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A New Approach to Quantum Cosmological Time: Dynamical Constants of Nature

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Abstract

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A New Approach to Quantum Cosmological Time: Dynamical Constants of Nature

by Bruno Alexandre

We promote the constants of nature to dynamical variables of which the canonical conjugates define a cosmological time. Each constant is connected to a different time. We extensively analyse the formalism behind this proposal, from single fluid to multi-fluid models. Specifically, we show that for the former the Hamiltonian constraint equation becomes an effective Schrodinger equation in the corresponding time, solved in the connection representation by monochromatic plane waves. We further show that for more than one fluid it is recommended to identify mini-superspace as a dispersive medium and we examine the motion of the peak of wave packets as well as the transition regions between dominating constants. Finally, we carry out original work, introducing a massless scalar field as a cosmological clock and simultaneously targeting the gravitational constant, G_N, to be the conjugate of another time variable. The aim is to explore a toy-model where we have more than one physical time in the same epoch. A double-time Schrodinger equation is obtained. We further analyse different types of possible solutions, with special focus on a fully semi-classical state in both constants, arriving at a two-time setting uncertainty principle. We close with a brief examination of a mixture between a scalar field and a radiation fluid in which three different clocks are present.

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Contents

List of Abbreviations

GR	**G**eneral **R**elativity
QFT	**Q**uantum **F**ield **T**heory
MSS	**M**ini-**s**uper**s**pace
WDW	**W**heeler-**d**e**W**itt
PDE	**P**artial **D**ifferential **E**quation
FRW	**F**riedmann **R**obertson **W**alker
EC	**E**instein **C**artan
SD	**S**elf **D**ual
MC	**M**aurer-**C**artan
CS	**C**hern-**S**imons

Chapter 1

Introduction

1.1 The Problem of Time

The Problem of Time is a very well known issue one has to deal with when trying to unify General Relativity and Quantum Theory [1, 2, 3, 4]. This appears as a result of the different interpretations of time in both theories. Einstein's theory of gravity is diffeomorphism invariant, i.e. one works with equations that transform covariantly under changes of spacetime coordinates, which leads to frame independent physical results. On the other hand in Quantum Mechanics the notion of time [5] originates from Newtonian physics, where time is absolute and external to the system. Hence, time cannot be seen as a physical observable as it is not represented by a quantum operator, being solely used to mark the evolution of the system.

Further conceptual issues arise from the problem of time [1]. For instance the status of causality and unitarity, the meaning of spacetime and the emergence of our classical universe from some primordial quantum event in the big-bang.

Thus, how can we re-introduce the notion of time in a quantum theory of gravity? A considerable amount of work has been done in order to answer this and many other questions [1]. There are currently three main approaches to this conundrum: identify time before quantisation, after quantisation or assuming time plays no fundamental role in the theory. In the first one, the canonical constraints are solved before quantisation. Time is identified as a functional of canonical variables and one looks for an effective Schrodinger equation. The second approach works in an opposite manner as the constraint equations are solved after quantising the system and time is identified only following this procedure. Also, the most important constraint is the WDW equation, whose solutions are functionals of the spatial metric. In the third scheme the aim is to treat time in the same way as in GR which consequently leads to a purely phenomenological way to grasp the notion of time with no fundamental conception in quantum theory. These schemes are extensively analysed in [1]. Different perspectives and discussions were also undertaken [6, 7, 8].

We should also note that most of the attempts to solve this issue are performed through finite dimensional models free of UV divergences. This is because quantising the field equations of GR in canonical quantum gravity leads to a non-renormalisable theory.

A rather interesting way to address the problem of time in GR is to identify events in a spacetime with the position of physical particles. For example, the value of a scalar field defined on a manifold M of a given particle is coordinate independent and so it is an observable. Notably, the value of the scalar field ϕ might be determined at that event where a group of fields takes on a given set of values. These fields can be expressed through matter distributions or they can be part of the gravitational field itself. Most approaches to the problem of time revolve around such notions. This underlines the important fact that a local time coordinate in spacetime is not enough to get a physical definition of time. For instance in the real world an observer measures time using the proper time along its worldline, which is a Lorentz scalar quantity and acts simultaneously in spacetime points as well as in the metric. More specifically, the proper time along the geodesic connecting the beginning and end points of the world-line is an intrinsic property if the points are labelled using internal coordinates. Labelling spacetime events using spatial reference frames and physical clocks is of great importance, both in the classical and quantum theories of relativity. For quantum theory, there is another relevant problem raised by the example of proper time described above. The calculation of a range of proper time uses the spacetime metric, and thus it is only meaningful when the equations of motion are solved [1]. Unlike in classical theory there are difficulties in any theory in which the geometry of spacetime has quantum fluctuations, having no fixed value and therefore time may become a quantum operator. This is a problematic concept that is not included in standard quantum theory.

1.2 The Cosmological Constant Problem

Another fundamental issue in physics is the values of the constants of nature [9]. Regardless of our ability to measure them, we still have no scientific method to explain or determine these numerical values.

The cosmological constant problem [10, 11, 12, 13, 14, 15] is by far the most significant one: the predicted value for the cosmological constant from vacuum energy contributions in quantum field theory is at least sixty orders of magnitude bigger than the observed value for this constant [15]. Simple QFT calculations show that this vacuum energy is at least $\rho_{\text{vac}} \gtrsim (\text{TeV})^4$. Furthermore, we cannot define ρ_{vac} to be our zero point energy because according to general covariance and the equivalence principle, all sorts of energy generate curvature and therefore, gravity. By Lorentz invariance, the energy-momentum tensor associated with this vacuum is $T_{\mu\nu} = -\rho_{\text{vac}} g_{\mu\nu}$. Looking at Einstein equations ($R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$) this is equivalent to defining a total vacuum energy, $\rho_T = \rho_{\text{vac}} + \frac{\Lambda}{8\pi G}$, which, according to observational measurements, is upper bounded by $\rho_T \lesssim (\text{meV})^4$. In this way, Λ needs to be fine-tuned by at least sixty decimal places. However, this is just the beginning of the problem [12]. The real issue is radiative instability rather than simply fine tuning, since when one performs perturbation theory to renormalize the vacuum energy, the amount of refinement one has applied to Λ is independent of the number of loop contributions in the calculations. This means that the bare cosmological constant needs to be retuned with extreme precision at each order in perturbation theory. Therefore, it is useless to, for example, fine tune the first loop contribution as it will be unstable in higher order loops. Indeed, this illustrates the high sensitivity of the vacuum energy regarding the unknown aspects of UV physics.

One attempt to solve the problem is to consider a gauge fixed version of GR known as Unimodular gravity [16, 10, 15]. This theory originates from the Einstein-Hilbert action subject to the constraint $\sqrt{-g} = 1$ and therefore $\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g} = 0$ where $g = \det g_{\mu\nu}$. This implies a reduction of the set of diffeomorphism transformations that are possible in GR. In infinitesimal form those can be written as $\delta g_{\mu\nu}(x) = \nabla_\mu k_\nu + \nabla_\nu k_\mu$, where k_μ is a gauge vector. Hence, by imposing the unimodular condition we get $-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} = \nabla_\mu k^\mu = 0$, which represents the reduced set of transverse diffeomorphism transformations.

The restricted Einstein-Hilbert plus matter action, S_M is then:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_M - \int d^4x\lambda(x)(\sqrt{-g} - 1) \tag{1.1}$$

where $\lambda(x)$ is a Lagrange multiplier. Variation with respect to this scalar gives the constraint $\sqrt{-g} = 1$. The metric field equation is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 8\pi G\lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.2}$$

where $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$. Taking the trace of (1.2) we obtain

$$\lambda = \frac{1}{4}\left(T + \frac{R}{8\pi G}\right), \tag{1.3}$$

and substituting this back in (1.2) we get the traceless Einstein equations:

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = 8\pi G\left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T\right). \tag{1.4}$$

Taking the divergence of (1.2) and considering both energy-momentum conservation, $\nabla_\mu T^{\mu\nu} = 0$, and the contracted Bianchi identities, $\nabla_\mu R^{\mu\nu} - \frac{1}{2}\nabla^\nu R = 0$, implies $\partial_\mu\lambda = 0$, i.e. λ is an integration constant. Thus, we found that the dynamics of this theory are equivalent to GR, with a cosmological constant given by $\Lambda = 8\pi G\lambda$. In this manner the integration constant λ, suffers from the same radiative instability as Λ in GR and therefore unimodular gravity does not bring any new perspective to the cosmological constant problem.

An important contribution to this problem is Weinberg's no go theorem which states that it is not possible to add extra fields to the matter action in order to correct the large vacuum energy without fine tuning. Hence, any attempt to solve the problem should explain how to get around it. Weinberg also described five approaches to the cosmological constant problem [14]. Some of them are supersymmetry, supergravity and superstrings, as well as quantum cosmology, the latter being the most promising one. All these approaches reveal that any possible solution to the cosmological constant problem will probably have a wide impact in multiple areas of physics and astronomy.

1.2.1 Sequestering the Vacuum Energy

A more promising way to address the problem, known as The Sequester, was recently suggested by Padilla and Kaloper [17, 18, 19]. We consider a coupling between GR and a quantum matter sector which contains the Standard Model. Λ, the usual cosmological "constant", now plays the role of a dynamical variable which is a spacetime constant but can be varied in the action. The aim is to have Λ cancelling the Standard Model vacuum energy. The *sequestring action* is then

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R - \Lambda\right] + S_m + \sigma\left(\frac{\Lambda}{\lambda^4\mu^4}\right), \tag{1.5}$$

where μ is a mass scale, $\sigma\left(\frac{\Lambda}{\lambda^4\mu^4}\right)$ is the sequestering function which is odd and differentiable and should be determined by phenomenology and λ is a dynamical variable that balances the scales between matter and the Planck mass. The matter action is given by

$$S_m = -\int d^4x\sqrt{-g}\lambda^4\mathcal{L}_m(\lambda^{-2}g^{\mu\nu}, \Psi) \tag{1.6}$$

and we can write $\tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}$, with the associated stress-energy tensor:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{-2\lambda^4}{\sqrt{-\tilde{g}}}\frac{\delta S_m}{\lambda^2 \delta\tilde{g}^{\mu\nu}} = \lambda^2 \tilde{T}_{\mu\nu}, \tag{1.7}$$

$$T^\mu_\nu = \lambda^4 \tilde{T}^\mu_\nu. \tag{1.8}$$

Therefore, varying the action with respect to the metric, λ and Λ we get the following equations of motion:

$$M_p^2 G^\mu_\nu = -\Lambda\delta^\mu_\nu + \lambda^4 \tilde{T}^\mu_\nu \tag{1.9}$$

$$\int d^4x\sqrt{-g} = \frac{1}{\lambda^4\mu^4}\sigma'\left(\frac{\Lambda}{\lambda^4\mu^4}\right) \tag{1.10}$$

$$\frac{4\Lambda}{\lambda^4\mu^4}\sigma'\left(\frac{\Lambda}{\lambda^4\mu^4}\right) = \int d^4x\sqrt{-g}T, \tag{1.11}$$

where $T = T^\mu_\mu$.

An important point that we can extract from (1.10) is that the spacetime volume should be finite, meaning the universe will end in a Big Crunch. σ is assumed to be differentiable and if the spacetime volume was infinite then we would need $\lambda = 0$, which cannot happen since it would mean that all particles in the Standard Model were massless.

