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Braneworlds and their mysteries

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Abstract

Braneworlds are fascinating objects. They provide an interesting framework within which we can explore whether our universe exists in a fundamentally higher dimensional spacetime. This dissertation is devoted to studying braneworlds and some of their aspects. We begin with an overview of extra-dimensional physics and especially the mathematical tools needed to investigate braneworlds. The effective Einstein field equations on the brane are derived, motivating the introduction of the Randall-Sundrum models. A detailed description of braneworld gravity is provided. We then move on to braneworld cosmology, where from the view point of a homogeneous and isotropic brane, we see quadratic deviations from the FLRW results from standard cosmology. We finally look at black holes, specifically some discussion into static, spherically symmetric braneworld black hole solutions. We assume an equation of state for the Weyl term in the Einstein equations, which encodes the effect of extra dimensions, and we employ this to attempt to classify some characteristics of these solutions. We finally discuss further interesting ideas, such as holography.

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Dedication

I dedicate this work to my parents, my Ammi & my Appa, and to my dear brother, Akshay.

Contents

Chapter 1

Introduction

1.1 The Current State of Affairs...

The Standard Model of particle physics (SM), regarded as an effective field theory (EFT) near the TeV scale, is one of theoretical physics' greatest triumphs, explaining various elementary particles and their fundamental interactions. It is a gauge quantum field theory obeying a set of rules, described by the finite dimensional Lie groups *SU(3)*$\times$*SU(2)*$\times$*U(1)*. SU(3) symmetries describe quantum chromodynamics (QCD), SU(2) describing the weak interaction and U(1)describing quantum electrodynamics (QED). It describes the existence of four bosons or force carriers that mediate interactions between fermions, namely: the photon for the electromagnetic interaction, gluons for the strong interaction and the $W^{\pm}$ and Z^{0} bosons for the weak interaction. Fermions or the ordinary matter particles can be distinguished into quarks and leptons, which interact with each other through the exchange of bosons. The unification of the electromagnetic and weak interactions by Glashow, Salam and Weinberg in 1979 **[1]** is often described as the birth of the Standard Model and the idea of unification. Grand Unified Theories (GUTs) that effectively unify all fundamental interactions are yet to be proven correct. The SM works remarkably within low energy regimes, as our present colliders are able to achieve these energies as they fall within the range of theoretical predictions.

The enigmatic existence of dark matter, the incompatibility of gravity with the SM, the hierarchy problem are among some of the plethora of unresolved and open questions, leading to the fact that the SM is still very incomplete and could be regarded as an effective field theory.

General Relativity (GR) is another remarkable theory and has been highly successful in its regime, but it is a classical theory. John Archibald Wheeler, a titan of modern theoretical physics and especially gravity, once remarked, *"Spacetime tells matter how to move; matter tells spacetime how to curve."* This beautiful quote captures the essence of the general theory of relativity, a geometrical theory that generalises the comprehension of spacetime provided by Special Relativity. **[3]** The central piece of GR is the Einstein Field Equations (EFE), which relate the distribution of energy and momentum to curvature and its manifestation as gravity. These equations are:

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = \tfrac{8\pi G}{c^4}T_{\mu\nu} \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, G being Newton's gravitational constant, c is the speed of light and $T_{\mu\nu}$ is the energy-momentum tensor.

GR breaks down in extremely strong gravitational fields when quantum effects are non-negligible, typically in spacetime singularities such as those found at the centre of black holes or at the Big Bang. We therefore require a quantum theory of gravity to further probe these problems. **[2]**

1.2 Extra Dimensions and Gravity in the Bulk

1.2.1 A bit of history on Extra Dimensions

Kaluza-Klein theory

Our observable universe is 3+1 dimensional. Soon after GR's appearance on the world stage, there were attempts to unify gravity with electromagnetism. Around a century ago, Kaluza and Klein theorised **[4; 5]** that by adding an extra dimension to space, one could unify electromagnetism and gravity. They used the metric tensor in five dimensional (5D) spacetime in which our universe was a 3+1 dimensional hypersurface, where you get a $\mathcal{M}^{1,3} \times S^1$ theory. To get a basic idea, consider the following - the vacuum Einstein equations in 4-D can be derived from varying the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\ R \tag{1.2}$$

where g is the determinant of the metric tensor, while the sourceless Maxwell equations can be derived from

$$S_{EM} = -\frac{1}{4} \int d^4x\ F_{\mu\nu} F^{\mu\nu} \tag{1.3}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor. Now, Kaluza's idea was to consider gravity in 5 dimensions, described the following action

$$S = \frac{1}{16\pi G^{(5)}} \int d^4x dy \sqrt{g^{(5)}}\ R^{(5)} \tag{1.4}$$

Here, y denotes the coordinate of the fifth dimension and the superscripts denote 5-D quantities. To ensure theoretical independence of the fifth dimension imposed a condition that the metric components are independent of y:

$$\frac{\partial g^{(5)}_{\mu\nu}}{\partial y} = 0 \tag{1.5}$$

Following which, writing out the metric as suggested:

$$g^{(5)}_{\mu\nu} = \phi^{-\frac{1}{3}} \begin{pmatrix} g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & 0 \end{pmatrix} \tag{1.6}$$

Usinig Eqn.1.6, the action in Eqn.1.4 becomes

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{g} \Big(R - \frac{1}{4} \phi F_{\mu\nu} F^{\mu\nu} - \frac{1}{6\phi^2} \partial_\mu \phi \partial^\mu \phi \Big) \tag{1.7}$$

The action in (1.7) describes 4-D gravity together with classical electromagnetism and a massless Klein-Gordon scalar field ϕ. (Refer to [4; 5] for the details)

This was a step towards a 'unified theory of all interactions' - although at the cost of an additional dimension. In recent decades, extra dimensions are an integral part of fundamental theoretical physics, with a burgeoning focus on making these dimensions consistent with our existing frameworks in physics, driving progress in string theory, particle phenomenology and cosmology.

Braneworlds and Domain Walls

Braneworlds are just a part of the general story and represent an interesting way to deal with extra dimensions. In the 1980's, Rubakov and Shaposhnikov **[6]** proposed an effective theory with a single extra dimension, in which SM fields are located in a 4-dimensional submanifold, a domain wall, essentially suggesting that ordinary matter is confined to a potential well which is narrow along the additional dimension(s), thereby localising matter on the brane. They conjectured that we live on a 4-dimensional brane embeddeded in a higher dimensional universe. This work laid the foundation for further extra dimensional models as well as *braneworld scenarios*. The basic idea of braneworld scenarios is that our observable universe could be a 3+1 dimensional hypersurface (the brane) embedded in a higher dimensional spacetime (the bulk). Only gravity can propagate freely into the bulk while all matter are confined onto the brane.

ADD Braneworlds

In March 1998, Arkani-Hamed, Dimopoulos and Dvali (ADD) presented a new framework where the universe has 6 dimensions, two of which are compactified in a 2-torus T^2**[7]**. Their model stipulated that SM fields were localised on the 4-dimensional throat (brane) of a vortex in 6 dimensions with a Pati-Salam gauge symmetry *SU(4)*×*SU(2)*×*SU(2)* in the bulk, which meant gravity was free to propagate into the bulk. The ADD model allowed for considerations of large extra dimensions, up to a millimeter. This model was proposed to solve the infamous *hierarchy problem*, which will be discussed shortly. Consider the Newtonian gravitational potential between two test masses on the brane, we see

$$V(r) \approx \begin{cases} \frac{m_1 m_2}{M_f^{2+n}} \frac{1}{r^{n+1}} & \mathrm{r} \ll \mathrm{R} \\ \frac{m_1 m_2}{M_f^{2+n}} \frac{1}{R^n r} & \mathrm{r} \gg \mathrm{R} \end{cases} \tag{1.8}$$

where R is the supposed size of the extra dimension(s) and M_f is the fundamental mass scale of gravity in the full (4+n)-D spacetime. We see for small separations r $\ll$ R, the potential is that of higher dimensional gravity and otherwise, gravity is insensitive to the extra dimension(s) and the potential behaves like our usual 4D gravity. Therefore, we say that an observer on the brane experiences an *effective* 4-D Planck scale given by

$$M_p^2 = M_f^{2+n} R^n \tag{1.9}$$

So, we see that the fundemantal scale can be much lower than the Planck mass, while still giving rise to a large effective Planck mass on the brane ($M_p \sim 10^{16}$TeV) due to the large volume of the extra dimensions.

1.2.2 Randall-Sundrum Models

In 1999, Lisa Randall and Raman Sundrum proposed two different kinds of extra dimensional models, RS-1 [8] and RS-2 [9]. In RS-1, the extra dimension is warped and compact, yielding a braneworld scenario. Here, our world is modeled as a (3+1)-dimensional brane embedded in a 5-dimensional bulk. Generally, a n-brane can be modeled as a submanifold of dimension n enclosing a (n+1) space. RS-2 on the other hand, assumes an extra dimension with infinite size, a negative cosmological constant in the bulk Λ, a tension λ on the brane making it a gravitating object and a Z_2 symmetry of the bulk with respect to the brane. The idea is that λ is set to a value that yields GR at low energies.

What is interesting, is the fact that although the RS model is an empirical braneworld setup, it can be related to string theory in several ways [10]. It is notionally similar to heterotic M-theory, in that the original RS model had two domain walls at the end of an interval, and this similarity varies calculationally when examining and comparing the gravitational spectrum of GR [11; 12]. A more interesting parallel is that with type IIB string theory, where the RS model can roughly be associated with the near horizon limit of a stack of D3 branes, which allows us to investigate the AdS/CFT correspondence [13]. The RS model is particularly useful as an explicit calculator for any theory with extra dimensions in which gravity can probe these hidden dimensions. Extra dimensions run the risk of creating unwanted addtional physics problems and require that we reproduce SM and GR physics at the necessary energy scales. With RS, the gravitational physics is self-consistent and calculable, which lets us investigate an array of interesting phenomenon in various settings.

Black holes are some of the most intriguing objects we can study within the braneworld framework of extra dimensions. From the Kaluza-Klein point of view, extra dimensions manifest as extra charges that black holes might carry from a 4D point of view [14; 15].

However, these require a black hole to be 'smeared' across the extra dimension along the extra dimension rather than be localised. Conversely, braneworld scenarios predict highly localised and strongly warped extra dimensions, implying that black hole physics for this gravitating brane is especially important. As stated above, we find braneworld black hole solutions interesting as they might be able to shed further light upon the RS model and the AdS/CFT correspondence. (Refer to [16; 17; 18; 19] for further information)

1.2.3 EFE in the bulk

To study gravity in these various multidimensional braneworld scenarios, it is necessary to establish how GR works in higher dimensional spaces - especially how the EFEs change considering the introduction of an extra dimension. It is well known that one can derive the EFE in vaccuum from the Einstein-Hilbert action. It is also possible to derive the full 4-dimensional EFE from the following action [1],

$$S^{(4)} = \int d^4x \sqrt{-g^{(4)}} \Big(\frac{R^{(4)} - 2\Lambda^{(4)}}{16\pi G^{(4)}} + \mathcal{L}_M^{(4)} \Big) \tag{1.10}$$

and we can attain the energy-momentum tensor from the matter action $S_M^{(4)} = \int d^4x \sqrt{-g^{(4)}} \mathcal{L}_M^{(4)}$ in the following manner [2] [20]:

$$T_{\mu\nu}^{(4)} = \frac{-2}{\sqrt{-g^{(4)}}} \frac{\delta S_M^{(4)}}{\delta g_{(4)}^{\mu\nu}} \tag{1.11}$$

One can also include the cosmological constant term in the matter Lagrangian, so that it can be explicitly considered as the energy density of the vacuum. If this is the only matter distribution we consider, then we have $\mathcal{L}_M^{(4)} = -\Lambda^{(4)}$. Using this, we can write a

[1] The superscript (4) indicates that the objects are 4-dimensional. We are setting c=1 but writing out factors of G.

[2] The 4D reduced Planck mass is defined in natural units as $\mathcal{M}^{(4)} = \frac{1}{\sqrt{8\pi G^{(4)}}}$

4-dimensional action as follows:

$$S^{(4)} = \frac{1}{2}\int d^4x\sqrt{-g^{(4)}}\left(\mathcal{M}^{(4)}\right)^2 R^4 + \int d^4x\sqrt{-g^{(4)}}(-\Lambda^{(4)}) \tag{1.12}$$

thereby, using Eq.(1.11), the energy-momentum tensor will be of the form $T_{\mu\nu} = \Lambda^{(4)} g_{\mu\nu}$, since $\Lambda^{(4)}$ is a constant.

Using similar arguments, we can generalise this gravitational action to an arbitrary dimension n,

$$S^{(n)} = \frac{1}{2\kappa^{(n)}}\int d^nx\sqrt{-g^{(n)}}R^{(n)} + \int d^nx\sqrt{-g^{(n)}}\Big(-\Lambda^{(n)}\Big) \tag{1.13}$$

where κ is the appropriate gravitational constant, adjusted using the n-dimensional Planck scale factor. The superscript (n) is meant to indicate the dimension of interest. Suppose we do consider n=5, we see that $\kappa = \frac{1}{2\mathcal{M}^3}$, which results in an action,

$$S = \int d^5x\sqrt{-g}\Big(\mathcal{M}^3 R - \Lambda\Big) \tag{1.14}$$

By varying the action of Eq.(1.14) with respect to the metric, we arrive at a 5D analogue of the EFE,

$$G_{AB} = R_{AB} - \tfrac{1}{2}Rg_{AB} = \tfrac{1}{2\mathcal{M}^3}T_{AB} \tag{1.15}$$

Note that even in 5D, the reintroduction of the cosmological constant in the matter action will still give us $T_{AB} = -\Lambda g_{AB}$, considering that there is no energy distribution beyond that of the vacuum.

1.3 Hierarchy Problem

The Hierarchy problem tends to be the overarching name that physicists use to refer to a number of different phenomenon, first of which being the discrepancy between the Higgs mass and the Planck mass. As one calculates the mass of the Higgs using QFT of the SM, it can be inferred that it receives contributions from all energy scales, up to the highest energy limit at which the SM is valid. An obvious choice for this energy is the Planck mass. This discrepancy between the electroweak scale ($\approx 100GeV$) and the Planck scale ($\approx 10^{18}GeV$) is referred to as *the hierarchy problem* **[21]**. Further details can be found at **[22]**.

We know that the Higgs potential can be written as:

$$V = m_H^2|H|^2 + \lambda|H|^4 \tag{1.16}$$

where V is the Higgs potential, H is the Higgs field, m$_H$ is the scalar mass of the Higgs boson and λ is a free parameter determined by the vacuum expectation value. For $\lambda > 0$ and $m_H^2|H|^2 < 0$, the vacuum expectation value will be nonzero, giving us $< H >= \sqrt{\frac{-m_H^2}{2\lambda}}$. With the observed mass of the Higgs at 125 GeV and $\langle H \rangle$ = 174 GeV, we find $m_H^2 = (-92.9GeV)^2$. Issues tend to arise when we consider the couplings of SM fermions to the Higgs fields, resulting in higher order corrections to m_H such as:

$$\Delta m_H^2 = -\frac{|\lambda_f|^2}{8\pi^2}\Lambda_{UV}^2 + .. \tag{1.17}$$

where λ_f is the Yukawa coupling to a SM fermion and Λ_{UV} is a cutoff to the matrix element, to prevent divergences. Since there is no physical mechanism within the framework of the SM yielding a small value of Λ_{UV} to arrive at the observed value of the mass of the Higgs boson. This implies that the SM is valid upto the Planck scale ($\Lambda_{UV} = \Lambda_{Planck}$), requiring very highly fine-tuned higher-order corrections, or it could also imply the existence of a new physical scale, at which BSM theories begin to dominate [21].

There are various approaches to solving this problem, but here we are interested in extra dimensional and braneworld scenarios. In layman's terms, a braneworld is a slice through the spacetime that we occupy. Viewing these extra dimensions perpendicular to our slice is not trivial as all of our standard physics is confined. We could study these extra dimensions by considering how gravity behaves [10]. The possibility that braneworlds can resolve the hierarchy problem via a geometric renormalisation of Newton's constant [7; 23; 24], indicates a chance for mini black hole productions at the LHC [25].

1.4 Outline of Contents

This project aims to focus on braneworlds and some of their related intricacies regarding gravitational collapse and black hole behaviour. **Chapter 1** is the introduction aiming to give an overview and motivation behind braneworld study. **Chapter 2** will largely deal with setting up some prerequisites to familiarise the reader with the needed toolkit to understand work in this area. It aims to deal with some ideas from GR and differential geometry to introduce braneworlds. The chapter ends with a description of non-perturbative gravity on the brane, where the idea is the use the tools from earlier in the chapter to project the 5D equations onto the brane to obtain the 4D effective Einstein equations.

Chapter 3 introduces the two Randall-Sundrum models and some of their intricacies. It also includes a brief summary of Anti-de Sitter (AdS) spacetime as we come to see how crucial AdS is to the study of braneworlds.

Chapter 4 is entirely about gravity and its behaviour on braneworlds. The section on non-perturbative gravity has been covered in Chapter 2, but it is briefly reviewed and expanded upon. Moving on, the idea is to develop the framework of linearised general relativity in braneworlds.

Chapter 5 and **Chapter 6** is about applying the various techniques to understand strong gravity on braneworlds, with **Chapter 5** focussing on braneworld cosmology and **Chapter 6** on braneworld black holes. Finally this dissertation is wrapped up in the conclusion in **Chapter 7**.

Chapter 2

'Braneworlds 101'

A proficiency in GR will be assumed going forward, although a familiarity with more formal differential geometry notions would be helpful. **[20; 28; 29]** would be the best places to brush up on this material. Seeing as braneworlds are not part of standard GR instruction, this chapter will expand upon a first course in GR. We begin by introducing some of the notions in the (d-1)+1 decomposition formalism of GR, and Israel's junction conditions, which we need to introduce braneworlds.[1] After completing a brief review of submanifolds and hypersurfaces, we will dive into some introductory ideas about braneworlds, mainly their general setup and the behaviour of gravity.

2.1 Geometry of Hypersurfaces

2.1.1 Induced metrics and Extrinsic Curvature

The (d-1)+1 decomposition of Einstein's equations is a generalisation of the 3+1 formalism of GR **[26; 27]**. Let us begin with a (d-1) hypersurface Σ embedded[2] in d-dimensional spacetime $\mathcal{M}$. As the brane has one dimension less than the bulk, it is said to be a *co-dimension one* hypersurface. We can define this (d-1) dimensional submani-

[1] d is the dimension of the whole spacetime and the metric signature will be (-,+,+,...,+)

[2] Σ can be thought of as the 'brane' in a braneworld.

fold Σ embedded in a manifold $\mathcal{M}$ via the d parametric equations

$$x^a = x^a(\phi^1, \phi^2, ..., \phi^{d-1}) \tag{2.1}$$

where ϕ^μ are the internal coordinates in the submanifold. Looking in the figure below, Σ divides $\mathcal{M}$ into two parts: $\mathcal{M}^\pm$

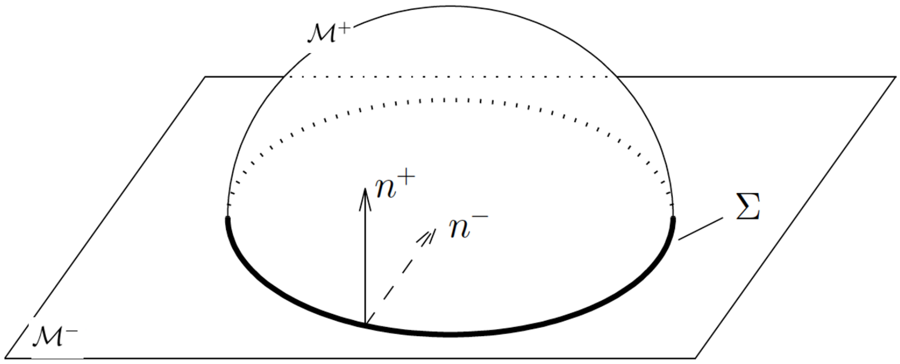

Figure 2.1: Hypersurface embedding in a higher dimensional spacetime. We see $\mathcal{M}$ is composed of $\mathcal{M}^+$ (the half sphere) and $\mathcal{M}^-$ (the plane). We see the intersection of $\mathcal{M}^\pm$ is the hypersurface Σ (the circular outline). Here we have $n^\pm$ as unit normal vectors of Σ.

Conventionally, n^+ *points into* $\mathcal{M}^+$, and n^- *points out of* $\mathcal{M}^-$. We see in Fig.(2.1), n^+ is not defined in $\mathcal{M}^-$ and n^- is not defined in $\mathcal{M}^+$. Generally speaking, Σ can be a hypersurface within a hypersurface *family* that will *locally* foliate spacetime. Consider a parameter τ that characterises this foliation, where τ is a constant and a specific hypersurface within our family. Based on the convention, we can see that the direction of $n^\pm$ is the same as the increasing direction of τ. We normalise n as:

$$n^\alpha n_\alpha = \begin{cases} +1 & \text{for when } \Sigma \text{ is the brane in the braneworld} \\ -1 & \text{for when } \Sigma \text{ is a constant hypersurface} \end{cases} \tag{2.2}$$

The (d-1) internal coordinates ϕ^μ can be eliminated from the d parametric equations to give our one constraint equation using τ:

$$\boldsymbol{\tau}(x^a) = \text{constant} \tag{2.3}$$

Naturally, we can define a unit normal vector on all points of Σ by

$$n_\mu = \frac{\boldsymbol{\nabla}_\mu \tau}{\sqrt{g^{\mu\nu}\boldsymbol{\nabla}_\mu \tau \boldsymbol{\nabla}_\nu \tau}} \tag{2.4}$$

with the appropriate signage as in Eqn.(2.2). $\boldsymbol{\nabla}$ is the covariant derivative associated to our given metric $g_{\mu\nu}$ (metric of $\mathcal{M}$). From this definition of a normal vector, one may define the *induced metric* on Σ, as

$$q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \tag{2.5}$$

$q_{\mu\nu}$ is the *intrinsic metric* on the brane, defined only on the hypersurface. $q_{\mu\nu}$ is a 4-D tensor (i.e. It lies in the tangent bundle of the brane as a manifold) and encodes all geometric information of the submanifold. We can accordingly construct familiar objects like the Riemann and Ricci tensor.

The induced metric in the form q^μ_ν behaves like a projection operator, between the tangent space of $\mathcal{M}$ and the tangent space of Σ. So, generally, the projection of a tensor to the tangent space to Σ is

$$T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} = q^{\mu_1}_{\rho_1} \ldots q^{\mu_k}_{\rho_k}\, q^{\sigma_1}_{\nu_1} \ldots q^{\sigma_l}_{\nu_l}\, T^{\rho_1 \ldots \rho_k}_{\sigma_1 \ldots \sigma_l} \tag{2.6}$$

To further understand this, let us consider a simple example - the projection of a vector v^μ, decomposed into its tangent and perpendicular components to Σ, $v^\mu = v^\mu_\parallel + v^\mu_\perp$.

Using Eq.(2.6), we can act on this vector with q^μ_ν

$$q^\mu_\nu v^\nu = (\delta^\mu_\nu - n^\mu n_\nu)(v^\nu_\parallel + v^\nu_\perp) = v^\mu_\parallel \tag{2.7}$$

Equipped with this, we can define a covariant derivative on the brane D_μ, simply projecting the covariant derivative $\boldsymbol{\nabla}_\mu$ of the bulk, to the brane,

$$D_\rho T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} = q^{\mu_1}_{\sigma_1} \ldots q^{\mu_k}_{\sigma_k}\; q^{\lambda_1}_{\nu_1} \ldots q^{\lambda_l}_{\nu_l} \left(q^\omega_\rho \boldsymbol{\nabla}_\omega \right) T^{\sigma_1 \ldots \sigma_k}_{\lambda_1 \ldots \lambda_l} \tag{2.8}$$

From Eq.(2.8), we can see that $D_\mu q_{\nu\rho} = 0$, showing that D_μ is a unique derivative operator associated with $q_{\nu\rho}$.

