$
$$= \int d^N\xi \int d^N\bar{\xi} \ \langle e^{-i\lambda_1(\bar{\xi}^1)^4} \delta(\phi^1 - \bar{\xi}^1) \ldots e^{-i\lambda_N(\bar{\xi}^N)^4} \delta(\phi^N - \bar{\xi}^N)$$
$$e^{i\lambda_N(\xi^N)^4} (\xi^N)^{n_N} \delta(\phi^N - \xi^N) \ldots e^{i\lambda_1(\xi^1)^4} (\xi^1)^{n_1} \delta(\phi^1 - \xi^1) \rangle_\Psi =$$
$$= \langle e^{-i\lambda_1(\phi_1)^4} \ldots e^{-i\lambda_N(\phi_N)^4} (\phi_N)^{n_N} e^{i\lambda_N(\phi_N)^4} \ldots (\phi_1)^{n_1} e^{i\lambda_1(\phi_1)^4} \rangle_\Psi =$$
$$= -i \ G_{\Lambda F}(n_1, \ldots, n_N) \tag{3.46}$$

It is important to notice that the product of field operators $\phi_1\phi_2\dots\phi_N$ does not depend on the choice of natural labelling. This means that if we choose another natural labelling and write down the product of field operators in this new total (linear) order, this new product will equal the original one due to spacelike commutativity.
A property of partial orders is that if causet A is a stem of causet B, then there is a natural labelling of causet B that labels the elements of the stem A first and then carries on labelling the other elements of B. This is fundamental to prove our conjecture.
Label the elements of stem A following $1\dots R$ and the ones of B with $1\dots R\dots N$
Define

$$\phi_x^{int} := e^{-i\lambda_1(\phi_1)^4} e^{-i\lambda_2(\phi_2)^4} \dots e^{-i\lambda_R(\phi_R)^4} \phi_x e^{i\lambda_R(\phi_R)^4} \dots e^{i\lambda_2(\phi_2)^4} e^{i\lambda_1(\phi_1)^4}.$$

Again, we use that

$$(\phi_x^{int})^n = e^{-i\lambda_1(\phi_1)^4} \dots e^{-i\lambda_R(\phi_R)^4} (\phi_x)^n e^{i\lambda_R(\phi_R)^4} \dots e^{i\lambda_1(\phi_1)^4}$$

Then,

$$\begin{aligned}
&\langle \mathcal{T}[(\phi_{R+1}^{int})^{n_{R+1}} (\phi_{R+2}^{int})^{n_{R+2}} \dots (\phi_N^{int})^{n_N}]\rangle_\Psi = \\
&= \langle \mathcal{T}[e^{-i\lambda_1(\phi_1)^4} \dots e^{-i\lambda_R(\phi_R)^4} (\phi_{R+1})^{n_{R+1}} e^{i\lambda_R(\phi_R)^4} \dots e^{i\lambda_1(\phi_1)^4} \\
&\quad e^{-i\lambda_1(\phi_1)^4} \dots e^{-i\lambda_R(\phi_R)^4} (\phi_{R+2})^{n_{R+2}} e^{i\lambda_R(\phi_R)^4} \dots e^{i\lambda_1(\phi_1)^4} \dots \\
&\quad e^{-i\lambda_1(\phi_1)^4} \dots e^{-i\lambda_R(\phi_R)^4} (\phi_N)^{n_N} e^{i\lambda_R(\phi_R)^4} \dots e^{i\lambda_1(\phi_1)^4}]\rangle_\Psi = \\
&= \langle \mathcal{T}[exp(-i\underline{\Lambda}\cdot\underline{\bar{\xi}}^4)(\phi_{R+1})^{n_{R+1}}(\phi_{R+2})^{n_{R+2}} \dots (\phi_N)^{n_N} exp(i\underline{\Lambda}\cdot\underline{\xi}^4)]\rangle_\Psi = \\
&= \int d^N\xi \int d^N\bar{\xi}\ \langle e^{-i\lambda_1(\bar{\xi}^1)^4}\delta(\phi^1-\bar{\xi}^1)\dots e^{-i\lambda_R(\bar{\xi}^R)^4}\delta(\phi^R-\bar{\xi}^R)\dots\delta(\phi^N-\bar{\xi}^N)(\xi^N)^{n_N}\delta(\phi^N-\xi^N) \\
&\dots(\xi^{R+2})^{n_{R+2}}\delta(\phi^{R+2}-\xi^{R+2})(\xi^{R+1})^{n_{R+1}}\delta(\phi^{R+1}-\xi^{R+1})e^{i\lambda_R(\xi^R)^4}\delta(\phi^R-\xi^R)\dots e^{i\lambda_1(\xi^1)^4}\delta(\phi^1-\xi^1)\rangle_\Psi = \\
&= \langle e^{-i\lambda_1(\phi_1)^4} \dots e^{-i\lambda_R(\phi_R)^4} (\phi_N)^{n_N} \dots (\phi_{R+2})^{n_{R+2}} (\phi_{R+1})^{n_{R+1}} e^{i\lambda_R(\phi_R)^4} \dots e^{i\lambda_1(\phi_1)^4}\rangle_\Psi = \\
&= -i\ G_{\Lambda F}(n_{R+1},\dots,n_N)
\end{aligned} \tag{3.47}$$

□

Lemma 3.5.2. *Only interaction points in the causal past of the causal set $\mathcal{C}$ influence the free propagation.*

Proof. Given the background causal set $\mathcal{C}$ with k elements, let us define $F(x) = \{y \in \mathcal{C}\,|\, y \succ x\}$ which is the (noninclusive) future of x with cardinality R and $P(x) = \{y \in \mathcal{C}\,|\, y \prec x\}$ which is the (noninclusive) past of x , with cardinality N- R. Time ordering is applied to all labels, $y_1 \succ \dots \succ y_R \succ x_1 \dots \succ x_k \succ y_{R+1} \dots \succ y_N$ Start with defining interacting operators including the future interaction points as

$$\phi_x^{int} := e^{-i\lambda_{y_1}(\phi_{y_1})^4} e^{-i\lambda_{y_2}(\phi_{y_2})^4} \dots e^{-i\lambda_{y_N}(\phi_{y_N})^4} \phi_x e^{i\lambda_{y_N}(\phi_{y_N})^4} \dots e^{i\lambda_{y_2}(\phi_{y_2})^4} e^{i\lambda_{y_1}(\phi_{y_1})^4}.$$

$$G_{\Lambda F}(x_1, x_2, \ldots, x_k) = i\langle \mathcal{T}[\phi^{int}_{x_1}\phi^{int}_{x_2}\ldots\phi^{int}_{x_k}]\rangle_\Psi =$$

$$= i\langle \mathcal{T} e^{-i\lambda_{y_N}(\phi_{y_N})^4}\ldots e^{-i\lambda_{y_1}(\phi_{y_1})^4}\phi_{x_1}e^{i\lambda_{y_1}(\phi_{y_1})^4}\ldots e^{i\lambda_{y_N}(\phi_{y_N})^4}$$
$$e^{-i\lambda_{y_N}(\phi_{y_N})^4}\ldots e^{-i\lambda_{y_1}(\phi_{y_1})^4}\phi_{x_2}e^{i\lambda_{y_1}(\phi_{y_1})^4}\ldots e^{i\lambda_{y_N}(\phi_{y_N})^4}\ldots$$
$$e^{-i\lambda_{y_N}(\phi_{y_N})^4}\ldots e^{-i\lambda_{y_1}(\phi_{y_1})^4}\phi_{x_k}e^{i\lambda_{y_1}(\phi_{y_1})^4}\ldots e^{i\lambda_{y_N}(\phi_{y_N})^4}\rangle_\Psi =$$