Combining (1.10) and (1.11) we can determine Λ and arrive at

$$\Lambda = \frac{1}{4}\langle T\rangle, \tag{1.12}$$

where $\langle T\rangle = \frac{\int d^4x\sqrt{-g}T}{\int d^4x\sqrt{-g}}$. Substituting this back in to (1.9) yields

$$M_p^2 G^\mu_\nu = T^\mu_\nu - \frac{1}{4}\langle T\rangle\delta^\mu_\nu. \tag{1.13}$$

The energy-momentum tensor can be separated in two parts: V_{vac} associated with the vacuum energy from the Standard Model loops and τ^μ_ν describing local excitations about this vacuum. In equation form, $T^\mu_\nu = -V_{\text{vac}}\delta^\mu_\nu + \tau^\mu_\nu$, and consequentially we have

$T = -4V_{\text{vac}} + \tau$. Thus, the vacuum energy drops out of the gravitational field equations which take the form:

$$M_p^2 G^\mu_\nu = \tau^\mu_\nu - \frac{1}{4}\delta^\mu_\nu \langle \tau \rangle. \tag{1.14}$$

This is exactly what we needed! The dynamics are no longer dependent on the vacuum energy and instead are only determined by matter excitations, which means that we have a radiatively stable effective cosmological constant, $\Lambda_{\text{eff}} = \frac{1}{4}\langle\tau\rangle$, no longer sensitive to loop corrections at any order in perturbation theory and stable enough to be trusted as a "yardstick for cosmology" [12]. This theory is still being developed and a significant amount of work still needs to be done. Kaloper and Padilla later suggested an improvement to this theory by changing to a manifestly local theory [20], the main differences being that the effective cosmological constant will be fixed by the ratio of two integration constants and the spacetime volume no longer needs to be finite. The manifestly local action is given by

$$\begin{aligned} S = \quad & \int d^4x\sqrt{-g}\left[\frac{\kappa^2(x)}{2}R - \Lambda(x) - \mathcal{L}_m(g^{\mu\nu},\Psi)\right] + \\ & + \frac{1}{4!}\int \left[\sigma\left(\frac{\Lambda(x)}{\mu^4}\right)F_{\mu\nu\alpha\beta} + \hat{\sigma}\left(\frac{\kappa^2(x)}{M_p^2}\right)\hat{F}_{\mu\nu\alpha\beta}\right] dx^\mu dx^\nu dx^\alpha dx^\beta, \end{aligned} \tag{1.15}$$

where $F_{\mu\nu\alpha\beta} = 4\partial_{[\mu}A_{\nu\alpha\beta]}$ and $\hat{F}_{\mu\nu\alpha\beta} = 4\partial_{[\mu}\hat{A}_{\nu\alpha\beta]}$ are two 4-forms, invariant under $A_{\mu\nu\alpha} \to A_{\mu\nu\alpha} + 3\partial_{[\mu}B_{\nu\alpha]}$ and $\hat{A}_{\mu\nu\alpha} \to \hat{A}_{\mu\nu\alpha} + 3\partial_{[\mu}\hat{B}_{\nu\alpha]}$ respectively. This guarantees that Λ and $\kappa^2 = M_p^2/\lambda^2$ are constants on shell. The functions σ and $\hat{\sigma}$ are arbitrary and smooth.

We can therefore compute the equations of motion for this action. Starting with the metric equations, using the Palatini identity

$$\delta R_{\mu\nu} = \nabla_\alpha \delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu \delta\Gamma^\alpha_{\alpha\mu} \tag{1.16}$$

which leads to $\delta R = R_{\mu\nu}\delta g^{\mu\nu} - g^{\mu\nu}\Box\delta g_{\mu\nu} + \nabla^\mu\nabla^\nu\delta g_{\mu\nu}$ we get

$$\kappa^2 G^\mu_\nu = (\nabla^\mu\nabla_\nu - \delta^\mu_\nu\Box)\kappa^2 + T^\mu_\nu - \Lambda\delta^\mu_\nu, \tag{1.17}$$

where $T_{\mu\nu}$ is again given by

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{1.18}$$

The rest of the field equations are obtained through variation of the action with respect to Λ, κ^2, $A_{\mu\nu\alpha}$ and $\hat{A}_{\mu\nu\alpha}$ yielding:

$$F_{\mu\nu\alpha\beta}\frac{\sigma'}{\mu^4} = \epsilon_{\mu\nu\alpha\beta} \tag{1.19}$$

$$\hat{F}_{\mu\nu\alpha\beta}\frac{\hat{\sigma}'}{M_p^2} = -\frac{R}{2}\epsilon_{\mu\nu\alpha\beta} \tag{1.20}$$

$$\frac{\sigma'}{\mu^4}\partial_\mu\Lambda = 0 \tag{1.21}$$

$$\frac{\hat{\sigma}'}{M_p^2}\partial_\mu\kappa^2 = 0, \tag{1.22}$$

where $\epsilon_{\mu\nu\alpha\beta} = \sqrt{-g}e_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor and $e_{\mu\nu\alpha\beta}$ is the Levi-Civita symbol. The last two equations show that Λ and κ^2 are integration constants. Tracing (1.17) and using (1.22) yields $\kappa^2 R = 4\Lambda - T$. Then, taking the spacetime average and combining this with (1.19) and (1.20) gives

$$\Lambda = \frac{1}{4}\langle T\rangle + \Delta\Lambda, \tag{1.23}$$

$$\Delta\Lambda = \frac{1}{4}\kappa^2\langle R\rangle = -\frac{\kappa^2\mu^4\hat{\sigma}'}{2M_p^2\sigma'}\frac{\int\hat{F}}{\int F} \tag{1.24}$$

and by substituting back in (1.17) we arrive at:

$$\kappa^2 G^\mu_\nu = T^\mu_\nu - \delta^\mu_\nu\Delta\Lambda - \frac{1}{4}\delta^\mu_\nu\langle T\rangle. \tag{1.25}$$

Considering again the same splitting of the stress-energy tensor, $T^{\mu}_{\nu} = -V_{\rm vac}\delta^{\mu}_{\nu} + \tau^{\mu}_{\nu}$, (1.23) takes the form

$$4\Lambda + 4V_{\rm vac} = \langle\tau\rangle + \kappa^2\langle R\rangle. \tag{1.26}$$

From the traced gravitational field equation we get $\kappa^2 R = 4\Lambda + 4V_{\rm vac} - \tau$ which together with (1.26) leads to

$$\kappa^2(R - \langle R\rangle) = -(\tau - \langle\tau\rangle). \tag{1.27}$$

Using again the equation for $T_{\mu\nu}$ and (1.27), (1.25) can be written as

$$\kappa^2\left(R^{\mu}_{\nu} - \frac{1}{4}\delta^{\mu}_{\nu}R\right) = \tau^{\mu}_{\nu} - \frac{1}{4}\delta^{\mu}_{\nu}\tau. \tag{1.28}$$

Finally, using the equations of motion for the 4-forms and taking the respective Hodge duals we obtain:

$$\star F = \frac{1}{4!}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu\alpha\beta} = -\frac{\mu^4}{\sigma'} = \langle\star F\rangle \tag{1.29}$$

$$\star\hat{F} = \frac{R}{2}\frac{M_p^2}{\hat{\sigma}'}, \tag{1.30}$$

from which we get:

$$\star\hat{F} - \langle\star\hat{F}\rangle = -\frac{M_p^2}{2\kappa^2\hat{\sigma}'}(\tau - \langle\tau\rangle). \tag{1.31}$$

Consequently, (1.24) can be written as

$$\Delta\Lambda = -\frac{\kappa^2}{2M_p^2}\hat{\sigma}'\langle\star\hat{F}\rangle. \tag{1.32}$$

We should now highlight some important aspects of this theory. From equations (1.29) and (1.31) we see that the form sectors are radiatively stable. Hence, according to equation (1.32) the same holds for $\Delta\Lambda$. In (1.26) the right hand side is stable, however we know that V_{vac} is not, which means that Λ will absorb large radiative corrections from the vacuum energy. Moreover, (1.27) and (1.28) are the gravitational field equations and they describe the same theory as GR except with the advantage that the UV sensitive part of the vacuum energy is now absorbed/cancelled by $\Delta\Lambda$. Also, the residual finite cosmological constant needs to be fixed by observational data and it is completely stable against any additional radiative corrections to V_{vac} without the need for fine tuning. Therefore, this theory is a consistent starting point for the definition of the Feynman path integral but there is still a broad range of topics to investigate such as inflation and phase transitions.

1.3 Overview

It has been recently suggested that the value of the constants of nature and the problem of time in GR could be somehow related [21]. Time is the physical way to describe changes whereas the constants of nature are by definition fixed parameters. Thus, one might wonder if these two completely different concepts could be in fact complementary quantum variables just like position and linear momentum.

In this work we promote these constants to phase space variables and we adopt their canonical conjugates as cosmological clocks. However, given the wide range of choices available [22], one should be careful in selecting a constant that provides a clear separation of space and time for a specific region of phase space. Thus, we end up having times conjugate to the cosmological constant Λ or the gravitational constant G_N, depending on the epoch in which we are living. Foremost, we need to pay special attention to multi-time situations since we have to deal with the handover from one clock to another during transition regions.

We will examine the problem in mini-superspace, i.e. an homogeneous and isotropic universe, in the connection representation. This requires starting from the Einstein-Cartan action and reduce it to MSS.

That said, it's important and pedagogical to start this work by introducing the EC formalism, which is done in the beginning of Chapter 2. We then introduce the Ashtekar variables, fundamental in the Hamiltonian formulation of GR, and the real Chern-Simons state, the latter being the solution to the real Hamiltonian constraint equation. To conclude this chapter we compute the real CS state for a Friedman-Robertson-Walker universe, which will be recovered later in Chapter 3 using a different approach.

Finally, in Chapter 3 we introduce and develop the necessary concepts to promote the constants of nature to off-shell dynamical variables, whose canonical conjugate variables will define a physical time. We start by considering a general single perfect fluid model. From the Einstein-Cartan action we obtain the Hamiltonian constraint, which becomes a Schrodinger equation in the time associated with the constant dominating the dynamics of the system. This equation can be solved in the connection representation by outgoing monochromatic plane waves. Subsequently, we generalize for a multi-fluid universe and so we need to deal with more than one time. In particular, we consider a mixture of dark energy and radiation since the calculations are simpler, but models with other elements display similar qualitative behaviour. Lastly, we consider a universe dominated by a real scalar field coupled to gravity by a "deconstantized" gravitational constant, the two quantities being associated with two distinct cosmological times that occur in the same epoch. This might bring some new implications to the table when compared to the previous multi-fluid models.

Chapter 2

Einstein-Cartan Formalism and the Real Chern-Simons State

In this chapter we briefly introduce the Einstein-Cartan formalism. We present the Cartan structure equations and use them to calculate the connection 1-form and the Riemann curvature 2-form for the FRW metric. This allows us to derive the EC action in mini-superspace which will be the starting point for the proposals we will develop throughout this work. We further introduce the new Ashtekar variables and the $3+1$ split [23] of the EC action that leads to the Hamiltonian formulation of GR [24]. Finally, we present the solution to the real Hamiltonian constraint, known as the Chern-Simons wave function, and compute it particularly for an homogeneous universe. This last quantity is important for the analysis done later in this thesis.

2.1 Einstein-Cartan Action in MSS

In this section we introduce the tetrad formalism in order to obtain the reduced form of the Einstein-Cartan action in MSS for the FRW metrics.

The transition from a general spacetime with metric $g_{\mu\nu}$ to flat Minkowski spacetime is done through the tetrad basis transformation [25]:

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}, \tag{2.1}$$

where we are using Latin indices for the Lorentz group and Greek indices for spacetime. We also assume $\det e^a_\mu \neq 0$, which means the previous relation is invertible. Then, $e^a = e^a_\mu dx^\mu$ are 1-forms that define a non coordinate basis in this space from which we can get the invariant line element in the form:

$$ds^2 = \eta_{ab} e^a \otimes e^b. \tag{2.2}$$

This description is quite practical for a diagonal metric as the tetrad components can be easily read off. Furthermore, the torsion 2-form is defined by the 1st Cartan equation:

$$T^a = De^a = de^a + \Gamma^a{}_b e^b, \tag{2.3}$$

where $\Gamma^a{}_b = \Gamma^a_{\mu\, b} dx^\mu$ is the connection 1-form associated with the exterior covariant derivative D and we are omitting the wedge product symbol ($\wedge$) between differential-forms. The last object we need for the EC action is the curvature 2-form defined by the 2nd Cartan equation:

$$R^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \Gamma^c{}_b. \tag{2.4}$$

Hence, we can now apply this formalism to the FRW metric [26], which is given by

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \left[\frac{dr^2}{K(r)^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right], \tag{2.5}$$

where $K(r) = \sqrt{1 - kr^2}$, $k = 0, -1, 1$ for a flat, closed or open universe respectively, $N(t)$ is the lapse function and $a(t)$ is the scale factor. The signature of the Minkowski metric should match the spacetime metric and for this reason we have $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$.