This brings us to the notion of *extrinsic curvature* of Σ, defined by

$$K_{\mu\nu} = D_\mu n_\nu = q^\rho_\mu q^\sigma_\nu \boldsymbol{\nabla}_\rho n_\sigma = q^\rho_\mu \boldsymbol{\nabla}_\rho n_\nu \tag{2.9}$$

We note that this quantity is symmetric $K_{\mu\nu} = K_{(\mu\nu)}$. It gives us the derivative of the normal vector in a direction tangential to Σ, characterising the extrinsic nature of the embedding, and so we can interpret it loosely as the 'bending' of Σ in $\mathcal{M}$. We can now define Christoffel symbols and the Riemann tensor in the usual way but just using $q_{\mu\nu}$. [20; 28; 29]

2.1.2 Gauss-Codazzi equations

The Gauss-Codazzi equations are important to relate and construct (n-1)-dimensional quantities, constructed from the induced metric $_{ab}$, from n-dimensional quantities constructed from the full metric g_{ab}. Basically, they project bulk quantities to the brane, and corrections are given by the extrinsic curvature. The curvature ${}^{(n-1)}R_{abc}{}^{d}\omega_d$ of Σ

defined via a dual vector field ω_a on Σ

$$ {}^{(n-1)}R_{abc}{}^{d}\omega_d = D_a D_b \omega_c - D_b D_a \omega_c \tag{2.10} $$

Using the definition in Eq.(2.8),

$$ \begin{aligned} D_a D_b \omega_c &= q_a{}^{f} q_b{}^{g} q_c{}^{h} \nabla_f \Big(q_g{}^{e} q_h{}^{d} \nabla_e \omega_d \Big) \\ &= q_a{}^{f} q_b{}^{e} q_c{}^{d} \nabla_f \nabla_e \omega_d + q_a{}^{f} q_b{}^{e} q_c{}^{h} \nabla_f\, q_h{}^{d} \nabla_e \omega_d + q_a{}^{f} q_b{}^{g} q_c{}^{d} \nabla_f\, q_g{}^{e} \nabla_e \omega_d \\ &= q_a{}^{f} q_b{}^{e} q_c{}^{d} \nabla_f \nabla_e \omega_d - q_b{}^{e} K_{ac}\, n^d \nabla_e \omega_d - q_c{}^{d} K_{ab}\, n^e \nabla_e \omega_d \\ &= q_a{}^{f} q_b{}^{e} q_c{}^{d} \nabla_f \nabla_e \omega_d + K_{ac} K_b{}^{d} \omega_d + K_{ab} K_c{}^{d} \omega_d \end{aligned} \tag{2.11} $$

Note: the third equality in Eqn.(2.11) follows from,

$$ q_a{}^{f} q_c{}^{h} \nabla_f q_h{}^{d} = q_a{}^{f} q_c{}^{h} \nabla_f \left(g_h{}^{d} - n_h n^d \right) = -K_{ac} n^d \tag{2.12} $$

And the fourth equality in Eqn.(2.11) is from,

$$ q_b{}^{e} n^d \nabla_e \omega_d = q_b{}^{e} \nabla_e (n^d \omega_d) - \omega_d\, q_b{}^{e} \nabla_e n^d = -\omega_d K_b{}^{d} \tag{2.13} $$

Now, as we antisymmetrise on a and b, the final term in Eqn.(2.11) vanishes as K_{ab} is symmetric, meaning Eqn.(2.10) becomes

$$ \begin{aligned} {}^{(n-1)}R_{abc}{}^{d}\omega_d &= 2 q^{f}_{[a} q^{e}_{b]} q_c{}^{d} \nabla_f \nabla_e \omega_d + 2 K_{c[a} K_{b]}{}^{d} \omega_d \\ &= 2 q_a{}^{f} q_b{}^{e} q_c{}^{k} \nabla_{[f} \nabla_{e]} \omega_k + 2 K_{c[a} K_{b]}{}^{d} \omega_d \\ &= q_a{}^{f} q_b{}^{e} q_c{}^{k} q_g{}^{d}\, {}^{(n)}R_{fek}{}^{g} \omega_d + K_{ac} K_b{}^{d} \omega_d - K_{bc} K_a{}^{d} \omega_d \end{aligned} \tag{2.14} $$

Here, we have used the definition of curavture ${}^{(n)}R_{fek}{}^{g}$ of $\mathcal{M}$,

$$2\nabla_{[f}\nabla_{e]}\omega_k = {}^{(n)}R_{fek}{}^{g} \tag{2.15}$$

Since ω_d is arbitrary, a simple comparison of Eqns.(2.14) and (2.15) gives us the **Gauss equation**:

$${}^{(n-1)}R_{abc}{}^{d} = q_a{}^{e} q_b{}^{f} q_c{}^{g} q_h{}^{d}\, {}^{(n)}R_{efg}{}^{h} + K_{ac}K_b{}^{d} - K_{bc}K_a{}^{d} \tag{2.16}$$

We can derive the Codazzi equation via a similar procedure. Combining Eqns.(2.8) and (2.9),

$$\begin{aligned} D_a K_{bc} &= q_a{}^{d} q_b{}^{e} q_c{}^{f}\, \nabla_d\Big(q_e{}^{g}\nabla_g n_f\Big) \\ &= q_a{}^{d} q_b{}^{g} q_c{}^{f}\, \nabla_d\nabla_g n_f + q_a{}^{d} q_b{}^{e} q_c{}^{f}\, \nabla_d\, q_e{}^{g}\nabla_g n_f \\ &= q_a{}^{d} q_b{}^{g} q_c{}^{f}\, \nabla_d\nabla_g n_f - K_{ab}\, n^g n_c{}^{f}\nabla_g n_f \quad \text{(Using Eqn.(2.12))} \end{aligned} \tag{2.17}$$

Now, antisymmetrising a and b,

$$\begin{aligned} D_{[a}K_{b]c} &= q^d_{a[} q^g_{b]} q_c{}^{f}\, \nabla_d\nabla_g n_f = q_a{}^{d} q_b{}^{g} q_c{}^{f}\, \nabla_{[d}\nabla_{g]} n_f \\ \implies D_a K_{bc} - D_b K_{ac} &= q_a{}^{d} q_b{}^{g} q_c{}^{f}\, {}^{(n)}R_{dgfe}\, n^e \quad \text{(Using Eqn.(2.15))} \end{aligned} \tag{2.18}$$

Working with Eqn.(2.18), we raise the c index and contract with a. This yields the **Codazzi equation**:

$$D_a K^a_b - D_b K = {}^{(n)}R_{cd}\, n^d\, q^c_b \tag{2.19}$$

Gaussian Normal Coordinates

Through each point p $\in \Sigma$, there exists a unique geodesic with a tangent vector n^a. Suppose we considered the chart $\{x^\mu\}$ on a portion of Σ containing p, and labelled the points in the bulk with the parameter y along the geodesic on which it lies. We then construct the coordinate system, $\{x^\mu, y\}$ known as the *Gaussian Normal Coordinates* [3] *(GNC)*, which is commonly employed in the study of braneworld scenarios. Now, we have $n_a dx^a = dy$ and then the metric takes a form

$$ds^2 = g_{ab}dx^a dx^b = q_{\mu\nu}dx^\mu dx^\nu + dy^2 \tag{2.20}$$

An important yet attractive feature of the GNC system is that it allows us to construct the extrinsic curvature in a much more natural way $K_{\mu\nu} = \frac{1}{2}\frac{\partial}{\partial y}q_{\mu\nu}$, which makes sense as we defined y tangentially to Σ. Without a loss of generality, one can choose the hypersurface to be located at y=0 in the GNC. The geodesics originating from Σ may encounter singularities, but within a certain neighbourhood of the surface, the GNC is well defined.

2.2 Decomposition of Einstein's Equations

Following a Gauss-Codazzi-Ricci decomposition [26; 27; 28; 29] (keeping d general), we can decompose the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ as follows

$$^{(d)}G_{\mu\nu}n^\mu n^\nu = \frac{1}{2}(-^{(d-1)}R + K^2 - K_{\mu\nu}K^{\mu\nu}) \quad \textit{(Note a} \pm \textit{sign in front of R from Eqn.(2.2))} \tag{2.21}$$

$$^{(d)}G_{\mu\nu}n^\nu q^\mu_\alpha = D_\alpha K - D_\mu K^\mu_\alpha \tag{2.22}$$

$$\begin{aligned} ^{(d)}G_{\mu\nu}q^\mu_\alpha q^\nu_\beta = &\frac{1}{\alpha}(\mathcal{L}_m K_{\alpha\beta} - q_{\alpha\beta}\mathcal{L}_m K) +^{(d-1)} G_{\alpha\beta} \\ &+ \left(2K_{\alpha\mu}K_\beta{}^\mu - KK_{\alpha\beta} + \frac{1}{2}q_{\alpha\beta}(K^2 + K_{\mu\nu}K^{\mu\nu})\right) \\ &+ \frac{1}{\alpha}(q_{\alpha\beta}D^\mu D_\mu\alpha - D_\alpha D_\beta\alpha) \end{aligned} \tag{2.23}$$

[3]Sometimes referred to as hypersurface orthogonal coordinates

$\mathcal{L}_m$ is the Lie derivative with respect to m_μ, where it is defined as $m_\mu = \frac{n_\mu}{\sqrt{g^{\mu\nu}\nabla_\mu\tau\nabla_\nu\tau}}$, from Eqn.(2.4). The super-indices characterise dimension as before.

Generalising Eqn.(1.15) to d-dimensional spacetime and then imposing it upon the above equations (2.21,2.22,2.23), we find:

$$8\pi G_d\,\lambda = \tfrac{1}{2}\Big(-^{(d-1)}\,R + K^2 - K_{\mu\nu}K^{\mu\nu}\Big) \tag{2.24}$$

$$8\pi G_d\,S_\alpha = D_\alpha K - D_\mu K^\mu_\alpha \tag{2.25}$$

$$\begin{aligned} 8\pi G_d\,S_{\alpha\beta} = {} & \frac{1}{\alpha}(\mathcal{L}_m K_{\alpha\beta} - q_{\alpha\beta}\mathcal{L}_m K) +^{(d-1)} G_{\alpha\beta} \\ & + \Big(2K_{\alpha\mu}K_\beta{}^\mu - KK_{\alpha\beta} + \frac{1}{2}q_{\alpha\beta}(K^2 + K_{\mu\nu}K^{\mu\nu})\Big) \\ & + \frac{1}{\alpha}(q_{\alpha\beta}D^\mu D_\mu\alpha - D_\alpha D_\beta\alpha) \end{aligned} \tag{2.26}$$

G_d is Newton's gravitational constant in d-dimensions. Here, $\alpha = \frac{1}{\sqrt{g^{\mu\nu}\nabla_\mu\tau\nabla_\nu\tau}}$ from Eqn.(2.4). We define $\lambda \equiv T_{\mu\nu}n^\mu n^\nu$, $S_\alpha \equiv T_{\mu\nu}n^\nu q^\nu_\alpha$ and $S_{\alpha\beta} \equiv T_{\mu\nu}q^\mu_\alpha q^\nu_\beta$. Using this, we can construct a decomposition as follows $T_{\mu\nu} = \lambda n_\mu n_\nu + n_\mu S_\nu + S_\mu n_\nu + S_{\mu\nu}$, which when we take a trace gives us T = S + λ, where $S \equiv S_{\mu\nu}q^{\mu\nu}$.

Eqn.(2.24) is called the *constraint equation*, Eqn.(2.25) is the *momentum constrain equation* and lastly Eqn.(2.26) is the *evolution equation* **[27]**. We can express Eqn.(2.26) alternatively as **[26; 27]**

$$\mathcal{L}_m K_{\alpha\beta} = D_\alpha D_\beta\alpha + \alpha\Big[-^{(d-1)}\,R_{\alpha\beta} + KK_{\alpha\beta} - 2K_{\alpha\mu}K_\beta{}^\mu + 8\pi G_d\Big(S_{\alpha\beta} - q_{\alpha\beta}\tfrac{S+\lambda}{d-2}\Big)\Big] \tag{2.27}$$

Extrinsic curvature can now be formulated as

$$\mathcal{L}_m q_{\alpha\beta} = -2\alpha K_{\alpha\beta} \tag{2.28}$$

Eqns.(2.27) and (2.28) give us a complete set of evolution equations where we use $q_{\mu\nu}$ and $K_{\mu\nu}$ as the fundamental variables. This formalism is sometimes referred to as the *ADM-York formalism of GR*.

2.2.1 Israel's Junction Conditions

Any physical system with boundary surfaces will require a formulation of proper junction conditions to deal with discontinuities across the surface. A well known example from GR is the matching of interior and exterior Schwarzschild metrics at the surface of a star **[28; 29; 31]**. In GR, if the stress-energy tensor is highly concentrated on Σ, it will result in discontinuities. Israel produced a geometric, covariant derivation of of junction conditions for treating such discontinuities **[30]**.

The *first junction condition* states that the intrinsic geometry of a hypersurface is well-defined, meaning that the induced metric obtained from $\mathcal{M}^{+}$ agrees with the one obtained from $\mathcal{M}^{-}$:

$$q^{+}_{\mu\nu} - q^{-}_{\mu\nu} = 0 = \hat{[} q_{\mu\nu} \hat{]} \tag{2.29}$$

(It is convenient to introduce some notation: $\hat{[} a \hat{]} = a^{+} - a^{-}$.)

We define $\mathcal{J}_{\mu\nu}$ as the singular part of the projected energy-momentum tensor on Σ, defined as

$$\mathcal{J}_{\mu\nu} \equiv \int_{0^-}^{0^+} S_{\mu\nu}\, dl \tag{2.30}$$

where $dl \equiv dt(\partial_t)^{\mu}\, n_{\mu} = \alpha dt$ is the proper length across Σ and for simplicity, l is set to 0 at Σ.

The *second junction condition* can be obtained by integrating Eqn.(2.26) over an infinitesimal layer of Σ **[30]**

Similarly, if we integrate Eqns. (2.24) and (2.25), we get

$$\mathcal{J}_\mu \equiv \int_{0^-}^{0^+} S_\mu \, dl = 0 \;\; ; \;\; \mathcal{P} \equiv \int_{0^-}^{0^+} \rho \, dl = 0 \tag{2.32}$$

These two relations are vanishing as the right hand sides of Eqns. (2.24) and (2.25), along with the non-$\mathcal{L}_m$ terms on the right hand side of Eqn.(2.24), are all well-defined and finite on both sides of the hypersurface.

Combining Eqns.(2.25) and (**??**), we get

$$D_\mu \mathcal{J}_\nu{}^\mu = -\hat{[}S_\nu\hat{]} \tag{2.33}$$

We can also define $\hat{\{}q_{\mu\nu}\hat{\}} \equiv q^+ + q^-$ from the relation for $\hat{[}q_{\mu\nu}\hat{]}$. A simple algebraic manipulation will show us that $\hat{[}q^2\hat{]} = \hat{[}q\hat{]}\hat{\{}q\hat{\}}$. Using which, Eqn.(2.24) can be rewritten as $8\pi G_d \; \hat{[}\lambda\hat{]} = \frac{1}{2}(\hat{[}K\hat{]}\hat{\{}K\hat{\}} - \hat{[}K_{\mu\nu}\hat{]}\hat{\{}K^{\mu\nu}\hat{\}})$. Combining this relation with Eqn.(**??**),

$$\hat{[}\lambda\hat{]} = \frac{1}{2}\Big(\frac{\mathcal{J}}{(2-d)}\hat{\{}K\hat{\}} - (\mathcal{J}_{\mu\nu} - q_{\mu\nu}\frac{\mathcal{J}}{(d-2)})\hat{\{}K^{\mu\nu}\hat{\}} \Big) = -\frac{1}{2}\mathcal{J}_{\mu\nu}\hat{\{}K^{\mu\nu}\hat{\}} \tag{2.34}$$

Eqns. (2.33) and (2.34) give us conservation laws for singular matter on Σ.

2.3 General Braneworld Formalism

For our purposes, we briefly define a class of braneworlds as 5D spacetime, where SM matter is confined on a (3+1)-dimensional brane Σ, but we allow gravity to propagate freely into the bulk. Essentially we imagine our universe to be this brane Σ, so it is natural to expect that the induced equations on the brane become Einstein's Field Equations within a certain limit. Using Gauss' equations (Eqn.(2.16) and d-dimensional EFEs, [32]

$$
\begin{aligned}
{}^{(d-1)}\mathcal{G}_{\alpha\beta} = \frac{d-3}{d-2}\, 8\pi G_d \Big\{ T_{\mu\nu} h^{\mu}_{\alpha}\, h^{\nu}_{\beta} + h_{\alpha\beta}\left(T_{\mu\nu} n^{\mu} n^{\nu} + \frac{T}{1-d} \right) \Big\} \\
+ (\mathcal{K}\mathcal{K}_{\alpha\beta} - \mathcal{K}_{\alpha\gamma}\mathcal{K}^{\gamma}_{\beta}) - \frac{1}{2} h_{\alpha\beta}(\mathcal{K}^2 - \mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu}) - E_{\alpha\beta}
\end{aligned}
\tag{2.35}
$$

where $T_{\alpha\beta}$ is the stress-energy tensor in d-dimensional spacetime and T is its trace. We define the tensor $E_{\mu\nu}$ as

$$
E_{\mu\nu} \equiv^{(d)} C_{\alpha\beta\gamma\sigma}\, n^{\alpha}\, n^{\gamma}\, h_{\mu}{}^{\beta}\, h_{\nu}{}^{\sigma} \tag{2.36}
$$

Where $C_{\alpha\beta\gamma\sigma}$ is the Weyl tensor defined by

$$
{}^{(d)}R_{\alpha\beta\gamma\sigma} =^{(d)} C_{\alpha\beta\gamma\sigma} + \frac{1}{d-2}\left(g_{\mu[\gamma}\, {}^{(d)}R_{\sigma]\nu} - g_{\nu[\gamma}\, {}^{(d)}R_{\sigma]\mu} \right) - \frac{1}{(d-1)(d-2)}\, {}^{(d)}R\; g_{\mu[\gamma}g_{\sigma]\nu} \tag{2.37}
$$

As seen from Eqn.(2.35), the matter content $T_{\mu\nu}$ needs to be specified.

2.3.1 Effective EFE on the brane

As we consider a 5D model, where gravity can leak into the bulk, how can we really ascertain that there is a fifth bulk dimension? Essentially, we wish to be able to differentiate between a (3+1)-dimensional brane embedded in a higher dimensional bulk and a (3+1)-dimensional universe independent of any embedding, as described by traditional GR. The idea is to present the effective EFE in the brane, by projecting the 5D EFE defined in the bulk onto the embedded brane. **[32]**

4D Einstein tensor

We will continue to use Σ as the 4-dimensional submanifold with induced metric $q_{\mu\nu}$, which is embedded in the 5-dimensional bulk $\mathcal{M}$ with a metric g_{AB}. We begin with the Guass equation, which lets use write the 4-dimensional Riemann tensor in terms of the

5-dimensional analogue,

$$^{(4)}R_{abc}{}^{d} ={}^{(5)} R_{efg}{}^{h} q_{a}{}^{e} q_{b}{}^{f} q_{c}{}^{g} q_{h}{}^{d} + K_{ac}K_{b}{}^{d} - K_{bc}K_{a}{}^{d} \tag{2.38}$$

Contracting Eq.(2.38) in its indices **b** and **d**, we get to the 4-dimensional Ricci tensor

$$\begin{aligned}
^{(4)}R_{ac} ={}^{(4)} R_{adc}{}^{d} = {}& {}^{(5)}R_{efg}{}^{h} q_{a}{}^{e} q_{c}{}^{g}\Big(q_{h}{}^{d} q_{d}{}^{f}\Big) + K_{ac}K_{d}{}^{d} - K_{dc}K_{a}{}^{d} \\
= {}& {}^{(5)}R_{efg}{}^{h} q_{a}{}^{e} q_{c}{}^{g}\Big(q_{h}{}^{f}\Big) + K_{ac}K - K_{dc}K_{a}{}^{d} \\
= {}& {}^{(5)}R_{efg}{}^{h} q_{a}{}^{e} q_{c}{}^{g}\Big(g_{h}{}^{f} - n_{h}n^{f}\Big) + K_{ac}K - K_{dc}K_{a}{}^{d} \\
= {}& \Big({}^{(5)}R_{efg}{}^{h} g_{h}{}^{f}\Big) q_{a}{}^{e} q_{c}{}^{g} -{}^{(5)} R_{efg}{}^{h}\, n_{h}n^{f} q_{a}{}^{e} q_{c}{}^{g} + K_{ac}K - K_{dc}K_{a}{}^{d} \\
\Rightarrow{}^{(4)}R_{ac} = {}& {}^{(5)}R_{eg}\, q_{a}{}^{e} q_{c}{}^{g} - \tilde{\psi}_{ac} + K_{ac}K - K_{dc}K_{a}{}^{d}
\end{aligned} \tag{2.39}$$

where $\tilde{\psi_{ac}} ={}^{(5)} R_{efg}{}^{h}\, n_{h}n^{f} q_{a}{}^{e} q_{c}{}^{g}$, and we used Eqn.(2.5) in the third equality. Repeating the contraction, we can obtain the 4-dimensional Ricci scalar

$$\begin{aligned}
^{(4)}R ={}^{(4)} R_{ac}q^{ac} = {}& {}^{(5)}R_{eg}\Big(q_{a}{}^{e} q_{c}{}^{g} q^{ac}\Big) - \tilde{\psi}_{ac}q^{ac} + \Big(q^{ac}K_{ac}\Big)K - K_{bc}\Big(K_{a}{}^{b} q_{ac}\Big) \\
= {}& {}^{(5)}R_{eg}\Big(q^{eg}\Big) - \tilde{\psi}_{ac}q^{ac} + (K)\,K - K_{bc}\Big(K^{bc}\Big) \\
\Rightarrow{}^{(4)}R = {}& {}^{(5)}R_{eg}\, q^{eg} - \tilde{\psi}_{eg}q^{eg} + K^{2} - K_{eg}K^{eg}
\end{aligned} \tag{2.40}$$

Plugging the above results into the 4-dimensional Einstein tensor

$$\begin{aligned}
{}^{(4)}G_{ac} ={}^{(4)} R_{ac} - \frac{1}{2}{}^{(5)}R q_{ac} \\
&= {}^{(5)}R_{eg}\, q_a{}^e q_c{}^g - \tilde{\psi}_{ac} + K_{ac}K - K_{dc}K_a{}^d \\
&- \frac{1}{2}\Big({}^{(5)}R_{eg}\, q^{eg} - \tilde{\psi}_{eg}q^{eg} + K^2 - K_{eg}K^{eg}\Big) q_{ac} \\
&= {}^{(5)}R_{eg}\, q_a{}^e q_c{}^g - \tilde{\psi}_{ac} + K_{ac}K - K_{dc}K_a{}^d - \frac{q_{ac}}{2}\Big(K^2 - K_{eg}K^{eg}\Big) \\
&- \frac{1}{2}\Big({}^{(5)}R_{eg}\, q^{eg}q_{ac} - \tilde{\psi}_{eg}q^{eg}q_{ac}\Big)
\end{aligned} \tag{2.41}$$

