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SUSY Localization and AdS Black Holes

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Abstract

Equivariant localization has a natural extension to supersymmetric field theories, which directly exploits the particular nature of supermanifolds to reduce dimensionality of an otherwise intractable path integral to a 1-loop exact form. This paper walks through pedagogically and historically interesting examples where localization naturally appears, performing calculations such as the twisted Dirac index. Subsequently after the general ideas have been revealed naturally using physics techniques in field theories, the specific description of the mathematics of equivariant cohomology, the cartan model, localization, and the Atiyah-Bott-Berline-Vergne formula are discussed. With the relevant structure of localization presented a direct analogy and extension to supersymmetric localization is presented. Concluding is a digression on AdS black holes with the intent of establishing readily apparent application of supersymmetric localization to make exact quantum entropy calculations or perform analysis of holography. Bekenstein-Hawking radiation, Wald entropy, AdS/CFT, and a quantum entropy formula for supersymmetric extremal black holes are reviewed.

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Chapter 1

Localization in 0 and 1 dimensions, a heuristic introduction

The application of the path integral in quantum theory has a been an incredible tool since Richard Feynman introduced it in 1948. It has become so ubiquitous that physicists never bat an eye at its definition until a mathematician exclaims just how incredible it is that we managed to calculate anything at all[1]. What has become even more surprising is that the path integral has proven more and more useful even as our theories have become more complicated. The path integral has not been used without difficulty however. In most cases calculations are carried out in expansions and approximations, often using typical techniques of Feynman diagrams and the stationary phase approximation. In the 1980's a very powerful formalism came about after solving the traditionally difficult calculations for a class of oscillatory integrals by Duistermaat and Heckman [1]. Subsequently, Berline-Vergne [2] and Atiyah-Bott [3] independently related this result to a generalization of integration of equivariant forms yielding a localization formula. Also taking place at the same time was the superstring revolution where many physicists had their eyes open for interesting and useful mathematical papers, especially ones involving Atiyah. Shortly after these results, the localization formula was put to use into particular topological and supersymmetric field theories for calculating the path integral exactly. Late into 1988, after several applications, Edward Witten formalized many of the localization formula's application to topological field theories [4] and after this followed many different proofs of index theorems in mathematics by physicists.

For many, these calculations were a novelty but for most the fact that one could compute the path integral exactly, even for interacting theories, was very promising and likely only fueled the superstring fervor. Up to that point, instances where the path integral could be computed exactly were few and far between. While localization had become a powerful tool in calucations

[1]Freeman Dyson was not discouraged!

for these limited class of theories it was not until 2007 when Pestun calculated Wilson loops on S^4 proving a conjecture relating Gaussian matrix models to $\mathcal{N}=2$ supersymmetric (SUSY) Yang-Mills theory, thus expanding the application of localization to non-topological theories [5]. Since then, localization has become a central tool in the study of supersymmetric theories on various backgrounds with explicit use in classifying results for holography and recently making exact calculations of quantum entropy of black holes. The value of supersymmetric localization was once esoteric, but in its modern applications it may prove a tenant going forward when calculating otherwise intractable path integrals, but may also prove to illuminate some structure in the class of theories where quantum gravity lives.

The purpose of this paper is to provide an honest heuristic introduction and very rough consolidation of material available on supersymmetric localization[2], ie to be as self contained as possible. To this end, the organization and style are intended to put pedagogy first. Therefore, this introduction has a inherent narrative that attempts to assume no more than a typical graduate physics education and only supplies technical details in line with the narrative or as consequential statements which should not directly impact a reader unfamiliar with the concepts. The idea will be to front-load as many of the relevant concepts and conclusions in self contained examples before discussing the formal mathematics and relationship to physics in the second chapter. Naturally, the initial discussion will be equally accessible to career physicists who are otherwise unfamiliar with one or more particular topic discussed here as well as physicists who find themselves on too far removed from these theoretical ideas but are still curious.

The roadmap to contextualize modern applications of supersymmetric localization will be rather simple. First, we will motivate localization with very simple examples in 0 and 1 dimensions where we will point out key features in localization that are integral to the localization arguments. Along the way we'll uncover familiar concepts which will turn out to be more fundamental than they first appear, for example the connection between Euler Classes/Characteristics, the Witten Index and it various realizations. With these examples as a baseline we will uncover the mathematics of Equivariant Cohomology and the Cartan Model, which are key in discussing how localization occurs for finite integrals. This mathematical basis will give us a good intuition on how to extend localization to the notorious path integral in the supersymmetric context and discover a robust structure to evaluate otherwise intractable infinite dimensional integrals. The last chapter will serve as a quick dive into black hole entropy and provide an open corridor whereby the application of localization is currently in use and will likely play a large role in years to come.

[2] A complete review of localization in QFTs is nearly 700 pages [6].

1.1 Localization in a simple 0-d Supersymmetric QFT

Much of the preliminaries discussed in this section take point from the Clay monograph, Mirror Symmetry by Hori et al [7]. There are two typical starting points for very basic localization; Supersymmetric Quantum Mechanics (SQM) and the stationary phase approximation. Both leverage familiarity, but operate on two sides of complexity and can fail to convey the power together with salient points in a convincing way[3]. This is important since all starting points on the subject are heuristic, so the approach favored here is an attempt to exemplify how localization might emerge and if it is applicable then demonstrate how useful it could be in certain contexts. In this way, the mathematics and applications introduced later on will have a similar flavor as the introduction as well as demonstrate how the heuristic can be extended without constantly pondering about how these arguments break down with a continuous spectrum or any other scenarios where the initial arguments might break down.

Let us begin by considering a 0 dimensional spacetime and a supersymmetric QFT with 1 boson, Grassmann-even ϕ, and 2 fermions, complex Grassmann-odd ψ and $\bar{\psi}$. This is the simplest nontrivial particle content with both fermions and bosons for a QFT we can consider since our action needs to be Grassmann-even. Since our space is 0 dimensional we have a trivial metric and thus the trivial representation of Lorentz group, i.e. the fields are scalar on $\mathbb{R}$ and we have a Lebesgue measure for the field space. Further, since there is no time direction it is elementary enough to conclude that the Hamiltonian is 0, and in some sense it is not appropriate to call H a Hamiltonian but will make sense in higher dimensions. Therefore, it must be the case that the anticommutator of the supercharges, generators, in the SUSY algebra yield $\{Q, \bar{Q}\} = H = 0$.

To construct our first path integral, we need to correctly identify an appropriate action and invariant measure. With the correct action we'll be able to state a sensible supersymmetric transformation and the corresponding supersymmetric algebra.

First let us choose our measure to be $d\phi d\psi d\bar{\psi}$ so that,

$$Z = \int d\phi d\psi d\bar{\psi}\; e^{-S}$$

$$\text{where, } S = S[\phi, \psi, \bar{\psi}].$$

We note that the action must be bosonic and then must conclude that S cannot depend on independent monomials of ψ or $\bar{\psi}$, nor powers of the fermions greater than 2 since $\chi^2 = \bar{\chi}^2 = 0$ for any Grassmann odd variable χ, thus the action should have the bosonic term $\propto \bar{\psi}\psi$. Additionally, to ensure the path integral converges properly, we may consider the superpotential W as a real

[3] The appropriate perspective here is that of a physicist whose direct interest would be to know how useful the mathematics is, given the vast landscape of interesting mathematics.

polynomial of the field ϕ, so that part of the action that depends only on ϕ may be written as,

$$S_\phi = \frac{1}{2}(\partial_\phi W)^2 = \frac{1}{2}(\partial W)^2.$$

The last consideration needed to complete the construction of the action is the supersymmetric transformations on the fields we have in our theory.

The basis of any supersymmetric transformation is a mapping of the bosonic fields to the fermionic fields, therefore a natural choice would be a linear combination of the fermions with a set of fermionic paramaters. Under that same logic, we would like the transformation acting on the fermions to yield some dependence on the bosonic field, and a great candidate by inspection is the superpotetial W in the form $\delta\psi \propto \bar{\epsilon}\partial W$. The sensible infinitesimal transformation on the fields is then,

$$\delta\phi = \epsilon\psi - \bar{\epsilon}\bar{\psi}$$

$$\delta\psi = \bar{\epsilon}\partial W \text{ and } \delta\bar{\psi} = \epsilon\partial W$$

$$\delta = \epsilon Q + \bar{\epsilon}\bar{Q} = \begin{bmatrix} 0 & \epsilon & -\bar{\epsilon} \\ \bar{\epsilon}\partial W & 0 & 0 \\ \epsilon\partial W & 0 & 0 \end{bmatrix}, \text{ where Q and } \bar{Q} \text{ are supercharges.}$$

Since we need $\delta S = 0$, the fermionic part of the action should be $\psi\bar{\psi}\,\partial^2 W$, leaving us to verify that the action under the following form is invariant, as required.

$$\begin{aligned} S &= \frac{1}{2}(\partial W)^2 + \psi\bar{\psi}\,\partial^2 W \\ \delta S &= \delta\frac{1}{2}\left(\frac{\partial W(\phi)}{\partial\phi}\right)^2 + (\delta\psi\bar{\psi}\partial^2 W + \psi\delta\bar{\psi}\partial^2 W) \\ &= \partial W\partial^2 W(\epsilon\psi - \bar{\epsilon}\bar{\psi}) + (\bar{\epsilon}\bar{\psi} + \psi\epsilon)\partial W\partial^2 W \\ &= \partial W\partial^2 W[\{\epsilon,\psi\} - \{\bar{\epsilon},\bar{\psi}\}] = 0, \text{ since Grassmann numbers anitcommute.} \end{aligned}$$

This prototypical approach to formulate the action to our desired theory only gets us so far, but it is a useful algorithm to reverse engineer when looking at a theory for the first time. The action being invariant is not enough for our theory to be consistent[4] since we also require that the measure also be invariant. Fortunately, this is manifestly true and easy to check by observing that $tr(\delta) = 0$ yielding $det\left(e^{-\delta}\right) = 1$, indicating the Jacobian of the transformation has determinant 1 and thus the measure is invariant. For higher dimensions, the process will be similar but will then

[4]Often the can also conlude the theory in non-anamolous.

require the superdeterminants and supertraces instead.

The construction of this action was quite simple, but not without issue. The demand that the generators be nilpotent constrains the equation of motion, $Q^2\bar{\psi} = \psi\partial^2 W = 0$, which only holds for the algebra on-shell in general. In fact, this issue can easily be worked around with a proper change of variables that is less intuitive than the above construction, but will prove useful in evaluating the structure of localization as a method. This highlighted issue will actually reveal itself to be a feature of localization and therefore it was worth stumbling upon.

With that caveat out of the way, we can finally write down the full partition function, albeit not properly normalized, for this theory as

$$Z = \int d\phi d\psi d\bar{\psi}\; e^{-\frac{1}{2}(\partial W)^2 - \psi\bar{\psi}\partial^2 W}.$$

1.1.1 Deforming the path integral

Our goal now is to perform some analysis on this partition function in a familiar way by introducing some deformation parameter λ, such that $W \mapsto \lambda W$, and therefore produce a generating functional

$$Z[\lambda] = \int d\phi d\psi d\bar{\psi} e^{-S_\lambda} = \int d\phi d\psi d\bar{\psi} e^{-\frac{\lambda^2}{2}(\partial W)^2 - \lambda\psi\bar{\psi}\partial^2 W}.$$

As usual with generating functionals we are encouraged take derivatives,

$$\begin{aligned}
\frac{d}{d\lambda} Z[\lambda] &= \int d\phi d\psi d\bar{\psi}\; \left(-\lambda(\partial W)^2 - \psi\bar{\psi}\partial^2 W\right)\; e^{-S_\lambda} \\
&= \int d\phi d\psi d\bar{\psi}\; (-Q_\lambda)(\partial W\bar{\psi})\; e^{-S_\lambda} \\
\text{notably since } S_\lambda \text{ is invariant,} \qquad &= -\int d\phi d\psi d\bar{\psi}\; Q_\lambda(\partial W\bar{\psi}\; e^{-S_\lambda}) \\
\text{with the EOM,} \qquad Q(\partial W\bar{\psi}) &= (Q\partial W)\bar{\psi} + \partial W Q\bar{\psi} = (Q^2\bar{\psi})\bar{\psi} + (\partial W)^2 \\
&= \psi\bar{\psi}\partial^2 W + (\partial W)^2 \\
\text{Taking } Q \text{ as a vector field,} \qquad \frac{d}{d\lambda} Z[\lambda] &= -\int d\phi d\psi d\bar{\psi}\; \left(\psi\partial_\phi + \lambda\partial W\partial_{\bar{\psi}}\right)\left(\partial W\bar{\psi}\; e^{-S_\lambda}\right) \\
&= -\int d\phi d\psi d\bar{\psi}\; \left[\psi\partial_\phi(\partial W\bar{\psi}\; e^{-S_\lambda}) - \lambda(\partial W)^2\partial_{\bar{\psi}}(\bar{\psi}\; e^{-S_\lambda})\right] \\
\alpha\int d\theta 1 = 0, \text{ the second term vanishes,} \qquad &= -\frac{d}{d\phi}\int d\phi d\psi d\bar{\psi}\; \left[\psi\bar{\psi}\partial W e^{-S_\lambda}\right]
\end{aligned}$$

It is worth commenting on the derivation above before concluding the point; in taking the derivative we discovered an operator insertion and thus effectively calculated a correlator, ableit trivial, but this point will be explored with cohomology[5] in the next chapter, but already on its' own suggests some possibility in calculating more complicated correlators exactly[6].

Continuing, given a well behaved path integral, we demand that our boundary terms vanish for $S(\phi) \to +\infty$ as $|\phi| \to \infty$, therefore the term above is 0 as it is a total derivative.

$$\frac{d}{d\lambda} Z[\lambda] = 0$$

Thus our partition function is independent of our deformation by λ, including $\lambda = 1$ which corresponds to our undeformed partition function, and we can easily generalize this discovery to $W \mapsto (1+\lambda)W$. The relevance of this fact is subtle, to appreciate it observe that we can redefine $S_\phi = \frac{1}{2}\lambda^2(\partial W)^2$ trivially and take limits,

$$e^{-S_\phi} = e^{-\frac{1}{2}\lambda^2(\partial W)^2} \to 0 \text{ as } \lambda \to \infty \text{ unless } \partial W = 0.$$

The key observation is that if we deform the action with sufficiently strong coupling λ, we discover fixed points $\partial W \neq 0$, which localizes the partition function about some sufficiently small neighborhood. This is a key feature of a superintegral as in normal approaches we would need to sum an infinite number of Feynman diagrams to rediscover the same analysis, but with this careful consideration we have essentially discovered that the semi-classical approximation is exact. We will visit this fact later as well as uncover the general structure of localization, but first we can take this analysis further and calculate exactly the partition function for this theory.