Thus, the tetrad basis associated with this metric is

$$e^0 = Ndt \tag{2.6}$$

$$e^1 = \frac{a}{K}dr \tag{2.7}$$

$$e^2 = ard\theta \tag{2.8}$$

$$e^3 = ar\sin\theta d\phi. \tag{2.9}$$

Using now the 1st Cartan equation for zero torsion, $T^a = 0$, and computing the exterior derivative of the tetrad basis

$$de^0 = 0 \tag{2.10}$$

$$de^1 = \frac{b}{a}e^0e^1 \tag{2.11}$$

$$de^2 = \frac{b}{a}e^0e^2 + \frac{K}{ar}e^1e^2 \tag{2.12}$$

$$de^3 = \frac{b}{a}e^0e^3 + \frac{K}{ar}e^1e^3 + \frac{\cot\theta}{ar}e^2e^3, \tag{2.13}$$

we can read out the components of the connection 1-form for this spacetime to get

$$\Gamma^i{}_0 = \frac{b}{a}e^i \tag{2.14}$$

$$\Gamma^2{}_1 = \frac{K}{ar}e^2 = Kd\theta \tag{2.15}$$

$$\Gamma^3{}_1 = \frac{K}{ar}e^3 = K\sin\theta d\phi \tag{2.16}$$

$$\Gamma^3{}_2 = \frac{\cot\theta}{ar}e^3 = \cos\theta d\phi, \tag{2.17}$$

with $b = \dot{a}/N$. These determine all the components since we have $\Gamma_{ab} = -\Gamma_{ba}$ and so, $\Gamma^i{}_0 = \Gamma^0{}_i$ and $\Gamma^i{}_j = -\Gamma^j{}_i$.

On the other hand, the 2nd Cartan equation tells us how to compute the curvature 2-form. With a bit of algebra and computing the exterior derivative of the connection

1-form

$$d\Gamma^1{}_0 = \frac{\dot{b}}{Na} e^0 e^1 \tag{2.18}$$
$$d\Gamma^2{}_0 = \frac{\dot{b}}{Na} e^0 e^2 + \frac{bK}{ra^2} e^1 e^2 \tag{2.19}$$
$$d\Gamma^3{}_0 = \frac{\dot{b}}{Na} e^0 e^3 + \frac{bK}{ra^2} e^1 e^3 + \frac{b\cot\theta}{ra^2} e^2 e^3 \tag{2.20}$$
$$d\Gamma^2{}_1 = -\frac{k}{a^2} e^1 e^2 \tag{2.21}$$
$$d\Gamma^3{}_1 = -\frac{k}{a^2} e^1 e^3 + \frac{K\cot\theta}{r^2 a^2} e^2 e^3 \tag{2.22}$$
$$d\Gamma^3{}_2 = -\frac{1}{a^2 r^2} e^2 e^3, \tag{2.23}$$

one can compute all the components $R^a{}_b$ and check that these can be written as

$$R^0{}_i = \frac{\dot{b}}{Na} e^0 e^i \tag{2.24}$$
$$R^i{}_j = \frac{b^2 + k}{a^2} e^i e^j. \tag{2.25}$$

The Einstein-Cartan action is [25]

$$S = \frac{1}{32\pi G_0} \int \epsilon_{abcd} \left(e^a e^b R^{cd} - \frac{\Lambda}{6} e^a e^b e^c e^d \right), \tag{2.26}$$

where G_0 is the fixed value of the gravitational constant G_N, ϵ_{abcd} is the Levi-Civita tensor and Λ is the cosmological constant.

Hence, using (2.24), (2.25) and considering the relations

$$\epsilon_{abcd} e^a e^b R^{cd} = 2\epsilon_{0ijk} e^0 e^i R^{jk} + 2\epsilon_{ij0k} e^i e^j R^{0k} = \tag{2.27}$$
$$= 2\left[\frac{b^2 + k}{a^2} + \frac{\dot{b}}{Na}\right] \epsilon_{0ijk} e^0 e^i e^j e^k \tag{2.28}$$

and

$$\epsilon_{abcd} e^a e^b e^c e^d = 4! e^0 e^1 e^2 e^3 = 4! \frac{Na^3}{K} r^2 \sin\theta d^4x, \tag{2.29}$$

we finally arrive at

$$S = \frac{3V_c}{8\pi G_0} \int dt \left[\dot{b}a^2 - Na\left(-b^2 - k + \frac{\Lambda}{3}a^2\right)\right], \tag{2.30}$$

where $V_c = \int drd\theta d\phi \frac{r^2 \sin\theta}{K}$ is the spatial volume of the universe. This is the EC action in MSS.

2.2 Real Chern-Simons State

The aim of this section is to derive the real Chern-Simons state [27] which is the solution to the real Hamiltonian constraint equation. We start by considering again the Einstein-Cartan action:

$$S = \frac{1}{32\pi G_0} \int \epsilon_{ABCD} \left(e^A e^B R^{CD} - \frac{\Lambda}{6} e^A e^B e^C e^D \right), \tag{2.31}$$

where the Lorentz group indices are now $A, B, C, \ldots$. We then introduce the Ashtekar Self-Dual $SU(2)$ connection [27, 28, 29, 30]:

$$A^i = \Gamma^i + iK^i, \tag{2.32}$$

where

$$K^i = \Gamma^{0i} \tag{2.33}$$

$$\Gamma^i = -\frac{1}{2}\epsilon^i{}_{jk}\Gamma^{jk} \tag{2.34}$$

and the indices $i, j, k, \ldots$ represent $SU(2)$ components, corresponding to the spatial part of the Lorentz group. The generators we will consider for this group are $t^i = -i\sigma^i/2$, where σ^i are the Pauli matrices. The generators obey the commutator $[t^i, t^j] = \epsilon^{ijk}t^k$ and the trace identities $\mathrm{Tr}(t^i t^j) = -\frac{1}{2}\delta^{ij}$ and $\mathrm{Tr}\big(t^i t^j t^k\big) = -\frac{1}{4}\epsilon^{ijk}$,where ϵ_{ijk} is the Levi-Civita symbol. Therefore, the connection can be written in the generator basis as

$A = A^i_a t^i dx^a$, where the indices $a, b, c, ...$ denote the spatial components of spacetime. The field strength tensor associated with A is $F = dA + AA$, which in components reads $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon_{ijk} A^j_a A^k_b$.

At this point, one might wonder if having a complex SD connection makes the theory also complex. However, this is not the case since the action is real and we can write everything in terms of the real variables K^i and Γ^i. This construction is useful in the sense that we can solve the real Hamiltonian constraint with a modified Chern-Simons state built from the Ashtekar connection.

Hence, the EC action subject to a $3+1$ split in the time gauge $e^0_a = 0$ takes the form [31]:

$$S = \frac{1}{16\pi G_0} \int dt d^3x \left[2 \operatorname{Im} \dot{A}^i_a E^a_i - (NH + N^a H_a + N_i G^i) \right], \tag{2.35}$$

where H, H_a and G^i are the Hamiltonian, diffeomorphism and Gauss constraints, respectively [31]. $E^a_i = \sqrt{h} e^a_i$ are the densitized inverse triads, with $h = \det h_{ij}$. h_{ij} is the spatial metric and e^a_i are the inverse triads. Furthermore, the real Hamiltonian constraint has the following form:

$$H = \epsilon^{ij}{}_k E^a_i E^b_j \left(\operatorname{Re} F^k_{ab} + \frac{\Lambda}{3} \epsilon_{abc} E^{ck} \right) = 0, \tag{2.36}$$

substituting the complex one given by

$$H = \epsilon^{ij}{}_k E^a_i E^b_j \left(F^k_{ab} + \frac{\Lambda}{3} \epsilon_{abc} E^{ck} \right) = 0. \tag{2.37}$$

From (2.35) we find the Poisson bracket for the canonical conjugate variables $\operatorname{Im} A^i_a$ and E^a_i

$$\{ \operatorname{Im} A^i_a(\boldsymbol{x}), E^b_j(\boldsymbol{y}) \} = 8\pi G_0 \delta^b_a \delta^i_j \delta(\boldsymbol{x} - \boldsymbol{y}), \tag{2.38}$$

which after quantisation leads to the commutator

$$[\operatorname{Im} A_a^i(\boldsymbol{x}), E_j^b(\boldsymbol{y})] = i l_p^2 \delta_a^b \delta_j^i \delta(\boldsymbol{x}-\boldsymbol{y}), \tag{2.39}$$

with $l_p = \sqrt{8\pi G_0 \hbar}$. Consequently we get

$$E_i^a(\boldsymbol{x}) = -i l_p^2 \frac{\delta}{\delta \operatorname{Im} A_a^i(\boldsymbol{x})}. \tag{2.40}$$

Thus, the constraint (2.36) in the $\operatorname{Im} A_a^i$ representation becomes:

$$\left(\operatorname{Re} B^{kc} - i l_p^2 \frac{\Lambda}{3} \frac{\delta}{\delta \operatorname{Im} A_c^k}\right)\psi = 0, \tag{2.41}$$

where $F_{ab}^i = \epsilon_{abc} B^{ci}$. The solution to this equation is

$$\psi_{CS}(A) = N \exp\left(\frac{3i}{l_p^2 \Lambda} \operatorname{Im} Y_{CS}\right), \tag{2.42}$$

where Y_{CS} is the Chern-Simons functional [32] given by

$$Y_{CS} = \int \operatorname{Tr}\left(AdA + \frac{2}{3}AAA\right) = -\frac{1}{2}\int\left(A^i dA^i + \frac{1}{3}\epsilon_{ijk}A^i A^j A^k\right). \tag{2.43}$$

This last one solves the complex Hamiltonian constraint (2.37) since we have:

$$\frac{\delta Y_{CS}}{\delta A_a^i} = -\frac{1}{2}\epsilon^{abc}F_{bc}^i = -B_i^a. \tag{2.44}$$

To show that (2.42) solves (2.41) we start by decomposing the Ashtekar connection in its real and imaginary parts [33] as $A^i = \alpha^i + i\beta^i$. Accordingly, the imaginary part of Y_{CS} is

$$\begin{aligned} \operatorname{Im} Y_{CS} \quad &= -\frac{1}{2}\int\left[\alpha^i d\beta^i + \beta^i d\alpha^i + \frac{1}{3}\epsilon_{ijk}(\alpha^i\beta^j\alpha^k + \beta^i\alpha^j\alpha^k + \alpha^i\alpha^j\beta^k - \beta^i\beta^j\beta^k)\right] \\ &= -\int\left[d\alpha^i\beta^i + \frac{1}{2}\epsilon_{ijk}\left(\beta^i\alpha^j\alpha^k - \frac{1}{3}\beta^i\beta^j\beta^k\right)\right] \end{aligned} \tag{2.45}$$

where from the first to the second line we computed an integration by parts in first term and discarded the boundary term. In this way, it is straightforward to check that

$$\frac{\delta \operatorname{Im} Y_{CS}}{\delta \beta_c^i} = -\frac{1}{2}\epsilon^{abc}[\partial_a \alpha_b^i - \partial_b \alpha_a^i + \epsilon_{ijk}(\alpha_a^j \alpha_b^k - \beta_a^j \beta_b^k)] \tag{2.46}$$

$$\Leftrightarrow \quad \frac{\delta \operatorname{Im} Y_{CS}}{\delta \operatorname{Im} A_c^i} = -\operatorname{Re} B_i^c, \tag{2.47}$$

which proves our point.

