Exploring the relation between multi-loop two-to-two parton scattering amplitudes in the Regge limit and tree-level string amplitudes

Auther: Yuyu Mo*†
Theory Group, Department of Physics, Imperial College London, London, SW7 2AZ, UK

Supervisor: Einan Gardi‡
Higgs Center, School of Physics and Astronomy, University of Edinburgh, Edinburgh, EH9 3FD, UK

Abstract

We make an ansatz that the hard part of 2-2 partonic scattering amplitude in the Regge limit could be expanded via the five-point closed string amplitude based on some mathematical features they shared. We then set up polynomial equations based on the ansatz. And we find that after partially solving the equation set, there are three inhomogeneous equations that do not have common zeros. Thus, the ansatz may need further generalization. Apart from this, we make a detailed discussion on deriving four-point and five-point tree-level closed string amplitude via performing the single value map on the corresponding one for open string tree level amplitude.

*ym719@ic.ac.uk
†Y.Y.Mo@sms.ed.ac.uk
‡einan.gardi@ed.ac.uk

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1 Introduction

For QCD scattering problems, e.g. for quarks and gluons, if the t-channel exchanging momentum becomes high enough ($-t \gg \Lambda^2_{\mathrm{QCD}}$), they are non-perturbative [7][8]. This is especially the case when the scattering angle is small, $1/\theta^2 = |s/(-t)| \gg 1$. This limit in the literature is also called *Regge limit*. Although perturbative expansion in QCD Regge limit scattering is not viable, we have BFKL framework [1][13] that offers an iterative calculation of expansion order by order and thus provides the building block for resuming high-energy Logarithm to all orders. [7] and [8] have applied BFKL formalism to two-to-two partonic scattering amplitude, and managed to resum infrared singular or infrared-renormalized amplitude on next leading logarithm accuracy, cf. (3.36) in [8] and (3.18) in [7].

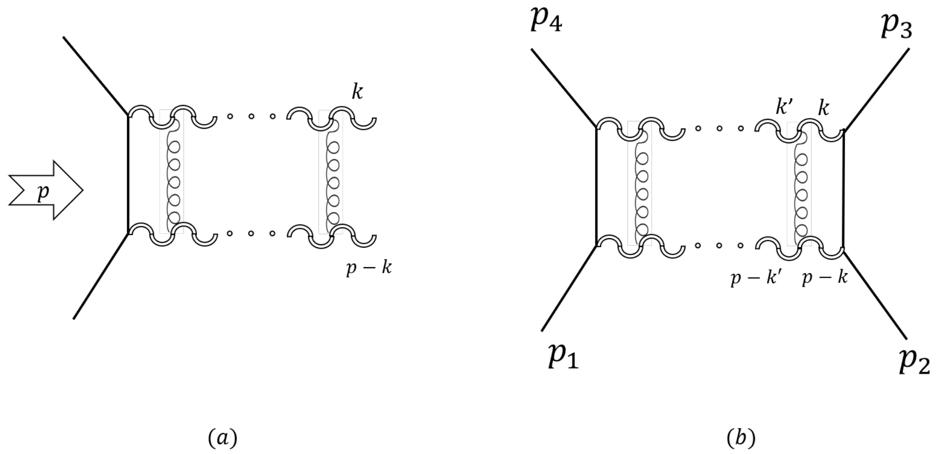

Figure 1: (a): Wave function diagram; (b): Scattering amplitude. ([8]Fig.2 & 3.).

However, there is still some part of the amplitude we can not do resum following this formalism, as one will see below.

For partonic scattering amplitude in *Regge limit*, one can split the amplitude $\mathcal{M}(s,t)$ into to odd and even part according to its symmetry of swapping s and u, $s \leftrightarrow u$:

$$\mathcal{M}^{(\pm)}(s,t) = \frac{1}{2}(\mathcal{M}(s,t) \pm \mathcal{M}(u,t)). \tag{1.1}$$

[6] further decomposes these amplitudes into real and imaginary coefficients.

We define $\mathbf{T}_k,\ k = 1,2,3,4$ to be the colour-charge operator with parton k, (see (b) in Figure 1). With

$$\begin{aligned}\mathbf{T}_s &= \mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{T}_3 - \mathbf{T}_4\\ \mathbf{T}_u &= \mathbf{T}_1 + \mathbf{T}_3 = -\mathbf{T}_2 - \mathbf{T}_4\\ \mathbf{T}_t &= \mathbf{T}_1 + \mathbf{T}_4 = -\mathbf{T}_2 - \mathbf{T}_3\end{aligned} \tag{1.2}$$

And

$$\mathbf{T}^2_{s-u} \equiv \frac{\mathbf{T}^2_s - \mathbf{T}^2_u}{2}. \tag{1.3}$$

Also for future reference (e.g. $\eta = C_1/C_2$)

$$\begin{aligned}C_1 &= 2C_A - \mathbf{T}^2_t,\\ C_2 &= C_A - \mathbf{T}^2_t,\end{aligned} \tag{1.4}$$

where C_A is in the *gluon Regge trajectory*: $\alpha_g(t) = \frac{\alpha_s}{\pi} C_A \alpha_g^{(1)}(t) + \mathcal{O}\left(\alpha_s^2\right)$ (see argument below (1.3) in [8]). The imaginary part of reduced even amplitude is given in [8] as

$$\mathcal{M}^{(+)}_{\mathrm{NLL}} \simeq i\pi \left[\frac{1}{2\epsilon}\frac{\alpha_s}{\pi} + \mathcal{O}\left(\alpha_s^2 L\right)\right] \mathbf{T}^2_{s-u} \mathcal{M}^{\mathrm{tree}} \tag{1.5}$$

where α_s is the coupling constant, and the signature even logarithms is wrote as $L \equiv \frac{1}{2}\left(\log\frac{-s-i0}{-t}+\log\frac{-u-i0}{-t}\right)$. All loop order expansion $\alpha_s^l L^{l-1}$ lies in $\mathcal{O}\left(\alpha_s^2 L\right)$, which we expand as

$$\hat{\mathcal{M}}_{\rm NLL}^{(+)}\left(\frac{s}{-t}\right)=\sum_{l=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^l L^{l-1}\hat{\mathcal{M}}_{\rm NLL}^{(+,l)}. \tag{1.6}$$

As a consequence of BFKL evolution[8], when the order is growing, the ladder graph structure becomes manifest (see (a) in Figure 1). The loop expansion of the reduced even amplitude $\hat{\mathcal{M}}_{\rm NLL}^{(+)}$ is calculated from integrating the free momentum k of the wave function $\Omega(p,k)$ as [8]

$$\hat{\mathcal{M}}_{\rm NLL}^{(+)}\left(\frac{s}{-t}\right)=-i\pi\int[{\rm D}k]\frac{p^2}{k^2(p-k)^2}\Omega(p,k)\mathbf{T}_{s-u}^2\mathcal{M}_{ij\to ij}^{(\rm tree)}, \tag{1.7}$$

where $[{\rm D}k]\equiv\frac{\pi}{B_0}\left(\frac{\mu^2}{4\pi e^{-\gamma_{\rm E}}}\right)^{\epsilon}\frac{{\rm d}^{2-2\epsilon}k}{(2\pi)^{2-2\epsilon}}$ and $B_0(\epsilon)=e^{\epsilon\gamma_{\rm E}}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$ from dimensional regularization. There are singularities in $\hat{\mathcal{M}}_{\rm NLL}^{(+)}\left(\frac{s}{-t}\right)$ which come from the integral, though the integrand Ω is finite[8]. One way of factorizing the singular part and the finite part $\hat{\mathcal{M}}_{\rm NLL}^{(+)}\left(\frac{s}{-t}\right)$ is to split the integrand Ω to soft and hard components proposed in [8]. The wave function $\Omega(p,k)$ can be decomposed as:

$$\Omega(p,k)=\Omega_{\rm s}(p,k)+\Omega_{\rm h}(p,k), \tag{1.8}$$

according to the criteria that the hard part got vanished in the soft limit: one of the external momenta tends to zero, i.e. $\lim_{k\to 0}\Omega_{\rm h}(p,k)=\lim_{k\to p}\Omega_{\rm h}(p,k)=0$. Substituting the splitting of wave function (1.8) into (1.7),

$$\hat{\mathcal{M}}_{\rm NLL}^{(+)}\left(\frac{s}{-t}\right)=\hat{\mathcal{M}}_{\rm NLL,s}^{(+)}\left(\frac{s}{-t}\right)+\hat{\mathcal{M}}_{\rm NLL,h}^{(+)}\left(\frac{s}{-t}\right). \tag{1.9}$$

With this splitting, there is no singularity when integrating Ω_h, and the corresponding hard part $\hat{\mathcal{M}}_{\rm NLL,h}^{(+)}\left(\frac{s}{-t}\right)$ in (1.9) is finite. Thus as one may see in (3.39) of [8], the hard part of the infrared-renormalized amplitude $\mathcal{H}_{\rm NLL,h}^{(+)}$ coincide with $\hat{\mathcal{M}}_{\rm NLL,h}^{(+)}$

$$\mathcal{H}_{\rm NLL,h}^{(+)}=\hat{\mathcal{M}}_{\rm NLL,h}^{(+)}, \tag{1.10}$$

and we won't distinguish them here.

And all the singularities lie in the integral of soft Ω_s and thus in $\hat{\mathcal{M}}_{\rm NLL,s}^{(+)}\left(\frac{s}{-t}\right)$. Moreover, one can do resumming for the singular and finite part of $\hat{\mathcal{M}}_{\rm NLL,s}^{(+)}\left(\frac{s}{-t}\right)$ [7][8], expressing them to a analytic functions valid at all orders. As for $\hat{\mathcal{M}}_{\rm NLL,h}^{(+)}\left(\frac{s}{-t}\right)$, however, there is no general expression of $\hat{\mathcal{M}}_{\rm NLL,h}^{(+)}\left(\frac{s}{-t}\right)$. But it has mathematical structures, which we will explain below, that enable us propose an ansatz that it may be expanded to all order via some finite terms of the order expansion of closed string tree level sphere integral.

As computed in (5.19) of [8], e.g.,

$$\begin{aligned}\hat{\mathcal{M}}_{\rm NLL,h}^{(+,3)} &= \frac{i\pi}{3!}\left\{\frac{3\zeta_3}{4}C_1C_2\right\}{\rm T}_{s-u}^2\mathcal{M}^{(\rm tree\ )},\\ \hat{\mathcal{M}}_{\rm NLL,h}^{(+,5)} &= \frac{i\pi}{5!}\left\{-\frac{5\zeta_5}{2}C_1^2C_2^2+\frac{45\zeta_5}{2}C_1C_2^3\right\}\mathbf{T}_{s-u}^2\mathcal{M}^{(\rm tree)},\end{aligned}$$

$\hat{\mathcal{M}}_{\rm NLL,h}^{(+)}\left(\frac{s}{-t}\right)$ has the number-theoretic properties that all the multiple zeta values are single valued (SVMZV) without any even zeta nunmber. For details of SVMZV, see [4][11][19]. In addition [8], l-loop $\hat{\mathcal{M}}_{\rm NLL,h}^{(+,l)}\left(\frac{s}{-t}\right)$ contains weight-l SVMZV or the product of SVMZVs whose total weight is l. With this we call $\hat{\mathcal{M}}_{\rm NLL,h}^{(+,l)}\left(\frac{s}{-t}\right)$ is of *uniform* *weight*. So $\hat{\mathcal{M}}^{(+,l)}_{\mathrm{NLL,h}}\left(\frac{s}{-t}\right)$ is of *single-valued* and *uniform weight*. Five-point tree level closed string amplitude also share the same properties (see the term in the bracket in (1.11) here or (6.24) in [18]):

$$\mathcal{M}(1,2,3,4,5) = \mathbf{A}^t\mathbf{S}_0\left(1+2\zeta_3\mathbf{M}_3+2\zeta_5\mathbf{M}_5...\right)\mathbf{A}_{YM}, \tag{1.11}$$

where in string theory, α is the coupling constant and $\mathbf{M}_3$ is of $O(\alpha^3)$, and $\mathbf{M}_5$ is of $O(\alpha^5)$, etc.

That's why we made an ansatz that the single-valued 5-point string disk amplitude could be the basis to expand the hard part of two-to-two partonic scattering amplitude in Regge limit. The rest of the paper is organised as follows: In Section 2, we introduce how to do single value map on harmonic poly-logarithms (HPL) or multiple poly-logarithms (MPL) with illuminative examples; in Section 3 we describe motivic single value map directly acting on multiple zeta values; Subsequently, Section 4 and Section 5 give examples, from baby model to complicated case, on deriving tree-level closed string amplitude from open string amplitude via single value map; In Section 6, we detailed state how we boil the ansatz down to equations, and find the inconsistency in solving the equation set; In Section 7, we briefly conclude the thesis by stating the result we have found, and describe how to further continue our study.

2 Single value map

2.1 Words and algebra

The single value map is more conveniently understood via alphabet perspective. We will first define the shuffle product and then state how to use it to construct a single value map. After that, some easy example will help to illustrate the mathematics.

Suppose we letters in alphabet: $a, b, c, d... \in \mathcal{C}$. The concatenation of letters gives us words, such as $a, ab, abc \in \mathcal{C}^*$. Meanwhile, introducing addition "+", which operates on two words by just adding them together as common addition, we could construct polynomials as $ab + cba + dabb$. Restricting the coefficients in rational number $\mathbb{Q}$, one could construct a $\mathbb{Q}$ free algebra $\mathbb{Q}\langle\mathcal{C}\rangle$ (also denoted as $\mathbb{Q}\langle a, b, c, ...\rangle$) [14]. Note that this algebra contains empty word ϵ as unit element.

Definition 2.1. The inner product or say, the duality, of words, is [14]

$$(u \mid v) = \delta^v_u, \quad u, v \in \mathcal{C}^*.$$

Definition 2.2. The right residual of word p w.r.t. q is defined via the inner product[14]

$$(p \triangleright q|z) = (p|qz) = \delta^{qz}_p \quad \forall z \in \mathcal{C}^*.$$

Example 2.1. *The most important example at this moment is*

$$pw \triangleright p = w \tag{2.1}$$

Here is the detail of (2.1)*:*

$$(pw \triangleright p|z) = (pw|pz) = \delta^{pz}_{pw} \quad \forall z \in \mathcal{C}^*. \tag{2.2}$$

The above equation vanishes unless $z = w$. So

$$pw \triangleright p = w. \tag{2.3}$$

2.2 Lexicographic order and Lyndon word

For this topic, we need to introduce an ordering for words.

Suppose we have a set of letters or an ordering, e.g., alphabet ordering $(a < b < c < d < ... < x < y < z)$. Given two words u, v, $u = u_1, u_2..., u_n$, $v = v_1, v_2..., v_n$,

1. when $m = n$, the number of word contained are the same, we will say $v > u$ if read from left to right, $u_1 = v_1,\ u_2 = v_2,\ ...u_{i-1} = v_{i-1}$ and $u_i < v_i$;

2. when $m \neq n$, we will first enlarge the shorter one, say u, by inserting $n-m$ smallest (smaller than any letter) "blank" letters in the end of u and thus u is the same length with v. Then performing the first step again, we should get the relation between u and v.

With the above Lexicographic order,

Definition 2.3. w is an Lyndon word, if for any splitting of w: $w = uv$ and $u, v \neq \emptyset$, we have $w < vu$.

We should also introduce the Lie bracket of the Lyndon word l:

Definition 2.4. Factorizing the longest Lyndon word u in $l = vu$, the lie bracket $[l]$ is [14]

$$\begin{aligned} [l] &= l \quad \text{for length } (l) = 1, \\ [l] &= [v, u]. \end{aligned}$$

Example 2.2. *Here are some simple examples of lie bracket* $(a < b)$

$$\begin{aligned} [a] &= a \\ [ab] &= ab - ba \\ [ababb] &= [[ab], [abb]] = [[ab], [abb]] = [[a, b], [[a, b], b]]. \end{aligned}$$

A polynomial f is some linear combination of words and can be expanded as

$$f = \sum_{w \in \mathcal{C}^*} (f \mid w) w. \tag{2.4}$$

To do the expansion, we need to introduce the shuffle product.

2.3 Shuffle product and word expansion

Consider words u and $v \in \mathcal{C}^*$ with construction $u = u_1 u_2 ... u_n, \ v = v_1 v_2 ... v_m$.

Definition 2.5. The shuffle of u and v, $u \text{ III } v$, is the sum of all permutations of the letters in u and v which preserves the original word orderings, i.e., the letter order $u_1, u_2, ..., u_n \in u$ will not change in the result of the shuffle and so will $v's$.

Example 2.3.

$$a \text{ III } bc = abc + bac + bca.$$

Note: the shuffle product is commutative and associative, (S.2.3 in[2])

$$\begin{aligned} u \text{ III } v &= v \text{ III } u \\ (u \text{ III } v) \text{ III } w &= u \text{ III } (v \text{ III } w). \end{aligned}$$

Before doing the Lyndon decomposition, we shall first introduce some basic concepts. Recall the expansion of polynomial for words (2.4).

Definition 2.6. The degree of f is defined by the maximum length of w which makes $(f|w) \neq 0$.

Example 2.4. *If* $f = ab$*, the degree is 2. If* $f = ab + abb$*, the degree is 3.*

We denote the set of all Lyndon words of length less than the degree of f as L_d. The decomposition of f takes the form [14]

$$f = \sum_{i=0}^{n} A_i \text{ III } l_{\max}^{\text{III} i}, \tag{2.5}$$

where l_{max} is the greatest word in L_d (e.q.(13) in [14]), and n is picked case by case, i.e., the length of $l_{\max}^{\text{III}\, n}$, may not transcend L_d. This is because one may arrive at vanishing A_k for $k > j$ if the length of $l_{\max}^{\text{III}\, j}$ already exceeds the degree of f. We shall also introduce the basic differential formula here. Recalling right residue (2.1), for shuffle product we have [14]

$$(f \text{ III } g) \triangleright p = (f \triangleright p) \text{ III } g + f \text{ III } (g \triangleright p).$$

Three words getting shuffled case:

$$\begin{aligned}(f \sqcup\!\sqcup g \sqcup\!\sqcup h) \triangleright p &= (f \sqcup\!\sqcup g \triangleright p) \sqcup\!\sqcup h + f \sqcup\!\sqcup g \sqcup\!\sqcup (h \triangleright p) \\ &= ((f \triangleright p) \sqcup\!\sqcup g + f \sqcup\!\sqcup (g \triangleright p)) \sqcup\!\sqcup h + f \sqcup\!\sqcup g \sqcup\!\sqcup (h \triangleright p) \\ &= g \sqcup\!\sqcup h \sqcup\!\sqcup (f \triangleright p) + f \sqcup\!\sqcup h \sqcup\!\sqcup (g \triangleright p) + f \sqcup\!\sqcup g \sqcup\!\sqcup (h \triangleright p).\end{aligned}$$

One can prove by induction that

$$(f_1 \sqcup\!\sqcup f_2 ... \sqcup\!\sqcup f_n) \triangleright p = \sum_{i=1}^{n} f_1 \sqcup\!\sqcup f_2 \sqcup\!\sqcup ...(f_i \triangleright p)... \sqcup\!\sqcup f_n. \tag{2.6}$$

(2.6) shows that

$$l_{\max}^{\sqcup\!\sqcup m} \triangleright l_{\max} = m \, l_{\max}^{\sqcup\!\sqcup m-1}.$$

Now we can do the right residue on f in (2.4). For instance,

$$f_1 = f \triangleright l_{\max} = \sum_{i=1}^{n} i A_i \sqcup\!\sqcup l_{\max}^{\sqcup\!\sqcup i-1} \tag{2.7}$$

$$f_2 = f \triangleright l_{\max}^2 = \sum_{i=2}^{n} i(i-1) A_i \sqcup\!\sqcup l_{\max}^{\sqcup\!\sqcup i-2}, \tag{2.8}$$

one can easily generate to the case when $l_{\max}$ becomes $[l_{\max}]$.

Here the A_i is obtained via right residual of Lie bracket of $l_{\max}$ and will not contain $l_{\max}$. Define $L'_d = L_d \backslash \{l_{\max}\}$. We can further decompose the A_i via $l'_{\max} \in L'_d$, where l'_{max} is the greatest word in L'_d. Examples 2.5 and 2.6 are given in the next Section 2.4.

2.4 Lyndon decomposition

In this section, we will decompose word to Lyndon word. Every word can be expressed by a combination of Lyndon Word [11]. This is a really powerful result as Lyndon decomposition can express some poly-logarithms with a fixed value to something we know. Moreover, it also simplifies the result of single value map, which can be seen below.

With preparation in 2.3, we are able to do Lyndon decomposition. We will use the list (2.9) (given in [14]) to decompose words that appear in MZV's.

$$\{\mathbf{0}, \mathbf{00001}, \mathbf{0001}, \mathbf{00011}, \mathbf{001}, \mathbf{00101}, \mathbf{0011}, \mathbf{00111}, \mathbf{01}, \mathbf{01011}, \mathbf{011}, \mathbf{0111}, \mathbf{01111}, \mathbf{1}\} \tag{2.9}$$

Let's see some examples: (the "bold front" numbers are letters.)

Example 2.5. *We will be following the process described at the end of Section 2.3. Let's look at Lyndon word list* (2.9) *from right to left (the greatest to the smallest), we can first assume:*

$$\mathbf{10} = A_2 \sqcup\!\sqcup \mathbf{1}^{\sqcup\!\sqcup \mathbf{2}} + A_1 \sqcup\!\sqcup \mathbf{1} + A_0.$$

We derive the expansion coefficient as follows:

$$\mathbf{10} \triangleright [\mathbf{1}] = \mathbf{10} \triangleright \mathbf{1} = \mathbf{0} = 2A_2 \sqcup\!\sqcup \mathbf{1} + A_1.$$

$$\mathbf{0} \triangleright \ \mathbf{1} = 0 = 2A_2$$

So

$$\begin{aligned} A_2 &= 0 \\ A_1 &= \mathbf{0}, \\ \textit{and} \quad A_0 &= -\mathbf{01}. \end{aligned}$$

To sum up:

$$\mathbf{10} = -\mathbf{01} + \mathbf{0} \sqcup\!\sqcup \mathbf{1}. \tag{2.10}$$

End of calculation.