$$= i\langle e^{-i\lambda_{y_N}(\phi_{y_N})^4}\ldots e^{-i\lambda_{y_1}(\phi_{y_1})^4}e^{i\lambda_{y_1}(\phi_{y_1})^4}\ldots e^{i\lambda_{y_R}(\phi_{y_R})^4}\phi_{x_1}\phi_{x_2}\ldots\phi_{x_k}e^{i\lambda_{y_{R+1}}(\phi_{y_{R+1}})^4}\ldots e^{i\lambda_{y_N}(\phi_{y_N})^4}\rangle_\Psi =$$

$$= i\langle e^{-i\lambda_{y_N}(\phi_{y_N})^4}\ldots e^{-i\lambda_{y_{R+1}}(\phi_{y_{R+1}})^4}\phi_{x_1}\phi_{x_2}\ldots\phi_{x_k}e^{i\lambda_{y_{R+1}}(\phi_{y_{R+1}})^4}\ldots e^{i\lambda_{y_N}(\phi_{y_N})^4}\rangle_\Psi \tag{3.48}$$

Indeed, the definition for ϕ^{int}_x is 3.43, where the only relevant interaction points are the one in the past of x. □

3.6 Series truncation

Lemma 3.6.1. *The number of contributions to the k-point function is finite, i.e there is a cut off in the perturbation expansion of the interaction terms.*

Proof. We start with treating the interaction as a perturbation expansion, considering an application of Baker-Campbell-Hausdorff (BCH) Formulae [Meh13]: given two operators A and B

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \ldots \frac{1}{n!}\underbrace{[A,[A,\ldots[A}_{\text{n A's}},B]..] + \ldots \tag{3.49}$$

For instance, for a weak interaction at y,

$$e^{-i\lambda\phi^n_y}\phi_x e^{i\lambda\phi^n_y} = \phi_x + [\phi_x, i\lambda\phi^n_y]$$
$$= \phi_x + n\ i\lambda i\Delta_{xy}\phi^{n-1}_y,$$

and by inserting $1 = e^{-i\lambda\phi^n_y}e^{i\lambda\phi^n_y}$ in between each ϕ_x, we have

$$e^{-i\lambda\phi^n_y}\phi^m_x e^{i\lambda\phi^n_y} = (\phi_x + n\ i\lambda i\Delta_{xy}\phi^{n-1}_y)^m\ . \tag{3.50}$$

Let us label the top point as 0, and points below as 1, 2, etc, in the opposite order to the causal order, i.e. $0 \succ 1 \succ 2 \succ 3...$, assuming that the coupling constant is λ through all the interaction region. Then, we want to evaluate the highest power of λ in

$$U^{-1}_{1,r}\phi_0 U_{1,r}\ , \tag{3.51}$$

where

$$U_{1,r} = e^{i\lambda\phi^n_r}e^{i\lambda\phi^n_{r-1}}...e^{i\lambda\phi^n_1}\ , \tag{3.52}$$

for $r > 1$, so that r indicates the sublevel below 0 where the earliest interaction can happen. Consider the first case, i.e. $r = 1$. We have

$$\begin{aligned} U_{1,1}^{-1}\phi_0 U_{1,1} &= n\ i\lambda i\Delta_{01}\phi_1^{n-1}... \\ &= O(\lambda)O(\phi^{n-1}) + ... \ . \end{aligned} \tag{3.53}$$

where "..." denotes all other terms involving smaller powers of λ. Now consider $r = 2$:

$$\begin{aligned} U_{1,2}^{-1}\phi_0 U_{1,2} &= n\ i\lambda i\Delta_{01}(U_{2,2}^{-1}\phi_1 U_{2,2})^{n-1}... \\ &= n\ i\lambda i\Delta_{01}(n\ i\lambda i\Delta_{12}\phi_2^{n-1} + ...)^{n-1} + ... \\ &= O(\lambda^{1+(n-1)})O(\phi^{(n-1)^2}) + ... \ . \end{aligned} \tag{3.54}$$

Next, for $r = 3$, we get

$$\begin{aligned} U_{1,3}^{-1}\phi_0 U_{1,3} &= U_{3,3}^{-1}(n\ i\lambda i\Delta_{01}(n\ i\lambda i\Delta_{12}\phi_2^{n-1} + ...)^{n-1})U_{3,3} + ... \\ &= n\ i\lambda i\Delta_{01}(n\ i\lambda i\Delta_{12}(U_{3,3}^{-1}\phi_2 U_{3,3})^{n-1} + ...)^{n-1} + ... \\ &= n\ i\lambda i\Delta_{01}(n\ i\lambda i\Delta_{12}(n\ i\lambda i\Delta_{23}\phi_3^{n-1} + ...)^{n-1} + ...)^{n-1} + ... \\ &= O(\lambda^{1+(n-1)+(n-1)^2})O(\phi^{(n-1)^3}) + ... \ . \end{aligned} \tag{3.55}$$

Hence, it should now be clear that the powers of λ follow the sum

$$S_{r,n} = \sum_{k=0}^{r-1}(n-1)^k = \frac{(n-1)^r - 1}{n-2} \ . \tag{3.56}$$

As a result, we obtained a formula to evaluate the highest power of λ for an interaction region with r sublevels, i.e.

$$U_{1,r}^{-1}\phi_0 U_{1,r} = O(\lambda^{S_{r,n}}) \ . \tag{3.57}$$

highlighting the order for which the expansion of ϕ^n theory truncates. The value of $S_{r,n}$ may be odd or even depending on n and r. If the interacting field is combined in some correlation function with some other fields, or some other power of fields, such that the highest order term has an odd number of fields in total, then that term will vanish under the expectation value. In this case, subtracting 1 off $S_{r,n}$ is necessary to calculate the highest power of λ present in the given correlation function.
For this subtraction of 1 to be legitimate, we need to know that a term of $O(\lambda^{S_{r,n}-1})$ appears in $U_{1,r}^{-1}\phi_0 U_{1,r}$. If there is no term, then no such term will be present under the expectation value. To see that there is such a term, consider the $r = 3$ example above:

$$\begin{aligned} U_{1,3}^{-1}\phi_0 U_{1,3} &= n\ i\lambda i\Delta_{01}(n\ i\lambda i\Delta_{12}(n\ i\lambda i\Delta_{23}\phi_3^{n-1} + ...)^{n-1} + ...)^{n-1} + ... \\ &= c\,\lambda^{1+(n-1)}(n\ i\lambda i\Delta_{23}\phi_3^{n-1} + \phi_2)^{(n-1)^2} + ... \ , \end{aligned} \tag{3.58}$$

for some constant c. For any r we have

$$U_{1,r}^{-1}\phi_0 U_{1,r} = c\,\lambda^{S_{r-1,n}}(c'\,\lambda\phi_r^{n-1} + \phi_{r-1})^{(n-1)^{r-1}} + ... \ , \tag{3.59}$$

for constants c and c'. The (non-commutative) binomial expansion of $(c'\,\lambda\phi_r^{n-1} + \phi_{r-1})^{(n-1)^{r-1}}$ on the RHS gives terms of order $O(1)$ up to $O(\lambda^{(n-1)^{r-1}})$. This includes a term of $O(\lambda^{(n-1)^{r-1}-1})$, which when combined with the powers of λ outside the brackets gives our desired order of $O(\lambda^{S_{r,n}-1})$. □

Then, taking the difference of consecutive powers as $S_{r,n} - S_{r-1,n}$, gives the maximum power of λ associated to the interaction point at the r-sublevel, i.e the maximum number of interactions at that vertex.

