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# Magnetic Monopole in the Pati-Salam Model

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**Abstract**

Magnetic monopole is a hypothetical particle that produce magnetic fields with non-zero divergence. However, none have been found experimentally. There is a mathematical solution of magnetic monopole discovered by 't Hooft and Polyakov by considering $SO(3)$ gauge theory [1, 2].

In this thesis, we will be focusing on finding magnetic monopoles in the Pati-Salam model. We are able to find the solution of a monopole similiar to the 't Hooft–Polyakov monopole in the Pati-Salam model. We found that its mass is around the Pati-Salam symmetry breaking scale.

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# Chapter 1

# Introduction

## 1.1 Magnetic monopole

One of the elementary properties of magnetism is that magnets are magnetic dipoles and magnetic field lines are always in loops. However, in 1974, a magnetic monopole in non-Abelian gauge theories were discovered by 't Hooft and Polyakov [1, 2]. The 't Hooft–Polyakov monopole is a topological soliton. It arises when the gauge group $SO(3)$, coupled to a scalar field, is spontaneously broken to $U(1)$ via Higgs mechanism.

Similar to electric charges, magnetic field lines start or end at magnetic monopoles. They interact strongly with electromagnetic fields. Classically, separated magnetic poles are forbidden in the formulation of electrodynamics as the magnetic field is given by the curl of the vector potential.

$$\vec{B} = \nabla \times \vec{A} \tag{1.1}$$

Divergence of curl of any vector field must vanish.

$$\nabla \cdot \left(\nabla \times \vec{V}\right) = 0 \tag{1.2}$$

So, magnetic monopoles cannot be described by vector potential in classical electrodynamics.

A particle with electric charge q produces electric field

$$\vec{E} = q\left(\frac{\vec{r}}{r^3}\right) \tag{1.3}$$

A charge moving in an electromagnetic field with velocity v would experience Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{1.4}$$

However, the reciprocity between electricity and magnetism suggests that if magnetic monopole exists, with magnetic charge g, it would produce magnetic field

$$\vec{B} = g\left(\frac{\vec{r}}{r^3}\right) \tag{1.5}$$

and it would experience "Lorentz force"

$$\vec{F} = g(\vec{B} - \vec{v} \times \vec{E}) \tag{1.6}$$

[3]

In 1931, Dirac proposed a vector field that could describe magnetic monopole [4]. Introduce the electromagnetic potential $\vec{B}$. Note that this is not the magnetic flux density we mentioned before. This electromagnetic potential $\vec{B}$ introduced by Dirac satisfies

$$\vec{E} = \nabla \times \vec{B} \tag{1.7.1}$$

$$\vec{H} = \frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \nabla \vec{B}_0 \tag{1.7.2}$$

[4], where $\vec{H}$ is the magnetic field strength. Note that the magnetic flux density and magnetic field strength differs by magnetization field.

Dirac suggested that it would result in magnetic charge being $\mu_0 = \frac{137}{2}e$. So, the attractive force between monopole and anti-monopole is $(137/2)^2 = 4962.25$ times the attractive force between proton and electron, so magnetic monopoles with opposite charge had never been seperated yet [4].

This formulation allows the existence of magnetic monopole. However, it does not describe electric charges. This means the dynamics of electric charge and magnetic charge cannot be described simultaneously in a simple $U(1)$ gauge theory. The solution to this is found to be non-abelian gauge theory e.g. $SO(3)$ or $SU(2)$. 't Hooft–Polyakov monopole was discovered as topological soliton by considering $SO(3)$ theory [1, 2].

Following the discovery of 't Hooft–Polyakov monopole, there have been searches and experiments to detect magnetic monopoles. Monopoles should be produced in particle accelerators experiments if the collision energy is higher than $2Mc^2$, where $M$ is the mass of the monopole. Magnetic monopoles are easy to detect as they interact strongly with electromagnetic field. Unlike other particles produced in accelerators, magnetic monopoles are stable as they do not decay. A monopole can only be destroyed when it meets an anti-monopole [3].

The Dirac quantization condition states that $qg = N/2$ [5, 6], where $q$ and $g$ are the electric charge and magnetic charge respectively, $N$ is an integer. The magnetic charge of the 't Hooft–Polyakov monopole is found to be $\frac{1}{e}$ [1]. In $SO(3)$ theory, $e$ is the elementary electric charge. This is in contradiction with the minimum charge allowed by Dirac quantization condition.

To resolve the contradiction of minimum electric and magnetic charge of a particle (carrying both and only electric and magnetic charge) allowed by Dirac quantization condition, we are searching for magnetic monopole in a theory where the elementary electric charge is $\frac{e}{2}$. We could consider a different gauge group. It is intuitive that if we want half the charge we got in $SO(3)$ theory, we should try $SU(2)$ instead as $SU(2)$ is a double cover of $SO(3)$. We are looking for magnetic monopoles in the Pati-Salam model since it is a candidate of grand unified theory, and it has $SU(2)$ supgroup spontaneously broken into $U(1)$.

## 1.2 Pati-Salam model

The Pati-Salam model $SU(4)_{PS} \times SU(2)_L \times SU(2)_R$ is an alternative of $SU(5)$ grand unification [7, 8]. Both the Pati-Salam group and $SU(5)$ are subgroup of $SO(10)$, and both can be further spontaneously broken to the Standard Model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ [7]. The subscript PS stands for Pati-Salam, C for colour, L for left-handed particle, R for right-handed particle, and Y for weak hyper-charge.

### 1.2.1 Symmetry breaking scale

At the moment, we only understand physics up to electroweak scale $\sim 100$GeV. We do not fully understand how Physics work at higher energy scale. It is possible that there are magnetic monopoles with mass above electroweak scale but below GUT scale. e.g. Pati-Salam breaking scale. The Pati-Salam model has a much lower symmetry breaking scale $\sim 1000$TeV compared to $SU(5) \sim 10^{16}$GeV [9]. Although LHC could only accelerate particles to a few TeV, the Pati-Salam energy scale is more realistic and achievable in near future. It is much more likely to be tested experimentally. This is why we are searching for magnetic monopoles in the Pati-Salam model in this project.

### 1.2.2 Leptons as the fourth color

In the Standard Model, leptons are treated as singlets of $SU(3)$. In the Pati-Salam model, the gauge symmetry $SU(3)$ of the 3 colors is extended to $SU(4)$ of 4 colors. The leptons are treated as the fourth color [8].

### 1.2.3 Left-right symmetry

The right-handed particles are also a doublet in the Pati-Salam model. Left-handed particles are in (4,2,1) representation, and the right-handed particles are in (4,1,2) representation [7]. The Pati-Salam group has several subgroups. The Standard Model group is one of them. Then, we requires $SU(4)_{PS} \times SU(2)_R$ spontaneously broken into $SU(3)_C \times U(1)_Y$. This breaks the Left-right symmetry. Right-handed particles in the Standard Model do not interact via weak force.

## 1.3 Outline of the project

In this project, we will follow the method in 't Hooft's paper [1]. In chapter 2, we first calculate the Pati-Salam symmetry breaking and deduce the vacuum manifold. From this, we will embed a 2 sphere into the vacuum manifold which will give rise to a monopole. In chapter 3, we will then write a suitable ansatz for the gauge connection, and derive the associated Euler-Lagrange equations. Lastly, we will evaluate the mass and charge of the monopole.

# Chapter 2

# Pati-Salam symmetry breaking

Magnetic monopoles arise when $SU(2)$ symmetry group is broken into $U(1)$. We are interested in the SSB process that the Pati-Salam group is spontaneouly broken into the Standard Model group. That requires $SU(4)_{PS} \times SU(2)_R \rightarrow SU(3)_C \times U(1)_Y$.

To find the residual symmetry group after SSB, we need to define a matrix [10]

$$S^{ab} = Q_0^\dagger \{t^a, t^b\} Q_0 \tag{2.1}$$

where $t^a$ are the generators, $Q_0$ is the vacuum expectation value of the scalar field.

$SU(4)$ group has 15 generators and $SU(2)$ group has 3 generators. So, $SU(4) \times SU(2)$ has 18 generators in total (see Appendix A).

Eigenvectors of $S^{ab}$ with zero eigenvalue corresponds to the unbroken generators, and eigenvectors with non-zero eigenvalue corresponds to the broken generators.

The $SU(4)$ and $SU(2)$ gauge group may have different coupling constants. We will re-scale the $SU(2)$ generators by $c = \frac{g_2}{g_4}$ such that the covarient derivative can be written as $D_\mu Q = \partial_\mu Q + ig_4 A_\mu^a t^a Q$.

The VEV required such that the residual symmetry group is $SU(3) \times U(1)$ is found to be $Q_0 = (0,0,0,0,0,0,0,v)$, where v is the field value at the minimum potential.

For VEV = $Q_0 = (0,0,0,0,0,0,0,v)$ (see appendix B.1),

$$S^{ab} = v^2 \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&3&0&0&\sqrt{6}c\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2c^2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2c^2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\sqrt{6}c&0&0&2c^2
\end{pmatrix} \tag{2.2}$$

The eigen-vectors are

$$\lambda_1 = 0 \qquad v_1 = (1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \tag{2.3.1}$$
$$\lambda_2 = 0 \qquad v_2 = (0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \tag{2.3.2}$$
$$\lambda_3 = 0 \qquad v_3 = (0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \tag{2.3.3}$$
$$\lambda_4 = 0 \qquad v_4 = (0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \tag{2.3.4}$$
$$\lambda_5 = 0 \qquad v_5 = (0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0) \tag{2.3.5}$$
$$\lambda_6 = 0 \qquad v_6 = (0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0) \tag{2.3.6}$$
$$\lambda_7 = 0 \qquad v_7 = (0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0) \tag{2.3.7}$$
$$\lambda_8 = 0 \qquad v_8 = (0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0) \tag{2.3.8}$$
$$\lambda_9 = 2v^2 \qquad v_9 = (0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0) \tag{2.3.9}$$
$$\lambda_{10} = 2v^2 \qquad v_{10} = (0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0) \tag{2.3.10}$$
$$\lambda_{11} = 2v^2 \qquad v_{11} = (0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0) \tag{2.3.11}$$
$$\lambda_{12} = 2v^2 \qquad v_{12} = (0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0) \tag{2.3.12}$$
$$\lambda_{13} = 2v^2 \qquad v_{13} = (0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0) \tag{2.3.13}$$
$$\lambda_{14} = 2v^2 \qquad v_{14} = (0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0) \tag{2.3.14}$$
$$\lambda_{15} = 0 \qquad v_{15} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,cos\gamma,0,0,-sin\gamma) \tag{2.3.15}$$
$$\lambda_{16} = 2c^2v^2 \qquad v_{16} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0) \tag{2.3.16}$$
$$\lambda_{17} = 2c^2v^2 \qquad v_{17} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0) \tag{2.3.17}$$
$$\lambda_{18} = (3+2c^2)v^2 \qquad v_{18} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,sin\gamma,0,0,cos\gamma) \tag{2.3.18}$$

where $sin\gamma = \frac{\sqrt{\frac{3}{2}}g_4}{\sqrt{g_2^2+\frac{3}{2}g_4^2}}$, $cos\gamma = \frac{g_2}{\sqrt{g_2^2+\frac{3}{2}g_4^2}}$

Define a new set of generators $T^a = v_{ab}t^b$. $T^1$ to $T^8$ are the unbroken $SU(4)$ generators, which form the residual $SU(3)$ group. $T^9$ to $T^{14}$ are the broken $SU(4)$ generators. $T^{16}$ and $T^{17}$ are the broken $SU(2)$ generators. $T^{15}$ is the residual $U(1)$ generator, which is a linear combination of $\sigma_z$ and $t^{15}$. $T^{18}$ is the linear combination orthonormal to $T^{15}$, which is broken. So, we can identify that the residual symmetry group is $SU(3) \times U(1)$

## 2.1 Vacuum manifold

In order to find the Lagrangian, we would have to parameterise the scalar field. Take the vacuum manifold as $G/H$, where $G$ is the gauge group and $H$ is the unbroken subgroup [10].

