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Dark Energy and Modified Gravity: A Dynamical Systems Approach

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Abstract

The mystery of dark energy has pervaded the field of cosmology for decades, and a satisfactory theoretical framework for describing its origins remains elusive. Models that aim to describe dark energy are relatively easy to propose, and split roughly into two categories: theories that introduce some field content, and theories that modify the gravitational interactions of the Universe. Once such a theory is formulated, its validity is sometimes far harder to comprehend, due to the relative complexity of the resulting field equations. The dynamical systems approach aims to alleviate this difficulty, by reformulating the field equations as a set of simpler differential equations, which permits the classification of the entire cosmic history within a given dark energy model. We will firstly examine models with added field content using dynamical systems tools; this will allow us to confirm the viability of each model without too many complicated calculations. We then make the natural progression towards theories of modified gravity, again with the hope of ruling out unfeasible models. We will also briefly make contact with observational restrictions on each theory of dark energy, which again is a task made easier in the dynamical systems framework. We thus conclude that the methods of dynamical systems, when applied to dark energy and cosmology more generally, are a powerful way of describing the Universe, and can be applied effectively across the entire field of theoretical cosmology.

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Contents

Introduction

The successes of Einstein's theory of General Relativity (GR) are almost unparalleled in the history of physics, and the cosmological applications of GR comprise a significant portion of cosmological literature produced in the past century. Technological advances since the formulation of GR have allowed for a dynamic interplay between observational and theoretical cosmology, with GR at the heart of a large majority of theoretical developments. This interplay became significant only shortly after the discovery of GR, as Einstein himself introduced a constant term into his field equations to produce a static universe, which at the time was believed to be the actual state of the Universe. Hubble's discovery [110] that the Universe was, in fact, expanding caused Einstein to eliminate the constant, dismissing its inclusion as his 'greatest blunder', or so the legend says.

The constant once again came to the forefront in the late 1990s, when it became clear that the rate of expansion of the universe was increasing [159, 166]. Insertion of the so-called *cosmological constant*, denoted by Λ, into the field equations [1]:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

yields such an accelerating universe. The term *dark energy* was then coined [111] as an all-encompassing expression for the unknown form of energy driving the acceleration. It is, in many ways, remarkable that GR could so readily accommodate for the observed acceleration of the Universe in such an elegant way, and further highlights the beauty of Einstein's theory.

The picture becomes rather less appealing when one examines the physical origin of the cosmological constant. Quantum Field Theory, another cornerstone of modern physics besides GR, predicts a value for the constant many times larger than the small value we observe. Also, we will see that the *constant* nature of Λ raises a significant concern regarding the initial conditions of the Universe. Naturally, we should then question whether the simple addition of Λ into the field equations is a sufficient description of reality, or whether something more subtle is required.

[1]Throughout this work, the convention $c = 1$ will be used.

The first step beyond the cosmological constant stems from the quite natural question: should the source of universal acceleration be some dynamical object, rather than the spacetime constant Λ? If this is the case, what type of object should it be? Perhaps inspired by the scalar field inflationary scenarios of the early 1980s [9, 103], the scalar field has been investigated as the source of dark energy from the moment we observed the acceleration of the Universe. Scalar fields are a natural candidate, since they are ubiquitous in the particle content of, for example, string theory and the Standard Model. They are also simple to work with in cosmological contexts; there is a well defined canonical action describing the scalar, which we can quite easily couple to gravity via the metric tensor $g_{\mu\nu}$. A natural extension of the canonical scalar is the non-canonical scalar, described by a non-standard action and with interesting consequences for cosmology.

Scalar field dark energy has been successful in some respects, while unsuccessful in others. This has understandably lead cosmologists to consider dark energy models utilising different types of fields, such as vectors and spinors. Such fields are fundamental in particle phsyics, as is the scalar, and are thus well worth considering as a potential candidate for dark energy.

There is an equally interesting possibility to describe dark energy, or more accurately, describe *away* dark energy. We could take the view that GR is, in fact, an incorrect theory at cosmological scales, and that dark energy is simply an artifact of our insufficient understanding of reality. The history of modified theories of gravity can be traced almost exactly to the years following the formulation of GR. As such, the cosmological implications of modified gravity comprise a broad and active area of research to this day. Therefore, a review of dark energy that hopes to be at all comprehensive should necessarily include an appraisal of modified gravity.

The aim of this thesis is to present a broad introduction to the numerous theories that can describe the acceleration of the Universe. An important question then arises: how can we study the viability of a cosmological model with regards to observational data? The field equations, in most cases derived from an action, are ostensibly solvable once the metric $g_{\mu\nu}$ has been defined. We should then be able to solve the resulting differential equations and predict the behaviour of any cosmological model.

In practice, however, the situation is rarely so simple. For models with even a small amount of complexity over (1), the field equations are highly complicated differential equations for which even numerical solutions are hard to obtain. This is where the theory of dynamical systems comes into play. The dynamical systems approach aims to reduce the field equations to a set of ordinary differential equations, termed an *autonomous system*, such that the entire cosmic history can be characterised. This is usually achieved by defining variables related to the standard cosmological parameters, such as the Hubble parameter H, and taking their derivative with respect to some timelike parameter. The variables used to define the system comprise the phase space, and the critical points are points in the phase space that are stationary, i.e. with vanishing derivative. If such a critical point is stable, it will act as an attractor; a point to which trajectories in the phase space will converge. The concepts such as phase space, critical points, and stability will be more formally defined in the next chapter, but it should hopefully already be clear that dynamical systems tools are a powerful method of studying dark energy models. For example, if we define an autonomous system describing a cosmological model, and the corresponding phase space exhibits a single attractor, we can calculate whether that point represents an accelerated phase of expansion, and therefore whether it is a viable framework for representing dark energy.

This thesis is structured so as to include the necessary basics of dynamical systems theory and GR, before considering the more advanced topics of dynamical dark energy and modified gravity. Chapter 1 is concerned with formally defining the dynamical system and the associated concept of linear stability theory. This will allow us to categorise the critical points of the system into those that are stable, unstable, or saddle points. We will then see that linear stability theory is inadequate in certain circumstances, and briefly outline two other methods of stability analysis that can be used instead. Examples are provided throughout this chapter to help elucidate the purely mathematical concepts that we define.