3.7 Generetaing functional

In the continuum theory, the source term is introduced into the action in the path integral expression, the single path integral, to define a generating functional of the source function. Then, k-functional derivatives of the generating functional w.r.t. the source give the "Feynman k-point functions": the free k-point function if there is no interaction term in the action and the interacting k-point function if there is an interaction. Now, we want to recover a similar structure in a discrete and finite framework, given a causet $(\mathcal{C}, \preceq)$ of cardinality N . We introduce two independent sources for ξ and $\bar{\xi}$, $\underline{J} = (J_1, J_2, \ldots J_N)$ and $\underline{\bar{J}} = (\bar{J}_1, \bar{J}_2, \ldots, \bar{J}_N)$, so that the decoherence functional gets multiplied by $e^{i\underline{J}\cdot\underline{\xi} - i\underline{\bar{J}}\cdot\underline{\bar{\xi}}}$. Then, the resulting object is a generating functional but *not* a decoherence functional defined as:

$$Z(\underline{J}, \underline{\bar{J}} \,|\, \Psi, \underline{\Lambda}) := \int d^N\xi \int d^N\bar{\xi} D(\xi, \bar{\xi} \,|\, \underline{\Lambda}) e^{i\underline{J}\cdot\underline{\xi}} e^{-i\underline{\bar{J}}\cdot\underline{\bar{\xi}}} = \\ \langle e^{-i\lambda_1(\phi_1)^4} e^{-i\bar{J}_1\phi_1} \ldots e^{-i\lambda_N(\phi_N)^4} e^{-i\bar{J}_N\phi_N} e^{i\lambda_N(\phi_N)^4} e^{iJ_N\phi_N} \ldots e^{i\lambda_1(\phi_1)^4} e^{iJ_1\phi_1} \rangle_\Psi. \tag{3.60}$$

We can note that since we showed that $D(\xi, \bar{\xi} \,|\, \Psi, \underline{\Lambda})$ is normalised, there is no need to divide the generating functional by $Z(0, 0)$.
Then, a derivative with respect to iJ_x brings down a ξ^x and w.r.t. $-i\bar{J}_x$ brings down a $\bar{\xi}$. Proceeding in this way, we can generate the Feynman k-point functions taking k J derivatives. Specifically, the expression 3.45 can be recovered up to a factor of i as one takes n_i derivatives w.r.t to iJ_i, and m_i derivatives w.r.t $-i\bar{J}_i$, and then sets all the J-parameters to 0.

Chapter 4

Evaluation

4.1 Two-point function

Now, we evaluate the two-point function

$$\langle \mathcal{T}\phi_x^{int}\phi_y^{int}\rangle_\Psi = \int d^N\xi \int d^N\bar{\xi}\ D(\xi,\bar{\xi}|\underline{\Delta})\xi^x\xi^y \tag{4.1}$$

as a perturbation series in λ and try to interpret the terms in the series as sums of Lorentzian spacetime "Feynman diagrams", using Wick's theorem.
Let us consider 3 interaction points totally ordered such as $x \prec 1 \prec 2 \prec 3 \prec y$, assuming that the interaction points are in the causal past of y and not in the causal past of x. As we fix the coupling constant to be equal in the interaction region, the expression of the two-point function becomes

$$\langle \mathcal{T}\phi_x^{int}\phi_y^{int}\rangle_\Psi = \langle e^{-i\lambda_1\phi_1^4}e^{-i\lambda_2\phi_2^4}e^{-i\lambda_3\phi_3^4}\phi_y e^{i\lambda_3\phi_3^4}e^{i\lambda_2\phi_2^4}e^{i\lambda_1\phi_1^4}\phi_x\rangle_\Psi. \tag{4.2}$$

Now, to treat it perturbatively, we can use the application of BCH 3.49. Then, the following commutator relation from [Pai12] will be frequently applied in the calculations :

$$[X_i^n, Y_j^m] = -\sum_{k=1}^{min(n,m)} \frac{(c_{ij})^k n!m!}{k!(m-k)!(n-k)!} X_i^{n-k}Y_j^{m-k}, \tag{4.3}$$

where in our evaluation $c_{ij} = -i\Delta_{ij}$. In particular, useful forms are:

$$\begin{aligned}[][\phi_i^3, \phi_j^4] &= 12i\Delta_{ij}\phi_i^2\phi_j^3 - 36(i\Delta_{ij})^2\phi_i\phi_j^2 + 24(i\Delta_{ij})^3\phi_j, \\ [\phi_i^2, \phi_j^4] &= 8i\Delta_{ij}\phi_i\phi_j^3 - 12(i\Delta_{ij})^2\phi_j^2.\end{aligned} \tag{4.4}$$

Finally, we notice that as we choose the order $j \prec i$, $\Delta_{ij} = K_{ij}^R$ since $K_{ij}^A = 0$, contributing as a retarded propagator.
Hence, we are ready to start our evaluation with ϕ_3 interaction, first commutator,

$$[\phi_y, (i\lambda)\phi_3^4] = 4(i\lambda)i\Delta_{y3}\phi_3^3 \tag{4.5}$$

Second term for ϕ_3 interaction

$$[4(i\lambda)i\Delta_{y3}\phi_3^3, (i\lambda)\phi_3^4] = 0. \tag{4.6}$$

First contribution for ϕ_2 interaction, using 4.4

$$[\phi_y + 4(i\lambda)i\Delta_{y3}\phi_3^3, (i\lambda)\phi_2^4] = 4(i\lambda)i\Delta_{y2}\phi_2^3 + 4(i\lambda)^2 i\Delta_{y3}(12i\Delta_{32}\phi_3^2\phi_2^3 - 36(i\Delta_{32})^2\phi_3\phi_2^2 + 24(i\Delta_{32})^3\phi_2). \tag{4.7}$$

Second commutator for ϕ_2 interaction, using 4.4

$$\begin{aligned}\frac{1}{2}[4(i\lambda)i\Delta_{y2}\phi_2^3 + 4(i\lambda)^2 i\Delta_{y3}(12i\Delta_{32}\phi_3^2\phi_2^3 - 36(i\Delta_{32})^2\phi_3\phi_2^2 + 24(i\Delta_{32})^3\phi_2), (i\lambda)\phi_2^4] = \\ \frac{1}{2}4(i\lambda)^3 i\Delta_{y3}(12i\Delta_{32}[\phi_3^2, \phi_2^4]\phi_2^3 - 36(i\Delta_{32})^2[\phi_3, \phi_2^4]\phi_2^2) = \\ -\frac{1}{2}4(i\lambda)^3 i\Delta_{y3}(-12\ 8\ (i\Delta_{32})^2\phi_3\phi_2^6 + 12\ 12\ (i\Delta_{32})^3\phi_2^5 + 36\ 4\ (i\Delta_{32})^3\phi_2^5).\end{aligned} \tag{4.8}$$

Third term for ϕ_2 interaction

$$\frac{1}{3!}4\ 12\ 8(i\lambda)^4 i\Delta_{y3}(i\Delta_{32})^2[\phi_3, \phi_2^4]\phi_2^6 = \frac{1}{3!}4\ 12\ 8\ 4(i\lambda)^4 i\Delta_{y3}(i\Delta_{32})^3\phi_2^9. \tag{4.9}$$