1.1.2 The 0-d analog of the Witten/Supersymmetric Index

Let us introduce a factor of $\frac{1}{\sqrt{2\pi}}$ to the partition function in anticipation of evaluating a Gaussian integral. We have yet to place any specific restrictions on the type of function W we want in our action, in general we might want to consider quasi-homogeneous functions, but here it will be sufficient to take W as a polynomial of degree N with $N-1$ critical points ϕ_c such that $\partial W\big|_{\phi_c} = 0$ and $\partial^2 W\big|_{\phi_c} \neq 0$; in other words the critical points are non-degenerate[7].

[5]This will be realized as a equivalence class.

[6]One such example will the an ordered sequence of operators or a Wilson line etc.

[7]In the right context W is called a Morse function.

In a typical expansion about a sufficiently small neighborhood around a single ϕ_c we find that,

$$
\begin{aligned}
W &= W(\phi_c) + \frac{a_c}{2}(\phi - \phi_c)^2 + ... \\
\partial W &= a_c(\phi - \phi_c)^2 + ... \\
\partial^2 W &= a_c + ... \\
\implies S &= \frac{a_c^2}{2}(\phi - \phi_c)^2 - a_c \psi \bar{\psi} + ... \\
\implies Z[W_c] &= \int \frac{d\phi d\psi d\bar{\psi}}{\sqrt{2\pi}} e^{-a_c^2 \frac{(\phi-\phi_c)^2 2}{2} - a_c \psi \bar{\psi} - ...} \\
&= \int \frac{d\phi d\psi d\bar{\psi}}{\sqrt{2\pi}} e^{-a_c^2 \frac{(\phi-\phi_c)^2}{2}} (1 - a_c \psi \bar{\psi}) e^{\mathcal{O}[(\phi-\phi_c)^2]} \\
&= a_c \int \frac{d\phi d\psi d\bar{\psi}}{\sqrt{2\pi}} e^{-a_c^2 \frac{(\phi-\phi_c)^2}{2}} \bar{\psi}\psi \\
&= a_c \int \frac{d\phi}{\sqrt{2\pi}} e^{-a_c^2 \frac{(\phi-\phi_c)^2}{2}} \\
&= \frac{a_c}{\sqrt{a_c^2}} = sgn\left(\partial^2 W\big|_{\phi_c}\right)
\end{aligned}
$$

Expanding about every critical point ϕ_c, and then summing over all the neighborhoods, thereby integrating over the entire domain, the partition function becomes,

$$
\implies Z[W] = \sum_{\phi_c : \partial W \phi_c = 0} sgn\left(\partial^2 W\big|_{\phi_c}\right).
$$

We know from preschool calculus that polynomials have alternating $\partial^2 W > 0$ and $\partial^2 W < 0$ for each root $\partial W\big|_{\phi_c} = 0$, and noting cancellation with pairs of alternating $sgn\left(\partial^2 W\big|_{\phi_c}\right)$ yields,

$$
\begin{aligned}
&\text{for } N \text{ odd, } N-1 \text{ is even} && \implies Z = sgn\left(\partial^2 W\big|_{\phi_c}\right) = 0 \\
&\text{for } N \text{ even, } N-1 \text{ is odd} && \implies Z = sgn\left(\partial^2 W\big|_{\phi_c}\right) = \pm 1.
\end{aligned}
$$

Where the sign of $Z[W_{\text{even}}]$ depends on $W(\phi) \to \pm\infty$ as $|\phi| \to \infty$. It is also worth noticing that we have implicitly used the deformation invariance we discovered, since the partition function is only sensitive to the critical points of the superpotential, which governs supersymmetric interactions. The above result, and the importance of the superpotential in this example, intuitively indicates that our partition function is counting some related quantity due to these supersymmetric interactions. With some reasoning, the likely interpretation is that we have counted the dimension of a subspace made up of ground states from the underlying Hilbert space subject to the preserved symmetries about these fixed points. We will see that similar calculations in higher dimensions, with a 1 dimensional SQFT, generates the Witten index and with the right perspective the Euler characteristic or more generally an Euler class. This counting realization is central to the application of SUSY localization, specifically in counting microstates of AdS black holes.

1.1.3 Landau-Ginzburg and the localization principle

We can take the previous example a little bit further and interpret some of the features we observe appropriately. As before, let us consider a theory with 2 fermions and 1 boson, but this time let us assume they are complex. In a similar fashion to the simple 0-d case we'll utilize the superpotential $W(\phi)$ to define our action and infinitesimal transformation in the following way,

$$\begin{aligned} &\delta\phi = \epsilon_1\psi_1 + \epsilon_2\psi_2 \qquad \delta\psi_1 = \epsilon_2\bar{\partial}\bar{W} \qquad \bar{\delta}\bar{\psi}_1 = \bar{\epsilon_2}\partial W \\ &\delta\bar{\phi} = \bar{\epsilon_1}\bar{\psi}_1 + \bar{\epsilon_2}\bar{\psi}_2 \qquad \delta\psi_2 = \epsilon_1\bar{\partial}\bar{W} \qquad \bar{\delta}\bar{\psi}_2 = \bar{\epsilon_1}\partial W \\ &\delta = \epsilon_1 Q_1 + \epsilon_2 Q_2 \qquad \bar{\delta} = \bar{\epsilon_1}\bar{Q}_1 + \bar{\epsilon_2}\bar{Q}_2 \end{aligned}$$

$$\begin{aligned} S[\phi, \psi_i] &= |\partial W|^2 + \partial^2 W\psi_1\psi_2 + \bar{\partial}^2\bar{W}\bar{\psi}_1\bar{\psi}_2 \\ \text{so that, } Z &= \int \frac{d\phi^2 d\psi^4}{2\pi} e^{-S[\phi,\psi_i]}. \end{aligned}$$

The transformations on the fields are the same as before but now with their conjugate pairs and again the anticommutation of the supercharges hold only when we demand the ψ_i equations of motion, $\partial^2 W = 0$. As mentioned before, we will perform a change a variables to highlight the properties of localization by recalling that we were able to integrate out the fermionic contribution to the integrand. Further, we have some interest in exploring the features of our theory about the fixed points and can leverage a transformation that explicitly highlights these neighborhoods. Therefore, let us perform a change a variables such that $S[\phi, \psi_i] \to S[\phi', 0]$,

$$\begin{aligned} S[\phi', 0] &= \left|\frac{\partial W}{\partial \phi'}\right|^2 \\ \text{with, } \phi' = \phi - \frac{\psi_1\psi_2}{\partial W} &\text{ and, } \chi_i = \frac{\psi_i}{\sqrt{\partial W}}. \end{aligned}$$

This transformation mimics the exact integration we performed in the simple 0-d case by changing the measure appropriately and deforming away from the critical points. Since this transformation is well defined away from $\partial W = 0$ and since the action depends only on ϕ' and not the new fermions, then under Berezin integration the partition function is trivially 0. In retrospect, for the simple 0-d case, we could have set $\epsilon_i = -\frac{\psi 1}{\partial W}$ to eliminate ψ_1. Further, this set of coordinates are singular near the critical points of W by design and thus this change of coordinates cannot be performed within a sufficiently small neighborhood of the critical points. Therefore, the partition function has non-zero contributions from these critical points as we discovered by brute force before, and our vanishing result held only for the domain modulo the critical points. With all this said, it may seem like the transformation we chose was just a convenience or a redefinition of the proper invariance we demand on the measure. The proper interpretation is to see that we are free to trade the supersymmetry tranformation for the fermions except when $\partial W = 0$ implies $\delta\psi = 0$, in other words the path integral localizes when the supersymmetric transformations on the fermions is 0. We will cover this in greater detail with more mathematical precision but the crux is this, the fermionic supersymmetry acts freely on the entire manifold except on some submanifolds where the action may vanish, and where this action vanishes we may characterize them by classifying them under an equivalence class as is natural. The challenge will be that in general these submanifolds or subspaces may not be compact and typical cohomological techniques are not sufficient. In conclusion here, we have come to a powerful interpretation of localization physically, but we have yet to calculate the partition function for this example.

We may proceed by expanding the superpotential around its critical points so that we have the familiar form and may use this to calculate the partition function again imposing the principle of localization;

$$
\begin{aligned}
W &= W(\phi'_c) + \frac{\alpha_c}{2}\left(\phi' - \phi'_c\right)^2 + ... \\
Z &= \int \frac{d\phi^2 d\psi^4}{2\pi} e^{-S[\phi,\psi_i]} \\
&= \sum_{\phi_c:\partial W\phi_c=0} \int \frac{d\phi^2 d\psi^4}{2\pi} \left|\alpha_c\right|^2 e^{-\left|\alpha_c(\phi'-\phi'_c)\right|^2} \psi_1\psi_2\bar{\psi}_1\bar{\psi}_2 \\
&= \sum_{\phi_c:\partial W\phi_c} \frac{\left|\alpha_c\right|^2}{\left|\alpha_c\right|^2} = \sum_{\phi_c:\partial W\phi_c} 1 = \text{number of crtitical points}
\end{aligned}
$$

This result is a Hail Mary for our previous assertion that we were effectively counting something. We will leave this point as just circumstantial evidence for now in order to digress to correlation functions. A great reason to study the Landau-Ginzburg model in this context is that in our calculation above we find that we are left with a 1 under the sum, and appropriate operator insertions should evaluate over the critical points with weight 1. To see this explicitly we need to remember that in order to utilize localization we need sufficient amount of supersymmetry for our fermionic fields. Mixed operators of ϕ and $\bar{\phi}$ won't preserve any supersymmetry, so we are left to consider holomorphic and anti-holomorphic operators. For those familiar with 2-d conformal field theory, this isn't a surprise as holormophicity is related to supersymmetry in Landau-Ginzburg, and will come into play under the hood when discussing sigma models. Our unnormalized correlation functions, for operator $\mathcal{O}$, that sufficiently preserve supersymmetry are,

$$
\begin{aligned}
\text{with } \bar{\delta},\ \langle \mathcal{O}(\phi) \rangle &= \int \frac{d\phi^2 d\psi^4}{2\pi} \mathcal{O}(\phi) e^{-S[\phi,\psi_i]} \\
&= \int \frac{d\phi d\bar{\phi}}{2\pi} \mathcal{O}(\phi) \left|\partial^2 W\right|^2 e^{-\frac{1}{2}\left|\frac{\partial W}{\partial \phi}\right|^2} \\
&= \sum_{\phi_c : \partial W \phi_c = 0} \mathcal{O}(\phi_c) \int \frac{d\phi d\bar{\phi}}{2\pi} \left|\partial^2 W\right|^2 e^{-\frac{1}{2}\left|\frac{\partial W}{\partial \phi}\right|^2} \\
&= \sum_{\phi_c : \partial W \phi_c = 0} \mathcal{O}(\phi_c) \\
\text{similarly with } \delta,\ \left\langle \mathcal{O}(\bar{\phi}) \right\rangle &= \sum_{\bar{\phi}_c : \bar{\partial} \bar{W} \bar{\phi}_c = 0} \mathcal{O}(\bar{\phi}_c)
\end{aligned}
$$

But there is an easier and more general way of constructing operators by recognizing that our operator insertions correspond to chiral fields. Take $\mathcal{O} = \bar{\delta}\Lambda$ or $\mathcal{O} = \bar{Q}\Lambda$, depending on your favorite notation, where Λ is a general operator, then notice that $\bar{\delta}^2 = 0$ for $\bar{\epsilon_1} = \bar{\epsilon_2}$ makes $\mathcal{O}$ a trivial chiral operator for any Λ. Correlation functions of this trivial chiral operators vanish as we calculated above and in the previous 0-d example, therefore they are inherently uninteresting. The interesting operators are chiral operators $\bar{Q}\mathcal{O} = 0$ that are not trivial, which naturally leads us to the chiral ring, a cohomological ring with "top form" $\bar{Q}\Lambda$, such that $\left\langle \mathcal{O} + \bar{Q}\Lambda \right\rangle = \langle \mathcal{O} \rangle$,

$$
H_{\bar{Q}} = \frac{\mathcal{O} : \bar{Q}\mathcal{O} = 0}{\mathcal{O} = \bar{Q}\Lambda}.
$$

We can now make additional connections to the group of symmetries involved in our theory and how they relate to the properties we are observing. We already saw that with arbitrarily large deformations, the bosonic term effectively muffles contributions everywhere except near the critical points. Normally, the contributions of the fermionic symmetries would vanish trivially due to invariance of the integral but this assumes the group of transformations act freely over the whole domain, in other words restricting to a smaller space yields trivial stabilizers. We have discovered that about these critical points is a locus which supplies a non-zero contribution, therefore the action of the group of fermionic symmetries is not free where the transformations of the fermions vanish. One way to explain localization here is observation that the integral only receives contributions from infinitesimal neighborhoods about these loci where the group action of fermionic symmetries is not free.

Up to this point we've considered theories with single variables of the fields, so for completeness it is rather simple to extend these examples to multivariable theories. For example the 0-d Laundau-Ginzburg theory we have above can be extended to a multivariable theory with the action,

$$S[\phi_i, \psi_1^i, \psi_2^i] = \sum_{i=1}^{N} |\partial_i W(\phi_1, ..., \phi_N)|^2 + \partial_i \partial_j W \psi_1^i \psi_2^j + \bar{\partial}_i \bar{\partial}_j \bar{W} \bar{\psi}_1^i \bar{\psi}_2^j.$$

These two 0-d examples are pretty common in the literature because they are extremely useful to keep in the back of one's mind when generalizing these ideas and taking them to the abstract limit. Additionally, these examples exemplify the generic technique of localization whilst illuminating physically relevant interpretations of various properties. As is common in physics, these techniques may seem like tricks at first but with proper interpretation they can dictate physical principles, or at least systematic properties of a class of theories that might be fundamental. This point is worth highlighting in the modern theoretical context as it can be difficult to understand how one area of research might be particularly exciting at one time or another. We will see that localization can serve to both inform physically relevant properties of various quantum field theories, but will also provide a valuable mathematical apparatus providing evidence of correspondences in the string/gauge/gravity landscape. Before taking a look at contemporary directions of localization, we will need to tackle the challenge of extending basic concepts in QM/QFT and SUSY to higher dimensions on curved spaces as well as addressing the complication of these loci not generically being compact manifolds. To remain faithful to the spirit of this chapter, we will encounter a very familiar example where a non-compact manifold appears when considering the stationary phase approximation, but tradition demands we visit all stops on our tour else we diverge too far from the standardized pedagogy.

1.2 Brief overview of SUSY QM, the Witten Index, and our first Index theorem

In order to motivate the application of deformation invariance and localization in supersymmetric QM, we will consider a 1-d analog of the 0-d theory we already analyzed. We will take a particularly geometric approach which will nicely tie in to the methods we need when considering curved spaces. It goes without saying that supersymmetric quantum mechanics, SQM, is a rich subject and can fill a thesis all by itself, let alone a text of this scope. Therefore, this section and the next will serve as condensed supplements that highlight the necessary components for working with a theory for our purposes, sadly at the cost of skipping over more subtle points that we will have to take for granted in order to extrapolate towards contemporary applications. Fortunately for most readers, familiarity will go a long way and any complications or lack of rigor may be substituted readily and verified where interest presents itself. For this purpose, the author encourages the reader to explore this subject in more depth in [7], [8], [9], [10], and [11].

