Imperial College London

Master's Thesis

Determining the Dimension of Causal Sets in Distinguishing Spacetimes

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Abstract

The Hawking-King-McCarthy-Malament (HKMM) theorem states that the causal structure of a distinguishing spacetime alone is enough to determine the geometry of the spacetime, up to conformal isometries. This motivates the treatment of the causal structure as a more fundamental object than spacetime itself. I investigate Causal Set Theory which treats spacetime on the Planck scale as a set of points endowed with a causal relation between them. I show that for simple strongly causal spacetimes one can deduce the dimension of the continuum spacetime from a causal set using the Myrheim-Meyer dimension estimator. I then explore how removing line segments from the spacetime affects the Myrheim-Meyer dimension and propose a method which counters some of these effects. I investigate one particular spacetime, which I call Spacetime Z, which is distinguishing but not strongly causal. I confirm that a causal set which has been sprinkled into Spacetime Z contains all of the information required to deduce the dimension of Spacetime Z.

Contents

Acknowledgements

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And, of course, the companionship of my friends and fellow course mates throughout the last year has been both comforting and reassuring. Thank you all!

Chapter 1

Introduction

1.1 Continuum spacetime

General Relativity tells us that spacetime is a continuous Lorentzian manifold with signature $(-, +, +, ..., +)$. In the continuum, manifolds have inherent properties which are well defined, for example the dimension and the metric of the manifold. The difference between Lorentzian and Riemannian manifolds is fundamental in understanding the causal structure of spacetime. Distances in Riemannian geometries are guaranteed to be positive definite and so cannot admit the correct causal structure. However, distances in Lorentzian manifolds can be positive, negative or null. This allows the causal structure to be characterised by the light cones, which are null geodesics from events in the spacetime [30]. An event A can only affect the set of events which lie in the future light cone of A. Similarly, A can only be effected by events which lie in the past light cone of A. All event outside of A's light cones can have no effect on A and A cannot affect those events. In this way the causal structure determines which events can physically influence other events [12].

1.2 Discrete spacetime

The singularities that General Relativity permits provides evidence that spacetime is not continuous but is instead discrete on some small scale [3]. Theories which consider spacetime as a discrete object have been considered by both physicists and philosophers alike [26, 22]. The well defined quantities that continuum manifolds have, such as dimension, are not well defined for a discrete spacetime. Therefore, an important validation for a discrete theory of spacetime is that it must somehow recover these continuous quantities. In the particular instance of dimension, Riemann suggested that the dimension of discrete space can be deduced by 'counting' [2, 3]. One technique proposed by Hausdorff defines dimension in terms of the *Hausdorff p-measure* [7, 15].

Definition 1.2.1 *The Hausdorff p-measure of a metric space X is defined with a countable collection of balls $B_i \in X$ each with diameter,* diam*, less than a small parameter ϵ such that the set of balls B_i covers X. The p-measure is then*

$$m_p(X) = \sup_{\epsilon > 0} \left[\inf \sum_{i=1}^{\infty} [\mathrm{diam}(B_i)]^p \right] . \tag{1.1}$$

Definition 1.2.2 *The Hausdorff dimension* HD*(X) of a metric space X is defined in terms of the Hausdorff p-measure $m_p(X)$ (definition 1.2.1) as*

$$\mathrm{HD}(X) = \sup \{p; m_p(X) = \infty\} = \inf \{p; m_p(X) = 0\} . \tag{1.2}$$

Riemann's counting idea is shown in the (positive definite) sum over the subsets in the metric space X. Since the sum is positive definite, this technique (and similar techniques) can only recover the dimension of Riemannian manifolds and cannot be used on the Lorentzian structure of General Relativity. The causal behaviour appears to play a vital role in Lorentzian geometry so is it possible that causality could be a basis for recovering continuous Lorentzian manifolds? In chapter 3 I demonstrate that counting the number of causal relations in discrete subsets is able to deduce the dimension of a Lorentzian manifold using the Myrheim-Meyer dimension estimator [19, 16].

1.3 Causal Sets

Causal Set Theory (CST) is a discrete theory of spacetime that replaces a continuous Lorentzian manifold with a finite set of points endowed with a "relation" between the points [5]. The discreteness scale for causal sets is on orders of the Planck length $\ell_p = \sqrt{G\hbar c^{-3}}$ [4, 28]. Causal Set Theory treats this discrete set, along with the order relations, as the fundamental make up of spacetime. The causal set then approximates a continuum manifold in the limit of a large number of points and on scales much greater than ℓ_p.

Definition 1.3.1 *A causal set, or causet, C is a finite set of N points with an order relation, denoted $\prec$, with the following properties [3]:*

1) if $x \prec y$ and $y \prec z$ then $x \prec z$ $\forall x, y, z \in C$ (transitivity);

2) $x \nprec x$ $\forall x \in C$ (acyclicity);

3) $\{z | x \prec z \prec y\}$ is finite for every pair of elements $x, y \in C$

Acyclicity is not required generally and breaking this condition gives rise to logical paradoxes where one's future can influence one's past [28]. In this thesis I have assumed that we live in a universe where such paradoxes do not exist, hence the requirement for acyclicity. Definition 1.3.1 item 3) is the statement that the number of elements in a subset of C is finite even in the 'large N limit'. Readers with a background in graph theory might be familiar with the concepts above and, in the vernacular of graph theory, a causal set is a *transitive directed acyclic graph.* A Directed Acyclic Graph (DAG) can be defined using only item 2) and item 3) but the additional transitivity condition makes this a transitive DAG.

1.3.1 The Hawking-King-McCarthy-Malament theorem

A strong motivation for studying Causal Set Theory comes from a theorem which was originally presented by Hawking, King and McCarthy [8]. The original HKM theory concerns two strongly causal spacetimes (M_1, g_1) and (M_2, g_2) that have dimension $d > 2$. It states that if there exists a homeomorphism $f : M_1 \to M_2$ between the spacetimes which is also a *causal isomorphism*, that is f and f^{-1} preserve causal order, then M_1 and M_2 are conformally isometric [20]. This theorem states that knowing just the causal structure of a spacetime can determine it's differential and topological structure up to a conformal factor [29]. An important development of this theorem was then made by Malament which relaxes the requirement for the two spacetimes to be strongly causal to being *distinguishing* [14]. The combined HKMM theorem is essential for deducing the topological information of spacetime from a causal set.

1.3.2 Discrete-continuum correspondence

For a causal set to be physically relevant it must approximate the Lorentzian spacetime required by General Relativity when the causal set becomes very dense. This relation can be encapsulated in the *discrete-continuum correspondence* which requires the definition of *Planck scale faithful embedding* as follows [3, 6]:

Definition 1.3.2 *A Planck scale faithful embedding from a causal set C to a continuous manifold M is an injective map $\phi : C \rightarrow M$ satisfying the properties:*

i) (Planck-scale uniform): The number of causal set elements embedded in any sufficiently large, physically nice region of M is approximately equal to the spacetime volume of the region in fundamental volume units.

ii) (Order-preserving): The elements $x, y \in C$ are ordered, $x \preceq y$, if and only if the embedding, ϕ, preserves the ordering; $\phi(x) \preceq \phi(y)$.

iii) The length scale on which the continuous geometry (M, g) fluctuates is much larger than the Planck length ℓ_p.

'Physically nice region' in this context means there are no Planck scale features and the region in question is approximately flat [6]. The requirement in definition 1.3.2 item i) explicitly uses Riemann's idea of counting to determine volume. Item ii) is a very important requirement that will be used later in chapter 5. It claims that the causal order of the embedded causal set must be the same as the order of the causal set itself. The discrete-continuum correspondence then goes as follows [6, 10]:

Lemma 1.3.1 *A causal set $(C, \prec)$ recovers the spacetime of General Relativity (M, g) if there exists a Planck scale faithful embedding.*

The discrete-continuum correspondence is fundamental for determining whether a causal set can recover the continuum spacetime.

1.4 My contribution

1.4.1 Deducing dimension

I show in chapter 3 that a causal set alone can deduce the dimension of the background manifold using a technique called the Myrheim-Meyer dimension estimator [19, 16]. I explain how to determine the causal order of causets in strongly causal Minkowski backgrounds and demonstrate that the Myrheim-Meyer dimension converges to the dimension of these spacetimes using two different interpretations. Similar work has been conducted in [16, 23, 1] although I also illustrate a different method for calculating the Myrheim-Meyer dimension. I detail the way that I have calculated the Myrheim-Meyer dimension which is useful for studying rectangular Minkowski spacetime and show where errors appear in my approach. Analysis of the apparent dimensional reduction or augmentation for small causets is conducted in [1, 17] however this thesis will not focus on these small causets. Instead I focus on the statistical properties of the causets, determining how well the dimension estimates represent the global spacetime. This is an important validation for Causal Set Theory as the HKMM theorem states that the topological and differentiable information of a spacetime can be recovered from it's causal structure alone.

1.4.2 Featureful spacetimes

In chapter 4 I discuss certain features that a spacetime can have and how the features effect the ability to determine it's causal structure. I show that using straight lines between pairs of points is not sufficient to induce the complete causal order of a causal set which has been sprinkled into one particular featureful spacetime. I illustrate the detrimental effects that the incomplete order has on the global properties of the causet. I then discuss an alternative method for determining the causal structure and argue that this method is more complete for causets sprinkled into featureful spacetimes.

1.4.3 Spacetime Z

Chapter 5 then combines the results of chapters 3 and 4 to investigate causets which have been sprinkled into a particular spacetime called Spacetime Z. This is a spacetime which is *distinguishing* but not strongly causal [9, 20] and is therefore useful for evaluating Malament's extension to the HKM theorem for causal sets which have been sprinkled into Spacetime Z. I repeat similar statistical analysis that is done in chapter 4 to show how frequently elements of the causet are sampled over and use this to show the Myrheim-Meyer dimension of Spacetime Z.

1.4.4 Causal set simulations

I have also developed a comprehensive Python library to enable me to run substantial causal set simulations efficiently. A significant amount of work went into designing useful, reusable and well tested modules for others to use and contribute to. The code for the project can be found here: `https://gitlab.com/awr.trumpet/causal_sets` and I encourage anyone that wants to develop my work to contribute.

Chapter 2

Causal definitions

This chapter contains the relevant definitions required to talk about the causal structure of a spacetime. These definitions can be found in numerous texts and in this chapter I frequently refer to texts [9, 12, 21] although alternative definitions can be found in [8]. I initially reiterate the definitions in the context of the continuum spacetime of General Relativity. Section 2.4 then makes the necessary adjustments and additions to the continuum definitions required for Causal Set Theory.

2.1 Causal relations in the continuum

In the continuum we assume that spacetime is a C^∞ manifold $\mathcal{M}$ with a global, Lorentzian, non-degenerate tensor field g of type $(0,2)$ with signature $(-,+,+,...,+)$ [21].

Definition 2.1.1 *A tangent vector X to any point $x \in \mathcal{M}$ is said to be:*

1) timelike, if $g_{\mu\nu}X^\mu X^\nu < 0$,

2) null, if $g_{\mu\nu}X^\mu X^\nu = 0$,

3) spacelike, if $g_{\mu\nu}X^\mu X^\nu > 0$.

To navigate the geometry the following definition is required

Definition 2.1.2 *A curve $\gamma : \Re \to \mathcal{M}$ is a smooth map which maps an open interval of the real numbers into the manifold.*

This definition requires a curve to have non-zero extent since it must map open intervals. Whilst this is a rather formal definition, the usual notion of a curve being a (not necessarily linear) line, parameterised by some real parameter in the manifold, applies.

Definition 2.1.3 *A curve, γ, is called timelike (spacelike, null) if the tangent vector at each point on the curve is everywhere timelike (spacelike, null) [9].*

Mathematically, a curve can follow any path in the manifold, it needn't remain timelike, spacelike or null. However, the class of curves which *do* remain timelike, spacelike or null are useful in physics, hence the more restrictive definition of 2.1.3. Throughout this thesis I make references to the following definition

Definition 2.1.4 *A causal curve is a curve that is everywhere non-spacelike.*

Definition 2.1.5 *A spacetime $(\mathcal{M}, g_{\mu\nu})$ is time orientable if for each point $p \in \mathcal{M}$ one can choose a consistent component of each timelike vector at p [21].*

An equivalent way to form this definition is that the same half of the light cone for each point must represent the future and the other half of the light cone the past [30]. Definition 2.1.3 together with definition 2.1.5 create the class of *future and past directed timelike curves* which represent the possible paths through the spacetime taken by massive objects.

2.1.1 Endpoints and extensibility

Taking our definition of curve from 2.1.2 then we can define endpoints on this curve:

Definition 2.1.6 *A point $p \in \mathcal{M}$ is a future (past) endpoint of a future (past) directed causal curve if $\exists t_0$ such that for every neighbourhood N of $p, \gamma(t) \in N \forall t > t_0$ $(t < t_0)$ [30].*

Physically this means, for future endpoints, that for all times t greater than some value t_0 the future directed causal curve at the value of t lies in a neighbourhood of the future endpoint p. For an infinitely big neighbourhood of p this is trivial since the causal curve will always be in this infinitely big neighbourhood but will not necessarily terminate. However, the key part to this definition is that p is only an endpoint if this condition holds for *every* neighbourhood of p. This includes an infinitely small neighbourhood around p and means that the causal curve must "terminate" arbitrarily close to p. An analogous interpretation can be given for past endpoints.

2.1.2 Chronology

The following three definitions of Chronology, Horismos and Causality have similar distinctions between past and future. For the sake of brevity I explicitly state the definitions with regards to the future and make the appropriate changes in the subsequent parentheses for the past. First I will begin with the definition of chronology.