2.2.1 Chern-Simons State for the FRW Metric

We now derive the Chern-Simons state and the EC action in mini-superspace. For the FRW metric presented in the last section, the SD connection (2.32) is given by

$$A^1 = \cos\theta d\phi + \frac{ib}{K}dr \tag{2.48}$$

$$A^2 = -K\sin\theta d\phi + ibr d\theta \tag{2.49}$$

$$A^3 = K d\theta + ibr\sin\theta d\phi \tag{2.50}$$

and the inverse densitized triad components are:

$$E_1^r = a^2 r^2 \sin\theta \tag{2.51}$$

$$E_2^\theta = \frac{a^2 r \sin\theta}{K} \tag{2.52}$$

$$E_3^\phi = \frac{a^2 r}{K}. \tag{2.53}$$

Hence, the first term in the EC action reads:

$$S = \frac{3V_c}{8\pi G_0}\int dt\dot{b}a^2 + ... \tag{2.54}$$

The only other non zero contribution in MSS is the Hamiltonian term:

$$H = \frac{1}{\sqrt{h}}\epsilon^{ij}{}_k E_i^a E_j^b \left(\operatorname{Re} F_{ab}^k + \frac{\Lambda}{3}\epsilon_{abc}E^{ck}\right), \tag{2.55}$$

which we can obtain through $F^i = dA^i + \frac{1}{2}\epsilon^{ijk}A^jA^k$ and whose components are:

$$F^1 = (-1 + K^2 - b^2r^2)\sin\theta d\theta d\phi \quad (2.56)$$

$$F^2 = \left(-K' + \frac{b^2r}{K}\right)\sin\theta drd\phi \quad (2.57)$$

$$F^3 = \left(K' - \frac{b^2r}{K}\right)drd\theta. \quad (2.58)$$

Putting all these together we get:

$$H = \frac{K}{a^3r^2\sin\theta}\left(2E_1^rE_2^\theta F_{r\theta}^3 - 2E_1^rE_3^\phi F_{r\phi}^2 + 2E_2^\theta E_3^\phi F_{\theta\phi}^1 + 2\Lambda E_1^rE_2^\theta E_3^\phi\right) = \quad (2.59)$$

$$= 6a\frac{r^2\sin\theta}{K}\left(-k - b^2 + \frac{\Lambda}{3}a^2\right) \quad (2.60)$$

and so we easily check the EC action in MSS to be (2.30).

On the other hand, to reduce the Chern-Simons state we need to compute (2.45). Starting with the first term we get:

$$d\alpha^i\beta^i = -\left(\frac{2bkr^2}{K} + \frac{b}{K}\right)\sin\theta drd\theta d\phi, \quad (2.61)$$

where the components $d\alpha^i$ are given by:

$$d\alpha^1 = -\sin\theta d\theta d\phi \quad (2.62)$$

$$d\alpha^2 = -K'\sin\theta drd\phi - K\cos\theta d\theta d\phi \quad (2.63)$$

$$d\alpha^3 = K'drd\theta. \quad (2.64)$$

The second one is obtained from

$$A^1A^2A^3 = \left(ibK - \frac{ib^3r^2}{K}\right)\sin\theta drd\theta d\phi. \quad (2.65)$$

Thus, putting both terms together we have

$$\operatorname{Im} Y_{CS} = V_c(b^3 + 3kb) \quad (2.66)$$

and finally the CS state in MSS:

$$\psi_{CS} = N \exp\left[\frac{3iV_c}{l_p^2 \Lambda}(b^3 + 3bk)\right]. \tag{2.67}$$

2.2.2 Chern-Simons State for a Closed FRW Universe

In this subsection we calculate the Chern-Simons state for a closed FRW universe, using the Maurer-Cartan forms that define the spatial unit 3-sphere. These obey $d\omega^i = \frac{1}{2}\epsilon^i{}_{jk}\omega^j\omega^k$ and are given by [33]:

$$\omega^1 = \cos\psi d\theta + \sin\psi \sin\theta d\phi \tag{2.68}$$

$$\omega^2 = \sin\psi d\theta - \cos\psi \sin\theta d\phi \tag{2.69}$$

$$\omega^3 = d\psi + \cos\theta d\phi. \tag{2.70}$$

Here $\psi \in (0, 4\pi)$, $\phi \in (0, 2\pi)$ and $\theta \in (0, \pi)$ are all angular coordinates in $\mathbb{S}^3$. The tetrad is

$$e^0 = N(t)dt \tag{2.71}$$

$$e^i = \frac{a(t)}{2}\omega^i \tag{2.72}$$

and its exterior derivative reads

$$de^i = \frac{b}{a}e^0 e^i + \frac{1}{a}\epsilon^i{}_{jk}e^j e^k, \tag{2.73}$$

which leads to the following connection 1-form:

$$\Gamma^i{}_0 = \frac{b}{2}\omega^i \tag{2.74}$$

$$\Gamma^i{}_j = \frac{1}{2}\epsilon^i{}_{jk}\omega^k. \tag{2.75}$$

Consequently, the Ashtekar connection is

$$A^i = (ib - 1)\,\frac{\omega^i}{2} \tag{2.76}$$

and together with

$$dA^i = \frac{1}{4}\,(ib - 1)\,\epsilon^i{}_{jk}\omega^j\omega^k \tag{2.77}$$

it is easy to compute the Chern-Simons functional (2.43):

$$Y_{CS} = V_c\left(ib^3 + 3ib - 2\right). \tag{2.78}$$

As a result, taking the imaginary part of the above equation, we obtain

$$\psi_{CS} = N\exp\left[\frac{3iV_c}{l_p^2\Lambda}\left(b^3 + 3b\right)\right], \tag{2.79}$$

where $V_c = 2\pi^2$ is the volume of $\mathbb{S}^3$. This agrees with (2.67) for $k = 1$, as expected.

Chapter 3

Dynamical Constants of Nature and Cosmological Time

We consider that cosmological time is the conjugate of the constants of nature [21, 34] which means that we have different definitions of time depending on the relevant constant controlling the dynamics in each epoch. To do so we promote these parameters to phase space variables that will be constants only as a result of the equations of motion. We start by introducing the necessary principles to define our physical clocks. We further derive in detail the cosmology of a single fluid universe and then generalize for multi-fluid models. For instance, we look closer at a universe made of radiation and dark energy. Finally, we present a scalar field as a cosmological clock coupled to gravity through a "deconstantized" gravitational constant.

3.1 Single Perfect Fluid Universe

Since we want to consider the universe as a mixture of perfect fluids we must generalize the action (2.26). We do the replacement $\frac{\Lambda}{6} \to \frac{4\pi G_0}{3}\mathcal{L}_M$ which yields

$$S = \frac{1}{32\pi G_0}\int \epsilon_{abcd}\left[e^a e^b R^{cd} + \frac{4\pi G_0}{3}\mathcal{L}_M e^a e^b e^c e^d\right], \tag{3.1}$$

where $\mathcal{L}_M = -\rho$ and ρ is the energy density given by

$$\rho = \sum_i \frac{C_i}{a^{3+3w_i}}. \tag{3.2}$$

C_i are constants, $a(t)$ is the scale factor and $w_i = 0, -1, 1/3$ for matter, "dark energy" or radiation. The reduction of the action to MSS done in the previous chapter leads to:

$$S = \frac{3V_c}{8\pi G_0} \int dt \left[a^2 \dot{b} - Na \left(-k - b^2 + \sum_i \frac{m_i}{a^{1+3w_i}} \right) \right], \tag{3.3}$$

with $m_i = \frac{8\pi G_0 C_i}{3}$. By the above we get that b and a^2 are canonical conjugate variables with the Poisson bracket $\{b, a^2\} = \frac{8\pi G_0}{3V_c}$, which upon quantization leads to the commutator $[b, a^2] = \frac{il_P^2}{3V_c}$, where $l_P = \sqrt{8\pi G_0 \hbar}$ is the reduced Planck length. For practical reasons and in analogy with the role of $\hbar$ in quantum mechanics we define $\mathfrak{h} = \frac{l_P^2}{3V_c}$. Furthermore the lapse function N is a Lagrange multiplier and so we get

$$\delta_N S = 0 \tag{3.4}$$

$$\Rightarrow \quad k + b^2 - \sum_i \frac{m_i}{a^{1+3w_i}} = 0. \tag{3.5}$$

Through a Legendre transformation

$$S = \int dt \left(\sum_i \dot{q}_i p_i - H \right) \tag{3.6}$$

we obtain the Hamiltonian for gravity plus matter

$$H = \frac{3V_c}{8\pi G_0} Na \left[-(k + b^2) + \sum_i \frac{m_i}{a^{1+3w_i}} \right] = \frac{3V_c}{8\pi G_0} Na\mathcal{H} \tag{3.7}$$

which from (3.5) gives the Hamiltonian constraint equation:

$$\mathcal{H} = 0. \tag{3.8}$$

We shall now examine each epoch separately, i.e. we consider only one i in the above sum at a time (and no sum implied by repeated i from now on). It is more convenient to write (3.8) in the approximate form

$$\mathcal{H} = H_0 - \alpha_i = 0, \tag{3.9}$$

$$H_0 = h_i(b)a^2 \tag{3.10}$$

where

$$h_i(b) = (b^2 + k)^{\frac{2}{1+3w_i}} \tag{3.11}$$

$$\alpha_i = m_i^{\frac{2}{1+3w_i}}. \tag{3.12}$$

The reason to do so originates from the fact that in the b representation $a^2 = -i\mathfrak{h}\partial_b$, which leads to a solvable differential equation in this representation. As we are considering the constants of nature to be the conjugate of time, we need to promote them to phase space variables with conjugate momenta p_i to be identified as time T_i. Thus, (3.3) gets an extra term:

$$S \to S + \frac{3V_c}{8\pi G_0}\int dt\dot{\alpha}_i p_i. \tag{3.13}$$

These α_i are then constants as a result of the equations of motion instead of being parameters set in stone:

$$\dot{\alpha}_i = \{\alpha_i, H\} = 0 \tag{3.14}$$

and classically they are nothing but integration constants like the well known models in unimodular gravity we introduced in Chapter 1. We also have the commutator

$$[\alpha_i, p_i] = i\mathfrak{h}, \tag{3.15}$$

from which we get an effective Schrodinger equation replacing $\alpha_i \to i\mathfrak{h}\partial_{T_i}$ $(T_i = p_i)$:

$$\left(H_0 - i\mathfrak{h}\partial_{T_i}\right)\psi = 0 \tag{3.16}$$

with monochromatic wave solution

$$\Rightarrow \psi(b, T_i; \alpha_i) = \psi_s(b; \alpha_i) e^{-\frac{i}{\mathfrak{h}}\alpha_i T_i}. \tag{3.17}$$

Here, ψ_s is the solution to the WDW equation, $\mathcal{H}\psi_s = 0$:

$$(H_0 - \alpha_i)\psi_s = 0 \tag{3.18}$$

$$\Leftrightarrow \left(-i\mathfrak{h} h_i(b)\partial_b - \alpha_i\right)\psi_s = 0 \tag{3.19}$$

which is easily solved by changing to a "linearizing spatial" variable

$$dX_i = \frac{db}{h_i(b)}, \tag{3.20}$$

$$\Rightarrow \quad \psi_s(b; \alpha_i) = N e^{\frac{i}{\mathfrak{h}}\alpha_i X_i(b)}. \tag{3.21}$$

These are not plane waves in b nor in m_i and so we don't have a linear dispersion relation in these variables. However, we do have the "linearizing" variables X_i and α_i in terms of which the solutions are plane waves moving with fixed speed (set to 1) in mini-superspace.