The first goal in this section will be to explore an invariant known as the Witten index[8] by examining the properties of SUSY in this context and then make contact with its relation to the path integral. After this we will expand on this relationship in order to discover a form of the Atiyah-Singer index theorem, originally done by [13] and [14], that together with Equivariant Cohomology essentially motivates localization as a tool.

1.2.1 SUSY and the Witten Index

In previous sections we relied on variations in field theoretic techniques as well as Grassmann numbers to apply supersymmetry, but here we may as well tread a little more carefully. In the 1d case of a $\mathcal{N}=1$ SUSY, the superalgebra of the graded super vector space is,

$$\{Q,\bar{Q}\} = 2H,$$

$$\{Q,Q\} = \{\bar{Q},\bar{Q}\} = 0$$

where, $\bar{Q} = Q^{\dagger}$ and H is now the Hamiltonian.

By considering an eigenstate decomposition of the Hamiltonian we may discover that the energy eigenvalues are non-negative since,

$$E = \frac{1}{2}\left(\|Q\left|E\right\rangle\|^2 + \|\bar{Q}\left|E\right\rangle\|^2\right) \geq 0$$

and in fact the ground state is a zero energy state where,

$$H\left|\psi\right\rangle = 0 \iff Q\left|\psi\right\rangle = \bar{Q}\left|\psi\right\rangle = 0 \text{ so, } \left|\psi\right\rangle \equiv \left|\Omega\right\rangle.$$

Thus, a correspondence exists between two states whereby they may transform into one another under Q and $\bar{Q}$ respectively. These states have opposite fermion numbers and split the Hilbert space,

$$\mathcal{H} = \mathcal{H}_{n,b} \oplus \mathcal{H}_{n,f}$$

where

$$\mathcal{H}_{n,b} \cong \mathcal{H}_{n,f} \ \ \forall n > 0$$

by these mappings. For $E=0$, the action of Q and $\bar{Q}$ annihilate the ground state, $\left|\Omega\right\rangle$, therefore $\mathcal{H}_{0,b}$ and $\mathcal{H}_{0,f}$ are not in general isomorphic. The difference in fermion number, or respective dimension, will encapsulate this.

Noting that $(-1)^F Q = -Q(-1)^F$, $F = \bar{\psi}\psi$ in the same way we define the number operator

[8] Named after Edward Witten, from his famous paper [12]

with the SHO, where F yields identity on fermionic states and 0 on bosonic states. Further, since $\bar{Q}$ is the adjoint of Q and we have a splitting of the Hilbert space into fermionic/bosonic subspaces that only differ in the ground state with respect to the fermion number,

$$tr_{\mathcal{H}}(-1)^F = \ker Q - \ker \bar{Q}.$$

For E_i, $i > 0$, we have that $n_b^{E_i} - n_f^{E_i} = 0$, then it must be the case that,

$$tr_{\mathcal{H}}\left((-1)^F e^{-\beta H}\right) = n_b^{\Omega} - n_f^{\Omega} = \dim \mathcal{H}_{0,b} - \dim \mathcal{H}_{0,f}.$$

This quantity is the Witten index, in fact this is the supertrace but we will discuss this when we connect this quantity to the path integral. For now, it is necessary to point out that for a discrete spectrum, ie a compact manifold, the Witten index is entirely topological. In other words, it is insensitive to the continuous changes to the parameters of a theory so long as the asymptotics of the Hamiltonian remain unchanged, here that means the index is independent of β as we saw in previous sections with respect to deformation invariance. Intuitively, deformations may add or subtract fermions/bosons in pairs keeping the difference in $n_b^{E_i} - n_f^{E_i}$ the same until the spectrum develops continuous bands. When this occurs the Witten index will jump and this is a phenomenon called wall-crossing, which studies the discontinuous change of such an invariant in string theory [15]. This phenomenon also becomes important in the IR limit for solutions of BPS black holes where the extremal limit no longer applies at the theory may not be well defined.

In some respects, this is the simplest example of a index we can consider in supersymmetric theories, where $\mathcal{N} = 1$. If we consider the $\frac{1}{2}\partial W^2$ theory, as we did in previous sections for this dimension and number of supercharges, we will find the essentially the same result, the path integral for the Witten index localizes to a neighborhood of constant maps to the critical points of W as a form of the Poincare-Hopf theorem [8] [16]. Therefore, it will be more interesting to examine other theoretical constructions, especially ones that lend themselves to geometrical/topological interpretations. Generically, the reasoning used to derive the Witten index can be done for $\mathcal{N} = n$ with the appropriate added indices and the condition $\left\{Q^i, Q^j\right\} = \left\{\bar{Q}^i, \bar{Q}^j\right\} = 0$. In this way, the Witten index together with the path integral will define index theorems for various dimensions, number of supercharges Q, and theories of interest. The index itself is a global invariant but may be expressed as an integral of local operators. Thus, the important feature of index theorems for physics is that they relate local operators to global features. This is accomplished by relating the ensemble of a density matrix as a partition function to the path integral of a Euclidean action.

For the rest of this section we will consider so called σ-models, which will lead nicely into the discussion of supergravity needed in subsequent chapters. σ-models are described by actions that relate the space of field configurations to a target space, often regarded as a spacetime. Our end goal for this section will be to describe a spinning particle moving on curved manifold using a σ-model and adding the appropriate fermions/SUSY.

1.2.2 Bosonic σ-model, relating the path integral to the partition function

We start by defining the map $\phi : \mathbb{R} \to M$, and with the metric, $g_{\mu\nu}$, on M we have the bosonic σ-model Lagrangian,

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{\phi}^{\mu} \dot{\phi}^{\nu}, \text{ where for } t \in \mathbb{R},\ \dot{\phi}^{\mu} = \frac{d\phi}{dt} \text{ and } g_{\mu\nu} = g_{\mu\nu}\left(x(t)\right).$$

This is clearly a diffeomorphic kinetic term and it describes the motion of a particle on the manifold. The wonderful feature of utilizing this geometric description, relating the worldline coordinates of an object to motion on a spacetime, is that upon proper quantization this appropriately describes the quantum mechanics of a relativistic particle in the 1-d case and quantum field theories of higher dimensional objects for $d > 1$. From here we will canonically quantize the theory and identify the corresponding Hilbert space. The conjugate momentum to ϕ^{μ} is,

$$p_{\mu} = \frac{\partial L}{\partial \dot{\phi}^{\mu}} = g_{\mu\nu} \dot{\phi}^{\nu} \text{ so that } H = \frac{1}{2} g^{\mu\nu} p_{\mu} p_{\nu}.$$

Given this we can construct the Hilbert space as the usual square integrable functions over M as $\mathcal{H} = L^2(\mathbb{R}, \sqrt{g} d^n\phi)$, here g is the determinant of the metric included to both normalize and provide a measure invariant inner product of wavefunctions. To touch base with quantum mechanics we need to push this a little further and identify operators that satisfy the Heisenberg algebra, $[\hat{\phi}^{\mu}, \hat{p_{\nu}}] = i\delta^{\mu}_{\nu}$, that respects non-constant g. The natural choice for $\hat{\phi}^{\mu}$ is just the action of ϕ^{μ} but choosing the naive $\hat{p}_{\nu} = -i\frac{\partial}{\partial \phi^{\mu}}$ does not yield a Hermitian operator under our inner product in general. To account for non-constant g we may simply add appropriate $f_{\mu}(\phi)$ that respects integration by parts as,

$$\hat{p}_{\nu} = -i\frac{\partial}{\partial \phi^{\mu}} + f_{\mu}(\phi) = -i\frac{\partial}{\partial \phi^{\mu}} - \frac{i}{4} g^{-1} \frac{\partial g}{\partial \phi^{\mu}} = -ig^{-\frac{1}{4}} \frac{\partial}{\partial \phi^{\mu}} g^{\frac{1}{4}}.$$

One additional complication is that the classical Hamiltonian we initially wrote down has products of $g_{\mu\nu}\left(x(t)\right)$ and ϕ_{μ}, therefore taking into consideration the commutation relations we have a typical ordering problem. To remedy this, we note that under our inner product of wavefunctions over M, with respect to the Hilbert space, we need only covariantize H so that it is globally defined. The form of $\hat{p}_{\nu}$ leads us to the compelling form of H,

$$H = -\frac{1}{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial\phi^\mu}\left(g^{\mu\nu}\sqrt{g}\frac{\partial}{\partial\phi^\nu}\right).$$

To the keen observer familiar in either harmonic analysis or differential geometry, one may recognize that this Hamiltonian is simply a Laplacian on M, $H = -\frac{\Delta}{2}$. This is a fantastic observation that reaffirms the relationship of σ-models and quantum theory, but also allows us to make a critical connection to statistical mechanics. Recall that the Schrödinger equation is given by,

$$i\partial_t\Psi = H\Psi, \text{ where solutions are time evolutions, } \Psi(t,\phi) = e^{-iHt}\Psi(t|_0,\phi).$$

To progress and make contact with the path integral it is necessary, or really just morally convenient, to take a Wick rotation to Euclidean signature as a form of analytic continuation taking $t = -i\tau$. Now the Schrödinger equation can be re-expressed as,

$$\partial_\tau\Psi = -H\Psi = \frac{1}{2}\Delta\Psi, \text{ and } \Psi(\tau,\phi) = e^{-\tau H}\Psi(\tau|_0,\phi).$$

This is just a diffusion equation, so we may generically use the heat kernel,

$$K_\beta(\phi_1,\phi_2) = \langle\phi_1|\, e^{-\beta H}\, |\phi_2\rangle\,,$$ [9]

to expand the energy spectrum and relate the canonical partition function to the path integral;

$$\begin{aligned}
Z(\beta) \equiv tr_{\mathcal{H}}\left(e^{-\beta H}\right) &= \int_{\mathbb{R}} d\phi\,\langle\phi|\, e^{-\beta H}\,|\phi\rangle \\
&= \int_{\mathbb{R}} d\phi K_\beta(\phi,\phi) \\
&= \int_{\mathbb{R}} d\phi \int_{\gamma_\beta[\phi,\phi]} \mathcal{D}\phi e^{-\int_0^\beta d\tau L} \\
&= \int_{\gamma_{s^1}} \mathcal{D}\phi e^{-\frac{1}{2}\int_0^\beta d\tau g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu} \\
&= \int_{\gamma_{s^1}} \mathcal{D}\phi e^{-S}
\end{aligned}$$

Since Feynman tells us that, $K_\beta(\phi_1,\phi_2) = \int_{\gamma_\beta[\phi(0),\phi(\beta)]}\mathcal{D}\phi e^{-\int_0^\beta d\tau L}$. In other words, we integrate over all periodic paths, γ_{s^1}, on M. Understanding this as an integral over a loop space will yield significant simplifications for regularizing the path integral via methods from [17] and [18], but now we may proceed with this theory and try to add supersymmetry.

[9] Explicitly, $K_\beta(\phi_1,\phi_2) = \frac{1}{\sqrt{2\pi\beta}}exp\left(\frac{-|\phi_1-\phi_2|^2}{2\beta}\right)$

1.2.3 de Rham σ-model

In flat space, without a magnetic background field, we expect a Dirac bilinear term in the Lagrangian like $L_\psi \sim g_{\mu\nu}\bar{\psi}^\mu\dot{\psi}^\nu = \left(\bar{\psi}, \dot{\psi}\right)$, we take care to note ψ is a fermion and not the wavefunction Ψ discussed earlier. On a Riemannian manifold manifold we may treat ψ as an element of the tangent space, $\psi(t) \in T_{\phi(t)}M$, and make the identification that ψ is a basis so that $\bar{\psi}^\mu \leftrightarrow d\phi^\mu$, so really ψ will transform as a vector[10]. We can then construct an appropriate Dirac bilinear term as,

$$\frac{i}{2}\left(\bar{\psi}, \nabla_t\dot{\psi}\right) = \frac{i}{2}g_{\mu\nu}\bar{\psi}^\mu\nabla_t\dot{\psi}^\nu = \frac{i}{2}g_{\mu\nu}\bar{\psi}^\mu\left(\dot{\psi}^\nu + \Gamma^\nu_{\rho\sigma}\dot{\phi}^\rho\psi^\sigma\right),$$
$$\text{where } \Gamma^\nu_{\rho\sigma} = \frac{1}{2}g^{\nu\lambda}\left(\partial_\rho g_{\lambda\sigma} + \partial_\sigma g_{\rho\lambda} - \partial_\lambda g_{\rho\sigma}\right).$$

Invariance with respect to coordinate transformations demands the $g_{\mu\nu}\bar{\psi}^\mu\Gamma^\nu_{\rho\sigma}\dot{\phi}^\rho\psi^\sigma$, but it is this term that will spoil supersymmetry if we bulldoze through by demanding $\delta\phi \propto \psi$ and $\delta\psi \propto \dot{\phi}$. The easiest way to see this is noticing that taking variations of the connection term will produce contributions roughly like $\sim\dot{\phi}\psi\psi\psi$ and although most other terms will cancel out, these will not. The last term we need turns out to be a curvature term, $R_{\mu\nu\rho\sigma}\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma$, and the variations on the curvature tensor itself cancel out due to the Bianchi identity. Originally the full Lagrangian for this model was determined by expanding a superfield into a superspace Lagrangian, but this amounts to the same amount of work and reproduces the same result though manifestly supersymmetric without trial and error. A rather clever way to come to the same conclusion is by use of Mathai-Quillen forms [18][11], which expresses the Euler class as a pullback of a cohomology class on a section of a vector bundle. This will play a role in the realization of equivariant localization for supersymmetric field theories where in contrast for this section we have elected to employ familiar techniques using the Hamiltonian, but without further ado the full Lagrangian for this theory is,

$$L = \frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{i}{2}g_{\mu\nu}\bar{\psi}^\mu\nabla_t\psi^\nu - \frac{1}{4}R_{\mu\nu\rho\sigma}\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma.$$

We can continue the analysis for the quantum theory as we did before but fortunately most of that work is done already and all we need do is analyze the fermionic sector. It is straightforward to read off that the conjugate momentum to ψ^μ is $p_{\psi^\mu} = -\frac{i}{2}\bar{\psi}^\nu g_{\nu\mu}$, therefore the anti-commutation relations are

[10] Formally they actually can define sections, here they are sections of TM restricted to a loop on M with periodic boundary conditions on $\phi(t)$. $T_\phi(LM) \cong \Gamma^\infty(\phi^*(TM) \otimes \mathbb{C})$

[11] A few equations will be given in the next chapter though without formal introduction.

$$\{\psi^\mu, \psi^\nu\} = \{\bar{\psi}^\mu, \bar{\psi}^\nu\} = 0$$
$$\{\psi^\mu, \bar{\psi}^\nu\} = g^{\mu\nu}.$$