Example 2.6. *For* **100***, from calculation in example 2.5, we may not have too many* $\mathbf{1}'s$ *in our assumption, as* n *in* $\mathbf{1}^{\shuffle\, n}$ *should not exceed the number of* **1** *in the word we want to expand. Looking at list* (2.9) *from right to left, and we may conjecture that*

$$\mathbf{100} = A_1 \shuffle \mathbf{1} + A_0. \tag{2.11}$$

Then

$$\mathbf{100} \triangleright [\mathbf{1}] = \mathbf{00} = A_1. \tag{2.12}$$

Now we further decompose A_1*.* A_1*is of length 2. Apart from* **1***, the greatest Lyndon word for length 2 is* **01***. However, notice that* A_1 *in* (2.12) *does not have any* **1** *or* **01***. So we may not use* **01** *to expand* A_1*. Further observation shows that what is left on the list* (2.9) *for length 2 is Lyndon word* **0***. As there are two* **0***'s in* A_1*, we can conjecture that*

$$A_1 = A_{21} \shuffle \mathbf{0}^{\shuffle\, 2} + A_{11} \shuffle \mathbf{0} + A_{01}. \tag{2.13}$$

We derive the coefficients A_{21}, A_{11}, A_{01} *here:*

$$A_1 \triangleright \mathbf{0} = \mathbf{00} \triangleright \mathbf{0} \quad = \quad \mathbf{0} = 2A_{21} \shuffle \mathbf{0} + A_{11}, \tag{2.14}$$
$$\mathbf{0} \triangleright \mathbf{0} \quad = \quad 1 = 2A_{21}. \tag{2.15}$$

So

$$A_{21} = \frac{1}{2}. \tag{2.16}$$

Since A_{21} *is already a number, we can omit the shuffle as,*

$$const \shuffle w = const \times w.$$

Substituting (2.16) *into* (2.14)*,* $A_{11} = 0, A_{01} = 0$*, so in* (2.11)

$$A_1 \quad = \quad A_{21}\mathbf{0} \shuffle \mathbf{0} = \frac{1}{2}\mathbf{0} \shuffle \mathbf{0}\ . \tag{2.17}$$

One can arrive at

$$A_0 \quad = \quad -\mathbf{010} - \mathbf{001}, \tag{2.18}$$

via expanding equation (2.11)*, and make both side equal.*

With (2.11) *and* (2.17) (2.18)*,*

$$\mathbf{100} = \frac{1}{2}\mathbf{0} \shuffle \mathbf{0} \shuffle \mathbf{1} - \mathbf{010} - \mathbf{001}. \tag{2.19}$$

Further proceed: decomposing A_0*. Reading the list* (2.9) *from right to left, the greatest word for length three containing only one* **1** *is* **01***, so*

$$A_0 = A_{10} \shuffle \mathbf{01} + A_{00} \tag{2.20}$$

As **001** *is already a Lyndon, we don't need to do any thing for this. So decompose* A_0 *only need decomposing* **010**

$$\mathbf{010} \triangleright [\mathbf{01}] = \mathbf{0}.$$

After fixing the constant term, **010** *reads*

$$\mathbf{010} = \mathbf{0} \shuffle \mathbf{01} - 2 \times \mathbf{001}. \tag{2.21}$$

Substitute (2.21) *into* (2.19)*,*

$$\mathbf{100} = \frac{1}{2}\mathbf{0} \shuffle \mathbf{0} \shuffle \mathbf{1} - \mathbf{0} \shuffle \mathbf{01} + \mathbf{001}. \tag{2.22}$$

End of calculation.

2.5 Single value map

Single value map on words is defined in [19]. For $u, v \in \mathcal{C}^*$

$$\text{sv } w = \sum_{uv=w} u \text{ III } \overline{\tilde{v}}. \tag{2.23}$$

Note: here we have used a different convention from [19], in order to be compatible with the result with `PolyLogTool` [11].

Some note about the notation: *Tilde* means the reversal of word while *bar* means the complex conjugate of words which corresponds to the complex conjugate of MZVs.(see below). The single value map is *linear* [11] (S.9.1).

Assuming that we only have **0** and **1** in the alphabet $\mathcal{C}$, we could have:

Example 2.7.

$$\begin{aligned} \text{sv}\mathbf{01} &= \text{ III } \overline{\widetilde{\mathbf{01}}} + \mathbf{0} \text{ III } \overline{\tilde{\mathbf{1}}} + \text{ III } \mathbf{01}. && (2.24) \\ \text{sv}\mathbf{001} &= \text{ III } \overline{\mathbf{100}} + \mathbf{0} \text{ III } \overline{\mathbf{10}} + \mathbf{00} \text{ III } \overline{\mathbf{1}} + \mathbf{001} \text{ III} && (2.25) \end{aligned}$$

Although we have this map on words, our ultimate purpose is to map them back to multiple poly-logarithms and then MZVs.

The multiple poly-logarithms is defined iteratively via integral below.

$$\begin{aligned} G(0; z) &= 1, \\ G(0; w; z) &= G(0; a_1 a_2 a_3 \dots a_n; z) = \int_0^z \frac{dt}{t - a_1} G(0; a_2 a_3 \dots a_n; t), && (2.26) \end{aligned}$$

where the second argument $a_1 a_2 ... a_n$ is a word, and others are number or variable. The remarkable duality is that they have a one-to-one correspondence with the word in w [11] (S.4.2).

$$w \Leftrightarrow G(0; w; z). \tag{2.27}$$

Note also that [11],

$$a \text{ III } b \Leftrightarrow G(0; a; z) G(0; b; z). \tag{2.28}$$

So from (2.23) and (2.28),

$$G_{\text{sv}}(0; w; z) = \sum_{uv=w} G(0; u; z) G(0; \tilde{v}; \bar{z}). \tag{2.29}$$

Example 2.8. *From* (2.24) *and* (2.25),

$$\begin{aligned} G_{\text{sv}}(01; z) &= G(0; 01; z) + G(0; 0; z) G(0; 1; \bar{z}) + G(0; 10; \bar{z}), && (2.30) \\ G_{\text{sv}}(001; z) &= G(0; 001; z) + G(0; 0; z) G(0; 10; \bar{z}) + G(0; 00; z) G(0; 1; \bar{z}) + G(0; 100; \bar{z}). && (2.31) \end{aligned}$$

Note: we have used: empty word $\epsilon \Leftrightarrow G(0; z) = 1$. One can also directly take single value map on multiple poly-logarithms. this time the sv map is not only linear but also preserves multiplicity, i.e. $\text{sv}G(0; w; z) G(0; u; z) = \text{sv}G(0; w; z) \text{sv}G(0; u; z)$.

There are also some important definitions that we shall list here:

Definition 2.7. Harmonic polylogarithms (HPL)

$$H(\underbrace{0\dots01}_{n_1}\underbrace{0\dots01}_{n_2}\cdots\underbrace{0\dots01}_{n_e}; z) \equiv (-1)^e G(0; \underbrace{0\dots01}_{n_1}\underbrace{0\dots01}_{n_2}\cdots\underbrace{0\dots01}_{n_e}; z). \tag{2.32}$$

Definition 2.8. Definition of mulit-zeta value (MZV):

$$\zeta_{n_1, n_2, \dots, n_e} \equiv (-1)^e G(0; \underbrace{0\dots01}_{n_1}\underbrace{0\dots01}_{n_2}\cdots\underbrace{0\dots01}_{n_e}; 1) = H(\underbrace{0\dots01}_{n_1}\underbrace{0\dots01}_{n_2}\cdots\underbrace{0\dots01}_{n_e}; 1). \tag{2.33}$$

2.6 Single valued map of ζ_2 and ζ_3

With equation (2.10), (2.30) and also the correspondence (2.28) between words and poly-logarithms, we can write our $G_{\rm sv}(01;z)$ as

$$G_{\rm sv}(01;z) \quad = \quad G(0;01;z) + G(0;0;z)G(0;1;\overline{z}) - G(0;01;\overline{z}) + G(0;0;\overline{z})G(0;1;\overline{z})$$

$$G_{\rm sv}(01;z) \quad = \quad G(0;0;z)G(0;1;\overline{z}) + G(0;0;\overline{z})G(0;1;\overline{z}) \tag{2.34}$$

If we want to know what's going on for the MZV, we have to set $z, \bar{z} \to 1$. This process has some subtleties that we should state here. (e.q.(3.2) in [11])

$$\lim_{z\to 1} G(0;0;z) = \lim_{z\to 1} \ln(z) = 0. \tag{2.35}$$

With (2.28),

$$G(0;\underbrace{0,...,0}_{n};1) = 0, \text{ for } n \geq 1. \tag{2.36}$$

$$G(0;1;1) = \int_0^1 \frac{dt}{t-1} \qquad \text{diverges.} \tag{2.37}$$

Note, however, we can renormalize $G(0;1;1)$ as $G(0;1;1) = 0$ (section3.3 in [10]).

Substituting (2.36)(2.37) into (2.34),

$$G_{\rm sv}(01;1) \quad = \quad 0. \tag{2.38}$$

With (2.33)

$$\text{sv}\zeta_2 = 0. \tag{2.39}$$

End of calculation

Let's go to $G_{\rm sv}(001,z)$. Substitute (2.22) (2.10) into (2.31)

$$\begin{aligned} G_{\rm sv}(001;z) \quad &= \quad G(0;001;z) + G(0;0;z)\left(-G(0;01;\overline{z}) + G(0;0;\overline{z})G(0;1;\overline{z})\right) + G(0;00;z)G(0;1;\overline{z}.) \\ &+ \quad G(0;00;\overline{z})G(0;1;\overline{z}) - G(0;0;\overline{z})G(0;01;\overline{z}) + G(0;001;\overline{z}) \end{aligned}$$

When $z, \bar{z} \to 1$,

$$G_{\rm sv}(001;1) = 2G(0;001;1) - 2G(0;0;1)G(0;01;1) + 4G(0;00;z)G(0;1;z). \tag{2.40}$$

With (2.33),

$$\text{sv}\zeta_3 = 2\zeta_3. \tag{2.41}$$

End of calculation

A more general proof of single valued single zeta value sv ζ_i will be given in the motivic single value map (see Example 3.1).

Remark 2.1. These are just very elementary example of doing single value map in alphabet perspective. If we go to a word of length of 4 and higher, i.e. 0010, 0011 etc, an auxiliary y-alphabet is included to do single value map in a sense of series expansion. See section 3.3 in [10] for more details. The next section will give a motivic version of single value map.

3 Single valued map in motivic contest

There is another effective way to construct a single value map directly on MZV's. To perform this, we may upgrade the MZVs to motivic version $\zeta^m_{...}$. There is a detailed description in section 3.2 in [5], but, in a word, we can view $\zeta^m_{...}$ and $\zeta_{...}$ as the same in doing single value map.

3.1 Duality

The ordinary MZVs span a rational vector space

$$\mathcal{Z} = \mathbb{Q}\left\langle \zeta_2, \zeta_3, \zeta_5, ...\right\rangle, \tag{3.1}$$

with basis and dimension conjectured in Table 1.

Introducing a new alphabet:

$$f_i \in \mathcal{F},\ \ i \in 2n+1, n \geq 1, \tag{3.2}$$

$\mathcal{F}$ together with f_2, the similar basis space structure is also manifest in Table 2 for space $\mathcal{H}^{\mathcal{MT}_+} \cong \mathcal{U}$, where

$$\mathcal{U} = \mathbb{Q}\left\langle \mathcal{F}\right\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\left[f_2\right]. \tag{3.3}$$

So Brown conjectured that they are not only similar but isomorphic (see the argument from (3.3) to (3.8) in [5] for more details):

$$\mathcal{Z} \cong \mathcal{H}^{\mathcal{MT}_+}. \tag{3.4}$$

Note: in Brown's original notation $\mathcal{U} = \mathbb{Q}\left\langle \mathcal{F}\right\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\left[f_2\right]$, but $\mathcal{U} \cong \mathcal{H}^{\mathcal{MT}_+}$. We just use $\mathcal{H}^{\mathcal{MT}_+}$ here for simplicity.

Weight N	1	2	3	4	5	6	7	8	...
Basis for $\mathcal{Z}_N$	$\emptyset$	$\zeta(2)$	$\zeta(3)$	$\zeta(2)^2$	$\zeta(5)$ $\zeta(3)\zeta(2)$	$\zeta(3)^2$ $\zeta(2)^3$	$\zeta(7)$ $\zeta(5)\zeta(2)$ $\zeta(3)\zeta(2)^2$	$\zeta(3,5)$ $\zeta(3)\zeta(5)$ $\zeta(3)^2\zeta(2)$ $\zeta(2)^4$	... ...
$\dim \mathcal{V}_N$	0	1	1	1	2	2	3	4	...

Table 1: Conjectural basis of vector space spanned by MZVs [5].

Weight N	1	2	3	4	5	6	7	8
Basis for $\mathcal{H}^{\mathcal{MT}_+}_N$	$\emptyset$	f_2	f_3	f_2^2	f_5 $f_3 f_2$	$f_3 \text{III} f_3$ f_2^3	f_7 $f_5 f_2$ $f_3 f_2^2$	$f_5 f_3$ $f_3 \text{III} f_5$ $f_3 \text{III} f_3 f_2$ f_2^4
dim	0	1	1	1	2	2	3	4

Table 2: Basis for vector space spanned by $\mathcal{F}$ [5].

Remark 3.1. Note also that: one may find in Table 1,2, the shuffle product for f_i basis corresponds to two zeta value basis times together, except for f_2 or ζ_2. The difference between f_2 or ζ_2 from others is that f_2 or ζ_2 is viewed as a constant[5].

Why we introduce the space $\mathcal{H}^{\mathcal{MT}_+}$? Because when a general ordinary MZVs $G(a_0; a_1, a_2, ..., a_n; a_{n+1})$, which span a ring $\mathcal{R}$, are upgraded into $G^m(a_0; a_1, a_2, ..., a_n; a_{n+1})$, the $G^m(a_0; a_1, a_2, ..., a_n; a_{n+1})$'s expand a space $\mathcal{H}$ which is isomorphic to $\mathcal{H}/(\zeta_2^m \mathcal{H}) \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta_2^m]$. One can see that $\mathcal{H}/(\zeta_2^m \mathcal{H}) \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta_2^m]$ is similar to (3.3), and indeed $\mathcal{H}$ can be embedded into $\mathcal{H}^{\mathcal{MT}_+}$. (Section 3.2 of [5]). Note also that there is also a ring homomorphic from $\mathcal{H}$ to $\mathcal{R}$ spanned by $G(a_0; a_1, a_2, ..., a_n; a_{n+1})$ [5].

Moreover, denoting the space of sum of module of weight less than or equal to N as $\mathcal{H}_{\leq N} = \oplus_{\leq N}\mathcal{H}$ and $\mathcal{H}^{\mathcal{MT}_{+}}{}_{\leq N} = \oplus_{\leq N}\mathcal{H}^{\mathcal{MT}_{+}}$, $\mathcal{H}_{\leq N}$ can be mapped into $\mathcal{H}^{\mathcal{MT}_{+}}{}_{\leq N}(\mathcal{U}_{\leq N})$ via an normalized isomorphic trivialization ϕ as ([5], [18]):

$$\begin{array}{rcl} \phi : \mathcal{H}_{\leq N} & \longrightarrow & \mathcal{H}^{\mathcal{MT}_{+}}{}_{\leq N}(\mathcal{U}_{\leq N}) \\ \zeta_2^m & \mapsto & f_2 \\ \zeta_i^m & \mapsto & f_i \quad \text{for } i = 2k+1, \ k = 1, 2, 3, ... \end{array}$$

Conclusion: so this means that given a specific weight N, one can say that

$$\mathcal{H}_{\leq N} \cong \mathcal{H}^{\mathcal{MT}_{+}}{}_{\leq N} \cong \mathcal{Z}_{\leq N}, \tag{3.5}$$

where the last $\cong$ we have used (3.4). So we can make the duality that $\zeta^m_{...} \Leftrightarrow \zeta_{...} \Leftrightarrow f_i ... f_j$.

The relation of all above spaces can be summarize in Figure 2.

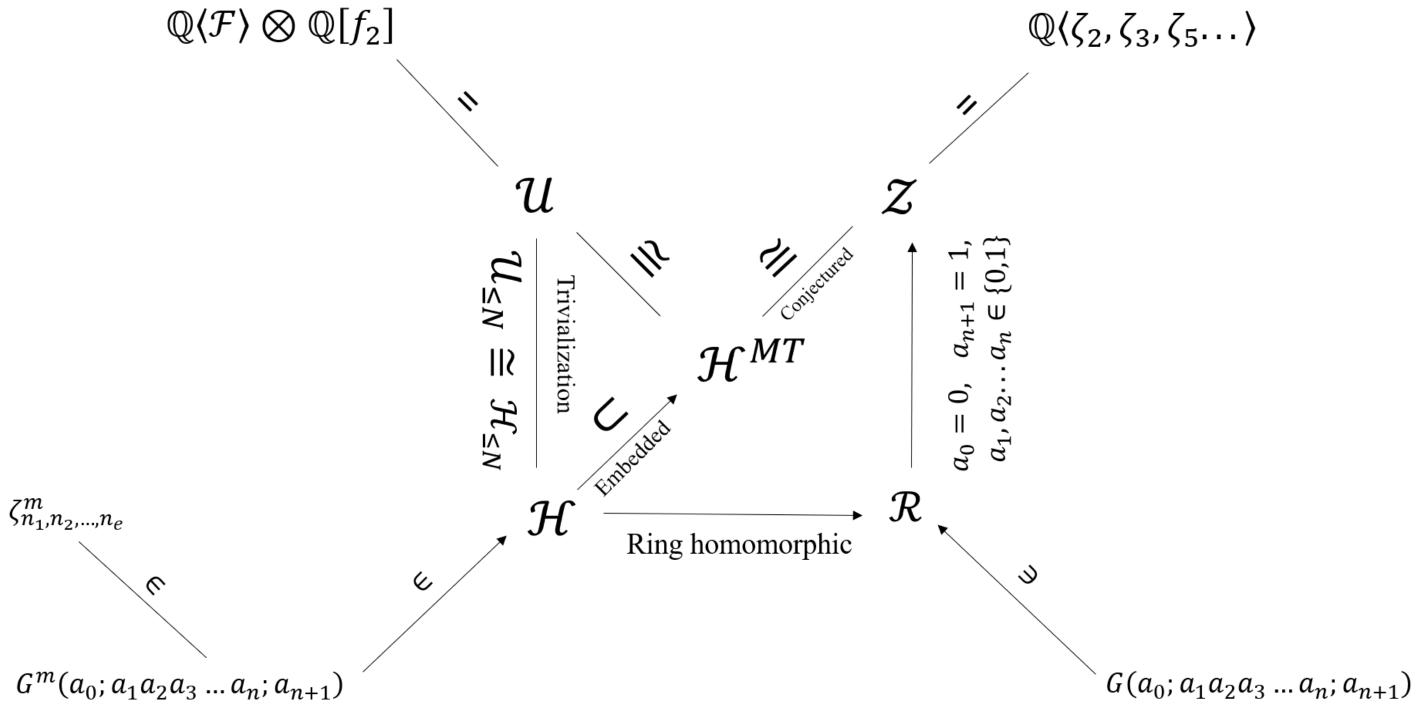

Figure 2: The relation for all the spaces mentioned in Subsection 3.1

3.2 Motivic single value map

Knowing that any structure in $\zeta^m_{...}$ will be inherited $\zeta_{...}$ is not enough. Doing Motivic Single Value Map requires also first decomposing a motivic MZV to Motivic Basis as in Table 2. Luckily, [5] gives us a very detailed description on how to map $\zeta^m_{..}$ to $f_i ... f_j$. See definition 4.3 4.4 and 4.6, also example 4.7 and Section 6 in [5]. We will just list what we will use here, which are given in [18].

$$\begin{aligned}
\phi\left(\zeta^m_{5,3,3}\right) &= -\frac{5}{2} f_5 \left(f_3 \text{III} f_3\right) + \frac{4}{7} f_5 f_2^3 - \frac{6}{5} f_7 f_2^2 - 45 f_9 f_2 \\
\phi\left(\zeta^m_{6,4,1,1}\right) &= \frac{1799}{18} f_9 f_3 - 32 f_7 f_3 f_2 + \frac{1133}{16} f_7 f_5 + 29 f_5 f_7 - 11 f_5^2 f_2 - \frac{16}{5} f_5 f_3 f_2^2 \\
&\quad + \frac{1}{3} f_3 \left(f_3 \text{ III } f_3 \text{ III } f_3\right) - \frac{799}{72} f_3 f_9 + 10 f_3 f_7 f_2 - \frac{1}{5} f_3 f_5 f_2^2 - \frac{36}{35} f_3^2 f_2^3 \\
\phi\left(\zeta^m_{5,5,3}\right) &= 25 f_5^2 f_3 - 10 f_9 f_2^2 - \frac{275}{2} f_{11} f_2 \\
\phi\left(\zeta^m_{7,3,3}\right) &= 30 f_5^2 f_3 - 7 f_7 \left(f_3 \text{ III } f_3\right) + \frac{32}{35} f_7 f_2^3 - \frac{56}{5} f_9 f_2^2 - \frac{407}{2} f_{11} f_2
\end{aligned}$$

With these examples, we are able to introduce the motivic single value map for $f_i \in F$ c.f. (3.2).
The single value map is quite the same as (2.23). However, there is no bar, (see e.q.(7.3) in [4]):

$$\mathrm{sv}w = \sum_{uv=w} u \text{ III } \tilde{v}. \text{ for } f_i \in \mathcal{F} \tag{3.6}$$

This is meaningful, because $f_i \in \mathcal{F}$ is directly related to MZVs. When we convert multi poly-logarithm to MZV's, we take $z \to 1$, so there is no difference between z or $\bar{z}$.

Example 3.1.

$$\begin{aligned} \mathrm{sv}f_2 &= 0; \\ \mathrm{sv}f_i &= f_i + \tilde{f}_i = 2f_i \quad \textit{for} \quad i = 2k+1,\ k = 1,2,3..., \end{aligned} \tag{3.7}$$

where the the reversal of a single word $\tilde{f}_i = f_i$. When we go back to MZVs, this becomes:

$$\begin{aligned} \mathrm{sv}\zeta_2 &= 0; \\ \mathrm{sv}\zeta_i &= 2\zeta_i \quad \textit{for} \quad i = 2k+1,\ k = 1,2,3..., \end{aligned} \tag{3.8}$$

Some more example is given in section 7.2 of [4]

We will construct more complicated motivic single value map in Section 5.2. So far we have complete our introduction of single value map. Next, we will go into some specific example in string theory.

4 A baby model for single value map in four-points amplitude

It is proved that the single value map bridges the disk integral for open string tree-level amplitude and sphere integral for closed string tree-level amplitude in string scattering [19]. The core of doing this is to calculate single-valued MZVs in the expression. We will give a very simple example $\mathrm{sv}Z_{4pt} = J_{4pt}$ to illustrate how this works. Consider the scattering of n massless particles. Choose a frame where $\sum_{i=1}^{n} k_i = 0$ and $k_i^2 = 0$. We then define the kinematic factor as [18][19]

$$s_{ij} := 2\alpha k_i \cdot k_j = s_{ji}, \quad s_{ij} \in \mathbf{R}, \tag{4.1}$$

with

$$s_{i,i} = 0, \quad \sum_{i=1}^{n} s_{ij} = 0 \ \forall \ j = 1,2,\ldots,n. \tag{4.2}$$

Note in high energy $k_i^2 = 0$, so $s_{ij} = 2\alpha k_i \cdot k_j$, $s_{ij} = 2\alpha k_i \cdot k_j$ are equivalent.