In Chapter 2, the basics of GR and its applications to cosmology are reviewed. The Einstein-Hilbert action is introduced, as this will play an important role in helping us define most of the more complicated theories considered in this thesis. The cosmological principle is used to write down a line element describing the Universe, and the basic quantities of differential geometry, such as the curvature tensor, are derived for the corresponding metric. We then explore the current canonical model of the Universe, the ΛCDM model, before discussing two of the most pertinent issues surrounding the cosmological constant. Lastly, the ΛCDM universe is studied using dynamical systems techniques. This provides a concrete example of the power of the dynamical systems approach, in the context of simple ΛCDM cosmology.

Chapter 3 deals with the most commonly studied model of dynamical dark energy: the scalar field. Firstly, we write down the action that defines the scalar field coupled minimally to gravity, and obtain the resulting field equations. A dynamical system is then constructed and the critical points are found, along with their stability properties. Then, there is a brief discussion on the various forms of scalar self-interaction potential that have been studied in the literature, before moving on to the non-canonical scalar models that are also well studied. The phenomenology of each model is considered, with close attention paid to the late-time dynamics. Lastly, we briefly depart from the dynamical systems path to investigate a generic class of scalar field Lagrangians that permit so-called *scaling solutions*, the details of which will be explained in due course.

Chapter 4 can, in some respects, be viewed as an extension of Chapter 3. Its primary focus is to define some of the models of dark energy that instead deal with non-scalar fields. Since the scalar is a zero-form, we consider one, two, and three-forms in turn, again utilising the dynamical systems tools and discussing the cosmological consequences for each model. Then, we consider spinor and Yang-Mills fields as dark energy sources, although in far less detail than the preceding n-forms. Instead, a (far from comprehensive) list of references is given with a brief description of each, should the reader be interested in the current literature on the subject.

In the final chapter, we examine some of the most prominent theories of modified gravity. Brans-Dicke and $f(R)$ gravity are studied in detail, again by using dynamical systems techniques. We also briefly describe the different formalisms for deriving the field equations in both cases, known as the first-order and second-order formalisms. Next, a more exotic form of of modified gravity, known as the RS2 braneworld model, is given the dynamical systems treatment. Lastly, we discuss massive gravity, teleparallel gravity, and Hořava-Lifschitz gravity, with references to the abundant dynamical systems literature for each model.

This thesis is intended to be a review on the vast subject of dynamical systems and dark energy. As such, the level of detail given in each section is mostly only enough to serve as an introduction to the topic at hand. In fact, most of the sections in this thesis could be extended to a self-contained, review-length work. For the reader interested in more detailed appraisals of each section, it is hoped that the list of references contained within can provide a starting point from which to build a deeper understanding of the topic.

Lastly, it should be noted that we have mostly neglected to discuss early universe cosmology, such as the inflationary epoch. This is intentional, as inflation itself comprises an expansive and rich area of research that, if included, could likely double the length of this work.

Conventions and Notation

- The metric signature $(-,+,+,+)$ will be used;
- $\partial_\mu \phi$ denotes the partial derivative of ϕ with respect to the corresponding spacetime coordinate, i.e. $\partial_\mu \phi = \frac{\partial \phi}{\partial x^\mu}$;
- The covariant derivative of a vector A^μ is defined by

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\sigma\mu} A^\sigma, \tag{2}$$

and the covariant derivative of a covector B_μ

$$\nabla_\mu B_\nu = \partial_\mu B_\nu - \Gamma^\sigma{}_{\nu\mu} B_\sigma, \tag{3}$$

where the $\Gamma^\sigma{}_{\nu\mu}$ are the components of the Levi-Civita connection.

Chapter 1

Dynamical Systems

We begin by defining a dynamical system as [195]

$$\dot{x} = f(x,t), \tag{1.1}$$

where x is an element of the *phase space*: $x \in U \subset \mathbb{R}^n$, the overdot on x denotes the derivative with respect to the parameter $t \in \mathbb{R}$, and the function f is a map: $f : U \to U$. In general, the above dynamical system can be viewed as a system of n differential equations, interpreting f as a vector field on $\mathbb{R}^n$. We denote a solution to (1.1) as $\psi(t)$.

We also define the the *critical points* as points $x_c \in U$ such that

$$f(x_c) = 0, \tag{1.2}$$

and the *stability* of said critical points as, informally speaking, a measure of whether a given trajectory starting close to x_c tends towards x_c.

Formally, we define two types of stability, as in [195]:

- **Lyapunov Stability**: x_c is Lyapunov stable if, for a given $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that, for any solution $\psi(t)$ of (1.1) satisfying $|x_c(t_0) - \psi(t_0)| < \delta$, then $|x_c(t) - \psi(t)| < \epsilon$ for $t > t_0$;

- **Asymptotic Stability**: x_c is asymptotically stable if it is Lyapunov stable, and for any solution $\psi(t)$ of (1.1), there exists a constant $a > 0$ such that if $|x_c(t_0) - \psi(t_0)| < a$, then $\lim_{t\to\infty} |x_c(t) - \psi(t)| = 0$.

1.1 Linear Stability Theory

In practice, once we have found a critical point or set of critical points for a system, we would like to understand the stability of each point and thus classify its behaviour. To this end, we linearise the system around the critical points by Taylor expanding each component of the vector field $f(x) = (f_1(x), ..., f_n(x))$

$$f_i(x) = f_i(x_c) + \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x_c) y_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 f_i}{\partial x_j^2}(x_c) y_j y_k + ..., \tag{1.3}$$

where $y = x - x_c$. We then neglect derivative terms that are second order or above, and define the *Jacobian* as

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}. \tag{1.4}$$

The eigenvalues of the Jacobian, evaluated at the critical points, are precisely the values that determine the stability of the system in the linear stability framework. This is summarised by the following theorem [195]: *if all the eigenvalues of J have negative real parts, then the critical point x_c is asymptotically stable.* We shall see this very clearly when we apply the above to some basic cosmological models, but for now it is worth giving a very simple example to fully elucidate some of the above concepts.

Consider the system

$$\dot{x} = x, \tag{1.5}$$

$$\dot{y} = -y + x^2, \tag{1.6}$$

which for which $(x_c, y_c) = (0, 0)$ is a critical point. The Jacobian evaulated at this point is then

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.7}$$

such that the eigenvalues of J are $\{-1, 1\}$. Thus, the origin $(0, 0)$ is neither stable nor unstable. It is instead a *saddle point*, which attracts trajectories in some directions while repelling trajectories in others.