### 2.1.1 $SU(2) \times SU(2) \to U(1)$

Let's consider a simpler case of $SU(2) \times SU(2) \to U(1)$ first. For simplicity, we assume the 2 $SU(2)$ groups have the same coupling constant.

For VEV $= (0,0,0,v)$ (see appendix B.2),

$$S^{ab} = v^2 \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{pmatrix} \tag{2.4}$$

The eigenvectors are

$$\lambda_1 = 4 \qquad v_1 = (0,0,1,0,0,1) \tag{2.5.1}$$
$$\lambda_2 = 2 \qquad v_2 = (1,0,0,0,0,0) \tag{2.5.2}$$
$$\lambda_3 = 2 \qquad v_3 = (0,1,0,0,0,0) \tag{2.5.3}$$
$$\lambda_4 = 2 \qquad v_4 = (0,0,0,1,0,0) \tag{2.5.4}$$
$$\lambda_5 = 2 \qquad v_5 = (0,0,0,0,1,0) \tag{2.5.5}$$
$$\lambda_6 = 0 \qquad v_6 = (0,0,-1,0,0,1) \tag{2.5.6}$$

There is only 1 unbroken generator. So we can identify that the residual symmetry group is $U(1)$. Define a new set of generators $T^a = v_{ab}t^b$.

Note that in the Pati-Salam model, We only need a 2-sphere embedding in the vacuum manifold to produce a magnetic monopole. We can parameterise any point on a 2-sphere with 2 parameters. So, in the $SU(2) \times SU(2)$ case, we chose only $T^1$ and $T^3$ when defining the 2-sphere. Note that the normalization does not matter when we consider the vacuum manifold.

The scalar field has 2 indices. We can represent it as a matrix, so that the first index transforms by multiplying $SU(2)$ matrix on the left, and the second index transforms by multiplying the transpose of $SU(2)$ matrix on the right.

The VEV can be factorized into $Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} = v \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$. With $\sigma_i$ being the $SU(2)$ generators, we can define the embeded 2-sphere in the vacuum manifold as $\chi = e^{ig\sigma_3\alpha_2} e^{ig\sigma_2\alpha_1} \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} e^{ig\sigma_3\alpha_2}$, where $\alpha_1$ and $\alpha_2$ are real parameters.

We require $\chi$ to satisfies $\chi(0,\theta) = \chi(2\pi,\theta)$. Fix the parameters $\alpha_1$ and $\alpha_2$ such that $\chi = v \begin{pmatrix} 0 & e^{i\phi} sin\frac{n\theta}{2} \\ 0 & cos\frac{n\theta}{2} \end{pmatrix}$, where $n$ is an integer. $\phi$ and $\theta$ are the azimuthal and polar angle respectively in spherical coordinates.

### 2.1.2 $SU(4) \times SU(2) \to SU(3) \times U(1)$

Back to the $SU(4) \times SU(2)$ case. As we only need a 2-sphere embedding in the vacuum manifold to produce a magnetic monopole, we chose to take only $T^{17}$ and $T^{18}$ when parameterising the scalar field instead of the full vacuum manifold.

We can represent the scalar field as

$$\begin{aligned}
\chi &= e^{ig_4 sin\gamma t^{15}\alpha_1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & v \end{pmatrix} e^{-ig_2 cos\gamma t^{17}\alpha_2} e^{ig_2 t^{18}\alpha_1} \\
&= v \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-\frac{3ig_4\alpha_1}{2k}} \end{pmatrix} \begin{pmatrix} sin(g_2\alpha_2) e^{\frac{i(g_2)^2\alpha_1}{g_4 k}} & cos(g_2\alpha_2) e^{\frac{-i(g_2)^2\alpha_1}{g_4 k}} \end{pmatrix} \\
&= v \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} sin(g_2\alpha_2) e^{\frac{i[(g_2)^2 - \frac{3}{2}(g_4)^2]\alpha_1}{g_4 k}} & cos(g_2\alpha_2) e^{\frac{-i[(g_2)^2 + \frac{3}{2}(g_4)^2]\alpha_1}{g_4 k}} \end{pmatrix}
\end{aligned} \tag{2.6}$$

where $k = \sqrt{g_2^2 + \frac{3}{2}g_4^2}$

We can rotate it by a global phase as there is unbroken $U(1)$ subgroup in the gauge group. The Lagrangian is invariant under this rotation. The scalar field becomes

$$\chi = v \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} sin(g_2\alpha_2) e^{\frac{2i(g_2)^2\alpha_1}{g_4 k}} & cos(g_2\alpha_2) \end{pmatrix} \tag{2.7}$$

Then, fix the parameters $\alpha_1$ and $\alpha_2$ such that

$$\chi = v \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} e^{i\phi} sin\frac{n\theta}{2} & cos\frac{n\theta}{2} \end{pmatrix} \tag{2.8}$$

This describes a 2-sphere embedding in the vacuum manifold. Next, we want to show that it cannot be deformed continuously to a point such that it has non-trivial homotopy which gives rise to magnetic monopoles. Consider the vector

$$\zeta^a = tr[\chi^\dagger \chi (\sigma^a)^T] \tag{2.9}$$

Calculate its components

$$\begin{aligned}
\zeta^1 &= v^2 tr \left[ \begin{pmatrix} 0 & 0 & 0 & e^{-i\phi} sin\frac{n\theta}{2} \\ 0 & 0 & 0 & cos\frac{n\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ e^{i\phi} sin\frac{n\theta}{2} & cos\frac{n\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \\
&= v^2 tr \left[ \begin{pmatrix} 0 & 0 & 0 & e^{-i\phi} sin\frac{n\theta}{2} \\ 0 & 0 & 0 & cos\frac{n\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ cos\frac{n\theta}{2} & e^{i\phi} sin\frac{n\theta}{2} \end{pmatrix} \right] \\
&= v^2 sin\frac{n\theta}{2} cos\frac{n\theta}{2} (e^{i\phi} + e^{-i\phi}) \\
&= 2v^2 sin\frac{n\theta}{2} cos\frac{n\theta}{2} cos\phi
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\zeta^2 &= v^2 tr \left[ \begin{pmatrix} 0 & 0 & 0 & e^{-i\phi} sin\frac{n\theta}{2} \\ 0 & 0 & 0 & cos\frac{n\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ e^{i\phi} sin\frac{n\theta}{2} & cos\frac{n\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] \\
&= iv^2 tr \left[ \begin{pmatrix} 0 & 0 & 0 & e^{-i\phi} sin\frac{n\theta}{2} \\ 0 & 0 & 0 & cos\frac{n\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ cos\frac{n\theta}{2} & -e^{i\phi} sin\frac{n\theta}{2} \end{pmatrix} \right] \\
&= iv^2 sin\frac{n\theta}{2} cos\frac{n\theta}{2} (e^{-i\phi} - e^{i\phi}) \\
&= 2v^2 sin\frac{n\theta}{2} cos\frac{n\theta}{2} sin\phi
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\zeta^3 &= v^2 tr \left[ \begin{pmatrix} 0 & 0 & 0 & e^{-i\phi} sin\frac{n\theta}{2} \\ 0 & 0 & 0 & cos\frac{n\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ e^{i\phi} sin\frac{n\theta}{2} & cos\frac{n\theta}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\
&= v^2 tr \left[ \begin{pmatrix} 0 & 0 & 0 & e^{-i\phi} sin\frac{n\theta}{2} \\ 0 & 0 & 0 & cos\frac{n\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ e^{i\phi} sin\frac{n\theta}{2} & -cos\frac{n\theta}{2} \end{pmatrix} \right] \\
&= v^2 \left( sin^2\frac{n\theta}{2} - cos^2\frac{n\theta}{2} \right)
\end{aligned} \tag{2.12}$$

Now we can calculate the length of this vector. Length of the vector $= \sqrt{\zeta^a \zeta^a}$

$$
\begin{aligned}
\zeta^a\zeta^a &= v^4\left[4sin^2\frac{n\theta}{2}cos^2\frac{n\theta}{2}cos^2\phi+4sin^2\frac{n\theta}{2}cos^2\frac{n\theta}{2}sin^2\phi+\left(sin^2\frac{n\theta}{2}-cos^2\frac{n\theta}{2}\right)^2\right]\\
&= v^4\left[4sin^2\frac{n\theta}{2}cos^2\frac{n\theta}{2}+\left(sin^2\frac{n\theta}{2}-cos^2\frac{n\theta}{2}\right)^2\right]\\
&= v^4\left(sin^2\frac{n\theta}{2}+cos^2\frac{n\theta}{2}\right)^2\\
&= v^4
\end{aligned}
\tag{2.13}
$$

So, the length of this vector is always $v^2$, which means every point on the manifold are equal-distant from the origin. It cannot go through the origin when moving from one point to another point without leaving the manifold. Thus, it cannot be deformed continuously to a point. It has non-trivial homotopy which give rise to magnetic monopoles we are searching for.

# Chapter 3

# 't Hooft–Polyakov monopole

In chapter 2, we found that a 2-sphere embedding in the vacuum manifold can be parameterized as

$$\chi = v \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} e^{i\phi} sin\frac{n\theta}{2} & cos\frac{n\theta}{2} \end{pmatrix} \tag{3.1}$$

We would focus on the $n = 2$ case here. Any representation of $SO(3)$ is equivalent to a representation of $SU(2)$. The fundamental representation of $SO(3)$ is equivalent to the the adjoint representation of $SU(2)$. It should give similar result as the 't Hooft–Polyakov monopole.

## 3.1 Lagrangian

We will work with the Lagrangian density following 't Hooft's recipe [1]. From this point, we omit the contribution from $SU(4)$ generators.