Let us simplify the terms in the final bracket in Eqn.(2.41)

$$\begin{aligned}
{}^{(5)}R_{eg}\, q^{eg}q_{ac} &= {}^{(5)}R_{eg}\, \Big(g^{eg} - n^e n^g\Big) q_{ac} \\
&= \Big({}^{(5)}R_{eg}\, g^{eg}\Big) q_{ac} -{}^{(5)} R_{eg}\, n^e n^g\, q_{ac} \\
&= {}^{(5)}R\Big(g_{eg}q_a{}^e q_c{}^g\Big) -{}^{(5)} R_{eg}\, n^e n^g\, q_{ac} \\
\Rightarrow{}^{(5)} R_{eg}\, q^{eg}q_{ac} &= {}^{(5)}R\, g_{eg}q_a{}^e q_c{}^g -{}^{(5)} R_{eg}\, n^e n^g\, q_{ac}
\end{aligned} \tag{2.42}$$

Similarly,

$$\tilde{\psi}_{eg}q^{eg}\, q_{ac} = {}^{(5)}R_{eg}\, n^e n^g\, q_{ac} \tag{2.43}$$

Plugging Eqns.(2.42) and (2.43) into (2.41)

$$\begin{aligned}
{}^{(4)}G_{ac} =& {}^{(5)}R_{eg}\, q_a{}^e q_c{}^g - \tilde{\psi}_{ac} + K_{ac}K - K_{dc}K_a{}^d - \frac{q_{ac}}{2}\Big(K^2 - K_{eg}K^{eg}\Big) \\
&- \frac{1}{2}\Big[\Big({}^{(5)}R\, g_{eg} q_a{}^e q_c{}^g - {}^{(5)}R_{eg}\, n^e n^g\, q_{ac}\Big) - \Big({}^{(5)}R_{eg}\, n^e n^g\, q_{ac}\Big)\Big] \\
=& {}^{(5)}R_{eg}\, q_a{}^e q_c{}^g - \tilde{\psi}_{ac} + K_{ac}K - K_{dc}K_a{}^d \frac{q_{ac}}{2}\Big(K^2 - K_{eg}K^{eg}\Big) \\
&- \frac{1}{2}\Big({}^{(5)}R\, g_{eg} q_a{}^e q_c{}^g\Big) + {}^{(5)}R_{eg}\, n^e n^g\, q_{ac} \\
\Rightarrow& {}^{(4)}G_{ac} = \Big({}^{(5)}R_{eg} - \frac{1}{2}{}^{(5)}R\, g_{eg}\Big) q_a{}^e q_c{}^g + {}^{(5)}R_{eg}\, n^e n^g\, q_{ac} \\
&+ K_{ac}K - K_{dc}K_a{}^d - \tilde{\psi}_{ac} - \frac{q_{ac}}{2}\Big(K^2 - K_{eg}K^{eg}\Big)
\end{aligned} \tag{2.44}$$

We see the 5D EFE in the brackets, which yields the 5D stress-energy tensor. Let us contract this equation,

$$\begin{aligned}
&\Big({}^{(5)}R_{eg} g^{eh}\Big) - \frac{1}{2}{}^{(5)}R\,\Big(g_{eg}g^{eg}\Big) = 8\pi G_5\Big(T_{eg}g^{eg}\Big) \\
&\Rightarrow {}^{(5)}R - \frac{5}{2}{}^{(5)}R = 8\pi G_5 T \\
&\Rightarrow {}^{(5)}R = \frac{-16\pi G_5}{3} T
\end{aligned} \tag{2.45}$$

The tensors have all been contracted in the usual way. Now, plugging (2.45) into the 5D EFE,

$$\begin{aligned}
&{}^{(5)}R_{eg} - \frac{1}{2}\Big(\frac{-16\pi G_5}{3}T\Big) g_{eg} = 8\pi G_5 T_{eg} \\
&\Rightarrow {}^{(5)}R_{eg} = 8\pi G_5\Big(T_{eg} - \frac{T}{3} g_{eg}\Big)
\end{aligned} \tag{2.46}$$

All this labour will soon pay off - we now substitute Eqns.(2.45) and (2.46) into the final result in Eqn.(2.44)

$$
\begin{aligned}
{}^{(4)}G_{ac} &= \left(8\pi G_5 T_{eg}\right) q_a{}^{e} q_c{}^{g} + \left[8\pi G_5\left(T_{eg} - \frac{T}{3} g_{eg}\right)\right] n^e n^g\, q_{ac} \\
&+ K_{ac}K - K_{dc}K_a{}^{d} - \tilde{\psi}_{ac} - \frac{q_{ac}}{2}\left(K^2 - K_{eg}K^{eg}\right) \\
&= 8\pi G_5 \left(T_{eg} q_a{}^{e} q_c{}^{g} + q_{ac}\left[T_{eg}n^e n^g - \frac{T}{3}\left(g_{eg}n^e n^g\right)\right]\right) \\
&+ K_{ac}K - K_{dc}K_a{}^{d} - \tilde{\psi}_{ac} - \frac{q_{ac}}{2}\left(K^2 - K_{eg}K^{eg}\right) \\
&\Rightarrow {}^{(4)}\, G_{ac} = 8\pi G_5 \left(T_{eg} q_a{}^{e} q_c{}^{g} + q_{ac}\left[T_{eg}n^e n^g - \frac{T}{3}\right]\right) \\
&+ K_{ac}K - K_{dc}K_a{}^{d} - \tilde{\psi}_{ac} - \frac{q_{ac}}{2}\left(K^2 - K_{eg}K^{eg}\right)
\end{aligned}
\tag{2.47}
$$

4D Stress-Energy tensor

We'd like to be able to write an equation for ${}^{(4)}G_{ac}$ only in terms of the energy-momentum tensor, for which, we need to write the extrinsic curvature K_{ac} and $\tilde{\psi}_{ac}$ in terms of T_{ac}. Recalling Eqn.(2.37), we set about decomposing our Riemann tensor into the Ricci tensor and scalar and the Weyl tensor,

$$
{}^{(5)}R_{efgd} = {}^{(5)}C_{efgd} + \tfrac{1}{3}\left(g_{e[g}\, {}^{(5)}R_{d]f} - g_{f[g}\, {}^{(d)}R_{d]e}\right) - \tfrac{1}{12}\, {}^{(d)}R\; g_{e[g}g_{d]f} \tag{2.48}
$$

Using our definition for $\tilde{\psi}_{ac}$,

$$
\begin{aligned}
\tilde{\psi}_{ac} &= {}^{(5)}R_{efg}{}^{h}\, n_h n^f q_a{}^e q_c{}^g \\
&= \left({}^{(5)}R_{efgd} g^{dh}\right) n_h n^f q_a{}^e q_c{}^g \\
&= \left({}^{(5)}R_{efgd}\right)\left(g^{dh}\, n_h\right) n^f q_a{}^e q_c{}^g \\
&= {}^{(5)}C_{efgd}\, n^d n^f q_a{}^e q_c{}^g + \frac{1}{3}\left(g_{e[g}\, {}^{(5)}R_{d]f} - g_{f[g}\, {}^{(d)}R_{d]e}\right) n^d n^f q_a{}^e q_c{}^g \\
&- \frac{1}{12}\left({}^{(d)}R\, g_{e[g} g_{d]f}\right) n^d n^f q_a{}^e q_c{}^g \\
&\Rightarrow \tilde{\psi}_{ac} = \psi_{ac} + \frac{1}{3}\left(g_{e[g}\, {}^{(5)}R_{d]f}\, n^d n^f q_a{}^e q_c{}^g\right) - \frac{1}{3}\left(g_{f[g}\, {}^{(d)}R_{d]e}\, n^d n^f q_a{}^e q_c{}^g\right) \\
&- \frac{1}{12}\left({}^{(d)}R\, g_{e[g} g_{d]f}\, n^d n^f q_a{}^e q_c{}^g\right)
\end{aligned}
\tag{2.49}
$$

The term $\psi_{ac} = {}^{(5)}C_{efgd}\, n^d n^f q_a{}^e q_c{}^g$ is the *electrical part of the Weyl tensor*; it is simply the projection of the 5-dimensional Weyl tensor onto the brane. The term in the first bracket in the final result in Eqn.(2.49) can be simplified using the energy-momentum tensor as in Eqn.(2.46),

$$
\begin{aligned}
&\left(g_{e[g}\, {}^{(5)}R_{d]f}\, n^d n^f q_a{}^e q_c{}^g\right) \\
&= g_{eg}\, {}^{(5)}R_{df}\, n^d n^f q_a{}^e q_c{}^g - g_{ed}\, {}^{(5)}R_{gf}\, n^d n^f q_a{}^e q_c{}^g \\
&= {}^{(5)}R_{df}\, n^d n^f \left(g_{eg}\, q_a{}^e q_c{}^g\right) - {}^{(5)}R_{gf}\, n^d n^f \left(q_a{}^e g_{ed}\right) q_c{}^g \\
&= \left[8\pi G_5\left(T_{df} - \frac{T}{3} g_{df}\right)\right] n^d n^f q_{ac} - {}^{(5)}R_{gf}\, n^d n^f \left(q_{ad}\right) q_c{}^g \\
&= 8\pi G_5\left(T_{df} - \frac{T}{3} g_{df}\right) n^d n^f q_{ac} - {}^{(5)}R_{gf}\, n^d n^f \left(g_{ad} - n_a n_d\right) q_c{}^g \\
&= 8\pi G_5\left(T_{df} - \frac{T}{3} g_{df}\right) n^d n^f q_{ac} - {}^{(5)}R_{gf}\left(n_a n^f - n_a n^f\right) q_c{}^g \\
&\Rightarrow \left(g_{e[g}\, {}^{(5)}R_{d]f}\, n^d n^f q_a{}^e q_c{}^g\right) = 8\pi G_5\left(T_{df} - \frac{T}{3} g_{df}\right) n^d n^f q_{ac}
\end{aligned}
\tag{2.50}
$$

We can now simplify the other terms accordingly,

$$\left(g_{f[g}\ {}^{(d)}R_{d]e}\ n^d n^f q_a{}^e q_c{}^g\right) = -8\pi G_5\left(T_{df} - \tfrac{T}{3}q_{df}\right) q_c{}^d q_a{}^f \tag{2.51}$$

$${}^{(d)}R\ g_{e[g}g_{d]f}\ n^d n^f q_a{}^e q_c{}^g = \tfrac{-16\pi G_5 T}{3} q_{ac} \tag{2.52}$$

Plugging the above relations back into Eqn.(2.49)

$$\begin{aligned}
\Rightarrow \tilde{\psi}_{ac} &= \psi_{ac} + \frac{1}{3}\left[\left(8\pi G_5\left(T_{df} - \frac{T}{3}g_{df}\right) n^d n^f q_{ac}\right) - \left(-8\pi G_5\left(T_{df} - \frac{T}{3}q_{df}\right) q_c{}^d q_a{}^f\right)\right] \\
&- \frac{1}{12}\left[\frac{-16\pi G_5 T}{3} q_{ac}\right] \\
&= \frac{8\pi G_5}{3}\left[T_{df} n^d n^f q_{ac} - \frac{T}{3}(g_{df} n^d n^f) q_{ac} + T_{df} q_c{}^d q_a{}^f - \frac{T}{3}(q_{df}\ q_c{}^d q_a{}^f)\right. \\
&+ \left(\frac{4\pi G_5 T}{9} q_{ac}\right) + \psi_{ac} \\
&\Rightarrow \tilde{\psi}_{ac} = \psi_{ac} + \frac{8\pi G_5}{3}\left[q_{ac}\left(T_{df} n^d n^f - \frac{T}{2}\right) + T_{df}\ q_c{}^d q_a{}^f\right]
\end{aligned} \tag{2.53}$$

Substituting this result back into Eqn.(2.47)

$$\begin{aligned}
{}^{(4)}G_{ac} &= 8\pi G_5\left(T_{eg} q_a{}^e q_c{}^g + q_{ac}\left[T_{eg} n^e n^g - \frac{T}{3})\right]\right) \\
&+ K_{ac}K - K_{dc}K_a{}^d - \frac{q_{ac}}{2}\left(K^2 - K_{eg}K^{eg}\right) \\
&- \left[\psi_{ac} + \frac{8\pi G_5}{3}\left[q_{ac}\left(T_{df} n^d n^f - \frac{T}{2}\right) + T_{df}\ q_c{}^d q_a{}^f\right]\right] \\
\Rightarrow {}^{(4)}G_{ac} &= \frac{16\pi G_5}{3}\left(T_{eg} q_a{}^e q_c{}^g + q_{ac}\left[T_{eg} n^e n^g - \frac{T}{4}\right]\right) \\
&+ K_{ac}K - K_{dc}K_a{}^d - \frac{q_{ac}}{2}\left(K^2 - K_{eg}K^{eg}\right) - \psi_{ac}
\end{aligned} \tag{2.54}$$

Brane Tension

Let us investigate further into stress-energy tensors of branes. Let us consider the *energy density* of the brane or commonly referred to as *brane tension* σ, essentially a sort of vacuum energy in the braneworld. We may have to alter the stress-energy tensor to account for this tension. It is also interesting to have additional fields on the brane, having a 4-dimensional stress-energy tensor η_{ac}. This prompts us to define $S_{ac} = \eta_{ac} - \sigma q_{ac}$, a energy-momentum tensor containing contributions from both the brane fields and brane tension. To define the 5D energy-momentum tensor, we also include the cosmological constant of the bulk, Λ_5 **[32]** :

$$T_{ac} = S_{ac}\delta(y) - \Lambda_5 g_{ac} = \Big(\eta_{ac} - \sigma q_{ac}\Big)\delta(y) - \Lambda_5 g_{ac} \tag{2.55}$$

whee we use a delta function of y to localise the brane contributions to T_{ac}. Now with this fancy new result, we can rewrite Eqn.(2.54) in a more convenient form:

$$T_{eg} q_a{}^e q_c{}^g = S_{eg} q_a{}^e q_c{}^g \delta(y) - \Lambda_5 g_{ac} \tag{2.56}$$

$$T_{eg} n^e n^g - \tfrac{T}{4} = \Big(S_{eg} n^e n^g - \tfrac{S}{4}\Big)\delta(y) + \tfrac{\Lambda_5}{4} \tag{2.57}$$

Using the above two, we can simplify the term in the brackets in Eqn.(2.54), $\Big(T_{eg} q_a{}^e q_c{}^g + q_{ac}\Big[T_{eg} n^e n^g - \tfrac{T}{4}\Big]\Big) = \tfrac{-3\Lambda_5}{4} q_{ac}$, allowing us to write Eqn.(2.54) as

$$^{(4)}G_{ac} = -4\pi G_5 q_{ac} + K_{ac}K - K_{dc}K_a{}^d - \tfrac{q_{ac}}{2}\Big(K^2 - K_{eg}K^{eg}\Big) - \psi_{ac} \tag{2.58}$$

We now need to express the extrinsic curvature tensor in terms of the stress-energy tensor. Contracting the 5D EFE, $G_{eg} = 8\pi G_5\big(S_{eg}\delta(y) - \Lambda_5 g_{eg}\big)$, gives us

$$\begin{aligned}\Big({}^{(5)}R_{eg} g^{eg}\Big) - \frac{1}{2}\,{}^{(5)}R\big(g_{eg}g^{eg}\big) &= 8\pi G_5\big(S_{eg}\delta(y) - \Lambda_5 g_{eg}\big)\\ \Rightarrow {}^{(5)}R - \frac{5}{2}{}^{(5)}R &= 8\pi G_5\big(S\delta(y) - 5\Lambda_5\big)\\ \Rightarrow {}^{(5)}R &= \frac{-16\pi G_5}{3}\big(S\delta(y) - 5\Lambda_5\big)\end{aligned} \tag{2.59}$$

The EFE now becomes,

$$\begin{aligned} {}^{(5)}R_{eg} - \frac{1}{2}\left[\frac{-16\pi G_5}{3}\big(S\delta(y) - 5\Lambda_5\big)\right]g_{eg} = 8\pi G_5\big(S\delta(y) - 5\Lambda_5\big) \\ \Rightarrow^{(5)} R_{eg} = 8\pi G_5\left[\left(S_{eg} - \frac{S}{3}g_{eg}\right)\delta(y) + \frac{16\pi G_5\Lambda_5}{3}\right] \end{aligned} \tag{2.60}$$

Recall that in Gaussian coordinates, the extrinsic curvature is given by $_{ac} = \frac{1}{2}\partial_y q_{ac}$. Also, $\partial_y K_{ac} = K_{cb}K_a{}^b$ - $\tilde{\psi}_{ac}$. Now, we rewrite the 4-dimensional Ricci tensor (Eqn.(2.39)) as

$$\begin{aligned} {}^{(4)}R_{ac} &= {}^{(5)}R_{eg}\, q_a{}^e q_c{}^g - \tilde{\psi}_{ac} + K_{ac}K - K_{dc}K_a{}^d \\ &= {}^{(5)}R_{eg}\, q_a{}^e q_c{}^g + \left(\partial_y K_{ac} - K_{cb}K_a{}^b\right) + K_{ac}K - K_{bc}K_a{}^b \\ \Rightarrow^{(5)} R_{eg}\, q_a{}^e q_c{}^g &= {}^{(4)}R_{ac} + 2K_{bc}K_a{}^b - K_{ac}K - \partial_y K_{ac} \\ \Rightarrow^{(5)} R_{eg}\, q_a{}^e q_c{}^g &= P_{ac} - \partial_y K_{ac} \end{aligned} \tag{2.61}$$

where $P_{ac} \equiv {}^{(4)}R_{ac} + 2K_{bc}K_a{}^b - K_{ac}K$. Comparing Eqns. (2.60) and (2.61),

$$\begin{aligned} P_{ac} - \partial_y K_{ac} &= 8\pi G_5\left[\left(S_{eg} - \frac{S}{3}g_{eg}\right)\delta(y) + \frac{16\pi G_5\Lambda_5}{3}\right] q_a{}^e q_c{}^g \\ &= 8\pi G_5\left[\left(\left(S_{eg}\, q_a{}^e q_c{}^g\right) - \frac{S}{3}\left(g_{eg}\, q_a{}^e q_c{}^g\right)\right)\delta(y) + \frac{2\Lambda_5}{3}\left(g_{eg}\, q_a{}^e q_c{}^g\right)\right] \\ \Rightarrow P_{ac} - \partial_y K_{ac} &= 8\pi G_5\left[\left(S_{ac} - \frac{S}{3}q_{ac}\right)\delta(y) + \frac{2\Lambda_5}{3}q_{ac}\right] \end{aligned} \tag{2.62}$$

Notice that in the above equation, K_{ac} remains undetermined. To do this, we have to integrate Eqn.(2.62) on the brane (y=0), in the interval $(-\epsilon, +\epsilon), \epsilon \to 0$, as $\epsilon \to 0$.

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon}\Big(P_{ac}-\frac{\mathrm{d}}{\mathrm{d}y}K_{ac}\Big)\,dy = 8\pi G_5\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon}\Big[\Big(S_{ac}-\frac{S}{3}q_{ac}\Big)\delta(y)+\frac{2\Lambda_5}{3}q_{ac}\Big]\,dy\\
&\quad\Rightarrow \lim_{\epsilon\to 0}\big(P_{ac}y|_{-\epsilon}^{+\epsilon}-K_{ac}|_{-\epsilon}^{+\epsilon}\big) = 8\pi G_5\Big(S_{ac}-\frac{S}{3}q_{ac}\Big)+\frac{2\Lambda_5}{3}q_{ac}\lim_{\epsilon\to 0}\big(y|_{-\epsilon}^{+\epsilon}\big)\\
&\quad\Rightarrow \lim_{\epsilon\to 0}\big(K_{ac}|_{-\epsilon}^{+\epsilon}\big) = -8\pi G_5\Big(S_{ac}-\frac{S}{3}q_{ac}\Big)\\
&\quad\Rightarrow \hat{[}K_{ac}\hat{]} = -8\pi G_5\Big(S_{ac}-\frac{S}{3}q_{ac}\Big)
\end{aligned} \tag{2.63}$$

where $K^{\pm}_{ac}$ denotes the extrinsic curvature in the direction of y. Considering the definition of extrinsic curvature, which contains a derivative term, we expect an invariance under $y \to -y$ given the symmetry $S^1/\mathbb{Z}^2$. Thus, we have the neat result of $K^+_{ac} \equiv K^-_{ac} \equiv K_{ac}$. Thus, we can simplify Eqn(2.63)

$$\begin{aligned}
K_{ac} &= -4\pi G_5\Big(S_{ac}-\frac{S}{3}q_{ac}\Big)\\
K &= q^{ac}K_{ac} = -4\pi G_5\Big(S-\frac{S}{3}(4)\Big) = \frac{4\pi G_5 S}{3}
\end{aligned} \tag{2.64}$$

Or, using the definition of S_{ac}

$$K_{ac} = -4\pi G_5\Big(\eta_{ac}+\frac{1}{3}(\sigma-\eta)\Big)q_{ac}$$

Recall also,

$$\begin{aligned}
S_{ac} &= \eta_{ac}-\sigma q_{ac}\\
S &= q^{ac}S_{ac} = \eta-4\sigma
\end{aligned} \tag{2.65}$$

Extrinsic Curvature

This now allows us to calculate the various terms in Eqn.(2.54) as we now have expressions for the extrinsic curvature in terms of the stress-energy tensor,

$$\begin{aligned}
K_{ac}K &= \Big[-4\pi G_5\Big(S_{ac}-\frac{S}{3}q_{ac}\Big)\Big]\Big(\frac{4\pi G_5 S}{3}\Big) = -\frac{16\pi^2G_5^2}{3}\,S\Big(S_ac-\frac{S}{3}q_{ac}\Big)\\
&= -\frac{16\pi^2G_5^2}{3}\big(\eta-4\sigma\big)\Big[\big(\eta_{ac}-\sigma q_{ac}\big)-\frac{\big(\eta-4\sigma\big)}{3}q_{ac}\Big]\\
&= -\frac{16\pi^2G_5^2}{3}\Big[-\frac{(\eta-4\sigma)^2}{3}q_{ac}+\big(\eta-4\sigma\big)\big(\eta_{ac}-\sigma q_{ac}\big)\Big]\\
&= \frac{16\pi^2G_5^2}{3}\Big[4\sigma\eta_{ac}+\frac{4\sigma^2q_{ac}}{3}-\frac{5\sigma\eta q_{ac}}{3}-\eta\eta_{ac}+\frac{\eta^3q_{ac}}{3}\Big]
\end{aligned} \tag{2.66}$$

Carrying out similar computations,

$$\begin{aligned}
K_{dc}K_a{}^{d} &= \Big[-4\pi G_5\Big(S_{dc}-\frac{S}{3}q_{dc}\Big)\Big]\Big[-4\pi G_5\Big(S_a^d-\frac{S}{3}q_a^d\Big)\Big]\\
&= 16\pi^2G_5^2\Big[\Big(\big(\eta_{dc}-\sigma q_{dc}\big)-\frac{\eta-4\sigma}{3}q_{dc}\Big)\Big]\Big[\Big(\big(\eta_a^d-\sigma q_a^d\big)-\frac{\eta-4\sigma}{3}q_a^d\Big)\Big]\\
&= 16\pi^2G_5^2\Big[\eta_{dc}\eta_a^d+\frac{2\big(\sigma-\eta\big)\eta_{ac}}{3}+\frac{(\sigma-\eta)^2q_{ac}}{9}\Big]
\end{aligned} \tag{2.67}$$

$$\begin{aligned}
K^2 &= \Big(\frac{4\pi G_5 S}{3}\Big)^2 = \Big(\frac{4\pi G_5\big(\eta-4\sigma\big)}{3}\Big)^2\\
&= \frac{16\pi^2G_5^2}{9}\big(\eta^2-8\sigma\eta+16\sigma^2\big)
\end{aligned} \tag{2.68}$$

$$\begin{aligned}
K_{eg}K^{eg} &= \Big[-4\pi G_5\Big(S_{eg}-\frac{S}{3}q_{eg}\Big)\Big]\Big[-4\pi G_5\Big(S^{eg}-\frac{S}{3}q^{eg}\Big)\Big]\\
&= 16\pi^2G_5^2\Big[\Big(\big(\eta_{eg}-\sigma q_{eg}\big)-\frac{\eta-4\sigma}{3}q_{eg}\Big)\Big]\Big[\Big(\big(\eta^{eg}-\sigma q^{eg}\big)-\frac{\eta-4\sigma}{3}q^{eg}\Big)\Big]\\
&= 16\pi^2G_5^2\Big[\eta_{eg}\eta^{eg}+\frac{2\big(\sigma-\eta\big)\eta}{3}+\frac{4\big(\sigma-\eta\big)^2}{9}\Big]
\end{aligned} \tag{2.69}$$

Effective EFE

We now have all the ingredients to obtain the EFE. We substitute the above results into Eqn.(2.54) and after some tedious yet straightforward algebra, we can write the *Effective EFE on the brane*, or the gravitational equations on the 3-brane **[32]**

$$^{(4)}G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + 8\pi G_4 \eta_{\mu\nu} + (8\pi G_5)^4 \pi_{\mu\nu} - \psi_{\mu\nu} \tag{2.70}$$

where,

$$\Lambda_4 = 4\pi G_5 \left(\Lambda_5 + \frac{4\pi G_5 \sigma^2}{3}\right) \tag{2.71}$$

$$G_4 = \frac{64\pi^2 G_5^2 \sigma}{48\pi} = \frac{4\pi\sigma G_5^2}{3} \tag{2.72}$$

$$\pi_{\mu\nu} = \frac{1}{4}\left(\frac{\eta\eta_{\mu\nu}}{3} - \frac{\eta^2 q_{\mu\nu}}{6} + \frac{\eta_{\alpha\beta}\eta^{\alpha\beta} q_{\mu\nu}}{2} - \eta_{\mu}{}^{\alpha}\eta_{\alpha\nu}\right) \tag{2.73}$$

Comparing the 5D EFE and the 4D EFE, we see that there are two new terms, $(8\pi G_5)^4 \pi_{\mu\nu}$ and $\psi_{\mu\nu}$. $\pi_{\mu\nu}$ is quadratic in $\eta_{\mu\nu}$ and is negligible at low energies. This only becomes dominating in cases where the energy matter-density in $\eta_{\mu\nu}$ tends to be larger than the brane tension σ, like the early universe. **[33; 34; 35]**. We could view this as a higher-energy correction, although we must note that this is not due to some higher derivative terms in the action. As discussed previously, $\psi_{\mu\nu}$ is part of the 5D Weyl tensor and carries non-local gravitational information from the bulk. It is a non-vanishing quantity if the bulk spacetime is different to an entirely AdS one. $\psi_{\mu\nu}$ is also negligible in low energy regimes, but it is larger than the quadratic terms in $\pi_{\mu\nu}$.