Moreover, as α_i is now a variable in phase space, we enlarge the space of solutions as we are allowed to have a superposition of these monochromatic solutions. The general solution is then given by

$$\psi(b, T_i) = \int d\alpha_i A(\alpha_i) e^{\frac{i}{\mathfrak{h}}\alpha_i (X_i(b) - T_i)}. \tag{3.22}$$

We can also note that the effective Schrodinger equation (3.16) is in fact a wave equation

$$(\partial_X + \partial_T)\psi = 0 \tag{3.23}$$

with retarded wave solutions:

$$\psi(b, T; \alpha) = F(X - T), \tag{3.24}$$

where F can be any function and the associated conserved current is given by $j^0 = j^1 = |\psi|^2$. The inner product can be defined as:

$$\langle\psi_1|\psi_2\rangle = \int dX \psi_1^*(b, T)\psi_2(b, T) \tag{3.25}$$

with the unitarity condition:

$$\partial_T\langle\psi_1|\psi_2\rangle = 0. \tag{3.26}$$

We can instead use the variable T to define the inner product, i.e.

$$\langle\psi_1|\psi_2\rangle = \int dT \psi_1^*(b, T)\psi_2(b, T) \tag{3.27}$$

which after substituting (3.22) yields

$$\langle\psi_1|\psi_2\rangle = \int d\alpha A_1^*(\alpha)A_2(\alpha). \tag{3.28}$$

3.1.1 Classical Equations of Motion

Regarding the classical Friedman equations, we note that the first one is nothing but the Hamiltonian constraint $H = 0$:

$$b^2 + k = \frac{m}{a^{1+3w}}, \tag{3.29}$$

which will take the usual form if we use the equation of motion for a with $N = 1$:

$$\dot{a} = \{a, H\} = Nb. \tag{3.30}$$

Furthermore, substituting the above equation (again with $N = 1$) in (3.33), we get the second Friedman equation

$$\ddot{a} = -a\frac{1+3w}{2}\frac{m}{a^{3(1+w)}}. \tag{3.31}$$

To describe the motion of the peak of the wave function, we must introduce the group velocity associated with each physical time, defined on $\{b, T_i\}$ space as

$$c_{gi} = \frac{db}{dT_i} = \frac{\dot{b}}{\dot{T}_i}. \tag{3.32}$$

Using the equations of motion for b

$$\dot{b} = \{b, H\} = -N\frac{1+3w}{2}\frac{\alpha^{\frac{1+3w}{2}}}{a^{2+3w}} \tag{3.33}$$

and for T_i

$$\dot{T}_i = \{T_i, H\} = -Na^{-3w_i}\frac{1+3w_i}{2}\alpha_i^{\frac{-1+3w_i}{2}} \tag{3.34}$$

we obtain

$$c_{gi} = \frac{\alpha}{a^2} = h(b) = \frac{db}{dX} = \frac{\dot{b}}{\dot{X}}, \tag{3.35}$$

which comparing with (3.32) shows that the classical trajectory for a single fluid is

$$\dot{X} = \dot{T}_i. \tag{3.36}$$

We should now highlight the following: radiation ($w = 1/3$) time is minus the conformal time

$$\dot{T}_r = -\frac{N}{a} \tag{3.37}$$

$$\Leftrightarrow \qquad dT_r = -\frac{N dt}{a}; \tag{3.38}$$

"matter" ($w = 0$) time is proportional to proper cosmological time (for $N = 1$)

$$\dot{T}_m = -\frac{1}{2m} \tag{3.39}$$
$$\Leftrightarrow \quad T_m = -\frac{1}{2m}t \tag{3.40}$$

and lambda ($w = -1$) time is proportional to Misner's volume time [35]

$$\dot{T}_\phi = \frac{Na^3\Lambda^2}{9}. \tag{3.41}$$

Thus, instead of using t as a time coordinate, we consider the physical times T_i, which are all a function of t, classically and on-shell, but quantum mechanically they are all independent variables.

It is now important to pause and analyse this new connection representation picture. Rather than using the conventional description with $a = a(t)$ we consider $b = b(T)$, which is completely equivalent from a classical point of view. Thus, expanding and contracting universes have, respectively, $b > 0$ and $b < 0$. A static universe is described by $b = 0$. Also, for a given single fluid content, b can either increase for $w < -1/3$, decrease for $w > -1/3$ or remain unchanged for $w = -1/3$. Hence, a bounce in b corresponds to a universe transitioning from a decelerated to an accelerated expansion.

3.2 Multi-time

In the case where we consider the model with multiple fluids (i.e. including the sum over i in (3.7)), we will have to deal with multiple times. Thus, we need to introduce the vectors $\boldsymbol{\alpha}$, representing all the constants associated with the fluids we are considering, and $\boldsymbol{T}$ which are their conjugate times. In this way, we will no longer get a Schrodinger equation like (3.16), but instead a PDE in all the different times obtained by doing the replacement:

$$\boldsymbol{\alpha} \rightarrow i\mathfrak{h}\partial_{\boldsymbol{T}}, \tag{3.42}$$

which leads to

$$\mathcal{H}[b, a^2, \boldsymbol{\alpha} \to i\mathfrak{h}\partial_{\boldsymbol{T}}]\psi = 0, \tag{3.43}$$

with general solution:

$$\psi(b, \boldsymbol{T}) = \int d\boldsymbol{\alpha} A(\boldsymbol{\alpha}) e^{-\frac{i}{\mathfrak{h}}\boldsymbol{\alpha}\cdot\boldsymbol{T}} \psi_s(b, \boldsymbol{\alpha}) \tag{3.44}$$

and $\psi_s(b, \alpha)$ solves the WDW equation with constant $\boldsymbol{\alpha}$ as before. It's important to note that, in general, this $\psi_s(b, \alpha)$ can be more complicated than the one in (3.21), having the general form:

$$\Rightarrow \psi_s(b; \boldsymbol{\alpha}) = N e^{\frac{i}{\mathfrak{h}} P(b, \boldsymbol{\alpha})}. \tag{3.45}$$

We now see that for the more realistic cases, we either no longer have a "linearizing" variable or if we do, it will not only be a function of b, but a function of α as well. Therefore, it is better to get back to the variables b and T and identify MSS as a dispersive medium, with dispersion relation given by:

$$\boldsymbol{\alpha} \cdot \boldsymbol{T} - P(b; \boldsymbol{\alpha}) = 0. \tag{3.46}$$

Assuming that $A(\boldsymbol{\alpha})$ is factorizable and peaked around $\boldsymbol{\alpha}_0$ we can write

$$P(b; \boldsymbol{\alpha}) = P(b; \boldsymbol{\alpha}_0) + \sum_i \frac{\partial P}{\partial \alpha_i}\bigg|_{\alpha_{0i}} (\alpha_i - \alpha_{i0}) + ... \tag{3.47}$$

which allows (3.44) to get the approximate form:

$$\psi(b; \boldsymbol{T}) \approx e^{\frac{i}{\mathfrak{h}}[P(b, \boldsymbol{\alpha}_0) - \boldsymbol{\alpha}_0 \cdot \boldsymbol{T}]} \prod_i \int d\alpha_i A(\alpha_i) e^{-\frac{i}{\mathfrak{h}}(\alpha_i - \alpha_{0i})(T_i - \frac{\partial P}{\partial \alpha_i})}, \tag{3.48}$$

where the first factor represents a monochromatic wave centred in $\boldsymbol{\alpha}_0$ and the second describes an envelope moving according to:

$$T_i = \left.\frac{\partial P}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}_0}. \tag{3.49}$$

Hence, by dotting this equation,

$$\dot{T}_i = \frac{\partial^2 P}{\partial b \partial \alpha_i}\dot{b}, \tag{3.50}$$

the group velocity defined on $\{b, T_i\}$ space:

$$c_{gi} = \left.\frac{db}{dT_i}\right|_{\boldsymbol{\alpha}_0} = \frac{\dot{b}}{\dot{T}_i} \tag{3.51}$$

can be written as

$$c_{gi} = \left(\frac{\partial^2 P}{\partial \alpha_i \partial b}\right)^{-1}. \tag{3.52}$$

3.3 Pure Λ Universe

We start by illustrating the above analysis for the cosmological constant Λ. The implications are a combination of results from unimodular gravity [10, 16] and the concept of Chern-Simons time [36] with a reinterpretation of the latter. In this case we have $w = -1$ and the Hamiltonian (3.7) takes the form:

$$H = \frac{3V_c}{8\pi G_0} Na\left[-(k+b^2) + \frac{\Lambda}{3}a^2\right]. \tag{3.53}$$

Hence, the Hamiltonian constraint equation (3.8) gives

$$\left[-(k+b^2) - \frac{il_P^2\Lambda}{9V_c}\partial_b\right]\psi_s = 0 \tag{3.54}$$

whose solution is the Chern-Simons state reduced to MSS (that we will recover next):

$$\psi_{CS} = N e^{i\frac{9V_c}{l_P^2\Lambda}\left(\frac{b^3}{3}+bk\right)}. \tag{3.55}$$

This is a pure phase which is the product of a "frequency" proportional to $1/\Lambda$ and the Chern-Simons time.

We can now apply the treatment introduced in the present chapter to (3.53) to write it in the same form as (3.9), from which it is straightforward to obtain

$$m_i = \frac{\Lambda}{3} \tag{3.56}$$

$$h_i(b) = (b^2 + k)^{-1} \tag{3.57}$$

$$\alpha_i = \frac{3}{\Lambda} \equiv \phi. \tag{3.58}$$

Following the same procedure we then have the commutator $[\phi, T_\phi] = \frac{il_P^2}{3V_c}$ and so we are left again with the same Schrodinger equation as (3.16), where $T_\phi = p_\phi$ is the conjugate momentum of ϕ identified as the time variable. The "spatial" part ψ_s will also obey equation (3.19) and so it is given by (3.21). Thus, the only thing left to do is to compute $X_i(b)$, still defined by (3.20):

$$X_i(b) = X_\phi = \int db(b^2 + k) = \frac{b^3}{3} + kb, \tag{3.59}$$

which is exactly the Chern-Simons functional in MSS and gives the Chern-Simons state we got in (3.55).

Our interpretation of Chern-Simons time is however different from that of Smolin and Soo [36, 37]. The full monochromatic solution is then

$$\psi(b, T_\phi) = N e^{\frac{i}{\hbar}\phi(X_\phi(b) - T_\phi)}. \tag{3.60}$$

Time evolution happens in terms of a time that is the conjugate of $3/\Lambda$, rather than the Chern-Simons functional. Also, $X_\phi(b) = \mathrm{Im}(Y_{CS})$ is a spatial variable, not a time.

Having enlarged our phase space by promoting Λ to a variable, we can now write the general solution as a superposition of these plane waves:

$$\psi(b, T_\phi) = \int d\phi A(\phi) e^{\frac{i}{\mathfrak{h}}\phi(X_\phi(b) - T_\phi)}. \tag{3.61}$$

We now consider three interesting cases for different amplitudes $A(\phi)$. First, we can have a completely undetermined ϕ, this being done with $A(\phi) = \frac{1}{2\pi}$, which leads to

$$\psi(b, T_\phi) = \delta(X_\phi - T_\phi). \tag{3.62}$$

This is interpreted as a light "ray" in MSS with a time T_ϕ fully fixed by X_ϕ. We thus see that a total delocalization in the cosmological constant is associated with an infinitely sharp clock. This solution was also interpreted as a conformal constraint in the parity-even branch of quasi-topological theories, in which Λ is allowed to vary as a result of adding a Gauss-Bonnet topological term [38, 39, 40].

On the other hand, if we consider a fully fixed Λ, $A(\phi) = \delta(\phi - \phi_0)$, we obtain:

$$\psi(b, T_\phi) = e^{\frac{i}{\mathfrak{h}}\phi_0(X_\phi(b) - T_\phi)}. \tag{3.63}$$

This is the Chern-Simons state in the usual EC theory, where Λ is fully fixed: an infinite plane wave moving at the speed of light, however completely delocalized such that time disappears. Thus, we highlight an important fact: *infinitely sharp constants are failed clocks.*

The last important case are coherent states centred around a fixed value of Lambda, given by a normal distribution:

$$A(\phi) = \frac{e^{-\frac{(\phi - \phi_0)^2}{4\sigma^2}}}{(2\pi\sigma^2)^{1/4}}. \tag{3.64}$$

Performing the integration in (3.61) we arrive at

$$\psi(b, T_\phi) = (8\pi\sigma^2)^{1/4} \exp\left[-\frac{\sigma^2}{\mathfrak{h}^2}(X_\phi(b) - T_\phi)^2 + \frac{i}{\mathfrak{h}}\phi_0(X_\phi(b) - T_\phi)\right], \tag{3.65}$$

from which we can read the saturated Heisenberg uncertainty principle:

$$\sigma_T \sigma_\alpha = \frac{\mathfrak{h}}{2}. \tag{3.66}$$

All the above analysis is equally valid for any single fluid universe (any α), only $X(b)$ will be different case by case.