Now we may build the Hilbert space by acting with $\bar{\psi}^\mu$'s on the ground state until all the fermionic states are excited, whereas ψ^μ annihilate the vacuum for all μ. We had previously made the identification that $\bar{\psi}^\mu$ was a differential form on the target space M, now looking at p_{ψ^μ} and their action on the vacuum this identification is nearly manifest. The bosonic part of the Hilbert space is still the space of square integrable functions on M, and multiplying with $\phi^\mu(t)$'s on the fermionic excitations produces a poly-form as an element of the Hilbert space, thus

$$\text{for } \Psi \in \mathcal{H},\ \Psi(x, \bar{\psi}) = f(\phi) + \alpha_\mu(\phi)\bar{\psi}^\mu + \beta_{\mu\nu}(\psi)\bar{\psi}^\mu\bar{\psi}^\nu + \ldots$$
$$\Rightarrow\ \mathcal{H} = \Omega^\bullet(M).$$

Remarkably, the theory is actually de Rham. The Hilbert space is the space of all possible square integrable[12] differential forms where also now we may conclude $\{Q, \bar{Q}\} = 2H = -\Delta$, the Laplace-Beltrami operator acting on all forms, not just the functions we found in the bosonic case.

$$H = \frac{1}{2}\left(Q\bar{Q} + \bar{Q}Q\right) = -\frac{1}{2}\left(dd^\dagger + d^\dagger d\right)$$
$$Q = d,\ \bar{Q} = d^\dagger.$$

Given this fact we may conclude by Hodge's theorem, $Harm^p(M) \cong H^p_{dR}(M)$, that the ground states of the theory is actually $H^\bullet_{dR}(M)$, ie the theory is a de Rham complex. To find representatives of a cohomology in the complex we only need find the zero-eigenstates of Δ, therefore if we find $H|\Omega\rangle \Rightarrow Q|\Omega\rangle = 0$, which is the same analysis we considered at the beginning of this section. The fermion operator on a state now counts the degree of the form and the parity operator $(-1)^F$ determines if the state is bosonic(even)/fermionic(odd). With this information we are now equipped to compute the Witten index directly in this topological context recalling that the dimension of $H^k(M)$ is the k'th Betti number $b_k(M)$ and the Euler characteristic is $\chi(M) = \sum_{k=0}^n (-1)^k b_k$. The $H|\Omega\rangle = 0$ states with degree k are elements of the cohomology with degree k and we may choose the basis of this cohomology to be the zero-eigenstates of H. Then it must be the case that the dimension of the Hilbert subspace with degree k form is the Betti number and we know that the Witten index does this counting with alternating sign due to parity,

[12] As inherited from the bosonic sector

$$tr\left((-1)^F e^{-\beta H}\right) = b_0 - b_1 + b_2 - b_3 + ... = \sum_{k=0}^{n}(-1)^k b_k = \chi(M)$$

Assuming that the path integral returns this value as well, we have found that the Witten index for this theory is the Gauss-Bonnet theorem with the caveat that the dimension of M is even, otherwise the integral vanishes trivially. This can be done using a rescaling and deformation invariance leaving only the curvature term under a Grassmannian integral so that one may integrate out the non-constant modes and fermionic zero modes to yield,

$$tr\left((-1)^F e^{-\beta H}\right) = \chi(M) = \frac{1}{(2\pi)^n}\int_M Pf(R).$$

Where R is the curvature 2-form to the tangent bundle on M [6]. The details of the calculation on the path integral side are similar to the theory we will deal with next but first we need to address the periodic boundary conditions required to define the trace.

If we consider the trace of the operator over a Hilbert space containing only fermions, eg the partition function, there will be a minus sign in the adjoint as $\left\langle -\bar{\psi}\right|$. So if we define the path integral from the heat kernel we would have to consider boundary conditions that are antiperiodic. If we construct an action of our de Rham SQM and wish to express the partition function as a path integral, the boundary conditions for the bosonic and fermionic parts need to be on equal footing. Fortunately, the parity operator on the adjoint $\left\langle -\bar{\psi}\right|(-1)^F = \left\langle \bar{\psi}\right|$, so the supertrace of the Hilbert space invokes periodic boundary conditions(PBC) on the integral,

$$Str_{\mathcal{H}}\left(e^{-\beta H}\right) \equiv tr_{\mathcal{H}}\left((-1)^F e^{-\beta H}\right) = \int_{PBC}\mathcal{D}\phi\mathcal{D}\phi\mathcal{D}\bar{\psi}e^{-S[\phi,\psi,\bar{\psi}]}.$$

This more or less reaffirms the construction we considered at the beginning of this section and makes it clear how to interpret the canonical partition functions, at least for the Euclidean action, and the connection/necessity for the Witten index.

1.2.4 $\mathcal{N} = \frac{1}{2}$ σ-model and the twisted Dirac index

The last goal of this section is to evaluate the path integral for a theory that evaluates to an important result in mathematics called the Atiyah-Singer index formula. It turns out that employing a modification to the supersymmetric σ-model we constructed we can recover a different finite integral over a topological index called the Dirac $\hat{A}$-genus, "A-roof genus," which is the index for the Dirac operator. The change necessary is to reduce $\mathcal{N} = 1$ to $\mathcal{N} = \frac{1}{2}$ by setting $\bar{\psi} = \psi$, ie we now use the fermion to describe the spin of a moving particle on a curved manifold. This is not quite all that we should do either because it would be far more enlightening to also include a background gauge field to impose a topological twist a la Witten, this way we yield a slightly more general form of the Atiyah-Singer index formula of the twisted Dirac operator. The simplest choice without causing too much headache is to consider a $U(1)$ gauge field, in other words a $U(1)$ principal bundle over M with connection A and field strength/curvature $F = dA$. Coupling the gauge field using the typical physics principle of writing all legal terms, here we gauge with $U(1)$ in the usual way and drop the Riemann curvature term due to the Bianchi identity, then the action is,

$$S[\phi, \psi] = \int dt \frac{1}{2} \left(g_{\mu\nu} \dot{\phi}^\mu \dot{\phi}^\nu + g_{\mu\nu} \psi^\mu \nabla_t \psi^\nu + 2\dot{\phi}^\mu A_\mu - \psi^\mu F_{\mu\nu} \psi^\nu \right).$$

In the de Rham SQM we did not formally write down the supersymmetric transformations[13] but it will prove useful to note what they are for $\mathcal{N} = \frac{1}{2}$ in order to follow the procedure we utilized in the 0-d analog. The obvious guess in the spirit of supersymmetry is,

$$\delta\phi^\mu = \psi^\mu, \text{ and } \delta\psi^\mu = \dot{\phi}^\mu \text{ where as expected for a moving particle } \delta^2 = \partial_t.$$

We need to take into account the periodic boundary conditions we put on the path integral to regulate the paths so that ϕ is in the loop space and path integral is integrated over the corresponding tangent bundle. The supercharge that accomplishes this is actually a equivariant exterior derivative, again identifying $\psi \leftrightarrow d\phi$, but less formally we can easily extract the transformations by considering functional differentiation restricted to loops in the loop space LM,

$$Q_{\dot{\phi}} = \int_{s^1} dt \left(\psi^\mu \frac{\delta}{\delta\phi^\mu} + \dot{\phi}^\mu \frac{\delta}{\delta\psi^\mu} \right) = d + \iota_{\dot{\phi}} \text{ and } Q_{\dot{\phi}}^2 = \int_{s^1} dt \frac{d}{dt}.$$

The next step is to find that the action is $Q_{\dot{\phi}}$-exact,

$$\begin{aligned}
Q_{\dot{\phi}} \hat{V}[\phi, \psi] &= Q_{\dot{\phi}} \int_{s^1} dt \left(\frac{1}{2} g_{\mu\nu} \dot{\phi}^\nu + A_\mu \right) \psi^\mu \\
&= \int_{s^1} dt' \left(\psi^\mu \frac{\delta}{\delta\phi^\mu} + \dot{\phi}^\mu \frac{\delta}{\delta\psi^\mu} \right) \int_{\gamma_{s^1}} dt \left(\frac{1}{2} g_{\mu\nu} \dot{\phi}^\nu + A_\mu \right) \psi^\mu \\
&= \int_{s^1} dt' \int_{\gamma_{s^1}} dt \left(\psi^\mu \frac{1}{2} \frac{\delta g_{\mu\nu} \dot{\phi}^\nu \psi^\mu}{\delta\phi^\mu} + \dot{\phi}^\mu \frac{1}{2} \frac{\delta g_{\mu\nu} \dot{\phi}^\nu \psi^\mu}{\delta\psi^\mu} + \psi^\mu \frac{\delta A_\mu \psi^\mu}{\delta\phi^\mu} + \dot{\phi}^\mu \frac{\delta A_\mu \psi^\mu}{\delta\psi^\mu} \right) \\
&= \int_{s^1} dt \left(\frac{1}{2} g_{\mu\nu} \psi^\mu \nabla_t \psi^\nu + \frac{1}{2} g_{\mu\nu} \dot{\phi}^\mu \dot{\phi}^\nu - \frac{1}{2} \psi^\mu F_{\mu\nu} \psi^\nu + \dot{\phi}^\mu A_\mu \right).
\end{aligned}$$

Using $\delta^2 = \partial_t$, $\delta g \sim \Gamma$ or alternatively $\partial_t(g_{\mu\nu}\psi^\mu) \sim \nabla_t \psi^\mu$, and $F = dA$.

[13] Those transformations will only differ by terms involving the connection.

Furthermore, S is $Q_{\dot{\phi}}$-closed since $\hat{V}[\phi,\psi]$ is invariant under $Q^2_{\dot{\phi}}$ and thus we may introduce an arbitrary parameter λ since $Q_{\dot{\phi}}\left|\Omega\right\rangle = 0$. Taking $\lambda \to \infty$, the fermionic integral vanishes under the path integral and the bosonic integral under the path integral localizes to constant paths where $\dot{\phi}^{\mu}(t) = 0$. These are zero-modes of $\delta^2 = \partial_t$ which generate the finite locus the integral localizes to. Crucial here is the observation that we can decompose the field configuration and measure into these zero-modes, constant modes with respect to time, and fluctuations which in a limit and expansion to local coordinates will cancel out.

$$\text{Take, } \phi(t) = \phi_0 + \tilde{\phi}(t) \text{ and } \psi(t) = \psi_0 + \tilde{\psi}(t)$$
$$\Rightarrow \mathcal{D}\phi\mathcal{D}\psi = d^{2n}\phi_0 d^{2n}\psi_0 \mathcal{D}\tilde{\phi}\mathcal{D}\tilde{\psi}$$
$$\text{rescaling the non-constant modes as } \left(\tilde{\phi},\tilde{\psi}\right) \to \left(\frac{\tilde{\phi}}{\sqrt{\lambda}},\frac{\tilde{\psi}}{\sqrt{\lambda}}\right).$$

The fermionic and bosonic measure transform with opposite weight and are therefore invariant, so expanding $\lambda S[\phi,\psi]$ to leading order in normal coordinates we have

$$\lambda S[\phi,\psi]|_{\lambda\to\infty} = \int_{s^1} dt \left(\frac{1}{2}g_{\mu\nu}(\phi_0)\dot{\tilde{\phi}}^{\mu}\dot{\tilde{\phi}}^{\nu} + \frac{1}{2}\eta_{ij}\tilde{\psi}^i\partial_t\tilde{\psi}i^j - \frac{1}{2}\psi_0^{\mu}F_{\mu\nu}(\phi_0)\psi_0^{\nu} + \frac{1}{2}R_{ij\mu\nu}\psi_0^i\psi_0^j\dot{\tilde{\phi}}^{\nu}\dot{\tilde{\phi}}^{\nu} + \mathcal{O}\left(\lambda^{-\frac{1}{2}}\right)\right)$$

Using generalized coordinates/vielbeins, it is convenient to recognize the curvature two form $\mathcal{R}_{\mu\nu} = \frac{1}{2}R_{ij\mu\nu}\psi_0^i\psi_0^j$ so that the last term is $\mathcal{R}_{\mu\nu}\dot{\tilde{\phi}}^{\nu}\dot{\tilde{\phi}}^{\nu}$. The fluctuation terms are quadratic and so the action can be decomposed into free actions over the fluctuations and the path integral over these terms becomes a Gaussian integral for each. The bosonic and fermionic fluctuations will be integrated out under a path integral as,

$$\int \mathcal{D}\tilde{\phi}e^{-\lambda S[\tilde{\phi}]} = \frac{1}{\sqrt{det\left(\delta^{\nu}_{\mu}\partial_t^2 + \mathcal{R}^{\nu}_{\mu}\partial_t\right)}} \text{ and, } \int \mathcal{D}\tilde{\psi}e^{-\lambda S[\tilde{\psi}]} = Pf(\partial_t)$$

$$\begin{aligned}
\int d\psi_1 d\psi_1^* ... d\psi_n d\psi_n^* e^{-\frac{1}{2}\psi^{*i}B_{ij}\psi^j} &= \frac{(-1)^n}{2^n n!}\int d\psi_1 d\psi_1^* ... d\psi_n d\psi_n^* \left(\psi^{*i_1}B_{i_1 j_1}\psi^{j_1}\right) ... \left(\psi^{*i_n}B_{i_n j_n}\psi^{j_n}\right) \\
&= \frac{1}{2^n n!}\int d\psi_1 ... d\psi_n d\psi_1^* ... d\psi_n^* \epsilon_{i_1 j_1 ... i_n j_n}\psi^{*i_1}B_{i_1 j_1}\psi^{j_1} ... \\
&= \frac{1}{2^n n!}\epsilon_{i_1 j_1 ... i_n j_n}B_{i_1 j_1} ... B_{i_n j_n} \\
&= Pf(B) = \pm\sqrt{det(B)} \text{ inverse to the result in even coordinates.}
\end{aligned}$$

Putting these together,

$$\int \mathcal{D}\tilde{\phi}\mathcal{D}\tilde{\psi}e^{-\lambda S[\tilde{\phi},\tilde{\psi}]} = \frac{Pf(\partial_t)}{\sqrt{det\left(\delta^\nu_\mu \partial_t^2 + \mathcal{R}^\nu_\mu \partial_t\right)}} = \frac{1}{\sqrt{det\left(\delta^\nu_\mu \partial_t + \mathcal{R}^\nu_\mu\right)}}.$$

Therefore the path integral becomes,

$$\int d^{2n}\phi_0 d^{2n}\psi_0 \mathcal{D}\tilde{\phi}\mathcal{D}\tilde{\psi}e^{-\lambda S[\phi_0,\psi_0,\tilde{\phi},\tilde{\psi}]} = \int d^{2n}\phi_0 d^{2n}\psi_0 \frac{e^{\lambda \frac{1}{2}\psi_0^\mu F_{\mu\nu}(\phi_0)\psi_0^\nu}}{\sqrt{det\left(\delta^\nu_\mu \partial_t + \mathcal{R}^\nu_\mu\right)}}.$$

To make headway on regularizing and evaluating the determinant term recall that $\phi(t)$ is a loop and the $\tilde{\phi(t)}$ are fluctuation of these loops so we may expand the eigenmodes of ∂_t on $\tilde{\phi(t)}$ as non-zero integers to the unit circle $e^{2\pi ikt}$, ie values $2\pi ik$ where $k \neq 0$. For the curvature term we may notice that since it is antisymmetric we may decompose it into 2x2 skew-diagonal antisymmetric blocks with eigenvalues $\pm\omega_i$, for n blocks.