Interestingly, $\psi_{\mu\nu}$ is constrained by the motion of matter on the brane. To show this, consider Eqns.(2.22) and (2.63). As in [32], we have:

$$D_\alpha K - D_\mu K_\alpha{}^{\mu} \propto D_\nu \eta_\mu{}^{\nu} = 0 \tag{2.74}$$

The contracted Bianchi identities $D^\mu\, {}^{(4)}G_{\mu\nu} = 0$ give us a relation

$$\begin{aligned} D^\mu \psi_{\mu\nu} &= K^{\rho\theta}\big(D_\nu K_{\rho\theta} - D_\theta K_{\nu\rho}\big) \\ &= 16\pi^2 G_5^2\Big[\eta^{\rho\theta}\big(D_\nu \eta_{\rho\theta} - D_\theta \eta_{\nu\rho}\big) + \frac{1}{3}\big(\eta_{\mu\nu} - q_{\mu\nu}\eta\big)D^\mu \eta\Big] \end{aligned} \tag{2.75}$$

We see that $\psi_{\mu\nu}$'s divergence is constrained by the matter term. This implies that the effective EFE in Eqn.(2.70) is not closed and one must also solve the 5D EFE for the bulk to get brane solutions. $\psi_{\mu\nu}$ is due to this non-closure and prompts a possible imposition of $\psi_{\mu\nu}$ on the brane. Essentially, a normal gravitational theory can be recovered if the brane tension is positive, where Einstein's GR can be recovered in the low energy limit.

Chapter 3

The Randall-Sundrum Model

This chapter will present the two Randall-Sundrum models, given their key conceptual importance in a range of ideas regarding braneworlds. Now that we have covered some background concepts needed to study braneworlds, we can now dive into the subject. As briefly mentioned in Section 1.2.2 and in **[8; 9]**, the RS model has one(or two) domain walls situated as submanifolds in an AdS spacetime. The story begins with a non-factorisable metric, a 4-dimensional metric scaled by a "*warp factor*", which is a function of the additional dimension **[8]** :

$$ds^2 = e^{-2kr_c\phi}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\phi^2 \tag{3.1}$$

where $\{x^\mu\}$ are the 4D spacetime coordinates, k is Planck order scale, ϕ is the coordinate for our extra dimension ($0 \leq \phi \leq \pi$), which is a finite coordinate set by the size of r_c. The idea here is that the spacetime is constructed such that there are 4D flat slices stacked along the fifth dimension. $\phi = 0$ indicates the presence of a domain wall - the braneworld, representing a flat Minkowski universe. **[10]**

The Israel equations (Section 2.2) are a frequently used tool in this area. The idea is to rewrite the 5D spacetime as a 4D base spacetime with coordinates $\{x^\mu\}$, plus a normal distance ϕ from the domain wall. x^μ remain constant along geodesics normal to the domain wall, forming a 5D coordinate system $\{x^\mu, \phi\}$. These coordinates are valid within r_c, a "compactification radius" prior to orbifolding **[8]**. It splits the tangent space into parallel and normal components and generally takes the form

$$ds^2 = q_{\mu\nu} dx^\mu dx^\nu - d\phi^2 \tag{3.2}$$

This form allows us to encapsulate the nontrivial geometrical aspects within $q_{\mu\nu}$, the 4-dimensional metric, making the fifth metric component 1 as ϕ is merely the proper distance from the brane.[1]

3.1 AdS Spacetime

Before we look into the specifics of the RS models, it is useful to take a slight detour and briefly review the Anti-de Sitter spacetime. It plays an important part in braneworld considerations, including the Randall-Sundrum braneworlds, and other related studies, such as the AdS/CFT correspondence. The AdS spacetime is a spacetime of *constant curvature*, which typically can be a solution to the EFE with a constant and negative cosmological constant. We will follow the presentation layed out in **[36]**.

Spaces with constant curvature

To start, consider a flat spacetime $\mathbb{E}^3$ with metric

$$ds^2 = dX^2 + dY^2 + dZ^2 \tag{3.3}$$

The sphere S^2 defined by a surface satisfying

$$X^2 + Y^2 + Z^2 = R^2 \tag{3.4}$$

[1] $n^a = \delta^a_z$ is the normal to the brane.

We can pull out familiar tools like spherical polar coordinates and solve this constraint. The surface of S^2 is a space of constant positive curvature. S^2 has an SO(3) invariance, as the surface in Eqn.(3.4) obeys the SO(3) invariance of the "ambient" $\mathbb{E}^3$ space. This makes S^2 homogeneous, as any point can be mapped to any other via a SO(3) transformation (or a 3D rotation).

The space with constant negative curvature is called *hyperbolic* space. In this case, H^2 is harder to visualise as we cannot embed it into $\mathbb{E}^3$, but we can however embed it into 3-dimensional Minkowski space. We define the hyperbolic space via

$$\begin{aligned} ds^2 &= -dZ^2 + dX^2 + dY^2 \\ -Z^2 &+ X^2 + Y^2 = -L^2 \end{aligned} \tag{3.5}$$

The Fig.(3.1) below shows this embedding. We also see a SO(1,2) invariance of the ambient Minkowski spacetime, similar to the positive curvature case. This shows the homogeneity of the space, as any point can be mapped to another via a SO(1,2) Lorentz transform.

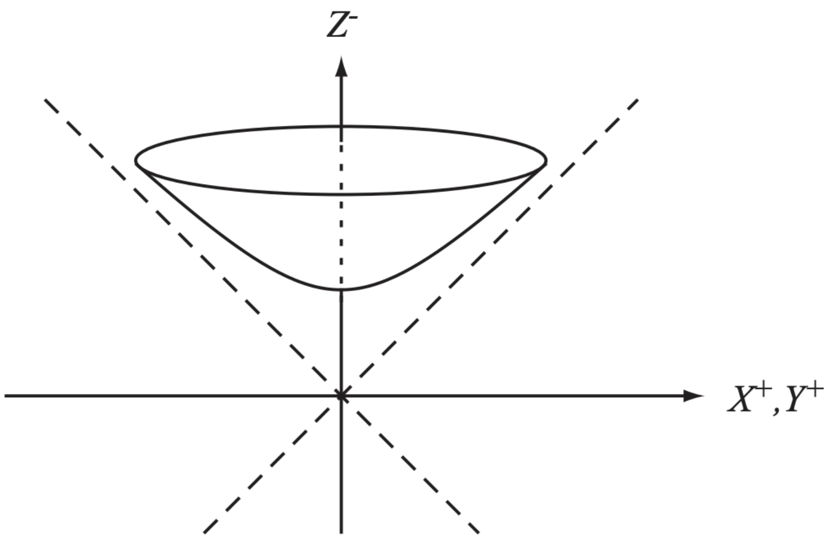

Figure 3.1: The embedding of H^2 into $\mathbb{R}^{1,2}$, where the superscripts refer to the signature of the ambient spacetime (+ being *spacelike*, - being *timelike*). As seen in **[36]**.

To solve the constraint equation, we use the following coordinate system

$$X = L \sinh\rho \cos\phi \ , \ \ Y = L \sinh\rho \sin\phi \ , \ \ Z = L \cosh\rho \tag{3.6}$$

yielding a metric

$$ds^2 = L^2 \big(d\rho^2 + \sinh\rho^2 d\phi^2 \big) \tag{3.7}$$

Spacetimes with constant curvature

Let us shift our focus to the more pertinent matter of spacetimes with constant curvature. Let us start by taking an ambient flat spacetime with two timelike directions, X and Z,

$$ds^2 = -dZ^2 - dX^2 + dY^2 \tag{3.8}$$

The 2D AdS spacetime, AdS_2, can be embedded into such a spacetime, defined by a surface

$$-Z^2 - X^2 + Y^2 = L^2 \tag{3.9}$$

where L is referred to as the *AdS radius*. AdS_2 has a SO(2,1) invariance of its ambient spacetime. We choose a set of coordinates

$$X = L \cosh\rho \sin\tau \ , \ \ Y = L \sinh\rho \ , \ \ Z = L \cosh\rho \cos\tau \tag{3.10}$$

yielding a metric

$$ds^2 = L^2 \big(d\rho^2 - \cosh\rho^2 d\tau^2 \big) \tag{3.11}$$

This coordinate system $\{\rho, \tau\}$, where $0 < \rho < \infty$ and $-\infty < \tau < \infty$, is called the *global coordinates system*, which dispels any ambiguity that despite embedding AdS_2 in a flat ambient spacetime with two timelike directions, AdS_2 only has one timelike direction.

Poincaré coordinates

A convenient coordinate system is the *Poincaré coordinate system* $\{t, r\}$, with $-\infty < t < \infty$ and $0 < r < \infty$. We define thiis system by

$$\begin{aligned} X &= Lrt \\ Y &= \frac{Lr}{2}\Big(-t^2 + \frac{1}{r^2} - 1 \Big) \\ Z &= \frac{Lr}{2}\Big(-t^2 + \frac{1}{r^2} + 1 \Big) \end{aligned} \tag{3.12}$$

wherein, the metric is now

$$ds^2 = -L^2 r^2 dt^2 + \tfrac{L^2}{r^2} dr^2 \tag{3.13}$$

Sometimes L is set to 1. An important point to note is that AdS spacetimes have a boundary, the *AdS boundary*. The existence of the boundary means that one must specify the boundary conditions on the AdS boundary to solve initial-value problems **[36]**. It is interesting to study the AdS boundary in Poincaré coordinates, located at $r \to \infty$ - refer to **[37]** for further details. For interest, this boundary plays a role in the study of the AdS/CFT correspondence, where it corresponds to specifying which external sources one must add in the gauge theory side.

Let us generalise the above results to higher dimensions. We simply add d spatial dimensions to the ambient spacetime,

$$ds^2_{d+3} = -dX_0^2 - dX_{d+2}^2 + dX_1^2 + \cdots + dX_{d+1}^2 \tag{3.14}$$

where the (d+2)-dimensional version of AdS spacetime is defined by

$$-X_0^2 - X_{d+2}^2 + X_1^2 + \cdots + X_{d+1}^2 = -L^2 \tag{3.15}$$

giving us in Poincaré coordinates

$$\begin{aligned} X_0 &= \frac{Lr}{2}\Big(x_i^2 - t^2 + \frac{1}{r^2} + 1\Big) \\ X_i &= Lrx_i \\ X_{d+1} &= \frac{Lr}{2}\Big(x_i^2 - t^2 + \frac{1}{r^2} - 1\Big) \\ X_{d+2} &= Lrt \end{aligned} \tag{3.16}$$

[2]giving us the following metric

$$ds^2 = -L^2r^2dt^2 + L^2r^2\delta_{ij}dx^idx^j + \tfrac{L^2}{r^2}dr^2 \tag{3.17}$$

Note here that δ_{ij} is the d-dimensional Kronecker delta. Additionally, AdS_{d+2} is SO(d+2,1) invariant of its ambient spacetime, making it a *maximally symmetric spacetime*, allowing us to write a Riemann tensor and associated quantities as follows

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -\frac{1}{L^2}\big(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\big) \\ R_{\alpha\beta} &= -\frac{d+1}{L^2}g_{\alpha\beta} \\ R &= -\frac{(d+1)(d+2)}{L^2} \end{aligned} \tag{3.18}$$

Thus, for AdS_{d+2},

$$R_{\alpha\beta} - \tfrac{1}{2}Rg_{\alpha\beta} = -\tfrac{d(d+1)}{L^2}g_{\alpha\beta} \tag{3.19}$$

which is the same as the vacuum EFE with a negative cosmological constant

$$\Lambda = -\tfrac{d(d+1)}{L^2} \tag{3.20}$$

[2]i runs from $1 \dots d$

For completeness, let us present the metric for AdS_5 spacetime

$$ds^2_{AdS_5} = -r^2 dt^2 + r^2\left(dx^2 + dy^2 + dz^2\right) + \tfrac{1}{r^2} dr^2 \tag{3.21}$$

AdS Black Holes

Black holes are possible in an AdS spacetime. Whilst, we will cover black holes on branes later on, the metric for a AdS-Schwarzschild black hole is presented

$$\begin{aligned} ds^2_{SAdS_5} &= -r^2 f(r) dt^2 + r^2\left(dx^2 + dy^2 + dz^2\right) + \frac{1}{r^2 f(r)} dr^2 \\ f(r) &= 1 - \left(\frac{r_s}{r}\right)^4 \end{aligned} \tag{3.22}$$

The horizon is located at $r = r_s$. Notice that the horizon extends infinitely in the (x,y,z) directions - a planar horizon. Also note that for $r_s = 0$, Eqn.(3.22) reduces to Eqn.(3.21).

3.2 RS I

The RS1 model introduces one extra dimension y [3] which is compactified in a $S^1/\mathbb{Z}^2$ orbifold (a circle folded across a diameter), S^1 being a circle of radius r_c and $\mathbb{Z}^2 = \{1, -1\}$ is the multiplicative parity group. Essentially, the extra dimension is a circle with both sides identified in an equivalence class. There are two fixed points in the orbifold $y = (0, \pi r_c)$, taken as the locations for two 3-branes extending in the $\{x^\mu\}$ directions, which form the boundary for our 5D bulk. It is important to note that r_c is independent of $\{x^\mu\}$ by 4D Poincaré invariance. A crucial aspect of this construction as stated previously, is that matter is confined to the brane whereas gravity can access the entire 5D bulk.

While it would be convenient, we cannot formally recover RS1 from M-theory. It is, however, worthwhile to consider that the preservation of Poincaré invariance leading to

[3] Following notation used in Chapter 2.

the emergence of certain relations in RS1, are the same as the 5D effective theory of the 11-dimensional theory on the orbifold $\mathbb{R}^{10} \times S^1/\mathbb{Z}^2$ as presented by Witten and Horava **[38]**.

EFE for RS1

The metric of the full 5D spacetime must be such that it is flat on the brane, since we are discarding any energy-momentum source that could result in a curvature, apart from vacuum energy. We demand a Poincaré invariance compatibility in the $\{x^\mu\}$ directions. In Gaussian normal coordinates, established on the brane (y=0), the metric is

$$\begin{aligned} ds^2 &= e^{-2A(y)} q_{\mu\nu} dx^\mu \otimes dx^\nu + dy^2 \\ g_{mn} &= e^{-2A(y)} q_{\mu\nu} + \delta^5_m \delta^5_n \end{aligned} \tag{3.23}$$

From the above equation, it is clear that $g_{mn} = g_{mn}(y)$, giving us $\partial_\mu g_{mn} = 0$. Let's move onto the non-vanishing Christoffel symbols

$$\begin{gathered} \Gamma^5_{\mu\nu} = -\frac{1}{2} g^{55} \big(\partial_5 g_{\mu\nu}\big) = -\frac{1}{2} \partial_y \big(e^{-2A} q_{\mu\nu}\big) = -\frac{1}{2} q_{\mu\nu} \big(- 2e^{-2A} \partial_y A\big) \\ \Rightarrow \Gamma^5_{\mu\nu} = g_{\mu\nu} \partial_y A \end{gathered} \tag{3.24}$$

$$\begin{gathered} \Gamma^\mu_{\nu 5} = \frac{1}{2} g^{\mu\rho} \big(\partial_5 g_{\rho\nu}\big) = \frac{1}{2} e^{2A} q^{\mu\rho} \partial_y \big(e^{-2A} q_{\rho\nu}\big) = \frac{1}{2} e^{2A} q^{\mu\rho} q_{\rho\nu} \big(- 2e^{-2A} \partial_y A\big) \\ \Rightarrow \Gamma^\mu_{\nu 5} = -\delta^\mu_\nu \partial_y A \end{gathered} \tag{3.25}$$

$\Big($The above symbols are calculated using $\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \big(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\big).\Big)$ To calculate the Ricci tensor components, we use this formula, $R_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\sigma_{\mu\sigma} + \Gamma^\sigma_{\sigma\lambda} \Gamma^\lambda_{\mu\nu} - \Gamma^\sigma_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$. We can see that $R_{\mu 5} = R_{5\mu} = 0$, leaving the remaining Ricci tensor components as

$$\begin{aligned}
R_{\mu\nu} &= \partial_5 \Gamma^5_{\mu\nu} + \Gamma^\sigma_{\sigma 5}\Gamma^5_{\mu\nu} - \Gamma^\sigma_{\nu 5}\Gamma^5_{\mu\sigma} - \Gamma^5_{\nu\rho}\Gamma^\rho_{\mu 5} \\
&= \partial_y\big(g_{\mu\nu}\partial_y A\big) + \big(-\delta^\sigma_\sigma \partial_y A\big)\big(g_{\mu\nu}\partial_y A\big) - \big(-\delta^\sigma_\nu \partial_y A\big)\big(g_{\mu\sigma}\partial_y A\big) - \big(-\delta^\rho_\mu \partial_y A\big)\big(g_{\nu\rho}\partial_y A\big) \\
&= \partial_y\big(e^{-2A} q_{\mu\nu}\partial_y A\big) - 4g_{\mu\nu}\big(\partial_y A\big)^2 + g_{\mu\nu}\big(\partial_y A\big)^2 + g_{\mu\nu}\big(\partial_y A\big)^2 \\
&= e^{-2A} q_{\mu\nu}\Big(-2\big(\partial_y A\big)^2 + \big(\partial^2_{yy} A\big)\Big) - 2g_{\mu\nu}\big(\partial_y A\big)^2 \\
\Rightarrow R_{\mu\nu} &= g_{\mu\nu}\Big(\big(\partial^2_{yy} A\big) - 4\big(\partial_y A\big)^2\Big) \\
\Rightarrow R_{55} &= 4\Big(\big(\partial^2_{yy} A\big) - 4\big(\partial_y A\big)^2\Big)
\end{aligned} \tag{3.26}$$

We now compute the Ricci scalar from the above result

$$\begin{aligned}
R &= g^{mn} R_{mn} = g^{\mu\nu} R_{\mu\nu} + g^{55} R_{55} \\
&= g^{\mu\nu}\Big[g_{\mu\nu}\Big(\big(\partial^2_{yy} A\big) - 4\big(\partial_y A\big)^2\Big)\Big] + 4\Big(\big(\partial^2_{yy} A\big) - 4\big(\partial_y A\big)^2\Big) \\
&= 4\big(\partial^2_{yy} A\big) - 16\big(\partial_y A\big)^2 + 4\big(\partial^2_{yy} A\big) - 4\big(\partial_y A\big)^2 \\
\Rightarrow R &= 4\Big(2\big(\partial^2_{yy} A\big) - 5\big(\partial_y A\big)^2\Big)
\end{aligned} \tag{3.27}$$

Finally, we can now compute the Einstein tensor $G_{mn} = G_{\mu\nu} + G_{55}$

$$\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \\
&= \Big(g_{\mu\nu}\Big(\big(\partial^2_{yy} A\big) - 4\big(\partial_y A\big)^2\Big)\Big) - \frac{1}{2}\Big(4\Big(2\big(\partial^2_{yy} A\big) - 5\big(\partial_y A\big)^2\Big)\Big) g_{\mu\nu} \\
\Rightarrow G_{\mu\nu} &= 3g_{\mu\nu}\Big(2\big(\partial_y A\big)^2 - \big(\partial^2_{yy} A\big)\Big)
\end{aligned} \tag{3.28}$$

And,

$$\begin{aligned}
G_{55} &= \Big(4\Big(\big(\partial^2_{yy} A\big) - 4\big(\partial_y A\big)^2\Big)\Big) - \frac{1}{2}\Big(4\Big(2\big(\partial^2_{yy} A\big) - 5\big(\partial_y A\big)^2\Big)\Big) g_{55} \\
\Rightarrow G_{55} &= 6\big(\partial_y A\big)^2
\end{aligned} \tag{3.29}$$

Now, we can use Eqn.(1.15), setting $T_{AB} = -\Lambda g_{AB}$. For the fifth dimension's component,

$$G_{55} = -\frac{\Lambda g_{55}}{2\mathcal{M}^3} \Rightarrow \partial_y A = \pm\sqrt{-\frac{\Lambda g_{55}}{12\mathcal{M}^3}} \tag{3.30}$$

Notice that since $\mathcal{M} > 0$, we need the warp factor term to represent an exponential decay of sorts, to allow for A(y) to be a real function, as seen in Fig.(3.2). This requires us imposing a condition that $\Lambda < 0$, an AdS spacetime, as discussed in Section(3.1). This implies that the bulk between the branes is a 5-dimensional AdS space, AdS_5. Let's now define $\kappa \equiv \sqrt{-\frac{\Lambda g_{55}}{12\mathcal{M}^3}}$, where $g_{55} = 1$. Thus, integrating Eqn.(3.30) gives us A(y)=$\pm\kappa y$. Recalling the orbifold symmetry, one has A(y) = $\kappa|y|$.