Define Beta function:

$$B(x,y) := \int_0^1 dt\ t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{4.3}$$

and C function:

$$C(a,b) := \int d^2z\ |z|^{2a-2}|1-z|^{2b-2} = \frac{\pi\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)}, \tag{4.4}$$

where we have applied the convention:$d^2z = dxdy$. For more details, see proof of (4.3) and (4.4) in Section 6 Appendix of [9]. The 4 point disk $Z_{4\mathrm{pt}}$ and 4 point sphere integral $J_{4\mathrm{pt}}$ for a specific choice of kinematic factors read:[19]

$$Z_{4\mathrm{pt}} := \int_0^1 \frac{\mathrm{d}z}{z} z^{s_{12}}(1-z)^{s_{23}} \tag{4.5}$$

$$J_{4\mathrm{pt}} := \frac{1}{\pi}\int_{\mathbf{C}} \frac{\mathrm{d}^2 z}{z\bar{z}(1-\bar{z})}|z|^{2s_{12}}|1-z|^{2s_{23}}. \tag{4.6}$$

We will now try to expand $Z_{4\mathrm{pt}}$ and $J_{4\mathrm{pt}}$:

$$\begin{aligned} Z_{4\mathrm{pt}} &= \int_0^1 \frac{\mathrm{d}z}{z} z^{s_{12}}(1-z)^{s_{23}} \\ &= \int_0^1 \mathrm{d}z z^{s_{12}-1}(1-z)^{1+s_{23}-1}. \end{aligned} \tag{4.7}$$

Comparing (4.7) with (4.3), we get

$$\begin{aligned} Z_{4\text{pt}} &= \frac{\Gamma(s_{12})\Gamma(s_{23}+1)}{\Gamma(1+s_{12}+s_{23})} \\ &= \frac{1}{s_{12}}\frac{\Gamma(s_{12}+1)\Gamma(s_{23}+1)}{\Gamma(1+s_{12}+s_{23})} \\ &= \frac{1}{s_{12}}\exp[\ln\Gamma(s_{12}+1)+\ln\Gamma(s_{23}+1)-\ln\Gamma(1+s_{12}+s_{23})] \end{aligned} \tag{4.8}$$

Using the identity $\log\Gamma(1+x) = -\gamma x + \sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(-x)^k$, we further get

$$\begin{aligned} Z_{4\text{pt}} &= \frac{1}{s_{12}}\exp\left[-\gamma(s_{12}+s_{23}-(s_{12}+s_{23})) + \sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(-1)^k\left[s_{12}^k+s_{23}^k-(s_{12}+s_{23})^k\right]\right] \\ &= \frac{1}{s_{12}}\exp\left[\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(-1)^k\left[s_{12}^k+s_{23}^k-(s_{12}+s_{23})^k\right]\right] \end{aligned} \tag{4.9}$$

So

$$Z_{4\text{pt}} = \frac{1}{s_{12}}\exp\left[\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(-1)^k\left[s_{12}^k+s_{23}^k-(s_{12}+s_{23})^k\right]\right] \tag{4.10}$$

Now let's take a look at $J_{4\text{pt}}$. We want $J_{4\text{pt}}$ to have similar form as $C(a,b)$ (4.4).

$$\begin{aligned} J_{4\text{pt}} &= \frac{1}{\pi}\int_{\mathbf{C}}\frac{\mathrm{d}^2 z}{z\bar{z}(1-\bar{z})}|z|^{2s_{12}}|1-z|^{2s_{23}} \\ &= \frac{1}{\pi}\int_{\mathbf{C}}\mathrm{d}^2 z\ |z|^{2s_{12}-2}|1-z|^{2s_{23}-2}(1-z). \end{aligned} \tag{4.11}$$

We may change the integrand to some integral with the use of the definition of Gamma function $|z|^{2a-2} = \frac{1}{\Gamma(1-a)}\int_0^\infty dt\ t^{-a}e^{-|z|^2 t}$, $|1-z|^{2b-2} = \frac{1}{\Gamma(1-b)}\int_0^\infty du u^{-b}e^{-|1-z|^2 u}$. For convenience, we will set $s_{12}=a$ $s_{23}=b$.

$$J_{4\text{pt}} = \frac{1}{\pi}\int\frac{d^2z du dt}{\Gamma(1-a)\Gamma(1-b)}t^{-a}u^{-b}e^{-|z|^2 t}e^{-|1-z|^2 u}(1-z) \tag{4.12}$$

If we take $z = x+\mathrm{i}y$, we have

$$\begin{aligned} J_{4\text{pt}} &= \frac{1}{\pi}\int\frac{dxdydudt}{\Gamma(1-a)\Gamma(1-b)}t^{-a}u^{-b}e^{-(t+u)\left(x^2+y^2\right)+2xu-u}(1-z) \\ &= \frac{1}{\pi}\int\frac{dxdydudt}{\Gamma(1-a)\Gamma(1-b)}t^{-a}u^{-b}\exp\left(-(t+u)\left[\left(x-\frac{u}{t+u}\right)^2+y^2\right]-u+\frac{u^2}{t+u}\right)(1-x-\mathrm{i}y) \\ &= \frac{1}{\pi}\int\frac{dxdydudt}{\Gamma(1-a)\Gamma(1-b)}t^{-a}u^{-b}\exp\left(-(t+u)\left[\left(x-\frac{u}{t+u}\right)^2+y^2\right]-u+\frac{u^2}{t+u}\right)(1-\frac{u}{t+u}-\left(x-\frac{u}{t+u}\right)-\mathrm{i}y) \\ &= \frac{1}{\pi}\int\frac{dxdydudt}{\Gamma(1-a)\Gamma(1-b)}t^{-a}u^{-b}\exp\left(-(t+u)\left[\left(x-\frac{u}{t+u}\right)^2+y^2\right]-u+\frac{u^2}{t+u}\right)(\frac{t}{t+u}-\left(x-\frac{u}{t+u}\right)-\mathrm{i}y). \end{aligned} \tag{4.13}$$

The purpose of the above is to cook a vanishing integral like $\int\mathrm{d}x\ \left(x-\frac{u}{t+u}\right)e^{-(t+u)\left(x-\frac{u}{t+u}\right)^2}=0$, $\int\mathrm{d}y\ ye^{-(t+u)y^2}=0$ and also Gaussian integral $\int\mathrm{d}x\ e^{-(t+u)\left(x-\frac{u}{t+u}\right)^2}=\sqrt{\frac{\pi}{t+u}}$, $\int\mathrm{d}y\ e^{-(t+u)y^2}=\sqrt{\frac{\pi}{t+u}}$.

So

$$\begin{aligned} J_{4\text{pt}} &= \frac{1}{\pi}\int\frac{dxdydudt}{\Gamma(1-a)\Gamma(1-b)}t^{-a}u^{-b}\exp\left(-(t+u)\left[\left(x-\frac{u}{t+u}\right)^2+y^2\right]-u+\frac{u^2}{t+u}\right)\frac{t}{t+u}, \\ &= \int\frac{dudt}{\Gamma(1-a)\Gamma(1-b)}t^{-a}u^{-b}\exp\left(-u+\frac{u^2}{t+u}\right)\frac{t}{(t+u)^2}, \\ &= \int\frac{dudt}{\Gamma(1-a)\Gamma(1-b)}t^{-a}u^{-b}\exp\left(-\frac{ut}{t+u}\right)\frac{t}{(t+u)^2}. \end{aligned} \tag{4.14}$$

We then change $(t,u) \to (\alpha,\beta)$ with $t=\alpha\beta, u=\alpha(1-\beta)$,$\alpha\in[0,\infty)$ and $\beta\in[0,1]$. The absolute value for the jacobian determinant is

$$\left|\begin{bmatrix} \frac{\partial t}{\partial\alpha} & \frac{\partial t}{\partial\beta} \\ \frac{\partial u}{\partial\alpha} & \frac{\partial u}{\partial\beta} \end{bmatrix}\right| = \left|\begin{bmatrix} \beta & \alpha \\ (1-\beta) & -\alpha \end{bmatrix}\right| = |-\alpha| = \alpha.$$

So

$$J_{4\text{pt}} = \frac{1}{\Gamma(1-a)\Gamma(1-b)}\int_0^\infty \mathrm{d}\alpha \int_0^1 \mathrm{d}\beta\ \alpha\alpha^{-a}\beta^{-a}\alpha^{-b}(1-\beta)^{-b}e^{-\beta(1-\beta)\alpha}\frac{\alpha\beta}{\alpha^2}. \tag{4.15}$$

Note the integral inside $J_{4\text{pt}}$ which we denote A, can be modified as:

$$\begin{aligned} A &= \int_0^\infty \mathrm{d}\alpha\ \alpha\alpha^{-a}\alpha^{-b}\frac{\alpha}{\alpha^2}e^{-\beta(1-\beta)\alpha}, \\ &= \int_0^\infty \mathrm{d}\alpha\ \alpha^{-a-b}e^{-\beta(1-\beta)\alpha}, \\ &= (\beta(1-\beta))^{a+b-1}\int_0^\infty \mathrm{d}\alpha\beta(1-\beta)\ (\alpha\beta(1-\beta))^{-a-b}e^{-\beta(1-\beta)\alpha}, \\ &= (\beta(1-\beta))^{a+b-1}\int_0^\infty \mathrm{d}\alpha\beta(1-\beta)\ (\alpha\beta(1-\beta))^{1-a-b-1}e^{-\beta(1-\beta)\alpha}, \\ &= (\beta(1-\beta))^{a+b-1}\Gamma(1-a-b). \end{aligned} \tag{4.16}$$

So

$$\begin{aligned} J_{4\text{pt}} &= \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)}\int_0^1 \mathrm{d}\beta\ \beta^{1-a}(1-\beta)^{-b}(\beta(1-\beta))^{a+b-1}, \\ &= \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)}\int_0^1 \mathrm{d}\beta\ \beta^{b+1-1}(1-\beta)^{a-1}, \\ &= \frac{\Gamma(1-a-b)\Gamma(1+b)\Gamma(a)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1+a+b)}. \end{aligned} \tag{4.17}$$

Note: if we have $c=-a-b$ then

$$J_{4\text{pt}} = \frac{\Gamma(1+c)\Gamma(1+b)\Gamma(a)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)}. \tag{4.18}$$

From momentum conservation, $k_1+k_2+k_3+k_4=0$, we have $k_4^2=0=2(k_1\cdot k_2+k_1\cdot k_3+k_2\cdot k_3)$, so $s_{12}+s_{13}+s_{23}=0$. Thus, the kinematic factor has the correspondence with a,b,c in (4.18). So we write the kinematic factors $s_{12}=a$ $s_{23}=b$, and then follow the same step of (4.9). We get

$$\begin{aligned} J_{4\text{pt}} &= \frac{\Gamma(1+s_{13})\Gamma(1+s_{23})\Gamma(s_{12})}{\Gamma(1-s_{12})\Gamma(1-s_{23})\Gamma(1-s_{13})}, \\ &= \frac{1}{s_{12}}\frac{\Gamma(1+s_{13})\Gamma(1+s_{23})\Gamma(1+s_{12})}{\Gamma(1-s_{12})\Gamma(1-s_{23})\Gamma(1-s_{13})}, \\ &= \frac{1}{s_{12}}\exp\left[-\gamma(s_{12}+s_{23}+s_{13}+s_{12}+s_{23}+s_{13})+\sum_{k=2}^\infty\frac{\zeta(k)}{k}\left((-1)^k({s_{12}}^k+{s_{23}}^k+{s_{13}}^k)-({s_{12}}^k+{s_{23}}^k+{s_{13}}^k)\right)\right], \\ &= \frac{1}{s_{12}}\exp\left[\sum_{k=1}^\infty\frac{\zeta(2k+1)}{2k+1}\left((-1)({s_{12}}^{2k+1}+{s_{23}}^{2k+1}+{s_{13}}^{2k+1})-({s_{12}}^{2k+1}+{s_{23}}^{2k+1}+{s_{13}}^{2k+1})\right)\right], \\ &= \frac{1}{s_{12}}\exp\left[\sum_{k=1}^\infty\frac{\zeta(2k+1)}{2k+1}\left(-2({s_{12}}^{2k+1}+{s_{23}}^{2k+1}+{s_{13}}^{2k+1})\right)\right], \\ &= \frac{1}{s_{12}}\exp\left[\sum_{k=1}^\infty\frac{\zeta(2k+1)}{2k+1}\left(-2({s_{12}}^{2k+1}+{s_{23}}^{2k+1}-(s_{12}+s_{23})^{2k+1})\right)\right]. \end{aligned} \tag{4.19}$$

So

$$J_{4\text{pt}} = \frac{1}{s_{12}} \exp\left[\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2k+1}\left(-2({s_{12}}^{2k+1} + {s_{23}}^{2k+1} - (s_{12}+s_{23})^{2k+1})\right)\right]. \tag{4.20}$$

As we know, the sv map for Riemann zeta values is[19]:

$$.\zeta_{\text{sv}}(2k) = 0, \quad \zeta_{\text{sv}}(2k+1) = 2\zeta(2k+1) \tag{4.21}$$

So form (4.10) and (4.21)

$$\begin{aligned} \text{sv } Z_{4\text{pt}} &= \frac{1}{s_{12}} \exp\left[\text{sv } \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}(-1)^k \left[s_{12}^k + s_{23}^k - (s_{12}+s_{23})^k\right]\right], \\ &= \frac{1}{s_{12}} \exp\left[\sum_{k=1}^{\infty} \frac{2\zeta(2k+1)}{2k+1}(-1)\left[s_{12}^{2k+1} + s_{23}^{2k+1} - (s_{12}+s_{23})^{2k+1}\right]\right]. \end{aligned} \tag{4.22}$$

Compare (4.22) with (4.20),

$$\text{sv } Z_{4\text{pt}} = J_{4\text{pt}}. \tag{4.23}$$

For those who may have the interest, there is a general proof of $\text{sv}Z = J$ in section 3.2 and section 3.3 of [19].

5 Single value map for 5 points

5.1 Hypergeometric function and scattering amplitude

The five-point tree-level open string scattering amplitude and tree-level closed string scattering amplitude is calculated in [18]. Here we would show how the 5-point disk amplitude and the 6-point are related via the single value map. Before we proceed, we would first introduce some formula related to hypergeometric function[12]. A hypergeometric function of parameter $a_1 ... a_p$, $b_1, ... b_q$ is defined as

Definition 5.1.

$${}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \cdot \frac{z^n}{n!}. \tag{5.1}$$

where $(a)_n$ means $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$ and $(a)_0 = 1$.

We could also have an integral representation: for Re $c >$ Re $b > 0$, hypergeometric function

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt. \tag{5.2}$$

And the recursion relation:

$$\begin{aligned} &{}_{p+1}F_{q+1}\left(\begin{matrix} a_1, a_2, \ldots, a_p, a_{p+1} \\ b_1, b_2, \ldots, b_q, b_{q+1} \end{matrix}; z\right) = \\ &\frac{\Gamma(b_{q+1})}{\Gamma(a_{p+1})\Gamma(b_{q+1}-a_{p+1})} \int_0^1 t^{a_{p+1}-1}(1-t)^{b_{q+1}-a_{p+1}-1} {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; zt\right) dt, \end{aligned} \tag{5.3}$$

for $\text{Re}\, b_{q+1} > \text{Re}\, a_{p+1} > 0$.

Let's go back to our disk integral. A general disk integral with colour indices τ and ρ is given by [19]

$$Z(\tau \mid \rho) := \int_{-\infty \le z_{\tau(1)} \le z_{\tau(2)} \le \ldots \le z_{\tau(n)} \le \infty} \frac{\mathrm{d}z_1 \mathrm{d}z_2 \ldots \mathrm{d}z_n}{\text{volSL}_2(\mathbb{R})} \frac{(-1)^{n-3} \prod_{1 \le i < j \le n} |z_i - z_j|^{s_{ij}}}{z_{\rho(1),\rho(2)} z_{\rho(2),\rho(3)} \cdots z_{\rho(n-1),\rho(n)} z_{\rho(n),\rho(1)}}, \tag{5.4}$$

where n is the number of external points. To fix the gauge freedom $\mathrm{volSL}_2(\mathbb{R})$, we have to set $z_{\tau(1)}=0, z_{\tau(n-1)}=1, z_{\tau(n)}=\infty$ and also inserting $|(z_{\tau(1)}-z_{\tau(n-1)})(z_{\tau(1)}-z_{\tau(n)})(z_{\tau(n-1)}-z_{\tau(n)})|$ in the numerator. So

$$
\begin{aligned}
&Z(1,2,3,4,5|1,2,5,3,4)\\
=&\lim_{z_5\to\infty}\int_{0\le z_2\le z_3\le 1}\mathrm{d}z_2\mathrm{d}z_3\frac{|(0-1)(0-z_5)(1-z_5)|\left(\begin{array}{c} z_2^{s_{12}}z_3^{s_{13}}|0-1|^{s_{14}}|z_2-z_3|^{s_{23}}|z_2-1|^{s_{24}}|z_3-1|^{s_{34}}\\ |z_1-z_5|^{s_{15}}|z_2-z_5|^{s_{25}}|z_3-z_5|^{s_{35}}|1-z_5|^{s_{45}}\end{array}\right)}{(-z_2)(z_2-z_5)(z_5-z_3)(z_3-1)(1-0)}\\
=&\int_0^1\mathrm{d}z_3\int_0^{z_3}\mathrm{d}z_2\frac{\infty^2 z_2^{s_{12}}z_3^{s13}(z_3-z_2)^{s_{23}}(1-z_2)^{s_{24}}(1-z_3)^{s_{34}}\infty^{s_{15}+s_{25}+s_{35}+s_{45}}}{(z_2)\infty^2(z_3-1)(1-0)}.
\end{aligned}
\tag{5.5}
$$

Note that from identity for Mandelstam variables (4.2), $\infty^{s_{15}+s_{25}+s_{35}+s_{45}}=\infty^0=1$, so there is no divergent in the integrand. With this,

$$
\begin{aligned}
Z(1,2,3,4,5\mid 1,2,5,3,4)&=\int_0^1\mathrm{d}z_3\int_0^{z_3}\mathrm{d}z_2\frac{z_2^{s_{12}}z_3^{s_{13}}(z_3-z_2)^{s_{23}}\,(1-z_2)^{s_{24}}\,(1-z_3)^{s_{34}}}{z_2\,(z_3-1)}\\
&=-\int_0^1\mathrm{d}z_3\int_0^1 z_3\mathrm{d}u\frac{u^{s_{12}}z_3^{s_{12}}z_3^{s_{13}}z_3^{s_{23}}\,(1-u)^{s_{23}}\,(1-z_3u)^{s_{24}}\,(1-z_3)^{s_{34}}}{z_3u\,(1-z_3)},
\end{aligned}
\tag{5.6}
$$

where for the second equation, we have changed the variable $z_2\to u:\ z_2=z_3u$, trying to construct an integration of hyper geometrical function like the one in (5.2). With this in mind,

$$
\begin{aligned}
Z(1,2,3,4,5\mid 1,2,5,3,4)&=-\int_0^1\mathrm{d}z_3\int_0^1\mathrm{d}u z_3^{s_{12}+s_{13}+s_{23}}(1-z_3)^{s_{34}-1}u^{s_{12}-1}(1-u)^{s_{23}}(1-uz_3)^{s_{24}},\\
&=-\int_0^1\mathrm{d}z_3\ z_3^{s_{12}+s_{13}+s_{23}}(1-z_3)^{s_{34}-1}\int_0^1\mathrm{d}u\ u^{s_{12}-1}(1-u)^{s_{23}}(1-uz_3)^{s_{24}}.
\end{aligned}
\tag{5.7}
$$

From observation, $b=s_{12}$ $c=s_{23}+s_{12}+1$, so we write:

$$
Z(1,2,3,4,5\mid 1,2,5,3,4)=-\int_0^1\mathrm{d}z_3\ z_3^{s_{12}+s_{13}+s_{23}}(1-z_3)^{s_{34}-1}\frac{\Gamma(s_{12})\Gamma(s_{23}+1)}{\Gamma(s_{23}+s_{12}+1)}{}_2F_1\left(\begin{array}{c}-s_{24},s_{12}\\ s_{12}+s_{23}+1\end{array};z_3\right).
\tag{5.8}
$$

Then comparing (5.8) with the recursive relation of hyper geometric function (5.3), we have $a_3=s_{12}+s_{13}+s_{23}+1,\ b_2=s_{12}+s_{13}+s_{23}+s_{34}+1$. So

$$
\begin{aligned}
Z(1,2,3,4,5\mid 1,2,5,3,4)=&-\frac{\Gamma(s_{12})\Gamma(s_{23}+1)\Gamma(s_{12}+s_{13}+s_{23}+1)\Gamma(s_{34})}{\Gamma(s_{23}+s_{12}+1)\Gamma(s_{12}+s_{13}+s_{23}+s_{34}+1)}\\
&\times{}_3F_2\left(\begin{array}{c}-s_{24},s_{12},s_{12}+s_{13}+s_{23}+1\\ s_{12}+s_{23}+1,s_{12}+s_{13}+s_{23}+s_{34}+1\end{array};z_3\right).
\end{aligned}
\tag{5.9}
$$

End of calculation

We may use the above five-point open amplitude. But currently the five-point open amplitude we are working on is in [18]:

Definition 5.2.

$$
\mathcal{A}(1,\ldots,N)=\sum_{\sigma\in S_{N-3}}A_{YM}\left(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N\right)F^{\sigma}_{(1,\ldots,N)}\left(s_{ij}\right)
\tag{5.10}
$$

where σ the permutation group element of S_{N-2}[18]. E.g.

$$
F^{(23\ldots N-2)}_{(1,\ldots,N)}\left(s_{ij}\right)=(-1)^{N-3}\int_{z_i<z_{i+1}}^{N-2}\prod_{j=2}^{N-2}dz_j\left(\prod_{i<l}|z_{il}|^{s_{il}}\right)\left\{\left(\prod_{k=2}^{[N/2]}\sum_{m=1}^{k-1}\frac{s_{mk}}{z_{mk}}\right)\left(\prod_{k=[N/2]+1}^{N-2}\sum_{n=k+1}^{N-1}\frac{s_{kn}}{z_{kn}}\right)\right\},
\tag{5.11}
$$

and other permutations are just swapping indices.