$$\begin{aligned}
det\left(\partial_t + \mathcal{R}^{(i)}\right) &= \prod_{k\neq 0}^{\infty} (2\pi ik + \omega_i)(2\pi ik - \omega_i) \\
&= \prod_{k\neq 0}^{\infty} (2\pi i)^2 \left(k + \frac{\omega_i}{2\pi i}\right)\left(k - \frac{\omega_i}{2\pi i}\right) \\
&= \prod_{k=1}^{\infty} (2\pi i)^4 k^4 \left(1 - \frac{\omega_i^2}{(2\pi ik)^2}\right) \\
&= \prod_{k=1}^{\infty} (2\pi ik)^4 \frac{\sinh\frac{\omega_i}{2}}{\omega_i/2}, \text{ where by Euler, } \frac{\sin(\theta)}{\theta} = \prod_{k=1}^{\infty}\left(1 - \frac{\theta^2}{\pi^2 k^2}\right) \\
&= \prod_{k=1}^{\infty} (-i)\frac{\sinh\frac{\omega_i}{2}}{\omega_i/2} \\
&= (-i)^n det\left(\frac{\sinh\frac{1}{2}\mathcal{R}}{\frac{1}{2}\mathcal{R}}\right)
\end{aligned}$$

Above we used analytic continuation of the Riemann zeta function to cancel out the $2\pi k$ factors by recognizing,

$$\prod_{k=1}^{\infty} k^4 = e^{-4\zeta'(0)} = e^{2log(2\pi)} = 4\pi^2 \text{ and } \prod_{k=1}^{\infty} (2\pi)^4 = \left(4\pi^2\right)^{2\zeta(0)} = \left(4\pi^2\right)^{-1}$$

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

The partition function becomes,

$$\begin{aligned}
Z &= \int d\phi_0 d\psi_0 e^{-\frac{i}{2}\psi_0^\mu F_{\mu\nu}(\phi_0)\psi_0^\nu} \sqrt{det\left(\frac{\sinh\frac{1}{2}\mathcal{R}}{\frac{1}{2}\mathcal{R}}\right)} \\
&= \int_M ch(F) \wedge \hat{A}(R).
\end{aligned}$$

Here $ch(F)$ is the Chern character of the $U(1)$ gauge field and the determinant term is the $\hat{A}$-genus. This is an integral over characteristic classes of the principal gauge group over M and is known as the Atiyah-Singer index theorem for a twisted spin complex. Below is a short list of resulting theorems via localization of a supersymmetric σ-model over a compact manifold [16][13].

Theory	Localization/Index
$\mathcal{N} = \frac{1}{2}$ Spin complex	Dirac Atiyah-Singer
$\mathcal{N} = 1$ de Rham complex	Gauss-Bonnet
$\mathcal{N} = 1$ Equivariant Morse w/ potential	Poincare-Hopf
$\mathcal{N} = 1$ Signature complex	Hirzebruch
$\mathcal{N} = 1$ Dolbeault complex	Riemann-Roch

In this example we have directly applied all the lessons learned throughout the chapter to find a very important result in mathematics. We have also related this result to a physical interpretation in the Witten index which formally plays a role is supersymmetry breaking and yields important information in classifying theories. In the next chapter we will formalize these concepts into relevant mathematics then discuss how to port them over to the suspersymmetric quantum field theory case.

Chapter 2

Nuts and Bolts of Localization, Some Mathematical Rigor

In this chapter we will explore the mathematics necessary to perform localization. Moreover we will uncover the algorithm and features of localization that we stumbled upon in our exploration in chapter 1. For the interested reader to explore more on these subjects, [19] and [16] are great resources on equivariant cohomology and [20] gives a nice introduction to localization and SUSY on curved backgrounds.

The first goal is to generalize the traditional notion of cohomology to take into account a group action. We generally would like to compute integrals with G as a symmetry and make an identification of points related by G, eg gauge transformations. The natural thing to do in this setting is to quotient the manifold by G. If G acts freely, ie has no fixed points, then de Rham cohomology is sufficient since M/G, space of orbits, is a smooth closed manifold[1], hence $H^\bullet(M/G)$ is well defined. This is too restrictive for many symmetries since it may be the case that M/G is not a smooth closed manifold because the G action generates fixed points[2]. A simple example of this is the circle action, $U(1)$, on the sphere S^2,

$$M = S^2 \Rightarrow ds^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

To generate $U(1)$ on the sphere we can consider circles, generated by the azimuthal angle ϕ, about the z-axis and identify these circles with polar angle θ and height as a projection onto z by $f_{U(1)} = f(\theta,\phi) = \cos\theta$. The integral one could consider of this action is a path integral analog as

[1]Compact and without boundary.
[2]Usually the resulting space is an orbifold.

the oscillatory integral which may be computed exactly,

$$
\begin{aligned}
Z_f(t) &= \int_{S^2} d^2x\sqrt{g}e^{itf(x)} \\
&= \int_0^{2\pi} d\phi \int_0^{\pi} sin(\theta)e^{it\cos\theta} \\
&= 2\pi \int_{-1}^{1} d(\cos\theta)e^{it\cos\theta} \\
&= \frac{2\pi i}{t}\left(-e^{it} + e^{-it}\right) \\
&= 4\pi\frac{\sin t}{t}.
\end{aligned}
$$

For a general oscillatory function in an introductory quantum mechanics course we would usually try to Taylor expand the phase about some isolated stationary points and find that by typical Gaussian integration the leading terms are a semiclassical approximation weighted by 1-loop determinants[3],

$$
Z_f(t) = \int_M d^n x\sqrt{g}e^{itf(x)} \sim \sum_{x_k:df(x_k)=0} \frac{e^{itf(x_k)}}{\sqrt{g^{-1}H_f(x_k)}} + \mathcal{O}(t^{-\frac{n}{2}-1}).
$$

Normally this is a necessary approximation since as $t \to \infty$ then usually $tf(x)$ is highly oscillatory, eg if $t \sim \frac{1}{\hbar}$. In the circle action example we found that the integral was computed exactly, in fact it was computed exactly as a sum of contributions from the two stationary points at the poles. This is a realization of a generic feature of localization in that it may be interpreted as a situation in which the stationary phase approximation becomes exact and is an example of Duistermaat-Heckman localization [1]. This feature arose from a bug, the quotient $S^2/U(1)$ is not an appropriate manifold but an interval, yet the fixed points due to this action gave us an exact result. The salient point here is that an interval is contractible to a point and we know that the cohomology of a point is the empty set, this construction is trivial and comparing it to the exact result from integration tells us our normal notion of cohomology is not enough.

2.1 Equivariant Cohomology

Let us consider an even dimensional Riemannian manifold without boundary[4] and a compact Lie group with action on M,

$$
G \times M \to M
$$

[3]Here $g^{-1}H_f(x_k)$ is the Morse index of f at x_k, and H is the Hessian.

[4]Smooth and closed might be sufficient in some cases but for SUSY localization these are the properties we want.

The action is a homomorphism that takes

$$\psi : G \to Diff(M),\ Diff(M) \equiv \text{diffeomorphisms } M \to M.$$

Formally we call this action free if for $(g, p) \to g \cdot p$, $g \in G$ and $p \in M$, then $g \cdot p = p \Rightarrow g = e$ $\forall p \in M$, in other words the stabilizer of G is the identity. If the action is free and G is compact we may motivate the definition,

$$H^\bullet_G \equiv H^\bullet(M/G) = \bigoplus_{k=0}^{n} H^k(M/G).$$

This definition is not quite right if we want to consider actions that are not free. To do this we can try and find a manifold that is homotopically equivalent to M where G is free on that manifold, then they will share the same cohomology. The universal bundle associated to G, denoted EG[5], will accomplish this in general. The universal bundle is the correct choice since G acts on EG without fixed points and EG is contractible. We may define the G-equivariant cohomology using the topological definition of cohomology as,

$$H^\bullet_G \equiv H^\bullet(M \times_G EG) = H^\bullet((M \times EG)/G).$$

The details of this formal definition are not important for our purposes since it does not lend itself to informing us how to perform calculations. Fortunately, the definition above does not depend on EG, so we have some freedom in constructing this cohomology algebraically, usually this choice is called a "model" or differential graded algebra[6]. There are three models typically treated in the literature; the Weil model, the Cartan model, and the BRST model[7]. We will introduce the Cartan model here which has a really straight forward relationship to the quantities and concepts utilized in SUSY localization. However, the starting point for all three models is understanding how to incorporate the group action into the space of differential forms that normally defines the cohomology.

2.1.1 The Cartan model

A common theme when dealing with spaces, especially topological spaces and manifolds, is to qualify how a structure works locally before lifting it to the whole space. This is also the right place for developing the Cartan model since, technical definitions aside, a fiber bundle is a manifold

[5] Formally, EG is a bundle with group G such that every local action G on M pullsback to a classifying space BG, ie a space that identifies EG/G as a way to topologically relate M's. For example the classifying space to S^1 is $\mathbb{C}P^\infty$ and the universal bundle is $ES^1 = S^\infty$, ie Hopf fibrations.

[6] A graded algebra that is also a differential complex, eg the de Rham complex.

[7] The Weil and Cartan models are isomorphic while the BRST model connects the two [21].

where locally we have a product of two topological spaces. The bundle itself can be thought of as attaching fibers at every point on the manifold, which is conceptually what we would like to do with a group action. So, at every point on the manifold we have our usual structure in the local coordinate patch and a tangent space. Therefore, locally understanding how G acts on the manifold amounts to understanding the appropriate maps of the infinitesimal group action given by the Lie algebra $\mathfrak{g}$, local elements on the manifold associated to G, and the differential forms on M, local elements of the manifold M.

The naive thing to do would be to tensor two respective dual spaces since they both have maps that take elements to the real numbers, for a real manifold, and thus construct differential forms on M valued in $\mathfrak{g}^*$. Imagine that we have the fiber construction and the principal G-bundle, $P \to M$, then there is a $\mathfrak{g}$-valued 1-form on P that is a connection, ω, to the tangent space T_pP mapping to the Lie algebra $\mathfrak{g}$. We can write a basis for elements in the Lie algebra as, $\chi = \chi^a T_a$, and χ^a are elements of the dual space $\mathfrak{g}^*$. With this we can define linear maps to the space of forms on the principal bundle P, $\Omega(P)$, but since G is a group fibered over M what we are really interested in are all possible combinations of forms[8]. This leads us to consider the symmetric algebra of $\mathfrak{g}^*$, whose elements are polynomials in χ^a, to "act" on the space of forms on M. In other words we take differential forms on M valued in $S(\mathfrak{g}^*)$, poly-forms $\alpha(\chi) \in \Omega^\bullet(M) \otimes S(\mathfrak{g}^*)$ under the normal wedge product of forms.

As in the construction of the de Rham cohomology, what we are really looking to do is construct a differential graded algebra. Therefore, we care about poly-forms which are G-invariant. The group action on the manifold M induces a vector field on M for each basis element in the group, and we can generate a flow in the usual way by the exponential map. Further, there is an induced action on the α that is the Lie derivative along the vector as well as an induced adjoint action on the $S(\mathfrak{g}^*)$ part by,

$$\mathcal{L}_{v_a}\alpha(\chi) + f^b_{ac}\chi^c \frac{\partial \alpha(\chi)}{\chi^b} = 0.$$

Here the equality is the condition that the form be G-invariant. The space of the G-invariant forms is denoted by,

$$\alpha(\chi) \in (\Omega^\bullet(M) \otimes S(\mathfrak{g}^*))^G$$

Finally the last ingredient is to assign a grading using the equivariant exterior derivative, or just equivariant differential for short, demanding equivariantly closed forms such that $d_{v_a}\alpha = 0$

[8]This point is a little tricky, formally we can define a homomorphism $S(\mathfrak{g}^*) \to \Omega(P)$ but really end up with is a composition of this homomorphism with the homomorphism of the exterior algebra to the space of forms on the principal bundle, this defines the Weil algebra which models total space EG. It is not trivial, but the construction desired here for the Cartan model amounts to setting $\omega = 0$. That said, it might be intuitive from a group perspective that the symmetric algebra is enough, Cartan likely did.

and equivariantly exact forms such that $\alpha = d_{v_a}\beta$,[9]

$$\begin{aligned}
d_{v_a} &: \Omega^\bullet(M) \otimes S(\mathfrak{g}^*) \to \Omega^\bullet(M) \otimes S(\mathfrak{g}^*) \\
d_{v_a} &= d + \chi^a \iota_{v_a} \\
\Rightarrow (d_{v_a}\alpha)_k &= d\alpha_{k-1} + \chi^a \iota_{v_a}\alpha_{k+1} \\
\alpha(\chi) &= \sum_{k=0}^{n} \alpha_k(\chi), \;\; \alpha_k \in \Omega^k(M).
\end{aligned}$$

We can see that this is necessary construction since the normal de Rham differential is of degree $+1$, the contraction is of degree -1, and so we need an overall degree of $+1$ for the equivariant differential from χ^a. Moreover, we must have for G-invariant α

$$\begin{aligned}
d_{v_a}^2 \alpha &= \chi^a (d\iota_{v_a} + \iota_{v_a} d)\alpha \\
&= \chi^a \mathcal{L}_{v_a}\alpha(\chi) \\
&= -\chi^a f_{ac}^b \chi^c \frac{\partial \alpha(\chi)}{\chi^b} \\
&= 0.
\end{aligned}$$

Using the Cartan homotopy formula for the second line and antisymmetry of the structure constants for the last line. This also implies that v_a is Killing[10]. Finally we may define the k-th equivariant cohomology,

$$\begin{aligned}
H_G^k(M) &= \frac{\ker d_G|_{\Omega_G^k(M)}}{\operatorname{Im} d_G|_{\Omega_G^{k-1}(M)}} \\
\Rightarrow H_G^\bullet(M) &= H\left(\left(\Omega^\bullet(M) \otimes S(\mathfrak{g}^*)\right)^G, d_G \right).
\end{aligned}$$

Finally, we can define integration over these forms in a close analogy to the usual de Rham cohomology. Observe that we have the following recursive relationship of forms,

$$(d_{v_a}\alpha)_k = d\alpha_{k-1} + \chi^a \iota_{v_a}\alpha_{k+1}$$

and if $d_{v_a}\alpha = 0$ then,

$$d\alpha_k = 0, \quad \iota_{v_a}\alpha_k = d\alpha_{k-2}, \quad \iota_{v_a}\alpha_{k-2} = d\alpha_{k-4}, \; ...$$

[9] There is a convention sign choice in the definition of d_{v_a}, and often physicists will choose degree of the coefficient to be 1 which is acceptable when $G = U(1)$, though will not always play a large role because it can be recovered when adding background fields or redefinitions.