A *hyperbolic point* is a critical point x_c such that none of the eigenvalues of J, evaluated at x_c, have zero real part. This is significant, as the process of linearisation described above is relevant and useful only for hyperbolic fixed points. Non-hyperbolic points require a different type of stability analysis entirely, Lyapunov theory being one such method.

1.2 Lyapunov Theory

Following the standard dynamical systems literature, we briefly discuss the Lyapunov method employed when linear stability theory is inadequate. The starting point is a theorem, the proof of which can be found in [195]:

Consider the system:

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n. \tag{1.8}$$

Let x_c be a critical point, and let $V : U \to \mathbb{R}$ be a differentiable, continuous function defined on a neighbourhood of x_c such that:

- $V(x_c) = 0$ and $V(x) > 0$ if $x \neq x_c$
- $\dot{V}(x) \leq 0$ in $U - \{x_c\}$.

Then x_c is stable. It is also asymptotically stable if

$$\dot{V}(x) < 0 \in U - \{x_c\}. \tag{1.9}$$

The function V is known as a *Lyapunov function*, and if U can be chosen to be all of $\mathbb{R}^n$, then x_c is *globally asymptotically stable.*

Note that the central difficulty of the Lyapunov method is finding a suitable Lyapunov function; there is no algorithmic way of finding the function and we instead rely on intuition and luck. Also note that the absence of a Lyapunov function does not imply instability.

As a brief example, consider [195]

$$\dot{x} = y, \qquad \dot{y} = -x + \epsilon x^2 y, \tag{1.10}$$

which has a non-hyperbolic critical point at $(0, 0)$. If we choose $V(x, y) = \frac{1}{2}(x^2 + y^2)$, we have

$$V(0, 0) = 0, \qquad V(x, y) > 0, \tag{1.11}$$

for any point (x, y) in the neighbourhood of the critical point. Also

$$\begin{aligned} \dot{V}(x, y) &= \nabla V(x, y) \cdot (\dot{x}, \dot{y}) && (1.12) \\ &= (x, y) \cdot (y, -x + \epsilon x^2 y) && (1.13) \\ &= \epsilon x^2 y^2, && (1.14) \end{aligned}$$

where $\cdot$ denotes the standard inner product, and we have used the chain rule in the first line. Thus, if $\epsilon < 0$, the critical point $(0, 0)$ is stable. It is also straightforward to see that we can choose $U = \mathbb{R}^2$, and so $(0, 0)$ is in fact globally asymptotically stable. We see that the strength of the Lyapunov method is its simplicity, but this is counteracted by the difficulty of finding a suitable form for V.

1.3 Centre Manifold Theory

Consider again the system (1.8). The theory of centre manifolds can be employed when one or more of the critical points of the system are non-hyperbolic. Once (1.8) is linearised, recall that the Jacobian J is an $n \times n$ matrix with n eigenvalues. We can then view $\mathbb{R}^n$ as the disjoint union of the subspaces $\mathbb{E}^s, \mathbb{E}^u$ and $\mathbb{E}^c$, where each is a space of eigenvectors with the following properties:

- $\mathbb{E}^s$ is the space of eigenvectors of J with eigenvalues that have a *negative* real part,
- $\mathbb{E}^u$ is the space of eigenvectors of J with eigenvalues that have a *positive* real part,
- $\mathbb{E}^c$ is the space of eigenvectors of J with eigenvalues that have *zero* real part.

Each subspace $\mathbb{E}$ is an *invariant subspace* [195] since, informally speaking, solutions in $\mathbb{R}^n$ with initial conditions entirely within one of the $\mathbb{E}$ spaces will remain within that space.

Consider the case when $\mathbb{E}^u$ for a non-hyperbolic point is empty, and recall that the linearised version of (1.8) can be written as

$$\dot{y} = Jy. \tag{1.15}$$

Then, note that there is always a coordinate transformation that allows us to write the above as

$$\dot{u} = A_c u + R_c(u, v), \tag{1.16}$$

$$\dot{v} = A_s v + R_s(u, v), \tag{1.17}$$

where $(u, v) \in \mathbb{R}^c \times \mathbb{R}^s$, A_s (A_c) is a matrix having eigenvalues with positive (zero) eigenvalue, and $R_s(u, v)$ and $R_c(u, v)$ both satisfy

$$R_{c,s}(0, 0) = 0, \qquad \nabla R_{c,s}(0, 0) = 0. \tag{1.18}$$

The reader is again referred to [195] for the details of the coordinate transformation. The centre manifold $W^c(0)$ is then defined as

$$W^c(0) = \{(u, v) \in \mathbb{R}^s \times \mathbb{R}^c | v = h(u), h(0) = 0, \nabla h(0) = 0, |u| < \delta\}, \tag{1.19}$$

for δ sufficiently small and h a regular function on $\mathbb{R}^c$.

In order to apply the above to a concrete dynamical system, we rely on three theorems given in [196]:

Theorem 1.1. *There exists a centre manifold for* (1.16)-(1.17)*, and the dynamics of* (1.16)-(1.17) *restricted to the centre manifold is given by*

$$\dot{z} = A_c z + R_c(z, h(z)), \qquad z \in \mathbb{R}^c, \tag{1.20}$$

for $|z|$ *sufficiently small.*

Theorem 1.2. *1. If the zero solution of* (1.20) *is stable (unstable), then the zero solution of* (1.16)-(1.17) *is also stable (unstable).*

2. If the zero solution of (1.20) *is stable and* (u, v) *is a solution of* (1.16)-(1.17) *with* $(u(0), v(0))$ *sufficiently small, then there is a solution* $z(t)$ *of* (1.20) *such that as* $t \to \infty$

$$u(t) = z(t) + \mathcal{O}(e^{-\gamma t}), \tag{1.21}$$

$$v(t) = h(z(t)) + \mathcal{O}(e^{-\gamma t}), \tag{1.22}$$

where $\gamma > 0$ *is some constant.*

Then recall that the centre manifold is defined such that $v = h(u)$. We can then differentiate v with respect to time to obtain

$$\dot{v} = \nabla h(u) \cdot \dot{u}, \tag{1.23}$$

into which we can substitute the expressions in (1.16)-(1.17) to find

$$A_s h(u) + R_s(u, h(u)) = \nabla h(u) \cdot (A_c u + R_c(u, h(u))). \tag{1.24}$$