For five points amplitude, according to (5.10), one may have

$$\begin{aligned}&\mathcal{A}(1,2,3,4,5)=A_{YM}(1,2,3,4,5)F^{(23)}_{(1,2,3,4,5)}+A_{YM}(1,3,2,4,5)F^{(32)}_{(1,2,3,4,5)}\\&\mathcal{A}(1,3,2,4,5)=\mathcal{A}(1,2,3,4,5)|_{2\iff 3}.\end{aligned}\tag{5.12}$$

Although (5.11) and (5.4) are different, one may arrive at similar integral expression for $F^{(23)}_{(1,2,3,4,5)}$ and $F^{(32)}_{(1,2,3,4,5)}$ via (5.11) and Euler or Selberg integrals[18]. Below integrals are just as (5.7).

$$F^{(23)}_{(1,2,3,4,5)} \;=\; s_{12}s_{34}\int_0^1 dx\int_0^1 dy x^{s_{45}}y^{s_{12}-1}(1-x)^{s_{34}-1}(1-y)^{s_{23}}(1-xy)^{s_{24}},\tag{5.13}$$

$$F^{(32)}_{(1,2,3,4,5)}=s_{13}s_{24}\int_0^1 dx\int_0^1 dy x^{s_{45}}y^{s_{12}}(1-x)^{s_{34}}(1-y)^{s_{23}}(1-xy)^{s_{24}-1}.\tag{5.14}$$

From procedure (5.7) to (5.9), we are able to arrive at

results

$$F^{(32)}_{(1,2,3,4,5)} \;=\; \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})\Gamma(1+s_{34})\Gamma(1+s_{45})}{\Gamma(1+s_{12}+s_{23})\Gamma(1+s_{34}+s_{45})}{}_3F_2\left[\begin{matrix}-s_{24},s_{12},1+s_{45}\\1+s_{12}+s_{23},1+s_{34}+s_{45}\end{matrix};1\right],\tag{5.15}$$

$$F^{(32)}_{(1,2,3,4,5)} \;=\; s_{13}s_{24}\frac{\Gamma(1+s_{12})\Gamma(1+s_{23})\Gamma(1+s_{34})\Gamma(1+s_{45})}{\Gamma(2+s_{12}+s_{23})\Gamma(2+s_{34}+s_{45})}{}_3F_2\left[\begin{matrix}1-s_{24},1+s_{12},1+s_{45}\\2+s_{12}+s_{23},2+s_{34}+s_{45}\end{matrix};1\right].\tag{5.16}$$

So with (5.15)(5.16)(5.12), we have an analytic 5 point disk amplitude $\mathcal{A}(1,2,3,4,5)$ and $\mathcal{A}(1,3,2,4,5)$.

$$\begin{aligned}&\mathcal{A}(1,2,3,4,5)=A_{YM}(1,2,3,4,5)\tfrac{\Gamma(1+s_{12})\Gamma(1+s_{23})\Gamma(1+s_{34})\Gamma(1+s_{45})}{\Gamma(1+s_{12}+s_{23})\Gamma(1+s_{34}+s_{45})}{}_3F_2\left[\begin{matrix}-s_{24},s_{12},1+s_{45}\\1+s_{12}+s_{23},1+s_{34}+s_{45}\end{matrix};1\right]\\&+A_{YM}(1,3,2,4,5)s_{13}s_{24}\tfrac{\Gamma(1+s_{12})\Gamma(1+s_{23})\Gamma(1+s_{34})\Gamma(1+s_{45})}{\Gamma(2+s_{12}+s_{23})\Gamma(2+s_{34}+s_{45})}{}_3F_2\left[\begin{matrix}1-s_{24},1+s_{12},1+s_{45}\\2+s_{12}+s_{23},2+s_{34}+s_{45}\end{matrix};1\right],\\&\mathcal{A}(1,3,2,4,5)=\mathcal{A}(1,2,3,4,5)|_{2\iff 3}.\end{aligned}\tag{5.17}$$

One can have a more compact form (useful in the next subsection):

$$\mathcal{A}_{5pt}=\left(\begin{matrix}\mathcal{A}(1,2,3,4,5)\\\mathcal{A}(1,3,2,4,5)\end{matrix}\right)=\left(\begin{matrix}F^{(23)}_{(1,2,3,4,5)} & F^{(32)}_{(1,2,3,4,5)}\\F^{(23)}_{(1,3,2,4,5)} & F^{(32)}_{(1,3,2,4,5)}\end{matrix}\right)\left(\begin{matrix}A_{YM}(1,2,3,4,5)\\A_{YM}(1,3,2,4,5)\end{matrix}\right).\tag{5.18}$$

End of results

In principle, performing single value map on amplitude $\mathcal{A}(1,2,3,4,5),\mathcal{A}(1,3,2,4,5)$ is a problem of $\mathrm{sv}F^{(23)}_{(1,2,3,4,5)}$ and $\mathrm{sv}F^{(32)}_{(1,2,3,4,5)}$. To do so, we need a formula as $\log\Gamma(1+x)=-\gamma x+\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(-x)^k$ in four point case, making MZVs manifest. Unfortunately, we don't have one for hypergeometric function. But there is an "intriguing observation" given in [18], which we will mention in next subsection.

5.2 Single value map on five-point disk amplitude

The 5 point single-valued open string tree-level scattering amplitude may appear to be a basis to expand hard part of a 2 to 2 partonic scattering amplitude in Regge limit, which is the problem we want to study in this paper. There is another advantage of doing the single value map: one may only compute the 5 point disk integral in string theory and the do the single value map to obtain a spherical one without calculating the complicated spherical integral. Let's now continue our calculation.We have calculated in Subsection 5.1, the compact form of five-point disk amplitude $\mathcal{A}_{5pt}$ (5.18). The "intriguing observation" is that the compact form can be expanded as:

$$\mathcal{A}_{5pt}=\mathbf{PQ}:\exp\left\{\sum_{n\geq 1}\zeta_{2n+1}\mathbf{M}_{2n+1}\right\}:\mathbf{A}_{YM},\tag{5.19}$$

in which MZVs are manifest. Note: $\mathbf{A}_{YM}=(A_{YM}(1,2,3,4,5),A_{YM}(1,3,2,4,5))^T$ is an irrelevant constant vector of single value map. The rest of $\mathcal{A}_{5pt}$ in (5.19) has three main part we should focus: $\mathbf{P}$, $\mathbf{Q}$ and $:\exp\left\{\sum_{n\geq 1}\zeta_{2n+1}\mathbf{M}_{2n+1}\right\}:$.

Each is a 2×2 matrix and the "normal order" we mean put the greatest index of $\mathbf{M}$ in the front. The expansion below will be better illustrate the above description.

$$: \exp \left\{ \sum_{n \geq 1} \zeta_{2n+1} \mathbf{M}_{2n+1} \right\} := \mathbf{1} + \zeta_3 \mathbf{M}_3 + \zeta_5 \mathbf{M}_5 + \zeta_3^2 \mathbf{M}_3^2 + \zeta_7 \mathbf{M}_7 + \zeta_3 \zeta_5 \mathbf{M}_5 \mathbf{M}_3 ..., \tag{5.20}$$

$$\mathbf{P} = \mathbf{1} + \sum_{n \geq 1} \zeta_2^n \mathbf{P}_{2n},$$

$$\mathbf{Q} = \mathbf{1} + \mathbf{Q}_8 + \mathbf{Q}_9 + \mathbf{Q}_{10} + \mathbf{Q}_{11} + \mathbf{Q}_{13} + ..., \tag{5.21}$$

The detailed value of $\mathbf{Q}$ is given in [18].

The single value map or sv in this chapter are the motivic version described in Section 3. We should mention here that the motivic single value map acting on specify MZV will also preserve multiplication:

$$\text{sv } \zeta_{n_1,n_2,...,n_k} \zeta_{m_1,m_2,...,m_l} = \text{sv } \zeta_{n_1,n_2,...,n_k} \text{ sv } \zeta_{m_1,m_2,...,m_l}. \tag{5.22}$$

So the single value map on $\mathcal{A}$ is

$$\text{sv } \mathcal{A}_{5pt} = (\text{sv } \mathbf{P}) \ (\text{sv } \mathbf{Q}) \left(\text{sv } : \exp \left\{ \sum_{n \geq 1} \zeta_{2n+1} \mathbf{M}_{2n+1} \right\} : \right) \mathbf{A}_{YM}. \tag{5.23}$$

In the below calculation, we will just omit $\mathbf{A}_{YM}$, but will recover it in the very last end.

5.3 Single value map on P Q and the Exponential term

Single valued P to all orders:

As $\mathbf{P}$ part only contains $\mathbf{1}$ and multiples of ζ_2, the single valued $\mathbf{P}$ is unit matrix $\mathbf{1}$.

$$\text{sv } \mathbf{P} = \mathbf{1} + \sum_{n \geq 1} \text{sv } \zeta_2^n \mathbf{P}_{2n} = \mathbf{1} \tag{5.24}$$

Single valued Exponential term to order 11:

$$\begin{aligned} \text{sv} : \exp \left\{ \sum_{n \geq 1} \zeta_{2n+1} \mathbf{M}_{2n+1} \right\} : \ &= \ \mathbf{1} + 2\mathbf{M}_3 \zeta_3 + 2\mathbf{M}_5 \zeta_5 + 2\mathbf{M}_3.\mathbf{M}_3 \zeta_3^2 + 2\mathbf{M}_7 \zeta_7 + 4\zeta_3 \zeta_5 \mathbf{M}_5.\mathbf{M}_3 \\ &+ \ \frac{4}{3} \mathbf{M}_3.\mathbf{M}_3.\mathbf{M}_3 \zeta_3^3 + 2\mathbf{M}_9 \zeta_9 + 4\zeta_3 \zeta_7 \mathbf{M}_7.\mathbf{M}_3 + 2\mathbf{M}_5.\mathbf{M}_5 \zeta_5^2 + 2\mathbf{M}_{11} \zeta_1 1 \\ &+ \ 4\zeta_3^2 \zeta_5 \mathbf{M}_5.\mathbf{M}_3.\mathbf{M}_3 + O(12). \end{aligned} \tag{5.25}$$

Calculations of sv Q to order 11

The most non-trivial bit of calculation is the single value map on $\mathbf{Q}$ where some complicated MZVs are presented. We will first give the result of the single value map and then state the details on how to get it.

Calculation of sv$\mathbf{Q}_8$:

$$\begin{aligned} \mathbf{Q}_8 \ &= \ \frac{1}{5} \zeta_{5,3} \left[\mathbf{M}_5, \mathbf{M}_3 \right], \\ \text{sv } \mathbf{Q}_8 \ &= \ \frac{1}{5} \left(\text{sv } \zeta_{5,3} \right) \left[\mathbf{M}_5, \mathbf{M}_3 \right] = -2\zeta_3 \zeta_5 \left[\mathbf{M}_5, \mathbf{M}_3 \right]. \end{aligned} \tag{5.26}$$

Here as one may see, we have to do the single value map on $\zeta_{5,3}$. As we described in Subsection 3.1, there is a one-to-one correspondence between $\zeta^m_{...} \leftrightarrow \zeta_{...} \leftrightarrow f_i...f_j$. So from [5] e.q. (6.2),

$$\begin{aligned}
\phi\left(\zeta^m_{5,3}\right) &= -5f_5f_3 \\
\mathrm{sv}\phi\left(\zeta^m_{5,3}\right) &= -5 \text{ III } f_3f_5 - 5f_5 \text{ III } f_3 - 5f_5f_3 \text{ III} \\
&= -5\times(2f_3f_5 + 2f_5f_3) \\
&= -10f_3 \text{ III } f_5
\end{aligned} \tag{5.27}$$

From Subsection 3.1, we know that (5.27) has already encoded the information for sv$\zeta_{5,3}$. Also from *Remark* 3.1, the shuffle of words in $\mathcal{F}$ corresponds to the produce of two MZVs:f_3 III $f_5 \leftrightarrow \zeta_3\zeta_5$, so

$$\mathrm{sv}\zeta_{5,3} = -10\zeta_3\zeta_5. \tag{5.28}$$

End of calculation of sv$\mathbf{Q}_8$:

Calculation of sv$\mathbf{Q}_9$:

$$\mathbf{Q}_9 = 0, \text{ so } \mathrm{sv}\mathbf{Q}_9 = 0. \tag{5.29}$$

End of calculation of sv$\mathbf{Q}_9$

Calculation of sv$\mathbf{Q}_{10}$:

$$\begin{aligned}
\mathbf{Q}_{10} &= \left\{\frac{3}{14}\zeta_5^2 + \frac{1}{14}\zeta_{7,3}\right\}[\mathbf{M}_7, \mathbf{M}_3], \\
\mathrm{sv}\ \mathbf{Q}_{10} &= \left\{\frac{3}{14}\left(\mathrm{sv}\ \zeta_5^2\right) + \frac{1}{14}\left(\mathrm{sv}\ \zeta_{7,3}\right)\right\}[\mathbf{M}_7, \mathbf{M}_3], \\
&= \left\{\frac{12}{14}\zeta_5^2 - \left(\frac{12}{14}\zeta_5^2 + 2\zeta_3\zeta_7\right)\right\}[\mathbf{M}_7, \mathbf{M}_3], \\
&= -2\zeta_3\zeta_7\,[\mathbf{M}_7, \mathbf{M}_3].
\end{aligned} \tag{5.30}$$

The detailed calculation of single valued $\zeta_{7,3}$ is given here. From [5] e.q. (6.3),

$$\phi\left(\zeta^m_{7,3}\right) = -14f_7f_3 - 6f_5f_5.$$

$$\begin{aligned}
\mathrm{sv}f_7f_3 &= \text{ III } f_3f_7 + f_7 \text{ III } f_3 + f_7f_3 \text{ III } = 2(f_3f_7 + f_7f_3) = 2f_3 \text{ III } f_7, \\
\text{Similarly } \mathrm{sv}f_5f_5 &= 2(f_5f_5 + f_5f_5) = 2f_5 \text{ III } f_5.
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{sv}\phi\left(\zeta^m_{7,3}\right) &= -28f_7 \text{ III } f_3 - 12f_5 \text{ III } f_5, \\
\mathrm{sv}\zeta_{7,3} &= -28\zeta_7\zeta_3 - 12\zeta_5^2
\end{aligned} \tag{5.31}$$

End of calculation of $\mathbf{Q}_{10}$

If we encounter some more sv $\zeta_{n,m}$ in the subsequent calculation, we will only offer the result directly.

Calculation of sv$\mathbf{Q}_{11}$:

$$\mathbf{Q}_{11} = \left\{9\zeta_2\zeta_9 + \frac{6}{25}\zeta_2^2\zeta_7 - \frac{4}{35}\zeta_2^3\zeta_5 + \frac{1}{5}\zeta_{5,3,3}\right\}[\mathbf{M}_3, [\mathbf{M}_5, \mathbf{M}_3]], \tag{5.32}$$

As all the first 3 terms have ζ_2 which vanishes after the map, we will only have

$$\mathrm{sv}\ \mathbf{Q}_{11} = \frac{1}{5}\mathrm{sv}\ \zeta_{5,3,3}\,[\mathbf{M}_3, [\mathbf{M}_5.\mathbf{M}_3]]. \tag{5.33}$$

As given in e.q.(4.28) in [18],

$$\phi\left(\zeta^m_{5,3,3}\right) = -\frac{5}{2} f_5\left(f_3 \text{III} f_3\right) + \frac{4}{7} f_5 f_2^3 - \frac{6}{5} f_7 f_2^2 - 45 f_9 f_2, \tag{5.34}$$

Here we have f_2 in or expression. As it corresponds to ζ_2, we will have sv...f_2... $= 0$, (see below).

$$\begin{aligned} \text{sv}\left(-\frac{5}{2} f_5\left(f_3 \text{III} f_3\right) + \frac{4}{7} f_5 f_2^3 - \frac{6}{5} f_7 f_2^2 - 45 f_9 f_2\right) &= \text{sv}\left(-\frac{5}{2} f_5\left(f_3 \text{III} f_3\right)\right), \\ &= -10\left(f_3 f_3 f_5 + f_3 f_5 f_3 + 2 f_5 f_3 f_3\right), \\ &= -5 f_5 \text{ III } f_3 \text{ III } f_3 - 5 f_5\left(f_3 \text{ III } f_3\right). \end{aligned} \tag{5.35}$$

The first term in (5.35) is easily identified with $\zeta_5\zeta_3^2$.

The second term will go into the $2\mathbf{Q}_{11}$. See below. We just copy equation (5.35) and add those ...f_2... terms back which we have erased due to sv map.

$$\begin{aligned} \text{sv}\left(-\frac{5}{2} f_5\left(f_3 \text{III} f_3\right)\right) &= -5 f_5 \text{ III } f_3 \text{ III } f_3 - 5 f_5\left(f_3 \text{ III } f_3\right) + 2\left(\frac{4}{7} f_5 f_2^3 - \frac{6}{5} f_7 f_2^2 - 45 f_9 f_2\right) \\ &\quad - 2\left(\frac{4}{7} f_5 f_2^3 - \frac{6}{5} f_7 f_2^2 - 45 f_9 f_2\right). \end{aligned} \tag{5.36}$$

As we know in subsection 3.1, the trivialization ϕ is an isomorphism.

We can consider the inversion map ϕ^{-1} to get MZVs expression based on (5.36). $\phi^{-1}(f_2) = \zeta_2^m$, $\phi^{-1}(f_i) = \zeta_i^m$, for $i = 2k+1,\ k = 1, 2, 3, \ldots$. We also have

$$\phi^{-1}\left(-\frac{5}{2} f_5\left(f_3 \text{III} f_3\right) + \frac{4}{7} f_5 f_2^3 - \frac{6}{5} f_7 f_2^2 - 45 f_9 f_2\right) = \zeta^m_{5,3,3}. \tag{5.37}$$

So one may see what we want to do in (5.36): we try to cook an MZV part in $\mathbf{Q}_{11}$ (something we know already). As you can see the last 4 terms on the first line of (5.36), they are just like 2×(5.34).

All in all, we have,

$$\text{sv}\zeta^m_{5,3,3} = -5\zeta_5^m\zeta_3^m\zeta_3^m + 2\zeta^m_{5,3,3} - \frac{8}{7}\zeta_5^m(\zeta_2^m)^3 + \frac{12}{5}\zeta_7^m(\zeta_2^m)^2 + 90\zeta_9^m\zeta_2^m. \tag{5.38}$$

The first term comes from f_5 III f_3 III f_3. And the second term comes from the rest of the first line of (5.36). The second line of (5.36) corresponds to the last 3 terms of (5.38).

The procedure for $\mathbf{Q}_{12}$ and $\mathbf{Q}_{13}$ are almost the same, except that there is a special term in $\mathbf{Q}_{13}$ that needs special care. Multiplying (5.38) by $\frac{1}{5}$, we have

$$\text{sv}\frac{1}{5}\zeta^m_{5,3,3} = -\zeta_5^m\zeta_3^m\zeta_3^m + \frac{2}{5}\zeta^m_{5,3,3} - \frac{8}{35}\zeta_5^m(\zeta_2^m)^3 + \frac{12}{25}\zeta_7^m(\zeta_2^m)^2 + 18\zeta_9^m\zeta_2^m. \tag{5.39}$$

Compare (5.32) and (5.39),

$$\begin{aligned} \text{sv } \mathbf{Q}_{11} &= \frac{1}{5}\text{sv } \zeta_{5,3,3}\left[\mathbf{M}_3, [\mathbf{M}_5.\mathbf{M}_3]\right], \\ &= 2\mathbf{Q}_{11} - \zeta_5^m\zeta_3^m\zeta_3^m\left[\mathbf{M}_3, [\mathbf{M}_5.\mathbf{M}_3]\right]. \end{aligned} \tag{5.40}$$

End of calculation of $\mathbf{Q}_{11}$

Calculation of sv$\mathcal{A}_{5pt}$, 11th order.