[10] Specifically if M is compact, $\mathcal{L}_v g = 0$

This formally grades the even and odd degrees into two separate recursive relations. This also implies that for a highest degree polyform that is equivariantly closed, α_k and α_{k-1} are closed in the usual de Rham construction. Moreover, if another form, β, is equivariantly exact then β_k and β_{k-1} are also exact. Thus by Stokes theorem,

$$\int_M \alpha = \int_M \alpha_{2l}, \ dim(M) = 2l$$
$$\int_M d_{v_a}\beta = \int_M d\beta_{2l-1} = 0$$
$$\Rightarrow \int_M (\alpha + d_{v_a}\beta) = \int_M \alpha.$$

Now that we have localization we can establish a dictionary between equivariant cohomology and supersymmetry and subsequently reveal the localization principles that appeared in chapter 1.

2.1.2 Cartan model for super manifolds and SUSY

The notion of equivariant cohomology in the Cartan model can be generalized/mapped to supersymmetric spaces for path integrals. There is a nice set of analogies that can be made but for the moment we will focus on relating the fields and equivariant differential to their supersymmetric quantities, later we will see how quantities in finite-dimensional localization can be related to the infinite-dimensional setting.

Let us start of by defining a supermanifold as a graded manifold. Convenient for the context of this paper is to grade the tangent bundle of M that has a group action G. We do this by assigning the degree of the tangent vectors as 1 so that we may regard the coordinates on M, ϕ^μ, as even and the corresponding 1-form coordinates, $dx^\mu \leftrightarrow \psi^\mu$, so that they are odd. We can then consider their transformations and integration of measure,

$$\tilde{\phi^\mu} = f^\mu(\phi) \to \tilde{\psi^\mu} = \frac{\partial f^\mu(\phi)}{\partial \phi^\nu}\psi^\nu.$$
$$\int d^n\phi d^n\psi = \int d^n\tilde{\phi} d^n\tilde{\psi}.$$

Carrying along naturally we may identify the equivariant differential forms,

$$\alpha(\phi, \psi) = \sum_k \frac{1}{k!}\alpha_{\mu_1\mu_2\ldots\mu_k}\psi^{\mu_1}\ldots\psi^{\mu_k}$$

We see the notion of "poly-forms" was the correct one as these are C^∞ on the graded tangent bundle and is isomorphic to the ring of differential forms in the cohomological sense. Furthermore, we can use the physics notion of a set of variations on fields to define a supersymmetry,

$$\delta\phi^\mu = \psi^\mu, \quad \text{and} \quad \delta\psi^\mu = \chi^a v_a^\mu$$
$$\delta\chi^a = 0$$
$$\delta \leftrightarrow d_{v_a}.$$

Finally to connect with $\alpha(\chi)$ as defined in the Cartan model, we relate the polyforms on the graded tangent bundle to $\alpha(\chi)$ in coordinates in an integral as,

$$Z[0] = \int_M \alpha(\chi) = \int_{T_{[1]}M} d^n\phi d^n\psi \, \alpha(\phi,\psi,\chi).$$

This integral, up to generalizations, will allow us to make apparent the localization arguments, specifically that G-equivariant integrals localize. This by itself will validate the mathematical features of the examples explored, especially the index theorems discussed in chapter 1, but also will prove as a good motivation to try and extend the tool to curved supersymmetry and non-compact spaces.[11]

2.2 Localization

In this section we will consider a $U(1)$ action, but will be able to generalize later. We will follow the same procedure outlined in chapter 1 and summarize the steps. We start with a deformed version of $Z[0]$ being motivated to make analogs to the path integral case. Take,

$$Z[t] = \int_{T_{[1]}M} d^n\phi d^n\psi \, \alpha(\phi,\psi,\chi) e^{-t\delta V(\chi)}$$

where V is nilpotent under δ, $\delta^2 V = 0$. This is an appropriate deformation since $\alpha(\phi,\psi,\chi)e^{-t\delta V(\chi)}$ is in the same equivariant cohomology class as α by definition of V. Observe that by taking a derivative with respect to t we discover $Z[t]$ is independent of t,

$$\begin{aligned}
\frac{d}{dt} Z[t] &= -\int_{T_{[1]}M} d^n\phi d^n\psi \, \delta V(\chi)\alpha(\phi,\psi,\chi)e^{-t\delta V(\chi)} \\
&= -\int_{T_{[1]}M} d^n\phi d^n\psi \, \delta\left(V(\chi)\alpha(\phi,\psi,\chi)e^{-t\delta V(\chi)}\right) \\
&= 0
\end{aligned}$$

[11]The infinite dimensional nature of the path integral already proves tenuos, extending this into the regime of path integrals over non-compact spaces proves challenging rigorously, but some insights have been made by physicists in recent years.

by Stokes theorem and $\delta^2 V = \delta\alpha = 0$. Now we take the $t \to \infty$ limit of $Z[t]$ to find that the integral vanishes unless $\delta V = 0$. Let us characterize potential space of saddle points by choosing an appropriate V,

$$
\begin{aligned}
V &= g_{\mu\nu}\psi^\mu \delta\psi^\nu = \chi g_{\mu\nu}\psi^\mu v^\nu \\
\delta^2 V &= \chi \mathcal{L}_v \chi g_{\mu\nu}\psi^\mu v^\nu = 0.
\end{aligned}
$$

Now we want to solve learn about $\delta V = 0$ under the integral,

$$
\begin{aligned}
\delta V &= \chi\left(\left(g_{\mu\nu}v^\mu\right)_{,\rho}\psi^\rho\psi^\nu + \chi g_{\mu\nu}v^\mu v^\nu\right) \\
\Rightarrow \lim_{t\to\infty} Z[t] &= \int_{T_{[1]}M} d^n\phi d^n\psi\, \alpha(\phi,\psi,\chi) e^{-t\chi\left(g_{\mu\nu}v^\mu\right)_{,\rho}\psi^\rho\psi^\nu - t\chi^2 g_{\mu\nu}v^\mu v^\nu}
\end{aligned}
$$

Thus the integral localizes to a fixed points of the group action such that $|v|^2 = 0 \to v = 0$, these will be the points where the integral does not vanish in the limit and will pick up contributions. Let us now consider a local patch about a fixed point with canonically flat metric. Since we are considering an even dimensional manifold, then we may consider the flat space as a product of $\mathbb{R}^2$ and $U(1)$ rotates each of these planes by a local vector field. Therefore we will have a sum of rotating vectors fields characterized by some paramater such that,

$$
v(\phi_p) = \sum_i^n \omega_i^p \left(x_i \partial_{y_i} - y_i \partial_{x_i}\right).
$$

Plugging this back into $V = \frac{\chi}{2}\sum \omega_i^p \left(x_i\psi_{y_i} - y_i\psi_{x_i}\right)$ and rescaling the same way we did for the σ-model, $\frac{1}{\sqrt{t}}$, the integral becomes around this specific fixed point is

$$
\begin{aligned}
\lim_{t\to\infty} Z[t] &= \int_{T_{[1]}M|_p} d^n\tilde{\phi} d^n\tilde{\psi}\ \alpha e^{-t\delta V} \\
&= \int_{T_{[1]}M|_p} d^n\tilde{\phi} d^n\tilde{\psi}\ \alpha_0|_p e^{-\chi\sum\left(\omega_i^p\psi_{x_i}\psi_{y_i}\right) - \frac{\chi^2}{2}\sum\left(\omega_i^p\right)^2\left(x_i^2 - y_i^2\right)} \\
&= (2\pi)^n \frac{\alpha_0|_p \prod \chi\omega_i^p}{\prod \chi^2\left(\omega_i^p\right)^2} \\
&= \left(\frac{2\pi}{\chi}\right)^n \frac{\alpha_0|_p}{\sqrt{det(v_p)}}.
\end{aligned}
$$

In the derivation above we used the fact that the only term in the α poly-form that can contribute is the term without factors of the rescaling of t, ie the bottom most form α_0. Similarly, the local coordinates in the exponential are quadratic so the recaling cancels. The last few lines we used Gaussian integration and interpreted $\prod\left(\omega_i^p\right)^2 = det(v_p)$. This result is the famous Atiyah-Bott-Berline-Vergne formula [2] [3]. In the derivation above we chose a particular V, but actually one may arrive at the same general result by constructing δV to lowest order in powers of ϕ and ψ

with symmetric and antisymmetric tensors respectively[12]. More succinctly, this result generalizes for sufficient choices of V.

Including all fixed points then we sum and quote the full result and generalization,

$$\int_{T_{[1]}M} d^n\phi d^n\psi\ \alpha = (2\pi)^n \sum_{\phi_k \in M_v} \frac{\alpha_0(\phi_k)}{\sqrt{\det\left(\partial_\mu v^\rho(\phi_k)\right)}}$$
$$\int_M \alpha(\chi) = \sum_F \int_F \frac{\alpha(\chi)}{e_N(F)(\chi)}.$$

In the last equation we generalize to a locus that may not be isolated, with a compact group action and equivariantly closed form. We also notice the appearance of the Euler form over the normal bundle F, which we mentioned was an important feature of index theorem whilst deriving the Dirac index. This realization is best derived[13] using the Mathai-Quillen form [18], briefly the summary of the formalism is as follows but will not be covered in detail further,

$$\begin{aligned}
\text{Regularized Euler number} \quad & \chi_s(E) = \int_M e_{s,\nabla}(E) \\
\text{Euler Form as a pullback} \quad & e_\nabla(E) = s^*\Phi_\nabla(E) \\
\text{Mathai-Quellin form} \quad & \Phi_\nabla(E) = (2\pi)^{-n} \int d\eta\ e^{-\frac{1}{2}\left(\xi^2 - \eta_a \Omega^{ab}_\nabla \eta_b + i\nabla\xi^a\eta_a\right)}.
\end{aligned}$$

There are two principle arguments that may be concluded from this calculation and understanding equivariant cohomology. First is that equivariantly closed forms on a manifold M are equivariantly exact on the complement of the locus. Secondly is that the integral of an equivariant closed form on a manifold M localizes to a locus of the action. These notions extend naturally to supersymmetric quantum field theories by fitting the equivariant cohomology scheme into it's formulation. We have already seen this algorithm demonstrated for a simple SQM but we will make the application and interpretation of equivariant cohomology more apparent in the next section.

[12]This is a common method in writing down the field configurations to an action if you know the type of symmetry your theory needs to have etc.

[13]For the application in SUSY QFT [6]

2.3 Supersymmetric Localization

Equivariant Localization	Supersymmetric Localization
$d_v = d + \iota_v$	$\mathcal{Q}$
$d_v^2 = \mathcal{L}_v$	$\mathcal{Q}^2 = B$, B is a boson
Even Polyforms	Bosons
Odd Polyforms	Fermions
$d_v\alpha = 0$	$\mathcal{Q}\mathcal{O} = 0$
$\int_M \alpha = \int_M \alpha e^{td_v\beta}$ s.t. $\mathcal{L}_v\beta = 0$	$\int \mathcal{D}\varphi\mathcal{O}e^{-S[\varphi]} = \int \mathcal{D}\varphi\mathcal{O}e^{-S[\varphi]-t\mathcal{Q}V[\varphi]}$ s.t. $BV[\varphi] = 0$
Locus M_v	Locus of field configurations
$\int_F e_N(F)$	1-loop Sdet

The extension of equivariant localization to SUSY localization is straight forward. The table above provides a concise map for all the quantities that we have computed or defined in this chapter and the previous. Given a supersymmetric action, $\mathcal{Q}S[\varphi] = 0$, where φ are configuration of fields in the theory and $\mathcal{Q}$ is a Grassmann odd supercharge such that it squares to a bosonic charge $\mathcal{Q}^2 = B$, the path integral is[14],

$$Z = \int \mathcal{D}\varphi \; e^{S[\varphi]}$$

If we have an operator that is a BPS[15] observable such that $\mathcal{Q}_{BPS}B = 0$, then the expectation value of the observable exactly computable and represents a $\mathcal{Q}$-coholomogy class of fermionic operator insertions,

$$\langle \mathcal{O}_{BPS} \rangle = \int \mathcal{D}\varphi \; \mathcal{O}_{BPS} \; e^{iS[\varphi]}$$

$$\langle \mathcal{O}_{BPS} \rangle = \langle \mathcal{O}_{BPS} + \mathcal{Q}\mathcal{O} \rangle \, .$$

Since $\langle \mathcal{O}_{BPS} \rangle$ depends only on the $\mathcal{Q}$-coholomogy then for a fermionic functional of fields such that $\mathcal{Q}^2 V = 0$ then by definition of anti-commuting Grassmann numbers[16],

$$\begin{aligned}
[\mathcal{O}_{BPS}] &= \left[\mathcal{O}_{BPS} \; e^{-\lambda \mathcal{Q}V}\right] \\
\Rightarrow \langle \mathcal{O}_{BPS} \rangle &= \int \mathcal{D}\varphi \; \mathcal{O}_{BPS} \; e^{-S[\varphi]-\lambda\mathcal{Q}V} \\
\frac{d}{d\lambda} \langle \mathcal{O}_{BPS} \rangle &= \int \mathcal{D}\varphi \; (\mathcal{Q}V)\mathcal{O}_{BPS} \; e^{-S[\varphi]-\lambda\mathcal{Q}V} \\
&= \int \mathcal{D}\varphi \; \mathcal{Q}\left(V\mathcal{O}_{BPS} \; e^{-S[\varphi]-\lambda\mathcal{Q}V}\right) \\
&= 0
\end{aligned}$$

[14] In Euclidean signature.
[15] This concept will be put into perspective next chapter but context will suffice for now.
[16] In an expansion of the exponential, they expansion will still be in the cohomology class.

Therefore, the deformed partition function is independent of the parameter λ, allowing us to make any conclusions we make hereafter to apply to the undeformed path integral $Z[0]$ and in general we may change lambda arbitrarily. This enables us to take the limit $\lambda \to \infty$ so that the integrand is dominated by the saddle point or locus of the localizing action functional $\mathcal{Q}V[\varphi]$ when $\mathcal{Q}V = 0$, which defines a localizing manifold or locus.

Now that we understand all the pieces and properties of supersymmetric localization, the last two pieces are a general recipe to compute the integral and identify the locus. The primary idea here is that when we employ the saddle point approximation we reduce the dimensional of the integral exactly to a smaller space defined by the localization locus. As we did in de Rham supersymmetric theory, we can expand our fields into constant modes corresponding to the locus and normal non-constant fluctuations

$$\varphi = \varphi_0 + \frac{\delta\varphi}{\sqrt{\lambda}}.$$

The calculation proceeds as normal and in the $\lambda \to \infty$ we find,

$$\begin{aligned}\langle \mathcal{O}_{BPS} \rangle &= \int \mathcal{D}\varphi_0 \; \mathcal{O}_{BPS}|_{\varphi_0} \; e^{-S[\varphi_0] - \frac{1}{2}\int \frac{\delta^2 S_{loc}[\varphi_0]}{\delta\varphi^2}(\delta\varphi)^2 \Big|_{\varphi_0}} \\ &= \int \mathcal{D}\varphi_0 \; \mathcal{O}_{BPS}|_{\varphi_0} \; e^{-S[\varphi_0]} \; Sdet\left(\frac{\delta^2 S_{loc}[\varphi_0]}{\delta\varphi^2}\right)^{-1}.\end{aligned}$$

Where the the superdeterminant term is a one-loop functional from the quadratic fluctuations of the fields orthogonal to the localizing manifold, usually this is denoted as $Z_{\text{1-loop}}(\varphi)$, and this tells us that our notion of an exact integral now becomes a 1-loop exact path integral. There is an additional caveat that may be used to great effect when performing calculations. Notice that we chose an arbitrary supercharge to define the set of BPS observable, if we run the algorithm above with two different supercharges in general it will be the case that the form of the integral and localization locus differ. So long as integrals themselves also differ by $\mathcal{Q}$-exact observables, which they do by construction, then the evaluation of the integral will be the same. It may be the case that a particular calculation is easier for one choice of observable compared to another, therefore it is a huge benefit to find the set of observables that produce 1-loop determinants that are tractable.