Let's propose the Lagrangian density

$$\mathscr{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} - \frac{1}{2} D_\mu Q_{i\alpha} (D^\mu Q_{i\alpha})^* - \frac{1}{2}\mu^2 Q^2_{i\alpha} - \frac{1}{8}\lambda^2 (Q^2_{i\alpha})^2 \tag{3.2}$$

where

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + e\epsilon_{abc} W^b_\mu W^c_\nu \tag{3.3}$$

is the field strength tensor. $W^a_\mu$ are the fields of gauge bosons. Note that we are using letter $e$ here to represent the coupling constant of $SU(2)_R$. If considering theory with other gauge group, $e$ is the coupling constant of the associated gauge group. Not to confuse with electroweak theory, where $e$ refers to the coupling constant of $U(1)_Y$, which is the charge of electron.

The parameter $\mu^2$ is chosen to be negative so that the scalar field has non-zero vacuum expectation value

$$\mu^2 = -\frac{1}{2}\lambda F^2 \tag{3.4}$$

$$< Q^2_a >= F^2 \tag{3.5}$$

This is not the full Pati-Salam Lagrangian density. For the Pati-Salam group, there are other gauge invarient terms that could be added to the Lagrangian. We ignored the contributions from the SU(4) part. This is sufficient to produce the magnetic monopole we are looking for.

Note that this is not the usual convention for $SU(N)$ theories. However, for $n = 2$, we can easily transform the variables from spherical polar to Cartesian. In this case, the theory does not involve complex numbers anymore.

For the scalar field, consider the ansatz

$$Q_{i\alpha} = \chi_{i\alpha}(\phi, \theta) Q(r) \tag{3.6}$$

By transformimg the coordinates from spherical to Cartesian.

$$\chi = v \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} e^{i\phi} sin\theta & cos\theta \end{pmatrix} = v \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} x+iy & z \end{pmatrix} = v \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ x+iy & z \end{pmatrix} \tag{3.7}$$

We can write the ansatz for the scalar field as

$$Q_a = r_a Q(r) \tag{3.8}$$

After regrouping (see appendix E), the Lagrangian density becomes

$$\mathscr{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} - \frac{1}{2} D_\mu Q_a (D^\mu Q_a)^* - \frac{1}{2}\mu^2 Q_a^2 - \frac{1}{8}\lambda (Q_a^2)^2 \tag{3.9}$$

The covarient derivative can be written as

$$D_u Q_a = \partial_u Q_a + e\epsilon_{abc} W^b_\mu Q_c \tag{3.10}$$

We also need an ansatz for the vector field. We will take it to be

$$W^a_\mu = \epsilon_{\mu ab} r_b W(r) \tag{3.11}$$

where $\epsilon_{\mu ab}$ is the Levi-Civita symbol if $\mu = 1, 2, 3$, and $\epsilon_{4ab} = 0$.

Because of the ansatz, the Largrangian density is spherically symmetric. We can apply the followings to simplify the Lagrangian density.

First, the transformation of coordinates

$$r_a = (x, y, z) = (rcos\phi sin\theta, rsin\phi sin\theta, rcos\theta) \tag{3.12}$$

The derivative transforms as

$$\partial_x F(r) = cos\phi sin\theta \partial_r F(r) \tag{3.13.1}$$
$$\partial_y F(r) = sin\phi sin\theta \partial_r F(r) \tag{3.13.2}$$
$$\partial_z F(r) = cos\theta \partial_r F(r) \tag{3.13.3}$$

for arbitrary function F(r).

Eqn (3.14) and (3.15) immediately tell us

$$r_a r_a = r^2 \tag{3.14.1}$$
$$r_a \partial_a F(r) = r\partial_r F(r) \tag{3.14.2}$$
$$\partial_a F(r) \partial_a F(r) = (\partial_r F(r))^2 \tag{3.14.3}$$

Note that the derivative of the coordinates is just

$$\partial_\mu r_a = \delta_{\mu a} \tag{3.15}$$

We also assume the system to be stationary, so

$$\partial_t Q(r) = 0 \tag{3.16.1}$$
$$\partial_t W(r) = 0 \tag{3.16.2}$$

Lastly, we will need this identity of Levi-Civita symbol

$$\epsilon_{abc}\epsilon_{ade} = \delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd} \tag{3.17}$$

Now, we can expend and simplify the terms in the Lagrangian density.

First, consider the field strength tensor term $-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a}$.

$$\begin{aligned}
-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a} =& -\frac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial^\mu W^{\nu a} - \partial^\nu W^{\mu a}) \\
&- \frac{e}{2}\epsilon_{abc}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)W^{\mu b}W^{\nu c} \\
&- \frac{e^2}{4}\epsilon_{abc}\epsilon_{ade}W^b_\mu W^c_\nu W^{\mu d}W^{\nu e}
\end{aligned} \tag{3.18}$$

It can be reduced to (see appendix C.1)

$$-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a} = -r^2\left(\frac{dW}{dr}\right)^2 - 4rW\frac{dW}{dr} - 6W^2 - 2er^2W^3 - \frac{1}{2}e^2r^4W^4 \tag{3.19}$$

Next, consider the covarient derivative term $-\frac{1}{2}D_\mu Q_{i\alpha}(D^\mu Q_{i\alpha})^*$.

$$\begin{aligned}
D_\mu Q_{i\alpha}(D^\mu Q_{i\alpha})^* =&[\partial_\mu(r_a Q(r)][\partial^\mu(r_a Q(r)] \\
&+ 2[\partial^\mu(r_a Q(r))e\epsilon_{abc}\epsilon_{\mu bf}r_f W(r)r_c Q(r)] \\
&+ e^2\epsilon_{abc}\epsilon_{ade}\epsilon_{\mu bf}\epsilon^{\mu dg}r_f r_g r_c r_e W(r)^2 Q(r)^2
\end{aligned} \tag{3.20}$$

It can be reduced to (see appendix C.2)

$$-\frac{1}{2}D_\mu Q_{i\alpha}(D^\mu Q_{i\alpha})^* = -\frac{1}{2}r^2\left(\frac{dQ}{dr}\right)^2 - rQ\frac{dQ}{dr} - \frac{3}{2}Q^2 - 2er^2WQ^2 - e^2r^4W^2Q^2 \tag{3.21}$$

We also have the potential terms

$$-\frac{1}{2}\mu^2 Q_a^2 = \frac{1}{4}\lambda F^2 r^2 Q^2 \tag{3.22}$$

$$-\frac{1}{8}\lambda(Q_a^2)^2 = -\frac{1}{8}\lambda r^4 Q^4 \tag{3.23}$$

A constant term (vacuum energy) is added to the Lagrangian density so that the Lagrangian is finite: $-\frac{1}{8}\lambda F^4$

So, the Lagrangian density becomes

$$\begin{aligned}
\mathscr{L} =& -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a} - \frac{1}{2}D_\mu Q_{i\alpha}(D^\mu Q_{i\alpha})^* - \frac{1}{2}\mu^2 Q_{i\alpha}^2 - \frac{1}{8}\lambda^2(Q_{i\alpha}^2)^2 + C \\
=& -r^2\left(\frac{dW}{dr}\right)^2 - 4rW\frac{dW}{dr} - 6W^2 - 2er^2W^3 - \frac{1}{2}e^2r^4W^4 \\
&- \frac{1}{2}r^2\left(\frac{dQ}{dr}\right)^2 - rQ\frac{dQ}{dr} - \frac{3}{2}Q^2 - 2er^2WQ^2 - e^2r^4W^2Q^2 \\
&+ \frac{1}{4}\lambda F^2r^2Q^2 - \frac{1}{8}\lambda r^4Q^4 - \frac{1}{8}\lambda F^4
\end{aligned} \tag{3.24}$$

The Lagrangian is then

$$\begin{aligned}
L =&4\pi\int_0^\infty r^2 dr\mathscr{L} \\
=&4\pi\int_0^\infty r^2 dr\left[-r^2\left(\frac{dW}{dr}\right)^2 - 4rW\frac{dW}{dr} - 6W^2 - 2er^2W^3 - \frac{1}{2}e^2r^4W^4\right. \\
&- \frac{1}{2}r^2\left(\frac{dQ}{dr}\right)^2 - rQ\frac{dQ}{dr} - \frac{3}{2}Q^2 - 2er^2WQ^2 - e^2r^4W^2Q^2 \\
&\left.+\frac{1}{4}\lambda F^2r^2Q^2 - \frac{1}{8}\lambda r^4Q^4 - \frac{1}{8}\lambda F^4\right]
\end{aligned} \tag{3.25}$$

## 3.2 Euler-Lagrange Equation

Since the system is stationary, the energy of the system $E = -L$ [1].

As in eqn 3.1 in ref [1], let's introduce the following dimensionless parameters:

$$w = W/F^2 e \tag{3.26.1}$$
$$q = Q/F^2 e \tag{3.26.2}$$
$$x = eFr \tag{3.26.3}$$
$$\beta = \lambda/e^2 = M_Q^2/M_W^2 \tag{3.26.4}$$

where $M_Q$ and $M_W$ are the mass of the associated scalar boson and vector gauge boson in the Pati-Salam model respectively. Note that $M_W$ is not the mass of W boson in electroweak theory.

The energy of the system is then

$$\begin{aligned} E =& \frac{4\pi M_W}{e^2}\int_0^\infty x^2 dx \left[ x^2 \left(\frac{dw}{dx}\right)^2 + 4xw\frac{dw}{dx} + 6w^2 + 2x^2w^3 + \frac{1}{2}x^4w^4 \right. \\ &+ \frac{1}{2}x^2\left(\frac{dq}{dx}\right)^2 + xq\frac{dq}{dx} + \frac{3}{2}q^2 + +2x^2wq^2 + x^4w^2q^2 \\ &\left. -\frac{1}{4}\beta x^2q^2 + \frac{1}{8}\beta x^4q^4 + \frac{1}{8}\beta \right] \end{aligned} \tag{3.27}$$

The mass of the monopole is given by the minima of E. So, we need to solve the E-L equations.

$$\frac{\partial E}{\partial q} - \partial_x\left(\frac{\partial E}{\partial(\partial_x q)}\right) = 0 \tag{3.28.1}$$
$$\frac{\partial E}{\partial w} - \partial_x\left(\frac{\partial E}{\partial(\partial_x w)}\right) = 0 \tag{3.28.2}$$

We need to evaluate $\frac{\partial E}{\partial q}$, $\partial_x\left(\frac{\partial E}{\partial(\partial_x q)}\right)$, $\frac{\partial E}{\partial w}$ and $\partial_x\left(\frac{\partial E}{\partial(\partial_x w)}\right)$.