Now we can expand our sv$\mathcal{A}_{5pt}$ to order 11: substituting (5.24),(5.25),(5.30)(5.40), we have

$$\begin{aligned}
\text{sv}\mathcal{A}_{5pt} &= \text{sv } \mathbf{P} \text{ sv } \left(1+\mathbf{Q}_8+\mathbf{Q}_9+\mathbf{Q}_{10}+\mathbf{Q}_{11}\right) : \exp\left\{\sum_{n\geq 1}\text{sv}\zeta_{2n+1}\mathbf{M}_{2n+1}\right\} :, \\
&= \left\{\begin{array}{l} \left(1-2\zeta_3\zeta_5\left[\mathbf{M}_5,\mathbf{M}_3\right]-2\zeta_3\zeta_7\left[\mathbf{M}_7,\mathbf{M}_3\right]+2\mathbf{Q}_{11}-\zeta_5^m\zeta_3^m\zeta_3^m\left[\mathbf{M}_3,\left[\mathbf{M}_5.\mathbf{M}_3\right]\right]\right)\times \\ \left(\begin{array}{l} 1+2\mathbf{M}_3\zeta_3+2\mathbf{M}_5\zeta_5+2\mathbf{M}_3\cdot\mathbf{M}_3\zeta_3^2+2\mathbf{M}_7\zeta_7+4\zeta_3\zeta_5\mathbf{M}_5\cdot\mathbf{M}_3 \\ +\quad \frac{4}{3}\mathbf{M}_3\cdot\mathbf{M}_3\cdot\mathbf{M}_3\zeta_3^3+2\mathbf{M}_9\zeta_9+4\zeta_3\zeta_7\mathbf{M}_7\cdot\mathbf{M}_3+2\mathbf{M}_5\cdot\mathbf{M}_5\zeta_5^2 \\ +\quad 4\zeta_3^2\zeta_5\mathbf{M}_5\cdot\mathbf{M}_3\cdot\mathbf{M}_3+2\mathbf{M}_{11}\zeta_{11}\end{array}\right)\end{array}\right\}, \\
&= \left\{\begin{array}{r} 1+2\mathbf{M}_3\zeta_3+2\mathbf{M}_5\zeta_5+2\mathbf{M}_3\cdot\mathbf{M}_3\zeta_3^2+2\mathbf{M}_7\zeta_7+2\zeta_3\zeta_5\left\{\mathbf{M}_5,\mathbf{M}_3\right\} \\ +\frac{4}{3}\mathbf{M}_3\cdot\mathbf{M}_3\cdot\mathbf{M}_3\zeta_3^3+2\mathbf{M}_9\zeta_9+2\zeta_3\zeta_7\left\{\mathbf{M}_7,\mathbf{M}_3\right\}+2\mathbf{M}_5\cdot\mathbf{M}_5\zeta_5^2+2\mathbf{Q}_{11} \\ -4\zeta_3\zeta_5\left[\mathbf{M}_5,\mathbf{M}_3\right]\mathbf{M}_3\zeta_3-\zeta_5\zeta_3\zeta_3\left[\mathbf{M}_3,\left[\mathbf{M}_5.\mathbf{M}_3\right]\right]+\quad 4\zeta_3^2\zeta_5\mathbf{M}_5\cdot\mathbf{M}_3\cdot\mathbf{M}_3+2\mathbf{M}_{11}\zeta_{11}\end{array}\right\}, \\
&= \left\{\begin{array}{r} 1+2\mathbf{M}_3\zeta_3+2\mathbf{M}_5\zeta_5+2\mathbf{M}_3\cdot\mathbf{M}_3\zeta_3^2+2\mathbf{M}_7\zeta_7+2\zeta_3\zeta_5\left\{\mathbf{M}_5,\mathbf{M}_3\right\} \\ +\frac{4}{3}\mathbf{M}_3\cdot\mathbf{M}_3\cdot\mathbf{M}_3\zeta_3^3+2\mathbf{M}_9\zeta_9+2\zeta_3\zeta_7\left\{\mathbf{M}_7,\mathbf{M}_3\right\}+2\mathbf{M}_5\cdot\mathbf{M}_5\zeta_5^2+2\mathbf{Q}_{11} \\ 4\zeta_3\zeta_5\mathbf{M}_3\mathbf{M}_5\mathbf{M}_3\zeta_3+2\mathbf{M}_{11}\zeta_{11}-\zeta_5\zeta_3\zeta_3\left\{2\mathbf{M}_3\mathbf{M}_5\mathbf{M}_3-\mathbf{M}_3\mathbf{M}_3\mathbf{M}_5-\mathbf{M}_5\mathbf{M}_3\mathbf{M}_3\right\}\end{array}\right\}, \\
&= \left\{\begin{array}{c} 1+2\mathbf{M}_3\zeta_3+2\mathbf{M}_5\zeta_5+2\mathbf{M}_3\cdot\mathbf{M}_3\zeta_3^2+2\mathbf{M}_7\zeta_7+2\zeta_3\zeta_5\left\{\mathbf{M}_5,\mathbf{M}_3\right\} \\ +\frac{4}{3}\mathbf{M}_3\cdot\mathbf{M}_3\cdot\mathbf{M}_3\zeta_3^3+2\mathbf{M}_9\zeta_9+2\zeta_3\zeta_7\left\{\mathbf{M}_7,\mathbf{M}_3\right\}+2\mathbf{M}_5\cdot\mathbf{M}_5\zeta_5^2+2\mathbf{Q}_{11} \\ 2\mathbf{M}_{11}\zeta_{11}+\zeta_5\zeta_3\zeta_3\left\{2\mathbf{M}_3\mathbf{M}_5\mathbf{M}_3+\mathbf{M}_3\mathbf{M}_3\mathbf{M}_5+\mathbf{M}_5\mathbf{M}_3\mathbf{M}_3\right\}\end{array}\right\}, \\
&= \left\{\begin{array}{c} 1+2\mathbf{M}_3\zeta_3+2\mathbf{M}_5\zeta_5+2\mathbf{M}_3^2\zeta_3^2+2\mathbf{M}_7\zeta_7+2\zeta_3\zeta_5\left\{\mathbf{M}_5,\mathbf{M}_3\right\} \\ +\frac{4}{3}\mathbf{M}_3\cdot\mathbf{M}_3\cdot\mathbf{M}_3\zeta_3^3+2\mathbf{M}_9\zeta_9+2\zeta_3\zeta_7\left\{\mathbf{M}_7,\mathbf{M}_3\right\}+2\mathbf{M}_5\cdot\mathbf{M}_5\zeta_5^2 \\ +2\mathbf{M}_{11}\zeta_{11}+2\mathbf{Q}_{11}+\zeta_5\zeta_3\zeta_3\left\{\mathbf{M}_3,\left\{\mathbf{M}_5,\mathbf{M}_3\right\}\right\}\end{array}\right\}. \qquad (5.41)
\end{aligned}$$

End of calculation of sv$\mathcal{A}_{5pt}$ to 11th order

Calculation of sv$\mathcal{A}_{5pt}$, 12th order.

Calculation of sv $\mathbf{Q}_{12}$.

Data of $\mathbf{Q}_{12}$ is from e.q.(3.17) in [18].

$$\begin{aligned}
\mathbf{Q}_{12} = &\left\{\frac{2}{9}\zeta_5\zeta_7+\frac{1}{27}\zeta_{9,3}\right\}\left[\mathbf{M}_9,\mathbf{M}_3\right] \\
&+\frac{48}{691}\left\{\frac{18}{35}\zeta_2^3\zeta_3^2+\frac{1}{5}\zeta_2^2\zeta_3\zeta_5-10\zeta_2\zeta_3\zeta_7-\frac{7}{2}\zeta_2\zeta_5^2-\frac{3}{5}\zeta_2^2\zeta_{5,3}-3\zeta_2\zeta_{7,3}\right. \qquad (5.42) \\
&\left.-\frac{1}{12}\zeta_3^4-\frac{467}{108}\zeta_5\zeta_7+\frac{799}{72}\zeta_3\zeta_9+\frac{2665}{648}\zeta_{9,3}+\zeta_{6,4,1,1}\right\}\left\{\left[\mathbf{M}_9,\mathbf{M}_3\right]-3\left[\mathbf{M}_7,\mathbf{M}_5\right]\right\}.
\end{aligned}$$

Cancelling those with ζ_2, we have

$$\text{sv}\mathbf{Q}_{12} = \text{sv}\left(\begin{array}{l}\left\{\frac{2}{9}\zeta_5\zeta_7+\frac{1}{27}\zeta_{9,3}\right\}\left[\mathbf{M}_9,\mathbf{M}_3\right]+ \\ \frac{48}{691}\left\{-\frac{1}{12}\zeta_3^4-\frac{467}{108}\zeta_5\zeta_7+\frac{799}{72}\zeta_3\zeta_9+\frac{2665}{648}\zeta_{9,3}+\zeta_{6,4,1,1}\right\}\left\{\left[\mathbf{M}_9,\mathbf{M}_3\right]-3\left[\mathbf{M}_7,\mathbf{M}_5\right]\right\}\end{array}\right). \qquad (5.43)$$

We then do the sv map on zeta values with odd subscript and also on $\zeta_{9,3}$ whose trivialization map value is from (4.35) in [18]. We won't give too much information on sv$\zeta_{9,3}$ here as one could easily do it when referencing to (4.35) in [18] and the previous method for doing sv$\zeta_{5,3}$ and sv$\zeta_{7,3}$.

$$\text{sv}\mathbf{Q}_{12} = \left(\begin{array}{l}\left\{\frac{8}{9}\zeta_5\zeta_7+\frac{1}{27}\left(-42\zeta_5\zeta_7-54\zeta_3\zeta_9\right)\right\}\left[\mathbf{M}_9,\mathbf{M}_3\right]+ \\ \frac{48}{691}\left\{-\frac{4}{3}\zeta_3^4-\frac{467}{27}\zeta_5\zeta_7+\frac{799}{18}\zeta_3\zeta_9+\frac{2665}{648}\left(-42\zeta_5\zeta_7-54\zeta_3\zeta_9\right)+\text{sv}\zeta_{6,4,1,1}\right\} \\ \left\{\left[\mathbf{M}_9,\mathbf{M}_3\right]-3\left[\mathbf{M}_7,\mathbf{M}_5\right]\right\}\end{array}\right) \qquad (5.44)$$

Now we do sv map on $\zeta^m_{6,4,1,1}$, Following the same procedure in calculation of $\mathbf{Q}_{10}$ or $\mathbf{Q}_{11}$, we first have from [18] (4.35):

$$\begin{aligned}\phi\left(\zeta^m_{6,4,1,1}\right) = & \frac{1799}{18} f_9 f_3 - 32 f_7 f_3 f_2 + \frac{1133}{16} f_7 f_5 + 29 f_5 f_7 - 11 f_5^2 f_2 - \frac{16}{5} f_5 f_3 f_2^2 \\ & + \frac{1}{3} f_3 \left(f_3 \text{ III } f_3 \text{ III } f_3\right) - \frac{799}{72} f_3 f_9 + 10 f_3 f_7 f_2 - \frac{1}{5} f_3 f_5 f_2^2 - \frac{36}{35} f_3^2 f_2^3.\end{aligned} \tag{5.45}$$

When we ignore all the ζ_2 terms and make all the words shuffled, we have

$$\begin{aligned}\text{sv}\phi\left(\zeta^m_{6,4,1,1}\right) = & \frac{1799}{9} f_9 \text{ III } f_3 + \frac{1133}{8} f_7 \text{ III } f_5 + 29 \times 2 f_5 \text{ III } f_7 \\ & + \frac{4}{3} f_3 \text{ III } f_3 \text{ III } f_3 \text{ III } f_3 - \frac{799}{36} f_3 \text{ III } f_9.\end{aligned} \tag{5.46}$$

Doing inverse map of trivialization and also making the correspondence of $\zeta^m_{\dots} \leftrightarrow \zeta_{\dots}$, we have

$$\begin{aligned}\text{sv}\left(\zeta_{6,4,1,1}\right) = & \frac{1799}{9} \zeta_9 \zeta_3 + \frac{1133}{8} \zeta_7 \zeta_5 + 29 \times 2 \zeta_5 \zeta_7 \\ & + \frac{4}{3} \zeta_3 \zeta_3 \zeta_3 \zeta_3 - \frac{799}{36} \zeta_3 \zeta_9.\end{aligned} \tag{5.47}$$

Substituting (5.47) into (5.44), we have

$$\text{sv}\mathbf{Q}_{12} \quad = \quad 2[\mathbf{M}_5, \mathbf{M}_7]\zeta_5\zeta_7 + 2[\mathbf{M}_3, \mathbf{M}_9]\zeta_3\zeta_9. \tag{5.48}$$

End of calculation of sv$\mathbf{Q}_{12}$

Expanding : $\exp\left\{\sum_{n\geq 1} \zeta_{2n+1}\mathbf{M}_{2n+1}\right\}$: to 12th order, we can then have sv$\mathcal{A}_{5pt}$ to 12th order:

$$\begin{aligned}&\text{sv}\mathcal{A}_{5pt} = \text{sv}\left(1 + \mathbf{Q}_8 + \mathbf{Q}_9 + \mathbf{Q}_{10} + \mathbf{Q}_{11} + \mathbf{Q}_{12}\right) : \exp\left\{\sum_{n\geq 1} \zeta_{2n+1}\mathbf{M}_{2n+1}\right\} :, \\ &= ...O(11) + \frac{2}{3}\mathbf{M}_3^4\zeta_3^4 + 4\mathbf{M}_7\mathbf{M}_5\zeta_5\zeta_7 + 4\mathbf{M}_9\mathbf{M}_3\zeta_3\zeta_9 + \text{sv}\mathbf{Q}_{12} + O(13), \\ &= ...O(11) + \frac{2}{3}\mathbf{M}_3^4\zeta_3^4 + 4\mathbf{M}_7\mathbf{M}_5\zeta_5\zeta_7 + 4\mathbf{M}_9\mathbf{M}_3\zeta_3\zeta_9 + 2[\mathbf{M}_5, \mathbf{M}_7]\zeta_5\zeta_7 + 2[\mathbf{M}_3, \mathbf{M}_9]\zeta_3\zeta_9 + O(13), \\ &= ...O(11) + \frac{2}{3}\mathbf{M}_3^4\zeta_3^4 + 2\left\{\mathbf{M}_5, \mathbf{M}_7\right\}\zeta_5\zeta_7 + 2\left\{\mathbf{M}_3, \mathbf{M}_9\right\}\zeta_3\zeta_9 + O(13).\end{aligned} \tag{5.49}$$

End of calculation of sv$\mathcal{A}_{5pt}$ to 12th order

Calculation of sv$\mathcal{A}_{5pt}$, 13th order.

Calculation of sv$\mathbf{Q}_{13}$

$$\begin{aligned}\mathbf{Q}_{13} = & \left\{\frac{11}{4}\zeta_2\zeta_{11} - \frac{2}{35}\zeta_2^2\zeta_9 - \frac{16}{245}\zeta_2^3\zeta_7 - \frac{3}{35}\zeta_{5,5,3} + \frac{1}{14}\zeta_{7,3,3}\right\}\left[\mathbf{M}_3, \left[\mathbf{M}_7, \mathbf{M}_3\right]\right], \\ & + \left\{\frac{11}{2}\zeta_2\zeta_{11} + \frac{2}{5}\zeta_2^2\zeta_9 + \frac{1}{5}\zeta_5\zeta_{5,3} + \frac{1}{25}\zeta_{5,5,3}\right\}\left[\mathbf{M}_5, \left[\mathbf{M}_5, \mathbf{M}_3\right]\right].\end{aligned} \tag{5.50}$$

Again, erasing all the ζ_2 bits, we have

$$\begin{aligned}\text{sv } \mathbf{Q}_{13} = & \left\{-\frac{3}{35}\text{sv } \zeta_{5,5,3} + \frac{1}{14}\text{sv } \zeta_{7,3,3}\right\}\left[\mathbf{M}_3, \left[\mathbf{M}_7, \mathbf{M}_3\right]\right] \\ & + \left\{\frac{1}{5}\text{sv}\zeta_5\text{sv}\zeta_{5,3} + \text{sv } \frac{1}{25}\zeta_{5,5,3}\right\}\left[\mathbf{M}_5, \left[\mathbf{M}_5, \mathbf{M}_3\right]\right].\end{aligned} \tag{5.51}$$

We obtain trivialization data for $\phi\left(\zeta^m_{5,5,3}\right)$ and $\phi\left(\zeta^m_{7,3,3}\right)$ from e.q.(4.42) in [18].

$$\phi\left(\zeta^m_{5,5,3}\right) = 25 f_5^2 f_3 - 10 f_9 f_2^2 - \frac{275}{2} f_{11} f_2 \tag{5.52}$$

Erasing f_2 bits in the trivialization and performing motivic sv map based on equation (3.6):

$$\begin{aligned} \mathrm{sv}\phi\left(\zeta^m_{5,5,3}\right) &= 25\ \mathrm{sv} f_5^2 f_3 \\ &= 25\left(4 f_3 f_5^2 + 2 f_5 f_3 f_5 + 2 f_5^2 f_3\right) \end{aligned} \tag{5.53}$$

Observing (5.52), we will modify (5.53) as:

$$\mathrm{sv}\phi\left(\zeta^m_{5,5,3}\right) \;=\; 25\left(2(f_3 f_5 \text{ III } f_5) + 2 f_5^2 f_3\right) - 2\left(10 f_9 f_2^2 + \frac{275}{2} f_{11} f_2\right) + 2\left(10 f_9 f_2^2 + \frac{275}{2} f_{11} f_2\right). \tag{5.54}$$

This is the reconstruction for MZV $\zeta_{5,5,3}$ of $\mathbf{Q}_{13}$. One may take a look at context around (5.36) and (5.37) for more details on how we reconstruct $\mathbf{Q}_{11}$. From inverse trivialization and equations (5.52)(5.54),

$$\mathrm{sv}\zeta^m_{5,5,3} \;=\; 2\zeta^m_{5,5,3} + \phi^{-1}(50 f_3 f_5 \text{ III } f_5) - 2(10\zeta_9^m(\zeta_2^m)^2 + \frac{275}{2}\zeta^m_{11}\zeta^m_2). \tag{5.55}$$

$\phi^{-1}(50 f_3 f_5 \text{ III } f_5)$ is a highly non-trivial term that we should illustrate below. Next we go to trivialization of $\zeta_{7,3,3}$:

$$\phi\left(\zeta^m_{7,3,3}\right) = 30 f_5^2 f_3 - 7 f_7\left(f_3 \text{ III } f_3\right) + \frac{32}{35} f_7 f_2^3 - \frac{56}{5} f_9 f_2^2 - \frac{407}{2} f_{11} f_2. \tag{5.56}$$

Again eliminating f_2 bits and doing sv map based on (3.6):

$$\begin{aligned} \mathrm{sv}\phi\left(\zeta^m_{7,3,3}\right) &= 30\left(2\left(f_3 f_5 \text{ III } f_5\right) + 2 f_5^2 f_3\right) - 14\left(4 f_7 f_3^2 + f_3^2 f_7 + f_3 f_7 f_3\right), \\ &= 30\left(2\left(f_3 f_5 \text{ III } f_5\right) + 2 f_5^2 f_3\right) - 14\left(f_7\left(f_3 \text{ III } f_3\right) + f_3 \text{ III } f_3 \text{ III } f_7\right) \\ &\quad +2\left(\frac{32}{35} f_7 f_2^3 - \frac{56}{5} f_9 f_2^2 - \frac{407}{2} f_{11} f_2\right) - 2\left(\frac{32}{35} f_7 f_2^3 - \frac{56}{5} f_9 f_2^2 - \frac{407}{2} f_{11} f_2\right), \end{aligned} \tag{5.57}$$

where the last equation is the reconstruction of $\zeta^m_{7,3,3}$ in $\mathbf{Q}_{13}$. So looking at (5.56) and (5.57), we have

$$\mathrm{sv}\zeta^m_{7,3,3} = 2\zeta^m_{7,3,3} + \zeta^m_7(\zeta^m_3)^2 + 60\phi^{-1}(f_3 f_5 \text{ III } f_5) - 2\left(\frac{32}{35}\zeta^m_7(\zeta^m_2)^3 - \frac{56}{5}\zeta^m_9(\zeta^m_2)^2 - \frac{407}{2}\zeta^m_{11}\zeta^m_2\right). \tag{5.58}$$

With $\mathrm{sv}\zeta_{5,3}$ (5.28) which we have calculated earlier, also (5.55), (5.58) and the duality between $\zeta^m_{\dots}$ and $\zeta_{\dots}$, we are able to write:

$$\mathrm{sv}\ \mathbf{Q}_{13} = \left(\begin{array}{l} \left\{\begin{array}{l} -\frac{3}{35}\times 2\ \zeta_{5,5,3} + \frac{1}{14}\times 2\ \zeta_{7,3,3} \\ -\frac{3\times 2}{35}\left(10\zeta_9\zeta_2^2 + \frac{275}{2}\zeta_{11}\zeta_2\right) - \frac{1}{14}\times 2\left(\frac{32}{35}\zeta_7\zeta_2^3 - \frac{56}{5}\zeta_9\zeta_2^2 - \frac{407}{2}\zeta_{11}\zeta_2\right) \\ -\frac{3}{35}\times 50\ \phi^{-1}\left(f_3 f_5 \text{ III } f_5\right) - \frac{1}{14}\times 14\zeta_3^3\zeta_7 + \frac{60}{14}\phi^{-1}\left(f_3 f_5 \text{ III } f_5\right) \end{array}\right\}\times \\ [\mathbf{M}_3,[\mathbf{M}_7,\mathbf{M}_3]] + \\ \left\{-4\zeta_5^2\zeta_3 + 2\times\frac{1}{25}\zeta_{5,5,3} + \frac{1}{25}\times 2\left(10\zeta_9\zeta_2^2 + \frac{275}{2}\zeta_{11}\zeta_2\right) + \frac{1}{25}\times 50\ \phi^{-1}\left(f_3 f_5 \text{ III } f_5\right)\right\}\times \\ [\mathbf{M}_5,[\mathbf{M}_5,\mathbf{M}_3]] \end{array}\right), \tag{5.59}$$

$$= \left(\begin{array}{l} \left\{\begin{array}{l} -\frac{3}{35}\times 2\ \zeta_{5,5,3} + \frac{1}{14}\times 2\ \zeta_{7,3,3} \\ -\frac{4}{35}\zeta_9\zeta_2^2 - \frac{32}{245}\zeta_7\zeta_2^3 + \frac{11}{2}\zeta_{11}\zeta_2 \\ -\frac{1}{14}\times 14\zeta_3^2\zeta_7 \end{array}\right\}[\mathbf{M}_3,[\mathbf{M}_7,\mathbf{M}_3]] \\ +\left\{-4\zeta_5^2\zeta_3 + 2\times\frac{1}{25}\zeta_{5,5,3} + \frac{4}{5}\zeta_9\zeta_2^2 + 11\zeta_{11}\zeta_2 + \frac{1}{25}\times 50\ \phi^{-1}\left(f_3 f_5 \text{ III } f_5\right)\right\}\times \\ [\mathbf{M}_5,[\mathbf{M}_5,\mathbf{M}_3]] \end{array}\right). \tag{5.60}$$

The $\phi^{-1}(f_3 f_5 \text{ III } f_5)$ seems weird. Although it gets cancelled in the coefficient of $[\mathbf{M}_3, [\mathbf{M}_7, \mathbf{M}_3]]$, it still remains in $[\mathbf{M}_5, [\mathbf{M}_5, \mathbf{M}_3]]$. It's not a bad news, as comparing (5.50) we may need this to reconstruct $\frac{2}{5}\zeta_{5,3}\zeta_5$ so as to recover $2\mathbf{Q}_{13}$. Next we will try to see what exactly $\phi^{-1}(f_3 f_5 \text{ III } f_5)$ is mapping to.