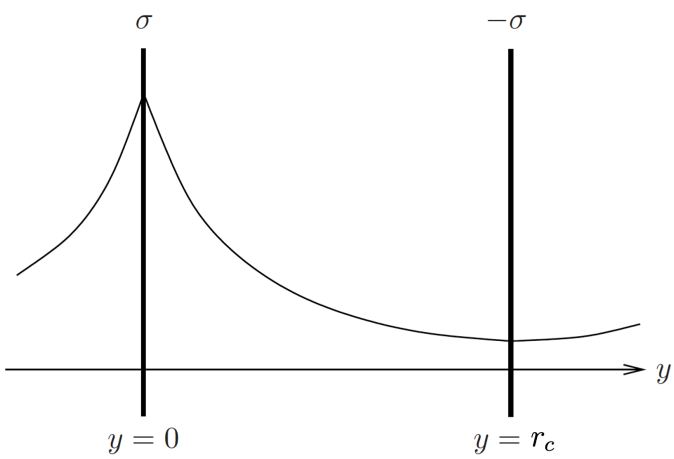

Figure 3.2: The behaviour of the warp factor in the RS1 model.

Brane tension in RS1

We have only looked at the extra component of the EFE - there may be additional information in the $\mu\nu$ components. Begin by taking the first derivative of A(y)

$$\frac{\mathrm{d}A(y)}{\mathrm{d}y} = \kappa\frac{\mathrm{d}|y|}{\mathrm{d}y} = \kappa\frac{y}{|y|} \tag{3.31}$$

It is also a simple observation that $(\frac{y}{|y|})^2 = 1, \forall y \in (-r_c, r_c)$. The derivative of $\frac{y}{|y|}$ is $2\delta(y)$, but since $y \in S^1/\mathbb{Z}^2$, we express the delta function within both bounds of our orbifold,

$$\kappa \frac{\mathrm{d}}{\mathrm{d}y}\frac{y}{|y|} = \frac{\mathrm{d}^2}{\mathrm{d}y^2}A(y) = 2\kappa\big(\delta(y) - \delta(y - r_c)\big) \tag{3.32}$$

Now, we proceed to check the above solution using the $\mu\nu$ components of the EFE,

$$\begin{aligned}
G_{\mu\nu} &= -\frac{\Lambda g_{\mu\nu}}{2\mathcal{M}^3} \\
\Rightarrow 3g_{\mu\nu}\Big(2\big(\partial_y A\big)^2 - \big(\partial^2_{yy}A\big)\Big) &= -\frac{\Lambda g_{\mu\nu}}{2\mathcal{M}^3} \\
\Rightarrow 3g_{\mu\nu}\Big(2\kappa^2\Big(\frac{y}{|y|}\Big)^2 - 2\kappa\big(\delta(y) - \delta(y - r_c)\big)\Big) &= -\frac{\Lambda g_{\mu\nu}}{2\mathcal{M}^3} \\
\Rightarrow (6\kappa^2)g_{\mu\nu} - 6\kappa\big(\delta(y) - \delta(y - r_c)\big)g_{\mu\nu} &= -\frac{\Lambda g_{\mu\nu}}{2\mathcal{M}^3}
\end{aligned} \tag{3.33}$$

One can easily identify that $6\kappa^2 = 6\big(\sqrt{-\Lambda/12\mathcal{M}^3}\big)^2 = -\Lambda/2\mathcal{M}^3$, whereas $-6\kappa\big(\delta(y) - \delta(y - r_c)\big)g_{\mu\nu} \neq 0$, meaning there is an absence of something in the $\mu\nu$ components of the 5D EFE. This absence or discrepancy is referred to as the *brane tension* from Section(2.3).

We deal with this, by adding to the action in Eqn.(1.14), the matter action of each brane $(y = 0, r_c)$ due to their respective brane tensions σ_1 and σ_2:

$$\begin{aligned}
S_1 &= \int d^4x dy \sqrt{-g}\big(-\sigma_1\big)\delta(y) = \int d^4x\sqrt{-q_1}\big(-\sigma_1\big) \\
S_2 &= \int d^4x dy \sqrt{-g}\big(-\sigma_2\big)\delta(y - r_c) = \int d^4x\sqrt{-q_2}\big(-\sigma_2\big)
\end{aligned} \tag{3.34}$$

Note, how a second derivative of A(y) when evaluated and integrated in each brane, leaves a 4-dimensional action. Also, q_1 and q_2 are the determinants of the induced metrics on either brane. Consider $S_M = \int d^5x\sqrt{-g}\big(-\Lambda - \sigma_1\delta(y) - \sigma_2\delta(y - r_c)\big)$, where both brane tensions are constant, the $\mu\nu$ components of the energy-momentum tensor becomes

$$T_{\mu\nu} = -\big(\Lambda + \sigma_1\delta(y) + \sigma_2\delta(y - r_c)\big)g_{\mu\nu} \tag{3.35}$$

Parameter Setting

Despite this new definition, Eqn.(3.30) holds water as brane tensions are not defined in the bulk - the extra dimensional component has no contributions from S_1 and S_2. So,

$$\begin{aligned} G_{\mu\nu} &= \left(-\frac{\Lambda}{2\mathcal{M}^3} - \frac{\sigma_1\delta(y)}{2\mathcal{M}^3} - \frac{\sigma_2\delta(y-r_c)}{2\mathcal{M}^3}\right) g_{\mu\nu} \\ \Rightarrow (6\kappa^2)g_{\mu\nu} - 6\kappa\big(\delta(y) + 6\kappa\delta(y-r_c)\big)g_{\mu\nu} &= \left(-\frac{\Lambda}{2\mathcal{M}^3} - \frac{\sigma_1\delta(y)}{2\mathcal{M}^3} - \frac{\sigma_2\delta(y-r_c)}{2\mathcal{M}^3}\right) g_{\mu\nu} \end{aligned} \tag{3.36}$$

Leading to,

$$-6\kappa = \tfrac{-\sigma_1}{2\mathcal{M}^3} \;\Rightarrow \sigma_1 = 12\kappa\mathcal{M}^3 \tag{3.37}$$

$$6\kappa = \tfrac{-\sigma_2}{2\mathcal{M}^3} \;\Rightarrow \sigma_2 = -12\kappa\mathcal{M}^3 \tag{3.38}$$

Giving us the result that the brane tensions are equal in magnitude but have opposite signs. Also,

$$\sigma^2 = (12)^2\Big(\tfrac{-\Lambda}{12\mathcal{M}^3}\Big)\big(\mathcal{M}^3\big)^2 \;\Rightarrow \Lambda = \tfrac{-\sigma^2}{12\mathcal{M}^3} \tag{3.39}$$

Eqn.(3.39) provides a *fine tuning* mechanism in the model, as σ the brane tension is a free parameter freely chosen to determine the high energy scale of the theory **[39]**. The negative cosmological constant prevents gravity from leaking into the extra dimension at low energies **[40]**. Now, finally we can write the full metric for the RS1 model

$$ds^2 = \exp\left(-2\sqrt{\tfrac{-\Lambda}{12\mathcal{M}^3}}|y|\right) q_{\mu\nu}dx^\mu \otimes dx^\nu + dy^2 \tag{3.40}$$

3.3 RS II

When we started talking about braneworlds, we stated that SM fields were localised on the brane but gravity could freely propagate into the bulk. This will worry a braneworld observer as Newton's inverse square law for gravity is a property of 4D gravity and is experimentally verified to be as low as $r \sim 0.2mm$. This issue is cleverly resolved if the extra dimension is small and compact, owing to a large mass gap between the zero graviton mode and the first Kaluza Klein mode. This ensures that gravity behaves four dimensionally, except at very high energies. Braneworld models showcase the idea that gravity can appear 5-dimensional even in low energies, thereby violating Newton's law. The RS2 is a subtle model that managed to produce Newton's law on the brane despite having an infinite extra dimension. This is due to the negative cosmological constant in the bulk.

RS2 Setup

To arrive at RS2, one begins with RS1 and extends the brane separation to infinity, such that we are left with a single brane of positive tension. The geometry is described by the metric in Eqn.(3.23) with $r_c \to \infty$. In Fig.(3.3), we can see the behaviour of the warp factor for the RS2 model. It has a peak at y = 0, indicating a positive tension for the brane there. Note a $\mathbb{Z}^2$ symmetry about y = 0.

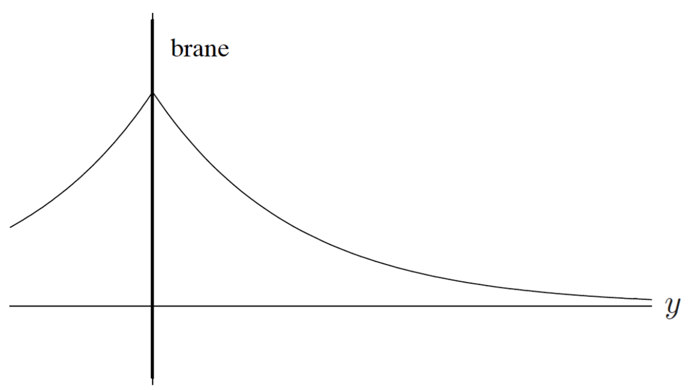

Figure 3.3: The behaviour of the warp factor in the RS2 model.

The size of the extra dimension is infinite. The bulk is empty except for a negative cosmological constant. Our stress tensor is therefore as we defined in Eqn.(2.55) :

$$T_{ac} = S_{ac}\delta(l) - \Lambda_5 g_{ac} = \big(\eta_{ac} - \sigma q_{ac}\big)\delta(l) - \Lambda_5 g_{ac} \tag{3.41}$$

where, (just to remind ourselves), Λ_5 is the cosmological constant in the bulk, l is the proper length defined only in a neighbourhood of the brane. The brane is located at y = 0. η_{ac} is the stress-energy tensor on the brane, σ is the brane tension and q_{ac} is the metric on the brane. Now, as a consequence of the $\mathbb{Z}^2$ symmetry, one gets $K^+_{ac} = -K^-_{ac}$. Israel's second junction condition becomes

$$K^+_{ac} = -K^-_{ac} = 4\pi G_5\Big(S_{ac} - q_{ac}\tfrac{S}{3}\Big) = 4\pi G_5\Big(\sigma\tfrac{q_{ac}}{3} + \eta_{ac} - q_{ac}\tfrac{\eta}{3}\Big) \tag{3.42}$$

The above equation relates K_{ac} with the distribution of matter on the brane, which can eliminate the extrinsic curvature terms in Eqn.(2.35). The term $E_{\alpha\beta}$ in Eqn.(2.35) which is related to the Weyl tensor, is associated with the geometry of the bulk and we cannot eliminate it so easily. But, this term only becomes relevant at higher energies **[32]**.

Evaluating Eqn.(2.35) on either side of the brane, performing the evaluation at $l \neq 0$ followed by taking the limit $l \to 0$. This analysis yields Einstein's equations at low energies, as seen in Section(2.3) and Eqns.(2.70 - 2.73), with some minor changes.

Parameter Setting

In order for G_4 to have the correct sign, Eqn.(2.72) requires $\sigma > 0$. According to Eqn.(2.71), Λ_4 the cosmological constant on the brane can be set to any value by just tuning the value of σ. If we set $\Lambda_4 = 0$, Eqn.(2.71) reduces to

$$\sigma = \frac{6}{l\sqrt{8\pi G_5}} \tag{3.43}$$

where l is the AdS radius in the bulk, where it relates to Λ_5 as follows $l = \sqrt{-\frac{6}{\Lambda_5}}$. Combining Eqns.(2.72) and (3.43)

$$G_4 = \frac{\sqrt{8\pi G_5}}{l} G_5 \tag{3.44}$$

In RS2 theory, l is a freely adjustable parameter, whose value we determine by experiment.

Vacuum Solution

There are classes of vacuuum solutions when matter is absent **[40; 41]**

$$ds^2 = e^{-2|y|/l}\left(q_{\mu\nu}dx^\mu dx^\nu + dy^2\right) \tag{3.45}$$

where $y \in (-\infty, \infty)$ is the extra dimension and the brane is located at y = 0. $q_{\mu\nu}$ has no y dependence, and can be any vaccuum solution from GR. Say, we define $z \equiv le^{y/l}$ when y is positive and $z \equiv le^{-y/l}$ when y is negative. This, in coordinate space corresponds to a two-to-one mapping $\pm y$ to $z = le^{|y|/l} \in [l, \infty)$. This is however only a superficial feature as $y \equiv -y$ due to the $\mathbb{Z}^2$ symmetry. Using coordinates (x^μ, z), Eqn.(3.45) becomes

$$ds^2 = \frac{l^2}{z^2}\left[(q_{\mu\nu}dx^\mu dx^\nu) + dz^2\right] \tag{3.46}$$

where $z \geq l$ covers the entire physical spacetime, now positioning our brane at $z = l$. The simplest case would be to let $q_{\mu\nu}$ be 4D Minkowski spacetime. Correspondingly, the 5D space is then a part of the Poincaré patch of AdS space. Interestingly, $q_{\mu\nu}$ can also take black hole solutions, leading to black string solutions in the brane, something which we will introduce later on.

Matter in RS2

Due to the nature of the RS2 braneworld setup, the'conservation laws' in Eqns.(2.33) and (2.34) become

$$\begin{aligned} \mathcal{D}_\mu S_\nu{}^\mu &= 0 \\ \hat{[\sigma]} &= -\frac{1}{2} S_{\mu\nu} \hat{\{K^{\mu\nu}\}} = 0 \end{aligned} \tag{3.47}$$

where $\mathcal{D}$ is the covariant derivative associated to $q_{\mu\nu}$. Eqn.(3.47) gives a conservation of matter on the brane. Since the brane tension part $\sigma q_{\mu\nu}$ satisfies this conservation law, we require the matter part on the brane $\eta_{\mu\nu}$ to also obey it. This is consistent with the equation of motion of matter on the brane, which takes its form from 4D GR. This is due to the fact that matter is trapped on the brane and cannot 'feel' the extra dimension.

Non-critical braneworlds

While RS2 agrees with Newton's law and other properties of 4D gravity, it contradicts a particularly crucial experimental observation. Various studies of supernovae suggest that the universe contains a small positive cosmological constant **[42; 43]**. RS2 has Minkowski on the brane with a vanishing cosmological constant. Let us focus on extending RS2 to allow for dS or AdS braneworlds.

Recall that we have demanded a compatibility with 4D Poincaré invariance, giving us Eqn.(3.46). We also saw that we could fine tune the brane tension σ against the bulk cosmological constant, Λ. We had the relations between them from Eqn.(3.39). This is referred to as the *criticality condition*, and as such, flat braneworlds satisfying it as referred to as critical. Now, let us generalise Eqn.(3.46) to allow for dS or AdS braneworlds

$$ds^2 = a^2(z)\big[(g_{\mu\nu} dx^\mu dx^\nu) + dz^2\big] \tag{3.48}$$

where $g_{\mu\nu}$ can be Minkowski, dS or AdS as per our choosing. The solutions to the bulk equations with the appropriate boundary conditions can be found in [44; 45; 46], although we shall aim to briefly go through the solutions here. Given, the ansatz in Eqn.(3.48), we need to solve the bulk equations with the cosmological constant $\Lambda = -6\kappa^2$, following which our solutions need to satisfy the boundary conditions imposed at the brane with positive brane tension σ at $z = 0$.

The bulk equations of motion are just given by EFE with the appropriate cosmological constant

$$R_{ab} - \tfrac{1}{2} R g_{ab} = -\Lambda_5 g_{ab} \tag{3.49}$$

Defining Λ_4 as the cosmological constant on the brane, gives us

$$\begin{aligned} \mu\nu: &\quad \frac{\Lambda_4}{a^2} - 3\Big(\frac{a'}{a}\Big)^2 - \frac{a''}{a} = -4\kappa^2 \\ zz: &\quad -4\frac{a''}{a} = -4\kappa^2 \end{aligned} \tag{3.50}$$

where the prime denotes a differentiation with respect to z. It is obvious to note that these equations will have three classes of solutions, depending on the signage of Λ_4.

$$a(z) = \begin{cases} \frac{1}{k}\sqrt{\frac{\Lambda_4}{3}} \sinh\left(\pm\kappa z + c\right) \text{ for } \Lambda_4 > 0 \\ \frac{1}{k}\sqrt{-\frac{\Lambda_4}{3}} \cosh\left(\pm\kappa z + c\right) \text{ for } \Lambda_4 < 0 \\ e^{\pm\kappa z + c} \text{ for } \Lambda_4 = 0 \end{cases} \tag{3.51}$$

(c is a constant of integration.)

The boundary conditions are given by the Israel junction conditions on the brane.

$$\hat{[}K_{ab}\hat{]} = -\tfrac{8\pi G_5}{3}\sigma q_{ab} \tag{3.52}$$

where q_{ab} is the induced metric on the brane. Given the specific form of the metric and the fact that we have a $\mathbb{Z}^2$ symmetry across the brane, we see

$$\frac{a'}{a}\Big|_{z=0^+} = -\frac{4\pi G_5}{3}\sigma = \tilde{\sigma} \tag{3.53}$$

Combining the above arguments with the fact that we have a positive brane tension

$$a(z) = \begin{cases} \frac{1}{\kappa}\sqrt{\frac{\Lambda_4}{3}}\sinh\left(-\kappa|z| + c\right) \text{ for } \Lambda_4 > 0 \\ \frac{1}{\kappa}\sqrt{-\frac{\Lambda_4}{3}}\cosh\left(\pm\kappa|z| + c\right) \text{ for } \Lambda_4 < 0 \\ e^{\pm\kappa|z|+c} \text{ for } \Lambda_4 = 0 \end{cases} \tag{3.54}$$

with the following conditions

$$\tilde{\sigma} = \begin{cases} \kappa\coth c > \kappa \text{ for } \Lambda_4 > 0 \\ \kappa\tanh c < \kappa \text{ for } \Lambda_4 < 0 \\ \kappa \text{ for } \Lambda_4 = 0 \end{cases} \tag{3.55}$$

We also have the freedom to set $a(0) = 1$, giving us

$$\begin{cases} \kappa = \sqrt{\frac{\Lambda_4}{3}}\sinh c \text{ for } \Lambda_4 > 0 \\ \kappa = \sqrt{-\frac{\Lambda_4}{3}}\cosh c \text{ for } \Lambda_4 < 0 \\ c = 0 \text{ for } \Lambda_4 = 0 \end{cases} \tag{3.56}$$

We can use Eqns.(3.54) and (3.55) to fix the cosmological constant to be

$$\Lambda_4 = 3\left(\tilde{\sigma}^2 - \kappa^2\right) \tag{3.57}$$

Summarising the above results,

$$\begin{cases} \text{Minkowski: } a(z) = e^{\pm\kappa|z|+c} \\ \\ \text{dS: } a(z) = \frac{1}{\kappa}\sqrt{\frac{\Lambda_4}{3}}\sinh\left(-\kappa|z|+c\right); \quad \kappa = \sqrt{\frac{\Lambda_4}{3}}\sinh c \\ \\ \text{AdS: } a(z) = \frac{1}{\kappa}\sqrt{-\frac{\Lambda_4}{3}}\cosh\left(-\kappa|z|+c\right); \quad \kappa = \sqrt{-\frac{\Lambda_4}{3}}\cosh c \end{cases} \tag{3.58}$$

When σ takes its critical value, we have $\tilde{\sigma} = \kappa$ and $\Lambda_4 \to 0$. AdS branes have the property where σ exceeds its critical value ($\tilde{\sigma} > \kappa$), therefore referred to as *subcritical* branes. dS on the other hand, the opposite is true, therefore it being referred to as a *supercritical* brane.

We have seen previously how gravity localised on braneworlds, dampening as they went further into the bulk as a result of the warp factor. Let us look at this behaviour for subcritical and supercritical branes. Consider in Figs.(3.4) and (3.5), the behaviour of the warp factor,

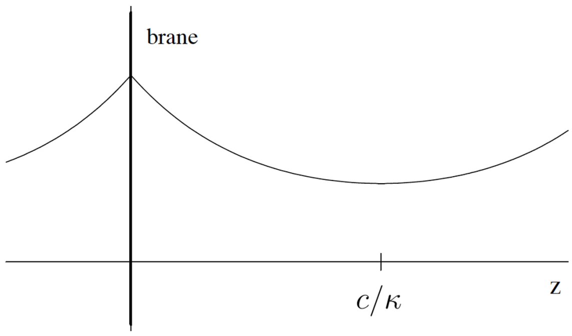

Figure 3.4: The behaviour of the warp factor for an AdS (subcritical) brane.

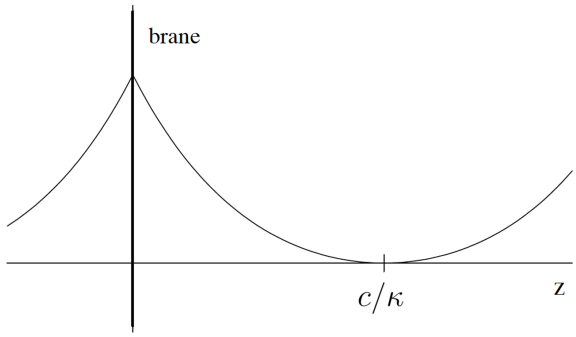

Figure 3.5: The behaviour of the warp factor for a dS (supercritical) brane.

We see a turnaround in the warp factor in both cases. Now, for the dS case, this corresponds to the dS horizon at the point where the warp factor vanishes altogether. It can be argued that dS branes are even more likely to exhibit 4D gravity than flat branes, due to a greater damping from the warp factor. (Refer to [47; 48; 49] for further details.) Unlike plain old RS2, we see a mass gap between the zero mode and the heavier modes in metric perturbations. Additionally, Newton's constant on the brane is proportional the brane tension σ, as opposed to κ.

The AdS situation is trickier. Near the brane, metric fluctuations act the same for flat, dS and AdS branes. The warp factor does not vanish at the turnaround point, rather it begins to grow past this point. Assuming that this point lies far away from the brane, we could believe that at low energies, we see gravity localisation. At finite temperatures, one could we even tuck this point behind a black hole horizon. [48] presents the case for localisation, despite the absence of a normalisable zero mode, for those interested.

Chapter 4

Gravity and Braneworlds

A basic requirement for any alternative theory of gravity, is its ability to reproduce Newton's law of gravitation in the appropriate limit. Braneworld gravity fails at the beginning of this endeavour, with 5D gravity producing a $1/r^3$ force law, as opposed to 4D familiar version. This is where the warp factor comes in to save the day, essentially 'squeezing' gravity around the brane so that gravity mostly spreads in the four directions parallel to the brane and Newton's law is recovered, despite the extra dimension.

Obviously, the empirical success of GR exceeds that of just reproducing Newton's laws in the weak field approximation for static sources and nonrelativistic masses. GR's big breakthrough came from predicting light bending around the Sun, confirmed by Eddington in 1919. GR has also had immense success in other experimental tests, such as perihelion precession of planetary orbits, radar time-delay experiments and so on. All these effects can be explained within the neat framework of linearised GR, where the assumption is that spacetime can be described as tiny perturbations about a background.

This chapter intends to focus on introducing the framework of linearised gravity but in braneworlds. We will see that GR results are recovered on the brane, with small corrections on account of the extra dimension. However, we also need to touch upon the problems that cannot be solved using the weak field limit, hence requiring non-perturbative gravity.