Definition 2.1.7 *The chronological future (past) of a point $p \in \mathcal{M}$, denoted by $I^+(p)$ ($I^-(p)$) is the set of points that can be reached from p by a future (past) directed time-like curve starting at p.*

This definition involves purely time-like curves and the chronological future is always an open set since it's compliment is closed [9]. It is common to also see the notation $p \ll q$ which is equivalent to saying 'p is in the chronological past of q' or $p \in I^-(q)$ [12].

2.1.3 Horismos

Definition 2.1.8 *The horismos future (past) of a point $p \in \mathcal{M}$ denoted by $E^+(p)$ ($E^-(p)$) is the set of points that can be reached from p on null geodesics from p.*

In this way the term horismos is rather apt since it is the *horizon* of points that can be reached by null geodesics from an initial point. $E^{\pm}(p)$ forms the boundary of $I^{\pm}(p)$ which I write as $E^{\pm}(p) = \mathrm{Bnd}\left[I^{\pm}(p)\right]$ using the $\pm$ as an abuse of notation to denote either future or past [9]. Again it is common to see the notation $p \to q$ which is equivalent to $p \in E^-(q)$.

2.1.4 Causality

The horismos and chronological futures (pasts) of a point are subsets of the causal future (past) of a point such that it can be thought of as the union of the two.

Definition 2.1.9 *The causal future (past) of point $p \in \mathcal{M}$, denoted $J^+(p)$ ($J^-(p)$) is the set of points that can be reached from p by a non-spacelike curve.*

Two points $p, q \in \mathcal{M}$ are *causally* related to each other if p is in either the causal future or past of q. $J^{\pm}$ can be defined in terms of $I^{\pm}$ and $E^{\pm}$ by

$$J^{\pm}(p) = I^{\pm}(p) \cup E^{\pm}(p) \tag{2.1}$$

which suggests the causal structure is made up of the chronological and horismos relations [9]. In texts it is common to see $p \prec q$ to represent $p \in J^-(q)$ but I will deliberately reserve this notation purely for the ordering relation between elements in a causal set and will always refer to continuum causal relations using the set $J^{\pm}$.

Another useful definition which uses the chronological relations is that of and Alexandrov interval [6, 18]:

Definition 2.1.10 *An Alexandrov interval $A[p,q]$ between two points $p, q \in \mathcal{M}$ where $p \in I^-(q)$ is a "causal diamond" consisting of $I^+(p) \cap I^-(q)$.*

Since $I^{\pm}$ is an open set this definition of the Alexandrov interval is also open and just contains the interior of the causal diamond. The Alexandrov interval can be considered to be a spacetime in it's own right in isolation from the spacetime from which is was created.

2.2 Hierarchy of causal spacetimes

When defining causal conditions on spacetimes there is a hierarchy of conditions with which a causal space can obey [18]. These imply increasingly restrictive conditions on the space-time in relation to it's causal structure. Here I will be focusing on *causal*, *past* ***or*** *future distinguishing*, *past* ***and*** *future distinguishing* and *strongly causal* spacetimes although there are other causal conditions one can impose.

2.2.1 Causal spacetime

The weakest condition discussed in this thesis is that of causality.

Definition 2.2.1 *A causal spacetime is a spacetime in which there are no closed causal curves [9].*

For causality to hold it means that for any point p in the spacetime, a causal curve cannot move away from p and re-intercept p again. The existence of such a curve would mean that one could perform some action at p and it's effect could change the initial action.

2.2.2 Past or future distinguishing spacetime

I now describe the condition for a spacetime to be individually future or past distinguishing. As before, the parentheses show the changes required to separate future and past respectively.

Definition 2.2.2 *A spacetime is future (past) distinguishing at $p \in \mathcal{M}$ if every neighbourhood N of p contains a (sub)neighbourhood U of p which no future (past) directed causal curve starting from p intersects U more than once [12].*

Physically this condition enforces that, on a path through p, once you pierce the boundary of the (sub)neighbourhood, U, you cannot re-pierce the boundary of U. This must hold for every neighbourhood of p and if this condition holds for all $p \in \mathcal{M}$ then the spacetime itself is said to be future or past distinguishing. An equivalent statement given in [9] is that if $I^+(q) = I^+(p)$ ($I^-(q) = I^-(p)$) then $q = p$ which can be interpreted as the chronological future (past) of each point in the spacetime must be distinct. Being future or past distinguishing is a stronger condition than being causal because it states that there is a neighbourhood of each point that causal curves cannot re-enter.

2.2.3 Past and future distinguishing spacetime

Definition 2.2.3 *A past and future distinguishing spacetime at $p \in \mathcal{M}$ is a spacetime that satisfies both past and future distinguishing conditions at p simultaneously.*

Such a spacetime is commonly referred to simply as 'distinguishing'. Figure 2.1 shows a spacetime which is only future distinguishing but not past distinguishing and so is therefore not generally distinguishing.

2.2.4 Strongly causal spacetime

Definition 2.2.4 *A strongly causal spacetime at $p \in \mathcal{M}$ is one such that every neighbourhood N of p contains a sub-neighbourhood U of p which no causal curve intersects more than once.*

The reason that being strongly causal is more restrictive than distinguishing is that in a strongly causal spacetime the causal curve in question merely starts in the sub-neighbourhood of p and need not pass through p itself.

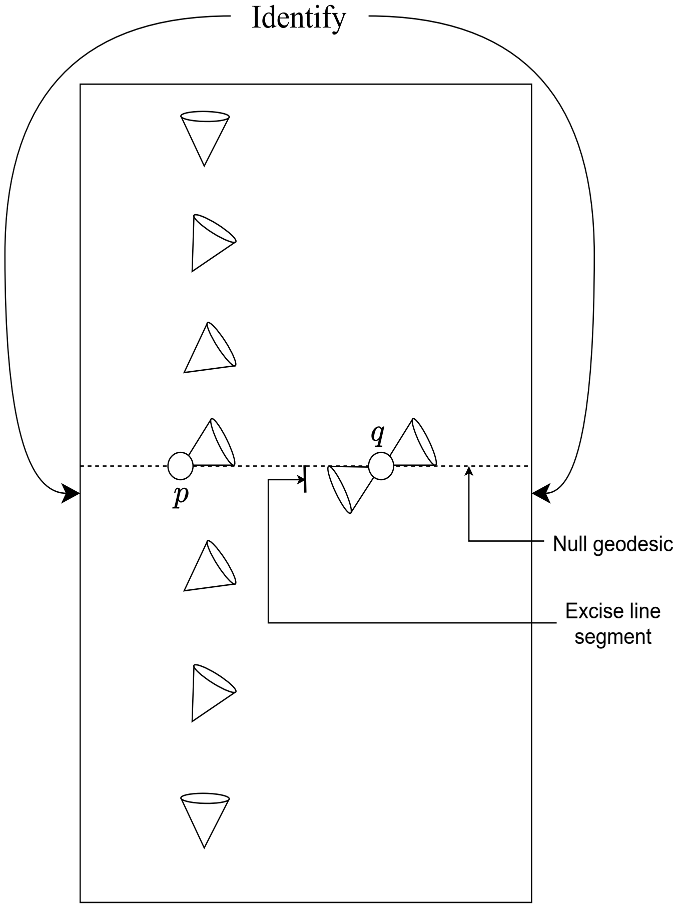
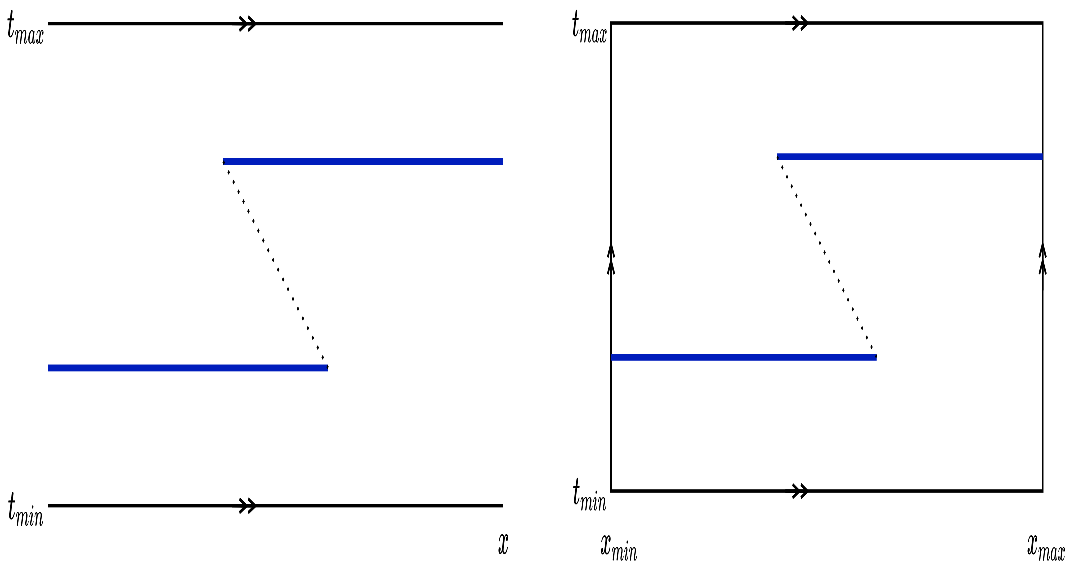


Figure 2.1: A diagram depicting a spacetime that is past distinguishing but not future distinguishing [21]. The excised line segment is enough to prevent any point on the highlighted null geodesic from having the same past as any other point. However the spacetime is not future distinguishing as every point on this null geodesic has the same future.

2.3 Strong causality violation

In this section I will focus on one particular spacetime, which I have called Spacetime Z, which is distinguishing but not strongly causal. A similar spacetime is described in [9] in figure 38 on page 193. Spacetime Z is a 1+1 Minkowski space that features two horizontal excised lines at different time coordinates. The excised lines overlap in such a way that the endpoints of the excised lines can be connected by a "characteristic null line", thus forming a Z shape. The spacetime depicted in [9] has a finite temporal axis but with it's boundaries identified and the spatial axis extends to infinity. Similar to this, Spacetime Z has a finite and identified temporal axis however the spatial axis is also finite and identified. To make the distinction clear I have shown both spacetimes in figure 2.2.



(a) (b)

Figure 2.2: Two similar figures both showing non-strongly causal spacetimes. The blue lines on the diagram represent excised lines and the dotted line is a null curve which connects the endpoints of the excised lines. The identification of the axis is indicated by a double headed arrow on the boundary of that axis. Figure 2.2a shows the spacetime from [9] in figure 38 on page 193. The temporal axis is identified forming a cylinder in time and the spatial axis, along with the excised lines, continue off to infinity. Spacetime Z (figure 2.2b) however has finite spatial axis between x_{min} and x_{max} but the boundaries are identified forming a cylinder in both space and time.

2.3.1 Enforcing distinguishability

The reason Spacetime Z is so interesting is because it is a spacetime that is distinguishing but not strongly causal. Figure 2.3 depicts a causal curve which is allowed to re-enter a sub-neighbourhood of a particular point which lies exactly on the characteristic null

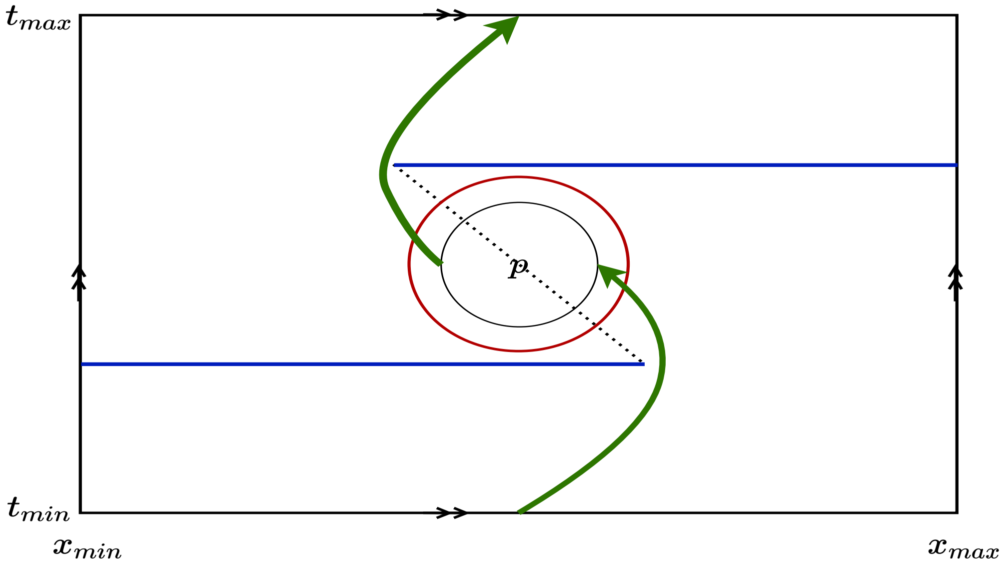


Figure 2.3: Figure depicting a causal path in Spacetime Z which breaks the strong causality condition. The temporal axis is identified which allows the curve to "wrap around" the spacetime. The solid blue lines are the two excised lines and the point p lies exactly on the characteristic null line which joins the endpoints of the excised lines. The larger red circle around p is a large neighbourhood of p and the smaller black circle a sub-neighbourhood. The green arrow then shows a future directed causal curve which starts just inside the sub-neighbourhood of p and wraps around to re-enter the sub-neighbourhood of p thus breaking definition 2.2.4.

line. In fact it is only the points that lie exactly on the characteristic null line which do not satisfy definition 2.2.4. The distinguishing nature of Spacetime Z is subject to a constraint however; if the temporal separation of the excised lines is too big then a causal curve will inevitably intersect one of the excised lines. Moving the excised lines further apart could mean that the spacetime becomes globally strongly causal and therefore we require a condition on the separation of the excised lines to avoid this happening.