At length $N = 13$ the basis (c.f. Table 1) for motivic MZVs reads ([18] (4.44)):

$$\begin{aligned}
\xi_{13} &= a_1\zeta^m_{7,3,3} + a_2\zeta^m_{5,5,3} + a_3\zeta^m_{13} + a_4\zeta^m_{7,3}\zeta^m_3 + a_5\zeta^m_{5,3}\zeta^m_5 + a_6\zeta^m_7\left(\zeta^m_3\right)^2 \\
&+ a_7\left(\zeta^m_5\right)^2\zeta^m_3 + a_8\zeta^m_{5,3,3}\zeta^m_2 + a_9\zeta^m_{5,3}\zeta^m_3\zeta^m_2 + a_{10}\zeta^m_{11}\zeta^m_2 + a_{11}\zeta^m_5\left(\zeta^m_3\right)^2\zeta^m_2 \\
&+ a_{12}\left(\zeta^m_3\right)^3\left(\zeta^m_2\right)^2 + a_{13}\zeta^m_9\left(\zeta^m_2\right)^2 + a_{14}\zeta^m_7\left(\zeta^m_2\right)^3 + a_{15}\zeta^m_5\left(\zeta^m_2\right)^4 + a_{16}\zeta^m_3\left(\zeta^m_2\right)^5 ,
\end{aligned} \tag{5.61}$$

where the coefficients are given by derivatives in (5.62) acting on the trivialisation $\phi(\xi_{13})$. For more details, see (4.24) and (4.24) in [18] also 4.1 in [3]. We gave a detailed example in (5.64) to illustrate how it works.

$$\begin{aligned}
D_1 &= \frac{1}{14}\left[\partial_3, \left[\partial_7, \partial_3\right]\right], D_2 = \frac{1}{25}\left[\partial_5, \left[\partial_5, \partial_3\right]\right] - \frac{3}{35}\left[\partial_3, \left[\partial_7, \partial_3\right]\right], D_3 = \partial_{13}, \\
D_4 &= \frac{1}{14}\left[\partial_7, \partial_3\right]\partial_3, D_5 = \frac{1}{5}\partial_5\left[\partial_5, \partial_3\right], D_6 = \frac{1}{2}\partial_7\partial_3^2, D_7 = \frac{3}{14}\left[\partial_7, \partial_3\right]\partial_3 + \frac{1}{2}\partial_5^2\partial_3, \\
D_8 &= \frac{1}{5}c_2\left[\partial_3, \left[\partial_5, \partial_3\right]\right], D_9 = \frac{1}{5}c_2\left[\partial_5, \partial_3\right]\partial_3, \\
D_{10} &= c_2\partial_{11} + \frac{11}{2}\left[\partial_5, \left[\partial_5, \partial_3\right]\right] + \frac{11}{4}\left[\partial_3, \left[\partial_7, \partial_3\right]\right], D_{11} = \frac{1}{2}c_2\partial_5\partial_3^2, D_{12} = \frac{1}{6}c_2^2\partial_3^3, \\
D_{13} &= c_2^2\partial_9 + 9c_2\left[\partial_3, \left[\partial_5, \partial_3\right]\right] + \frac{2}{5}\left[\partial_5, \left[\partial_5, \partial_3\right]\right] - \frac{2}{35}\left[\partial_3, \left[\partial_7, \partial_3\right]\right], \\
D_{14} &= c_2^3\partial_7 + \frac{6}{25}c_2\left[\partial_3, \left[\partial_5, \partial_3\right]\right] - \frac{16}{245}\left[\partial_3, \left[\partial_7, \partial_3\right]\right], \\
D_{15} &= c_2^4\partial_5 - \frac{4}{25}c_2\left[\partial_3, \left[\partial_5, \partial_3\right]\right], D_{16} = c_2^5\partial_3,
\end{aligned} \tag{5.62}$$

where the derivatives in (5.62) are defined as [5] [18]:

$$\partial_{2n+1}\left(f_{i_1}\dots f_{i_r}\right) = \begin{cases} f_{i_2}\dots f_{i_r}, & i_1 = 2n+1 \\ 0, & \text{otherwise} . \end{cases} \tag{5.63}$$

Let's return to $\phi^{-1}(f_3 f_5 \text{ III } f_5)$. We already have a word $f_3 f_5 \text{ III } f_5$. So in order to reconstruct $\zeta_{5,3}\zeta_5$, we let D_5 act on it:

$$\begin{aligned}
D_5\left(f_3 f_5 \text{ III } f_5\right) &= \frac{1}{5}\partial_5\left[\partial_5, \partial_3\right]\left(f_3 f_5 \text{ III } f_5\right) = \frac{1}{5}\left(\partial_5\partial_5\partial_3\left(f_3 f_5 \text{ III } f_5\right) - \partial_5\partial_3\partial_5\left(f_3 f_5 \text{ III } f_5\right)\right), \\
&= \frac{1}{5}\left(\partial_5\partial_5\left(\partial_3 f_3 f_5 \text{ III } f_5\right) + \partial_5\partial_5\left(f_3 f_5 \text{ III } \partial_3 f_5\right) - \partial_5\partial_3\left(\partial_5 f_3 f_5 \text{ III } f_5\right) - \partial_5\partial_3\left(f_3 f_5 \text{ III } \partial_5 f_5\right)\right), \\
&= \frac{1}{5}\left(\partial_5\partial_5\left(\partial_3 f_3 f_5 \text{ III } f_5\right) - \partial_5\partial_3\left(f_3 f_5 \text{ III } \partial_5 f_5\right)\right), \\
&= \frac{1}{5}\left(\partial_5\partial_5\left(f_5 \text{ III } f_5\right) - \partial_5\partial_3\left(f_3 f_5\right)\right), \\
&= \frac{1}{5}\left(\partial_5\left(\partial_5 f_5 \text{ III } f_5\right) + \partial_5\left(f_5 \text{ III } \partial_5 f_5\right) - 1\right), \\
&= \frac{1}{5}\left(\partial_5 f_5 + \partial_5 f_5 - 1\right) = \frac{1}{5}.
\end{aligned} \tag{5.64}$$

So

$$f_3 f_5 \text{ III } f_5 \to \dots \frac{1}{5}\zeta_{5,3}\zeta_5 \dots . \tag{5.65}$$

One may notice that there are ... in (5.65). We can not exclude the possibility that other coefficients are non-zero.

$$\begin{aligned}
D_2 &= \frac{1}{25}\left[\partial_5, \left[\partial_5, \partial_3\right]\right], D_7 = \frac{3}{14}\left[\partial_7, \partial_3\right]\partial_3 + \frac{1}{2}\partial_5^2\partial_3, \\
D_{10} &= c_2\partial_{11} + \frac{11}{2}\left[\partial_5, \left[\partial_5, \partial_3\right]\right] + \frac{11}{4}\left[\partial_3, \left[\partial_7, \partial_3\right]\right], \\
D_{13} &= c_2^2\partial_9 + 9c_2\left[\partial_3, \left[\partial_5, \partial_3\right]\right] + \frac{2}{5}\left[\partial_5, \left[\partial_5, \partial_3\right]\right] - \frac{2}{35}\left[\partial_3, \left[\partial_7, \partial_3\right]\right] .
\end{aligned} \tag{5.66}$$

What we get is:

$$D_2\left(f_3 f_5 \text{ III } f_5\right)=0, \quad D_7\left(f_3 f_5 \text{ III } f_5\right)=1, \quad D_{10}\left(f_3 f_5 \text{ III } f_5\right)=0, \quad D_{13}\left(f_3 f_5 \text{ III } f_5\right)=0. \tag{5.67}$$

So with (5.67), by comparing equation (5.62) (5.61),

$$a_7=1. \tag{5.68}$$

So (5.65) becomes:

$$f_3 f_5 \text{ III } f_5 \rightarrow \frac{1}{5}\zeta_{5,3}\zeta_5+\zeta_5^2\zeta_3 \tag{5.69}$$

Plugging (5.69) into (5.60), one exactly obtains $\frac{2}{5}\zeta_{5,3}\zeta_5$ and thus $2\mathbf{Q}_{13}$:

$$\begin{aligned}
\text{sv}\mathbf{Q}_{13} &= \left\{\begin{array}{l} -\frac{3}{35}\times 2\ \zeta_{5,5,3}+\frac{1}{14}\times 2\ \zeta_{7,3,3} \\ -\frac{4}{35}\zeta_9\zeta_2^2-\frac{32}{245}\zeta_7\zeta_2^3+\frac{11}{2}\zeta_{11}\zeta_2 \\ -\frac{1}{14}\times 14\zeta_3^2\zeta_7 \end{array}\right\}[\mathbf{M}_3,[\mathbf{M}_7,\mathbf{M}_3]] \\
&\quad +\left\{-4\zeta_5^2\zeta_3+2\times\frac{1}{25}\zeta_{5,5,3}+\frac{4}{5}\zeta_9\zeta_2^2+11\zeta_{11}\zeta_2+\frac{2}{5}\zeta_{5,3}\zeta_5+2\zeta_5^2\zeta_3\right\}[\mathbf{M}_5,[\mathbf{M}_5,\mathbf{M}_3]]. \\
&= 2\mathbf{Q}_{13}-\zeta_3^2\zeta_7[\mathbf{M}_3,[\mathbf{M}_7,\mathbf{M}_3]]-2\zeta_5^2\zeta_3[\mathbf{M}_5,[\mathbf{M}_5,\mathbf{M}_3]].
\end{aligned} \tag{5.70}$$

End of calculation of $\text{sv}\mathbf{Q}_{13}$

We then expand : $\exp\left\{\sum_{n\geq 1}\zeta_{2n+1}\mathbf{M}_{2n+1}\right\}$: to order 13. The single value map on $\mathcal{A}_{5pt}$ reads:

$$\begin{aligned}
& \text{sv}\left(1+\mathbf{Q}_8+\mathbf{Q}_9+\mathbf{Q}_{10}+\mathbf{Q}_{11}+\mathbf{Q}_{12}+\mathbf{Q}_{13}\right):\exp\left\{\sum_{n\geq 1}\zeta_{2n+1}\mathbf{M}_{2n+1}\right\}:, \\
= & \ldots+O(12)+2\mathbf{Q}_{13}-\zeta_3^2\zeta_7[\mathbf{M}_3,[\mathbf{M}_7,\mathbf{M}_3]]-2\zeta_3\zeta_5^2[\mathbf{M}_5,[\mathbf{M}_5,\mathbf{M}_3]]+ \\
& +4\mathbf{M}_7\mathbf{M}_3^2\zeta_3^2\zeta_7+2\mathbf{M}_{13}\zeta_{13}+\text{sv}\mathbf{Q}_8\times 2\zeta_5\mathbf{M}_5+\text{sv}\mathbf{Q}_{10}\times 2\zeta_3\mathbf{M}_3+O(14), \\
= & \ldots+O(12)+2\mathbf{Q}_{13}-\zeta_3^2\zeta_7[\mathbf{M}_3,[\mathbf{M}_7,\mathbf{M}_3]]-2\zeta_5^2\zeta_3[\mathbf{M}_5,[\mathbf{M}_5,\mathbf{M}_3]]+4\mathbf{M}_5^2\mathbf{M}_3\zeta_3\zeta_5^2 \\
& +4\mathbf{M}_7\mathbf{M}_3^2\zeta_3^2\zeta_7+2\mathbf{M}_{13}\zeta_{13}+(-2\zeta_3\zeta_5[\mathbf{M}_5,\mathbf{M}_3])\times 2\zeta_5\mathbf{M}_5+(-2\zeta_3\zeta_7[\mathbf{M}_7,\mathbf{M}_3])\times 2\zeta_3\mathbf{M}_3+O(14), \\
= & \ldots+O(12)+2\mathbf{Q}_{13}+2\zeta_5^2\zeta_3\left\{\mathbf{M}_5^2,\mathbf{M}_3\right\}+\zeta_3^2\zeta_7\{\mathbf{M}_3,\{\mathbf{M}_7,\mathbf{M}_3\}\}+2\mathbf{M}_{13}\zeta_{13}+O(14).
\end{aligned} \tag{5.71}$$

End of calculation of $\mathcal{A}_{5pt}$, 13th order

Final result:

$$\begin{aligned}
\text{sv}\mathcal{A}_{5pt} = & \Big(\mathbf{1}+2\mathbf{M}_3\zeta_3+2\mathbf{M}_5\zeta_5+2\mathbf{M}_3^2\zeta_3^2+2\mathbf{M}_7\zeta_7+2\zeta_3\zeta_5\{\mathbf{M}_5,\mathbf{M}_3\}+\frac{4}{3}\mathbf{M}_3\cdot\mathbf{M}_3\cdot\mathbf{M}_3\zeta_3^3+2\mathbf{M}_9\zeta_9 \\
& +2\zeta_3\zeta_7\{\mathbf{M}_7,\mathbf{M}_3\}+2\mathbf{M}_5\cdot\mathbf{M}_5\zeta_5^2+2\mathbf{M}_{11}\zeta_{11}+2\mathbf{Q}_{11}+\zeta_5\zeta_3\zeta_3\{\mathbf{M}_3,\{\mathbf{M}_5,\mathbf{M}_3\}\}\frac{2}{3}\mathbf{M}_3^4\zeta_3^4 \\
& +2\{\mathbf{M}_5,\mathbf{M}_7\}\zeta_5\zeta_7+2\{\mathbf{M}_3,\mathbf{M}_9\}\zeta_3\zeta_9 2\mathbf{Q}_{13}+2\zeta_5^2\zeta_3\left\{\mathbf{M}_5^2,\mathbf{M}_3\right\}+\zeta_3^2\zeta_7\{\mathbf{M}_3,\{\mathbf{M}_7,\mathbf{M}_3\}\} \\
& +2\mathbf{M}_{13}\zeta_{13}+O(14)\Big)\mathbf{A}_{YM}.
\end{aligned} \tag{5.72}$$

This coincides with *single-valued* and *uniform weighted* part of five-point spherical integral (6.24) in [18]. A more compact form reads:

$$\text{sv}\mathcal{A}_{5pt} = \mathbf{F}\mathbf{A}_{YM}. \tag{5.73}$$

Finally, the result useful in our paper is

$$\begin{aligned}
\mathbf{F} = & \Big(\mathbf{1}+2\mathbf{M}_3\zeta_3+2\mathbf{M}_5\zeta_5+2\mathbf{M}_3^2\zeta_3^2+2\mathbf{M}_7\zeta_7+2\zeta_3\zeta_5\{\mathbf{M}_5,\mathbf{M}_3\}+\frac{4}{3}\mathbf{M}_3\cdot\mathbf{M}_3\cdot\mathbf{M}_3\zeta_3^3+2\mathbf{M}_9\zeta_9 \\
& +2\zeta_3\zeta_7\{\mathbf{M}_7,\mathbf{M}_3\}+2\mathbf{M}_5\cdot\mathbf{M}_5\zeta_5^2+2\mathbf{M}_{11}\zeta_{11}+2\mathbf{Q}_{11}+\zeta_5\zeta_3\zeta_3\{\mathbf{M}_3,\{\mathbf{M}_5,\mathbf{M}_3\}\}\frac{2}{3}\mathbf{M}_3^4\zeta_3^4 \\
& +2\{\mathbf{M}_5,\mathbf{M}_7\}\zeta_5\zeta_7+2\{\mathbf{M}_3,\mathbf{M}_9\}\zeta_3\zeta_9 2\mathbf{Q}_{13}+2\zeta_5^2\zeta_3\left\{\mathbf{M}_5^2,\mathbf{M}_3\right\}+\zeta_3^2\zeta_7\{\mathbf{M}_3,\{\mathbf{M}_7,\mathbf{M}_3\}\} \\
& +2\mathbf{M}_{13}\zeta_{13}+O(14)\Big).
\end{aligned} \tag{5.74}$$

6 Expanding hard part of scattering amplitude with single-valued string five-point tree-level disk amplitude

We would first introduce some basic information of polynomial analysing as used in this Section. Then we are going to describe how we cook equations by matching coupling constant weight by weight and $\eta = -C_1/C_2$(see (1.4)) order by order. After this, we are going to partially solve the equation set. Some discussion will be given based on the result we have.

6.1 Polynomial and ideal

Denote a polynomial ring as $\mathbb{P}$. Introducing an variable ordering: $x_1 < x_2 < \ldots < x_n$, we will give some basic concept of polynomials,[15][20]. We define a *term* t in the ring $\mathbb{P}$:

$$\text{Term: } t = B(c_1, c_2, \ldots, c_n) x_1^{c_1} x_2^{c_2} \ldots x_n^{c_n},$$

where $B(c_1, c_2, \ldots, c_n)$ is the coefficient of $x_1^{c_1} x_2^{c_2} \ldots x_n^{c_n}$ depending on the power of each variable. A more compact form would be as

$$t = B(\mathbf{c})\mathbf{x}^{\mathbf{c}},$$

where $\mathbf{c} = (c_1, c_2, ..., c_n)$ and $\mathbf{x}^{\mathbf{c}} = x_1^{c_1} x_2^{c_2} \ldots x_n^{c_n}$.

A polynomial F is defined as,

$$\text{Polynomial: } F = \sum_{\mathbf{c}} B(\mathbf{c}) x^{\mathbf{c}}.$$

Since we have the ordered variable $x_1 < x_2 < ... < x_n$, we can introduce the leading variables of F.

$$\text{Leading Variable} \quad LV(F) = \text{The greatest variable w.r.t. Lexicographic order (see def. in Section 2.2).} \tag{6.1}$$

With the leading variable of F, we are able to define the class of F.

$$\text{Class of } F \quad CL(F) = \text{The subscript of the greatest variable w.r.t. Lexicographic order.} \tag{6.2}$$

The 'power' of a variable in a term t is called 'degree'.

$$\text{The degree of term } t \text{ w.r.t.} x_r \quad DG(t, x_r) \quad = \quad \text{The power of } x_r \text{ in term } t.$$

A polynomial F contains some terms, $F = t_1 + t_2 + ... + t_m$, and each term t_i has a degree DG_i for a given x_r. The degree of x_r for F is thus defined as the maximum degree for a term t in F could have for x_r.

$$\text{The degree of } F \text{ w.r.t. } x_r \quad DG(F, x_r) \quad = \quad \max\{DG(t, x_r), \text{ for } t \in F\}. \tag{6.3}$$

As we have defined leading variable (6.1) and also the degree of a polynomial F (6.3), we can define leading degree for F as

$$\text{The leading degree of } F \quad LDG(F) \quad = \quad DG(F, LV(F)). \tag{6.4}$$

So far as we have introduced leading degree (6.4) and class of F (6.2), we could ask what is the coefficient of $x_{CL(F)}^{LDG(F)}$ in F, which is also called the initial of F:

$$\text{Leading coefficient of } F \text{ (initial of } F) \quad INI(F) \quad = \quad \text{Coefficient}[F, x_{CL(F)}^{LDG(F)}].$$

Example 6.1. $F_1 = 3x_1^5 x_2^3 x_3^2 + x_1^6 x_2 + \frac{1}{2} x_3^3$ *then* $CL(F) = 3$. $LV(F_1) = x_3$, $DG(F_1, x_2) = 3$, $LDG(F_1) = 3$, $INI(F_1) = \frac{1}{2}$.

As we know from linear algebra that $1, x, x^2, x^3, ...x^{n-1}$ form a basis of n-dim vector space. Here suppose we have a polynomial set $\mathcal{F} = \{F_1, F_2, ..., F_l\}$. It also forms a basis. We can use some polynomials in ring $\mathbb{P}$ to construct ideal generated by $\mathcal{F}$ denoted as $\langle\mathcal{F}\rangle$:

$$\langle F\rangle := \{P_1F_1 + P_2F_2 + ...P_nF_n : \forall P_1, P_2, ..., P_l \in \mathbb{P}\}\,. \tag{6.5}$$

Note: the solution structure of ideal and the original polynomial set reads[15][20]:

$$\mathrm{Zero}(\langle\mathcal{F}\rangle) = \mathrm{Zero}(\mathcal{F}). \tag{6.6}$$

For the spirit of elimination, one may see the importance of an ideal: we could modify the polynomial, and see if there is a possibility of eliminating some F_i. If so, this can simplify the equations without losing any information on zeros.

On the other hand, the ideal of $\mathcal{F}$ can also extend our polynomial set to more complicated ones. However, there is a interesting extension: if $\exists Q_1, Q_2...Q_n \in \mathbb{P}$ s.t. $Q_1F_1 + Q_2F_2 + ...Q_nF_n = 1$, then $\mathrm{Zero}(\mathcal{F}) = \emptyset$,[15]. This conclusion is from Hilbert's Nullstellensatz.

Hilbert's Nullstellensatz:[15][20],

$$\mathrm{Zero}(\mathcal{F}) = \emptyset, \text{iff } 1 \in \langle\mathcal{F}\rangle\,. \tag{6.7}$$

Remark 6.1. Why $1 \in \langle\mathcal{F}\rangle$ means there is no solution? As we mentioned in (6.6), $\mathrm{Zero}(\langle\mathcal{F}\rangle) = \mathrm{Zero}(\mathcal{F})$, so $1 \in \langle\mathcal{F}\rangle$ means that we can reduce the polynomial ideal

$$\langle\mathcal{F}\rangle = \langle 1, F_1', ...F_m'\rangle = \langle 1\rangle \tag{6.8}$$

where the second equation, we have changed our generator in the ideal of $\langle\mathcal{F}\rangle$ and make a $'1'$ by action $Q_1F_1 + Q_2F_2 + ...Q_nF_n = 1$ for $Q_1, Q_2...Q_n \in \mathbb{P}$. The last equation comes from that all the polynomials in $\mathbb{P}$ is in the ideal $\langle 1\rangle$. And clearly $\mathrm{Zero}(\langle 1\rangle) = \emptyset$. Thus, $\mathrm{Zero}(\mathcal{F}) = \mathrm{Zero}(\langle\mathcal{F}\rangle) = \mathrm{Zero}(\langle 1\rangle) = \emptyset$.

So far we have completed a very brief introduction of polynomial analysis. Let's expand the hard part (6.11) of two-to-two partonic scattering amplitude in Regge limit with the single-valued five-point string disk amplitude.(5.72)

6.2 Solving ansatz

As we have seen that the single value map would change the disk amplitude to spherical one. Now we will try to consider the *single-valued uniform-weighted* five-point tree-level closed string amplitude(e.q.(6.24) [18]) as a basis to expand the hard part of $2 \to 2$ parton scattering amplitude in Regge limit which is also *single-valued and of uniform weight* (see e.q.(5.19) in[8]). For the mathematical details, see introduction 1.