$$\begin{aligned} \frac{\partial E}{\partial q} =& \frac{4\pi M_W}{e^2}x^2(xq' + 3q + 4x^2qw + 2x^4w^2q - \frac{1}{2}\beta x^2q + \frac{1}{2}\beta x^4q^3) \\ =& \frac{4\pi M_W}{e^2}(x^3q' + 3x^2q + 4x^4qw + 2x^6w^2q - \frac{1}{2}\beta x^4q + \frac{1}{2}\beta x^6q^3) \end{aligned} \tag{3.29}$$

$$\begin{aligned} \frac{\partial E}{\partial(\partial_x q)} =& \frac{4\pi M_W}{e^2}x^2(x^2q' + xq) = \frac{4\pi M_W}{e^2}(x^4q' + x^3q) \\ \partial_x\left(\frac{\partial E}{\partial(\partial_x q)}\right) =& \frac{4\pi M_W}{e^2}(5x^3q' + 3x^2q + x^4q'') \end{aligned} \tag{3.30}$$

$$\begin{aligned} \frac{\partial E}{\partial w} =& \frac{4\pi M_W}{e^2}x^2(4xw' + 12w + 6x^2w^2 + 2x^4w^3 + 2x^2q^2 + 2x^4wq^2) \\ =& \frac{4\pi M_W}{e^2}(4x^3w' + 12x^2w + 6x^4w^2 + 2x^6w^3 + 2x^4q^2 + 2x^6wq^2) \end{aligned} \tag{3.31}$$

$$\begin{aligned} \frac{\partial E}{\partial(\partial_x w)} =& \frac{4\pi M_W}{e^2}x^2(2x^2w' + 4xw) = \frac{4\pi M_W}{e^2}(2x^4w' + 4x^3w) \\ \partial_x\left(\frac{\partial E}{\partial(\partial_x w)}\right) =& \frac{4\pi M_W}{e^2}(12x^3w' + 2x^4w'' + 12x^2w) \end{aligned} \tag{3.32}$$

So, the E-L equation for q is

$$0 = -4x^3q' + 4x^4qw + 2x^6w^2q - \frac{1}{2}\beta x^4q + \frac{1}{2}\beta x^6q^3 - x^4q'' \tag{3.33}$$

And the E-L equation for w is

$$0 = -8x^3w' + 6x^4w^2 + 2x^6w^3 + 2x^4q^2 + 2x^6wq^2 - 2x^4w'' \tag{3.34}$$

## 3.3 The mass of the monopole

Solving the equations numerically with NDSolve in Mathematica. 4 boundary conditions are required to solve a system of 2 second-order differential equations.

According to page 92 of ref. [11], to avoid singularity, the boundary conditions at x = 0 are

$$q(0) = 0 \tag{3.35.1}$$
$$w(0) = 0 \tag{3.35.2}$$

In page 280 of 't Hooft's paper, it is shown that the boundary conditions at infinity are

$$Q(r) = F/r \tag{3.36.1}$$
$$W(r) = \frac{-1}{er^2} \tag{3.36.2}$$

[1]

This is required such that the energy is finite (page 92 of [11]).

With the re-scaling presented in (3.26), the boundary conditions at infinity become

$$q(x) = 1/x \tag{3.37.1}$$
$$w(x) = -1/x^2 \tag{3.37.2}$$

Re-scale q and w again so that the equations are easier to solve by shooting method.

$$q(x) \rightarrow q(x)/x \tag{3.38.1}$$
$$w(x) \rightarrow w(x)/x^2 \tag{3.38.2}$$

After the re-scaling in (3.38), the boundary conditions at infinity becomes

$$q(x) = 1 \tag{3.39.1}$$
$$w(x) = -1 \tag{3.39.2}$$

However, these are not good boundary conditions for numerical methods. At $x = 0$, numerical calculations will involve numbers divided by zero. So, we evaluate the BCs at $x = 0.000001$ instead. This would not affect the result significantly if the value we pick is close enough to zero.

Mathematica is not good at solving equations with BCs at 2 ends. So, we are using shooting method to find the value of $q'(0.000001)$ and $w'(0.000001)$ which gives the 2 BCs in (3.39).

The functions are very sensitive to the initial conditions. It is difficult to numerically solve the function very accurately. We could find solutions that converge to 1 and -1 very quickly, but then become unstable and blow up to infinity at large x. There are limits in precision due to the approximation on the boundary conditions. It could not be improved by fine tuning the parameters. So, when evaluating the integral, we replaced the "tail" of $q(r)$ and $w(r)$ by constant 1 and -1 respectively when the function is very close to that value. This would be a good approximation as we know that the functions should converge.

Here are the plots of the solutions.

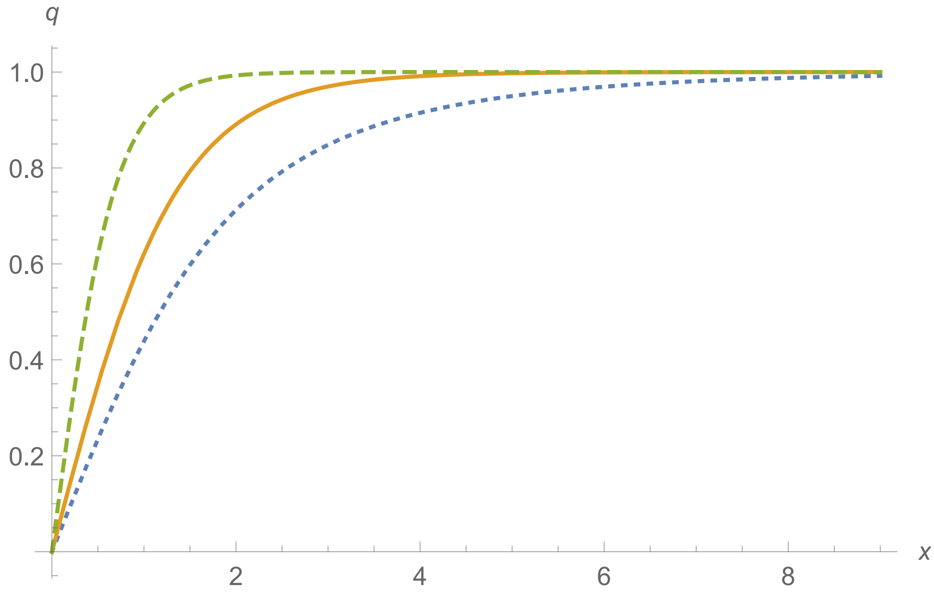



Fig.1

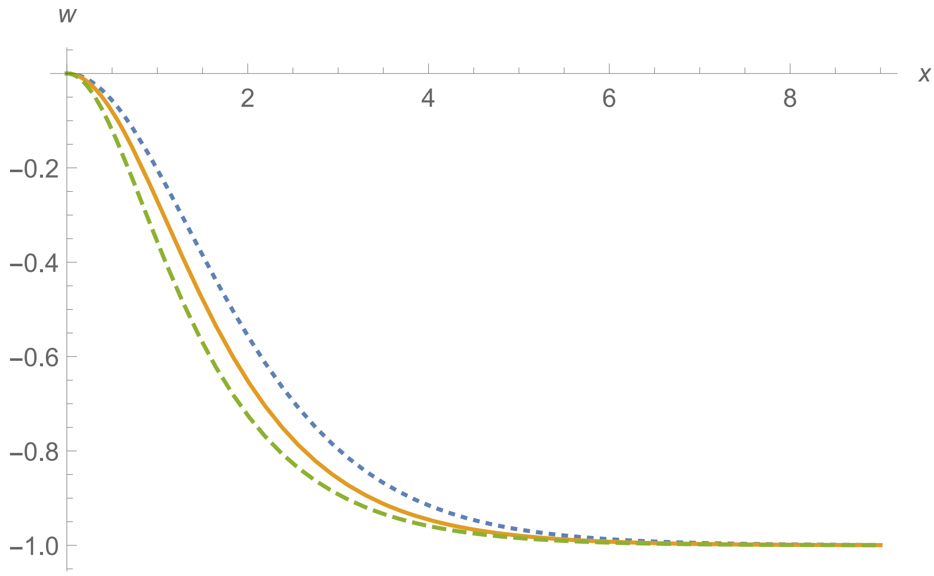



Fig.2

The functions q(r) and w(r) for $\beta = 0.1$ (dotted curves), $\beta = 1$ (solid curves), and $\beta = 10$ (dashed curves).

The mass of the monopole is found to be (see Appendix D)

$$M_m = \frac{4\pi M_W}{e^2} C(\beta) \tag{3.40}$$

where $C(\beta) = 1.1062$ for $\beta = 0.1$, $C(\beta) = 1.2377$ for $\beta = 1$, and $C(\beta) = 1.4332$ for $\beta = 10$.

This result agrees with the result in ref. [1]. $C(\beta) = 1.1$ for $\beta = 0.1$, and $C(\beta) = 1.44$ for $\beta = 10$ (the author noted that it maybe slight too large).

## 3.4 Magnetic charge

To find the total magnetic flux, we need the electromagnetic tensor $F_{\mu\nu}$. Here are the details from 't Hooft (page 281 of ref. [1]).

Let's propose that

$$F_{\mu\nu} = \frac{1}{|Q|} Q_a G^a_{\mu\nu} - \frac{1}{e|Q|^3} \epsilon_{abc} Q_a (D_\mu Q_b)(D_\nu Q_c) \tag{3.41}$$

After a gauge rotation, the scalar field can be rotated to $Q_a = |Q|(0,0,1)$, then we have $F_{\mu\nu} = \partial_\mu W^3_\nu - \partial_\nu W^3_\mu$.

This satisfies the usual Maxwell equations everywhere except $Q_a = 0$. From (3.41), combined with (3.3) and (3.10), we get

$$Q_a G^a_{\mu\nu} = -\frac{F}{er^3}\epsilon_{\mu\nu a} r_a \tag{3.42}$$

$$D_u Q_a = \partial_u Q_a + e\epsilon_{abc} W^b_\mu Q_c = 0 \tag{3.43}$$

So, the electromagnetic tensor becomes

$$F_{\mu\nu} = -\frac{1}{er^3}\epsilon_{\mu\nu a} r_a \tag{3.44}$$

In classical electromagnetism, the B-field is given by

$$B_i = -\frac{1}{2}\epsilon_{ijk} F^{jk} = r_i/er^3 \tag{3.45}$$

The total magnetic flux is then $4\pi/e$. Magnetic charge $g = 1/e$.

# Chapter 4

# Discussions

Recall the Dirac quantization condition $qg = N/2$. 't Hooft suggested that the monopole found having magnetic charge $g = 1/e$ satisfies the Dirac's condition $qg = 1/2$ with isospin 1/2 representation in $SU(2)$ group describing particles with charges $\pm\frac{1}{2}e$ (page 283 of ref. [1]). But in $SO(3)$ gauge theory, it has double the minimum magnetic charge.

However, recall that when we choose the embeded 2-sphere in the vacuum manifold at the beginning of chapter 3, we chose the $n = 2$ case. This means a full $SU(2)$ rotation actually wraps around the 2-sphere twice (as $SU(2)$ is a double cover of $SO(3)$). If the $n = 1$ case also gives a soliton solution, that would give a monopole with half the magnetic charge but the minimum electric charge allowed for such a monopole would be $e$. It is possible that there exist a family of magnetic monopoles, with magnetic charge $\frac{n}{2e}$ in $SU(2)$ theory, if there are solutions for general $n$.

The mass of the monopole we found is about the Pati-Salam breaking scale. At least it is much lower than $SU(5)$ GUT scale. There are chances to be able to produce it in accelerators in the future.

Mathematically, it is perfectly fine to have magnetic monopoles existing. However, there may be other reasons prohibiting the existence of magnetic monopole.