2.3.2 Determining the separation of the excised lines

We want to determine the necessary requirement for Spacetime Z to be distinguishing but not strongly causal. In words, this is the statement that the excised lines must not be "too far" separated. To algebraically quantify "too far" I ask the reader to refer to figure 2.4 for the algebraic quantities used in this derivation. A causal curve must be able to start arbitrarily close to p travel away and return arbitrarily close to p which can be summarised as

$$\Delta x > x_p + \frac{a}{2}\tan(\theta) \tag{2.2}$$

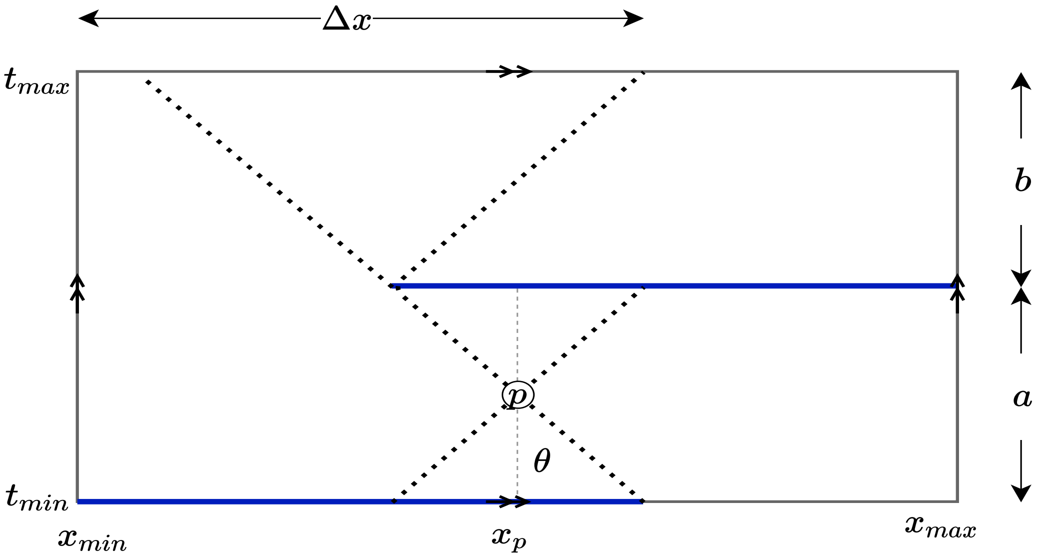


Figure 2.4: Diagram showing an annotated Spacetime Z which is related to figure 2.2b by a change of coordinates The blue lines indicate lines that have been excised from the spacetime. Both the temporal and spatial boundaries are identified as indicated by the double arrows. A point p has been marked which lies on the midpoint between the two excised lines and the dotted lines around p show the null cones from p. A causal curve following exactly the null cones is the limiting condition which satisfied the strong causality condition.

where Δx is the minimum distance along the x axis that our causal curve has to reach in order to avoid the excised line. It is then possible to find another expression for Δx by considering the path that starts at p, travels along the future directed null curve from p to the end point of the upper excised line and then travels along the future null curve to reach Δx[1]. This can be written as

$$\Delta x = x_p + \left(b - \frac{a}{2}\right)\tan(\theta). \tag{2.3}$$

We can now eliminate Δx by substituting equation (2.3) into equation (2.2) to get $b > a$. Finally, using that $a + b = t_{max} - t_{min}$, we get the condition for the separation of the excised lines,

$$a < \frac{t_{max} - t_{min}}{2}. \tag{2.4}$$

This says that Spacetime Z is distinguishing but not strongly causal as long as the distance between the excised lines is less than half the total temporal distance of the spacetime.

[1]Remember that we cannot travel along the null lines because we would hit the endpoints of the excised line. Instead we can get arbitrarily close to the null lines.

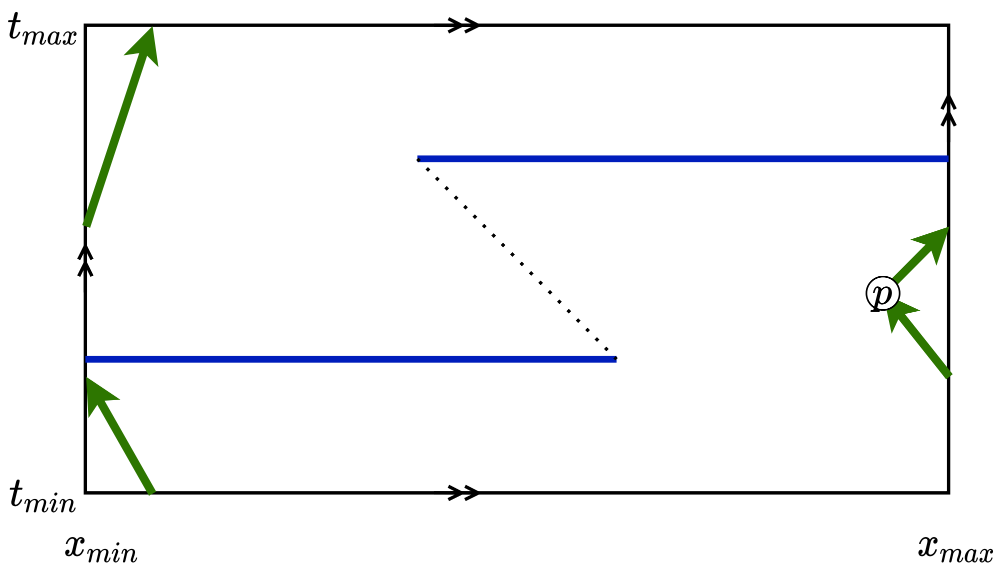


Figure 2.5: Diagram of Spacetime Z with a closed causal curve indicated by the green arrows. The line starts at p and uses the identification of the spatial axis to avoid hitting the excised line. It then continues upwards to wrap around in time and then uses the spatial axis again to meet back at p. The existence of such a curve means that Spacetime Z as it is does not even satisfy the basic causality condition

2.3.3 Non-causal paths in Spacetime Z

One very important difference between the spacetime proposed by [9] and Spacetime Z is that Spacetime Z is finite and both the temporal and spatial boundaries are identified. This presents a problem; as it is Spacetime Z is not causal! It is possible to trace causal paths that use the identification of the spatial boundaries to skip around the excised lines and use the identification of the temporal axis to create a closed timelike curve, as in figure 2.5. This doesn't mean we have to abandon Spacetime Z however because there is a small correction that can be made to it which is shown in figure 2.6. This modified Spacetime Z is strictly distinguishing as it is not possible to create closed causal curves. For the purposes of this thesis I still regard the original Spacetime Z as distinguishing and use it to run computational simulations. The non-causal paths are difficult to reproduce on a computer so according to the simulations the unmodified version is still distinguishing.

2.4 Amendments for causal sets

As mentioned at the start of this chapter, all of the definitions given so far have been for a continuum spacetime. This section will make amendments to the above definitions as well as providing additional definitions to be able to define and describe causal sets.

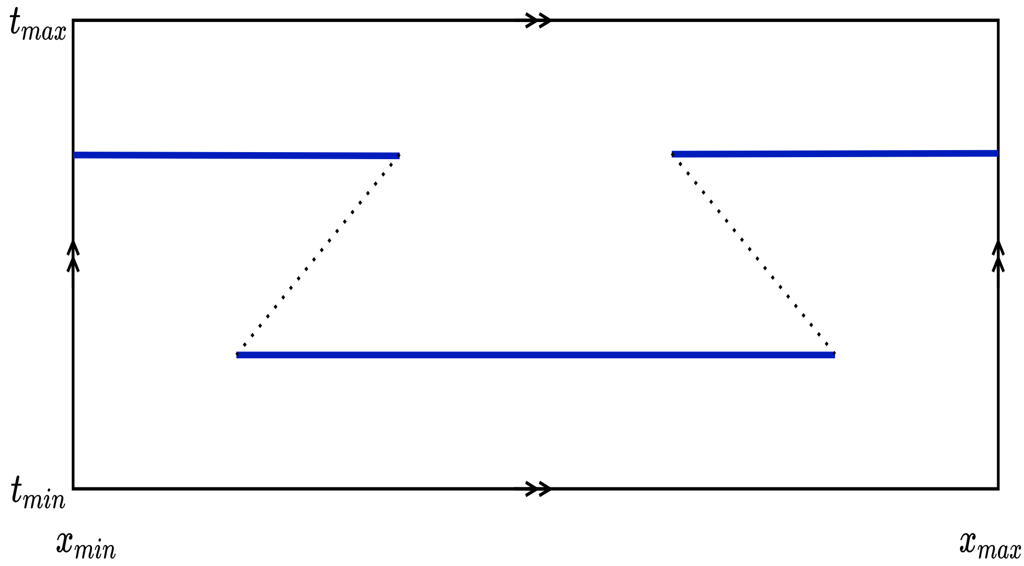


Figure 2.6: A diagram showing a modified Spacetime Z which is required to keep the spacetime distinguishing and not allow any closed causal curves unlike in the original spacetime. There are now two characteristic null lines which make a shape similar to a jigsaw puzzle piece.

2.4.1 The Hauptvermutung

As stated in section 1.3, the central idea of Causal Set Theory is that the continuum nature of the spacetime of General Relativity is an approximation of a more fundamental object, a causal set. The continuum spacetime can only be recovered if the causal set $(C, \prec)$ can be faithfully embedded into a GR spacetime (M, g) [3]. The Hauptvermutung, meaning "central conjecture", further states that this embedding must be unique up to diffeomorphisms. If two spacetimes (M_1, g_1) and (M_2, g_2) both embed $(C, \prec)$ by $\phi_1 : C \rightarrow (M_1, g_1)$ and $\phi_2 : C \rightarrow (M_2, g_2)$ then there exists a diffeomorphism which maps M_1 to M_2, meaning g_1 and g_2 are approximately isometric [3]. Ensuring that the Hauptvermutung holds means that discrete-continuum correspondence also holds and this is essential for determining the causal order of the causal set [6].

2.4.2 Poisson sprinkling

How does one allocate points to create a causal set? One could consider the collection of points to have a lattice structure. However, a lattice spacetime is not generally Lorentz covariant as one can perform a boost such that the spacetime has a high density of points along one axis and a low density of point on another [28]. Instead consider dividing the spacetime into regularly spaced regions of volume ϵ and set the probability of placing a point in each of the volumes to some fixed value δ. Then the probability of putting n points into a spacetime region of volume v follows the binomial distribution. In the limit of the binomial distribution, $\epsilon \to 0$, the probability $P_v(n)$ of placing n elements into a spacetime region of volume v is given by the Poisson distribution [26]

$$P_v(n) = \frac{(\rho_c v)^n}{n!} e^{-\rho_c v}. \tag{2.5}$$

ρ_c is an adjustable *discreteness* scale which represents the scale on which spacetime is considered discrete [29]. The expected number of points $\langle N \rangle$ in the whole spacetime volume V is related to the discreteness scale in such a way that $\langle N \rangle = \rho_c V$ [23, 24]. This means that the coordinates of each point in the spacetime can be chosen uniformly at random covering the whole spacetime volume. This process of dividing spacetime and randomly assigning points in this way is commonly called a Poisson process. The Poisson process is the only way achieve a random sprinkling of points which is also Lorentz covariant [25].

2.4.3 Causal order

After creating a sprinkling of causal set elements using the Poisson process what's left is to establish the causal order of the sprinkling to form a causal set. This is where the discrete-continuum correspondence becomes vitally useful. Of particular importance is definition 1.3.2 item ii) which states that the causal order between any two points in C must be the same as the causal relation between those points after embedding them in the continuum spacetime. In practice this means the causal order on C can be inherited directly from it's faithful embedding in (M, g). This is used extensively in my causal set simulations described later in chapter 3.

2.4.4 Causal intervals

In analogy to definition 2.1.10 we can define an interval in a causal set which I call a causal interval [28]:

Definition 2.4.1 *The causal interval $I[p,q] \subset C$ between two points $p, q \in C$ for which $p \prec q$ is the set of points $\{z \in C | p \prec z \prec q\}$.*

Since $I[p,q]$ is a subset of C, all of the elements in $I[p,q]$ are also elements of the causal set C. One important distinction between a causal interval and an Alexandrov interval is that the causal intervals *cannot* be considered as it's own causal set. The causal order of the interval must be inherited from the causal set from which the interval was created. In practice this means you have to know the causal order of the whole causet before creating causal intervals. This is discussed further in section 3.2.1.

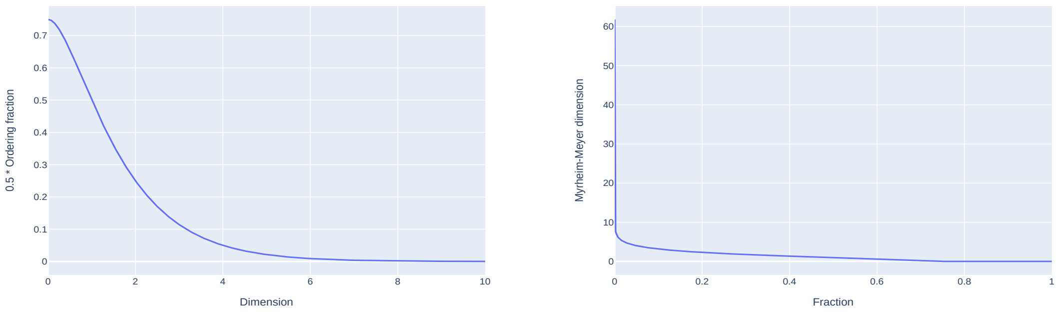
Chapter 3

Deducing dimension from causal sets

This chapter aims to show if that a causal set C which faithfully embeds into a continuum spacetime (M, g) contains all of the necessary information required to deduce the dimension of (M, g). There are two popular techniques for estimating the dimension based on a causal set: the Myrheim-Meyer dimension estimator [19, 16] and midpoint scaling [26]. I will only discuss the former in this thesis as it has been already been established that the Myrheim-Meyer dimension estimator converges to the dimension of the continuum spacetime for large causal intervals in [16, 23]. I use a different approach to calculating the Myrheim-Meyer dimension and confirm that for a 1+1 Minkowski space the Myrheim-Meyer dimension converges to two. Section 3.2 then investigates how frequently each causal set element contributes to the dimension estimates and I demonstrate errors that can occur close to the spatial boundaries.