Again, at the next iteration, the series truncate. First contribution for ϕ_1 interaction (only up to λ^3), estimating term by term

$$\begin{aligned}[\phi_y + 4(i\lambda)i\Delta_{y3}\phi_3^3 + 4(i\lambda)i\Delta_{y2}\phi_2^3, (i\lambda)\phi_1^4] = \\ 4(i\lambda)i\Delta_{y1}\phi_1^3 + 4(i\lambda)^2 i\Delta_{y3}(12i\Delta_{31}\phi_3^2\phi_1^3 - 36(i\Delta_{31})^2\phi_3\phi_1^2 + \\ 24(i\Delta_{31})^3\phi_1) + 4(i\lambda)^2 i\Delta_{y2}(12i\Delta_{21}\phi_2^2\phi_1^3 - 36(i\Delta_{21})^2\phi_2\phi_1^2 + 24(i\Delta_{21})^3\phi_1),\end{aligned}$$

$$\begin{aligned}[4\ 12(i\lambda)^2 i\Delta_{y3}i\Delta_{32}\phi_3^2\phi_2^3, (i\lambda)\phi_1^4] = 4\ 12(i\lambda)^3 i\Delta_{y3}i\Delta_{32}([\phi_3^2, \phi_1^4]\phi_2^3 + \phi_3^2[\phi_2^3, \phi_1^4] = \\ -4\ 12(i\lambda)^3 i\Delta_{y3}i\Delta_{32}(-8i\Delta_{31}\phi_3\phi_1^3\phi_2^3 + 12(i\Delta_{31})^2\phi_1^2\phi_2^3 \\ -12i\Delta_{21}\phi_3^2\phi_2^2\phi_1^3 + 36(i\Delta_{21})^2\phi_3^2\phi_2\phi_1^2 - 24(i\Delta_{21})^3\phi_3^2\phi_1) = \\ -4\ 12(i\lambda)^3 i\Delta_{y3}i\Delta_{32}(-8i\Delta_{31}\phi_3([\phi_1^3, \phi_2^3] + \phi_2^3\phi_1^3) + \\ 12(i\Delta_{31})^2([\phi_1^2, \phi_2^3] + \phi_2^3\phi_1^2) - 12i\Delta_{21}\phi_3^2\phi_2^2\phi_1^3 + 36(i\Delta_{21})^2\phi_3^2\phi_2\phi_1^2 - 24(i\Delta_{21})^3\phi_3^2\phi_1) = \\ -4\ 12(i\lambda)^3 i\Delta_{y3}i\Delta_{32}(-8i\Delta_{31}\phi_3(-9i\Delta_{21}\phi_2^2\phi_1^2 + 18(i\Delta_{21})^2\phi_2\phi_1 - 6(i\Delta_{21})^3 + \phi_2^3\phi_1^3) + \\ 12(i\Delta_{31})^2(6(i\Delta_{21})^2\phi_2 - 6i\Delta_{21}\phi_2^2\phi_1 + \phi_2^3\phi_1^2) - 12i\Delta_{21}\phi_3^2\phi_2^2\phi_1^3 + 36(i\Delta_{21})^2\phi_3^2\phi_2\phi_1^2 - 24(i\Delta_{21})^3\phi_3^2\phi_1),\end{aligned}$$

$$\begin{aligned}[4(i\lambda)^2 i\Delta_{y3}(-36(i\Delta_{32})^2\phi_3\phi_2^2 + 24(i\Delta_{32})^3\phi_2), (i\lambda)\phi_1^4] = \\ 4(i\lambda)^3 i\Delta_{y3}(-36(i\Delta_{32})^2[\phi_3, \phi_1^4]\phi_2^2 - 36(i\Delta_{32})^2\phi_3[\phi_2^2, \phi_1^4] + 24(i\Delta_{32})^3[\phi_2, \phi_1^4]) = \\ 4(i\lambda)^3 i\Delta_{y3}(-36\ 4\ (i\Delta_{32})^2 i\Delta_{31}\phi_1^3\phi_2^2 - 36\ 8(i\Delta_)^2 i\Delta_{21}\phi_3\phi_2\phi_1^3 + \\ 12\ 36(i\Delta_{32})^2(i\Delta_{21})^2\phi_3\phi_1^2 + 24\ 4(i\Delta_{32})^3 i\Delta_{21}\phi_1^3) = \\ 4(i\lambda)^3 i\Delta_{y3}(-36\ 4\ (i\Delta_{32})^2 i\Delta_{31}([\phi_1^3, \phi_2^2] + \phi_2^2\phi_1^3) - \\ -36\ 8(i\Delta_{32})^2 i\Delta_{21}\phi_3\phi_2\phi_1^3 + 12\ 36(i\Delta_{32})^2(i\Delta_{21})^2\phi_3\phi_1^2 + 24\ 4(i\Delta_{32})^3 i\Delta_{21}\phi_1^3) = \\ -4(i\lambda)^3 i\Delta_{y3}(36\ 4\ (i\Delta_{32})^2 i\Delta_{31}(-6i\Delta_{21}\phi_2\phi_1^2 + 6(i\Delta_{21})^2\phi_1 + \phi_2^2\phi_1^3) + \\ 36\ 8(i\Delta_{32})^2 i\Delta_{21}\phi_3\phi_2\phi_1^3 - 12\ 36(i\Delta_{32})^2(i\Delta_{21})^2\phi_3\phi_1^2 - 24\ 4(i\Delta_{32})^3 i\Delta_{21}\phi_1^3).\end{aligned} \tag{4.10}$$

Second term for ϕ_1 interaction (only up to λ^3)

$$\frac{1}{2}4(i\lambda)^2 i\Delta_{y3}(12i\Delta_{31}[\phi_3^2,(i\lambda)\phi_1^4]\phi_1^3 - 36(i\Delta_{31})^2[\phi_3,(i\lambda)\phi_1^4]\phi_1^2) =$$
$$\frac{1}{2}4(i\lambda)^3 i\Delta_{y3}(12i\Delta_{31}(8i\Delta_{31}\phi_3\phi_1^6 - 12(i\Delta_{31})^2\phi_1^5) - 36\ 4(i\Delta_{31})^3\phi_1^5),$$
$$\frac{1}{2}4(i\lambda)^2 i\Delta_{y2}(12i\Delta_{21}[\phi_2^2,(i\lambda)\phi_1^4]\phi_1^3 - 36(i\Delta_{21})^2[\phi_2,(i\lambda)\phi_1^4]\phi_1^2) =$$
$$\frac{1}{2}4(i\lambda)^3 i\Delta_{y2}(12i\Delta_{21}(8i\Delta_{21}\phi_2\phi_1^6 - 12(i\Delta_{21})^2\phi_1^5) - 36\ 4(i\Delta_{21})^3\phi_1^5). \tag{4.11}$$

Applying 3.56 for n=4 and r=3, the series is expected to truncate at order λ^{13}. However, following up with the calculation, we notice that this term is the expectation value of an odd number of fields. Hence, the highest order contribution is with λ^{12}. We can also evaluate the maximum number of interactions at each vertex, i.e 8 for ϕ_1, 3 for ϕ_2 and 1 for ϕ_3.