There has been attempts to detect magnetic monopoles directly. In 1982, an experiment detecting moving magnetic charge with a superconductive ring sucessfully detected a jump in current by exactly the same amount that would be generated by a magnetic monopole passing through the ring [12]. However, later experiments have not been able to reproduce the same result. So, it is believed to be caused by other effects [3].

If magnetic monopoles existed in the history of our universe, they have to be formed before inflation so that they are diluted to a very low density [3]. If they are formed after inflation, we would be able to detect the relic radiation produced, and they would not have all annhilated [13].

# Appendix A

# $SU(4) \times SU(2)$ generators

## A.1 $SU(4)$ generators

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

[14]

$$t^i = \lambda_i \otimes \mathbb{1}_2$$

## A.2 $SU(2)$ generators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad [14]$$

$$t^{16} = \mathbb{1}_4 \otimes \sigma_x \quad t^{17} = \mathbb{1}_4 \otimes \sigma_y \quad t^{18} = \mathbb{1}_4 \otimes \sigma_z$$

# Appendix B

# $S^{ab}$ matrix and eigenvectors

## B.1 $SU(4) \times SU(2) \rightarrow SU(3) \times U(1)$

In[113]:=

$$\texttt{Sigma1T} = c\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\texttt{Sigma2T} = c\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$$

$$\texttt{Sigma3T} = c\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\texttt{lumbda1} = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\texttt{lumbda2} = \begin{pmatrix}0&-I&0&0\\I&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\texttt{lumbda3} = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\texttt{lumbda4} = \begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\texttt{lumbda5} = \begin{pmatrix}0&0&-I&0\\0&0&0&0\\I&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\texttt{lumbda6} = \begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$$

$$\texttt{lumbda7} = \begin{pmatrix}0&0&0&0\\0&0&-I&0\\0&I&0&0\\0&0&0&0\end{pmatrix}$$

$$\texttt{lumbda8} = (1/(\texttt{Sqrt[3]}))\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-2&0\\0&0&0&0\end{pmatrix}$$

$$\texttt{lumbda9} = \begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}$$

$$\texttt{lumbda10} = \begin{pmatrix}0&0&0&-I\\0&0&0&0\\0&0&0&0\\I&0&0&0\end{pmatrix}$$

$$\texttt{lumbda11} = \begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix}$$

$$\texttt{lumbda12} = \begin{pmatrix}0&0&0&0\\0&0&0&-I\\0&0&0&0\\0&I&0&0\end{pmatrix}$$



$$\texttt{lumbda13} = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

$$\texttt{lumbda14} = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-I\\0&0&I&0\end{pmatrix}$$

$$\texttt{lumbda15} = (1/(\texttt{Sqrt[6]}))\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{pmatrix}$$

Out[113]= {{0, c}, {c, 0}}

Out[114]= {{0, i c}, {-i c, 0}}

Out[115]= {{c, 0}, {0, -c}}

Out[116]= {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[117]= {{0, -i, 0, 0}, {i, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[118]= {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[119]= {{0, 0, 1, 0}, {0, 0, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 0}}

Out[120]= {{0, 0, -i, 0}, {0, 0, 0, 0}, {i, 0, 0, 0}, {0, 0, 0, 0}}

Out[121]= {{0, 0, 0, 0}, {0, 0, 1, 0}, {0, 1, 0, 0}, {0, 0, 0, 0}}

Out[122]= {{0, 0, 0, 0}, {0, 0, -i, 0}, {0, i, 0, 0}, {0, 0, 0, 0}}

Out[123]= $\left\{\left\{\frac{1}{\sqrt{3}}, 0, 0, 0\right\}, \left\{0, \frac{1}{\sqrt{3}}, 0, 0\right\}, \left\{0, 0, -\frac{2}{\sqrt{3}}, 0\right\}, \{0, 0, 0, 0\}\right\}$

Out[124]= {{0, 0, 0, 1}, {0, 0, 0, 0}, {0, 0, 0, 0}, {1, 0, 0, 0}}

Out[125]= {{0, 0, 0, -i}, {0, 0, 0, 0}, {0, 0, 0, 0}, {i, 0, 0, 0}}

Out[126]= {{0, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 0, 0}, {0, 1, 0, 0}}

Out[127]= {{0, 0, 0, 0}, {0, 0, 0, -i}, {0, 0, 0, 0}, {0, i, 0, 0}}

Out[128]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}}

Out[129]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, -i}, {0, 0, i, 0}}

Out[130]= $\left\{\left\{\frac{1}{\sqrt{6}}, 0, 0, 0\right\}, \left\{0, \frac{1}{\sqrt{6}}, 0, 0\right\}, \left\{0, 0, \frac{1}{\sqrt{6}}, 0\right\}, \left\{0, 0, 0, -\sqrt{\frac{3}{2}}\right\}\right\}$

```
In[131]:= VEV = (0 0
                 0 0
                 0 0
                 0 1)
           X[1] =  lumbda1.VEV
           X[2] =  lumbda2.VEV
           X[3] =  lumbda3.VEV
           X[4] =  lumbda4.VEV
           X[5] =  lumbda5.VEV
           X[6] =  lumbda6.VEV
           X[7] =  lumbda7.VEV
           X[8] =  lumbda8.VEV
           X[9] =  lumbda9.VEV
           X[10] =  lumbda10.VEV
           X[11] =  lumbda11.VEV
           X[12] =  lumbda12.VEV
           X[13] =  lumbda13.VEV
           X[14] =  lumbda14.VEV
           X[15] =   lumbda15.VEV
           X[16] = VEV.Sigma1T
           X[17] = VEV.Sigma2T
           X[18] = VEV.Sigma3T
Out[131]= {{0, 0}, {0, 0}, {0, 0}, {0, 1}}
Out[132]= {{0, 0}, {0, 0}, {0, 0}, {0, 0}}
Out[133]= {{0, 0}, {0, 0}, {0, 0}, {0, 0}}
Out[134]= {{0, 0}, {0, 0}, {0, 0}, {0, 0}}
Out[135]= {{0, 0}, {0, 0}, {0, 0}, {0, 0}}
Out[136]= {{0, 0}, {0, 0}, {0, 0}, {0, 0}}
Out[137]= {{0, 0}, {0, 0}, {0, 0}, {0, 0}}
Out[138]= {{0, 0}, {0, 0}, {0, 0}, {0, 0}}
Out[139]= {{0, 0}, {0, 0}, {0, 0}, {0, 0}}
Out[140]= {{0, 1}, {0, 0}, {0, 0}, {0, 0}}
Out[141]= {{0, -i}, {0, 0}, {0, 0}, {0, 0}}
Out[142]= {{0, 0}, {0, 1}, {0, 0}, {0, 0}}
Out[143]= {{0, 0}, {0, -i}, {0, 0}, {0, 0}}
Out[144]= {{0, 0}, {0, 0}, {0, 1}, {0, 0}}
Out[145]= {{0, 0}, {0, 0}, {0, -i}, {0, 0}}
Out[146]= {{0, 0}, {0, 0}, {0, 0}, {0, -√(3/2)}}
Out[147]= {{0, 0}, {0, 0}, {0, 0}, {c, 0}}
Out[148]= {{0, 0}, {0, 0}, {0, 0}, {-i c, 0}}
Out[149]= {{0, 0}, {0, 0}, {0, 0}, {0, -c}}
```



```
In[150]:=
     G = Table[ConjugateTranspose[X[a]], {a, 1, 18}]
     F = Table[X[b], {b, 1, 18}]

     H = Table[Tr[G[[x]].F[[y]] + G[[y]].F[[x]]], {x, 1, 18}, {y, 1, 18}]
     MatrixForm[H]

Out[150]= {{{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}},
  {{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}},
  {{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}},
  {{0, 0, 0, 0}, {1, 0, 0, 0}}, {{0, 0, 0, 0}, {i, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 1, 0, 0}},
  {{0, 0, 0, 0}, {0, i, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 1, 0}}, {{0, 0, 0, 0}, {0, 0, i, 0}},
  {{0, 0, 0, 0}, {0, 0, 0, -√(3/2)}}, {{0, 0, 0, Conjugate[c]}, {0, 0, 0, 0}},
  {{0, 0, 0, i Conjugate[c]}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, -Conjugate[c]}}}
Out[151]= {{{0, 0}, {0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}, {0, 0}},
  {{0, 0}, {0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}, {0, 0}},
  {{0, 0}, {0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}, {0, 0}},
  {{0, 0}, {0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}, {0, 0}},
  {{0, 1}, {0, 0}, {0, 0}, {0, 0}}, {{0, -i}, {0, 0}, {0, 0}, {0, 0}},
  {{0, 0}, {0, 1}, {0, 0}, {0, 0}}, {{0, 0}, {0, -i}, {0, 0}, {0, 0}},
  {{0, 0}, {0, 0}, {0, 1}, {0, 0}}, {{0, 0}, {0, 0}, {0, -i}, {0, 0}},
  {{0, 0}, {0, 0}, {0, 0}, {0, -√(3/2)}}, {{0, 0}, {0, 0}, {0, 0}, {c, 0}},
  {{0, 0}, {0, 0}, {0, 0}, {-i c, 0}}, {{0, 0}, {0, 0}, {0, 0}, {0, -c}}}
Out[152]= {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 0, 0, √(3/2) c + √(3/2) Conjugate[c]},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2 c Conjugate[c], 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2 c Conjugate[c], 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, √(3/2) c + √(3/2) Conjugate[c], 0, 0, 2 c Conjugate[c]}}
```



```
Out[153]//MatrixForm=
(0 0 0 0 0 0 0 0 0 0 0 0 0 0   0                                0                 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0   0                                0                 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0   0                                0                 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0   0                                0                 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0   0                                0                 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0   0                                0                 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0   0                                0                 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0   0                                0                 0
 0 0 0 0 0 0 0 0 2 0 0 0 0 0   0                                0                 0
 0 0 0 0 0 0 0 0 0 2 0 0 0 0   0                                0                 0
 0 0 0 0 0 0 0 0 0 0 2 0 0 0   0                                0                 0
 0 0 0 0 0 0 0 0 0 0 0 2 0 0   0                                0                 0
 0 0 0 0 0 0 0 0 0 0 0 0 2 0   0                                0                 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 2   0                                0                 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0   3                                0                 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0   0                                2 c Conjugate[c]  0
 0 0 0 0 0 0 0 0 0 0 0 0 0 0   0                                0                 2 c Conjugat
 0 0 0 0 0 0 0 0 0 0 0 0 0 0   √(3/2) c + √(3/2) Conjugate[c]   0                 0)

Out[•]= {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3/4, 0, 0, √(3/2)/2},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, √(3/2)/2, 0, 0, 1/2}}
```



```
Out[•]= {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 27, 0, 0, 3 √(3/2)},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/2, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3 √(3/2), 0, 0, 1/2}}
```