3.4 Radiation Domination

For the case of a radiation dominated universe the equation of state is $w = 1/3$. Thus, substituting w back in equations (3.11) and (3.12) and keeping $m_i = m$ as a constant one gets

$$h_i(b) = b^2 + k \tag{3.67}$$

$$\alpha_i = m. \tag{3.68}$$

Once again we need to solve both the effective Schrodinger and the WDW equations and, the only object that differs from the previous section, is the function $X_i(b)$ which is now given by

$$X_i(b) = X_r = \int \frac{db}{b^2 + k} = \frac{\arctan\left(\frac{b}{\sqrt{k}}\right)}{\sqrt{k}}, \quad k > 0 \tag{3.69}$$

$$= -\frac{1}{b}, \quad k = 0 \tag{3.70}$$

$$= -\frac{\operatorname{arctanh}(\frac{b}{\sqrt{|k|}})}{\sqrt{|k|}}, \quad k < 0. \tag{3.71}$$

This should be seen as the equivalent to the Chern-Simons functional but for a radiation dominated universe. The general solution is again a superposition of monochromatic plane waves

$$\psi(b) = \int dm A(m) e^{i\frac{3V_c}{l_P^2} m(X_r(b) - T_r)}. \tag{3.72}$$

3.5 Λ plus Radiation

We consider now a model with two fluids, dark energy ($w = -1$) and radiation ($w = 1/3$). Hence, the Hamiltonian (3.7) takes the form

$$H = \frac{3V_c}{8\pi G_0} N a \left(-g + \frac{a^2}{\phi} + \frac{m}{a^2} \right) = 0, \tag{3.73}$$

where

$$\phi = \frac{3}{\Lambda} \tag{3.74}$$

$$\boldsymbol{\alpha} = (\phi, m) \tag{3.75}$$

$$g = k + b^2 \tag{3.76}$$

and with the usual Hamiltonian constraint:

$$\mathcal{H} = 0. \tag{3.77}$$

From this we can solve for a^2 to obtain

$$\Rightarrow \quad -ga^2 + \frac{a^4}{\phi} + m = 0 \tag{3.78}$$

$$\Rightarrow \quad a_\pm^2 = \frac{g \pm \sqrt{g^2 - 4m/\phi}}{2/\phi}, \tag{3.79}$$

which allows us to write the Hamiltonian constraint in the form of two constraints:

$$\mathcal{H} = \frac{1}{\phi}(a^2 - a_+^2)(a^2 - a_-^2) = 0 \tag{3.80}$$

$$\Rightarrow \quad \mathcal{H}_\pm = h_\pm(b; m/\phi)a^2 - \phi = 0, \tag{3.81}$$

$$h_\pm(b; m/\phi) = \frac{2}{g \pm \sqrt{g^2 - \frac{4}{3}\Lambda m}}. \tag{3.82}$$

As opposed to the single fluid case, H_0 now depends on the constants $\boldsymbol{\alpha}$. Having the Hamiltonian written in this form we have everything we need as in the previous cases to get ψ_s through

$$\left(-i\mathfrak{h} h_\pm(b)\partial_b - \phi\right)\psi_{s\pm} = 0 \tag{3.83}$$

$$\Rightarrow \quad \psi_{s\pm}(b; m/\phi) = N e^{\frac{i}{\mathfrak{h}}\phi X_\pm(b)} \tag{3.84}$$

with $X_\pm$ given by

$$X_\pm(b; \Lambda m) = \int db \frac{1}{2}\left(g \pm \sqrt{g^2 - 4m/\phi}\right). \tag{3.85}$$

Furthermore, a^2 must be real which means that

$$g^2 \geq \frac{4m}{\phi}. \tag{3.86}$$

Taking now the limit $g^2 >> 4m/\phi$, we get in first order, for each branch $+/-$

$$X_+ \approx \int dbg = X_\phi \tag{3.87}$$

$$X_- \approx \frac{m}{\phi}\int \frac{db}{b^2 + k} = \frac{m}{\phi}X_r. \tag{3.88}$$

Substituting these in $\psi_{s\pm}$ shows that, at this order, each branch corresponds to pure Λ or pure radiation domination:

$$\psi_{s+}(b; m/\phi) \approx N e^{\frac{i}{\mathfrak{h}}\phi X_\phi(b)} \tag{3.89}$$

$$\psi_{s-}(b; m/\phi) \approx N e^{\frac{i}{\mathfrak{h}} m X_r(b)}. \tag{3.90}$$

We can see that ψ_s is a piecewise function where each branch corresponds to each dominating constant and the respective X_i. Thus, for $g > 0$ (expanding universe), we have a transition from a decelerated ($\dot{b} < 0$) to an accelerated ($\dot{b} > 0$) expansion manifested by a "bounce" in b space at $g^2 \approx 4m/\phi$.

Let us now examine what happens when we consider the next order terms in the expansion. Therefore, we have:

$$X_+ \approx \int db \left(g - \frac{m}{g\phi}\right) = X_\phi - \frac{m}{\phi} X_r \tag{3.91}$$

$$X_- \approx \frac{m}{\phi} \int \frac{db}{g} + \frac{m^2}{\phi^2} \int \frac{db}{g^3} = \frac{m}{\phi} X_r + \frac{m^2}{\phi^2} \int \frac{db}{g^3}. \tag{3.92}$$

This yields the following branches for the wave function:

$$\psi_+(b; m/\phi, T_1, T_2) = N e^{\frac{i}{\mathfrak{h}}[\phi(X_\phi(b) - T_1) - m(X_r(b) + T_2)]} \tag{3.93}$$

$$\psi_-(b; m/\phi, T_1, T_2) = N e^{\frac{i}{\mathfrak{h}}\left[\phi\left(\frac{m^2}{\phi^2} \int \frac{db}{g^3} - T_1\right) + m(X_r(b) - T_2)\right]}. \tag{3.94}$$

We can now observe that deep in the Lambda epoch the wave function factorizes (as long as the amplitude $A(\boldsymbol{\alpha})$ factorizes) and so we can write

$$\psi_+(b; m/\phi, T_1, T_2) = F_1(X_\phi - T_1) F_2(X_r + T_2). \tag{3.95}$$

Thus, the two times are quantum mechanically independent and the classical trajectories for each plane wave are $\dot{X}_\phi = \dot{T}_1$ and $\dot{X}_r = -\dot{T}_2$. This factorization does not happen in the radiation epoch and we no longer have a plane wave in the minority clock (Λ time). Nevertheless, the wave packet's peak still follows the correct classical trajectory, as we can see through the group velocity:

$$c_g^{-1} = \frac{\partial^2}{\partial\phi\partial b} \left(\frac{m^2}{\phi} \int \frac{db}{g^3}\right) = \frac{\dot{T}_1}{\dot{b}} \tag{3.96}$$

that leads (using the equation of motion for T_1) to

$$\dot{b} = -N\frac{m}{a^3}, \tag{3.97}$$

which is the second Friedman equation for a radiation universe.

We can now compute exactly the group velocities associated with each cosmological time for a peaked distribution. The function $P(b, \boldsymbol{\alpha})$ defined generally before is now

$$P_{\pm} = \phi X_{\pm} \tag{3.98}$$

and the group speeds are:

$$c_{g1} = \frac{\dot{b}}{\dot{T}_1}\bigg|_{\text{peak}} = \left(\frac{\partial^2 P}{\partial\phi\partial b}\right)^{-1} \tag{3.99}$$

$$c_{g2} = \frac{\dot{b}}{\dot{T}_2}\bigg|_{\text{peak}} = \left(\frac{\partial^2 P}{\partial m\partial b}\right)^{-1}. \tag{3.100}$$

After computing these derivatives and using the following relations

$$h_{\pm} = \frac{\phi}{a^2} \tag{3.101}$$

$$g = \frac{a^2}{\phi} + \frac{m}{a^2} \tag{3.102}$$

$$\pm\sqrt{g^2 - 4m/\phi} = \frac{a^2}{\phi} - \frac{m}{a^2}, \tag{3.103}$$

we arrive at

$$\frac{\partial^2 P}{\partial\phi\partial b} = \frac{1}{h_{\pm}} \pm \frac{m}{\phi}\frac{1}{\sqrt{g^2 - 4m/\phi}} = \frac{a^4/\phi^2}{a^2/\phi - m/a^2} \tag{3.104}$$

$$\frac{\partial^2 P}{\partial m\partial b} = \pm\frac{1}{\sqrt{g^2 - 4m/\phi}} = -\frac{1}{a^2/\phi - m/a^2} \tag{3.105}$$

and so

$$c_{g1} = \frac{\phi^2}{a^4}\left(\frac{a^2}{\phi} - \frac{m}{a^2}\right) \tag{3.106}$$

$$c_{g2} = -\frac{a^2}{\phi} + \frac{m}{a^2}. \tag{3.107}$$

Indeed we see that the group velocity with respect to each T_i is the same for both branches and, using

$$\dot{T}_1 = \{T_1, H\} = N\frac{a^3}{\phi^2} \tag{3.108}$$

$$\dot{T}_2 = \{T_2, H\} = -\frac{N}{a}, \tag{3.109}$$

we check that the peak moves according to the classical trajectory

$$\dot{b} = \{b, H\} = N\left(\frac{a}{\phi} - \frac{m}{a^3}\right), \tag{3.110}$$

as we just recover the second Friedman equation for this universe.

3.6 Scalar Field as a Cosmological Clock

Let us now consider that the matter action is given by a massless scalar field [41]:

$$S = \frac{1}{32\pi G_0}\int[\epsilon_{abcd}e^a e^b R^{cd} - 16\pi G_N d\phi(\star d\phi)], \tag{3.111}$$

where G_0 is the fixed gravitational constant, G_N is the gravitational coupling to matter up to be "deconstantized" and ϕ is the scalar field. $\star d\phi = \frac{1}{3!}\epsilon_{\mu\nu\alpha\beta}\partial^\mu\phi dx^\nu dx^\alpha dx^\beta$ is the Hodge dual of $d\phi$. Reducing to MSS, the action takes the form:

$$S = \frac{3V_c}{8\pi G_0}\int dt\left[\dot{b}a^2 + Na(b^2+k) + \frac{1}{2}\frac{8\pi G_N}{3}\dot{\phi}^2\frac{a^3}{N}\right] = \tag{3.112}$$

$$= \frac{3V_c}{8\pi G_0}\int dt\left[\dot{b}a^2 + \pi_\phi\dot{\phi} - Na\left(-(b^2+k) + \frac{1}{2}\frac{3}{8\pi G_N a^4}\pi_\phi^2\right)\right] \tag{3.113}$$

with

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \tilde{G}_N \dot{\phi} \frac{a^3}{N}, \tag{3.114}$$

$$\tilde{G}_N \equiv \frac{8\pi G_N}{3}. \tag{3.115}$$

Thus, once again we get the Hamiltonian constraint

$$\mathcal{H} = -(b^2 + k) + \frac{\pi_\phi^2}{2a^4 \tilde{G}_N} = 0, \tag{3.116}$$

which can be simplified to the usual form

$$\Rightarrow \qquad h(b)a^2 - \alpha_1 \alpha_2 = 0 \tag{3.117}$$

where

$$h(b) = \sqrt{k + b^2} \tag{3.118}$$

$$\alpha_1 = \frac{1}{\sqrt{2\tilde{G}_N}} \tag{3.119}$$

$$\alpha_2 = \pi_\phi. \tag{3.120}$$

In order to have α_1 as a phase space variable the action (3.113) gets an extra term:

$$S \to S + \frac{3V_c}{8\pi G_0} \int dt \dot{\alpha}_1 T_1 \tag{3.121}$$

and T_1 is the time conjugate to α_1. Also, as ϕ is the conjugate variable to π_ϕ, we have the scalar field being identified as a time variable $T_2 = -\phi$.