Additionally, it is generally convenient to work with an localization action defined by a supercharge where one knows already how the fields respond to the action of said charge, this of particular importance when extending these localization argument to supergravity or curved SUSY as working off-shell allows one to avoid imposing equations of motion. Based upon the analysis above, this means that the additional terms in the total deformation of the action will not affect the supersymmetric transformations of the fields.

In conclusion, the choice of the localization is free and also one only need choose one supercharge to make a computation. We can make the previous arguments for choosing the right fermionic function more concrete by using the appropriate inner product so that

$$V = \sum_i \langle \mathcal{Q}\varphi_i, \varphi_i \rangle$$

$$\Rightarrow \mathcal{Q}V = 0 \quad \Leftrightarrow \quad \mathcal{Q}\varphi_i = 0$$

These equations are manifestly BPS and these fields manifestly belong to the BPS locus.

2.4 Curved backgrounds

This section will discuss schematically how to put rigid supersymmetry on curved backgrounds in order to utilize localization. We have a recipe on how to extend equivariant localization to supersymmetric localization, where in the finite dimensional case we can perform calculations on a compact curved manifold. The extension proposed above does not tell us what to do with the supersymmetry on these curved spaces and in fact supersymmetry may not be compatible with a particular space of interest. Therefore the task will be to describe an approach to put supersymmetry on a desired background if it is possible in the first place. There are a few techniques that may be employed, a popular one by Witten is to twist a theory so that it becomes topological. Another one is to try and solve the issue with imposing the symmetries themselves on the curved manifold by demanding the fermions that depend on spacetime symmetries, ie are in the gravity multiplet, vanish under SUSY variation in some limit.

A general method put forth by [22] is to either identify a supergravity with the correct supersymmetry for the matter content or build a supergravity theory with couplings and corresponding Killing spinor equation, then take a rigid limit $G_N \to 0$. In this limit we send the Newton's constant to zero, removing gravity and fixing the metric for the desired space while keeping the background fields also fixed. These backgrounds must solve the generalized Killing spinor equations and the gravitini must go to a fixed value in the limit since the supergravity variations are paramaterized by the gravitini. Therefore, if we solve Killing spinor equation $\delta\Psi = 0$, Ψ are the gravitini, then the gravitini solutions determine the rigid supersymmetry.

In the end this amounts to identifying the correct background fields a priori or putting them in by hand, taking a rigid limit, then solving for the gravitini first order differential equations. After this is accomplished one imposes minimal coupling and determines the inherited supersymmetry transformations on the fields, especially the supercurrent corresponding to a gravitino which couples linearly in the linearized Lagrangian. With this Lagrangian, appropriate supersymmetry, background fields, and metric then one may proceed with supersymmetric localization.

Chapter 3

Black Holes, AdS/CFT, and Entropy

It should go without saying that quantum gravity is a subject of great interest and presents huge challenges. Finding a theory of quantum gravity has led theoretical physicists to the theories of strings and extended objects in higher dimensions as well as modified theories of Einstein's General Relativity(GR). If one naively tries to quantize GR using typical perturbative method one quickly realizes that it is non-renormalizable, in our spacetime at least. One of the main testing grounds for a theory of quantum gravity involves explaining the microscopic properties of black hole thermodynamics. Research into these properties has been very exciting over the past 20 years and for the hopeful seems to indicate we might be able to make some headway on understanding aspects of quantum gravity in our universe. This chapter will provide a rough overview and relationship of key ideas involved in contemporary research. Specifically, these ideas will be developed at a very general scale in order to concisely discuss the relevance supersymmetric localization has been in recent years to either give exact quantum entropy of black holes or provide "empirical" evidence for AdS/CFT and holography. Each of these subjects is vast all on their own and a comprehensive review would contain an analysis of different toy models rather than a exploration of the key ideas.

The narrative here will be to discuss a simple way in which AdS arises when studying black hole solutions and relating the necessary limits to necessary consequences in both supersymmetry and a general setting. Following this will be a very short discussion on the AdS/CFT correspondence to give context to the theme of this paper, though it is a subject that deserves much care especially with regards to ideas in string theory and the formulation of conformal field theories. In the final parts of the chapter will be a discussion on black hole thermodynamics and some basic ideas necessary to begin adding quantum corrections, namely we reproduce the Wald entropy. Shortly following this will be discussion on a form of entropy that is approachable from the context of supersymmetric localization, namely the entropy function given by a Wilson line on the boundary.

3.1 AdS black holes

Classical black holes are a class of solutions in General Relativity that are often characterized by the appearance of an event horizon[1] and a coordinate singularity that also gives rise to singular curvature. The horizon defines a boundary where the force of gravity overcomes any possibility for a particle to accelerate out of the region beyond the horizon. The Einstein-Hilbert action is given by,

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}(R-\Lambda) + L_{matter}\right)$$

Which gives rise to the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{1}{2}\Lambda g_{\mu\nu} = G_{\mu\nu} + \frac{1}{2}\Lambda g_{\mu\nu} = 16\pi G T_{\mu\nu}.$$

The simplest black hole solution to these equations is the Schwarzchild black hole [23], which is the the spherically symmetric solution to the vacuum Einstein field equations,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

This geometry describes a spherical black hole of mass M with curvature singularity at $r = 0$ and event horizon at $r = 2M$ which is a resolvable coordinate singularity. The black hole described by this metric is static and asymptotically flat[2], therefore in some sense it is too simple in the context of our universe. One could then consider a solution by adding matter with a Maxwell Lagrangian so that we have a charged black hole, here $L_{Maxwell} = L_{matter} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, and the solution is given by Reissner-Nordström [24]

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{r^2}dt^2 + \frac{r^2}{(r-r_+)(r-r_-)}dr^2 + r^2 d\Omega_2^2$$
$$r_\pm = M \pm \sqrt{M^2 - (q_e^2 + q_m^2)}.$$

Where $r_\pm$ are the coordinate singularities, therefore if we take the extremal[3] limit $M = Q$, motivated by the desire to construct BPS black holes in SUSY, then the horizon is the extremal radius where we define $\rho = r_\pm = M = Q$ so that metric becomes,

$$ds^2 = -\frac{(r-\rho)^2}{r^2}dt^2 + \frac{r^2}{(r-\rho)^2}dr^2 + r^2 d\Omega_2^2$$

[1]It is an unfortunate aspect of black holes that the definition of a horizon is teleological, though one hopes that this is being or will be resolved.

[2]This can be made rigorous by Birkhoff's theorem.

[3]One may regard this as the zero temperature limit.

This lets us observe that there are three regions characterized by two horizons $r_\pm$ and in the extremal limit these horizons combine. If we instead take limit close to r_+ changing coordinates as

$$\rho = \lambda^{-1}(r - r_+), \quad \text{and} \quad \tau = \frac{\lambda}{r_+^2} t$$

$$\text{as } \lambda \to 0 \Rightarrow \quad ds^2 = r_+^2 \left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2} \right) + r_+^2 d\Omega_2^2$$

This realization of the limit reveals a $AdS_2 \times S^2$, the first part is AdS_2 and the second is the 2-sphere, geometry which inherits a $SO(2,1)$ symmetry not in the original solution. Specifically, the extremal solution has regions that are approximately this $AdS_2 \times S^2$ and in the limit described above this approximation becomes exact at the horizon. The inheritance of the additional symmetry is general a feature of extremal limits and near horizon behavior of these limits where essentially the solution picks up an "enhancement" of symmetry. In the context of SUSY black holes, a solution to a supergravity theory, usually only preserves some number of the SUSY charges. For example in 4d $\mathcal{N} = 2$ supergravity with vector and scalar fields, the blackhole solution is said to be $\frac{1}{2}$-BPS, meaning it only preserves half of the original supersymmetry[4]. In the limit described above, the black hole solution picks up additional symmetry which enhances the $\frac{1}{2}$-BPS to a fully supersymmetric solution and this is known as the BPS attractor mechanism [25][5], and in situation without SUSY it is known as the AdS attractor mechanism [26]. Applying these extremal limits and imposing the enhanced symmetry in the supergravity theory one can recover Bekenstein-Hawking entropy of the extremal black hole using variational techniques and the entropy functional[6]. This will be more clear after we briefly cover AdS/CFT, but this calculation will allow one to compare entropy results and as well as sub-leading corrections between a theory of gravity and a corresponding quantum theory.

It should be stated that the Reissner-Nordström solution is likely not a physically relevant solution and cannot be observed in our universe, so why should the coincidental appearance of AdS mean anything? There are two primary reasons to study AdS black holes, the first is the AdS/CFT correspondence and the second is the conjecture that all black holes that posses an extremal limit are AdS_2 in the near horizon geometry up to a compact component, such as S^2 in the 4d $\mathcal{N} = 2$ Reissner-Nordström black hole. It is thought that this includes not only supersymmetric black holes but also non-supersymmetric black holes [27].

To finish this section we will look at a consequence of extremality in SUSY black holes with a sufficiently symmetric additional component, such as the 2-sphere. Extending the previous example actually proves simple in the extremal limit if we impose the enhanced symmetries that occur at

[4]Specifically it preserves 4 of the total 8 real supercharges.
[5]One could also use these results to assert that any SUSY black hole must also be extremal.
[6]This functional can be interpreted as the effective action near horizon limit.

the horizon. So let us impose $SO(2,1)\times SO(3)$ on the fields in the near horizon limit, then the gauge fields corresponding to the charge of the black hole have a constant electric field strength, e^i, due to AdS^2 and constant magnetic flux due to S^2. Therefore the scalar fields take constant values, ϕ^a, and the fermions contribution to the Lagrangian vanish and thus the Lagrangian become a function,[7]

$$\begin{aligned}\mathcal{L}[g, \text{gauge fields, scalars, fermions}] \;\; \rightarrow \;\; \mathcal{L}^{(2d)} &= \int d\theta d\phi \alpha \sin\theta \mathcal{L} = 4\pi\alpha\mathcal{L}\left(q_i, p^i; \alpha, e^i, \phi^a\right) \\ S &= \alpha\mathcal{L}^{(2d)} \\ \mathcal{E}\left(q_i, p^i; \alpha, e^i, \phi^a\right) &\equiv 2\pi\left(e^i p_i - \alpha\mathcal{L}^{(2d)}\right).\end{aligned}$$

Where α is the scaling factor for the geometry, q and p are the electric and magnetic charges respectively, $\mathcal{E}\left(q_i, p^i; \alpha, e^i, \phi^a\right)$ is the Legendre transform of the action which generates the classical entropy function[8]. The metric used above and the fields are,

$$\begin{aligned} ds^2 &= \alpha\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + \alpha d\Omega_2^2 \\ F^i_{rt} &= e^i, \quad \text{and} \quad F^i_{\theta\phi} = p^i \sin\theta. \end{aligned}$$

The equations of motion for this theory correspond to the vanishing variation of the entropy function by the Bianchi identity, which is just a variational principle. With this function one can compute the classical entropy of the black hole as the entropy function evaluated at this limit [26].

3.1.1 AdS/CFT and the holographic principle

The AdS/CFT correspondence is an idea proposed by Juan Maldacena in [28]. The treatment here will be very surface level as the subject has filled books and journals with papers over the past 20 years. The fundamental idea is that

$$CFT_d \leftrightarrow AdS_{d+1}\times F$$

where CFT is a strongly coupled quantum field theory without gravity, CFT is short for conformal field theory, and the $AdS_{d+1}\times F$ is a theory of quantum gravity with a compact fiber F. The quantum gravity theory in question is typically a string theory. This relationship is made, when under the large-N limit of the CFT[9] recovers the classical limit, in other words the Planck length is much smaller than the curvature scale of the AdS.

[7]Evaluated on the near horizon geometry and integrated over the sphere.
[8]The idea here is to charges in terms of the field strength and scale parameter.
[9]Large N limit is the idea of a limit for a matrix model of fields that correspond to NxN matrices and under a gauge group such as $SU(N)$.

This correspondence can also be regarded as a form of holography which makes a relationship between a theory on the boundary of a spacetime, a quantum field theory, and the fields that exist in the bulk of that spacetime. This relationship has over the years accrued a number of results calculated on both sides of the correspondence which generally agree, which is surprising since the starting point for all the calculation and formulation of the theories involved seem like they have nothing to do with one another. When comparing a relationship by this correspondence an equality and dictionary for certain objects is defined. For example, it can be shown that an operator on the boundary is a field in the bulk in a certain limit. This allows one to make computations on the boundary, ie in the CFT using operator product expansions and rigorous mathematics involving bi-holomorphic functions, and take a limit to describe a field in the gravity theory.

Often times the gravity theory in question is a compactification[10] of a superstring theory vacuum solution from M-theory, which describes various extended objects that when analysed on their corresponding worldvolume yields a quantum theory based upon a central extension of a Lie algebra. More importantly is the assertion that,

$$\begin{aligned}
\mathcal{H}_{AdS} &= \mathcal{H}_{CFT} \\
Z_{AdS}[J_{boundary}] &= Z_{CFT}[J_{CFT}] \\
g_{\mu\nu} &= g^{class}_{\mu\nu} + h^{quantum}_{\mu\nu}
\end{aligned}$$

which corresponds to a large N limit which yields the semi-classical approximation that recovers in the last line solutions to Einstein's equations. Moreover, under this approximation we obtain schematically

$$\begin{aligned}
Z_{AdS} &= det(\bullet)e^{iS_{gravity}|_{g^{class}_{\mu\nu}}} \\
log\,(Z_{AdS}) &= S_{gravity} + 1/N \;\text{ loop corrections.}
\end{aligned}$$

If we take care to add boundary terms, such as Gibbons-Hawking terms, and apply and appropriate cut off we can discover additional contributions used in comparing theories in the form of counter terms which go like local divergences we are happy to see in QFTs[11]. Finally, what is especially striking is that when one considers a thermal partition function, ie at finite temperature in the Euclidean signature, one can recover a black hole solution on the AdS side which has an entropy on the CFT that agrees with Bekenstein-Hawking. This last point is the one of the bigger motivations to studying AdS/CFT in the context of this paper, and in fact localization if properly used on the boundary of the AdS, which is the conformal boundary and is typically $S^{d-1} \times S^1$, can be

[10]Reduction of dimension by taking the space as a composite spaces with tori or circles, for example.