3.1 Myrheim-Meyer dimension estimator

The Myrheim-Meyer dimension estimator uses only the information of a causal set to estimate the dimension of the background manifold. The information that is required is the number of points N and the number of causal relations R in the causal set. By the number of causal relations we mean pairs $p_i, p_j \in C$ such that $p_i \prec p_j$. By Lorentz invariance, the quantities N and R can only depend on the volume and the dimension of the interval (which is approximately equal to the volume and the dimension of the



(a) (b)

Figure 3.1: Plots related to calculating the Myrheim-Meyer dimension. Figure 3.1a shows the right hand side of equation (3.1) which half what is called the *ordering fraction* [19, 17]. This function is monotonically decreasing and so can be inverted to produce figure 3.1b. The ordering fraction asymptotes to zero as the dimension increases which corresponds to an infinite value of the Myrheim-Meyer dimension for very small fractions.

background manifold) [27]. That gives the following equation

$$\frac{\langle R \rangle}{\langle N \rangle^2} = \frac{\Gamma(d+1)\Gamma(\frac{d}{2})}{4\Gamma(\frac{3d}{2})} \tag{3.1}$$

and the Myrheim-Meyer dimension is the value d for which equation (3.1) holds true [1]. The angular brackets appear because of the randomness of the Poisson process so above holds true only for the expectation values of N and R. The right hand side of equation (3.1) is a monotonically decreasing function so can be numerically inverted and an value for d computed. Figure 3.1a shows the value of the right hand side of equation (3.1) for increasing values of d and figure 3.1b shows it's numerical inverse.

The work done previously in [16, 1, 23, 17] uses a fixed sprinkling into a Minkowski diamond to calculate the Myrheim-Meyer. They then take multiple Alexandrov intervals of fixed sizes inside the Minkowski diamond and calculate the Myrheim-Meyer dimension inside this Alexandrov interval. They average over the many Alexandrov intervals of different sizes to account for the randomness of the Poisson process. To explore rectangular spacetimes, such as Spacetime Z, I decided to take a different approach. I initially sprinkled a causal set into a rectangular spacetime and selected two random points $p, q \in C$ from the causal set. If $p \not\prec q$ then I discarded these points. If, however, $p \prec q$ I formed a causal interval $I[p, q]$ between the two points. I did this multiple times and calculated the Myrheim-Meyer dimension by counting the number of points and causal relations in each interval.

3.1.1 Determining the causal order

Crucial to calculating the Myrheim-Meyer dimension is to know the number of causal relations in each interval and for that one needs to know the causal order. A useful way to represent the global causal order of a causal set is to use the causal matrix which is defined using pairs of points $p_i, p_j \in C$ as [11]

$$C_{ij} = \begin{cases} 1 & \text{if}\, p_i \prec p_j \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

From this global causal matrix, each interval then samples only the pairs of points which are contained within the interval to build up a sub-causal matrix. The number of causal relations in a given interval then is simply the number of non-zero elements of the sub-causal matrix. The question then is to determine whether $p_i \prec p_j$ and this is where one can use the discrete-continuum correspondence. We know that if a causal set faithfully embeds into a continuum spacetime (M, g) then the causal order of the causet must be preserved by the embedding [6]. The causal order of the embedded points must also be the same as the continuum causal relations between the embedded points. That means that if there is a future directed causal curve connecting the embedding of p_i and p_j then $p_i \prec p_j$. For a causal set C which has been sprinkled into an empty Minkowski space we can determine the causal order using the metric g [1]. Using a two dimensional Minkowski space as an example, the metric is $ds^2 = -dt^2 + dx^2$ and the proper time is $d\tau^2 = -ds^2$. The proper time between two points $p_i, p_j \in C$ can be calculated as

$$d\tau_{ij}^2 = \left(p_i^0 - p_j^0\right)^2 - \left(p_i^1 - p_j^1\right)^2 \tag{3.3}$$

and the causal matrix becomes

$$C_{ij}^{SL} = \begin{cases} 1 & \text{if}\, d\tau_{ij}^2 >= 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

This is not the only way of determining the causal order on the causal set C. There only needs to exist one causal path connecting the two points for them to be causally ordered, that path needn't be the straight line between them. In section 4.2 I describe an

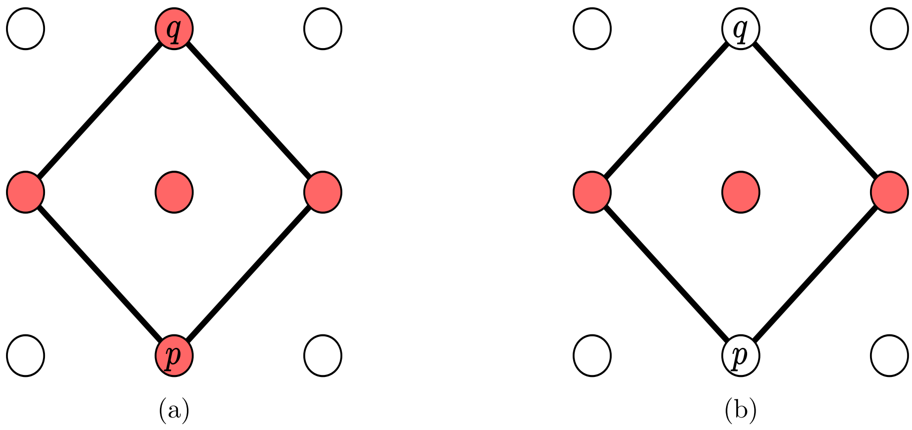


Figure 3.2: Two figures showing a discrete lattice spacetime with different types of intervals defined by points p and q where $p \prec q$. The red points are the points which are within the causal interval defined by the two points p and q. Figure 3.2a includes the extrema of the causal interval and is therefore called an *inclusive* causal interval. Figure 3.2b shows both p and q excluded from the interval and is therefore called an *exclusive* causal interval.

alternative way for determining the causal order using the null geodesics from each of the embedded causal set elements.

3.1.2 Inclusive or Exclusive intervals

The definition of a causal interval $I[p, q]$ contains an ambiguity because it does not enforce that the endpoints p, q are included in the interval. These two different types of intervals, where $p, q \in I[p, q]$ or $p, q \notin I[p, q]$, I call *inclusive* and *exclusive* causal intervals respectively. An inclusive interval is guaranteed to contain at least one more causal relation than an exclusive interval but only two more points. By equation (3.1), we therefore always expect the Myrheim-Meyer dimension of an inclusive interval to be lower than that of the equivalent[1] exclusive interval. Since the MM dimension scales as N^2, distinguishing between the two cases is highly irrelevant in the large N limit. However the difference *is* evident when calculating the Myrheim-Meyer dimension of smaller causal sets. To demonstrate the significance of this, figure 3.2 shows an idealised 1+1 Minkowski lattice spacetime[2] with an inclusive and exclusive causal interval highlighted. For the inclusive interval, the left hand side of equation (3.1) is well defined; the number of points is five and the number of causal relations is seven. Inverting the right hand side of equation (3.1) then gives the MM dimension of this interval as ≈1.848. However, for the exclusive inter-

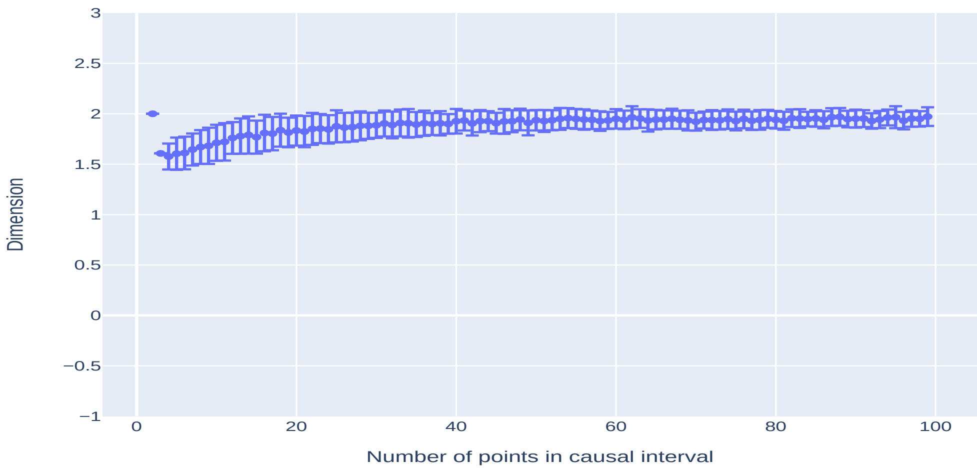
[1]Equivalent in this sense means that the interior of both inclusive and exclusive intervals contain exactly the same points, the only difference between the two is the respective inclusion or exclusion of the endpoints.

[2]Lattices are never used in causal sets but I am making an idealisation for the purposes of illustration. Similar problems can arise from a Poisson sprinkling.

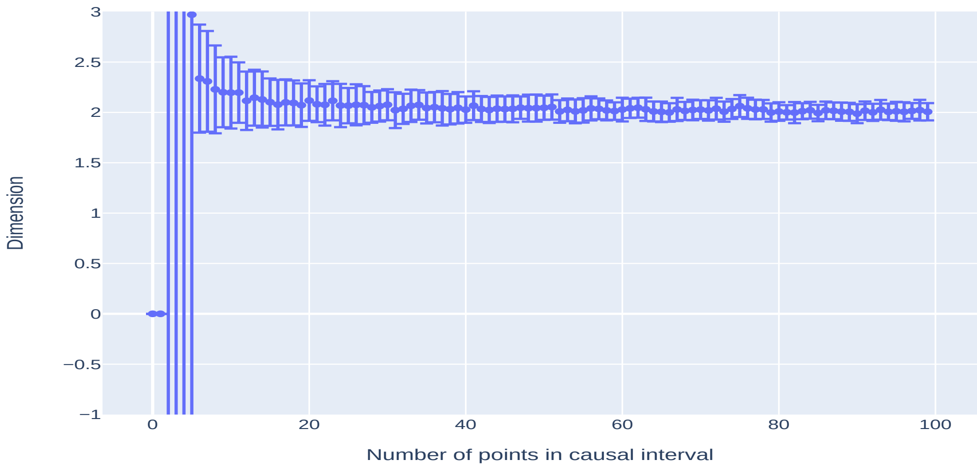
val there are three points in the interval but zero causal relations between them. Using figure 3.1b gives the MM dimension of this interval to be ∞! This impacts how the Myrheim-Meyer dimension converges to the background dimension when sampling many intervals over the causal set. To illustrate this I randomly sprinkled a causal set $\approx$1000 points into a 1+1 dimensional Minkowski diamond and deduced the causal order. I then formed 10,000 inclusive causal intervals and 10,000 exclusive causal intervals from random pairs of points within the causet. I counted the number of points contained in each interval and calculated the Myrheim-Meyer dimension of each interval which I then plot in figure 3.3. The blue dots in the diagram represent the mean value for the dimension and the error bars show ± 1 standard deviation. As expected, both diagrams converge to the background manifold dimension but using an inclusive causal interval causes the dimension to converge from below and the dimension of exclusive intervals converge from above. For causal intervals which contain a low number of points both types of interval show large differences from the background manifold dimension and this is studied more in [1, 17]. For this thesis I will only focus on intervals with more than twenty points for which the Myrheim-Meyer dimension begins to converge.

Since an exclusive causal interval does not include the endpoints of the interval, it is more akin to the previous work in [16, 1, 23, 17]. For this reason for the rest of the paper I use exclusive causal intervals in my analysis of other causal sets.

Despite there being $\approx$1000 points in the original sprinkling, when calculating the Myrheim-Meyer dimension I discarded any intervals which contained more than 100 points. Figure 3.4 shows that there is a very small probability of picking intervals with sizes comparable to the size of the original causet. This means that more statistical anomalies arise when calculating the MM dimension for larger intervals and because of this these intervals alone do not show convergence to the background dimension. However, the probability of picking a large interval scales with the number of points in the original causet so in the large N limit these anomalies seemingly disappear.



(a)



(b)

Figure 3.3: Figures showing the Myrheim-Meyer dimension of different sizes of causal intervals. The background manifold for both plots was a 1+1 Minkowski diamond sprinkled with ≈1000 points according to the Poisson process. Figure 3.3a uses inclusive causal intervals when sampling across the spacetime and figure 3.3b uses exclusive intervals. Each plot makes 10,000 samples of the spacetime. The points on the plot show the mean dimension from all of the different intervals of the given size and the error bars indicate ±1 standard deviation from the mean.