We can rearrange this to write

$$\nabla h(u) \cdot (A_c u + R_c(u, h(u))) - A_s h(u) - R_s(u, h(u)) \equiv \mathcal{N}(h(u)) = 0. \tag{1.25}$$

In theory this gives us a valid differential equation for $h(u)$, but in practice we employ an approximation given by the final relevant theorem:

Theorem 1.3. *Let* $\phi : \mathbb{R}^c \to \mathbb{R}^s$ *be a map with* $\phi(0) = \nabla\phi(0) = 0$ *such that*

$$\mathcal{N}(\phi(u)) = \mathcal{O}(|u|^q) \text{ as } x \to 0, \tag{1.26}$$

for $q > 1$. *Then*

$$|h(u) - \phi(u)| = \mathcal{O}(|u|^q) \text{ as } x \to 0. \tag{1.27}$$

This implies that we do not need to find an exact solution to (1.25); the approximate solution is sufficient as it encodes precisely the same stability properties for a given critical point.

As an example, consider the system [196]

$$\dot{x} = x^2 y - x^5, \tag{1.28}$$

$$\dot{y} = -y + x^2, \tag{1.29}$$

which has a critical point at $(x_c, y_c) = (0, 0)$, and a corresponding Jacobian given by

$$J = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.30}$$

from which we deduce that $(0, 0)$ is a non-hyperbolic point. The system is in the correct form of (1.16) and (1.17), with $A_c = 0$, $A_s = -1$, $R_c(x, y) = x^2 y - x^5$, $R_s(x, y) = x^2$, so we do not need to transform the system in any way. If we assume a centre manifold of the form

$$h(x) = ax^2 + bx^3 + \mathcal{O}(x^4), \tag{1.31}$$

(1.25) becomes

$$\mathcal{N}(h(x)) \equiv (2ax + 3bx^2 + \mathcal{O}(x^3))(ax^4 + bx^5 + \mathcal{O}(x^6)) + (a - 1)x^2 + bx^3 + \mathcal{O}(x^4) = 0. \tag{1.32}$$

The coefficients of each power of x must vanish. Thus, to third order we have $a = 1$, $b = 0$, and the centre manifold is given by

$$h(x) = x^2 + \mathcal{O}(x^4), \tag{1.33}$$

We can then restrict the dynamics to the centre manifold by using this expression for $h(x)$ in (1.20), which yields

$$\dot{x} = x^4 + \mathcal{O}(x^5). \tag{1.34}$$

Thus, if x is sufficiently small, $\dot{x}$ is positive and the origin is unstable.

Chapter 2

A Review of General Relativity and Cosmology

We begin with the Einstein-Hilbert action in the presence of matter, without a cosmological constant:

$$S = S_{EH} + S_m = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} + \mathcal{L}_m\right), \tag{2.1}$$

where $\kappa^2 = 8\pi G$, [1] R is the Ricci scalar, g is the determinant of the metric, and $\mathcal{L}_m$ is the Lagrangian describing the matter content of the Unvierse. Varying (2.1) with respect to $g^{\mu\nu}$ yields the sourced Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{2.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, and $T_{\mu\nu}$ is the stress-energy tensor for the matter source, given by

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}, \tag{2.3}$$

where $S_m = \int d^4x\mathcal{L}_m$.

Following the cosmological principle, we search for a metric that can describe a homogeneous and isotropic universe, i.e a metric that exhibits maximal spatial symmetry. The prime candidate for such a metric is the *Friedmann-Lemaître-Robertson-Walker* (FLRW) metric, whose line element is given by

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin\theta^2 d\phi^2\right), \tag{2.4}$$

where $a(t)$ is the scale factor, the coordinate t is the cosmic time, (r, θ, ϕ) are the standard comoving coordinates, and k is the spatial curvature, taking the values $k = 0, -1, +1$. A

[1] It is equally as common in the literature to set $\kappa = 8\pi G$, or $\kappa = 1$.

discussion on the derivation and maximal symmetry properties of this metric can be found in any standard textbook on GR or cosmology, including [51].

From (2.4) we can calculate the curvature tensor, $R^{\alpha}{}_{\beta\mu\nu}$, from its definition in terms of the *Christoffel symbol* $\Gamma^{\alpha}{}_{\mu\nu}$:

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}{}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\beta}, \tag{2.5}$$

where

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_{\mu}g_{\nu\beta} + \partial_{\nu}g_{\mu\beta} - \partial_{\beta}g_{\mu\nu}\right). \tag{2.6}$$

We can then calculate $R_{\mu\nu}$ via

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}, \tag{2.7}$$

and R via

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{2.8}$$

We thus have an explicit expression for the left hand side of (2.2), which we supplement with a suitable expression for $T_{\mu\nu}$. A suitable choice is to model the matter content of the universe as a perfect fluid, which is at rest in comoving coordinates. The four-velocity of the fluid is then given by $u^{\mu} = (-1, 0, 0, 0)$ and the energy-momentum tensor is

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + pg_{\mu\nu}, \tag{2.9}$$

where ρ is the energy density and p is the pressure exerted by the fluid. We then define an *equation of state*:

$$p = w\rho, \tag{2.10}$$

where w is the *equation of state (EoS) parameter*. For a dust-like perfect fluid, $w_m = 0$, while for a relativistic fluid $w_r = \frac{1}{3}$. We also note the constraint on w from the *dominant energy condition* [104], which implies that $|w| \leq 1$.

Using (2.2) and (2.9) we obtain the Friedmann equations [83, 84]:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2\rho}{3} - \frac{k}{a^2} = H^2, \tag{2.11}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}(\rho + 3p), \tag{2.12}$$

where we have defined the *Hubble parameter*: $H = \frac{\dot{a}}{a}$. Note that the acceleration condition $\ddot{a} > 0$ implies that the universe will accelerate only when $w < -\frac{1}{3}$, (this will be true even once we have introduced more complicated models involving more than one fluid and extra fields). Thus, a universe comprised entirely of either matter, radiation, or a combination of both will never expand at an accelerating rate. Also of note is the *continuity equation*, derived by imposing

$\nabla_\mu T^{\mu\nu} = 0$:

$$\frac{\dot{\rho}}{\rho} = -3H(1+w), \tag{2.13}$$

which we can integrate to obtain

$$\rho \propto a^{-3(1+w)}. \tag{2.14}$$

Then note that for dust-like ($w_m = 0$) and relativistic ($w_r = \frac{1}{3}$) matter, we have $\rho_m \propto a^{-3}$ and $\rho_r \propto a^{-4}$ respectively. This is our first example of a characteristic of any model involving both matter and radiation; since the matter energy density decreases at a slower rate than the radiation density, matter will necessarily be dominant at some point in the evolution of the universe, even if its initial density is much lower than that of radiation.