[11]Suprisingly, the coefficients of the log divergences agree quite well on both sides of the correspondence and might suggest a deeper universal interpretation, which is a good motivation when studying the microstructure of black holes in this context.

computed exactly[12]. The examples treated in this text have largely been topological, but in 2007 Pestun showed that supersymmetric localization could be done non-toplogically by coupling to a supergravity theory on S^4, this had immediate implications for the applications on other more complicated manifolds with additional fibering or complicated twisting [5]. What is a hopeful conclusion and extension of the entropy formula to classical statistical mechanics is the assertion that the entropy of the black hole is counting microstates of the dual CFT. This leads us into a short discussion on black hole entropy.

3.2 Black holes have temperature

In 1972 Jacob Bekenstein argued that black holes must have entropy, otherwise if a mass would fall into a black hole and the black hole did not have entropy, then the 2nd law of thermodynamics would be violated [29][13]. The arguments is that the if the object falls into the black hole then entropy would be lost from the universe and change would be negative. To compensate for this, one can generalize that there must be a positive contribution from the change of entropy in the black hole.

$$0 \leq \delta S_{tot} = \delta S_{out} + \delta S_{BH}$$

Subsequently in 1974/1975 Stephen Hawking [30] [31] proved through a difficult semi-classical calculation that black holes have entropy proportional to their $A/4$, area. Assume we have a Schwarzchild black hole, then

$$\begin{aligned} T &= \frac{\hbar}{2\pi k_B}\left(\frac{GM}{R^2}\right) = \frac{\hbar}{4\pi k_B}\left(\frac{1}{R}\right) \\ dS_{BH} &= \frac{1}{T}dM_{BH} = \frac{k_B}{\hbar G}2\pi R dR \\ \Rightarrow S_{BH} &= \frac{c^3 k_B A}{G\hbar} = \frac{k_B A}{4l_p^2}. \end{aligned}$$

Surprisingly this statement implies that black holes are almost perfect black bodies which radiates at a temperature proportional to their surface gravity. This was surprising since as a particle gets closer and closer to a black hole it would take more and more acceleration to escape until at the horizon the acceleration would approach infinity. Therefore, what is the physics governing the radiation of the black hole when particles cannot be emitted from it based up simple analysis of the solutions? The rough answer given by Hawking is that particles very close to the

[12]The S^1 is the timelike component which must be taken with care because it implies closed timelike curves, in carrying out a solution one usually tries to replace this factor with a disk.

[13]One could argue that the real insight is that black holes must have finite temperature instead because if one carries out a calculation classically in the frame of a far away observer then a black hole must have infinite entropy.

horizon pair produce due to quantum fluctuations of the vacuum, though this answer is still semi-classical at best. The idea is that one of the pair would escape as radiation while the other would fall in. Given that we have the 2nd law of thermodynamics the full first law of thermodynamics to the black hole would be,

$$\delta M = T_{BH}\delta\frac{A}{4} + \omega\delta J + \mu\delta Q.$$

The observation that since entropy is an extensive quantity, in that it increases with volume of the system, one might be puzzled to see an area law for the entropy. This is another reason why AdS/CFT and holography seem like such promising ideas to learn more about quantum gravity, we have a quantity that is typically related to a volume of a space being related to the boundary[14]. To add more complications to the story, the entropy equation for the black hole is a function that is measure by an observer at infinity and only depends on macroscopic observables such as mass, charge, spin[15].

The natural thing to do after discovering that black holes have entropy would be to try and write down the classical statistical mechanics interpretation by Boltzmann in terms of the microscopic degree of freedom, ie microstates of the black hole, the right answer turns out to be

$$S_{BH} = k_B \log d + ...$$

Discovering the subleading corrections is generally a difficult problem because the equation put forth by Hawking applies to all black holes, and while having a universal property to compare black holes by this means that the corrections are defined in an ultraviolet limit of gravity that we do not have at the scale of l_p^2. In other words, understanding the corrections by considering different types of black holes will vanish when taking a thermodynamic limit, which recall takes $n \to \infty$ and the volume $V \to \infty$ so that the microscopic side of the equation matches the macroscopic. Despite the difficulty of quantizing the Einstein-Hilbert action, some low energy corrections to the black hole Boltzmann equation have been found which are consistent with low energy effective theories of quantum gravity, eg string theory. If we assume that A is the only large parameter then these corrections look like, setting fundamental constants to unity,

$$S = \frac{A}{4} + \alpha_0 \log A + \frac{\alpha_1}{A} + \frac{\alpha_2}{A^2} + ...$$

However, when one modifies gravity in this way the resulting expansion in string coupling and powers of the curvature tensor violate the laws of thermodynamics of a black hole. So the idea is to understand how these laws get modified especially under higher derivative terms in the action.

[14]Most striking to the author in this case is that the complicated microstructure of the black hole is somehow encoded on the boundary, 1 dimensional lower than one might otherwise expect.

[15]This is consequence of the "No Hair Theorem."

This leads us to consider field variations in a gravitational Lagrangian,

$$\delta\left(\sqrt{-g}\mathcal{L}\right) = \sqrt{-g}E\delta\Phi + \sqrt{-g}\nabla_\mu\Theta^\mu(\delta\Phi)$$

We now proceed as usual and consider a symmetry of the action, then generically the Lagrangian should be invariant or a total derivative, ie $\delta\left(\sqrt{-g}\mathcal{L}\right) = \sqrt{-g}\nabla_\mu n^\mu$. Leading us to define a Noether current,

$$J^\mu = \Theta^{mu}\delta\Phi - n^\mu,$$

which satisfies $\nabla_\mu J^\mu = 0$ when $E = 0$. Using this we may define the Wald entropy using these current corresponding to invariance under diffeomorphism as an integral with boundary corresponding to the boundary at infinity and the horizon of the black hole with induced metric h [32],

$$S_{Wald} = 2\pi\oint_H d^2x\sqrt{h}\epsilon_{\mu\nu}\mathcal{Q}, \quad \mathcal{Q} = \oint_{\partial\Sigma} d^2x\sqrt{h}\epsilon_{\mu\nu}\mathcal{Q}^{\mu\nu}$$

If one considers a simple deformation to quadratic order on the Einstein-Hilbert action we can see that the Wald entropy characterizes the corrections by

$$\begin{aligned} S = -\frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R+\alpha R^2\right) \;\Rightarrow \mathcal{Q}^{\mu\nu\gamma\rho} = \frac{\delta\mathcal{L}}{\delta R_{\mu\nu\gamma\rho}} &= \frac{1}{32\pi}(1+2\alpha R)\left(g^{\mu\rho}g^{\nu\gamma} - g^{\mu\gamma}g^{\nu\rho}\right) \\ S_{Wald} &= \frac{1}{4}\oint_H d^2x\sqrt{h}\,(1+2\alpha R) \\ &= \frac{A}{4} + \frac{\alpha}{2}\oint_H d^2x\sqrt{h}R \end{aligned}$$

Therefore we can see explicitly this expression from Wald gives us the Bekenstein-Hawking entropy as well as the correction term corresponding to the deformation. The potential issue with this form of the entropy is that is capture strongly the local effective action due to the implementation of diffeomorphism invariance, but in general one could expect non-local corrections to a quantum theory of gravity. It should not be understated that it is quite powerful in that we can incorporate higher derivative terms and identifying the terms in the previous calculation it is manifest that the area law solution with event horizon S^2, the Schwarzschild black hole, which means that the Wald entropy does capture quantum corrections to the entropy. Thus, the Wald entropy generalizes Bekenstein-Hawking entropy in just the way we need it to.

There is quite a bit more to the story regarding black hole thermodynamics and a large set of results for the quantum side of black hole thermodynamics coming from results in string theory. This treatment here avoids many of the details involved in string theory intentionally to be as self-contained as possible and to go any further would require a large diversion away from the goal in seeing the relevance of localization. We will now connect the ideas we covered from the AdS black hole story and discuss the consequences thermodynamics has on these black holes.

3.2.1 A brief window into quantum entropy

The near horizon geometry of a charged BPS black hole is asymptotically AdS_2, and so using the ideas put forth the AdS/CFT we can use holography to explore the black hole's quantum entropy. Using the argument put forward in the AdS black hole section together with the discussion on the Wald entropy, we find

$$\begin{aligned} & S_{Wald} = -8\pi \int d\theta d\phi \frac{\partial S}{\partial R_{rtrt}} \sqrt{-g_{rr} g_{tt}} \\ \mathcal{L}^{(2d)} = \int d\theta d\phi \sqrt{-g} \mathcal{L}|_{horizon} \quad & \rightarrow S_{Wald}(q,p) = \mathcal{E}(p,q)|_{attractor} \end{aligned} \tag{3.1}$$

Where the attractor mechanism in this context may be accomplished by extremizing with respect to the parameters, ie taking first derivatives of $\mathcal{E} = 0$. To extend this to quantum entropy takes a little work in including quantum fluctuation of string fields, but essentially what happens is that the variational principle becomes a functional integral over all the fields demanding that they asymptote to the attractor configuration spelled out by extremization.

To accomplish this formally one may calculate the partition function of AdS_2 of a string theory on AdS_2 and a compact component. This by itself presents a challenge as AdS_2 itself is not compact, but in the spirit of QFT a finite part can be extracted via regularization applying the idea of duality to the CFT[16]. Schematically the computation goes as follows.

First we rotate the metric to Euclidean signature by taking $t \rightarrow i\theta$,

$$ds^2 = \alpha \frac{(r-1)^2}{r^2} d\theta^2 + \frac{r^2}{(r-1)^2} dr^2 + \alpha r^2 d\Omega_2^2.$$

Now we also need to impose conditions for $r \rightarrow \infty$[17],

$$\begin{aligned} g_{\mu\nu} \rightarrow g_{\mu\nu} = g_{\mu\nu}^{classical} + \mathcal{O}\left(\frac{1}{r^2}\right) \quad & \Rightarrow ds^2_{AdS_2} = \alpha(r^2 + \mathcal{O}(1)) d\theta^2 + \left(\frac{1}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right)\right) dr^2 \\ A^i_\mu \rightarrow A^i_\mu = A^{i,classical}_\mu + \mathcal{O}\left(\frac{1}{r}\right) \quad & \Rightarrow A^i_\mu = -ie^i(r + \mathcal{O}(1)) \\ \phi^a \rightarrow \phi^a = \text{constant} + \mathcal{O}\left(\frac{1}{r}\right) \quad & \Rightarrow \phi^a = \text{constant} + \frac{1}{r} \end{aligned}$$

[16]This algorithm is generically called holographic renormalization.

[17]These conditions end up being equivalent to choosing a microcanonical ensemble

Then we find a renomalized action as,

$$S_{renorm} = S_{bulk} - iq_i \oint A^i + S_{boundary}$$
$$S_{bulk} = C_0 r_0 + C_1 + \mathcal{O}\left(\frac{1}{r}\right)$$
$$-iq_i \oint A^i = C_0' r_0 + C_1' + \mathcal{O}\left(\frac{1}{r}\right).$$

Since the Wilson line contribution and the bulk contribution are linearly divergent we only need to make sure that we choose an appropriate boundary as a cut off that cancels these divergences. Finally, the result is the path integral over gauge fields over a loop on the boundary in the form of a Wilson line,

$$Z_{AdS_2} = \int \mathcal{D}\varphi \; e^{-S_{renorm}[\varphi]}$$
$$\Rightarrow \quad e^{S_{BH}^{grav}} \equiv W(q,p) = \left\langle exp\left[-iq_i \oint A^i\right]\right\rangle_{AdS_2}^{finite} [33]$$

Employing techniques in holographic renormalization, described above, this quantity agrees with the Wald entropy in the semi-classical limit[18]. The point here is a little deeper, this regularized Wilson line on the boundary of AdS_2 is the degeneracy needed in the Boltzman equation to relate microscopic degrees of freedom to the macroscopic degrees of freedom. Holography requires,

$$W(q,p) = d(q,p).$$

Where $d(q,p)$ is the degeneracy of states. Therefore, the path integral must evaluate to an integer. Moreover, the degeneracy must also be an index defined by bosonic degrees of freedom on the boundary, based on the limits we took in the AdS section, and because it is a degeneracy it must also be positive. The quantum entropy path integral as stated still seems to pose particular issues as one might hope to try and take a perturbative approach. The problem is that a perturbatation is limited since it is often asymptotic and there is no real way to expect to get an integer out because the precision is bounded. Therefore, one might hope that supersymmetric localization might be applicable here. In fact it is, but it involves formulating an off shell formulation of $\mathcal{N} = 2$ supergravity and to take a rigid limits to freeze the background metric and auxiliary fields. The full derivation and discussion are beyond the scope of this paper but for completeness the solutions are given by scalar field X^i and auxiliary fields Y^{r^2} about some off-shell instantons in AdS_2 with a complicated action[19]. The results on both sides of the correpondence

[18]One defined a large "radial" cutoff

[19]There is a result in terms of the 1-loop determinant and the Nekrasov instanton partition function, showing a direct application of SUGRA localization[34].

3.3 Conclusion

In this text we set out to give a treatment of supersymmetric localization and fully describe how to apply the tool in a supersymmetric setting. We built up on basic concepts applying more and more insight until we were finally able to derive one of the first non-trivial results of supersymmetric localization, the twisted Dirac index. Upon examining equivariant cohomology and the Atiyah-Bott formula and reapplying the insight gained from testing these ideas in simple theories, we were able to fully extend equivariant localization to supersymmetric localization in a robust way. The formulation reveals that despite huge difficulties in both mathematically defining a path integral and generally solve it, if one imposes the correct conditions the computations can simplify or become exact. This reveals an underlying connection between modern geometry and algebraic topology to the fundamental approaches of both the path integral and physical mathematics. Moreover, though not explored deeply here, is a the ever imposing notion that there might be truly fundamental consequences of string theory and modern mathematics in our universe, especially regarding the geometric Langlands correspondence[20]. Moreover, there are deep connections to number theory and quantum entropy which by extension might imply some important insights to supersymmetric gauge theories, gravity, duality, and the path integral. That said, these problems are only mentioned in passing as they do not necessarily translate to physical theories in our reality.

A promising avenue of pursuit is exploration of quantum gravity through the lens of quantum entropy, and supersymmetric localization provides a key tool in identifying the correct theories and concepts needed to describe it. Of particular interest right now are theories involving near AdS_2/CFT_1, SYK for example, which might play a key role in the story of quantum entropy given the current understanding of the near horizon geometry of extremal black holes. Further, if holography proves truly relevant in our reality and if there is a way to either observe extremal black holes or alternatively explain a mechanism, akin to effective field theory and symmetry breaking, that makes them fundamental but not observable. Even if it turns out that these black holes and correspondences are no more than toy models, the power they lend themselves to calculating quantum corrections to the Bekenstein-Hawking entropy should still say something fundamental about our universe.

[20]Though many physicists would find this motivation ill-posed from the point of view of Karl Popper and that mathematical beauty is a poor motivator for choosing one direction over another. The opposing argument should be that exploring the entire space of physical theories is important in order to understand the ones we have, and mathematics is the arena in which that is best accomplished outside impossibly expensive experiments.

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