In[154]:= **Eigensystem[H]**

Out[154]=

$$\Big\{\Big\{2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 2\,c\,\text{Conjugate}[c],$$
$$2\,c\,\text{Conjugate}[c], \frac{1}{2}\left(3+2\,c\,\text{Conjugate}[c]-\sqrt{\left(3+2\,c^2\right)\,\left(3+2\,\text{Conjugate}[c]^2\right)}\right),$$
$$\frac{1}{2}\left(3+2\,c\,\text{Conjugate}[c]+\sqrt{\left(3+2\,c^2\right)\,\left(3+2\,\text{Conjugate}[c]^2\right)}\right)\Big\},$$

```
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
```

$$\Big\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$-\frac{-3+2\,c\,\text{Conjugate}[c]+\sqrt{\left(3+2\,c^2\right)\,\left(3+2\,\text{Conjugate}[c]^2\right)}}{\sqrt{6}\,\left(c+\text{Conjugate}[c]\right)}, 0, 0, 1\Big\}, \Big\{0, 0, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, 0, 0, -\frac{-3+2\,c\,\text{Conjugate}[c]-\sqrt{\left(3+2\,c^2\right)\,\left(3+2\,\text{Conjugate}[c]^2\right)}}{\sqrt{6}\,\left(c+\text{Conjugate}[c]\right)}, 0, 0, 1\Big\}\Big\}\Big\}$$

In[ ]:=

In[ ]:=

## B.2 $SU(2) \times SU(2) \rightarrow U(1)$

```
In[•]:= Sigma1T = ( 0  1 )
                  ( 1  0 )
        Sigma2T = (  0  I )
                  ( -I  0 )
        Sigma3T = ( 1   0 )
                  ( 0  -1 )

        Sigma1 = ( 0  1 )
                 ( 1  0 )
        Sigma2 = ( 0  -I )
                 ( I   0 )
        Sigma3 = ( 1   0 )
                 ( 0  -1 )

Out[•]= {{0, 1}, {1, 0}}
Out[•]= {{0, i}, {-i, 0}}
Out[•]= {{1, 0}, {0, -1}}
Out[•]= {{0, 1}, {1, 0}}
Out[•]= {{0, -i}, {i, 0}}
Out[•]= {{1, 0}, {0, -1}}

In[•]:= VEV = ( 0  0 )
              ( 0  1 )
        X[1] =  Sigma1.VEV
        X[2] =  Sigma2.VEV
        X[3] =  Sigma3.VEV
        X[4] = VEV.Sigma1T
        X[5] = VEV.Sigma2T
        X[6] = VEV.Sigma3T
Out[•]= {{0, 0}, {0, 1}}
Out[•]= {{0, 1}, {0, 0}}
Out[•]= {{0, -i}, {0, 0}}
Out[•]= {{0, 0}, {0, -1}}
Out[•]= {{0, 0}, {1, 0}}
Out[•]= {{0, 0}, {-i, 0}}
Out[•]= {{0, 0}, {0, -1}}
```



```
In[•]:= G = Table[ConjugateTranspose[X[a]], {a, 1, 6}]
        F = Table[X[b], {b, 1, 6}]

        H = Table[Tr[G[[x]].F[[y]] + G[[y]].F[[x]]], {x, 1, 6}, {y, 1, 6}]
        MatrixForm[H]

Out[•]= {{{0, 0}, {1, 0}}, {{0, 0}, {i, 0}}, {{0, 0}, {0, -1}},
         {{0, 1}, {0, 0}}, {{0, i}, {0, 0}}, {{0, 0}, {0, -1}}}

Out[•]= {{{0, 1}, {0, 0}}, {{0, -i}, {0, 0}}, {{0, 0}, {0, -1}},
         {{0, 0}, {1, 0}}, {{0, 0}, {-i, 0}}, {{0, 0}, {0, -1}}}

Out[•]= {{2, 0, 0, 0, 0, 0}, {0, 2, 0, 0, 0, 0}, {0, 0, 2, 0, 0, 2},
         {0, 0, 0, 2, 0, 0}, {0, 0, 0, 0, 2, 0}, {0, 0, 2, 0, 0, 2}}

Out[•]//MatrixForm=
        ( 2  0  0  0  0  0 )
        ( 0  2  0  0  0  0 )
        ( 0  0  2  0  0  2 )
        ( 0  0  0  2  0  0 )
        ( 0  0  0  0  2  0 )
        ( 0  0  2  0  0  2 )

In[•]:= Eigensystem[H]

Out[•]= {{4, 2, 2, 2, 2, 0}, {{0, 0, 1, 0, 0, 1}, {0, 0, 0, 0, 1, 0},
          {0, 0, 0, 1, 0, 0}, {0, 1, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 1}}}

In[•]:=

In[•]:=
```

# Appendix C

# Derivations of terms in the Lagrangian

## C.1 Field strength tensor term

$$
\begin{aligned}
-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a} = &-\frac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial^\mu W^{\nu a} - \partial^\nu W^{\mu a}) \\
&-\frac{e}{2}\epsilon_{abc}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)W^{\mu b}W^{\nu c} \\
&-\frac{e^2}{4}\epsilon_{abc}\epsilon_{ade}W^b_\mu W^c_\nu W^{\mu d}W^{\nu e}
\end{aligned} \tag{C.1}
$$

The first term

$$
\begin{aligned}
&(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial^\mu W^{\nu a} - \partial^\nu W^{\mu a}) \\
=&[\partial_\mu(\epsilon_{\nu ab}r_b W(r)) - \partial_\nu(\epsilon_{\mu ab}r_b W(r))] \times [\partial^\mu(\epsilon^{\nu ac}r_c W(r)) - \partial^\nu(\epsilon^{\mu ac}r_c W(r))] \\
=&[\epsilon_{\nu ab}\partial_\mu r_b W(r) + \epsilon_{\nu ab}r_b\partial_\mu W(r) - \epsilon_{\mu ab}\partial_\nu r_b W(r) - \epsilon_{\mu ab}r_b\partial_\nu W(r)] \\
&\times [\epsilon^{\nu ac}\partial^\mu r_c W(r) + \epsilon^{\nu ac}r_c\partial^\mu W(r) - \epsilon^{\mu ac}\partial^\nu r_c W(r) - \epsilon^{\mu ac}r_c\partial^\nu W(r)] \\
=&2\times 2\delta_{bc}(\partial_i r_b W(r) + r_b\partial_i W(r)) \times (\partial_i r_c W(r) + r_c\partial_i W(r)) \\
&-2(\delta_{\nu\mu}\delta_{bc} - \delta_{\nu c}\delta_{b\mu}) \times (\partial_\mu r_b W(r) + r_b\partial_\mu W(r)) \times (\partial^\nu r_c W(r) + r_c\partial^\nu W(r)) \\
=&4(\delta_{ic}W(r) + r_c\partial_i W(r)) \times (\delta_{ic}W(r) + r_c\partial_i W(r)) \\
&-2(\partial_i r_c W(r) + r_c\partial_i W(r)) \times (\partial_i r_c W(r) + r_c\partial_i W(r)) \\
&+2(\partial_i r_i W(r) + r_i\partial_i W(r)) \times (\partial_c r_c W(r) + r_c\partial_c W(r)) \\
=&2(3W(r)^2 + 2r_i W(r)\partial_i W(r) + r^2(\partial_i(r))^2) \\
&+2(3W(r) + r_i\partial_i W(r)) \times (3W(r) + r_c\partial_c W(r)) \\
=&2(3W(r)^2 + 2rW(r)\partial_r W(r) + r^2(\partial_r(r))^2) \\
&+2(9W(r)^2 + 6rW(r)\partial_r W(r) + r^2(\partial_r(r))^2) \\
=&24W(r)^2 + 16rW(r)\partial_r W(r) + 4r^2(\partial_r W(r))^2
\end{aligned} \tag{C.2}
$$

The second term

$$
\begin{aligned}
&\epsilon_{abc}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)W^{\mu b}W^{\nu c}\\
=&\epsilon_{abc}[\epsilon_{\nu ad}(\partial_\mu r_d W(r) + r_d\partial_\mu W(r)) - \epsilon_{\mu ad}(\partial_\nu r_d W(r) + r_d\partial_\nu W(r))]\\
&\times \epsilon^{\mu be}\epsilon^{\nu cf} r_e r_f W(r)^2\\
=&\epsilon_{abc}(\delta_{ac}\delta_{df} - \delta_{af}\delta_{dc})[\partial_i r_d W(r) + r_d\partial_i W(r)]\epsilon^{ibe} r_e r_f W(r)^2\\
&- \epsilon_{abc}(\delta_{ab}\delta_{de} - \delta_{ae}\delta_{db})[\partial_i r_d W(r) + r_d\partial_i W(r)]\epsilon^{icf} r_e r_f W(r)^2\\
=&[(\delta_{ai}\delta_{ce} - \delta_{ae}\delta_{ci})(\delta_{ac}\delta_{df} - \delta_{af}\delta_{dc}) + (\delta_{ai}\delta_{bf} - \delta_{af}\delta_{bi})(\delta_{ab}\delta_{de} - \delta_{ae}\delta_{db})]\\
&\times [\partial_i r_d W(r) + r_d\partial_i W(r)] r_e r_f W(r)^2\\
=&[(\delta_{ie}\delta_{df} - \delta_{if}\delta_{de} - \delta_{ie}\delta_{df} + \delta_{ef}\delta_{id}) + (\delta_{if}\delta_{de} - \delta_{ie}\delta_{df} - \delta_{if}\delta_{de} + \delta_{ef}\delta_{id})]\\
&\times [\partial_i r_d W(r) + r_d\partial_i W(r)] r_e r_f W(r)^2\\
=&-2[\partial_f r_e W(r) + r_e\partial_f W(r)] r_e r_f W(r)^2\\
&+ 2[\partial_i r_i W(r) + r_i\partial_i W(r)] r^2 W(r)^2\\
=&-2[r^2 W(r) + r^3\partial_r W(r)]W(r)^2\\
&+ 2[3W(r) + r\partial_r W(r)] r^2 W(r)^2\\
=&4r^2 W(r)^3
\end{aligned}
\tag{C.3}
$$

The third term

$$
\begin{aligned}
&\epsilon_{abc}\epsilon_{ade} W^b_\mu W^c_\nu W^{\mu d} W^{\nu e}\\
=&(\delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd})\epsilon_{ibf}\epsilon_{jcg}\epsilon_{idh}\epsilon_{jek} r_f r_g r_h r_k W(r)^4\\
=&(\delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd})(\delta_{bd}\delta_{fh} - \delta_{bh}\delta_{fd})(\delta_{ce}\delta_{gk} - \delta_{ck}\delta_{ge}) r_f r_g r_h r_k W(r)^4\\
=&(3\delta_{ce}\delta_{fh} - \delta_{ce}\delta_{fh} - \delta_{ce}\delta_{fh} + \delta_{eh}\delta_{cf})(\delta_{ce}\delta_{gk} - \delta_{ck}\delta_{ge}) r_f r_g r_h r_k W(r)^4\\
=&(3\delta_{fh}\delta_{gk} - \delta_{fh}\delta_{gk} - \delta_{fh}\delta_{gk} + \delta_{kf}\delta_{hg}) r_f r_g r_h r_k W(r)^4\\
=&2r^4 W(r)^4
\end{aligned}
\tag{C.4}
$$