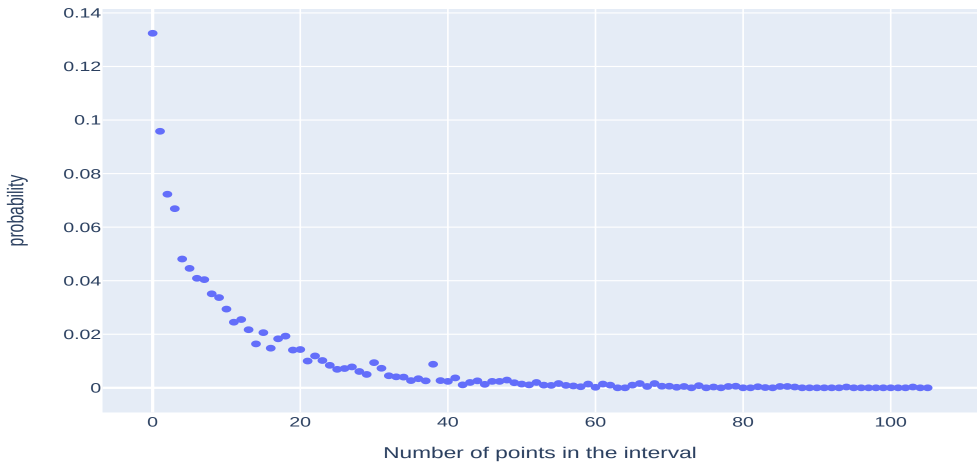


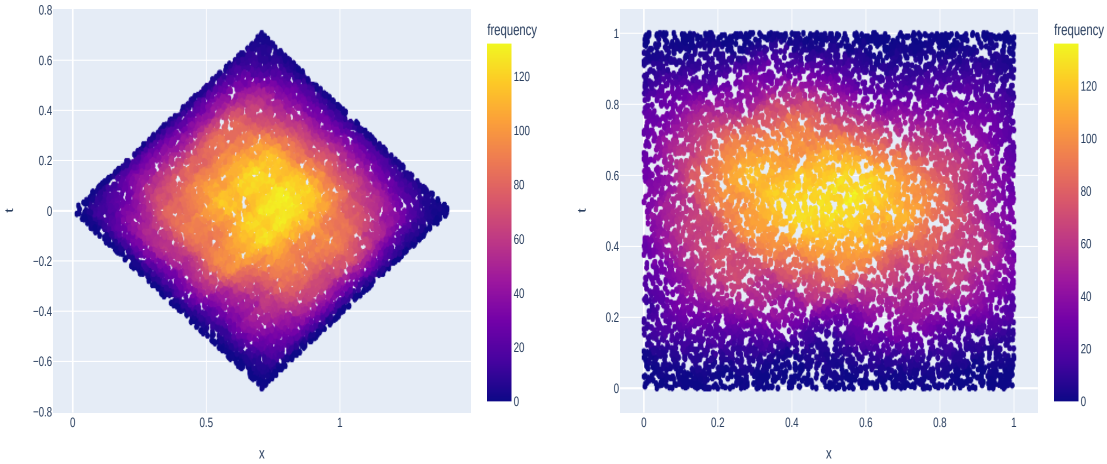
Figure 3.4: Figure which show the probability that a random exclusive interval from a causet contains a certain number of points. The initial causet had ≈100 points sprinkled into a 1+1 Minkowski causal diamond. I then created 10,000 exclusive causal intervals from the initial causet and registered the number of points in each interval. The plot shows that it is most likely that any one of those 10,000 intervals would contain a small number of points.

3.2 Frequency of occurrence within an interval

Having showed that the Myrheim-Meyer dimension estimator does converge to the dimension of the background manifold, I was then curious to see how well a sampling of intervals covers the whole causet. To do this I made a sprinkling of ≈10,000 points into two different 1+1 Minkowski spacetimes. One spacetime was the 1+1 Minkowski diamond I used when calculating the Myrheim-Meyer dimension in section 3.1 which is bounded by $-\sqrt{2}/2 \leq t \leq \sqrt{2}/2$ and $0 \leq x \leq \sqrt{2}$. I also investigated a rectangular spacetime bounded by $0 \leq t \leq 1$ and $0 \leq x \leq 1$. From each of the sprinklings I made 1000 random inclusive causal intervals and counted the number of times each point in the causet appeared in one of the intervals. Using this I can determine which areas of the causet are sampled from most frequently and I have highlighted these areas in figure 3.5. The plots show that points which lie in the centre of the causal set are sampled from most frequently.

3.2.1 Errors in a rectangular spacetime

As mentioned in section 2.4.4, the order relation between each pair of points in a causal interval must be the same as the order of those points in the global causet. For a simple 1+1 dimensional spacetime which does not identify the boundaries (as in figure 3.5) one



(a) (b)

Figure 3.5: Figure 3.5a shows the same Minkowski diamond used in section 3.1 when calculating the Myrheim-Meyer dimension and figure 3.5b shows a regular Minkowski space, bounded by $0 \leq t, x \leq 1$. Both spacetimes contain a sprinkling of ≈10,000 causal set elements. I made 1000 inclusive causal intervals over each of the sprinklings and the colour scale shows how frequently each point appeared in an interval.

can accurately calculate the causal order of an interval in isolation from the larger causet. This is not the case however for spacetimes which do identify the boundaries as only the global causal order reveals the wrap around. To demonstrate this I make the following definition for causal sets which have been sprinkled into a 1+1 dimensional Minkowski spacetime:

Definition 3.2.1 *A complete causal interval is one such that both the intersection points of the future light cones from p with the past light cones of q can be embedded into the continuum manifold.*

The exact intersection points may not be an elements of the causal set but it should be possible to embed the intersection points into the spacetime. Otherwise, if an interval is "cut-off" by the spatial boundaries it is called incomplete. See figure 3.6 for examples.

Consider the specific example of a spacetime with it's spatial boundaries identified and sprinkle a causal set into it. Every causal interval created from this sprinkling must be complete and therefore should not discriminate against points which are close to the spatial boundaries. To show this I sprinkled ≈10,000 points into such a spacetime and sampled 1000 random causal intervals, ensuring that each interval inherited the global causal order. I then recalculated how many times each point in the global causet was inside a causal interval and figure 3.7 correctly illustrates a uniform frequency band across the centre of the causet.

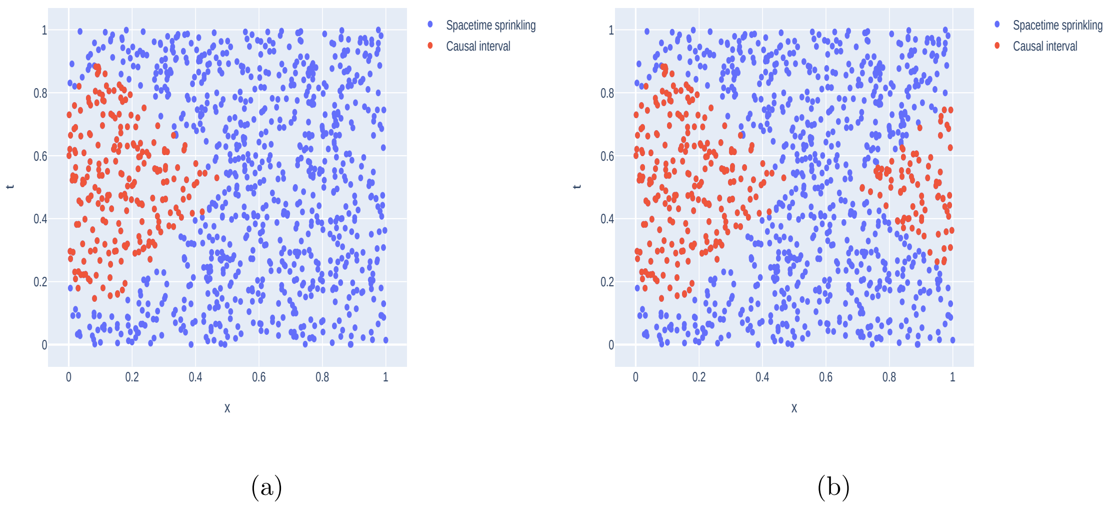


Figure 3.6: Two figures depicting the same spacetime sprinkling and a causal interval between the same two points in a 1+1 Minkowski spacetime. The past and future point of the causal interval are $(0.1, 0.1)$ and $(0.9, 0.1)$ respectively for both plots. Figure 3.6a shows the interval when the x boundary is not identified and figure 3.6b does have the boundary identified, thus allowing the causal interval to continue onto the other side of the spacetime.

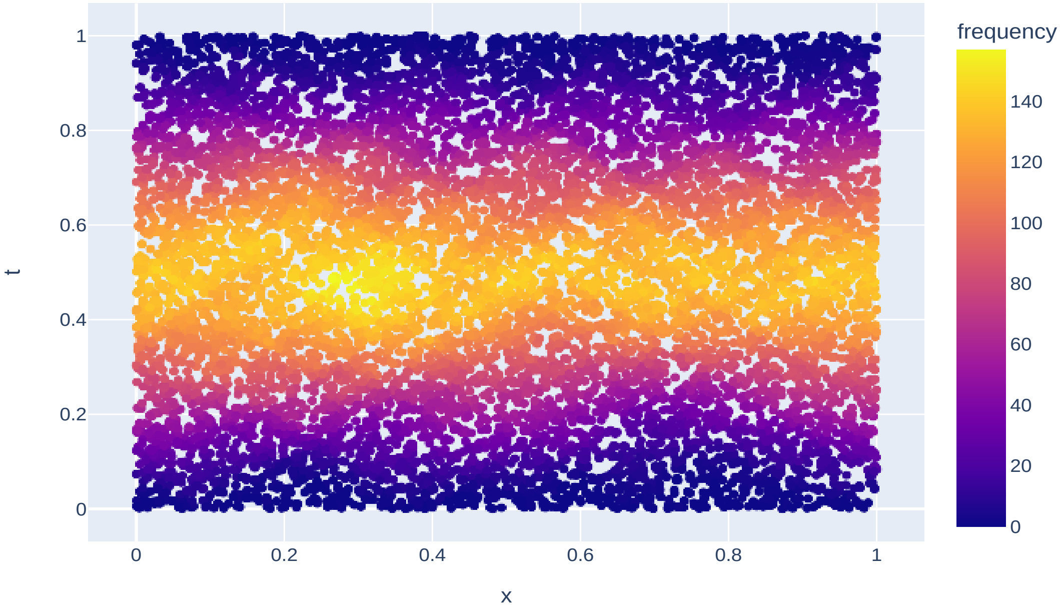


Figure 3.7: Figure depicting a random Poisson sprinkling of ≈10,000 elements into a rectangular 1+1 Minkowski spacetime that has the spatial boundaries identified. The colour map, similar to figure 3.5, is generated by sampling 1000 causal intervals from the sprinkling. It is vital that the intervals inherit the global causal order to realise the identification of the spatial boundaries. The result of this is the uniform colour band through the middle of the causet meaning that the causal intervals do not discriminate against points close to the spatial boundaries.

Chapter 4

Causal intervals in featureful spacetimes

In this chapter I investigate how identifying the temporal boundaries and removing line segments from a spacetime affects the causal structure. I demonstrate the consequences that these particular features have on a causal set which has been sprinkled into such spacetimes. Most notably this includes the ability to determine the causal order of the causet and therefore it's ability to be embedded in a continuum spacetime. I propose a new technique for calculating the global causal order of causets which have been sprinkled into featureful spacetimes. I claim that this new technique more reliably determines the causal order for these causets than the method described in section 3.1.1.

4.1 Excised lines

First let us consider a causet which has been sprinkled into a 1+1 dimensional Minkowski spacetime that excises a horizontal line though the centre of the spacetime, as in figure 4.1. To begin to evaluate whether such a causet can be faithfully embedded into a continuum spacetime we must determine the causal order of the causet. The excised line plays an important role in the causal order as it prevents causal relations between certain pairs of points. The straight line method as outlined in section 3.1.1 needs to be modified slightly to account for the excised line. Consider two points $p, q \in C$ which are separated by the excised line. To calculate the causal order we use the straight line method which would determine that $p \not\prec q$ because the straight line between the two intersects the excised line.

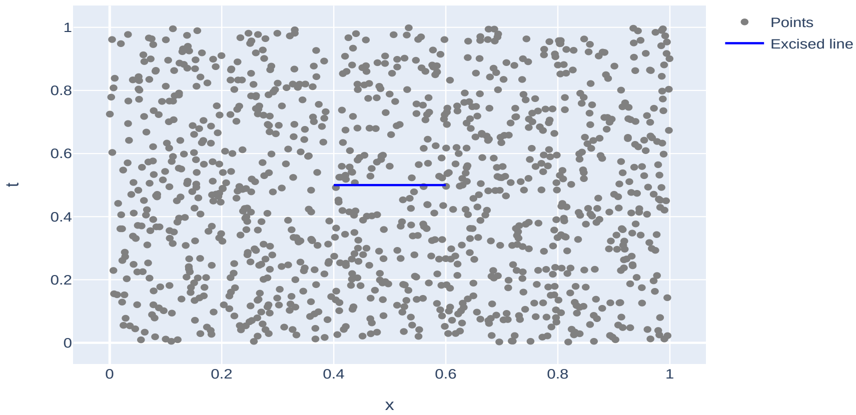


Figure 4.1: Figure showing a 1+1 Minkowski spacetime with a line removed from the spacetime which is shown in blue. Approximately 1000 points have then been sprinkled into the spacetime via a Poisson process. The excised line plays an important role in determining the causal order of this sprinkling because it prevents some points below the excised line from causally preceding points above the excised line.