2.1 Our Universe

Current observational evidence indicates that our Universe is flat or almost flat, expanding with an accelerated rate of expansion, and dominated by dark energy and dark matter [2]. The presence of a negative pressure dark energy component motivates the introduction of the cosmological constant, Λ, into the field equations (1), since a universe dominated by the cosmological constant produces the desired accelerated expansion, as we shall see. The canonical model of such a universe will hereafter be referred to as the Λ*CDM universe*.

The field equations with the Λ term introduced and with $k = 0$ yield the modified Friedmann equations:

$$H^2 = \frac{\kappa^2\rho}{3} + \frac{\Lambda}{3}, \tag{2.15}$$

$$2\dot{H} + 3H^2 = -\kappa^2 p + \Lambda, \tag{2.16}$$

where we note that we can view Λ as a contribution to the overall energy density via $\rho_\Lambda = \frac{\Lambda}{\kappa^2}$, with a corresponding equation of state $p_\Lambda = w_\Lambda \rho_\Lambda$. Assuming that the Λ term dominates, which is a valid assumption at late times, we can neglect contributions to the energy density from other sources and set $\rho = 0$; this is the so-called *De Sitter Universe*. Then, from (2.15) we immediately obtain $a \propto e^{t\sqrt{\frac{\Lambda}{3}}}$, i.e. the scale factor increases exponentially in time. Thus, $\ddot{a} > 0$ during any period in which Λ dominates.

A more realistic model, of course, includes some combination of matter and the cosmological constant. We can use the expression for ρ in (2.14) in (2.15) to obtain a differential equation for a. This yields

$$a \propto (\sinh(At))^{\frac{2}{3(w+1)}}, \tag{2.17}$$

where A is some constant. The asymptotic behaviour of a defined by the above equation is correct, as it yields de Sitter expansion as $t \to \infty$, and matter domination as $t \to 0$.

2.1.1 Problems with the Cosmological Constant

The cosmological constant is an attractive candidate for modelling the acceleration of the universe; it is a simple addition to the Einstein field equations, and in a very quick calculation we have seen that in the limit $t \to \infty$, the scale factor expansion is accelerating, as required by observation.

The first, and perhaps most significant, problem we encounter when considering the cosmological constant is the catastrophical disagreement between the value of ρ_Λ calculated from observation, and the vacuum energy density predicted by quantum field theory.

Roughly, we have [194] $\rho_\Lambda \approx 10^{-47}\text{GeV}^4$, while the sum of energies from vacuum fluctuations of a given field is $\rho_{QFT} \approx 10^{71}\text{GeV}^4$. This is a discrepancy of almost 120 orders of magnitude, and constitutes perhaps one of the worst disagreements between theory and experiment in the history of physics. A more thorough treatment of the problem [154] ultimately leads us to the conclusion that the *renormalised* value of Λ_{QFT}, the value we will actually measure, is highly sensitive to the mass of other particles. Perhaps even more worrying is that the counterterm, used to cancel divergences in our evaluation of the renormalised Λ_{QFT}, must be extremely finely tuned to obtain the small measured value of Λ. This could perhaps be overlooked if we only consider the one-loop contribution, but as soon as we include loops at higher order the extreme fine-tuning must be performed *ad infinitum*. This is the statement that the cosmological constant is *radiatively unstable* [154]; any fine-tuning performed at a given order is unstable at higher orders.

The second significant problem with Λ is the so-called *coincidence problem*. Put simply, the problem highlights the fact that the current energy densities of dark energy and dark matter are of the same order, i.e $\frac{\rho_{DE}}{\rho_{DM}} \approx \mathcal{O}(1)$. In most models of the universe, this situation is indeed highly coincidental, as a very specific set of initial conditions is required to yield the correct relative energy densities in the present epoch. Some argue that the coincidence is a non-issue; it only becomes problematic under the assumption that we could have existed, with equal likelihood, during any epoch of the cosmic history. This is of course not true, since our existence relies on structure that was not present in the early universe, as well as gravitationally bound systems which may not exist in the future. This also leads us naturally to the anthropic argument: our ability to observe the relative energy densities in the dark sector is entirely reliant on the fact that $\frac{\rho_{DE}}{\rho_{DM}} \approx \mathcal{O}(1)$. In effect, this places an upper bound on the current dark energy density, and can be seen as a partial resolution to the coincidence problem. In this thesis we will focus on models which resolve the coincidence problem without resorting to anthropic arguments, but the reader is referred to [191] for an in-depth discussion on solutions to the problem.

2.2 ΛCDM as a Dynamical System

We now utilise the dynamical systems tools introduced in 1 to study the ΛCDM universe. First consider (2.15) and (2.16) with two contributions to ρ, from both matter and radiation, denoted ρ_m and ρ_r respectively. Rearranging, we obtain

$$3H^2 = \kappa^2(\rho_m + \rho_r) + \Lambda, \tag{2.18}$$

$$2\dot{H} + 3H^2 = -\frac{1}{3}\kappa^2\rho_r + \Lambda, \tag{2.19}$$

where we have used the equation of state for both fluids. As is standard in the literature [33], we introduce the variables

$$x = \Omega_m = \frac{\kappa^2\rho_m}{3H^2}, \qquad y = \Omega_r = \frac{\kappa^2\rho_r}{3H^2}, \qquad \Omega_\Lambda = \frac{\Lambda}{3H^2}, \tag{2.20}$$

and note that (2.18) reduces to a simple constraint:

$$x + y + \Omega_\Lambda = 1. \tag{2.21}$$

This is significant, as we can trivially replace Ω_Λ with an expression involving just x and y. Then, to write down the ΛCDM dynamical system, we take the derivative of x and y with respect to the dimensionless variable $\eta = \ln(a)$ [33], denoted with a prime:

$$x' = \frac{dx}{d\eta} = x(3x + 4y - 3), \tag{2.22}$$

$$y' = \frac{dy}{d\eta} = y(3x + 4y - 4), \tag{2.23}$$

where we have used the constraint (2.21), rearranged (2.19) to obtain an expression for $\frac{\dot{H}}{H^2}$, and noted that $d\eta = Hdt$. These steps are mostly straightforward and hence omitted. We have arrived at our first example of a cosmological dynamical system, and we will now apply the methods of linear stability analysis to investigate the system's behaviour.