Collecting the results, we have

$$
-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a} = -r^2\left(\frac{dW}{dr}\right)^2 - 4rW\frac{dW}{dr} - 6W^2 - 2er^2W^3 - \frac{1}{2}e^2r^4W^4 \tag{C.5}
$$

## C.2 Covarient derivative term

$$
\begin{aligned}
D_\mu Q_{i\alpha}(D^\mu Q_{i\alpha})^* =&[\partial_\mu(r_a Q(r))][\partial^\mu(r_a Q(r))] + 2[\partial^\mu(r_a Q(r))e\epsilon_{abc}\epsilon_{\mu bf} r_f W(r) r_c Q(r)]\\
&+ e^2\epsilon_{abc}\epsilon_{ade}\epsilon_{\mu bf}\epsilon^{\mu dg} r_f r_g r_c r_e W(r)^2 Q(r)^2
\end{aligned}
\tag{C.6}
$$

The first term

$$
\begin{aligned}
[\partial_\mu(r_a Q(r))][\partial^\mu(r_a Q(r))] =&[(\partial_\mu Q(r))r_a + (\partial_\mu r_a)Q(r)][(\partial^\mu Q(r))r_a + (\partial^\mu r_a)Q(r)]\\
=&(\partial_\mu Q(r))(\partial^\mu Q(r)) r_a r_a + 2(\partial_\mu Q(r)) r_a(\partial^\mu r_a)Q(r) + (\partial_\mu r_a)(\partial^\mu r_a)Q(r)^2\\
=&-(\partial_r Q(r))^2 r^2 - 2(\partial_i Q(r)) r_a\delta_{ia}Q(r) - \delta_{ia}\delta_{ia}Q(r)^2\\
=&-(\partial_r Q(r))^2 r^2 - 2(\partial_r Q(r)) rQ(r) - 3Q(r)^2
\end{aligned}
\tag{C.7}
$$

The second term

$$
\begin{aligned}
&\partial^\mu(r_a Q(r))e\epsilon_{abc}\epsilon_{\mu bf} r_f W(r) r_c Q(r)\\
=&-[(\partial_i r_a)Q(r) + (\partial_i Q(r))r_a]e(\delta_{ai}\delta_{cf} - \delta_{af}\delta_{ci}) r_f r_c W(r)Q(r)\\
=&-[(\partial_i r_i)Q(r) + (\partial_i Q(r))r_i]er^2 W(r)Q(r) + [(\partial_c r_f)Q(r) + (\partial_c Q(r))r_f]er_f r_c W(r)Q(r)\\
=&[-3Q(r) - r(\partial_r Q(r)) + Q(r) + r(\partial_r Q(r))]er^2 W(r)Q(r)\\
=&-2er^2 W(r)W(r)^2
\end{aligned}
\tag{C.8}
$$

The third term

$$
\begin{aligned}
& e^2\epsilon_{abc}\epsilon_{ade}\epsilon_{\mu bf}\epsilon^{\mu dg}r_f r_g r_c r_e W(r)^2 Q(r)^2 \\
=& -e^2(\delta_{bd}\delta_{ce}-\delta_{be}\delta_{cd})(\delta_{bd}\delta_{fg}-\delta_{bg}\delta_{fd})r_f r_g r_c r_e W(r)^2 Q(r)^2 \\
=& -e^2(\delta_{bd}\delta_{ce}-\delta_{be}\delta_{cd})(r^2\delta_{bd}-r_d r_b)r_c r_e W(r)^2 Q(r)^2 \\
=& -e^2(3r^2-r^2)r^2 W(r)^2 Q(r)^2 + (r^4-r^4)W(r)^2 Q(r)^2 \\
=& -2e^2 r^4 W(r)^2 Q(r)^2
\end{aligned}
\tag{C.9}
$$

Collecting the results, we have

$$
-\frac{1}{2}D_\mu Q_{i\alpha}(D^\mu Q_{i\alpha})^* = -\frac{1}{2}r^2\left(\frac{dQ}{dr}\right)^2 - rQ\frac{dQ}{dr} - \frac{3}{2}Q^2 - 2er^2WQ^2 - e^2r^4W^2Q^2 \tag{C.10}
$$

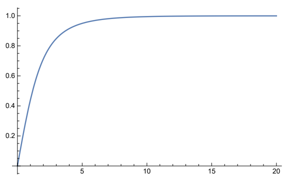
# Appendix D

# Mathematica Programmes - Numerical solution

## D.1 $\beta = 0.1$

```
In[104]:= a = 20
          b = 0.1
          c = 0.000001
          h = 1.44633
          k = -0.00000071437728038174599837057 5

          Sol = NDSolve[
            {-4 x^3 * (q'[x] / x - q[x] / x^2) + 4 x * w[x] × q[x] + 2 x * w[x]^2 × q[x] - 1/2 * b * x^3 *
                q[x] + 1/2 b * x^3 * q[x]^3 - x^4 * (q''[x] / x - 2 q'[x] / x^2 + 2 q[x] / x^3) == 0,
             -8 x^3 * (w'[x] / x^2 - 2 w[x] / x^3) + 6 w[x]^2 + 2 w[x]^3 + 2 x^2 * q[x]^2 +
                2 x^2 * w[x] × q[x]^2 - 2 x^4 (w''[x] / x^2 - 4 w'[x] / x^3 + 6 w[x] / x^4) == 0,
             q'[c] == h, q[c] == 0, w'[c] == k, w[c] == 0}, {w, q}, {x, c, a}]

          Plot[{q[x] /. Sol}, {x, c, a}, PlotRange -> Full]
          Plot[{w[x] /. Sol}, {x, c, a}, PlotRange -> Full]
          Plot[{q'[x] /. Sol}, {x, c, a}, PlotRange -> Full]
          Plot[{w'[x] /. Sol}, {x, c, a}, PlotRange -> Full]

Out[104]= 20

Out[105]= 0.1

Out[106]= 1. × 10^-6

Out[107]= 1.44633

Out[108]= -7.14377280381745998370 58 × 10^-7

Out[109]= {{w → InterpolatingFunction[ Domain: {{1. × 10^-6, 20.}} Output: scalar ],
            q → InterpolatingFunction[ Domain: {{1. × 10^-6, 20.}} Output: scalar ]}}
```

Out[110]=

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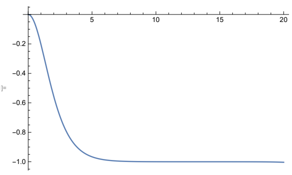
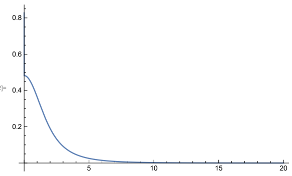
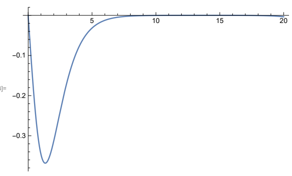
Out[111]=




Out[112]=




Out[113]=

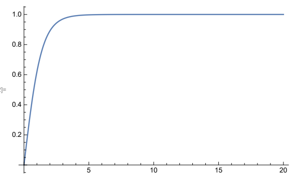


```
In[114]:= q0[x] = (q[x] /. Sol)
q1[x] = (q'[x] /. Sol)
w0[x] = (w[x] /. Sol)
w1[x] = (w'[x] /. Sol)

k1 = NIntegrate[
  x^2 (x^2 * (w1[x] / x^2 - 2 w0[x] / x^3) ^2 + 4 / x * w0[x] * (w1[x] / x^2 - 2 w0[x] / x^3) +
     6 w0[x] ^2 / x^4 + 2 w0[x] ^3 / x^4 + (1 / 2) x^-4 * w0[x] ^4 +
     (1 / 2) x^2 * (q1[x] / x - q0[x] / x^2) ^2 + q0[x] * (q1[x] / x - q0[x] / x^2) +
     (3 / 2) q0[x] ^2 / x^2 + 2 x^-2 * w0[x] * q0[x] ^2 + x^-2 * w0[x] ^2 * q0[x] ^2 -
     (1 / 4) b * q0[x] ^2 + (1 / 8) b * q0[x] ^4 + (1 / 8) b), {x, 0.000001, 18}]

Out[114]= {InterpolatingFunction[ Domain: {{1. × 10^-6, 20.}} Output: scalar ] [x] }

Out[115]= {InterpolatingFunction[ Domain: {{1. × 10^-6, 20.}} Output: scalar ] [x] }

Out[116]= {InterpolatingFunction[ Domain: {{1. × 10^-6, 20.}} Output: scalar ] [x] }

Out[117]= {InterpolatingFunction[ Domain: {{1. × 10^-6, 20.}} Output: scalar ] [x] }

Out[118]= {1.07841}


In[119]:= q0[x] = 1
q1[x] = 0
w0[x] = -1
w1[x] = 0

k2 = NIntegrate[
  x^2 (x^2 * (w1[x] / x^2 - 2 w0[x] / x^3) ^2 + 4 / x * w0[x] * (w1[x] / x^2 - 2 w0[x] / x^3) +
     6 w0[x] ^2 / x^4 + 2 w0[x] ^3 / x^4 + (1 / 2) x^-4 * w0[x] ^4 +
     (1 / 2) x^2 * (q1[x] / x - q0[x] / x^2) ^2 + q0[x] * (q1[x] / x - q0[x] / x^2) +
     (3 / 2) q0[x] ^2 / x^2 + 2 x^-2 * w0[x] * q0[x] ^2 + x^-2 * w0[x] ^2 * q0[x] ^2 -
     (1 / 4) b * q0[x] ^2 + (1 / 8) b * q0[x] ^4 + (1 / 8) b), {x, 18, 1000000}]
Out[119]= 1

Out[120]= 0

Out[121]= -1

Out[122]= 0

Out[123]= 0.0277773

In[124]:= k3 = k1 + k2
Out[124]= {1.10619}
```

## D.2 $\beta = 1$

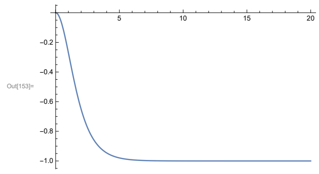
```
In[146]:= a = 20
b = 1
c = 0.000001
h = 2.19544259333844
k = -0.0000010227161293549
Sol = NDSolve[
  {-4 x^3 * (q'[x] / x - q[x] / x^2) + 4 x * w[x] × q[x] + 2 x * w[x]^2 × q[x] - 1 / 2 * b * x^3 *
      q[x] + 1 / 2 b * x^3 * q[x]^3 - x^4 * (q''[x] / x - 2 q'[x] / x^2 + 2 q[x] / x^3) == 0,
   -8 x^3 * (w'[x] / x^2 - 2 w[x] / x^3) + 6 w[x]^2 + 2 w[x]^3 + 2 x^2 * q[x]^2 +
     2 x^2 * w[x] × q[x]^2 - 2 x^4 (w''[x] / x^2 - 4 w'[x] / x^3 + 6 w[x] / x^4) == 0,
   q'[c] == h, q[c] == 0, w'[c] == k, w[c] == 0}, {w, q}, {x, c, a}]

Plot[{q[x] /. Sol}, {x, c, a}, PlotRange -> Full]
Plot[{w[x] /. Sol}, {x, c, a}, PlotRange -> Full]
Plot[{q'[x] /. Sol}, {x, c, a}, PlotRange -> Full]
Plot[{w'[x] /. Sol}, {x, c, a}, PlotRange -> Full]

Out[146]= 20

Out[147]= 1

Out[148]= 1. × 10^-6

Out[149]= 2.19544

Out[150]= -1.02272 × 10^-6

Out[151]= {{w → InterpolatingFunction[ Domain: {{1. × 10^-6, 20.}} Output: scalar ],

  q → InterpolatingFunction[ Domain: {{1. × 10^-6, 20.}} Output: scalar ]}}
```