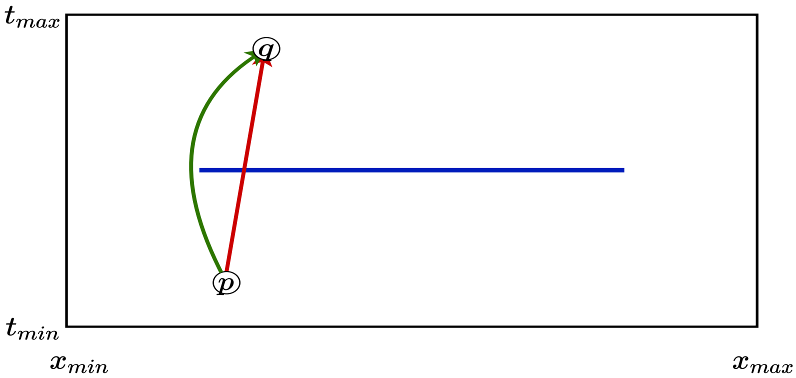
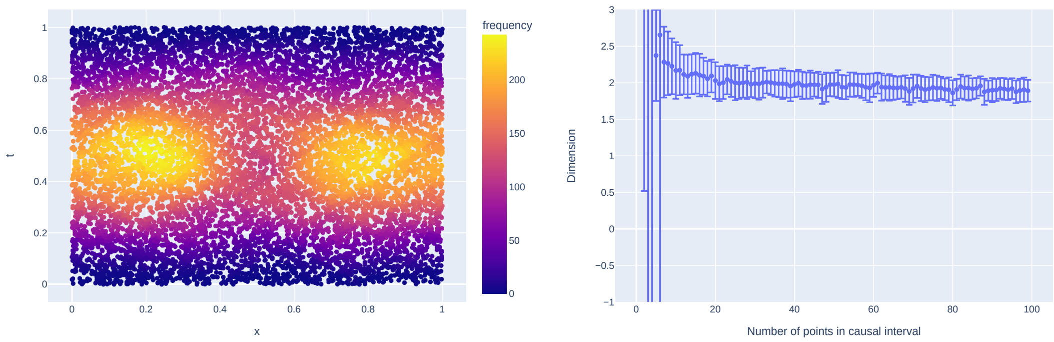


Figure 4.2: Figures showing a continuum 1+1 Minkowski spacetime with an excised line shown in blue. The points in the diagrams represent embedded causal set elements. In figure 4.2 the causal order of p and q has been determined using the straight line method meaning that $p \not\prec q$. After embedding p and q however, one can follow the causal curve shown in green which connects the two embedded points. The existence of such a causal path shows that $p \in J^-(q)$ and therefore causal set is not a faithful embedding.

Now if one faithfully embeds these two points into the continuum such that the order between them is preserved then we see that this order does not match the causal structure of the continuum. In the continuum, one may be able to construct a causal curve between the embedded points such that $p \in J^-(q)$. Such a scenario has been depicted in figure 4.2. Clearly we cannot use the straight line method to accurately determine the correct causal order of the causet.

I investigated how much the excised line affected how often elements of the causet were picked in a random sampling of causal intervals when using the straight line method to determine the causal order. I started by sprinkling ≈10,000 points into a 1+1 Minkowski spacetime which has the horizontal line between the coordinates $(0.5, 0.4)$ and $(0.5, 0.6)$ removed. Using the technique described earlier I sampled 1000 exclusive causal intervals across the causet and counted the number of times each element of the causet was contained within an interval. I plotted this using a colour map which is shown in figure 4.3a. The diagram highlights that the areas close to the spatial boundaries, which the excised line doesn't extend into, are sampled the most. Areas above and below the excised line are scarcely sampled from which is expected because the straight line method determines there are fewer causal relations in this area.

I also wanted to see if such a causal set could still accurately deduce the dimension of the background manifold. As shown above, the excised line has an effect on the points that are contained in certain intervals, with those points which straddle the excised line less likely to be included in an interval. The excised line therefore changes the value of the Myrheim-Meyer dimension for intervals which overlap the excised line. The right hand side of equation (3.1), $\langle R\rangle/\langle N\rangle^2$, scales by N^2 with $R \propto \mathcal{O}(N)$ the subdominant term for large intervals. Smaller intervals are less likely to be effected by the excised line because there is less chance that a small interval will overlap the excised line. However, for larger intervals which straddle the excised line there are fewer points contained in the interval when compared to an empty spacetime so we expect that the Myrheim-Meyer dimension is lower. To show that this was the case I sprinkled ≈1000 elements into the 1+1 dimensional Minkowski spacetime that has the same horizontal line excised as in figure 4.3a. I again used the straight line method to determine the causal order of the causet. I then sampled 10,000 exclusive causal intervals randomly and calculated the Myrheim-Meyer dimension of each interval. Figure 4.3b shows a plot of the average Myrheim-Meyer dimension against



(a) (b)

Figure 4.3: Figures showing the effects that an excised line has the causal order of a causal set. Figure 4.3a shows a sprinkling of ≈10,000 points into the spacetime described by figure 4.1. I then sample 1000 causal intervals across the sprinkling and the heat map shows the number of times each point is contained within one of the intervals. Figure 4.3b is generated from the same spacetime with approximately 1000 causal set elements. I then make 10,000 causal intervals from this sprinkling and calculate the Myrheim-Meyer dimension of each interval. I plot the mean dimension against the size of the intervals with the error bars representing ± 1 standard deviation from the mean. The Myrheim-Meyer dimension approaches 2 for relatively small intervals but for larger intervals the standard deviation increases and the mean dimension decreases.

the size of the causal interval and the error bars show $\pm 1\sigma$. As expected, for small intervals the Myrheim-Meyer dimension begins to approach the correct value of two. As the size of the intervals increase the average MM dimension begins to decrease below two and the deviation does not appear to converge. However the deviation does consistently overlap two which shows this causet is well approximated by a 1+1 dimensional Minkowski space.

4.2 Using the null line intersections

I now propose a new method which more correctly calculates the causal order of a causal set which has been sprinkled into particular featureful spacetimes. The causal order of an embedded causet should agree with the continuum causal relations between embedded points. In the continuum, the light cones determine the causal structure [13, 30] so is it possible to use a similar structure to determine the causal order of a causet? I claim that it *is* possible and I show that such a method is more reliable than using straight lines when calculating the causal order of a particular spacetime. I stress that this method does *not* work for a general spacetimes but it does work for previous examples and, importantly,

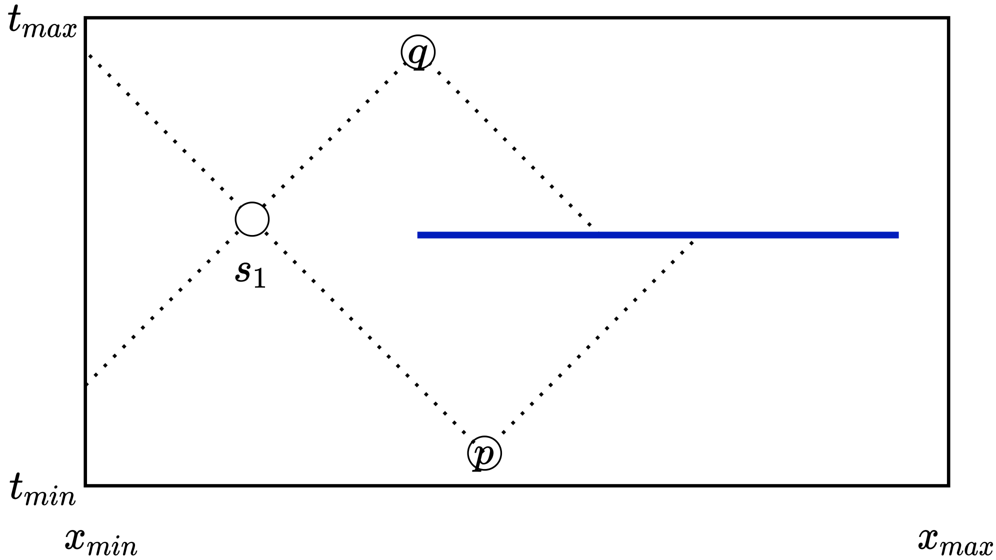


Figure 4.4: Figure which shows a continuum 1+1 Minkowski spacetime with an excised line shown in blue. The points p and q represent causal set elements which have been embedded into this spacetime. The dotted lines show the future null cone of p and the past null cone of q. The other point s_1 is the intersection point of the future null line from p with the past null line from q. The excised line in prevents the other intersection point from the positive gradient null line of p with the negative gradient null line from q from existing. However because of the existence of s_1 (and it is between p and q) we can still conclude that $p \in J^-(q)$.

for Spacetime Z.

4.2.1 Infinite Minkowski space

Consider a causal set which has been sprinkled into an empty infinite 1+1 dimensional Minkowski space. Pick two random points $p, q \in C$ with $p^0 < q^0$ and we want to determine whether $p \prec q$. We also want the order between p and q to agree with the causal relation between $\phi(p)$ and $\phi(q)$ into a continuum spacetime, where ϕ is the embedding. In the continuum one can draw the null cones extending $\phi(p)$ and $\phi(q)$. Since the spacetime is infinite the future null cones of $\phi(p)$ will intersect the past null cones of $\phi(q)$ at some point. If the intersection points, which I call s_1 and s_2 are such that $\phi(p)^0 < s_1^0, s_2^0 < \phi(q)^0$ then $\phi(p) \in J^-(\phi(q))$. This is because any timelike curve between $\phi(p)$ and $\phi(q)$ in the spacetime can continuously reparameterised to follow arbitrarily close to the null lines and still remain timelike [13, 30, 9]. We can then use this to determine the causal order of the points in the causet such that, if the above holds, $p \prec q$. Crucially, the causal order as determined using the null line method agrees with the causal order as determined by the straight line method in an infinite 1+1 dimensional Minkowski space.

4.2.2 Finite Minkowski space

A slight adjustment must be made to the null line method if instead the background manifold is a finite 1+1 dimensional Minkowski space. For "skinny" spacetimes which are taller than they are wide there is no way to adjust the null line method to work globally and this is one example where the null line method fails. Instead consider a sufficiently wide spacetime where the space is larger than time. Again we embed the points $p, q \in C$ with ϕ and extend the null lines from $\phi(p)$ and $\phi(q)$ off to the edge of the spacetime. There is now no guarantee that both intersection points of the null lines are in the spacetime, however the existence of one of them is enough to determine that $\phi(p) \in J^-(\phi(q))$. If neither intersection point exists, or an intersection point is not temporally between $\phi(p)$ and $\phi(q)$ then $\phi(p)$ and $\phi(q)$ are not causally related. This means that for $p, q \in C$, $p \not\prec q$. Again this agrees with the causal order between p and q determined using the straight line method.

4.2.3 Spacetimes with excised lines

In order to study spacetimes with excised lines, it is necessary to make another adjustment to the null line method. Consider the spacetime shown in figure 4.1 which has a horizontal line excised in the middle of the spacetime. Select two points $p, q \in C$ from a causet C which has been sprinkled into the spacetime, such that p is close to the origin and q is in the upper left region of the spacetime. Drawing the null lines from the embedding of the p and q shows that neither intersection point exists in the spacetime; one of the intersections lies outside of the spacetime and the other is prevented by the excised line. For the null line method to work in this situation we must identify the spatial boundaries of the spacetime so that the null lines can wrap around in space. This will still cause errors for very large intervals as even after wrapping around the spacetime the intersection of the null lines will be blocked. However, as shown by figure 3.7, the vast majority of causal intervals sampled elements from C are not in these incorrect regions so these errors should be insignificant in the analysis of C.

With that adjustment to the null line method I want to justify it's use in causal sets with excised lines. The benefit that the null line method presents over the straight line method is that there are two possible paths which one can trace to determine whether two elements of the causal set are causally related. Consider again points $p, q \in C$ straddling an excised line in the spacetime as in figure 4.4. The straight line method would incorrectly determine that $p \not\prec q$. However, after embedding p and q, an intersection of the null lines does exist and it is temporally between p and q. Therefore the null line method would correctly determine that $p \prec q$. It is also important that the null line method correctly identifies unrelated points. The following conditions determine whether $p \not\prec q \, \forall p, q \in C$ using a faithful embedding ϕ:

1. The intersection points of the null cones from $\phi(p)$ and $\phi(q)$ are not temporally between p and q OR
2. A single excised line intersects both the future directed null lines from $\phi(p)$ OR
3. A single excised line intersects both the past directed null lines from $\phi(q)$ OR
4. A single excised line intersects both the future directed positive gradient null line from $\phi(p)$ and the past directed positive gradient null line from $\phi(q)$ OR
5. A single excised line intersects both the future directed negative gradient null line from $\phi(p)$ and the past directed negative gradient null line from $\phi(q)$.

If any of these conditions are met then we conclude that $p \not\prec q$. As we will see later, these are sufficient conditions to investigate Spacetime Z which is the overall aim of this thesis.

4.2.4 The effectiveness of the null line method

I now show whether a causal order determined using the null line method is well approximated by a 1+1 dimensional Minkowski background that contains an excised line. If we use the null line method to determine the global causal order of the causet then the causal matrix becomes

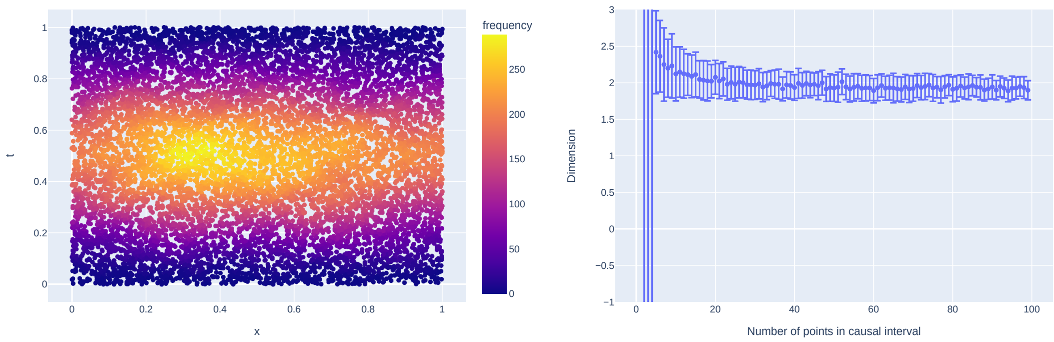
$$C_{ij}^{NL} = \begin{cases} 1 & \text{if one can draw at least one null line from } p_i \text{ and } p_j \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

To make a fair comparison between the null line method and the straight line method I repeated the same analysis conducted in section 4.1. I begin by showing how frequently each element of a causal set is included in a sampling of exclusive causal intervals. I sprinkled ≈10,000 causal set elements into a spacetime with a horizontal excised line between $(0.5, 0.4)$ and $(0.5, 0.6)$. I then calculated the causal matrix using the null line method, which now showed more causal relations than when calculating the causal matrix using the straight line method. I then sampled 1000 causal intervals over the causet and counted the number of times each element was contained in an interval. This generated the plot shown in figure 4.3a. The plot reveals that, using the null line method, there is no discrimination against points surrounding the excised line and causal set essentially becomes blind to the excised line.