6.2.1 Setting up equations

The spirit here is that we only match the part that are *single-valued uniform-weighted*, e.g. $\mathbf{F}$ in (5.72). Other constant part or common part,e.g. $\mathbf{A}_{YM}$ in single-valued disk amplitude or $i\pi$ (irrational an imaginary bit) and $\mathbf{T}_{s-u}^2\mathcal{M}^{(\text{tree})}$ in (6.11) will NOT be included in our equations. In below (**RHS** part), one may see that there is also an overall colour factor C_2^{k-1} which will also be excluded in $\hat{\mathcal{M}}_{\text{NLL,h}}^{(+,l)}$

RHS

$\hat{\mathcal{M}}_{\text{NLL,h}}^{(+,l)}$'s are integrated from the hard part of the wave function Ω_h(see (1.8)), which is derived order by order via BFKL equation:[8]

$$\frac{d}{dL}\Omega_h(p,k) = \frac{\alpha_s B_0(\epsilon)}{\pi}\hat{H}\Omega_h(p,k), \tag{6.9}$$

where the hamiltonian $\hat{H}$ reads,

$$\begin{aligned}\hat{H} &= \left(2C_A - \mathbf{T}_t^2\right)\hat{H}_{\mathrm{i}} + \left(C_A - \mathbf{T}_t^2\right)\hat{H}_{\mathrm{m}},\\ &= C_1\hat{H}_{\mathrm{i}} + C_2\hat{H}_{\mathrm{m}}.\end{aligned} \tag{6.10}$$

So we can have $\hat{\mathcal{M}}^{(+,l)}_{\rm NLL,h}$'s expressed in terms of C_1, C_2 ((5.19) in [8]):

$$\begin{aligned}
\hat{\mathcal{M}}^{(+,1)}_{\rm NLL,h} &= 0, \\
\hat{\mathcal{M}}^{(+,2)}_{\rm NLL,h} &= 0, \\
\hat{\mathcal{M}}^{(+,3)}_{\rm NLL,h} &= \frac{i\pi}{3!}\left\{\frac{3\zeta_3}{4}C_1C_2\right\}\mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree\ )}, \\
\hat{\mathcal{M}}^{(+,4)}_{\rm NLL,h} &= 0, \\
\hat{\mathcal{M}}^{(+,5)}_{\rm NLL,h} &= \frac{i\pi}{5!}\left\{-\frac{5\zeta_5}{2}C_1^2C_2^2+\frac{45\zeta_5}{2}C_1C_2^3\right\}\mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree\ )}, \\
\hat{\mathcal{M}}^{(+,6)}_{\rm NLL,h} &= \frac{i\pi}{6!}\left\{\frac{39\zeta_3^2}{16}C_1^3C_2^2-\frac{45\zeta_3^2}{2}C_1^2C_2^3+\frac{225\zeta_3^2}{2}C_1C_2^4\right\}\mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree\ )}, \\
\hat{\mathcal{M}}^{(+,7)}_{\rm NLL,h} &= \frac{i\pi}{7!}\left\{-\frac{2135\zeta}{256}C_1^4C_2^2+\frac{30135\zeta}{256}C_1^3C_2^3-\frac{20111\zeta_7}{32}C_1^2C_2^4+\frac{6111\zeta_7}{4}C_1C_2^5\right\}\mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree\ )}, \\
\hat{\mathcal{M}}^{(+,8)}_{\rm NLL,h} &= \frac{i\pi}{8!}\left\{\frac{611\zeta_3\zeta_5}{32}C_1^5C_2^2-\frac{643\zeta_3\zeta_5}{2}C_1^4C_2^3+\frac{8597\zeta_3\zeta_5}{4}C_1^3C_2^4-7086\zeta_3\zeta_5C_1^2C_2^5+13230\zeta_3\zeta_5C_1C_2^6\right\}\times \\
&\quad \mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree\ )}.
\end{aligned} \tag{6.11}$$

where for definition of colour factor C_1 and C_2, (see (1.4)).

The superscript $(+,l)$ for $\hat{\mathcal{M}}^{(+,l)}_{\rm NLL,h}$ denotes the l loops order, or also the lth weight of α_s. One can easily see what we mean from below equation:

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL}\left(\frac{s}{-t}\right)=\sum_{l=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^l L^{l-1}\hat{\mathcal{M}}^{(+,l)}_{\rm NLL}. \tag{6.12}$$

We make some further modification before obtaining weight matching equations.

We first ignore $i\pi$ (irrational and imaginary bit) and $\mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree\ )}$. These are the common part of $\hat{\mathcal{M}}^{(+,l)}_{\rm NLL}$ and are not possible to be expanded by $\mathbf{F}$ in (5.74). Then what is left is the dimensionful expression in the curly bracket in $\hat{\mathcal{M}}^{(+,l)}_{\rm NLL}$ which contain colour factors C_1, C_2. We then divide the curly bracket terms in $\hat{\mathcal{M}}^{(+,l)}_{\rm NLL}$ by C_2^{l-1} to make them dimensionless. What we get are

$$\begin{aligned}
\mathcal{M}^{(1)}_{\rm h} &= 0, \\
\mathcal{M}^{(2)}_{\rm h} &= 0, \\
\mathcal{M}^{(3)}_{\rm h} &= \frac{1}{3!}\left\{\frac{3\zeta_3}{4}\frac{C_1}{C_2}\right\}, \\
\mathcal{M}^{(4)}_{\rm h} &= 0, \\
\mathcal{M}^{(5)}_{\rm h} &= \frac{1}{5!}\left\{-\frac{5\zeta_5}{2}\left(\frac{C_1}{C_2}\right)^2+\frac{45\zeta_5}{2}\frac{C_1}{C_2}\right\}, \\
\mathcal{M}^{(6)}_{\rm h} &= \frac{1}{6!}\left\{\frac{39\zeta_3^2}{16}\left(\frac{C_1}{C_2}\right)^3-\frac{45\zeta_3^2}{2}\left(\frac{C_1}{C_2}\right)^2+\frac{225\zeta_3^2}{2}\frac{C_1}{C_2}\right\}, \\
\mathcal{M}^{(7)}_{\rm h} &= \frac{1}{7!}\left\{-\frac{2135\zeta_7}{256}\left(\frac{C_1}{C_2}\right)^4+\frac{30135\zeta_7}{256}\left(\frac{C_1}{C_2}\right)^3-\frac{20111\zeta_7}{32}\left(\frac{C_1}{C_2}\right)^2+\frac{6111\zeta_7}{4}\frac{C_1}{C_2}\right\} \\
\mathcal{M}^{(8)}_{\rm h} &= \frac{1}{8!}\left\{\frac{611\zeta_3\zeta 5}{32}\left(\frac{C_1}{C_2}\right)^5-\frac{643\zeta_3\zeta_5}{2}\left(\frac{C_1}{C_2}\right)^4+\frac{8597\zeta_3\zeta_5}{4}\left(\frac{C_1}{C_2}\right)^3-7086\zeta_3\zeta_5\left(\frac{C_1}{C_2}\right)^2+13230\zeta_3\zeta_5\frac{C_1}{C_2}\right\}.
\end{aligned} \tag{6.13}$$

We observe that all odd or even power of $\frac{C_1}{C_2}$ shares the same sign separately. We then set $\eta=-\frac{C_1}{C_2}$. After factorizing an overall minus sign, we have all the rest bits positive definite (See below).

$$\begin{aligned}
H[0] &= 0, \\
H[1] &= 0, \\
H[2] &= 0, \\
H[3] &= -\frac{3}{4\times 3!}\zeta_3\eta, \\
H[4] &= 0, \\
H[5] &= -\frac{5}{2\times 5!}\zeta_5\eta(\eta+9), \\
H[6] &= -\frac{3}{16\times 6!}\zeta_3^2\eta\left(13\eta^2+120\eta+600\right), \\
H[7] &= -\frac{7}{256\times 7!}\zeta_7\eta\left(305\eta^3+4305\eta^2+22984\eta+55872\right), \\
H[8] &= -\frac{1}{32\times 8!}\zeta_3\zeta_5\eta\left(611\eta^4+10288\eta^3+68776\eta^2+226752\eta+423360\right), \\
H[9] &= -\frac{1}{4096\times 9!}\eta\left[192\zeta_3^3\left(199\eta^5+3816\eta^4+29958\eta^3+123892\eta^2+265776\eta+411264\right),\right. \\
&+ \left.\zeta_9\left(262143\eta^5+5135424\eta^4+41853124\eta^3+181984832\eta^2+446510272\eta+589248000\right)\right], \\
\dots & \qquad (6.14)
\end{aligned}$$

where the i index in $H[i]$ denotes the weight of α_s (See (6.12)), and weight 0 is assumed to vanish. Now we have complete setting equations on the right hand side.

LHS

For the five-point single valued disk amplitude we have just calculated (5.72), we will only use the 2×2 matrix part $\mathbf{F}$ while the $\mathbf{A}_{YM}$ is not of our interest. We substitute each $\mathbf{M}_i(s_{jk})$ into $\mathbf{F}$ (sourced from [16]) where $\mathbf{M}_i$ is of α weight i. This is because each entry in $\mathbf{M}_i$ has i s_{jk}'s multiplying together, and each s_{jk} has α of weight one (recall our definition of s_{ij} in (4.1)). This is also the case for $\mathbf{Q}_m$ which is of α weight m.

So we now get an expression for $\mathbf{F}$ of α weighting from 0 to 13.

$$\begin{aligned}
\mathbf{F} = &\Big(\mathbf{1}+2\mathbf{M}_3\zeta_3+2\mathbf{M}_5\zeta_5+2\mathbf{M}_3^2\zeta_3^2+2\mathbf{M}_7\zeta_7+2\zeta_3\zeta_5\{\mathbf{M}_5,\mathbf{M}_3\}+\frac{4}{3}\mathbf{M}_3\cdot\mathbf{M}_3\cdot\mathbf{M}_3\zeta_3^3+2\mathbf{M}_9\zeta_9 \\
&+2\zeta_3\zeta_7\{\mathbf{M}_7,\mathbf{M}_3\}+2\mathbf{M}_5\cdot\mathbf{M}_5\zeta_5^2+2\mathbf{M}_{11}\zeta_{11}+2\mathbf{Q}_{11}+\zeta_5\zeta_3\zeta_3\{\mathbf{M}_3,\{\mathbf{M}_5,\mathbf{M}_3\}\}+\frac{2}{3}\mathbf{M}_3^4\zeta_3^4 \\
&+2\{\mathbf{M}_5,\mathbf{M}_7\}\zeta_5\zeta_7+2\{\mathbf{M}_3,\mathbf{M}_9\}\zeta_3\zeta_9+2\mathbf{Q}_{13}+2\zeta_5^2\zeta_3\left\{\mathbf{M}_5^2,\mathbf{M}_3\right\}+\zeta_3^2\zeta_7\{\mathbf{M}_3,\{\mathbf{M}_7,\mathbf{M}_3\}\} \\
&+2\mathbf{M}_{13}\zeta_{13}+O(14)\Big). \qquad (6.15)
\end{aligned}$$

Remark 6.2. Note that in (6.15), the weight can be easily seen from the subscript of $\mathbf{M}$ by adding subscripts together, e.g. $\left\{\mathbf{M}_5^2,\mathbf{M}_3\right\}$ has α of weight 13, $(5\times 2+3)$. This information is also stored in ζ subscripts, e.g. $\zeta_5^2\zeta_3$ corresponds to the term of weight 13.

As we see in (6.15), each weight of α is distinguished clearly, for example $2\mathbf{M}_3^2\zeta_3^2$ is of weight 6; $2\zeta_3\zeta_5\{\mathbf{M}_5,\mathbf{M}_3\}$ is of weight 8; $\zeta_3^2\zeta_7\{\mathbf{M}_3,\{\mathbf{M}_7,\mathbf{M}_3\}\}$, 13, etc. We will denote a general one as

$$\mathbf{F}^k(\zeta_{[\dots]}) = \begin{pmatrix} F_1^k(\zeta_{[\dots]}) & F_2^k(\zeta_{[\dots]}) \\ F_3^k(\zeta_{[\dots]}) & F_4^k(\zeta_{[\dots]}) \end{pmatrix}\times\zeta_{[\dots]}, \qquad (6.16)$$

for α of weight k, and a ζ configuration $\zeta_{[\dots]}$, (e.g. $\zeta_3,\zeta_3\zeta_5,\zeta_3^4$... etc).

Here k is the sum of all the subscripts of ζ configuration. For example

Example 6.2.

$$\mathbf{F}^{13}(\zeta_5^2\zeta_3) = \begin{pmatrix} \left\{\boldsymbol{M}_5^2,\boldsymbol{M}_3\right\}_{1,1} & \left\{\boldsymbol{M}_5^2,\boldsymbol{M}_3\right\}_{1,2} \\ \left\{\boldsymbol{M}_5^2,\boldsymbol{M}_3\right\}_{2,1} & \left\{\boldsymbol{M}_5^2,\boldsymbol{M}_3\right\}_{2,2} \end{pmatrix}\times\zeta_5^2\zeta_3. \qquad (6.17)$$

We set all dynamic variables s_{jk} in $\mathbf{F}$ as:

Definition 6.1. $s_{i(i+1)} \equiv s_i$ ($s_{51} = s_{15} \equiv s_5$) and

$$s_i = \alpha(a_i + b_i\eta), \tag{6.18}$$

where a_i, b_i are rational numbers because we are expanding (6.14) in which numerical coefficients are all rational. Note: the α in $s_i = \alpha(a_i + b_i\eta)$ is suppressed in actual calculation as we know which weight we are working on (see Remark 6.2).

We have now make the η manifest in $\mathbf{F}$. As one may find in (6.11), the hard amplitudes are just functions, while the single-valued disk amplitude is a two by two matrix $\mathbf{F}$ (6.15). With this observation we make use of all four entries of $\mathbf{F}$ as basis and expand the corresponding function in (6.14). To do so, one may need four rational coefficients A_1, A_2, A_3, A_4 as defined in (6.19) as a linear rational function of η:

$$A_i = A_{i0} + A_{i1}\eta. \tag{6.19}$$

We take each entry $F_i^k(\zeta_{[...]})$ in (6.16), (e.g. $\left\{\mathbf{M}_5^2, \mathbf{M}_3\right\}_{i,j}$ in example 6.2) to be a basis for the coefficient of $\zeta_{[...]}$ configuration at weight k, i.e.,

$$\left(A_1F_1^k(\zeta_{[...]}) + A_2F_2^k(\zeta_{[...]}) + A_3F_3^k(\zeta_{[...]}) + A_4F_4^k(\zeta_{[...]})\right)\zeta_{[...]}. \tag{6.20}$$

The corresponding coefficient for a given $\zeta_{[...]}$ configuration (at given α weight) on **RHS** (6.14) should equal to (6.20) on the **LHS**. The equation building process in this paragraph can be summarised as follows:

$$\left(A_1F_1^k(\zeta_{[...]}) + A_2F_2^k(\zeta_{[...]}) + A_3F_3^k(\zeta_{[...]}) + A_4F_4^k(\zeta_{[...]})\right) = \text{Coefficients}[H[k], \zeta_{[...]}]. \tag{6.21}$$

Remark 6.3. Here we have used an implicit correspondence that the expression at given weight k of α_s in partonic scattering should be the same as weight k of α in string scattering.

And with the above weight matching equation, we are able to arrive at a polynomial of variable η in both sides. We further match the coefficients of η at given order m:

$$O\left[\left(A_1F_1^k(\zeta_{[...]}) + A_2F_2^k(\zeta_{[...]}) + A_3F_3^k(\zeta_{[...]}) + A_4F_4^k(\zeta_{[...]})\right), \{\eta^m\}\right] = O\left[\text{Coefficients}[H[k], \zeta_{[...]}], \{\eta^m\}\right] \tag{6.22}$$

With $A_k = A_{k0} + A_{k1}\eta$ for $k = 1,2,3,4$ and also 5 kinematic ansatz $s_i = a_i + b_i\eta$ for $i = 1,2,3,4,5$, (see(6.18)), we will get a equation set of 18 variables $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5, A_{10}, A_{20}, A_{30}, A_{40}, A_{11}, A_{21}, A_{31}, A_{41}$. The equation structure is as in Table 3, where E#1($\eta^{\#2}$) means the equation of $O(\alpha, \#1)$ for $O(\eta, \#2)$.

$O(\alpha,0)$:	E0(η^0)	E0(η^1)									
$O(\alpha,3)$:	E3(η^0)	E3(η^1)	E3(η^2)	E3(η^3)	E3(η^4)						
$O(\alpha,5)$:	E5(η^0)	E5(η^1)	E5(η^2)	E5(η^3)	E5(η^4)	E5(η^5)	E5(η^6)				
$O(\alpha,6)$:	E6(η^0)	E6(η^1)	E6(η^2)	E6(η^3)	E6(η^4)	E6(η^5)	E6(η^6)	E6(η^7)			
$O(\alpha,7)$:	E7(η^0)	E7(η^1)	E7(η^2)	E7(η^3)	E7(η^4)	E7(η^5)	E7(η^6)	E7(η^7)	E7(η^8)		
$O(\alpha,8)$:	E8(η^0)	E8(η^1)	E8(η^2)	E8(η^3)	E8(η^4)	E8(η^5)	E8(η^6)	E8(η^7)	E8(η^8)	E8(η^9)	
$\zeta_3^3O(\alpha,9)$:	E9(η^0)	E9(η^1)	E9(η^2)	E9(η^3)	E9(η^4)	E9(η^5)	E9(η^6)	E9(η^7)	E9(η^8)	E9(η^9)	E9(η^{10})
$\zeta_9O(\alpha,9)$:	E9(η^0)′	E9(η^1)′	E9(η^2)′	E9(η^3)′	E9(η^4)′	E9(η^5)′	E9(η^6)′	E9(η^7)′	E9(η^8)′	E9(η^9)′	E9(η^{10})′

Table 3: The red group (with variables $A, a's$) and orange group (with variables $A, b's$) are of rank 3 w.r.t $A's$ individually. Variables $A, a, b's$ are coupled in homogeneous equations for subleading order of η, (green group), subsubleading order of η, (black group), and subsubsubleading order of η, (gray group). Red, orange groups, together with the E3(η^3) are of rank 7. Adding the inhomogeneous one E3(η^1), one can finally determine all the $A's$ with respect to $a's$ and $b's$.

I will list some simple equations here:

$$\begin{aligned}
\text{E0}(\eta^0) &= A_{10}+A_{40}=0, &(6.23)\\
\text{E0}(\eta^1) &= A_{11}+A_{41}=0, &(6.24)\\
\text{E3}(\eta^0) &= -2(a_2^3A20+a_2^2a_3(2A_{20}-A_{40})+a_1^2(A_{20}a_3-a_3A_{30}+a_2(A_{20}-A_{40})-a_3A_{40}+A_{10}(a_3-a_5)-A_{20}a_5)\\
&\quad -a_4(A_{10}a_3(a_3+a_4)+A_{40}a_5(a_4+a_5)+A_{20}(a_3-a_5)(a_3+a_4+a_5))+a_1(A_{20}a_3^2-a_3^2A_{30}+\\
&\quad 2a_3A_{30}a_4+a_2^2(2A_{20}-A_{40})-a_3^2A_{40}+2a_3a_4A_{40}+2a_3A_{30}a_5+2a_3A_{40}a_5-A_{20}a_5^2+\\
&\quad a_2(2A_{10}a_3+3A_{20}a_3-a_3A_{30}-2a_3A_{40}+2a_4A_{40}-A_{20}a_5)+A_{10}(a_3^2-a_5^2))-\\
&\quad a_2(A_{40}(a_3^2-2a_3a_5-2a_4a_5)+A_{20}(-a_3^2+a_3a_4+a_4^2+a_4a_5+a_5^2)))=0, &(6.25)\\
\text{E3}(\eta^4) &= (-A_{11}b_1^2b_3+A_{31}b_1^2b_3-2A_{11}b_1b_2b_3+\\
&\quad A_{31}b_1b_2b_3-A_{11}b_1b_3^2+A_{31}b_1b_3^2-2A_{31}b_1b_3b_4+A_{11}b_3^2b_4+A_{11}b_3b_4^2+A_{11}b_1^2b_5-2A_{31}b_1b_3b_5+A_{11}b_1b_5^2-\\
&\quad A_{21}(b_1+b_2-b_4)(b_2+b_3-b_5)(b_1+b_2+b_3+b_4+b_5)+A_{41}(b_2^2b_3+b_1^2(b_2+b_3)+b_4b_5(b_4+b_5)+\\
&\quad b_2(b_3^2-2b_3b_5-2b_4b_5)+b_1(b_2^2+2b_2(b_3-b_4)+b_3(b_3-2(b_4+b_5)))))=0. &(6.26)
\end{aligned}$$

We can make some observations of the equations:

1.all the variables in red ones of the first column of Table 3 are $a's$ and $A'_{i0}s$;

2.all the variables in orange ones of the greatest η order of Table 3 are $b's$ and $A'_{i1}s$.

3. all the equations are linear in $A's$.

Pick the first 3 or 4 equations in red sector, say E0(η^0) ,E3(η^0),E5(η^0), or adding E6(η^0). We have solutions (6.27) for A_{i0}. With this the rest of equations of red sector is automatically satisfied which was checked up to α weight 12. This shows that the red sector is of rank 3. Here is the solution of A_{i0}:

$$\begin{aligned}
A_{20} &= -\frac{A_{10}\left((-a_3+a_5)\,{a_4}^2+(a_3-a_5)\,(a_1+a_2-a_5)\,a_4+a_5\,a_1\,(a_2+a_3-a_5)\right)}{(a_4+a_5)\,(a_2+a_3-a_5)\,(a_1+a_2-a_4)},\\
A_{30} &= \frac{((a_2\,a_4+a_3\,(a_4+a_5))\,a_1+a_2\,a_5\,(a_3+a_4))\,A_{10}}{a_1\,a_3\,(a_4+a_5)}, \qquad (6.27)\\
A_{40} &= -A_{10}.
\end{aligned}$$

With (6.27), we are able to eliminate 3 capital $A's$ by expressing them via A_{10} and other $a's$ and $b's$.

Remark 6.4. It is excited to find such structure as we want to solve only 18 (finite) variables (see our setting for A_i and s_i). If the red sector is of infinite rank (each equation itself is independent of others), we would need infinite variables which is not viable. We can now mention our expectation of the whole equation set Table 3: all of them are of rank 18. If we are able to find 18 independent equations and solve $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5, A_{10}, A_{20}, A_{30}, A_{40}, A_{11}, A_{21}, A_{31}, A_{41}$, the rest of all the equations will automatically be satisfied! Unfortunately, we don't seem to have such beautiful structure and solution here. See below.

For orange sector, we also use the first 3 or 4 equations, say E0(η^1), E3(η^4),E5(η^6) , or adding E6(η^7), to solve and thus have solution (6.29). Equations in orange sector are also of rank 3, i.e. the rest of the equation in orange sector are automatically satisfied. Here is the solution of A_{i1}:

$$\begin{aligned}
A_{21} &= -\frac{A_{11}\left((-b_3+b_5)\,{b_4}^2+(b_3-b_5)\,(b_1+b_2-b_5)\,b_4+b_5\,b_1\,(b_2+b_3-b_5)\right)}{(b_4+b_5)\,(b_2+b_3-b_5)\,(b_1+b_2-b_4)}\\
A_{31} &= \frac{((b_3\,(b_4+b_5)+b_2\,b_4)\,b_1+b_2\,b_5\,(b_3+b_4))\,A_{11}}{b_1\,b_3\,(b_4+b_5)}, \qquad (6.28)\\
A_{41} &= -A_{11}.
\end{aligned}$$

So we are able to eliminate another 3 capital $A's$ by expressing them via A_{11} and other $a's$ and $b's$ in (6.29).

We can further use the homogeneous E3(η^3) in subleading green sector, to eliminate one capital A. This is because when we substituting all the current solutions (6.27) (6.28) into (6.29) (see below), the only remaining variables are A_{10} and A_{11}. So then, E3(η^3) becomes $A_{10}f(\mathbf{a},\mathbf{b})+A_{11}g(\mathbf{a},\mathbf{b})=0$, and thus, the eliminating A_{10} or A_{11}.