Out[152]=

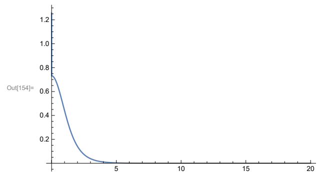




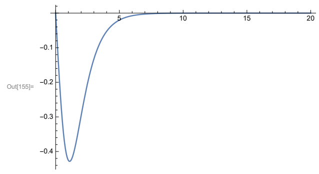
Out[153]=



Out[154]=



Out[155]=





```
In[ ]:= q0[x] = (q[x] /. Sol)
q1[x] = (q'[x] /. Sol)
w0[x] = (w[x] /. Sol)
w1[x] = (w'[x] /. Sol)

k1 = NIntegrate[
  x^2 (x^2 * (w1[x] / x^2 - 2 w0[x] / x^3)^2 + 4 / x * w0[x] * (w1[x] / x^2 - 2 w0[x] / x^3) +
      6 w0[x]^2 / x^4 + 2 w0[x]^3 / x^4 + (1 / 2) x^-4 * w0[x]^4 +
      (1 / 2) x^2 * (q1[x] / x - q0[x] / x^2)^2 + q0[x] * (q1[x] / x - q0[x] / x^2) +
      (3 / 2) q0[x]^2 / x^2 + 2 x^-2 * w0[x] * q0[x]^2 + x^-2 * w0[x]^2 * q0[x]^2 -
      (1 / 4) b * q0[x]^2 + (1 / 8) b * q0[x]^4 + (1 / 8) b), {x, 0.000001, 20}]

Out[ ]= {InterpolatingFunction[ Domain: {{1. × 10^-6, 32.}} Output: scalar ][x]}

Out[ ]= {InterpolatingFunction[ Domain: {{1. × 10^-6, 32.}} Output: scalar ][x]}

Out[ ]= {InterpolatingFunction[ Domain: {{1. × 10^-6, 32.}} Output: scalar ][x]}

Out[ ]= {InterpolatingFunction[ Domain: {{1. × 10^-6, 32.}} Output: scalar ][x]}

Out[ ]= {1.2127}

In[ ]:=
q0[x] = 1
q1[x] = 0
w0[x] = -1
w1[x] = 0

k2 = NIntegrate[
  x^2 (x^2 * (w1[x] / x^2 - 2 w0[x] / x^3)^2 + 4 / x * w0[x] * (w1[x] / x^2 - 2 w0[x] / x^3) +
      6 w0[x]^2 / x^4 + 2 w0[x]^3 / x^4 + (1 / 2) x^-4 * w0[x]^4 +
      (1 / 2) x^2 * (q1[x] / x - q0[x] / x^2)^2 + q0[x] * (q1[x] / x - q0[x] / x^2) +
      (3 / 2) q0[x]^2 / x^2 + 2 x^-2 * w0[x] * q0[x]^2 + x^-2 * w0[x]^2 * q0[x]^2 -
      (1 / 4) b * q0[x]^2 + (1 / 8) b * q0[x]^4 + (1 / 8) b), {x, 20, 1000000}]

Out[ ]= 1

Out[ ]= 0

Out[ ]= -1

Out[ ]= 0

Out[ ]= 0.0249995

In[ ]:= k3 = k1 + k2

Out[ ]= {1.2377}
```

## D.3 $\beta = 10$

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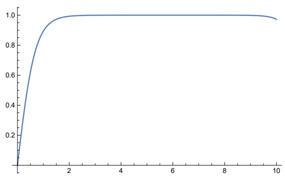
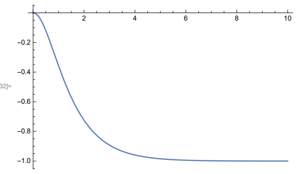
```
In[125]:= a = 10
b = 10
c = 0.000001
h = 4.3628479920
k = -0.0000016318645585234
Sol = NDSolve[
  {-4 x^3 * (q'[x] / x - q[x] / x^2) + 4 x * w[x] × q[x] + 2 x * w[x]^2 × q[x] - 1/2 * b * x^3 *
      q[x] + 1/2 b * x^3 * q[x]^3 - x^4 * (q''[x] / x - 2 q'[x] / x^2 + 2 q[x] / x^3) == 0,
   -8 x^3 * (w'[x] / x^2 - 2 w[x] / x^3) + 6 w[x]^2 + 2 w[x]^3 + 2 x^2 * q[x]^2 +
     2 x^2 * w[x] × q[x]^2 - 2 x^4 (w''[x] / x^2 - 4 w'[x] / x^3 + 6 w[x] / x^4) == 0,
   q'[c] == h, q[c] == 0, w'[c] == k, w[c] == 0}, {w, q}, {x, c, a}]

Plot[{q[x] /. Sol}, {x, c, a}, PlotRange -> Full]
Plot[{w[x] /. Sol}, {x, c, a}, PlotRange -> Full]
Plot[{q'[x] /. Sol}, {x, c, a}, PlotRange -> Full]
Plot[{w'[x] /. Sol}, {x, c, a}, PlotRange -> Full]
Out[125]= 10
Out[126]= 10
Out[127]= 1. × 10^-6
Out[128]= 4.36285
Out[129]= -1.63186 × 10^-6
Out[130]= {{w → InterpolatingFunction[ Domain: {{1. × 10^-6, 10.}} Output: scalar ],
  q → InterpolatingFunction[ Domain: {{1. × 10^-6, 10.}} Output: scalar ]}}
```

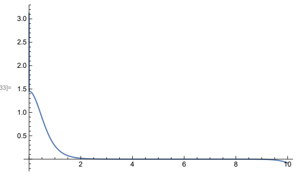
Out[131]=



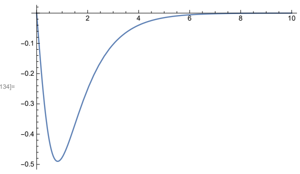

Out[132]=




Out[133]=




Out[134]=

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```
In[135]:= q0[x] = (q[x] /. Sol)
q1[x] = (q'[x] /. Sol)
w0[x] = (w[x] /. Sol)
w1[x] = (w'[x] /. Sol)

k1 = NIntegrate[
  x^2 (x^2 * (w1[x] / x^2 - 2 w0[x] / x^3)^2 + 4 / x * w0[x] * (w1[x] / x^2 - 2 w0[x] / x^3) +
     6 w0[x]^2 / x^4 + 2 w0[x]^3 / x^4 + (1/2) x^-4 * w0[x]^4 +
     (1/2) x^2 * (q1[x] / x - q0[x] / x^2)^2 + q0[x] * (q1[x] / x - q0[x] / x^2) +
     (3/2) q0[x]^2 / x^2 + 2 x^-2 * w0[x] * q0[x]^2 + x^-2 * w0[x]^2 * q0[x]^2 -
     (1/4) b * q0[x]^2 + (1/8) b * q0[x]^4 + (1/8) b), {x, 0.000001, 8.5}]

Out[135]= {InterpolatingFunction[ Domain: {{1. × 10^-6, 10.}} Output: scalar ][x]}
Out[136]= {InterpolatingFunction[ Domain: {{1. × 10^-6, 10.}} Output: scalar ][x]}
Out[137]= {InterpolatingFunction[ Domain: {{1. × 10^-6, 10.}} Output: scalar ][x]}
Out[138]= {InterpolatingFunction[ Domain: {{1. × 10^-6, 10.}} Output: scalar ][x]}
Out[139]= {1.37431}

In[140]:= q0[x] = 1
q1[x] = 0
w0[x] = -1
w1[x] = 0

k2 = NIntegrate[
  x^2 (x^2 * (w1[x] / x^2 - 2 w0[x] / x^3)^2 + 4 / x * w0[x] * (w1[x] / x^2 - 2 w0[x] / x^3) +
     6 w0[x]^2 / x^4 + 2 w0[x]^3 / x^4 + (1/2) x^-4 * w0[x]^4 +
     (1/2) x^2 * (q1[x] / x - q0[x] / x^2)^2 + q0[x] * (q1[x] / x - q0[x] / x^2) +
     (3/2) q0[x]^2 / x^2 + 2 x^-2 * w0[x] * q0[x]^2 + x^-2 * w0[x]^2 * q0[x]^2 -
     (1/4) b * q0[x]^2 + (1/8) b * q0[x]^4 + (1/8) b), {x, 8.5, 1000000}]
Out[140]= 1
Out[141]= 0
Out[142]= -1
Out[143]= 0
Out[144]= 0.058823

In[145]:= k3 = k1 + k2
Out[145]= {1.43313}
```

# Appendix E

# Regrouping the terms of scalar field

First, we can ignore the $SU(4)$ part for the scalar field as

$$\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 1 \tag{E.1}$$

For the $SU(2)$ part, it would become much simpler to work in $SO(3)$ representation instead. It is obvious that

$$\begin{pmatrix} x+iy & z \end{pmatrix} \begin{pmatrix} x-iy \\ z \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r^2 \tag{E.2}$$

So,

$$Q_{i\alpha}Q_{i\alpha} = Q_aQ_a \tag{E.3}$$

We also can write the covarient derivative in $SO(3)$ representation. The covarient derivative in $SO(3)$ is given by

$$\begin{aligned} D_\mu Q_a =& \partial_\mu(r_aQ(r)) + e \begin{pmatrix} 0 & -W_{3\mu} & W_{2\mu} \\ W_{3\mu} & 0 & -W_{1\mu} \\ -W_{2\mu} & W_{1\mu} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} Q(r) \\ =& \partial_\mu(r_aQ(r)) + e\epsilon_{abc}W^b_\mu r_cQ(r) \\ =& \partial_\mu Q_a + e\epsilon_{abc}W^b_\mu Q_c \end{aligned} \tag{E.4}$$

It is valid to replace the covarient derivative of $SU(2)$ with the expression in (E.4) because $SU(2)$ and $SO(3)$ are just different representations of 3-d rotations, which are equivalent. $(x, y, z)$ rotated by $SO(3)$ matrix would give the same result as $(x + iy, z)$ rotated by $SU(2)$ matrix with same parameters. The partial derivative terms would contract similarly as in (E.2).

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