4.1 Non-perturbative gravity

We'll start here as this description is more straightforward and one that we have carried out in Section (2.3.1). The summary of the story is in the effective EFE on the brane, as seen in Eqns.(2.70 - 2.73). They are stated below for reference

$$^{(4)}G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + 8\pi G_4 \eta_{\mu\nu} + (8\pi G_5)^4 \pi_{\mu\nu} - \psi_{\mu\nu} \tag{4.1}$$

where,

$$\Lambda_4 = 4\pi G_5 \Big(\Lambda_5 + \tfrac{4\pi G_5 \sigma^2}{3} \Big) \tag{4.2}$$

$$G_4 = \frac{64\pi^2 G_5^2 \sigma}{48\pi} = \frac{4\pi\sigma G_5^2}{3} \tag{4.3}$$

$$\pi_{\mu\nu} = \frac{1}{4}\left(\frac{\eta\eta_{\mu\nu}}{3} - \frac{\eta^2 q_{\mu\nu}}{6} + \frac{\eta_{\alpha\beta}\eta^{\alpha\beta} q_{\mu\nu}}{2} - \eta_{\mu}{}^{\alpha}\eta_{\alpha\nu} \right) \tag{4.4}$$

One can develop a (1+3) - covariant analysis of Eqn.(2.70) using the viewpoint of a brane-bound observer **[50]**. Consider a general decomposition of $\psi_{\mu\nu}$ with respect to a 4-velocity field u^μ:

$$\psi_{\mu\nu} = -\Big[U\Big(u_\mu u_\nu + \tfrac{1}{3} h_{\mu\nu} \Big) + 2p_{(\mu} u_{\nu)} + P_{\mu\nu} \Big] \tag{4.5}$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projects orthogonally to u^μ. This allows for a consideration such that $\psi_{\mu\nu}$ is an effective 'Weyl fluid', with a non-local energy density U, momentum density p_μ and an anisotropic stress tensor $P_{\mu\nu}$. These are given by

$$\begin{aligned} U &= -\psi_{\mu\nu}u^\mu u^\nu \\ p_\mu &= h_\mu{}^\alpha \psi_{\alpha\beta}u^\beta \\ P_{\mu\nu} &= \left[\frac{1}{3}h_{\mu\nu}h^{\alpha\beta} - h_{(\mu}{}^\alpha h_{\nu)}{}^\beta\right] \end{aligned} \tag{4.6}$$

Note that $\psi_{\mu\nu}$ is in Planck units as it is derived from the bulk. To compare U, p_μ and $P_{\mu\nu}$ with the *physical* energy-momentum of 4D matter, we need to rescale by $1/8\pi G_4$. There exist evolution equations for U and p_μ, but not for $P_{\mu\nu}$ **[50]**. This is due to the fact that $P_{\mu\nu}$ incorporates gravitational modes of the 5D graviton, which cannot be predicted by braneworld observers. In special cases, such as the Friedmann-Lemaître-Robertson-Walker (FLRW) brane, $P_{\mu\nu}$ vanishes due to symmetry reasons, thus closing the EFE on the brane. In general, however, the system of equations will not be closed.

4.2 Linearised Gravity

We will follow the work in linearised braneworld gravity by Garriga and Tanaka **[51; 52]**. Let us quickly look at perturbation theory in GR before we begin **[10]**.

Perturbation theory in GR

In GR, classical perturbations involve metric perturbations

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \tag{4.7}$$

around a given background solution. There are 3 (useful) points to remember when doing this.

1. $h_{\mu\nu}$ is a perturbation and therefore needs to be *small*.

2. Gauge freedom: GR has a large gauge group. Physics is invariant under general coordinate transformations and $h_{\mu\nu}$ enjoys multiple gauge degrees of freedom.

3.We need the *Lichnerowicz operator*, the perturbation of the Ricci tensor.

$$\delta R_{\mu\nu} = -\tfrac{1}{2}\boldsymbol{\nabla}^2 h_{\mu\nu} - R_{\mu\nu\rho\sigma}h^{\rho\sigma} + R^{\rho}_{(\mu}h_{\nu)\rho} + \boldsymbol{\nabla}_{\mu(}\boldsymbol{\nabla}^{\rho}h_{\nu)\rho} = -\tfrac{1}{2}\Delta_L h_{\mu\nu} \tag{4.8}$$

Consider perturbations $h_{\mu\nu}$ about the RS background, modifying Eqn.(3.45)

$$ds^2 = \left(e^{-2|y|/l}\eta_{\mu\nu} + h_{\mu\nu}\right)dx^{\mu}dx^{\nu} + dy^2 \tag{4.9}$$

Note that we are employing hypersurface orthogonal coordinates here, $h_4 = 0$, reducing the 15 functions h_{ab} to 10 functions $h_{\mu\nu}$. These are not unique and we can still make coordinate transformations that shift the brane while preserving this. We can somewhat justify this by considering the following change of coordinates $x^{\mu} \to x^{\mu} + \eta^{\mu}(x^{\nu})$ to Eqn.(4.7). This leads to metric change of the form, $g_{\mu\nu} \to g_{\mu\nu} + \boldsymbol{\nabla}_{\mu}\eta_{\nu} + \boldsymbol{\nabla}_{\nu}\eta_{\mu}$, from which it is possible to show that we can choose the five functions η^{μ} so to make the coordinate choice.

Linearising the bulk EFE in Eqn.(3.49) in the perturbations $h_{\mu\nu}$

$$R_{44} + \tfrac{4}{l^2} = -\tfrac{1}{2}\partial_y\left(e^{2|y|/l}\partial_y h\right) = 0 \tag{4.10}$$

$$R_{\mu 4} = \tfrac{1}{2}\partial_y\left[e^{2|y|/l}\left(\partial_\nu h_{\mu}{}^{\nu} - \partial_\mu h\right)\right] = 0 \tag{4.11}$$

$$R_{\mu\nu} + \frac{4}{l^2} g_{\mu\nu} = \frac{1}{2} e^{2|y|/l} \Big(2\partial_\rho \partial_{(\nu} h_{\mu)}{}^{\rho} - \Box^{(4)} h_{\mu\nu} - \partial_\mu \partial_\nu h \Big) - \frac{1}{2} \partial_y^2 h_{\mu\nu} + \frac{2}{l^2} h_{\mu\nu} + \eta_{\mu\nu} \Big(\frac{h}{l^2} + \frac{\partial_y h}{2l} \Big) = 0 \tag{4.12}$$

where h is the determinant of $h_{\mu\nu}$ and $\Box^{(4)} \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the 4-dimensional Laplacian. Notice, Eqns.(4.10,4.11,4.12) are just simplifications of the *Lichnerowicz operator* applied to a perturbation, $\Delta_L h_{\mu\nu} = 0$.

Let us choose the transverse, traceless Randall Sundrum gauge (captures the 5 polarisations of the 5D graviton in 5 independent $h_{\mu\nu}$ components) **[9]** everywhere in the bulk, defined by

$$h_{a4} = 0\,;\ \ \partial_\nu h_\mu{}^{\nu} = h = 0 \tag{4.13}$$

This gauge trivially solves Eqns.(4.10) and (4.11). Eqn.(4.12) gets reduced to a decoupled field equation for perturbations

$$\Big(e^{2|y|/l}\, \Box^{(4)} + \partial_y^2 - \tfrac{4}{l^2} \Big) h_{\mu\nu} = 0 \tag{4.14}$$

The boundary conditions are given by Israel's junction conditions at the brane. However, in general, choosing this gauge in the bulk, matter in the brane causes it to bend, displacing it from $y = 0$. We can see this in Fig.(4.1).

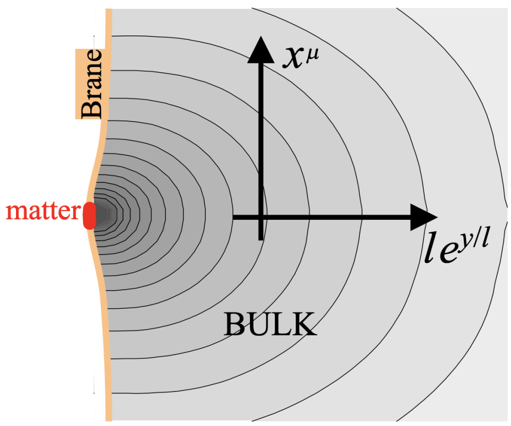

Figure 4.1: Gravitational field of a matter source in the RS gauge, showing brane bending. As seen in **[51]**.

We get around this problem by using the Gaussian normal gauge (coordinates $(\bar{x}^\mu, z)$, defined by $\bar{h}_{a4} = 0$ fixed at $z = 0$.), and then transforming between the two. Working on the positive side of the brane and assuming $\mathbb{Z}^2$ symmetry, this gives us

$$\Delta K_{\mu\nu} = -\tfrac{2}{l}\eta_{\mu\nu} + \partial_z \bar{h}_{\mu\nu} = -8\pi G_5 \Big(S_{\mu\nu} - \tfrac{1}{3} q^0_{\mu\nu} S \Big), \ (z = 0^+) \tag{4.15}$$

where $q^0_{\mu\nu} = q_{\mu\nu}(z=0) = (\eta_{\mu\nu} + \bar{h}_{\mu\nu}$ is the induced metric on the brane. $S_{\mu\nu}$ is the stress-energy tensor on the brane that can be split into brane tension and energy-momentum on the brane (like before in Eqn.(2.55):

$$S_{\mu\nu} = T_{\mu\nu} - \sigma q^0_{\mu\nu} \tag{4.16}$$

Plugging the above into Eqn.(4.15)

$$\begin{aligned} -\frac{2}{l}\eta_{\mu\nu} + \partial_z \bar{h}_{\mu\nu} &= -8\pi G_5 \Big((T_{\mu\nu} - \sigma q^0_{\mu\nu}) - \frac{1}{3} q^0_{\mu\nu} S \Big) \\ \Rightarrow \Big(\frac{2}{l} + \partial_z \Big) \bar{h}_{\mu\nu} &= -8\pi G_5 \Big(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu} T \Big), \ (z = 0^+) \end{aligned} \tag{4.17}$$

We now have to transform this condition back into the RS gauge, to ensure compatibility with Eqn.(4.14). Consider the following transformation:

$$z = y + \xi^4(x^a) \tag{4.18}$$

$$\bar{x}^\mu = x^\mu + \xi^\mu(x^a) \tag{4.19}$$

By imposing the requirement that the metric from Eqn.(4.9) takes its specific form in both gauges requires $\bar{h}_{44} = 0$ and $\bar{h}_{\mu 4} = 0$, giving us the following

$$\begin{aligned} \xi^4 &= \xi^4(x^\nu) \\ \xi^\mu &= -\frac{l}{2} e^{2|y|/l} \eta^{\mu\rho} \partial_\rho \xi^4((x^\nu) + A^\mu(x^\nu) \end{aligned} \tag{4.20}$$

where the functions $\xi^4(x^\nu)$ and $A^\mu(x^\nu)$ are independent of y. It now follows that, we can relate the two perturbations in the two coordinate systems as follows

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - l\partial_\mu \partial_\nu \xi^4 - \tfrac{2}{l} e^{-2|y|/l} \eta_{\mu\nu} \xi^4 + e^{-2|y|/l} \partial_{(\mu} \eta_{\nu)\delta} A^\delta \tag{4.21}$$

Inserting the above relation into Eqn.(4.17), the junction condition becomes

$$\left(\tfrac{2}{l} + \partial_y\right) h_{\mu\nu} = -\Sigma_{\mu\nu} \quad (y = 0^+) \tag{4.22}$$

where

$$\Sigma_{\mu\nu} = 8\pi G_5 \left(T_{\mu\nu} - \tfrac{1}{3}\eta_{\mu\nu} T\right) + 2\partial_\mu \partial_\nu \xi^4 \tag{4.23}$$

is the effective source term in the RS gauge including the effect of brane bending (the brane is displaced to $y = -\xi^4$). h being 0 in the RS gauge implies $\Sigma_\mu{}^\mu = 0$, so we can determine $\xi^4(x^\nu)$ from Eqn.(4.23),

$$\begin{aligned} 0 &= \Sigma_\mu{}^\mu = \Sigma_{\mu\nu}\eta^{\nu\mu} \\ &= 8\pi G_5 \left(T_{\mu\nu}\eta^{\nu\mu} - \frac{1}{3}\eta_{\mu\nu} T \eta^{\nu\mu}\right) + 2\partial_\mu \partial_\nu \xi^4 \eta^{\nu\mu} \\ &= 8\pi G_5 \left(T - \frac{4}{3} T\right) + 2\Box^{(4)} \xi^4 \\ \Rightarrow \Box^{(4)} \xi^4 &= \frac{4\pi G_5}{3} T \end{aligned} \tag{4.24}$$

which explicitly shows the the brane is bent by matter residing on it.

We now focus on combining Eqns.(4.14) and (4.22) into a single equation. We do this by including delta functions at the discontinuity, leading to equations of motion for perturbations in the RS gauge.

$$\left[e^{2|y|/l}\,\Box^{(4)} + \partial_y^2 - \tfrac{4}{l^2} + \tfrac{4}{l}\delta(y)\right]h_{\mu\nu} = -2\Sigma_{\mu\nu}\delta(y) \tag{4.25}$$

Eqn.(4.25) yields the vacuum solution for Eqn.(4.14) for $y \neq 0$, whereas integration across $y = 0$ gives us the junction condition Eqn.(4.22). Now for solving this equation, we define $G(x, y; x', y')$ to be the retarded 5D Green's function, which satisfies

$$\left[e^{2|y|/l}\,\Box^{(4)} + \partial_y^2 - \tfrac{4}{l^2} + \tfrac{4}{l}\delta(y)\right]G(x, y; x', y') = \delta^{(4)}(x - x')\delta^{(4)}(y - y') \tag{4.26}$$

with the solution to Eqn.(4.25) given by

$$\begin{aligned} h_{\mu\nu} &= -2\int d^4x'dy'G(x, y; x', y')\Sigma_{\mu\nu}(x')\delta(y') \\ &= -2\int d^4x'dy'G(x, y; x', 0)\Sigma_{\mu\nu}(x') \end{aligned} \tag{4.27}$$

To solve Eqn.(4.26) for the Green's function, we need a second boundary condition alongside the junction condition on the brane. For this, we take that perturbations remain bounded at the AdS horizon $y = \infty$. (Refer to **[53; 54; 55]** for more on this.) Now, we can construct a full Green's function using a complete set of eigenstates (Sturm-Liouville tools) **[9]**:

$$G(x, y; x', y') = \int \frac{d^4k}{(2\pi)^4} e^{ik_\mu(x^\mu - x'^\mu)} \left[\frac{e^{(|y|+|y'|)/l}l^{-1}}{\mathbf{k}^2 - (\omega + i\epsilon)^2} + \int_0^\infty dm \frac{u_m(y)u_m(y')}{m^2 + \mathbf{k}^2 - (\omega + i\epsilon)^2}\right] \tag{4.28}$$

where $k^\mu = (\omega, \mathbf{k})$ and,

$$u_m(y) = \frac{\sqrt{ml/2}\left[J_1(ml)Y_2(mle^{|y|/l}) - Y_1(ml)J_2(mle^{|y|/l})\right]}{\sqrt{J_1(ml)^2 + Y_1(ml)^2}} \tag{4.29}$$

J_n, Y_n are Bessel functions of order n. The first part of Eqn.(4.28) corresponds to a graviton zero mode and the second term is a continuum of KK modes.

Let us look at how we can use Sturm Liouville theory to obtain the above two relations. Let us begin by reintroducing a negative tension brane at $y = y_c$, such that it acts as a regulator and we can impose an additional boundary condition

$$(\partial_y + 2/l)|_{y=0^+}\ h_{\mu\nu} = 0 \tag{4.30}$$

This will place a new constraint on Green's function, so we modify Eqn.(4.26)

$$\left[e^{2|y|/l}\,\Box^{(4)} + \partial_y^2 - \tfrac{4}{l^2} + \tfrac{4}{l}\delta(y) - \tfrac{4}{l}\delta(y-y_c)\right]G(x,y;x',y') = \delta^{(4)}(x-x')\delta^{(4)}(y-y') \tag{4.31}$$

Now, let us take a Fourier transform on the above relation with respect to x^μ

$$\left[-e^{2|y|/l}k^2 + \partial_y^2 - \tfrac{4}{l^2} + \tfrac{4}{l}\delta(y) - \tfrac{4}{l}\delta(y-y_c)\right]\tilde{G}(k;y,y') = \delta^{(4)}(y-y') \tag{4.32}$$

where we define

$$\tilde{G}(k;y,y') = \int d^4x e^{-ik_\mu(x^\mu - x'^\mu)}G(x,y;x',y') \tag{4.33}$$

For $y \neq y'$, the Green's function will satisfy

$$(\partial_z^2 - \tfrac{4}{l^2})\tilde{G} = k^2 e^{2|y|/l}\tilde{G} \tag{4.34}$$

with the following boundary conditions

$$\begin{cases} (\partial_y + 2/l)\tilde{G} = 0 \quad \text{for} \quad y = 0^+ \\ (\partial_y + 2/l)\tilde{G} = 0 \quad \text{for} \quad y = y_c \end{cases} \tag{4.35}$$

We wish to find eigenstates $u_m(y)$ for our problem, with eigenvalues: $k^2 = -m^2$.

The zero mode eigenstate is

$$u_0(y) = N_0 e^{-2|y|/l} \tag{4.36}$$

where N_0 is a normalisation constant. To determine massive eigenstates, we will have to change variables to $z = mle^{|y|/l}$. This changes Eqn.(4.34) to Bessel's equation with n=2 **[56]**

$$\left[z^2\partial_z^2 + z\partial_z + (z^2 - 4)\right]\tilde{G} = 0 \tag{4.37}$$

with the following boundary conditions

$$\begin{cases} (z\partial_z + 2)\tilde{G} = 0 & \text{for} \;\; z = ml \\ (z\partial_z + 2)\tilde{G} = 0 & \text{for} \;\; z = z_c = mle^{y_c/l} \end{cases} \tag{4.38}$$

If we look in **[56]**, we see that Eqn.(4.37) has solutions $J_2(z)$ and $Y_2(z)$ which satisy the following recurrence relations

$$\begin{aligned} (z\partial_z + 2)J_2(z) &= zJ_1(z) \\ (z\partial_z + 2)Y_2(z) &= zY_1(z) \end{aligned} \tag{4.39}$$

which lets us deduce that the massive eigenstates are given by

$$u_m(y) = N_m\left[J_1(ml)Y_2(z) - Y_1(ml)J_2(y)\right] \tag{4.40}$$

where N_m is the normalisation constant in this case. An important point to note is that the boundary condition at $y = y_c$ is only satisfied for quantised values of m satisfying

$$\left[J_1(ml)Y_1(mle^{y_c/l}) - Y_1(ml)J_1(mle^{y_c/l})\right] = 0 \tag{4.41}$$

For large y, the asymptotic behaviour of Bessel's functions is given by

$$
\begin{aligned}
J_n(mle^{y/l}) &\sim \sqrt{\frac{2e^{-y/l}}{l\pi m}} \cos\left(mle^{-y/l} - \frac{n\pi}{2} - \frac{\pi}{4}\right) \\
Y_n(mle^{y/l}) &\sim \sqrt{\frac{2e^{-y/l}}{l\pi m}} \sin\left(mle^{-y/l} - \frac{n\pi}{2} - \frac{\pi}{4}\right)
\end{aligned}
\tag{4.42}
$$

As our regulator approaches infinity, Eqns.(4.41) and (4.42) imply that m is quantised in units of $\pi e^{-y_c/l}l$. The normalisation constants can be determined by

$$\int_{-y_c}^{y_c} dy\; e^{2|y|/l} u_m(y) u_n(y) = \delta_{mn} \tag{4.43}$$

The zero mode normalisation constant is simply

$$N_0^2 = \tfrac{1}{l}(1 - e^{-2y_c/l})^{-1} \tag{4.44}$$

The case for heavy modes is slightly more subtle. However, we note that for large y_c, the dominant contribution to the integral in Eqn.(4.43) lies in the region where $|y| = y_c$. Using the asymptotic behaviour relations

$$N_m^2 = \frac{\pi m}{2} e^{-y_c/l} \left[J_1(ml)^2 + Y_1(ml)^2\right]^{-1} + \mathcal{O}(e^{-2y_c/l} \tag{4.45}$$

The Fourier transformed Green's function now satisfies

$$\left(\partial_z^2 - \tfrac{4}{l^2} - k^2 e^{2|y|/l}\right)\tilde{G} = \delta(y - y') \tag{4.46}$$

This can be expressed in terms of a complete set of eigenstates

$$\tilde{G}(k; y, y') = -\frac{u_0(y)u_0(y')}{k^2} - \sum_m \frac{u_m(y)u_m(y')}{m^2 + k^2} \tag{4.47}$$

where to ensure $k^2 \neq -m^2$, we add a small imaginary part in the "time" direction(as in QFT), i.e. $k^\mu = (\omega + i\epsilon, \mathbf{k})$. Now, one removes the regulator by letting $y_c \to \infty$, wherein the quantisation in m vanishes, giving us a continuum limit

$$\sum_m \frac{u_m(y)u_m(y')}{m^2+k^2} \rightarrow \int_0^\infty dm \lim_{y_c\rightarrow\infty} \frac{1}{\pi e^{-y_c/l}/l}\left(\frac{u_m(y)u_m(y')}{m^2+k^2}\right) \tag{4.48}$$

Inverting the Fourier transform Eqn.(4.33) will give us the desired Eqns.(4.28) and (4.29).

For reasons of simplicity, we will look at the stationary case, for which the Green's function is

$$G(\mathbf{x},y;\mathbf{x}',y) = \int_{-\infty}^{\infty} dt' G(x,y;x',y') \tag{4.49}$$

The long distance behaviour of gravity is determined by the Green's function in the limit $|\mathbf{x}-\mathbf{x}'| \gg l$. The opposite limit simply yields $G(\mathbf{x},0;\mathbf{x}',0) \propto 1/r^2$, showing the fact that gravity becomes 5-dimensional at short distances. Considering both points on the brane and taking suitable expansions of the Bessel functions

$$G(\mathbf{x},0;\mathbf{x}',0) \approx -\tfrac{1}{4\pi l r}\left(1+\tfrac{l^2}{2r^2}+\ldots\right) \tag{4.50}$$

where we set $r = |\mathbf{x}-\mathbf{x}'|$. One can see that the zero mode gives the 4D1/r behaviour whereas the KK modes induce a subleading correction term proportional to $(l/r)^2$. For a source on the wall (y' = 0), the leading behaviour for large r and large y is given by **[51]**

$$G(\mathbf{x},0;\mathbf{x}',0) \approx -\frac{e^{-3|y|/l}}{8\pi l}\left[\frac{2e^{-2|y|/l}r^2+3l^2}{\left(e^{-2|y|/l}r^2+3l^2\right)^{3/2}}\right] \tag{4.51}$$

showing a steep perturbation decay away from the brane, as seen in Fig.(4.1).

Seeing as we are interested in the perturbation on the brane, it is convenient to switch back to GNC. Eqn.(4.21) becomes

$$h_{\mu\nu} = h_{\mu\nu}^{(m)} + h_{\mu\nu}^{(\xi)} + l\partial_\mu\partial_\nu\xi^4 + \tfrac{2}{l}e^{-2|y|/l}\eta_{\mu\nu}\xi^4 - e^{-2|y|/l}\partial_{(\mu}\eta_{\nu)\delta}A^\delta \tag{4.52}$$

where $h_{\mu\nu}$ has been split into matter and brane-bending parts, using Eqns.(4.23) and (4.27)

$$\begin{aligned} h^{(m)}_{\mu\nu} &= -16\pi G_5 \int d^4x' G(x,y;x',0)\Big(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\Big) \\ h^{(\xi)}_{\mu\nu} &= -4\int d^4x' G(x,y;x',0)\partial_\mu\partial_\nu\xi^4 \end{aligned} \tag{4.53}$$

If we choose $A^\mu(x^\nu)$ correctly, we can avail the remaining gauge freedom and simplify Eqn.(4.52) by setting all terms except the first and fourth term to zero. Finally, by settiing y = 0, we arrive at a rather naive looking formula for the perturbation on the brane:

$$\bar{h}_{\mu\nu} = -16\pi G_5 \int d^4x' G(x,0;x',0)\Big(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\Big) + \frac{2}{l}\eta_{\mu\nu}\xi^4 \tag{4.54}$$

where one can compute $G(x,0;x',0)$ from Eqn.(4.28) and ξ^4 can be determined using Eqn.(4.24).