Here we should make note of another important dynamical systems concept: the *invariant submanifold*. Informally, any flow on the phase space which lies on an invariant submanifold will always remain on that submanifold. For most linearised systems such as (2.22)-(2.23), the invariant submanifolds represent sections of the phase space that cannot be crossed, and thus divide the space into regions that are not joined by any orbit. For (2.22)-(2.23) there are two invariant spaces: the lines $x = 0$ and $y = 0$. Physically, this implies that vacuum solutions (with $\Omega_m = \Omega_r = 0$) will always remain vacuum solutions.

We now calculate the critical points of the system, given by finding simultaneous solutions to $x' = 0, y' = 0$. These are $(0, 0)$, $(0, 1)$ and $(1, 0)$, labelled $\mathcal{O}, \mathcal{A}$, and $\mathcal{B}$ respectively. We then find an expression for the Jacobian and evaluate it at the critical points:

$$J = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix} = \begin{pmatrix} 6x + 4y - 3 & 4x \\ 3y & 3x + 8y - 4 \end{pmatrix}. \tag{2.24}$$

Thus:

$$J_{\mathcal{O}} = \begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix}, \qquad J_{\mathcal{A}} = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}, \qquad J_{\mathcal{B}} = \begin{pmatrix} 3 & 4 \\ 0 & -1 \end{pmatrix}, \tag{2.25}$$

where the subscript on J indicates which critical point is being used to evaluate the Jacobian. The two eigenvalues of each matrix are $(-3, -4)$, $(1, 4)$, and $(3, -1)$ for $\mathcal{O}, \mathcal{A}$, and $\mathcal{B}$ respectively. Thus, we see that $\mathcal{O}$ is a stable fixed point, $\mathcal{A}$ is unstable, and $\mathcal{B}$ is a saddle point.

In Figure 2.1, we plot numerical solutions to the system (2.22), (2.23), and can clearly see the stability properties of the three fixed points. Note that we have included only the physical range $x > 0, y > 0$, and that the constraint (2.21) is represented by the fact that all the dynamics of the system are enclosed by the triangle joining $\mathcal{O}, \mathcal{A}$, and $\mathcal{B}$. Figure 2.1 also visually encodes the expected behaviour of the ΛCDM model, that is; a universe that is at first dominated by radiation will transition to a matter-dominated epoch, to be followed by dark energy domination at late times. This is also clear from Figure 2.2, where we have plotted the density parameter for each cosmological component against η. Note that in Figure 2.2, we have set the initial conditions such that $\Omega_{\Lambda(0)} \approx 0.7$. This highlights the fact that fine-tuning is required in the ΛCDM model to match the current observed energy densities, and that the coincidence problem is an intrinsic feature of the model.

FIGURE 2.1: ΛCDM system.

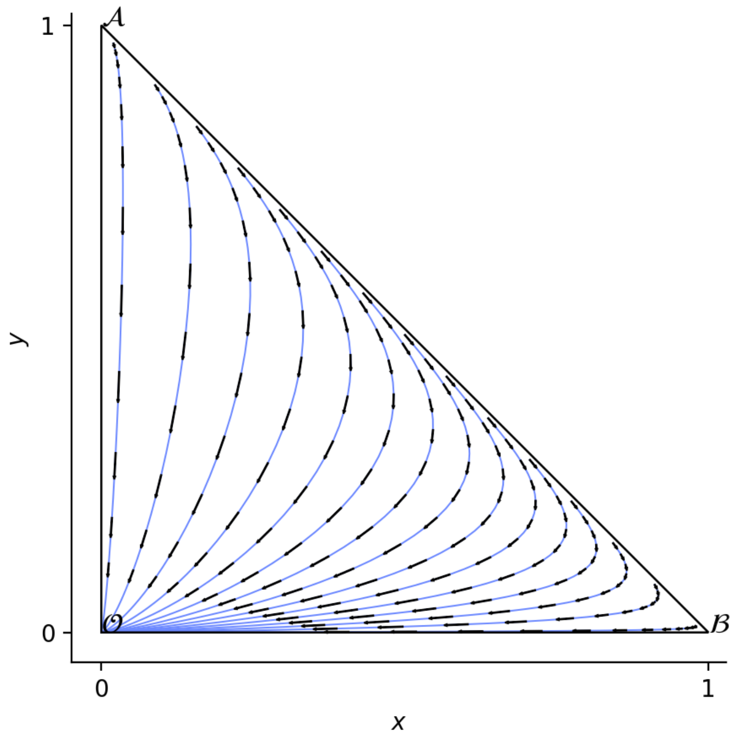
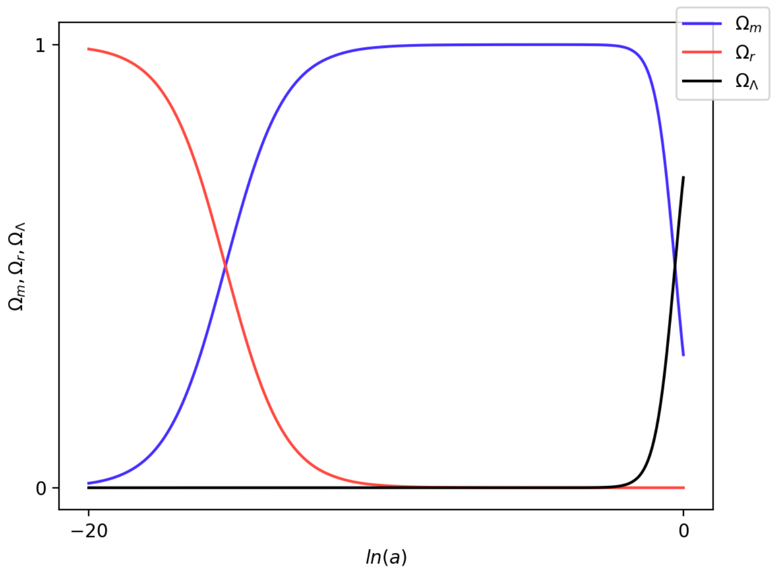


FIGURE 2.2: Energy density evolution for ΛCDM.