So adding E3(η^3), we have eliminated seven capital $A's$(combining (6.27)(6.29)(6.30)).

$$\begin{aligned}\text{E3}(\eta^3) \;=\;& A_{20}b_2^3 + 2a_1A_{21}b_2^2 + 2A_{21}a_3b_2^2 - a_1A_{41}b_2^2 - a_3A_{41}b_2^2 + 2A_{20}b_1b_2^2 - A_{40}b_1b_2^2 + 2A_{20}b_3b_2^2 - A_{40}b_3b_2^2 + A_{20}b_1^2b_2 \\ & -A_{40}b_1^2b_2 + A_{20}b_3^2b_2 - A_{40}b_3^2b_2 - A_{20}b_4^2b_2 - A_{20}b_5^2b_2 + 2a_1A_{21}b_1b_2 + 3A_{21}a_3b_1b_2 - a_3A_{31}b_1b_2 - 2a_1A_{41}b_1b_2 \\ & \ldots\ldots\ldots\ldots \\ & 2A_{40}b_1b_3b_5 + 2A_{21}a_4b_4b_5 - 2a_4A_{41}b_4b_5 + 2A_{21}a_5b_4b_5 - 2A_{41}a_5b_4b_5 + A_{11} \\ & \left(a_1b_3^2 - a_4b_3^2 + 2a_1b_1b_3 + 2a_1b_2b_3 - 2a_4b_4b_3 - a_1b_5^2 + a_3\left(b_1^2 + 2(b_2+b_3)b_1 - b_4(2b_3+b_4)\right) - 2a_1b_1b_5 - \right. \\ & a_5b_1(b_1+2b_5)) + a_2\left((2A_{11} - A_{31})b_1b_3 - A_{41}\left(b_1^2 + 2(b_2+b_3-b_4)b_1 + b_3^2 + 2b_2b_3 - 2b_3b_5 - 2b_4b_5\right) + \right. \\ & \left.\left. A_{21}\left(b_1^2 + (4b_2+3b_3-b_5)b_1 + 3b_2^2 + b_3^2 - b_4^2 - b_5^2 + 4b_2b_3 - b_3b_4 - b_4b_5\right)\right)\right) = 0 \end{aligned} \tag{6.29}$$

$$A_{11} = \frac{-A_{10}b_1b_3(b_1+b_2-b_4)(b_4+b_5)(b_5-b_2-b_3)(a_1(((a_3-a_5)a_4 + \ldots + (a_3-a_5)a_5)(-b_3+b_5)a_1a_3)a_2)}{((a_1+a_2-a_4)\ldots(-b_3+b_5)b_1(b_1-b_4)b_3)a_3)}. \tag{6.30}$$

Could we further use homogeneous equations in green sector, black sector and gray sector to eliminate the rest one capital A? No we can't. It is not because that the current solutions for $A's$ (all other 7 A's expressed by A_{10}) make them already identities, but the remaining one A, say A_{10} already become an overall factor of those homogeneous. See example 6.3. If we solve them w.r.t. A_{10}, we will get zero for all capital $A's$ which contradicts our assumptions. And the vanishing solution won't satisfy inhomogeneous equations e.g. E3(η^1) (6.32).

Example 6.3. *Here is what we get when we substituting* (6.27)(6.29)(6.30) *into remaining homogeneous equations*

$$\text{E5}(\eta^6) = \frac{A_{10}\times(h(\boldsymbol{a},\boldsymbol{b}))}{t(\boldsymbol{a},\boldsymbol{b})} = 0, \quad \text{E3}(\eta^3) = \frac{A_{10}\times(h'(\boldsymbol{a},\boldsymbol{b}))}{t'(\boldsymbol{a},\boldsymbol{b})} = 0, \tag{6.31}$$

where $\boldsymbol{a} = (a_1,a_2,a_3,a_4,a_5)$, $\boldsymbol{b} = (b_1,b_2,b_3,b_4,b_5)$, h, h', t, t' *are just functions of* $\boldsymbol{a}, \boldsymbol{b}$.

So we cannot pursue further with inhomogeneous equations to eliminate all A's. And we should now make use of those inhomogeneous ones.

$$\begin{aligned}\text{E3}(\eta^1) \;=\;& 1/8 + 2(-(a_2^3A_{21}) + A_{11}a_3^2a_4 + A_{21}a_3^2a_4 + A_{11}a_3a_4^2 + A_{21}a_3a_4^2 - A_{21}a_4^2a_5 + a_4^2A_{41}a_5 - A_{21}a_4a_5^2 + \\ & a_4A_{41}a_5^2 - A_{10}a_3^2b_1 - A_{20}a_3^2b_1 + a_3^2A_{30}b_1 - 2a_3A_{30}a_4b_1 + a_3^2A_{40}b_1 - 2a_3a_4A_{40}b_1 - 2a_3A_{30}a_5b_1 \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & a_2(a_3^2A_{41} - 2a_4A_{41}a_5 + A_{21}(-a_3^2 + a_3a_4 + a_4^2 + a_4a_5 + a_5^2) - 2a_4A_{40}b_1 + A_{20}a_5b_1 + A_{20}a_4b_3 - 2A_{40}a_5b_3 \\ & +2A_{20}a_4b_4 + A_{20}a_5b_4 - 2A_{40}a_5b_4 + A_{20}a_4b_5 - 2a_4A_{40}b_5 + 2A_{20}a_5b_5 + a_3(-2A_{41}a_5 \\ & -2A_{10}b_1 - 3A_{20}b_1 + A_{30}b_1 + 2A_{40}b_1 - 4A_{20}b_2 + 2A_{40}b_2 - 2A_{20}b_3 + 2A_{40}b_3 + A_{20}b_4 - 2A_{40}b_5))) = 0. \end{aligned} \tag{6.32}$$

We combine

$$\left\{\text{E0}(\eta^0), \text{E3}(\eta^0), \text{E5}(\eta^0), \text{E0}(\eta^1), \text{E3}(\eta^4), \text{E5}(\eta^6), \text{E3}(\eta^3), \text{E3}(\eta^1)\right\}. \tag{6.33}$$

And eliminate all $A's$ and express them with $a's, b's$, e.g. $\{A_{10}(\mathbf{a},\mathbf{b}), A_{20}(\mathbf{a},\mathbf{b})\ldots A_{41}(\mathbf{a},\mathbf{b})\}$. These lead to some really interesting expression. After the submission of $\{A_{10}(\mathbf{a},\mathbf{b}), A_{20}(\mathbf{a},\mathbf{b})\ldots A_{41}(\mathbf{a},\mathbf{b})\}$. We have $a's$ decouple with $b's$ in ζ_3 related equations:

E6(η^1)$|_{A(a,b)}$ for ζ_3^2; E9(η^1)$|_{A(a,b)}$ for ζ_3^3; E12(η^1)$|_{A(a,b)}$ for ζ_3^4.

$$\begin{aligned}\text{E6}(\eta^1)|_{A(a,b)} \;=\;& (-4\,a_2 - 4\,a_5)\,{a_1}^2 + \left(-4\,{a_2}^2 + 8\,a_2\,a_4 + (8\,a_4 + 8\,a_5)\,a_3 - 4\,{a_5}^2\right)a_1 - 4\,{a_2}^2a_3 \\ & + \left(-4\,{a_3}^2 + 8\,a_3\,a_5 + 8\,a_4\,a_5\right)a_2 - 4\,{a_3}^2a_4 - 4\,a_3\,{a_4}^2 - 4\,{a_4}^2a_5 - 4\,a_4\,{a_5}^2 + 5 = 0, \\ \text{E9}(\eta^1)|_{A(a,b)} \;=\;& \left(-80\,{a_2}^2 - 80\,a_5\,a_2 - 80\,{a_5}^2\right){a_1}^4 + \left(-160\,{a_2}^3 + (320\,a_4 - 80\,a_5)\,{a_2}^2\right. \\ & + \left((320\,a_4 + 160\,a_5)\,a_3 + 160\,a_4\,a_5 - 80\,{a_5}^2\right)a_2 + 320\,a_5\,\left(a_3\,(a_4 + a_5) - 1/2\,{a_5}^2\right)\Big)\,{a_1}^3 \\ & +\ldots\ldots\ldots\ldots \\ & -\,2\,a_4\,a_5)\,a_2 - 80\,{a_3}^4{a_4}^2 - 160\,{a_3}^3{a_4}^3 + \left(-80\,{a_4}^4 - 80\,{a_4}^3a_5 - 80\,{a_4}^2{a_5}^2\right){a_3}^2 \\ & -80\,{a_4}^3a_5\,(a_4 + a_5)\,a_3 - 80\,{a_4}^4{a_5}^2 - 160\,{a_4}^3{a_5}^3 - 80\,{a_4}^2{a_5}^4 + 51 = 0, \end{aligned} \tag{6.34}$$

$$\begin{aligned}\text{E12}(\eta^1)|_{A(a,b)} \;=\;& -(a_2+a_5)(a_2^2+a_5^2)a_1^6 + (-3a_2^4 + (6a_4 - 2a_5)a_2^3 + ((6a_4 + 2a_5)a_3 + \\ & 4a_4a_5 - 2a_5^2)a_2^2 + ((8a_4a_5 + 4a_5^2)a_3 + 2a_5^2(a_4 - a_5))a_2 + 6a_5^2(a_4 + a_5)a_3 - 3a_5^4)a_1^5 \\ & \ldots\ldots\ldots \\ & (-3a_4^5 - a_4^4a_5 - a_4^3a_5^2)a_3^4 + (-a_4^6 - 2a_4^5a_5 - 2a_4^4a_5^2)a_3^3 + (-a_4^6a_5 - 2a_4^5a_5^2 - \\ & 2a_4^4a_5^3 - a_4^3a_5^4)a_3^2 - a_4^4a_5^2(a_4 + a_5)^2a_3 - a_4^6a_5^3 - 3a_4^5a_5^4 - 3a_4^4a_5^5 - a_4^3a_5^6 + 119429640 = 0. \end{aligned} \tag{6.35}$$

We can proceed with these three equations and see what happens.

6.2.2 Evidence that the ansatz should be generalised

Here are the equations we have used:

$O(\alpha,0)$:	$\mathrm{E0}(\eta^0)$	$\mathrm{E0}(\eta^1)$									
$O(\alpha,3)$:	$\mathrm{E3}(\eta^0)$	$\mathrm{E3}(\eta^1)$	$\mathrm{E3}(\eta^2)$	$\mathrm{E3}(\eta^3)$	$\mathrm{E3}(\eta^4)$						
$O(\alpha,5)$:	$\mathrm{E5}(\eta^0)$	$\mathrm{E5}(\eta^1)$	$\mathrm{E5}(\eta^2)$	$\mathrm{E5}(\eta^3)$	$\mathrm{E5}(\eta^4)$	$\mathrm{E5}(\eta^5)$	$\mathrm{E5}(\eta^6)$				
$O(\alpha,6)$:	$\mathrm{E6}(\eta^0)$	$\mathrm{E6}(\eta^1)$	$\mathrm{E6}(\eta^2)$	$\mathrm{E6}(\eta^3)$	$\mathrm{E6}(\eta^4)$	$\mathrm{E6}(\eta^5)$	$\mathrm{E6}(\eta^6)$	$\mathrm{E6}(\eta^7)$			
$O(\alpha,7)$:	$\mathrm{E7}(\eta^0)$	$\mathrm{E7}(\eta^1)$	$\mathrm{E7}(\eta^2)$	$\mathrm{E7}(\eta^3)$	$\mathrm{E7}(\eta^4)$	$\mathrm{E7}(\eta^5)$	$\mathrm{E7}(\eta^6)$	$\mathrm{E7}(\eta^7)$	$\mathrm{E7}(\eta^8)$		
$O(\alpha,8)$:	$\mathrm{E8}(\eta^0)$	$\mathrm{E8}(\eta^1)$	$\mathrm{E8}(\eta^2)$	$\mathrm{E8}(\eta^3)$	$\mathrm{E8}(\eta^4)$	$\mathrm{E8}(\eta^5)$	$\mathrm{E8}(\eta^6)$	$\mathrm{E8}(\eta^7)$	$\mathrm{E8}(\eta^8)$	$\mathrm{E8}(\eta^9)$	
$\zeta_3^3 O(\alpha,9)$:	$\mathrm{E9}(\eta^0)$	$\mathrm{E9}(\eta^1)$	$\mathrm{E9}(\eta^2)$	$\mathrm{E9}(\eta^3)$	$\mathrm{E9}(\eta^4)$	$\mathrm{E9}(\eta^5)$	$\mathrm{E9}(\eta^6)$	$\mathrm{E9}(\eta^7)$	$\mathrm{E9}(\eta^8)$	$\mathrm{E9}(\eta^9)$	$\mathrm{E9}(\eta^{10})$
$\zeta_9 O(\alpha,9)$:	$\mathrm{E9}(\eta^0)'$	$\mathrm{E9}(\eta^1)'$	$\mathrm{E9}(\eta^2)'$	$\mathrm{E9}(\eta^3)'$	$\mathrm{E9}(\eta^4)'$	$\mathrm{E9}(\eta^5)'$	$\mathrm{E9}(\eta^6)'$	$\mathrm{E9}(\eta^7)'$	$\mathrm{E9}(\eta^8)'$	$\mathrm{E9}(\eta^9)'$	$\mathrm{E9}(\eta^{10})'$
$\zeta_3^4 O(\alpha,12)$:		$\mathrm{E12}(\eta^1)$									$\mathrm{E12}(\eta^{12})$

Figure 3: The yellow denote the equations we have used, including some automatically satisfied.

Although $\mathrm{E6}(\eta^1)|_{A(a,b)}$, $\mathrm{E9}(\eta^1)|_{A(a,b)}$, $\mathrm{E12}(\eta^1)|_{A(a,b)}$ for ζ_3^4, get decoupled with $b's$, they don't yield a solution. As they are all polynomials of $(a_1, a_2, a_3, a_4, a_5)$, we can construct the polynomial ideal (see (6.5)) to study the property of zeros. In another words, we use all the possible polynomials P_1, P_2, P_3 with variables a_1, a_2, a_3, a_4, a_5 to make the set of $P\mathrm{E6}(\eta^1)|_{A(a,b)} + P_2\mathrm{E9}(\eta^1)|_{A(a,b)} + P_3\mathrm{E12}(\eta^1)|_{A(a,b)}$:

$$\left\langle \mathrm{E6}(\eta^1)|_{A(a,b)}, \mathrm{E9}(\eta^1)|_{A(a,b)}, \mathrm{E12}(\eta^1)|_{A(a,b)} \right\rangle . \tag{6.36}$$

Unfortunately we have (from *PolynomialIdeal in Maple*, *Solve in Mathematica* and *Singular*)

$$1 \in \left\langle \mathrm{E6}(\eta^1)|_{A(a,b)}, \mathrm{E9}(\eta^1)|_{A(a,b)}, \mathrm{E12}(\eta^1)|_{A(a,b)} \right\rangle , \tag{6.37}$$

which according to (6.7)that there is not solution.
So $\mathrm{E6}(\eta^1)|_{A(a,b)}, \mathrm{E9}(\eta^1)|_{A(a,b)}, \mathrm{E12}(\eta^1)|_{A(a,b)}$ are inconsistent polynomial equations and will not yield common zeros. This is the evidence showing that our ansatz of expanding hard part of scattering amplitude with single-valued string 5-point disk amplitude needs further generalization.

7 Conclusion and discussion

We have found that in tree-level string scattering, spherical integral in four-point closed string scattering amplitude to all order can be derived via the single value map of disk integral in four-point open string scattering amplitude. Furthermore, we find that the *single valued uniform weighted* matrix part of five-point closed string scattering amplitude to 13 order can be derived via single value map of the matrix part of the disk integral for five-point open string scattering amplitude. On the process of deriving these, we have detailed studied single value map and relevant operations.

We make an ansatz that single-valued five-point open string scattering amplitude could be the basis to expand the hard part of two-to-two partonic scattering amplitude in Regge limit and set up equations to solve the coefficients. We eventually find that there is inconsistency in the solution which may lead to generalizing the ansatz.

For future arrangement, although we find that $\mathrm{E6}(\eta^1)|_{A(a,b)}, \mathrm{E9}(\eta^1)|_{A(a,b)}, \mathrm{E12}(\eta^1)|_{A(a,b)}$ do not have common zeros, we still need to further check before getting into generalization of ansatz. One immediate step, based on current observation, is to directly check the consistency of first column of blue sector without solving any variable in advance. As in remark 6.4, we need 18 equations of 18 rank to solve $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5, A_{10}, A_{20}, A_{30}, A_{40}, A_{11}, A_{21}, A_{31}, A_{41}$. The inhomogeneous blue sector has a great possibility that the equations we pick here are of different rank (unlike those cf. red sector, orange sector), but may also result in inconsistent equations. So one may get more persuasive result from further analysis on this sector. However, there is a technical problem in blue sector. As one get in deeper in blue sector, i.e. second column and third column, equation become extremely long and a PC with 24GB RAM will not able to handle this using *Maple, Mathematica or Singular*. As most of the cluster is not supporting these software, we may, in the future, try to make C++ code to make the equation analysing available on super computer.

If the above analysis in blue sector confirms with current finding, we may consider adding another five point spherical amplitude. As in (5.12), we only consider string disk amplitude $\mathcal{A}(1,2,3,4,5)$ and $\mathcal{A}(1,3,2,4,5)$ so far. We could consider other permutations of $1,2,3,4,5$ in $\mathcal{A}$: $\mathcal{A}(p_1,p_2,p_3,p_4,p_5)$. If the resulting closed string amplitude for $\mathcal{A}(p_1,p_2,p_3,p_4,p_5)$ after the single value map also has *uniform weight* properties, we may add it to our ansatz by setting $B_i = B_{i0} + \eta B_{i1}$ and also $s_k = c_k + \eta b_k$ for this new closed string scattering amplitude basis. If all five-point amplitudes do not work, we may consider six-point further. Furthermore, to greatly speed up the analysing and solving polynomial equations, we may also systematically study and apply finite field method [15][17] in the future.

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References

[1] I.I. Balitsky and L.N. Lipatov. “The Pomeranchuk Singularity in Quantum Chromodynamics”. In: *Sov. J. Nucl. Phys.* 28 (1978), pp. 822–829.

[2] M Bigotte et al. “Lyndon words and shuffle algebras for generating the coloured multiple zeta values relations tables”. In: *Theoretical computer science* 273.1-2 (2002), pp. 271–282.

[3] Francis Brown. *On the decomposition of motivic multiple zeta values.* 2011. arXiv: `1102.1310 [math.NT]`.

[4] Francis Brown. *Single-valued periods and multiple zeta values.* 2013. arXiv: `1309.5309 [math.NT]`.

[5] Francis CS Brown et al. “On the decomposition of motivic multiple zeta values”. In: *Galois–Teichmüller theory and arithmetic geometry.* Mathematical Society of Japan. 2012, pp. 31–58.

[6] Simon Caron-Huot, Einan Gardi, and Leonardo Vernazza. “Two-parton scattering in the high-energy limit”. In: *Journal of High Energy Physics* 2017.6 (June 2017). ISSN: 1029-8479. DOI: `10.1007/jhep06(2017)016`. URL: `http://dx.doi.org/10.1007/JHEP06(2017)016`.

[7] Simon Caron-Huot et al. “Infrared singularities of QCD scattering amplitudes in the Regge limit to all orders”. In: *Journal of High Energy Physics* 2018.3 (Mar. 2018). ISSN: 1029-8479. DOI: `10.1007/jhep03(2018)098`. URL: `http://dx.doi.org/10.1007/JHEP03(2018)098`.

[8] Simon Caron-Huot et al. *Two-parton scattering amplitudes in the Regge limit to high loop orders.* 2020. arXiv: `2006.01267 [hep-ph]`.

[9] D.Tong. *David Tong: Lectures on String Theory.* URL: `https://www.damtp.cam.ac.uk/user/tong/string/six.pdf`.

[10] Lance J. Dixon, Claude Duhr, and Jeffrey Pennington. “Single-valued harmonic polylogarithms and the multi-Regge limit”. In: *Journal of High Energy Physics* 2012.10 (Oct. 2012). ISSN: 1029-8479. DOI: `10.1007/jhep10(2012)074`. URL: `http://dx.doi.org/10.1007/JHEP10(2012)074`.

[11] Claude Duhr and Falko Dulat. “PolyLogTools — polylogs for the masses”. In: *Journal of High Energy Physics* 2019.8 (Aug. 2019). ISSN: 1029-8479. DOI: `10.1007/jhep08(2019)135`. URL: `http://dx.doi.org/10.1007/JHEP08(2019)135`.

[12] *Hypergeometric functions.* URL: `https://homepage.tudelft.nl/11r49/documents/wi4006/hyper.pdf`.

[13] E.A. Kuraev, L.N. Lipatov, and Victor S. Fadin. “The Pomeranchuk Singularity in Nonabelian Gauge Theories”. In: *Sov. Phys. JETP* 45 (1977), pp. 199–204.

[14] Hoang Ngoc Minh and Michel Petitot. “Lyndon words, polylogarithms and the Riemann zeta function”. In: *Discrete Mathematics* 217.1-3 (2000), pp. 273–292.

[15] Chenqi Mou. “Solving Polynomial Systems over Finite Fields: Algorithms, Implementation and Applications”. Theses. Université Pierre et Marie Curie, May 2013. URL: `https://tel.archives-ouvertes.fr/tel-01110887`.

[16] S. Stieberger O. Schlotterer. *Alpha’-expansion of open superstring amplitudes.* URL: `https://wwwth.mpp.mpg.de/members/stieberg/mzv/index.html`.

[17] Tiziano Peraro. “Scattering amplitudes over finite fields and multivariate functional reconstruction”. In: *Journal of High Energy Physics* 2016.12 (Dec. 2016). ISSN: 1029-8479. DOI: `10.1007/jhep12(2016)030`. URL: `http://dx.doi.org/10.1007/JHEP12(2016)030`.

[18] O Schlotterer and S Stieberger. “Motivic multiple zeta values and superstring amplitudes”. In: *Journal of Physics A: Mathematical and Theoretical* 46.47 (Nov. 2013), p. 475401. ISSN: 1751-8121. DOI: `10.1088/1751-8113/46/47/475401`. URL: `http://dx.doi.org/10.1088/1751-8113/46/47/475401`.

[19] Oliver Schlotterer and Oliver Schnetz. “Closed strings as single-valued open strings: a genus-zero derivation”. In: *Journal of Physics A: Mathematical and Theoretical* 52.4 (Jan. 2019), p. 045401. ISSN: 1751-8121. DOI: `10.1088/1751-8121/aaea14`. URL: `http://dx.doi.org/10.1088/1751-8121/aaea14`.

[20] D. Wang, G.E. Collins, and B. Buchberger. *Elimination Methods.* Texts & Monographs in Symbolic Computation. Springer, 2001. ISBN: 9783211832417. URL: `https://books.google.com/books?id=7gwLTNwRhPMC`.