I also wanted to see how the null line method influenced the Myrheim-Meyer dimension. We still expect that the excised line decreases the number of points and the number of relations in intervals straddled by the excised line. However, the null line method should include more points and more causal relations than the straight line method for such intervals. To make a fair comparison to the straight line method I again sprinkle ≈1000 points in to the 1+1 dimensional Minkowski spacetime described above and use the null line method to determine the causal order of the causet. I then used the causal order to sample 10,000 exclusive causal intervals. Calculating the Myrheim-Meyer dimension of each interval generates the plot in figure 4.5b. The blue dots represent the average dimension of each size of interval and the error bars are again $\pm 1\sigma$. For smaller intervals the effects of the excised line are minimal and the dimension begins to approach two. For larger intervals the average dimension again falls below the background dimension of two which reflects that the excised line reduces the number of points that are in intervals close to it. However the deviation consistently overlaps two and therefore we conclude that this causet alone is able to deduce the dimension of the background manifold.

4.3 Excised line and temporal axis identification

I now describe another particular subtlety which arises when the temporal boundary is identified, making a cylinder in time. A particularly useful property for rectangular spacetimes that do not allow wrap around in time or space is that, if calculated as an independent causal set, the causal order of an interval is the same as the causal order of the larger causal set. Practically this means that in simulations it is an easy task to calculate the causal order of any given interval as it can be determined in isolation. As talked about in section 3.2.1, identifying the spatial boundaries is a small extension one can make. Not so



(a) (b)

Figure 4.5: Figures depicting results from causets where the causal order has been determined using the null line method. The background manifold is a 1+1 dimensional Minkowski space that excises a horizontal line between $(0.5, 0.4)$ and $(0.5, 0.6)$. Figure 4.5a shows a dense sprinkling of $\approx$10,000 points into the spacetime. From this sprinkling I create 1000 exclusive causal intervals and the colour shows how frequently each point is included in one of the intervals. The region in the centre of the causal set is sampled from less frequently than the edges but in comparison to figure 4.3a points in this region are picked more frequently. Figure 4.5b shows the Myrheim-Meyer dimension for 10,000 random intervals across the causet of density $\approx$1000 points. In comparison to figure 4.3b the standard deviation, indicated by the error bars, stays more compact.

simple, however, is the identification of the temporal axis. Now there is no causal interval in the causet which can be considered independent of the larger causet; indeed one has to calculate the causal order of an entire causet before creating causal intervals from it. A sprinkling into a non-causal spacetime would break item 2) of definition 1.3.1 and so an order relation cannot be introduced on this sprinkling. Introducing excised lines into the spacetime can enforce the acyclicity property on sprinklings in such a spacetime however determining the causal order of the sprinkling becomes highly non-trivial. One could of course consider each combination of pairs of points in the causet and, through a series of conditions, determine whether they are causally ordered or not. The computational running time to calculate the global causal order in this way scales quadratically with the number of points in the causet. Investigations into even sparse causal sets becomes computationally expensive and time consuming. Ideally one would like a vectorisable algorithm which utilises modern CPU SIMD arrays. Unfortunately such algorithms do not exist generally but may be considered for particular spacetimes such as for Spacetime Z. It is possible to write a vectorised algorithm that uses the null line method which can also be applied to Spacetime Z.

Chapter 5

Deducing the dimension of Spacetime Z

In this chapter I analyse Spacetime Z. I first prove that using the null line method within this spacetime determines the continuum causal structure of the spacetime. I then describe an algorithm that uses the null line method to determine the causal matrix of a causal set that has been sprinkled into Spacetime Z. Using the causal matrix I then proceed to sample exclusive causal intervals on the causet to calculate the Myrheim-Meyer dimension. I show that this causal set does approximate a 1+1 dimensional spacetime.

5.1 Proof of the null line method in Spacetime Z

In this section I want to prove that we can use the null line method for Spacetime Z, which only works because of the particular structure of the excised lines. For this proof I will rely heavily on figure 5.1 which shows Spacetime Z split into several regions. Each of the regions are defined by the boundaries of the spacetime and important null lines though the spacetime. Regions A and B are divided by the characteristic null line of Spacetime Z so it is of extra importance that these regions have the correct causal structure. The null line that divides regions B and C is the continuation of the null line that passes through the endpoint of the upper excised line after the temporal wrap around. For the proof I use two causal set elements which have been embedded into the continuum spacetime and I call the embedded points p and q.

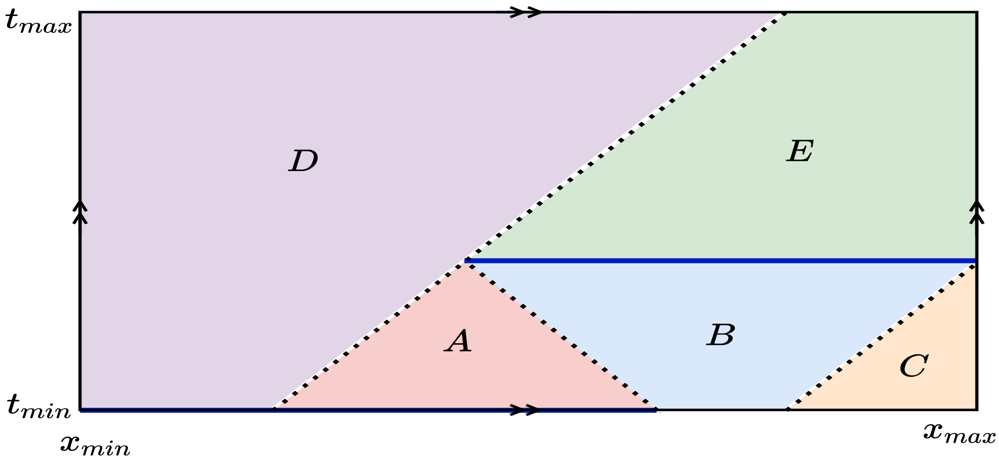


Figure 5.1

5.1.1 Local Minkowski space

Much of the following proof relies on dividing the spacetime up into sub-regions which locally look like Minkowski space. Since the null method works for wide Minkowski spaces, it works in the following scenarios. For example, a causal curve which starts in a region cannot use the time wrap around to re-enter the same region. As such each of the individual regions look like a bounded Minkowski space and any pair of points p, q which are both in A, B, C, D or E are only causally related if they are causally related in the associated Minkowski space.

5.1.2 Causally disjoint regions

For some regions of Spacetime Z there is no causal curve which can travel between them. In this sense they are "causally disjoint".

$p \in A, q \in E$

For this case there is no causal curve that can join p and q so A and E are causally disjoint

$p \in A, q \in C$

Since region C can be considered as the "temporal continuation" of region E the same argument can be applied for points in A and C.

$p \in D, q \in C$

The null line which is the boundary of D is also the boundary of C after the null line wraps around the time axis. For a causal curve to connect region C and D using the wrap around in time it would have to cross this null line which is not possible. There are paths that can use the wrap around in space to travel from C to D however the null line method algorithm described later in section 5.2.2 does not take into account these paths and so we can treat them as causally disjoint.

5.1.3 Conditional causality

For the other choices of points we have to be a bit more careful because the causal relations depend on the exact positioning of the points. For the following I divide the large regions in to smaller sub-regions using imaginary null lines from various points. The descriptions of these null lines are given in terms of a start point and a nautical direction with respect to the diagram.

$p \in D, q \in E$

The majority of D and E are causally disjoint, the only difficult region is the rectangular patch which is bisected by the null line which joins them. This small rectangular region again looks locally like Minkowski space since it is bounded by the excised lines and so no curve can wrap around in time. Therefore p and q are only causally related if they are related in Minkowski space.

$p \in A, q \in D$

Since there is no way for a future directed causal curve to leave D and enter A, the temporal identification does not effect the causal relations. However causal curves can leave A and enter D. Consider an imaginary null line which starts from the lower left corner of A and heads north west to the boundary of Spacetime Z. This sub-region $\cup A$ can be considered as Minkowski space.

$p \in D, q \in B$

The majority of B and D are causally disjoint because of the lower excised line which prevents many of the future directed curves which start in D from leaving D. For the small sub-region which is defined by a null line which runs parallel to the null line which bounds D this is not true. In this region causal curves are allowed to enter B. This small sub-region $\cup B$ can be treated as Minkowski space.

$p \in E, q \in B \cup C$

As mentioned previously, C can be considered as the "temporal continuation" of E and therefore $E \cup C$ can be considered as Minkowski. There are also sections of B that causal curves starting in E can also reach and are considered here because every such causal curve must also pass through C. If q is in the triangular region of B defined by the null line which divides B and C and another imaginary null line heading north west from the lower left corner of C then it is possible for p to be causally related to it. This small extension into B, along with $E \cup C$, can be considered as Minkowski so the causal relations are the same as for Minkowski space.

$p \in A, q \in B$

There is one small problem region in this case which is for points which lie on the characteristic null line which divides A and B. Consider the sub-region of B which is defined by the null line starting from the endpoint of the lower excised line heading north east until it intersects the upper excised line. This sub-region $\cup A$ is another section of this spacetime which locally looks Minkowski so it's causal relations can be inherited from that of Minkowski space. Much of this is redundant though and only includes relations which lie exactly on the characteristic null line. Any point in the interior of A can reach any point in the interior of B because of the temporal identification so any of these points are causally related.

5.2 Determining the global causal order in Spacetime Z

I now describe the algorithm that I use to determine the causal order of causal sets which have been sprinkled into Spacetime Z[1]. The algorithm is incomplete although I have put measures in place to reduce the impact of the errors that this causes.

[1]The code which implements this algorithm can be found in my causal set package which is published at `https://gitlab.com/awr.trumpet/causal_sets`.

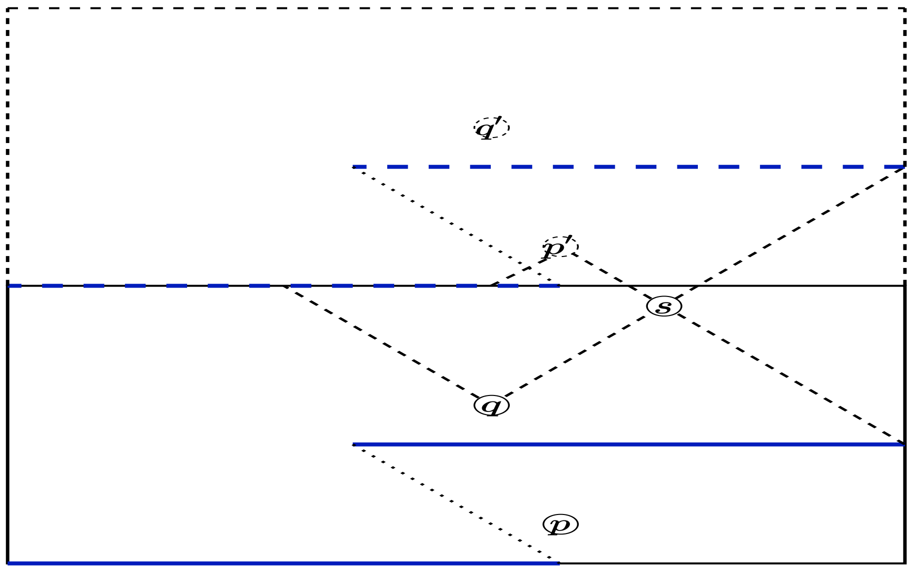


Figure 5.2: Figure showing two causal set element p and q which are embedded in Spacetime Z and the image of p and q (p', q') as well as the image of the excised lines. Considering just the original spacetime (the lower diagram with solid lines) without identifying the boundaries, there is no order between p and q because there is not causal path which connects them. However by identifying the time axis we see that the future null cone of q intersects the past null cone of p' and therefore we conclude that $q \prec p$.

5.2.1 Algorithm outline

The algorithm works by first considering the spacetime without temporal identification. It is important to correctly determine the causal order of a causet in such a spacetime because if two points are ordered without wrapping around in time, then they are still ordered after wrapping around. To accurately calculate the causal order in the unidentified version I first ignore the excised lines and treat the spacetime as an empty 1+1 Minkowski spacetime for which we know the null line method works. I then consider each excised line in Spacetime Z in turn to remove those relations that are blocked by each excised line.

Now I consider the temporal identification which adds causal relations. To do this I "map" an image of the causet above the original (see figure 5.2). I then repeat the procedure outlined above for this new larger spacetime, first determining the causal relations without the excised lines and then considering the excised lines in turn. The difference is now that I look for the causal relations between the original points and their mapped counterparts in the image. This will then tell whether a causal path can be traced between points using the temporal identification.

5.2.2 Unfinished business

One thing that I am yet to implement in this algorithm is to consider points which are causally related by using the spatial wrap around. This is because including such paths would also include the non-causal paths described earlier in section 2.3.3. The errors that come from not including these additional relations can be put down to edge effects and their contribution still needs to be calculated. However if the spacetime is sufficiently wide then the contribution from the edge effects would be small. To minimise the edge effects I only sprinkle the causet into a smaller region of the spacetime which still covers the characteristic null line. This way I do not effect the main feature of Spacetime Z.

5.2.3 Subtleties in the algorithm

I want to highlight a small subtlety with the algorithm. Let us again consider $p \in A, q \in B$ and the picture drawn in figure 5.3. Here the two points in the original spacetime are space-like to each other however using the mapped spacetime a causal curve can be connected from q to p'. Both the future directed positive gradient null line from q and the past directed positive gradient from p' intersect an excised line; wouldn't the algorithm determine them to be unrelated? It does not for exactly this reason! As stated in section 4.2.3 a *single* excised line must intersect both of the positive or negative gradient null lines for the points to be deemed unrelated. If, as in this case, there are two different excised lines which intersect the null lines the the causal relation is undetermined. By considering each excised line in turn in the algorithm I ensure that relations for cases similar to this one are not calculated incorrectly.