Chapter 3

Dynamical Dark Energy: The Scalar Field

We have seen that the cosmological constant provides a largely satisfactory model for the observed Universe, but that it also raises both theoretical and anthropic concerns regarding its value and origin (see Section 2.1.1). Alternative models that would alleviate (either partially or wholly) the cosmological constant problem(s) have been investigated since the problem was discovered [194]. One such model is that of a canonical scalar field that is weakly coupled to gravity, hereafter referred to as *quintessence*. The dynamical scalar field eases both the fine-tuning and coincidence problems that plague the cosmological constant, as a large range of initial conditions can lead to similar late-time behaviour. Also, we will see that solutions exist whereby the energy density of dark energy scales with that of the background fluid for a period of time, thus alleviating the coincidence problem.

We begin with the action for an interacting scalar field weakly coupled to gravity:

$$S_\phi = -\int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right), \tag{3.1}$$

then take the variation S_ϕ with respect to $g^{\mu\nu}$ to obtain the energy-momentum tensor for the scalar field:

$$T^{(\phi)}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - g_{\mu\nu}V(\phi). \tag{3.2}$$

Combining the Einstein-Hilbert term, the background fluid term (which could be matter, radiation, or a combination of both), and the scalar field term yields the total action: $S = S_{EH} + S_B + S_\phi$, from which we can derive the field equations in the presence of a scalar field:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2\left(T^B_{\mu\nu} + T^{(\phi)}_{\mu\nu}\right). \tag{3.3}$$

Using the FLRW metric (2.4) in a flat ($k = 0$) universe, we obtain the modified Friedmann equations

$$3H^2 = \kappa^2 \left(\rho + \frac{1}{2}\dot{\phi}^2 + V(\phi) \right), \tag{3.4}$$

$$2\dot{H} + 3H^2 = -\kappa^2 \left(p + \frac{1}{2}\dot{\phi}^2 - V(\phi) \right), \tag{3.5}$$

where ρ and p are, as before, the energy density and pressure of the barotropic fluid, respectively. We also note that an isotropic and homogeneous scalar field has an energy density, ρ_ϕ, and a pressure, p_ϕ, respectively defined by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{3.6}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{3.7}$$

from which we can deduce the form of the dynamical equation of state parameter

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{3.8}$$

3.1 EN Variables

We are now in a position to choose appropriate variables that will represent our dynamical system, in a process similar to that carried out in Section 2.2. The standard choice, as in [56] and [57], is

$$x = \frac{\kappa\dot{\phi}}{\sqrt{6}H}, \qquad y = \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \qquad \lambda = -\frac{V_{,\phi}}{\kappa V}, \qquad \Gamma = \frac{V V_{,\phi\phi}}{V_{,\phi}^2}, \tag{3.9}$$

where $V_{,\phi} = \frac{dV}{d\phi}$.

Armed with the definitions above, we can then specify a form for the potential $V(\phi)$ and gain considerable insight into the dynamics of the scalar field model. In [56] and [156], two potentials are explored; these are $V = V_0 e^{-\alpha\kappa\phi}$ and $V \propto \phi^{-a}$ respectively. Both models produce "Λ-like" accelerated behaviour at late times, as well as matter and radiation-dominated epochs at early times. We will instead pursue a more general approach, developed in [73] and [207], where we do not specify a potential as the starting point for the model. We instead draw as many conclusions as possible about quintessential dynamics with a generic potential, and only then make contact with specific potentials. We will see that, as we would hope, the generic approach includes the dynamics of models with a concrete form for $V(\phi)$, while also including results that are hidden when we specify the potential.

We again wish to find the derivative of the EN variables with respect to $\eta = \ln(a)$, which yields the following system

$$x' = -3x + \frac{\sqrt{6}}{2}y^2\lambda + \frac{3}{2}x\left[x^2(1-w) + (1+w)(1-y^2)\right], \tag{3.10}$$

$$y' = -\frac{\sqrt{6}}{2}xy\lambda + \frac{3}{2}y\left[x^2(1-w) + (1+w)(1-y^2)\right], \tag{3.11}$$

$$\lambda' = -\sqrt{6}x(\Gamma - 1)\lambda^2 = -\sqrt{6}xf(\lambda)\lambda^2, \tag{3.12}$$

where w is the equation of state parameter for the fluid. We have also used the fact that the Friedmann constraint (3.4) can be written as $1 = \Omega + x^2 + y^2$, where $\Omega = \frac{\kappa^2\rho}{3H^2}$, and that (3.5) can be rewritten to give

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}\left[x^2(1-w) + (1+w)(1-y^2)\right]. \tag{3.13}$$

The system represented by (3.10)-(3.12) is not closed, as Γ depends explicitly on ϕ. We note, however, that if we assume that the function $\lambda(\phi)$ is invertible, we can express Γ purely as a function of λ, and (3.10)-(3.12) becomes a closed, autonomous dynamical system.

We can briefly make contact with the discussion in [56] by noting that the exponential potential corresponds to the case $\Gamma = 1$, $\lambda' = 0$. Thus, our generic treatment leads to an extra dimension in the phase space, and potentially includes solutions that are excluded when we specify a potential.

3.2 Critical Points and Stability

The critical points of the system are again given by finding simultaneous solutions to $x' = 0, y' = 0, \lambda' = 0$. We omit the details of this calculation and simply give the critical points in Table 3.1. We have also calculated the Jacobian matrix, and evaluated its eigenvalues at each point, as in Section 2.2 - the results of the stability analysis are given in Table 3.2. To further understand the cosmological implications of each critical point, we calculate w_ϕ and the effective equation of state, $w_e = \frac{p_{tot}}{\rho_{tot}}$ for every point, noting that

$$w_\phi = \frac{x^2 - y^2}{x^2 + y^2}, \tag{3.14}$$

and

$$w_e = x^2 - y^2 + w(1 - x^2 - y^2). \tag{3.15}$$

p_{tot} and ρ_{tot} are the total pressure and total energy density respectively. In both Table 3.1 and Table 3.2, we have denoted λ_* as any value of λ for which $f(\lambda_*) = 0$, i.e a zero of the function f. λ_a is an arbitrary value of λ, $f'(\lambda) = \frac{\partial f}{\partial \lambda}$, and $A = \sqrt{24(w+1)^2 - (9(w+1) - 2)\lambda_*^2}$.