In the representation diagonalizing b, T_1 and T_2 we therefore transform the Hamiltonian constraint, not in a timeless WDW equation, but in a double-time Schrodinger equation:

$$\left(-i\mathfrak{h} h(b)\partial_b + \mathfrak{h}^2 \frac{\partial^2}{\partial T_1 \partial T_2} \right) \psi = 0. \tag{3.122}$$

The solution to this equation is then the general wave function written as a superposition of plane waves:

$$\psi(b;T_1,T_2) = \int d\alpha_1 d\alpha_2 A(\alpha_1,\alpha_2) e^{\frac{i}{\mathfrak{h}}[X(b)\alpha_1\alpha_2 - \alpha_1 T_1 - \alpha_2 T_2]}, \tag{3.123}$$

where

$$X(b) = \int \frac{db}{h(b)} \qquad = \text{arctanh}\frac{b}{\sqrt{b^2+k}}, \quad k>0 \tag{3.124}$$

$$= \log b, \quad k=0 \tag{3.125}$$

$$= \text{arctanh}\frac{b}{\sqrt{b^2-|k|}}, \quad k<0 \tag{3.126}$$

is the linearizing variable for $w=1$.

The equations of motion for ϕ and π_ϕ are

$$\dot{\phi} = \{\phi, H\} = \frac{N\pi_\phi}{\tilde{G}_N a^3} \tag{3.127}$$

$$\dot{\pi}_\phi = \{\pi_\phi, H\} = 0, \tag{3.128}$$

where from the second equation we see that α_2 is a conserved quantity, as expected since H is not time dependent. The same is true for α_1.

The group velocities with respect to each physical time are defined as

$$c_{gi} = \frac{db}{dT_i} = \frac{\dot{b}}{\dot{T}_i} \tag{3.129}$$

and using the equations of motion for both b and T_i we obtain:

$$\dot{b} = \{b, H\} = -2N\frac{\alpha_1^2\alpha_2^2}{a^5} \tag{3.130}$$

$$\dot{T}_1 = \{T_1, H\} = -2N\frac{\alpha_1\alpha_2^2}{a^3} \tag{3.131}$$

$$\dot{T}_2 = \{T_2, H\} = -2N\frac{\alpha_1^2\alpha_2}{a^3}, \tag{3.132}$$

which leads to

$$c_{g1} = \frac{\alpha_1}{a^2} \tag{3.133}$$

$$c_{g2} = \frac{\alpha_2}{a^2}. \tag{3.134}$$

On the other hand, we can use the dispersion relation for the medium given by

$$P(b, \alpha_1, \alpha_2) - \boldsymbol{\alpha} \cdot \boldsymbol{T} = 0, \tag{3.135}$$

where for the case in (3.123), $P(b, \alpha_1, \alpha_2)$ is simply

$$P(b, \alpha_1, \alpha_2) = X(b)\alpha_1\alpha_2, \tag{3.136}$$

to obtain the group speed introduced before:

$$c_{gi} = \left(\frac{\partial^2 P}{\partial \alpha_i \partial b}\right)^{-1}. \tag{3.137}$$

Hence, computing the derivatives and using (3.117), we check the previous equations obtained for these quantities:

$$c_{g1} = \frac{h(b)}{\alpha_2} = \frac{\alpha_1}{a^2} \tag{3.138}$$

$$c_{g2} = \frac{h(b)}{\alpha_1} = \frac{\alpha_2}{a^2}. \tag{3.139}$$

Recalling that $dX = \frac{db}{h(b)}$, the previous equations take the form

$$c_{g1} = \frac{\dot{b}}{\alpha_2 \dot{X}} \tag{3.140}$$

$$c_{g2} = \frac{\dot{b}}{\alpha_1 \dot{X}}, \tag{3.141}$$

which by comparison with (3.129) leads to

$$\dot{T}_1 = \alpha_2 \dot{X} \tag{3.142}$$

$$\dot{T}_2 = \alpha_1 \dot{X} \tag{3.143}$$

$$\Rightarrow \quad \dot{T}_1 = \frac{\alpha_2}{\alpha_1}\dot{T}_2 \tag{3.144}$$

and so classically, T_1 and T_2 are not independent time variables. Nonetheless, quantum mechanically they still describe two different times. This also shows that the motion of the peak of either coherent packets follows the classical equations of motion.

3.6.1 Classical Equations

For a massless scalar field the stress-energy tensor is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi \tag{3.145}$$

and for a perfect fluid we have

$$T_{\mu\nu} = (P+\rho)u_\mu u_\nu + P g_{\mu\nu}, \tag{3.146}$$

where P is the pressure, ρ the energy density and $u^\mu = (1,0,0,0)$ is the proper velocity of the fluid in the cosmological frame. Comparing these expressions we see that both P and ρ are the same

$$P = \rho = \frac{\dot{\phi}^2}{2}, \tag{3.147}$$

which shows that we can treat a scalar field as a perfect fluid with equation of state $w = 1$.

Thus, for the first Friedmann equation we get

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \tilde{G}_N \rho \tag{3.148}$$

$$\Rightarrow \quad b^2 + k = \tilde{G}_N \frac{\dot{\phi}^2}{2} a^2 \tag{3.149}$$

which is nothing but the Hamiltonian constraint we got in (3.116).

The second Friedmann equation is then equivalent to the equation of motion we found for b (3.130):

$$\frac{\ddot{a}}{a} = -\frac{1}{2}\tilde{G}_N(\rho + 3P) \tag{3.150}$$

$$\Rightarrow \quad \dot{b} = -\tilde{G}_N \dot{\phi}^2 a \tag{3.151}$$

and confirms that the peak of the wave function follows the classical trajectory.

3.6.2 Solutions

This setting allows us to explore the quantum implications of multi-time whenever two times can be used in the same region. The Heisenberg uncertainty principle is a purely kinematical result involving any canonical pair, and as such we have for a two-time, two-constant setting:

$$\sigma_{Ti}\sigma_{\alpha i} \geq \frac{\mathfrak{h}}{2}. \tag{3.152}$$

The question, however, is how the dynamical solutions to the theory make these two sets of uncertainties interact: Can we saturate them simultaneously? Or do multiple clocks get in each other's way? Rather than providing a formal analysis we present a number of physically obvious solutions and examine how they relate to the Heisenberg uncertainty bounds.

For example, we can consider the extreme case of 2 failed clocks, i.e. of two infinitely sharp constants:

$$A(\alpha_1, \alpha_2) = \delta(\alpha_1 - \alpha_{10})\delta(\alpha_2 - \alpha_{20}). \tag{3.153}$$

Unsurprisingly, this leads to the original plane waves:

$$\psi(b, T_1, T_2) = N e^{\frac{i}{\mathfrak{h}}[X(b)\alpha_{10}\alpha_{20} - \alpha_{10}T_1 - \alpha_{20}T_2]}. \tag{3.154}$$

We can also consider the case of perfect localization in one of the times (i.e. a uniform distribution in the associated constant):

$$A(\alpha_1, \alpha_2) = \frac{1}{2\pi} A(\alpha_2) \tag{3.155}$$

$$\Rightarrow \quad \psi(b, T_1, T_2) = \frac{-il_P^2}{6\pi V_c X} A\left(\frac{T_1}{X}\right) e^{-\frac{i}{\mathfrak{h}}\frac{T_1 T_2}{X}} \tag{3.156}$$

where $A(\alpha_2)$ can then be a normal distribution.

Then, we can consider the case of a semi-classical state in one clock and a failed clock in the other:

$$A(\alpha_1, \alpha_2) = \delta(\alpha_1 - \alpha_{10})\sqrt{\boldsymbol{N}(\alpha_{20}, \sigma)}. \tag{3.157}$$

Integrating, leads to the usual coherent state solution, i.e. the product of a plane wave centered on $\boldsymbol{\alpha}_0$ and a Gaussian envelope:

$$\psi(b, T_1, T_2) = N e^{-\frac{\sigma^2}{\mathfrak{h}^2}(\alpha_{10}X - T_2)^2 + \frac{i}{\mathfrak{h}}[\alpha_{20}\alpha_{10}X - \alpha_{10}T_1 - \alpha_{20}T_2]} \tag{3.158}$$

$$= \psi(b, T_1, T_2; \boldsymbol{\alpha}_0) e^{-\frac{\sigma^2}{\mathfrak{h}^2}(\alpha_{10}X - T_2)^2}. \tag{3.159}$$

The only novelty is that the fixed constant α_{10} (i.e. the failed clock) now acts as the inverse of the speed of propagation of the wave packet in the semi-classical time and space. This solution saturates, with equally spread uncertainties, the standard uncertainty principle in the variable with the coherent amplitude:

$$\sigma_{T2}^2 = \sigma_{\alpha 2}^2 = \frac{\mathfrak{h}}{2}. \tag{3.160}$$

All of this is unsurprising because having an infinitely sharp constant is the same as not having deconstantized that constant at all, so that its conjugate is a failed, or uniformly distributed clock. We could of course have chosen either of the two clocks as the failed clock.

Finally we consider the fully semiclassical case of the product of two Gaussians:

$$A(\alpha_1, \alpha_2) = \sqrt{\boldsymbol{N}(\alpha_{10}, \sigma_1)}\sqrt{\boldsymbol{N}(\alpha_{20}, \sigma_2)}. \tag{3.161}$$

Using

$$\int d^n x e^{-\frac{1}{2}A_{ij}x^i x^j + B_i x^i} = \sqrt{\frac{(2\pi)^n}{\det(A)}} e^{\frac{1}{2}B^i (A^{-1})_{ij} B^j} \tag{3.162}$$

with

$$A_{ij} = \begin{bmatrix} \frac{1}{2\sigma_1^2} & \frac{-iX}{\mathfrak{h}} \\ \\ \frac{-iX}{\mathfrak{h}} & \frac{1}{2\sigma_2^2} \end{bmatrix} \tag{3.163}$$

and

$$B_i = \frac{\alpha_{i0}}{2\sigma_i^2} - \frac{i}{\mathfrak{h}} T_i \tag{3.164}$$

we arrive at:

$$\begin{aligned} \psi(b, T_1, T_2) = \mathfrak{h}\sqrt{\frac{8\pi\sigma_1\sigma_2}{\mathfrak{h}^2 + 4\sigma_1^2\sigma_2^2 X^2}} \exp\left[-\sum_{i=1}^{2} \frac{\alpha_{i0}^2}{4\sigma_i^2}\right] \cdot \\ \cdot \exp\left[\frac{1}{2\det(A)}\left(\frac{1}{2\sigma_2^2}\left(\frac{\alpha_{10}}{2\sigma_1^2} - \frac{iT_1}{\mathfrak{h}}\right)^2 + \frac{1}{2\sigma_1^2}\left(\frac{\alpha_{20}}{2\sigma_2^2} - \frac{iT_2}{\mathfrak{h}}\right)^2\right)\right] \cdot \\ \cdot \exp\left[\frac{iX}{\mathfrak{h}\det(A)}\left(\frac{\alpha_{10}}{2\sigma_1^2} - \frac{iT_1}{\mathfrak{h}}\right)\left(\frac{\alpha_{20}}{2\sigma_2^2} - \frac{iT_2}{\mathfrak{h}}\right)\right], \end{aligned} \tag{3.165}$$

with $\det(A) = \frac{1}{4\sigma_1^2\sigma_2^2} + \frac{X^2}{\mathfrak{h}^2}$. This can be repackaged as

$$\psi(b, T_1, T_2) = \sqrt{\frac{8\pi\sigma_1\sigma_2\mathfrak{h}^2}{\mathfrak{h}^2 + 4\sigma_1^2\sigma_2^2X^2}} \exp\left[-\frac{\sigma_1^2(T_1 - \alpha_{20}X)^2 + \sigma_2^2(T_2 - \alpha_{10}X)^2}{\mathfrak{h}^2 + 4\sigma_1^2\sigma_2^2X^2}\right] \cdot$$
$$\cdot \exp\left[\frac{i\mathfrak{h}}{\mathfrak{h}^2 + 4\sigma_1^2\sigma_2^2X^2}\left(\alpha_{10}\alpha_{20}X - \alpha_{10}T_1 - \alpha_{20}T_2 - \frac{4\sigma_1^2\sigma_2^2}{\mathfrak{h}^2}T_1T_2X\right)\right] \tag{3.166}$$

with the interesting result that although the wave function does not factor, the cross terms are all in its phase (suggesting interesting entanglement effect), so that the probabilities do factor nonetheless.