4.1.1 Diagrams

Now, starting from the previous calculations, we develop a pictorial representation for $x \prec 1 \prec 2 \prec y$, this means excluding an interaction point from the above results. We draw the contributions in terms of retarded and Feynman propagators, arising from the possible Wick contractions. Let $a \bullet\!\!-\!\!\!-\!\!\bullet b = \overleftarrow{W}^{ba}$ represent a Wick contraction, while $a \bullet\!\!-\!\!\!\twoheadrightarrow\!\!\!-\!\!\bullet b = -i\Delta_{ba} = c_{ba}$

Order λ^1

$$c_{y2}\langle\phi_2^3\phi_x\rangle_\Psi + c_{y1}\langle\phi_1^3\phi_x\rangle_\Psi \tag{4.12}$$

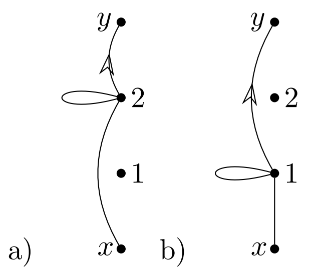


Figure 4.1: Diagrams at order λ of the two-point function with 2 points in the interaction region. b) is the only diagram for which there is no a decorated leg from the element immediately below y to y. Contribution 4.12

Order λ^2

$$c_{y2}c_{21}\langle\phi_2^2\phi_1^3\phi_x\rangle_\Psi + c_{y2}c_{21}^2\langle\phi_2\phi_1^2\phi_x\rangle_\Psi + c_{y2}c_{21}^3\langle\phi_1\phi_x\rangle_\Psi \tag{4.13}$$

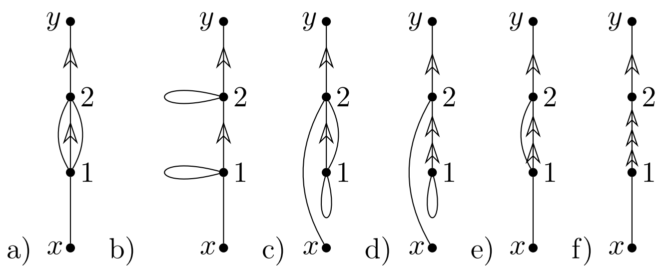


Figure 4.2: Diagrams at order λ^2 of the two-point function with 2 points in the interaction region. Contribution 4.13

Order λ^3

$$c_{y2}c_{21}^2\langle\phi_2\phi_1^6\phi_x\rangle_\Psi + c_{y2}c_{21}^3\langle\phi_1^5\phi_x\rangle_\Psi \tag{4.14}$$

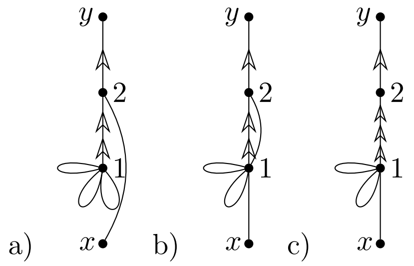


Figure 4.3: Diagrams at order λ^3 of the two-point function with 2 points in the interaction region. Contribution 4.14

Order λ^4

$$c_{y2}c_{21}^3\langle\phi_1^9\phi_x\rangle_\Psi \tag{4.15}$$

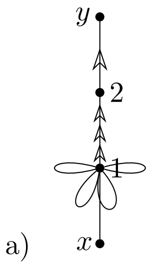
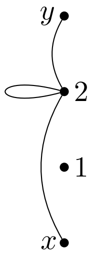
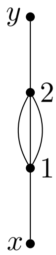
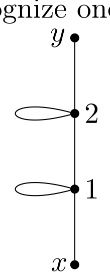
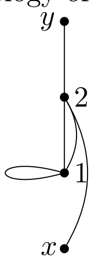
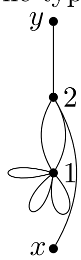
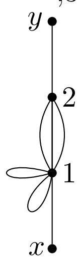
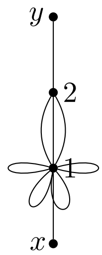


Figure 4.4: Diagram at order λ^4 of the two-point function with 2 points in the interaction region. Contribution 4.15

Hence, we can recognize one topology of the λ of the type  ,3 topologically distinct diagrams of the λ^2 i.e  ,  and  2 for λ^3  and  and 1 for λ^4 

Now, we draw diagrams for $x \prec 1 \prec 2 \prec 3 \prec y$
For λ^1 and λ^2, the diagrams are the same as in the previous case: we choose to omit 1 or 2 or 3 and then draw the figure on the remaining elements.
Order λ^3
Again, we consider pictorial representations for 2 interaction points, excluding one vertex: there are 3×3 of them. In addition, new diagrams arise involving all three interaction vertices. The resulting graphs always have a decorated leg connecting 3 and y and so this is omitted in the following representations.