There is also a region for which this subtlety is a disadvantage and incorrectly determines the causal relations. Causal set elements that lie either side of the null line which separates D and E will be shown as causally related by the null line method as a single excised line between these points also does not match all of the conditions required in section 4.2.3. One suggestion to fix this is to locate the point where the null lines re-enter the spacetime after wrapping around in time and tracing the null lines from this point. I have not done this but I minimise the issue by making the separation of the excised line very close to the limit for which this spacetime becomes strongly causal, as described in section 2.3.2. This also has the added bonus that we study the maximal effects of the non-strongly causal nature of the spacetime.

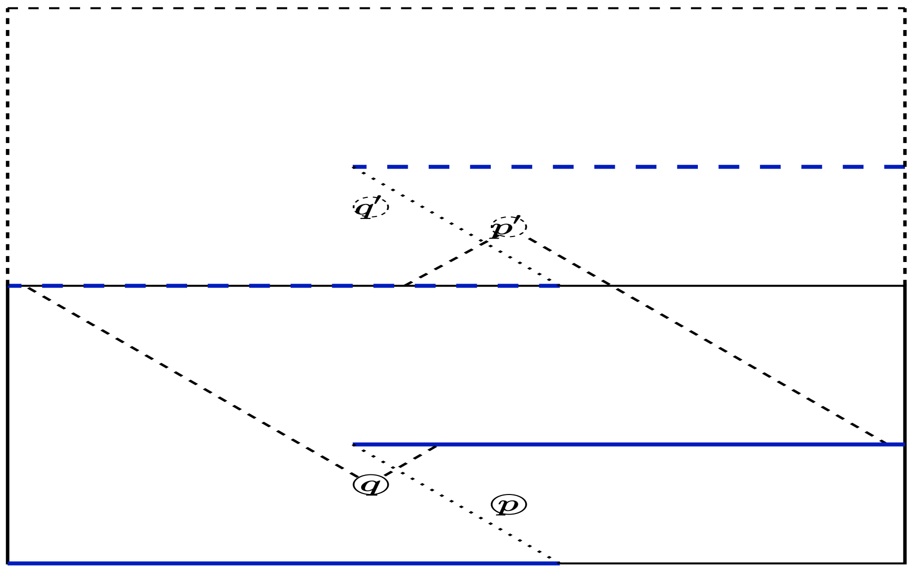


Figure 5.3: Figure showing Spacetime Z with two embedded causal set elements p and q and the image of the same spacetime with the mapped points p' and q'. A single excised line between q and p' does not satisfy all the conditions in section 2.3.2 and therefore this means that the null line method deems $q \in J^-(p')$. This is what is expected of two points which straddle the characteristic null line of Spacetime Z. It is particularly important to study these points because the closer they get to the characteristic null line the more they show the non-strongly causal nature of this spacetime.

5.3 Causal sets in Spacetime Z

Now that I am able to determine the causal order of causets which are sprinkled into Spacetime Z I want to see whether the causal set alone can deduce the dimension of Spacetime Z. To do this I created Spacetime Z with the spatial boundaries going between -4 and 4 and the temporal boundaries between 0 and 1 with both boundaries being identified. The coordinates of the endpoints of the lower excised line were $(0, -4)$ and $(0, 0.2)$. I then set the separation of the lines to be 0.4999 (close to 0.5) and calculated the coordinates of the endpoints of the upper excised line which are $(-0.2999, 0.4999)$. I then made sprinklings into a slightly smaller region; between -3 and 3 in space and 0 and 1 in time. This is to avoid the edge effects which I described in section 5.2.2 and focus more on the main part of the spacetime, the Z! This setup can be seen in figure 5.4. Once the points have been sprinkled I determine the causal matrix of the whole causet, which represents the causal order, using the null line method.

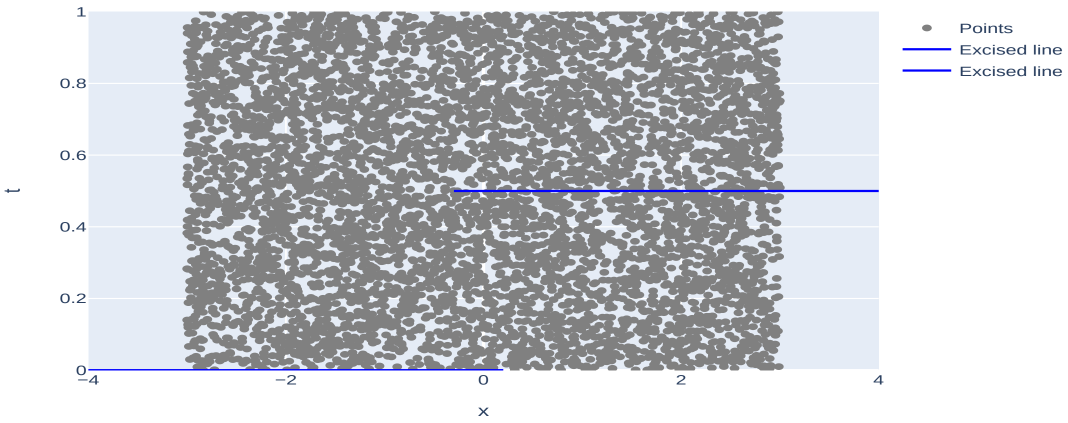


Figure 5.4: Figure which shows a causal set of unit volume density of ≈1000 points sprinkled into Spacetime Z. I use the shifted version of the spacetime so one excised line is on the bottom of the diagram and the other is a separation of 0.4999 away from the first in time.

5.3.1 Coverage of the sprinklings

To see how frequently points are sampled from the causet I repeat similar analysis done in previous chapters. I then created Spacetime Z as described above and sampled 10,000 random exclusive causal intervals across the causet. As detailed previously, because of the structure of Spacetime Z it is important that each interval inherits the global causal order. Counting the number of times each element was contained in an interval produced the plot in figure 5.5. As the diagram shows, the points which are sampled from most frequently are away from the excised lines, where the space has the same order locally as Minkowski space. It should be noted that even those points close to the excised lines are still sampled but less frequently than the other regions.

5.3.2 Myrheim-Meyer dimension of Spacetime Z

Now I evaluate whether the information contained in the causet is enough to deduce the dimension of Spacetime Z. For this I again sprinkle a causal set of ≈1000 elements into the spacetime within the spatial region of -3 to 3. I then created 3000 random exclusive intervals from across the causet and calculated the Myrheim-Meyer dimension for each one. Figure 5.6 shows the mean dimension of the interval against the number of points contained in the interval with the error bars again representing one standard deviation

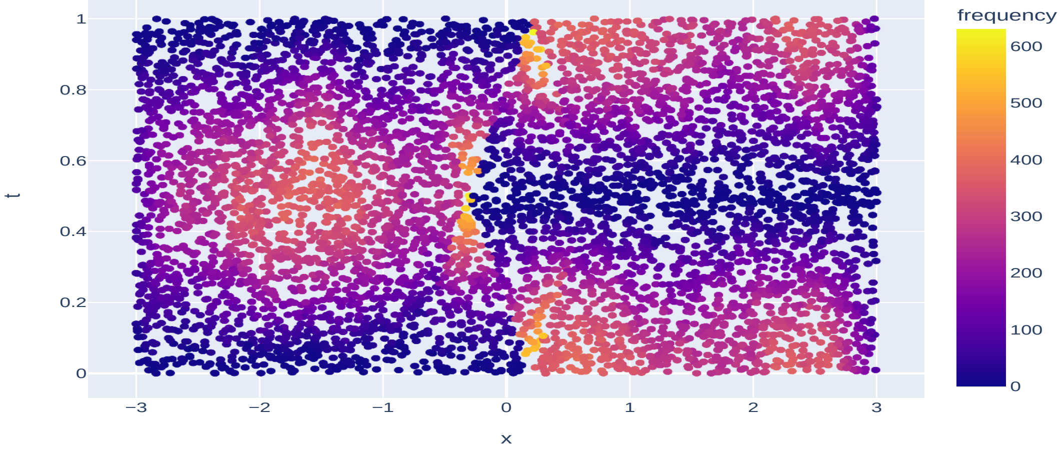


Figure 5.5: Figure showing the sprinkling from figure 5.4 into Spacetime Z where the range of the spatial axis only shows the sprinkling. 10,000 random exclusive intervals have been sampled across this sprinkling and the frequency with which each point is contained in an is shown as a colour map. The frequency is uniform in the areas that locally look like Minkowski space, with fewer points being sampled close to the excised lines.

from the mean. In comparison to an empty Minkowski spacetime shown in figure 3.3b there is a larger deviation from the mean for larger causal intervals. However the standard deviations seem to overlap the background manifold dimension of 2 and the mean values converge to 2 as well. This demonstrates that a causet which has been sprinkled into Spacetime Z can accurately deduce the dimension of Spacetime Z using the Myrheim-Meyer dimension estimator.

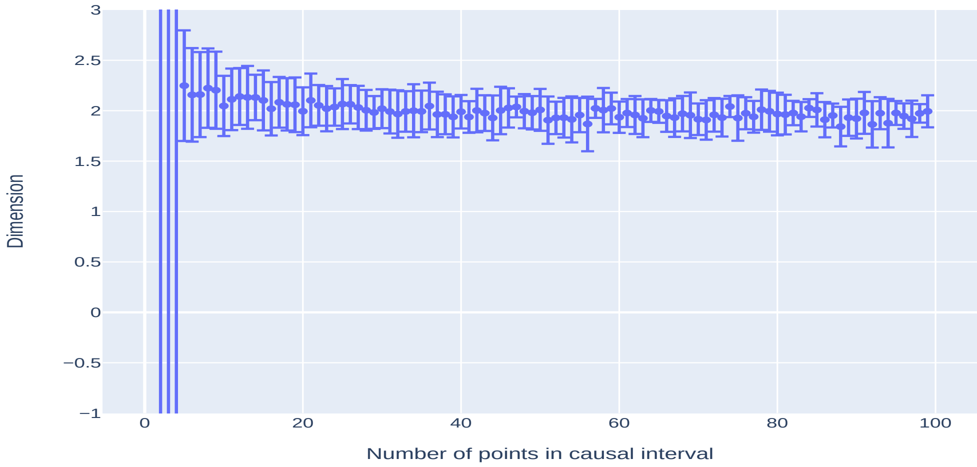


Figure 5.6: Plot which shows the Myrheim-Meyer dimension of a causet of unit volume density ≈1000 sprinkled into Spacetime Z. 3000 random exclusive causal intervals were sampled from across the causet and the Myrheim-Meyer dimension of each of them calculated. The plot shows the various sizes of the intervals with the blue dots representing the mean dimension for each size and the error bars show $\pm 1\sigma$ from the mean.

Chapter 6

Conclusion

6.1 Further work

There is a lot of scope for further work based on the findings of this thesis. The null line method algorithm described in section 4.2 still causes a few erroneous regions for sprinklings into Spacetime Z which I have highlighted in section 5.2.2. It would be interesting to see the contributions due to boundary effects once the spacial identification has been properly accounted for. Another area of interest is to vary the separation of the excised lines and see the effects this has on the Myrheim-Meyer dimension for Spacetime Z.

The work in this thesis has also only concentrated on causets in 1+1 dimensional spacetimes, including for Spacetime Z. The HKMM theorem however requires the dimension of the spacetime to be strictly greater than two [8, 14]. Future work could therefore involve generalising the work in chapter 5 to look at a higher dimensional Spacetime Z, where the excised lines become planes or hyperplanes. If one is able to deduce the dimension and other topological information from causets sprinkled into a higher dimensional spacetime it further indicates that a causal set is the fundamental make up of spacetime.

Mentioned earlier in section 2.3.3, Spacetime Z as it has been described in this thesis does not obey the basic causality condition of definition 2.1.9. However this closed causal curves have been ignored because they do not show up in the algorithms for determining the causal order which, in a simulation, makes Spacetime Z distinguishing. The modification to Spacetime Z to not allow the closed causal curves has been shown in figure 2.6 and further work could examine causal sets sprinkled into this “puzzle spacetime”.

6.2 Closing remarks

In this thesis I have presented Causal Set Theory as a theory of discrete spacetime [27, 4]. I outline the Hawking-King-McCarthy-Malament theorem [8, 14] which is one strong motivation for studying Causal Set Theory. It implies that, under certain conditions, the causal structure of spacetime alone is enough to recover the topological and differentiable structure of spacetime [20]. I describe continuum spacetimes which exhibit different causal properties including the main focus of this thesis, Spacetime Z which is distinguishing but not strongly causal [21]. I then show the differences between a continuum spacetime and a causal set, making necessary definitions to distinguish the two. I describe one method of determining the causal order on a causet which has been sprinkled into a strongly causal spacetime. Using the causal order, I demonstrate how one can deduce the dimension of the background manifold using the Myrheim-Meyer dimension estimator. I confirm results already shown in [16, 23, 1] for a causal sets sprinkled into a 1+1 dimensional Minkowski spacetime. I also investigate how frequently each causal set element is represented when calculating the Myrheim-Meyer dimension. I then describe featureful spacetimes and the problems they present for causets which are sprinkled into them. I propose a new method for determining the causal order in such causets that uses the light cone structure of the embedded points. I show that this new method alleviates some of the problems that arise in certain featureful spacetimes and illustrate that it is a valid method. I prove that the new method can be used in Spacetime Z and describe a particular implementation for calculating the causal order of a causal set which has been sprinkled into Spacetime Z. Using the causal order, I calculate the Myrheim-Meyer dimension of random causal intervals and show that the Myrheim-Meyer dimension converges to two. This shows that a causal set which has been sprinkled into Spacetime Z contains all of the information required to deduce the dimension of Spacetime Z.

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