Thus, we arrive at a modified saturated Heisenberg uncertainty relation:

$$\sigma_{Ti}^2\sigma_i^2 = \frac{\mathfrak{h}^2}{4} + \sigma_1^2\sigma_2^2X^2 \tag{3.167}$$

which clearly reduces to the usual (3.152) should one of the $\sigma = 0$ (i.e. one of the clocks be a failed clock). Indeed we prove that in that situation the failed clock may still be a coherent infinitely squeezed state, saturating the uncertainty relations. However the situation is totally different when two clocks are at work, as we see. In fact there is an excess-uncertainty in the uncertainty relations of both clocks/constants due to their simultaneous use, which only vanishes when either $X = 0$ or we give up on one of the clocks.

Thus, the moral of the story is that having more than one clock only makes the uncertainty worse, the more so the more the other clock becomes a perfect clock, and the larger the X^2. The naive:

$$\sigma_1^2 = \sigma_2^2 = \frac{\mathfrak{h}}{2} \tag{3.168}$$

leads to:

$$\sigma_{Ti}^2 = \frac{\mathfrak{h}}{2}(1 + X^2) \tag{3.169}$$

with inevitable $X(b)$-dependent squeezing.

3.7 Mixture of a Scalar Field and Radiation

We now focus on another multi-fluid problem considering a mixture of radiation and a massless scalar field. In light of what we have done before, we only need to modify the action (3.113) by adding the term

$$S \to S - \frac{3V_c}{8\pi G_0}\int dtNa\frac{m}{a^{1+3w}}, \tag{3.170}$$

where m is another constant that will be promoted to a phase space variable, and for radiation $w = 1/3$.

Thus, the Hamiltonian is

$$H = Na\left(-g + \frac{m}{a^2} + \frac{\alpha_1^2\alpha_2^2}{a^4}\right) = 0, \tag{3.171}$$

with $g = b^2 + k$ and α_1 and α_2 are defined by (3.119) and (3.120), respectively.

Solving (3.171) for a^2 we obtain two constraint equations:

$$a^2 = \frac{m}{2g}\left(1 \pm \sqrt{1 + \frac{4g\alpha_1^2\alpha_2^2}{m^2}}\right) \tag{3.172}$$

$$\Rightarrow \qquad h_\pm(b)a^2 - m = 0, \tag{3.173}$$

where

$$h_\pm(b) = \frac{2g}{1 \pm \sqrt{1 + \frac{4g\alpha_1^2\alpha_2^2}{m^2}}}. \tag{3.174}$$

Hence, we get two general solutions for the wave function given by

$$\psi_\pm(b, T_r, T_1, T_2) = \int dmd\alpha_1 d\alpha_2 A(m, \alpha_1, \alpha_2)e^{\frac{i}{\hbar}(mX_\pm(b) - mT_r - \alpha_1 T_1 - \alpha_2 T_2)} \tag{3.175}$$

with

$$X_\pm(b) = \int \frac{db}{h_\pm(b)} = \int \frac{1}{2g}\left[1 \pm \sqrt{1+\frac{4g\alpha_1^2\alpha_2^2}{m^2}}\right] db. \tag{3.176}$$

It is now easy to check that deep in the radiation epoch, i.e. $m >> \alpha_1, \alpha_2$, we obtain only one solution in first order written as $X_+ = X_r = \int \frac{db}{g}$, which perfectly agrees with the case of the single radiation fluid. However, considering higher order terms in the expansion:

$$\sqrt{1+\frac{4g\alpha_1^2\alpha_2^2}{m^2}} \approx 1 + \frac{2g\alpha_1^2\alpha_2^2}{m^2} - \frac{2g^2\alpha_1^4\alpha_2^4}{m^4}, \tag{3.177}$$

which leads to

$$X_+ \approx X_r + \frac{\alpha_1^2\alpha_2^2}{m^2}b - \frac{\alpha_1^4\alpha_2^4}{m^4}X_\phi \tag{3.178}$$

$$X_- \approx -\frac{\alpha_1^2\alpha_2^2}{m^2}b + \frac{\alpha_1^4\alpha_2^4}{m^4}X_\phi \tag{3.179}$$

where

$$X_\phi = \int g db, \tag{3.180}$$

we start to observe in X a structure similar to the case of dark energy and radiation domination with an extra term linear in b. The wave functions for each branch have then the approximate form:

$$\psi_+(b, \boldsymbol{T}; \boldsymbol{\alpha}) \approx N_+ \exp\left[\frac{i}{\hbar}\left(mX_r + \frac{\alpha_1^2\alpha_2^2}{m}b - \frac{\alpha_1^4\alpha_2^4}{m^3}X_\phi - \boldsymbol{\alpha}\cdot\boldsymbol{T}\right)\right] \tag{3.181}$$

$$\psi_-(b, \boldsymbol{T}; \boldsymbol{\alpha}) \approx N_- \exp\left[\frac{i}{\hbar}\left(-\frac{\alpha_1^2\alpha_2^2}{m}b + \frac{\alpha_1^4\alpha_2^4}{m^3}X_\phi - \boldsymbol{\alpha}\cdot\boldsymbol{T}\right)\right] \tag{3.182}$$

which do not factorize. Here $\boldsymbol{\alpha} = (m, \alpha_1, \alpha_2)$. We also observe that in the radiation epoch we have a plane wave for the radiation clock since $\psi_+ \propto e^{\frac{i}{\hbar}m(X_r - T_r)}$. However, this does not happen for the minority clocks T_1 and T_2. As for the ψ_- branch, we do not find plane waves for any of the three clocks.

Last but not least, we can check that for a peaked distribution the peak moves according to the classical trajectory. Hence, considering the function $P(m, \alpha_1, \alpha_2, b)$ from the dispersion relation of this medium given by

$$P_\pm = mX_\pm, \tag{3.183}$$

we can compute the group velocities associated with each physical time, by evaluating the derivatives of $P_\pm$ in each peak $\boldsymbol{\alpha}_0$:

$$\frac{\partial^2 P_\pm}{\partial m \partial b} = \frac{a^4}{ma^2 + 2\alpha_1^2\alpha_2^2} = \frac{1}{c_{gm}} \tag{3.184}$$

$$\frac{\partial^2 P_\pm}{\partial \alpha_1 \partial b} = \frac{2\alpha_1\alpha_2^2 a^2}{ma^2 + 2\alpha_1^2\alpha_2^2} = \frac{1}{c_{g1}} \tag{3.185}$$

$$\frac{\partial^2 P_\pm}{\partial \alpha_2 \partial b} = \frac{2\alpha_2\alpha_1^2 a^2}{ma^2 + 2\alpha_1^2\alpha_2^2} = \frac{1}{c_{g2}}. \tag{3.186}$$

To do this, we used the relations

$$g = \frac{m}{a^2} + \frac{\alpha_1^2\alpha_2^2}{a^4} \tag{3.187}$$

$$\pm\sqrt{1 + \frac{4g\alpha_1^2\alpha_2^2}{m^2}} = 1 + \frac{2\alpha_1^2\alpha_2^2}{ma^2} \tag{3.188}$$

that follow from the Hamiltonian constraint equation (3.171). Computing then the equations of motion for b

$$\dot{b} = \{b, H\} = -N\frac{ma^2 + 2\alpha_1^2\alpha_2^2}{a^5} \tag{3.189}$$

and for each one of the physical times, i.e. radiation, gravitational constant and scalar field

$$\dot{T}_r = -\frac{N}{a} \tag{3.190}$$
$$\dot{T}_1 = -\frac{2N\alpha_1\alpha_2^2}{a^3} \tag{3.191}$$
$$\dot{T}_2 = -\frac{2N\alpha_2\alpha_1^2}{a^3}, \tag{3.192}$$

it is straightforward to check the corresponding group velocities:

$$c_{gm} = \frac{\dot{b}}{\dot{T}_r} = \frac{ma^2 + 2\alpha_1^2\alpha_2^2}{a^4} \tag{3.193}$$
$$c_{g1} = \frac{\dot{b}}{\dot{T}_1} = \frac{ma^2 + 2\alpha_1^2\alpha_2^2}{2\alpha_1\alpha_2^2 a^2} \tag{3.194}$$
$$c_{g2} = \frac{\dot{b}}{\dot{T}_2} = \frac{ma^2 + 2\alpha_1^2\alpha_2^2}{2\alpha_2\alpha_1^2 a^2}. \tag{3.195}$$

Therefore, just like in the model with dark energy and radiation, this shows that the peak of both branches, $\pm$, moves with the same group velocity and follows the classical trajectory.

Conclusions and Outlook

Recapitulating, throughout this thesis we worked towards a new way to describe time in quantum cosmology. Broadening our conceptual horizons, we promote the constants of nature to dynamical variables. Associated with each constant one gets a space of states built from superpositions of waves propagating in a dispersive medium where the connection plays the role of space and time is the momentum conjugate to the constant. For the simple model with a single fluid it is possible to find linearizing variables that lead to plane waves moving with a fixed velocity. This provides a good clock with a simple inner product and definition of unitarity.

We further introduced the necessary formalism for a multi-fluid universe in which MSS is identified as a dispersive medium. For instance, resorting to a compound of Λ and radiation, we showed that the dominant fluid always generates a good clock. However, the minority clock is usually troublesome since we cannot always find linearizing variables and, even if we do, they will be different functions of the targeted constant comparing to the original one, i.e. if the wave function is coherent in one variable it will not be in another. Nonetheless, the peak of the wave function follows the classical equations of motion.

Finally, we brought some novelties to the table. We adopted a real scalar field as a cosmological clock together with a deconstantized gravitational constant as an attempt to explore what happens when we have more than one physical clock "at the same time". Consequently, the usual Hamiltonian constraint becomes a double-time Schrodinger equation, solved by a superposition of plane waves. We examined different kinds of solutions, the most interesting one being a fully semi-classical state in both variables, for which we do not have a factorization of the wave function, leading to quantum entanglement between the two time variables. In this case, we obtained a modified saturated Heisenberg uncertainty relation, dependent on the uncertainty of both constants and on the linearizing spatial variable $X(b)$. Surprisingly, working with two clocks at the same time does not make our lives easier, since this increases the uncertainty associated with each clock. Hence, working with just one clock is advisable.

With all that said, it is clear that some more work should be done following this thesis. A few important questions are still not answered. For instance, we can take a closer look at what happens in a reflection in connection space, i.e. when the universe transits from a contraction to an expansion period, in a multi-fluid universe. This examination involves going beyond the semi-classical states we already considered. Foremost, we are now shifting from matter to Lambda domination and so, it would be interesting to employ this new approach we presented to a matter-Lambda universe. This is however more complex since the Hamiltonian of this model cannot be simplified to the pleasant approximate form ($H = h(b)a^2 - \alpha$) we used throughout the work.

Regarding the problem with a scalar field and a deconstantized G_N, further analysis is required so as to understand the origin of the excessive uncertainty in the Heisenberg principle when two clocks are at play. Another route that should be pursued would be the search for distributions for the constants that saturate the uncertainty principle without exceeding it. This model can also be generalized to a perfect fluid with equation of state $p = w\rho$. For instance, we can characterize such fluid with a K-essence model staying close to the scalar field description by taking a power of the kinetic term in its action [41].

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