$$c_{32}c_{21}\langle\phi_3^2\phi_2^2\phi_1^3\phi_x\rangle_\Psi \tag{4.16}$$

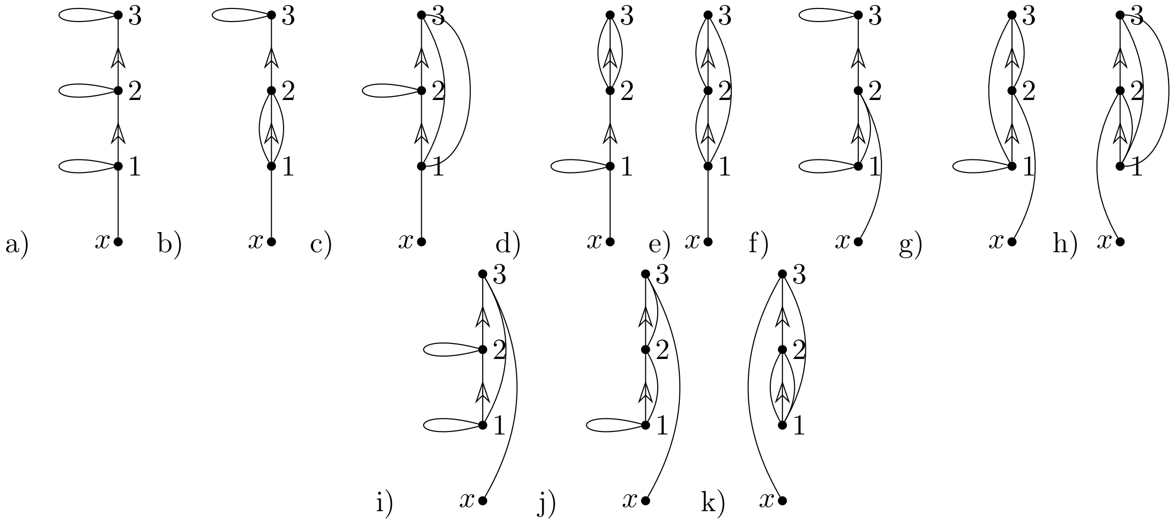


Figure 4.5: Diagrams at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.16

$$c_{32}c_{21}^2\langle\phi_3^2\phi_2\phi_1^2\phi_x\rangle_\Psi \tag{4.17}$$

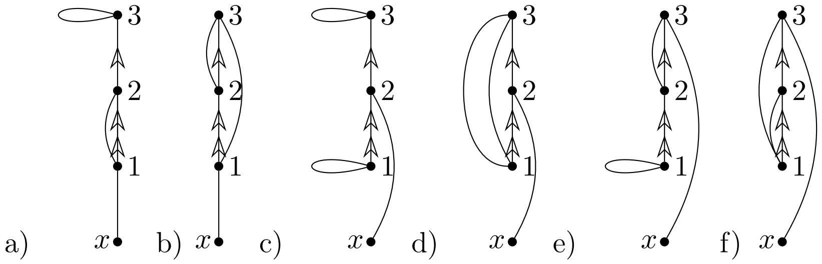


Figure 4.6: Diagrams at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.17

$$c_{32}c_{21}^3\langle\phi_3^2\phi_1\phi_x\rangle_\Psi \tag{4.18}$$

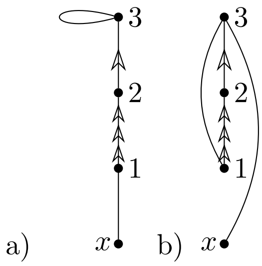


Figure 4.7: Diagrams at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.18

$$c_{32}c_{31}c_{21}\langle\phi_3\phi_2^2\phi_1^2\phi_x\rangle_\Psi \tag{4.19}$$

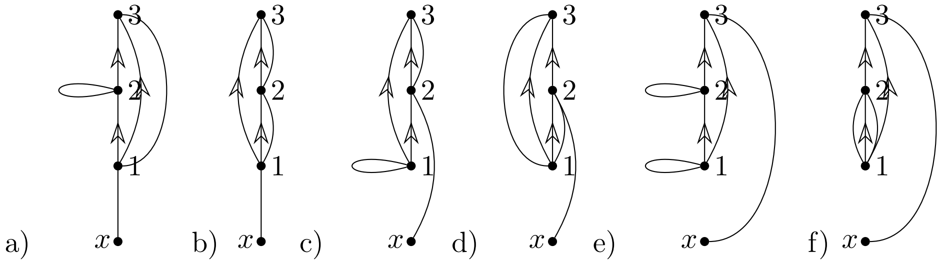


Figure 4.8: Diagrams at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.19

$$c_{32}c_{31}c_{21}^2\langle\phi_3\phi_2\phi_1\phi_x\rangle_\Psi \tag{4.20}$$

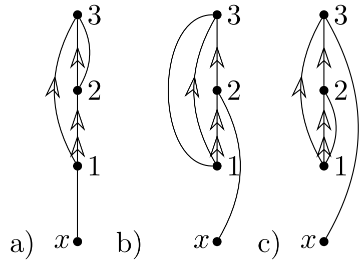


Figure 4.9: Diagrams at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.20

$$c_{32}c_{31}c_{21}^3\langle\phi_3\phi_x\rangle_\Psi \tag{4.21}$$

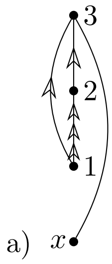


Figure 4.10: Diagram at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.21

$$c_{32}c_{31}\langle\phi_3\phi_2^3\phi_1^3\phi_x\rangle_\Psi \tag{4.22}$$

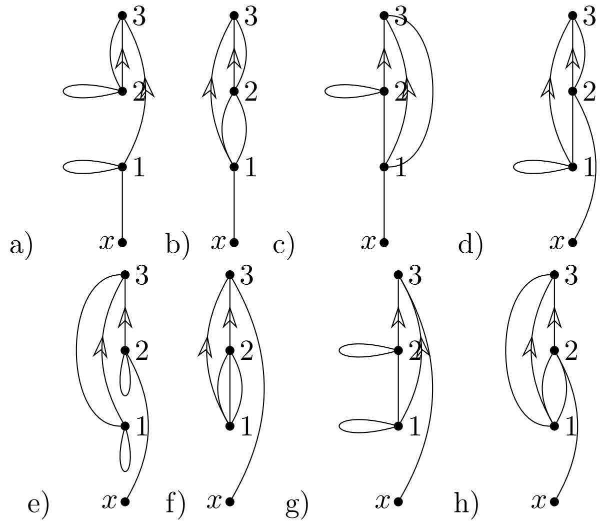


Figure 4.11: Diagrams at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.22

$$c_{32}c_{31}^2c_{21}\langle\phi_2^2\phi_1\phi_x\rangle_\Psi \tag{4.23}$$

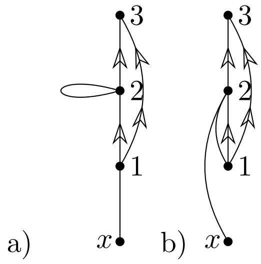


Figure 4.12: Diagrams at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.23

$$c_{32}c_{31}^2c_{21}^2\langle\phi_2\phi_x\rangle_\Psi \tag{4.24}$$

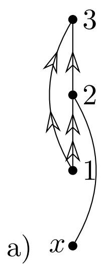


Figure 4.13: Diagram at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.24

$$c_{32}c_{31}^2\langle\phi_2^3\phi_1^2\phi_x\rangle_\Psi \tag{4.25}$$

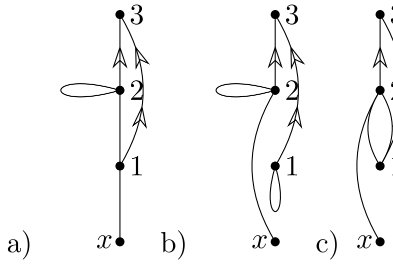


Figure 4.14: Diagrams at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.25

$$c_{32}^2c_{21}\langle\phi_3\phi_2\phi_1^3\phi_x\rangle_\Psi \tag{4.26}$$

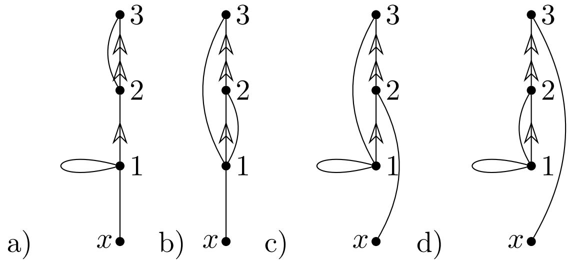


Figure 4.15: Diagrams at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.26

$$c_{32}^2c_{21}^2\langle\phi_3\phi_1^2\phi_x\rangle_\Psi \tag{4.27}$$

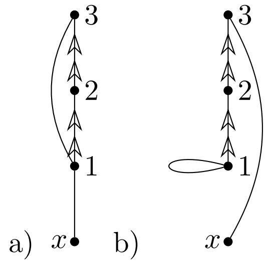


Figure 4.16: Diagrams at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.27

$$c_{32}^2c_{21}c_{31}\langle\phi_2\phi_1^2\phi_x\rangle_\Psi \tag{4.28}$$

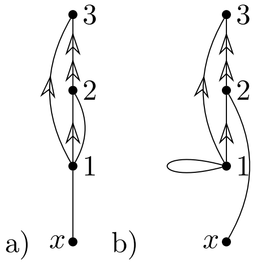


Figure 4.17: Diagrams at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.28

$$c_{32}^2 c_{21}^2 c_{31} \langle \phi_1 \phi_x \rangle_\Psi \tag{4.29}$$

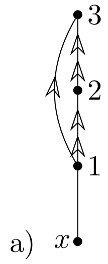


Figure 4.18: Diagram at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.29

$$c_{32}^2 c_{31} \langle \phi_2^2 \phi_1^3 \phi_x \rangle_\Psi \tag{4.30}$$

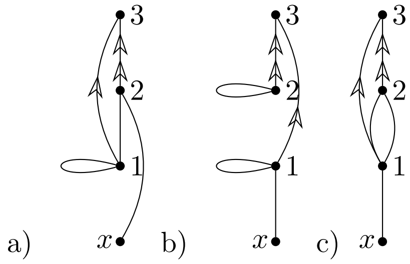


Figure 4.19: Diagrams at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.30

$$c_{32}^3 c_{21} \langle \phi_1^3 \phi_x \rangle_\Psi \tag{4.31}$$

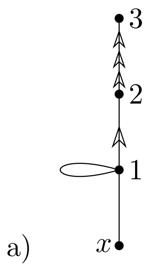
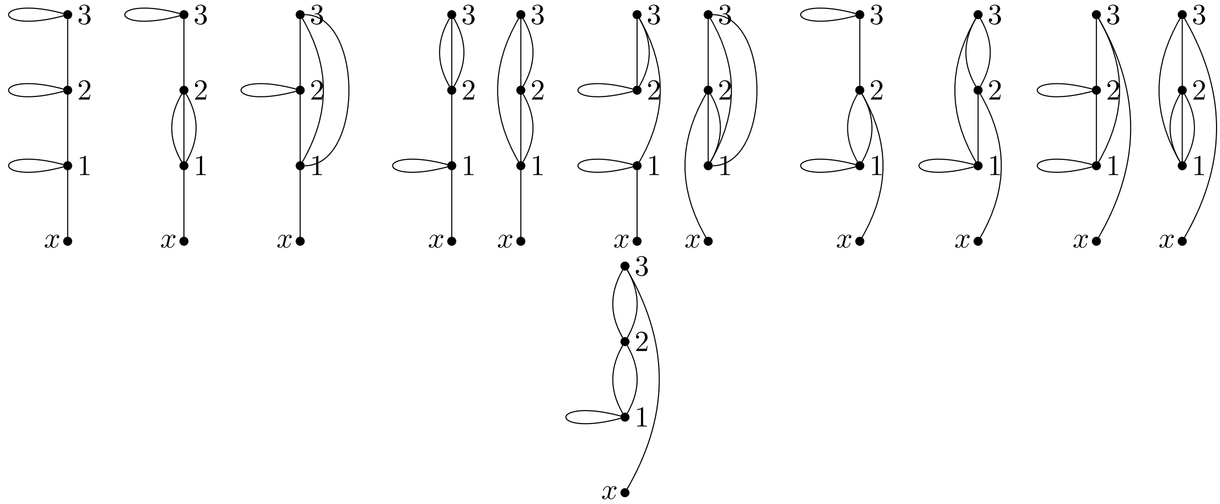


Figure 4.20: Diagram at order λ^3 of the two-point function with 3 points in the interaction region. Contribution 4.31

To conclude, there are 12 topologically distinct diagrams involving all 3 interaction vertices with y connecting to the top element in each case, i.e.



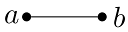
Chapter 5

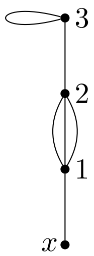
Results and Conclusion

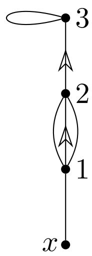
5.1 Modified Feynman rules

Now, we want to summarize our calculations for the two-point function, so that we can find a general pattern between diagrams. Our aim is to assign a mathematical expression to allowed diagrams to estimate the contributions at any order λ for the $x \prec 1 \prec 2 \cdots \prec m \prec y$ chain in ϕ^4 interacting theory without doing any calculation.

5.1.1 Terminology

Call  an F-leg or F-line, where F stands for Feynman, while  a Δ-leg or Δ-line.
Call a vertex with lines an interaction vertex.