4.2.1 The graviton propagator

It will soon become clear that Eqn.(4.54) reproduces the results of linearised GR for a stationary source on the brane, with minor corrections arising due to the KK modes. The KK modes in Eqn.(4.28) are only relevant at higher energies. So, at lower energies, the Green's function is dominated by the zero mode, and so we consider a zero mode truncation

$$G(x,0;x',0) = \frac{1}{l\Box^{(4)}} \tag{4.55}$$

where

$$\frac{1}{\Box^{(4)}} = -\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik_\mu(x^\mu - x'^\mu}}{\mathbf{k}^2 - (\omega + i\epsilon)^2} \tag{4.56}$$

is the massless scalar Green's function for 4D Minkowski space. The matter part of the brane perturbation is

$$h_{\mu\nu}^{(m)} = -\frac{16\pi G_5}{l}\int d^4x' \frac{1}{\Box^{(4)}}\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\right) \tag{4.57}$$

Note, however, this is *not* the propagator for a massless 4D graviton, should be a 1/2 instead of 1/3. This is a manifestation of the van Dam-Veltman-Zakharov discontinuity **[57; 58]**, which states that the mass approaches 0 limit of a graviton propagator does not coincide with the massless graviton propagator as the number of polarisation fields do not match. An extra 4D scalar field, a polarisation state which is contained in the 5D graviton propagato, persists in the massless limit. This could be problematic as such an incorrect tensor structure for the propagator results in a contradicting prediction of light bending as opposed to the prediction by GR **[57]**.

The good news is, we were careful as we included the effects of brane bending, descibed by the scalar field $\xi^4(x^\mu)$, given by the solution to Eqn.(4.24):

$$\xi^4 = \frac{4\pi G_5}{3}\int d^4x' \frac{1}{\Box^4}T \tag{4.58}$$

Plugging in Eqns.(4.57) and (4.58) into Eqn.(4.54), we find the full metric perturbation on the brane:

$$\bar{h}_{\mu\nu} = -\frac{16\pi G_5}{l}\int d^4x' \frac{1}{\Box^{(4)}}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right) \tag{4.59}$$

We see that the extra polarisation state has been compensated by the brane bending effect, giving us the correct tensorial structure for the 4D massless graviton propagator. Hence, linearised GR is recovered on the brane at lower energies.

4.2.2 Newtonian potentials on the brane

To understand the effects of KK modes, we now compute the perturbation for a point mass M at rest on the brane, with a stress-energy tensor

$$T_{\mu\nu} = M\delta^{(3)}(\mathbf{x})\delta^0_\mu\delta^0_\nu \tag{4.60}$$

Using Eqn.(4.24) to calculate the brane bending function

$$\Box^{(4)}\xi^4 = -\frac{4\pi G_5}{3}M\delta^{(3)}(\mathbf{x}) \tag{4.61}$$

For a time-independent solution, $\Box^{(4)}$ just reduces to the 3D Laplacian, giving us

$$\xi^4 = \frac{G_5 M}{3|\mathbf{x}|} \tag{4.62}$$

The matter part of the perturbation is computed using Eqn.(4.53) and Eqn.(4.50):

$$\begin{aligned} h^{(m)}_{\mu\nu} &= 16\pi G_5 \int d^3\mathbf{x}' \frac{1}{4\pi l|\mathbf{x}-\mathbf{x}'|}\left(1+\frac{l^2}{2|\mathbf{x}-\mathbf{x}'|^2}\right)\left(\delta^0_\mu\delta^0_\nu + \frac{1}{3}\eta_{\mu\nu}\right)M\delta^{(3)}(\mathbf{x}) \\ &= \frac{4G_5 M}{l|\mathbf{x}|}\left(1+\frac{l^2}{2r^2}\right)\left(\delta^0_\mu\delta^0_\nu + \frac{1}{3}\eta_{\mu\nu}\right) \end{aligned} \tag{4.63}$$

If we combined Eqns.(4.62) and (4.63) in Eqn.(4.54), we get a metric perturbation on the brane

$$\bar{h}_{\mu\nu} = \frac{2G_5 M}{l|\mathbf{x}|}\left[\left(1+\frac{l^2}{3r^2}\right)\eta_{\mu\nu} + \left(2+\frac{l^2}{r^2}\right)\delta^0_\mu\delta^0_\nu\right] \tag{4.64}$$

with components

$$\begin{cases} \bar{h}_{00} = \frac{2G_4 M}{|\mathbf{x}|}\left(1+\frac{2l^2}{3r^2}\right) \\ \bar{h}_{ij} = \frac{2G_4 M}{|\mathbf{x}|}\left(1+\frac{l^2}{3r^2}\right)\delta_{ij} \end{cases} \tag{4.65}$$

Hence, we recover the Schwarzschild metric to leading order, with minor corrections due to KK modes. The Newtonian potential is given by

$$\phi = \frac{1}{2}\bar{h}_{00} = \frac{G_4 M}{|\mathbf{x}|}\left(1 + \frac{2l^2}{3r^2}\right) \tag{4.66}$$

which does NOT contradict experimental tests of Newton's inverse square law, given $l 0.1 mm$ (the distance down to which Newton's law has been tested.) This also gives a lower bound for brane tension σ(1 TeV)^4.

Chapter 5

Braneworld Cosmology

We have looked at how the RS models provide a radical new way of thinking about our universe and the possible existence of extra dimensions. If this extra dimension is warped and AdS, then it can be infinitely large and still exhibit gravity localisation. We have also looked at how to generalise the RS2 model to include subcritical and supercritical braneworlds.

To better understand these ideas, we have to go a step further. The previous chapter dealt with weak gravity, but we have to also think about strong gravity, like cosmology and black holes. Previously, we looked at 5D bulks with a $\mathbb{Z}^2$ symmetry about a brane of codimension one. Now, we will aim to cover n-dimensional bulk spacetimes, occasionally relaxing $\mathbb{Z}^2$ symmetry. For work on generalising to branes of higher codimension, refer to **[59; 60; 61]** as this work will not touch upon this subject.

Previously, we also made an assumption that perturbations about the background spacetime were small: the energy-momentum tensor due to additional matter on the brane was much smaller than the assumed brane tension. Unfortunately, we cannot be so heavily reliant on perturbative physics, as it will not always give us the full story. There are two main approaches to physics on the brane by examining their cosmology: *brane based* approach and *bulk based* approach. We will see that these are equivalent, with each approach having its advantages depending on our setup.

5.1 Brane based Approach

This approach employs a non-perturbative approach, which we looked at in Section(2.3.1). For a cosmological brane, one has to ask whether there are surfaces of lower dimensionality which have the interpretation of an expanding universe.

5.1.1 A Friedmann-Lemaître-Robertson-Walker brane

Assuming the bulk spacetime has a negative cosmological constant with no additional fields,

$$\Lambda_n = -\tfrac{1}{2}(n-1)(n-2)k_n^2\ ; \quad T_{ab} \equiv 0 \tag{5.1}$$

where k_n is the inverse AdS length in n-dimensions. The cosmological constant on the brane is given as

$$\Lambda_{n-1} = \tfrac{1}{2}(n-2)(n-3)[\sigma^2 - k_n^2] \tag{5.2}$$

For a study of cosmology, it is important to examine the behaviour of an FLRW braneworld, which shows homogeneity and isotropy

$$ds_{n-1}^2 = h_{mn}dx^m dx^n = -dt^2 + \tilde{a}^2(t)d\mathbf{x}_k^2 \tag{5.3}$$

where $d\mathbf{x}_k^2$ is the metric on a (n-2)-dimensional Euclidean space X, of constant curvature $k = 0, \pm 1$

$$X = \begin{cases} S^{n-2} \text{ for } k = 1 \\ H^{n-2} \text{ for } k = -1 \\ \mathbb{R}^{n-2} \text{ for } k = 0 \end{cases} \tag{5.4}$$

$\tilde{a}(t)$ represents the scale factor of our braneworld. Let us assume that matter on the brane is modelled by a perfect fluid of pressure p and energy density ρ

$$T_{mn} = (\rho + p)u_m u_n + p h_{mn} \tag{5.5}$$

where u_m is the 4-velocity of the fluid.[1] The braneworld generalisation of the Friedmann equation that follows from Eqn.(2.70) is the found to be **[33; 34; 35]**

$$\begin{aligned} H^2 &= \phi - \frac{k}{\tilde{a}} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho + \left(\frac{4\pi G_n}{n-2}\right)^2 \rho^2 \\ \dot{H} &= \frac{k}{\tilde{a}} - \frac{8\pi G_{n-1}}{(n-3)}(\rho + p) - (n-2)\left(\frac{4\pi G_n}{n-2}\right)^2 \rho(\rho + p) \end{aligned} \tag{5.6}$$

where we define the Hubble parameter $H = \dot{\tilde{a}}/\tilde{a}$ (dot meaning differentiate with respect to t), $\phi = [\sigma^2 - k_n^2]$. We can see that these are not the standard FLRW equations as they contain quadratic terms in p and ρ. Braneworld cosmology is therefore different to standard cosmology. This unconventional behaviour was first noted in **[33]**. Notice, that for large values of the scale factor, we can recover standard cosmology, as we can ignore the non-linear density terms.

5.2 Bulk based Approach

We briefly saw the limitations of the brane based approach, choosing to impose a $\mathbb{Z}^2$ symmetry across the brane, ignoring the possibility of non-zero Weyl terms. This was due to the fact that we were working on a static brane in a dynamic bulk. The bulk based approach is the opposite - a dynamic brane in a static bulk. This allows us to include non-$\mathbb{Z}^2$ symmetric branes and non-zero Weyl terms. The drawback is that we will only be covering FLRW branes and cannot exploit the generalisation provided in Eqn.(2.70).

[1] We avoid difficulties with ψ_{mn} by setting it to 0, which corresponds to a *pure* AdS space in the bulk.

5.2.1 Generalised Birkhoff's Theorem

The thought of a static bulk does lead one to think - Birkhoff's theorem **[62; 63]**. To bridge the gap between the two approaches to braneworld cosmology, we will prove attempt to prove a generalised version of this theorem, first done in 5D **[64]**.

The most general metric compatible with this symmetry is **[65]**

$$ds^2 = A^{\frac{2}{n-2}} dx_k^2 + e^{2\chi} A^{-\frac{n-3}{n-2}} (-dt^2 + dz^2) \tag{5.7}$$

where A and χ are functions of t and z that are yet to be determined. Here, we are using the fact that the rest of the metric is 2D and therefore conformally flat. Without a loss of generality, it is safe to say that the brane is located at z = 0.

We also assume that the bulk contains no additional matter. Recall the bulk Einstein equations, ${}^{(n)}G_{ab} = (n)G_{ab} - \frac{1}{2}\,{}^{(n)}R g_{ab} = -\Lambda_n g_{ab} + 8\pi G_n T_{ab}$, into which we insert our metric ansatz. We arrive at the following equations,

$$\begin{aligned}
&\partial_{tt}^2 A - \partial_{zz}^2 A = \left[2\Lambda_n A^{\frac{1}{n-2}} - (n-2)(n-3)kA^{-\frac{1}{n-2}}\right] e^{2\chi} \\
&\partial_{tt}^2 \chi - \partial_{zz}^2 \chi = \left[\frac{\Lambda_n}{n-2} A^{-\frac{n-3}{n-2}} + \frac{n-3}{2} kA^{-\frac{n-1}{n-2}}\right] e^{2\chi} \\
&\partial_{tt}^2 A + \partial_{zz}^2 A = 2\big(\partial_z \chi \partial_z A + \partial_t \chi \partial_t A\big) \\
&\frac{\partial^2 A}{\partial t \partial z} = \partial_z \chi \partial_t A + \partial_t \chi \partial_z A
\end{aligned} \tag{5.8}$$

Now is a good time to switch to a more convenient set of coordinates: lightcone coordinates,

$$u = \tfrac{t-z}{2} \; ; \; v = \tfrac{t+z}{2} \tag{5.9}$$

leading to

$$\begin{aligned}
\frac{\partial^2 A}{\partial u \partial v} &= \left[2\Lambda_n A^{\frac{1}{n-2}} - (n-2)(n-3)kA^{-\frac{1}{n-2}}\right]e^{2\chi} \\
\frac{\partial^2 \chi}{\partial u \partial v} &= \left[\frac{\Lambda_n}{n-2}A^{-\frac{n-3}{n-2}} + \frac{n-3}{2}kA^{-\frac{n-1}{n-2}}\right]e^{2\chi} \\
2\partial_u\chi\partial_u A &= \partial_u\Big((\partial_u A)\big[\ln(\partial_u A)\big]\Big) \\
2\partial_v\chi\partial_v A &= \partial_v\Big((\partial_v A)\big[\ln(\partial_v A)\big]\Big)
\end{aligned} \tag{5.10}$$

The latter two equations in Eqn.(5.10) seem innocuous and we can easily integrate them:

$$\begin{aligned}
&\text{Case 1 : A is a constant} \\
&\text{Case 2} : A = A(u)\ ; \quad e^{2\chi} = \frac{\mathrm{d}A(u)}{\mathrm{d}u}\frac{\mathrm{d}V(v)}{\mathrm{d}v} \\
&\text{Case 3} : A = A(v)\ ; \quad e^{2\chi} = \frac{\mathrm{d}A(v)}{\mathrm{d}v}\frac{\mathrm{d}U(u)}{\mathrm{d}u} \\
&\text{Case 4} : A = A(u,v)\ ; \quad e^{2\chi} = \partial_u A\frac{\mathrm{d}V(v)}{\mathrm{d}v} = \partial_v A\frac{\mathrm{d}U(u)}{\mathrm{d}u}
\end{aligned} \tag{5.11}$$

where U(u) and V(v) are arbitrary non-zero functions of u and v respectively. We are not interested in Cases 1-3 (imnply $\Lambda_n = k = 0$). We focus on Case 4, for which we can see

$$A = A\big(U(u) + V(v)\big)\ ;\ e^{2\chi} = A'U'V' \tag{5.12}$$

Plugging into the first equation in Eqn.(5.10), gives us an ODE

$$\begin{aligned}
&A'' - \left[2\Lambda_n A^{\frac{1}{n-2}} - (n-2)(n-3)kA^{-\frac{1}{n-2}}\right]A' = 0 \\
\Rightarrow\ &A' - 2\Big(\frac{n-2}{n-1}\Big)\Lambda_n A^{\frac{n-1}{n-2}} + (n-2)^2 kA^{\frac{n-3}{n-2}} = (n-2)^2 c
\end{aligned} \tag{5.13}$$

Notice, the second equation in Eqn.(5.10) just gives the derivative of the ODE, and is satisfied automatically. Let us now impose jump conditions on the brane. To do this, we again assume homogeneity and isotropy

$$\begin{aligned} S_{mn} &= -\sigma h_{mn} + T_{mn} \\ T_{mn} &= (\rho + p)u_m u_n + p h_{mn} \end{aligned} \tag{5.14}$$

When we have a $\mathbb{Z}^2$ symmetry across the brane at z = 0, Israel's equations (2.63) give

$$4\pi G_n(\sigma + \rho) = -e^{-\chi} A^{-\frac{1}{2}(\frac{n-1}{n-2})} \partial_z A = \frac{1}{2} e^{-\chi} A^{-\frac{1}{2}(\frac{n-1}{n-2})} [U' - V'] A' \tag{5.15}$$

$$\begin{aligned} & 4\pi G_n \left[\frac{n-3}{n-2}(\sigma + \rho) - \sigma + \rho \right] = -\partial_z \left[e^{-\chi} A^{\frac{1}{2}(\frac{n-3}{n-2})} \right] \\ & = \frac{1}{4} e^{-\chi} A^{\frac{1}{2}(\frac{n-3}{n-2})} \left[(V' - U') \left(\frac{A''}{A} - \left(\frac{n-3}{n-2} \right) \frac{A'}{A} \right) + \frac{V''}{V} - \frac{U''}{U} \right] \end{aligned} \tag{5.16}$$

We can use Eqn.(5.13) to eliminate A' and A''. Let us make the following coordinate transformation, which will leave the boundary conditions at the brane unchanged.

$$u \to f(u) \qquad v \to f(v) \tag{5.17}$$

This symmetry is related to the t-z plane conformal symmetry. To get rid of this unphysical gauge freedom, we simply choose f = V, therefore setting V = v. Now, onto the issue of the U(u) physical degree of freedom. Setting

$$\tilde{a} = A^{\frac{1}{n-2}} \; ; \quad T = (n-2)(v - U) \tag{5.18}$$

we see that the bulk metric can be written as

$$ds_n^2 = -g(\tilde{a})dT^2 + \tfrac{d\tilde{a}^2}{f(\tilde{a})} + \tilde{a}^2 dx_k^2 \tag{5.19}$$

where

$$f(\tilde{a}) = -\frac{\tilde{a}'}{n-2} = -\frac{A' A^{-\frac{n-3}{n-2}}}{(n-2)^2} \tag{5.20}$$

From Eqn.(5.13)

$$f(\tilde{a}) = -\frac{2\Lambda_n}{(n-1)(n-2)}\tilde{a}^2 + k - \frac{c}{\tilde{a}^{n-3}} \tag{5.21}$$

For c¿0, the general metric in Eqn.(5.19) takes a form of the Schwarzschild black hole in flat, dS or AdS space, depending on the value of Λ_n. Given that our starting point was that our braneworld had a spatial geometry with constant curvature, we indeed have 'proved' a general version of Birkhoff's theorem. We assumed that the bulk physics was described by pure Einstein gravity with a cosmological constant, but proofs have been carried out for more complicated setups, such as Einstein-Maxwell **[66]** and Gauss-Bonnet **[67]**. We have shown that one can express the bulk geometry in a static form given by Eqn.(5.19), although we can longer say we have a static brane at z = 0. It is now a dynamic brane with a complicated trajectory. Ida **[68]** was the first to study braneworld cosmology in this perspective, but moving branes in a static AdS bulk were considered earlier by Klaus **[69]**.

5.2.2 A dynamic brane in the bulk

Now that we have bridged the gap from the brane based approach to braneworld cosmology, let us now further generalise Ida's bulk based approach. We will come to see that we allow ourselves far more flexibility regarding the structure of the bulk space-time, by transferring dynamics of our system from the bulk to the brane. We no longer assume $\mathbb{Z}^2$ symmetry and also potentially consider the changing of the cosmological constant on either side.

Let us begin by taking the general static solution Eqn.(5.19) to the EFE with cosmological constant Λ_n. To construct the solution, we treat the brane as the boundary of the bulk.

$$X^a = (\mathbf{x}^\mu, t(\tau), a(\tau)) \tag{5.22}$$

Now, we patch this bulk spacetime (labelled with '-') with another appropriate bulk (labelled with '+') with the same boundary valye $a(\tau)$. Here, we set τ to correspond to the proper time with respect to an observer comoving with the brane, which imposes the condition

$$-h^{\pm}\dot{t}_{\pm}^2 + \frac{\dot{Z}^2}{h^{\pm}} = -1 \tag{5.23}$$

so, whichever side of the brane you are on, the induced metric takes the FLRW form

$$ds_{n-1}^2 = h_{ab}dx^a dx^b = -d\tau^2 + a^2(\tau)d\mathbf{x}_k^2 \tag{5.24}$$

where $a(\tau)$ is taken to be the scale factor. The bulk metric is continuous across the brane and note that t can be discontinuous at the brane, because neither g_{ab} and h_{ab} explicitly depend on it.

In order to produce the type of brane we need, the bulk spacetimes need to be patched such that Israel's equations are satisfied. When defining the extrinsic curvature, we need some information about the outward normal.

$$n_a^{\pm} = \epsilon_{\pm}(\mathbf{0}, -\dot{a(\tau)}, \dot{t}_{\pm}(\tau)) \tag{5.25}$$

We define the extrinsic curvature $K_{ab} = h_a^c h_b^d \boldsymbol{\nabla}_{(c} n_{d)}$. We first find that

$$K_{\mu\nu} = \boldsymbol{\nabla}_{(\mu} n_{\nu)} = -\Gamma^a_{\mu\nu} n_a = \frac{\epsilon h \dot{t}}{a(\tau)} h_{\mu\nu} \tag{5.26}$$

The components of $\partial/\partial\tau$ are given by

$$\tau^a = (\mathbf{0}, \dot{a(\tau)}, \dot{t}_{\pm}(\tau)) \tag{5.27}$$

, which is normal to n^a. The last non-zero component of extrsinc curvature is

$$\begin{aligned} K_{\tau\tau} &= -\tau^a n_b \boldsymbol{\nabla}_a \tau^b = -n_c(\dot{\tau}^c + \Gamma^c_{ab}\tau^a\tau^b) \\ &= \epsilon\dot{a}\Big[\ddot{t} + \frac{h'}{h}\dot{t}\dot{a}\Big] - \epsilon\dot{t}\Big[\ddot{a} + (\frac{h'}{2})h\dot{t}^2 - (\frac{h'}{2})\frac{\dot{a}^2}{h}\Big] \\ &= \frac{\ddot{a} + \frac{1}{2}h'}{\epsilon h\dot{t}} \end{aligned} \tag{5.28}$$

Using the above relations, the Israel equations now yield

$$\begin{aligned} \frac{1}{2a}\hat{\{}\epsilon h\dot{t}\hat{\}} &= \frac{4\pi G_n}{n-2}(\sigma + \rho) \\ \frac{1}{2}\hat{\Big\{}\frac{\ddot{a} + \frac{1}{2}h'}{\epsilon h\dot{t}}\hat{\Big\}} &= \frac{4\pi G_n}{n-2}\big[\sigma - (n-3)\rho - (n-2)p\big] \end{aligned} \tag{5.29}$$

We refer to the Gauss-Codazzi equations, with the understanding that it is valid on both sides of the brane, and $G^{\pm}_{ab} = -\Lambda^{\pm}_n g^{\pm}_{ab}$. Now, taking the difference between the '+' and '-' equations,

$$-\Delta\Lambda_n = \tfrac{1}{2}\Big(\hat{\{}K\hat{\}}\Delta K - \hat{\{}K_{ab}\hat{\}}\Delta K^{ab}\Big) \tag{5.30}$$

Plugging in the values for extrinsic curvature, Eqn.(5.28) into the above relation

$$-\Delta\Lambda_n = 4\pi G_n(n-2)(\sigma - \rho)\frac{\Delta[\epsilon h\dot{t}]}{a} + 4\pi G_n(\sigma + \rho)\left[\frac{\ddot{a}+\frac{1}{2}h'}{\epsilon h\dot{t}}\right] \tag{5.31}$$

Finally, after some tedious algebra (which is omitted here), simplifying Eqns.(5.29) and (**??**), we arrive at these expressions for the scale factor.

$$\dot{a}^2 = -\tfrac{1}{2}\hat{\{}h\hat{\}} + \left[\tfrac{4\pi G_n}{n-2}(\sigma+\rho)a\right]^2 + \left[\tfrac{(n-2)\Delta h}{16\pi G_n}(\sigma+\rho)a\right]^2 \tag{5.32}$$

$$\begin{aligned}\ddot{a} = &-\frac{1}{4}\hat{\{}h'\hat{\}} - \Big(\frac{4\pi G_n}{n-2}\Big)^2(\sigma+\rho)\big[-\sigma+(n-3)\rho+(n-2)p\big]a \\ &+\left(\frac{(n-2)\Delta h}{16\pi G_n(\sigma+\rho)a}\right)^2\left[\frac{-\sigma+(n-3)\rho+(n-2)p}{(\sigma+\rho)a}\right] \\ &+\left(\frac{(n-2)}{16\pi G_n(\sigma+\rho)a}\right)^2\Delta h\Delta h'\end{aligned} \tag{5.33}$$

Note, for the above two equations to be consistent, we require the conservation of energy to hold on the brane:

$$\dot{\rho} = -(n-2)\frac{\dot{a}}{a}(\rho+p) \tag{5.34}$$

Here, we have seen the bulk based approach in action. We found the cosmological evolution equations Eqn.(5.32) and (5.33) for the brane without assuming the $\mathbb{Z}^2$ symmetry. This is particularly useful when considering braneworlds with differing cosmological constants on either side of the brane. Also, by considering general values of h, we allow the bulk Weyl tensor on either side to be zero.