 is called a self loop, while  is an infinity loop or double-self loop.
Call a diagram D={T, dec} where T is the topological diagram and dec is the Δ decorations on T.

For instance, T= 

and dec= 

5.1.2 The value of each diagram

These rules allow to estimate the value of each diagram, giving to each symbol a mathematical correspondence.

1. Each interaction vertex contributes a factor $(-i\lambda)^t$ where t is the "number of interactions" at the vertex.

2. Each F-leg connecting $i \prec j$ contributes $\overleftarrow{W}_{ji}$

3. Each Δ-leg connecting $i \prec j$ contributes $-i\Delta_{ji}$

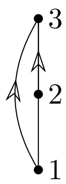
4. Each diagram is weighted by a combinatorial factor. See next section 5.1.3

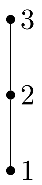
5.1.3 Combinatorial factors rules

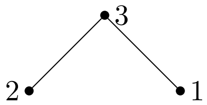
Let's call a vertex with k interactions (i.e. 4k legs) a k-interaction-vertex, (k-IV). In calculating the combinatorial factor of a diagram, each k-IV must be considered as a set of k separate vertices, from each of which there must be a Δ-path upwards to y, which does not need to be unique. The separated vertices are considered as indistinguishable, i.e. identical.
EXAMPLES
So, for instance, for 2 interaction vertices at order λ^3 in 4.3 a), there are two Δ-legs up from the 2-multivertex and by this rule, one of the legs must go up from one separated vertex and the other leg from the other separated vertex.
The result we would expect is $(1/2 \cdot 48 \cdot 8)\cdot$ Wick factor from $\langle \phi_2 \phi_1^6 \phi_x \rangle$.
Firstly, separate out the vertex 1 into 1A and 1B.
Then, the counting for diagram gives (4 for "y attaches to 2") · (3-choose-2 for the 2 legs of 2 that will attach to 1A and 1B) · (4 for leg_1 of 2 to attach to 1A) · (4 for leg_2 of 2 to attach to 1B) · (Wick factor from $\langle \phi_2 \phi_1^6 \phi_x \rangle$ which joins x to the diagram and wires the remaining legs together). No factor for the decoration which is unique. Hence, $4 \cdot 3 \cdot 4 \cdot 4 = 48 \cdot 4$, which is correct.
Now, we look at λ^4 diagram for 2 interaction vertices 4.4. The contribution we want to recover is $(1/6) \cdot 48 \cdot 8 \cdot 4\cdot$ (Wick factor from $\langle \phi_1^9 \phi_x \rangle$). Again, firstly separate out the vertex 1 into 1A and 1B and 1C. Then, from the diagram we get (4 for y attaches to vertex 2) · (4 for 1A) · (4 for 1B) · (4 for 1C) · (Wick factor from $\langle \phi_1^9 \phi_x \rangle$) $= 16 \cdot 16 \cdot$ (Wick factor from $\langle \phi_1^9 \phi_x \rangle$) which is what we expected.

5.1.4 More terminology

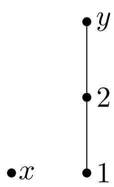
The interaction vertex set (IVS) of a diagram is the set of interaction vertices.
The vertex set (VS) of a diagram is the set of interaction vertices and $\{x, y\}$. Hence, VS= IVS $\cup \{x, y\}$
The DO is the order induced on the IVS (or VS) by the restriction of the causal set order to the IVS (VS).
The Δ-DO is the order $\prec_{Delta}$ on the IVS defined by: $i \prec_{Delta} j$ if there is a Δ-leg from i to j and take the *transitive closure.*

DO and Δ-DO are different orders. For instance, considering 4.11 c), 

is 

in DO and 

is in Δ-DO.

Another example is 4.3 a), which is a total order in DO but 

in Δ-DO.

5.1.5 Diagram rules

The following rules can be applied to select the diagrams which exist and contribute at order λ and above. Hence, we can construct graphical representations for all perturbative corrections in the expansion.

1. y is above *exactly one* interaction vertex in the Δ-DO. The vertex, V_y, is the unique maximal

element of IVS on Δ- DO.

2. In the Δ-DO there is an "upward chain of Δ-legs" from each vertex in IVS to V_y (and hence up to y), which does not need to be unique, as shown by 5.1.

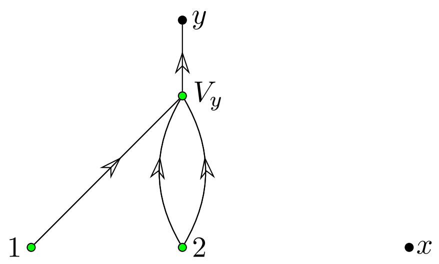


Figure 5.1: This figure highlights rule 2 applied to the contribution 4.30, where the green vertices are in IVS

3. There is a leg, exactly one, from x to an interaction vertex

4. The total number of legs (Δ or F) from an interaction vertex is 4η where $\eta \geqslant 1$, $\eta \in \mathbb{N}$. t is the number of interactions at the vertex.

5. There is a limit on how many interactions can happen at a given vertex in IVS depending on its "sublevel below y", see 3.6. At sublevel r, the maximum number of interactions η_r is given by

$$\eta_r = \frac{(n-1)^r - (n-1)^{r-1}}{n-2}$$

Accordingly, there is a cut off in the series at finite order of λ and so a finite number of diagrams contributing to the expression.

Note: We took into account elements between x and y which are totally ordered in the underlying causal set order. However, our rules include all the cases in which the elements are not totally ordered, since spacelike contributions would give vanishing Δ-legs, leading to *a general underlying order*.
In addition, x can precede or be spacelike to any other element because x is connected by a F-leg to an interaction point, which does not require a causal order. Hence, the correlation between x and y is evaluated considering how y is affected by interactions in its past.

5.2 Similar approaches

In [RDC14], Dickinson et al. develop relativistic quantum field theories highlighting the role of causality in particle dynamics. Their approach contrasts the usual “in-out” formalism where amplitudes consist of Feynman propagators and sources and detectors are treated on the same footing. On the other hand, they distinguish the detection and the production source evaluating an “in-in” expectation value instead of the usual expression for the scattering matrix. With this method, the retarded propagator plays a prominent role, leading to manifestly causal results. In particular, they evaluate the expectation value of fields in the presence of an external source, showing that an unbroken chain of retarded propagators transmits the effect of the source to the detector. Although Feynman propagators

contribute, they never break the causal link from source to detector.
Our procedure is in agreement with their method, although some discrepancies can be analysed. A major difference with our calculation is that they estimate $\langle \mathcal{T}\phi_y^{int}\rangle$ in ϕ^3 theory given the interaction in the source region in the continuum, instead of the two-point function $\langle \mathcal{T}\phi_y^{int}\phi_x^{int}\rangle$, with the interaction region in the causal past of y with discrete spacetime. As a result, their contributing series does not truncate since it is always possible to add more points in the interaction region of a continuum spacetime. Furthermore, they introduce a new interaction vertex at each iteration in the perturbation expansion, allowing just one interaction at each vertex and assuming that the contribution from the commutator relation with the same field at the next iteration gives a vanishing term. We would recover their results, relabelling all the exceeding contributions at each vertex. In addition, they discard vacuum diagrams, since they are ruled out by the sensitivity of the detector. Such loop diagrams are fundamental in our results, so a further calculation with the source and detector set up would have helped the comparison.

In the interest of studying non-linearities during the inflationary era, an alternative diagrammatic representation of the in-in formalism and the associated rules are given by Marcello Musso in [Mus13]. Starting with an iterative solution of the equation of motion for the interacting operator field, higher order corrections are obtained in terms of lower order ones and retarded Green functions. Then, the resulting correlation functions are shown to be analogous to the local n-point functions computed in the in-in formalism along a close time path. Developing the diagrammatic framework, he first deals with open graphs, i.e. a general procedure to represent the perturbation operators expansion. Then, closed graphs are assembled from open trees joining the free legs, expressing correlation functions. Hence, a new version of Feynman rules is introduced to evaluate loop correlation functions involving retarded Green functions and two-field expectation values without doing any calculation. However, most of the diagrammatic contributions includes divergences in the integrals over three-momentum, demanding the renormalization of the effective potential and highlighting that this approach still deals with continuum spacetime.

5.3 Further work

What follow involves directions of further work and open questions on the research topic.

- Suppose we choose Ψ to be a two particle state, i.e. the SJ state acted on by two single-particle-SJ-mode-packet creation operators. We can define a decoherence functional for that state which is Gaussian. Since the SJ modes are a "complete set", we can find a superposition of SJ modes that corresponds to "an incoming wave packet with momentum peaked around some momentum"[Sor17]. Then, we can choose the state in the decoherence functional to be a state representing two incoming packets whose spacetime support intersects in the spacetime region where the interaction is turned on. Is this an alternative to deal with scattering or do we recover the same results as in our method, i.e calculating "Feynman n-point functions" as perturbative expansions in λ and summing over Feynman diagrams and then plug the appropriate n-point function(s) into a scattering calculation?

- Let us build a scenario. Fix a physical lab of a particular physical size-duration, i.e. a 4 volume, and choose two points x and y that we want to calculate the two-point function of. Let $x \prec y$ and let R be an interaction region of a fixed physical size in the Alexandrov interval between x and y. Then, we sprinkle into this interaction region at different densities ρ and we assume the interaction happens at every sprinkled point in R. Adding x and y to our sprinkled points forms our causet $\mathcal{C}$. We have a mass m and a coupling λ. Then, for each ρ, m and λ we can calculate the two-point function $\langle \mathcal{T}\phi_y^{int}\phi_x^{int}\rangle$. Comments:
  1. If ρ is too small it will not be able to capture the physics of a field of mass m and coupling λ.
  2. If we keep ρ fixed (and big enough for that m and λ) and calculate the two-point function for

different sprinklings, what is the mean and and what are the fluctuations?
3. What happens if we keep m and λ fixed and increase ρ?
4. Is there a "continuum limit"? Is there a sequence of theories (ρ_k, m_k, λ_k) such that $\rho_k \to \infty$ as $k \to \infty$ while the 2 point function tends to a meaningful expression in the limit?
5. It seems reasonable that the sprinkling density ρ in R sets the physical "scale" at which the calculation is being done in the *effective field theory* sense i.e. it cuts off the physics at smaller physical scales. Can we have an evidence of it?
6. Are the higher order terms in the λ expansion small so that perturbation expansion makes sense? Is the λ^2 term smaller than the λ^3 term etc.? We note that Δ_{ab} in 4-D goes like $\sqrt{\rho}$ from 2.10, while what are the values of $\overleftarrow{W}$ entries?

- The main point that needs further discussion is how to deal with *loops at vertices*. A loop at vertex i gives a factor W_{ii} which is finite in the causal set theory but singular in the continuum. In ϕ^4 there are two diagrams in the case of two interaction points at order λ as shown in 4.1 and so there are N diagrams for N interaction points, always assuming that the interaction points are only in the causal past of y and not of x. So the sum over all diagrams at $O(\lambda)$ is a sum of that "diagram-type" over all interaction points. In the perturbative ϕ^4 theory in a Lorentzian manifold, the $O(\lambda)$ contribution to the interacting two-point function is given by the same diagram. In the continuum, the mathematical expression for the loop at i is the divergent factor W_{ii} with point i integrated over the whole interaction region to the causal past of y.
Each loop contributes the same to the diagram in the continuum when the state is the usual vacuum state no matter where the interaction point is since in that case $W_{ii} = W_{jj}$. However, for the SJ state in the causal set, the diagonal terms of W are not constant.
Finally, we notice that there is a difference between single loops at vertices and double loops. A double loop can be removed and the remaining diagram is still a valid diagram with one less power of λ. On the contrary, a single loop cannot be so removed.

- An interesting application of this work is to use the interacting two-point correlations to compute the entanglement entropy (up to first order in perturbation theory), as described in [YCZ20].
The scenario involves particular circumstances where there is limited access to the spacetime where a quantum field is defined. A loss occurs in the information that was enclosed in the n-point functions involving all points in the region. This loss of information is measured by entanglement entropy. For instance, a black hole is the prototypical and most studied example of a background spacetime where all the correlation information of a quantum field are not accessible.

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