Chapter 6

Braneworld Black Holes

The linearised gravity result for an isolated mass and the brane cosmology metric suggest a somewhat deeper importance to braneworlds and black holes **[10]**. The Newtonian potential corrections coincide exactly with the 1-loop corrections to the graviton progagator **[19]**. The cosmological dark radiation term in the brane Friedmann equation corresponds to the energy density of a conformal field theory at the Hawking temperature of a black hole **[16]**. These clues, and analogies with lower dimensional branes, have led to Emparan, Fabbri and Kaloper's black hole *holographic conjecture* **[70]**, which states that if we can find a classical solution to the RS model, then the braneworld can be interpreted as a 4D quantum corrected spacetime. If we considered black holes, then we end up with a quantum corrected black hole. It is crucial in the braneworld context to find black hole solutions, both to shed further light on braneworld gravity, and also, observations of black holes might provide an exciting means of testing possible extra dimensions.

Ideally, it would be very useful if we had a full 5D solution describing a black hole localised on the brane. A slicing of a 5D accelerating black hole metric, known as the *C-metric* **[71]** would give a solution. This is simply due to the Poincaré coordinate system used to chart the brane, is from the bulk perspective, an accelerating patch covering a part of AdS space (think Rindler coordinates in Minkowski spacetime). Therefore, a black hole must also be accelerating in order to 'keep up' with the brane and remain localised on it.

This chapter will begin by discussing the general system of equations for this problem, followed by an analysis of Chamblin, Hawking and Reall's paper on braneworld black holes, it being the first attempt to find black holes on a RS brane. We will also look into some stability analysis of this solution. Following which, several different solutions, making various assumptions are also presented, and finally a brief look into the approximate methods to solving braneworld black holes, using the brane and bulk approaches.

6.1 Basic Equations

The general static, spherically symmetric metric on the brane can be written as

$$ds^2 = -A^2(r)dt^2 + B^2(r)dr^2 + C^2(r)d\Omega_{II}^2 \tag{6.1}$$

where $d\Omega_{II}^2 = d\theta^2 + sin^2\theta d\phi^2$ is the metric of the unit 2-sphere. As we saw previously, we cannot naively connect spherical symmetry with staticity and apply Birkhoff's theorem. **Note**, there appears to be no consensus on whether gravitational collapse on a brane results in a brane results in the formation of a black hole that is time-dependent or static **[**72; 73; 74; 75; 76**]**. By making the assumption that the metric is static, we adopt a viewpoint that there exists a 5D solution analogous to the 4D C-metric, possessing a timelike Killing vector, implying it can be sliced in a manner as to create a static 4D braneworld black hole.

The metric in (6.1) has a feature that we may enjoy, which is that we can choose our radial coordinate r, quite arbitrarily. Say we wanted to choose the *area gauge*, where $C = r$ as the area surrounding the black hole behaves in the usual way: $\mathcal{A}(r) = 4\pi r^2$. This is a subtle argument which will not be dealt with here but essentially one should not make the restrictive choice of $C = r$ as it runs into complications with energy conditions and turning points in the area function.

The vacuum brane field equations from Eqn.(2.70), with $\Lambda_4 = 0$ are

$$G_{\mu\nu} = -\psi_{\mu\nu} \tag{6.2}$$

where $\psi_{\mu}{}^{\mu} = \nabla_{\mu}\psi^{\mu\nu} = \psi_{[\mu\nu]} = 0$. While $\psi_{\mu\nu}$ is unknown from the brane point of view, it can generally be decomposed with respect to a 4-velocity as in Eqn.(4.5). Static symmetry implies $p_{\mu} = 0$ and $P_{\mu\nu} = P(r_{\mu}r_{\nu} - \frac{1}{3}h_{\mu\nu}$ where r^{μ} is a radial vector, further reducing to

$$-\psi_{\mu\nu} = U(r)\Big(u_{\mu}u_{\nu} + \tfrac{1}{3}h_{\mu\nu}\Big) + P(r)\Big(r_{\mu}r_{\nu} - \tfrac{1}{3}h_{\mu\nu}\Big) \tag{6.3}$$

In an inertial frame $u^{\mu} = \frac{1}{A}(1,0,0,0)$, then Eqn.(6.2) simplifies to:

$$\begin{aligned}
G^t_t &= -\frac{1}{C^2} + \frac{1}{B^2}\left[2\frac{C''}{C} - 2\frac{B'C'}{BC} + \frac{C'^2}{C^2}\right] = -U \\
G^t_t &= -\frac{1}{C^2} + \frac{1}{B^2}\left[2\frac{A'C'}{AC} + \frac{C'^2}{C^2}\right] = \frac{1}{3}(U+2P) \\
G^\theta_\theta = G^\phi_\phi &= \frac{1}{B^2}\left[\frac{C''}{C} + \frac{A''}{A} + \frac{A'C'}{AC} - \frac{A'B'}{AB} - \frac{B'C'}{BC}\right] = \frac{1}{3}(U-P)
\end{aligned} \tag{6.4}$$

where $A' = \frac{\partial A}{\partial r}$. A useful alternative in this situation is the conservation of Weyl 'angular momentum':

$$(U+2P)' + 2\tfrac{A'}{A}(2U+P) + 6P\tfrac{C'}{C} = 0 \tag{6.5}$$

The above system of equations contains 3 independent equations with 4 unknowns. This implies that we have to make an assumption about $\psi_{\mu\nu}$ or $g_{\mu\nu}$ to close the system and obtain a black hole solution.

6.2 CHR Black Hole

Chamblin, Hawking and Reall (CHR) **[41]** were the first to attempt to find a black hole on a RS brane, by replacing the Minkowski metric in Eqn.(3.45) with the Schwarzschild metric. Note, we can replace $\eta_{\mu\nu}$ with *any* Ricci-flat metric. (Refer to **[77]** for further details on Ricci flat branes). The CHR metric looks like:

$$ds^2 = a^2(z)\left[\left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2 d\Omega_{II}^2\right] - dz^2 \qquad (6.6)$$

This is the only known exact solution that looks like a black hole from the brane point of view. However, it does not correspond to our expectations of brane black holes. If matter is confined to a brane, one typically expects any gravitational effects to be localised near the brane. Intuitively, even for a collapsed star, we expect that while the horizon might extend into the bulk, it should ideally be localised near the brane the singularity should 'not' extend out into the bulk. This solution has $\psi_{\mu\nu} = 0$, so the brane geometry receives no corrections from bulk gravitational effects. Each 4D slice of $y =$ constant, has the induced geometry of the Schwarzschild metric and we see a line singularity at $r = 0$, extending all along z. The CHR black string extends all the way out to the AdS horizon, and here the black hole horizon becomes singular.

Consider the 5D Riemann tensor

$$R_{abcd}R^{abcd} = \tfrac{40}{l^2} + \tfrac{48M^2}{r^6}e^{4|z|/l} \qquad (6.7)$$

which also shows the unsatisfactory behaviour that as $y \to \infty$, the curvature diverges! There is however, another problem with the CHR black string - it suffers from a classical instability **[78]**.

6.2.1 Gregory-Laflamme Instability

Black string instabilities were first discovered in vaccuum, [79; 80] for the KK black string:

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega_{II}^2 - dz^2 \tag{6.8}$$

This scenario has a cylindrical event horizon with an entropy $4\pi G_N M^2$. Assuming a KK compactification scale of L_{KK},a 5D black hole with the same mass as the black string has an entropy $8\sqrt{2\pi L_{KK} G_N} M^{3/2}/3\sqrt{3}$. Therefore, for small enough masses (relative to the compactification scale), a black string has lower entropy than a 5D black hole, implying that the string should be perturbatively or nonperturbatively unstable.

The existence of the instability is confirmed via the Lichnerowicz equation

$$\nabla^2 h^{ab} + 2R_{cd}{}^{ab} h^{cd} = 0 \tag{6.9}$$

To show an unstable solution, we can find any instability and a simple s-mode has the form [10]

$$h^{ab} = e^{i\mu z} e^{\Omega t} \begin{bmatrix} H^{tt}(h) & h(r) & 0 & 0 & 0 \\ h & H^{rr}(h) & 0 & 0 & 0 \\ 0 & 0 & K(h) & 0 & 0 \\ 0 & 0 & 0 & K/\sin\theta^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.10}$$

This is a physical mode, since any gauge degree of freedom will have to be 4D, thus satisfying a massless 4D Lichnerowicz equation, whereas this mode satisfies a massive 4D Lichnerowicz equation. This causes the horizon to ripple as seen in Fig.(6.1)

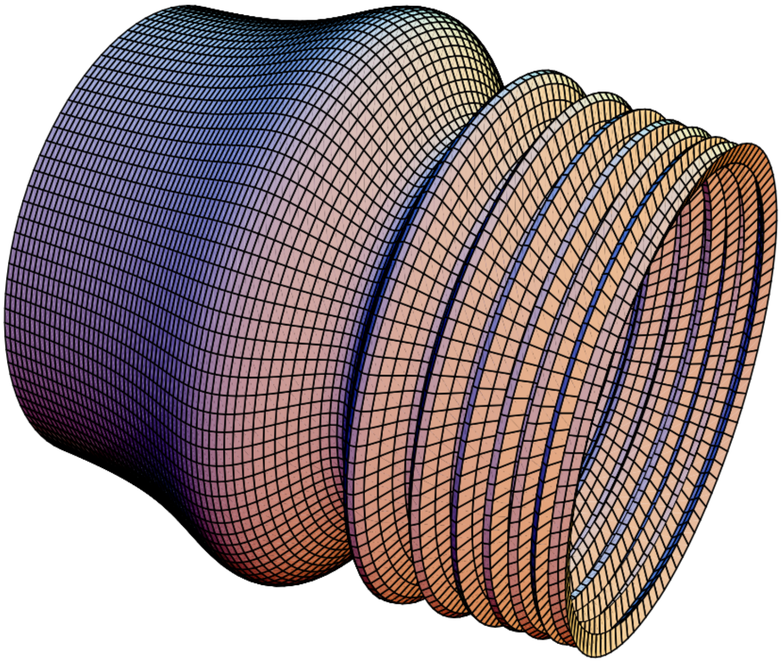

Figure 6.1: The event horizon of a perturbed AdS black string. As seen in **[78]**.

For the CHR black string, the presence of the negative bulk cosmological constant could change some features of this analysis, but ultimately, the crucial feature of the black string instability is that is a purely 4D massive Transverse Trace Free mode a.k.a. the RS gauge is satisfied. Working out the perturbations for the CHR black string background:

$$\left(\left(\boldsymbol{\nabla}^{(4)}\right)^2 h_{\mu\nu} + 2R^{(4)}_{\mu\nu\lambda\rho}h^{\lambda\rho}\right) - \left(e^{-4|z|/l}\left(e^{2|z|/l}h_{\mu\nu}\right)'\right)' = 0 \tag{6.11}$$

This simply means that we can take the standard KK instability and substitute the appropriate massive z-independent eigenfunction: $h_{\mu\nu}(z) = \chi_{\mu\nu}u_m(z)$, such that $\chi_{\mu\nu}$ satisfies

$$\left(\Delta_L^{(4)} + m^2\right)\chi_{\mu\nu} = 0 \tag{6.12}$$

where $\Delta_L^{(4)}$ is the 4D Schwarzschild-Lichnerowicz operator. Basically, we have the same sort of 4D instability, but a different z-dependence appropriate to the RS background. This instability causes ripples with ever-increasing frequency towards the AdS the horizon, like in the black string case. We see this below in Fig.(6.2).

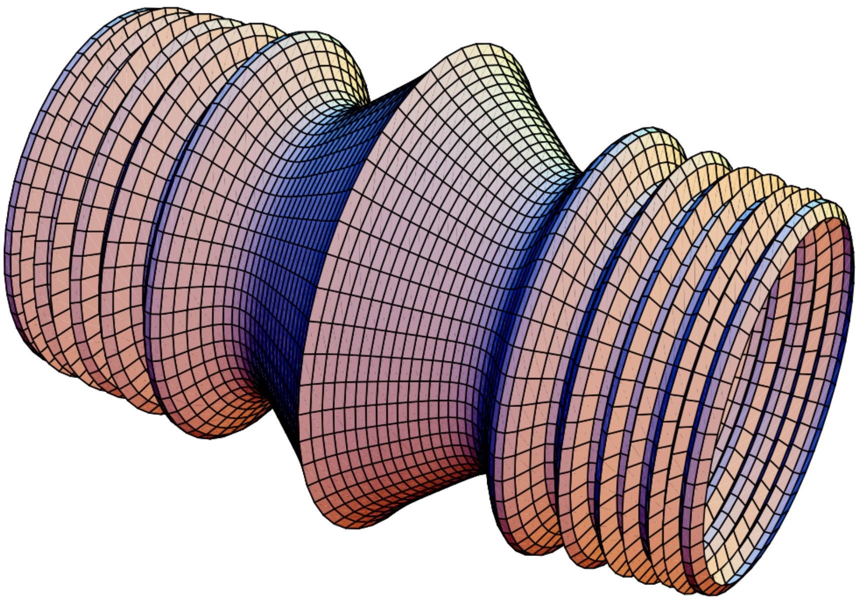

Figure 6.2: The event horizon of a perturbed RS black string. The brane is at the central cusp. As seen in **[78]**.

We could introduce a second brane to cut off the black string instability and mitigate the issue of curvature divergence. It has been shown that the back-reaction of bulk gravity waves onto the brane results in observational signatures that are distinctive from standard GR **[81]**.

6.3 Solutions: Standard Examples

6.3.1 The linearised weak field metric

The linearised weak field metric for a static source on the brane was derived in Section(4.2). In the area gauge, the metric is as follows:

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{4Ml^2}{3r^3}\right)dt^2 + \left(1 - \frac{2M}{r} - \frac{2Ml^2}{r^3}\right)^{-1} dr^2 + r^2 d\Omega_{II}^2 \quad (6.13)$$

Due to the trace free property of $\psi_{\mu\nu}$, we require a valid solution to have a vanishing Ricci scalar. The Ricci scalar for the above metric is

$$R = 2l^2 M^2 \frac{\left[-42l^2r^3 - 9r^5 + 4M\left(8l^4 + 18l^2r^2 + 3r^4\right)\right]}{r^5\left(4l^2M + 6Mr^2 - 3r^3\right)^2} \quad (6.14)$$

Since Eqn.(6.13) is derived in the linearised approximation (neglects terms of order M^2 or higher), we expect $R = 0$ upto $\mathcal{O}(M)$. However, Eqn.(6.14) shows us that R does not vanish beyond linear order, implying that the metric in (6.13) can only be a solution to the field equations in the linearised far-field limit. It is not a valid solution for the entire horizon exterior of a braneworld black hole. The solution is obtained by solving the 5D linearised equations, assuming only the bulk is asymptotically AdS and the perturbations are bounded at the AdS horizon. Therefore, one would expect a metric for a static, spherically symmetric black hole on the brane should be expected to have this form.

The Weyl tensor components for this metric are:

$$\begin{aligned} \psi_t{}^t &= -\frac{4Ml^2}{r^5} \\ \psi_r{}^r &= -\frac{2Ml^2}{r^5} \\ \psi_\phi{}^\phi &= \psi_\theta{}^\theta = \frac{3Ml^2}{r^5} \end{aligned} \quad (6.15)$$

or, in terms of U and P in Eqn.(6.3)

$$U = -\tfrac{4}{5}P = -\tfrac{4Ml^2}{r^5} \tag{6.16}$$

Here, terms are calculated to linear order in M, consistent with the linearised approximation used in deriving Eqn.(6.13). We note that non-zero $\psi_{\mu\nu}$ is responsible for the Schwarzschild metric corrections in (6.13). Therefore, a correct black holes solution should also have non-zero $\psi_{\mu\nu}$, if it is to agree with (6.13) in the weak field limit.

6.3.2 The tidal Reissner-Nordström black hole

The Weyl term $\psi_{\mu\nu}$ being antisymmetric and trace-free, has the same algebraic symmetries as the stress-energy tensor of GR, allowing us to make a formal correspondence $-\psi_{\mu\nu} \leftrightarrow T^{em}_{\mu\nu}$. Therefore, Einstein-Maxwell solutions in GR actually result in vacuum braneworld solutions, which Dadhich *et al.* **[82]** used to write the tidal Reissner-Nordström (tidal RN) solution:

$$ds^2 = -\Big(1 - \frac{2M}{r} + \frac{Q}{r^2}\Big)dt^2 + \Big(1 - \frac{2M}{r} + \frac{Q}{r^2}\Big)^{-1}dr^2 + r^2 d\Omega^2_{II} \tag{6.17}$$

This metric also solves the general solution to the brane field equations using an asumption that $A(r) = 1/B(r)$.

The solution Eqn.(6.17) has the form of the Reissner-Nordström solution of GR, but we do not have an electric field on the brane. Q is a *tidal* charge parameter arising from the gravitational field of the bulk. Unlike GR, Q can be both positive and negative.
For the positive case, the tidal RN metric has qualitatively the same properties as RN geometry: two horizons, both of which lie inside the Schwarzschild horizon, and the

singularity at $r = 0$ is timelike. In the case of $Q < 0$, we have one horizon lying outside the Schwarzschild horizon and you have a spacelike singularity at the centre.

The negative Q is the more natural case as it makes a positive contribution to the gravitational potential, thereby strengthening the gravitational field. This is expected as the tidal charge arises from the source mass M on the brane **[82]**.

The Weyl tensor for this solution is

$$\psi_t{}^t = \psi_r{}^r = \psi_\phi{}^\phi = \psi_\theta{}^\theta = \frac{Q}{r^4} \tag{6.18}$$

or in terms of U and P in Eqn.(6.3)

$$U = -\tfrac{1}{2}P = \tfrac{Q}{r^4} \tag{6.19}$$

The tidal-RN metric does not satisfy the weak limit Eqn.(6.13), therefore cannot describe the entire spacetime around a braneworld black hole. Let us, however,briefly look into this solutions when considering *small* black holes on the brane. For a black hole that has a horizon size much smaller than the AdS length scale, the AdS curvature has very little effect on the geometry. This leaves behind a spacetime with an induced metric of 5D Schwarzschild-Tangherlini metric **[83]**:

$$ds^2 = -\left(1 - \tfrac{r_h^2}{r^2}\right)dt^2 + \left(1 - \tfrac{r_h^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_{II}^2 \tag{6.20}$$

Therefore, the tidal-RN metric displays the correct 5D behaviour of gravity at short distances, where the Q/r^2 term dominates and the metric approximates the 5D induced Schwarzschild-Tangherlini metric. If one looks carefully, the 5D Schwarzschild-Tangherlini metric is a special case of the tidal-RN metric with $M = 0$ and $Q = -r_h^2$. Therefore, the tidal-RN metric could be a good approximation for the strong field regime for small black holes.

6.3.3 Black Strings

Another common example, but this was covered in Section(6.2) when discussing the CHR Black Hole.

6.3.4 Solutions assuming a metric

The above approaches to finding various braneworld black hole solutions were all guided by the form of the Schwarzschild metric in standard GR. So, the obvious next attempt would be to find braneworld black hole solutions to fix $A(r)$ or $B(r)$ to take a Schwarzschild form and then attempt to solve the field equations for the other. This leads to solutions of the form **[84]**:

$$ds^2 = -\left[\left(1+\epsilon\right)\sqrt{1-\tfrac{2M}{r}}-\epsilon\right]^2 dt^2 + \left(1-\tfrac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_{II}^2 \tag{6.21}$$

$$ds^2 = -\left(1-\tfrac{2M}{r}\right)dt^2 + \frac{\left(1-\frac{3M}{2r}\right)}{\left((1-\frac{2M}{r}\right)\left((1-\frac{r_0}{r}\right)} dr^2 + r^2 d\Omega_{II}^2 \tag{6.22}$$

Refer to **[84]** for a detailed discussion on the properties of these equations. It is important to note that, they do not satisfy the far-field limit, hence cannot be considered satisfactory black hole solutions.

Chapter 7

Conclusion

We have been on a long and meandering journey through braneworlds, and can't but wonder whether these things exist in Nature or not. It would be nice to answer whether we live on a brane or not? The paper by A.Lukas *et al.* **[12]** is a very interesting and seminal paper on our universe as a domain wall and how it fits into brane models. It is unlikely that the RS models can accurately describe the structure of our universe, they are incredible 'toy models' to gain further insights. RS models do not account for supersymmetry, which although yet to be discovered, is commonly thought to exist. Furthermore, M theory posits additional dimensions which we have to accept if we choose to go ahead with M theory. Despite all this, the RS models while being simple, have provided new avenues for research into areas like alternatives to compactification, braneworld holography and so on.

Chapter 1, we set the stage for extra dimensions and models like the Randall-Sundrum model by taking a trip down memory lane, seeing how the study of extra dimensions has evolved. **Chapter 2** dives into some mathematical tools needed to understand and set up braneworlds, ideas such as Israel's junction conditions and the Gauss-Codazzi equations, which we derived. We finish with a detailed derivation of the effective EFE on the brane, by using the Gauss-Codazzi equations. This did give us a Weyl term which is non-local, causing the equations to not be closed. **Chapter 3** introduced the

Randall-Sundrum models and investigated their behaviour, including subcritical and supercritical branes.

Chapter 4 is where we dived deeper into braneworld gravity, presenting a detailed setup of the framework of linearised GR. We also showed how we have covered the non-perturbative approach to braneworld gravity and how that is used to solve various problems. **Chapter 5**, we discussed cosmology on the brane. The most intriguing feature was the quadratic energy-momentum terms that appeared in the FLRW equations. We can usually neglect the effect these terms play at low density, but they become important if the universe was very small at some time. It is also interesting to mention that certain braneworld cosmology setups do not possess a Big Bang singularity. Finally, **Chapter 6** was where we discussed some static, spherically symmetric braneworld Black Hole solutions solutions. Ideally, a full 5D solution describing a black hole localised on the brane would be desirable. Such a solution would be given by a suitable slicing of the 5D C-metric, however this has not yet been found and we are unsure of its very existence. We also looked into CHR black holes and the associated instability scenarios.

The story with braneworlds goes on and on. Braneworld holography is an area that can warrant its own dissertation; rather for that matter any of the topics I touched upon. There is numerous work being done on holography, black holes, cosmology, localisation techniques, the list goes on. In contemporary physics, the most common link made with extra dimensions are theories of quantum gravity and unification, especially string theory. Braneworlds can and will provide useful insights to these more fundamental theories, and open up an exciting prospect of testing these ideas via astrophysical means. The existence of extra dimensions is surely a tantalising possibility, and if ever confirmed, would surely be one of the greatest triumps in physics.

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