Table 3.1: Cosmological parameters and stability of each critical point.

Label	(x_c, y_c, λ_c)	w_ϕ	w_e	Existence
$\mathcal{O}$	$(0,0,0)$	Undefined	w	Always
$\mathcal{A}_\pm$	$(\pm 1,0,0)$	1	1	$\lambda^2 f(\lambda)=0$ at $\lambda=0$
$\mathcal{B}$	$(0,1,0)$	-1	-1	Always
$\mathcal{C}$	$(0,0,\lambda_a))$	Undefined	w	Always
$\mathcal{D}$	$(0,0,\lambda_*))$	Undefined	w	Always
$\mathcal{E}_\pm$	$(\pm 1,0,\lambda_*))$	1	1	$\lambda^2 f(\lambda)=0$ at $\lambda=0$
$\mathcal{F}$	$(\frac{\lambda_*}{\sqrt{6}}, \sqrt{1-\frac{\lambda_*^2}{6}}, \lambda_*))$	$\frac{\lambda_*^2}{3}-1$	$\frac{\lambda_*^2}{3}-1$	$\lambda_*^2 < 6$
$\mathcal{G}$	$(\frac{\sqrt{6}(w+1)}{2\lambda_*}, \frac{\sqrt{6(1-w^2)}}{2\lambda_*}, \lambda_*))$	w	w	$3(w+1) \leq \lambda_*^2$

Table 3.2: Critical points and Jacobian eigenvalues for the general scalar field system

Label	(x_c, y_c, λ_c)	Eigenvalues of Jacobian	Stability
$\mathcal{O}$	$(0,0,0)$	$\{0, \frac{3}{2}(w \pm 1)\}$	Saddle for $w<1$, unstable otherwise
$\mathcal{A}_\pm$	$(\pm 1,0,0)$	$\{0,3,3(1-w)\}$	Unstable
$\mathcal{B}$	$(0,1,0)$	$\{0,-3,-3(w+1)\}$	Stable if $f(0)>0$
$\mathcal{C}$	$(0,0,\lambda_a))$	$\{0, \frac{3}{2}(w \pm 1)\}$	Saddle for $w<1$, unstable otherwise
$\mathcal{D}$	$(0,0,\lambda_*))$	$\{0, \frac{3}{2}(w \pm 1)\}$	Saddle for $w<1$, unstable otherwise
$\mathcal{E}_\pm$	$(\pm 1,0,\lambda_*))$	$\{\mp\sqrt{6} f'(\lambda_*)\lambda_*^2, \frac{1}{2}(6 \mp \sqrt{6}\lambda_*), 3(1-w)\}$	Saddle or unstable
$\mathcal{F}$	$(\frac{\lambda_*}{\sqrt{6}}, \sqrt{1-\frac{\lambda_*^2}{6}}, \lambda_*))$	$\{\lambda_*^2 - 3(w+1), \frac{\lambda_*^2}{2}-3, -\lambda_*^3 f'(\lambda_*)\}$	(3.16)
$\mathcal{G}$	$(\frac{\sqrt{6}(w+1)}{2\lambda_*}, \frac{\sqrt{6(1-w^2)}}{2\lambda_*}, \lambda_*))$	$\{-3\lambda_*(w+1)f'(\lambda_*), \frac{3}{4}(w-1) \pm \frac{3\sqrt{1-w}}{4\lambda_*}A\}$	(3.17)

Point $\mathcal{F}$ is stable if

$$\lambda_*^2 < 3(w+1), \qquad \lambda_* f'(\lambda_*) > 0, \tag{3.16}$$

while $\mathcal{G}$ is stable if

$$\frac{24(w+1)^2}{9(w+1)-2} > \lambda_*^2 > 3(w+1). \tag{3.17}$$

3.2.1 Stability of Non-Hyperbolic Points

Note the presence of the non-hyperbolic point $\mathcal{B}$, whose non-vanishing eigenvalues are negative. To fully understand the stability of this point we must employ a method beyond linear stability analysis.

We will reproduce some of the results of [73], to provide a concrete example of the application of centre manifold theory to a real cosmological problem. We begin by making a coordinate transformation $\bar{y} = y - 1$ such that the system (3.10)-(3.12) becomes

$$\lambda' = -\sqrt{6}x(\Gamma - 1)\lambda^2 = -\sqrt{6}xf(\lambda)\lambda^2, \tag{3.18}$$

$$x' = -3x + \frac{\sqrt{6}\lambda}{2}(\bar{y}+1)^2 + \frac{3x}{2}\left(x^2(1-w) - \bar{y}^2(1+w) - 2\bar{y}(1+w)\right), \tag{3.19}$$

$$\bar{y}' = -\frac{\sqrt{6}}{2}\lambda x(\bar{y}-1) + \frac{3}{2}\left(x^2\bar{y}(1-w) - 2\bar{y}(1+w) - \bar{y}^3 - \bar{y}^2(3+w) + x^2(1-w)\right). \tag{3.20}$$

The Jacobian for this system evaluated at point $\mathcal{B}$ is

$$J_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\sqrt{6}}{2} & -3 & 0 \\ 0 & 0 & -3(w+1) \end{pmatrix}. \tag{3.21}$$

If we then construct a matrix $\mathcal{M}$ whose columns are the eigenvectors of $J_{\mathcal{B}}$:

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\sqrt{6}}{6} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.22}$$

the product matrix $\mathcal{M}J_{\mathcal{B}}\mathcal{M}^{-1}$ will be diagonal and in the correct form to apply the centre manifold theorem. If we consider the coordinate transformation

$$\begin{pmatrix} \tilde{\lambda} \\ \tilde{x} \\ \tilde{\bar{y}} \end{pmatrix} = \mathcal{M}^{-1} \begin{pmatrix} \lambda \\ x \\ \bar{y} \end{pmatrix}, \tag{3.23}$$

the dynamics restricted to the centre manifold (and evaluated specifically at point $\mathcal{B}$) are given by

$$\tilde{\lambda}' = \lambda' = -\lambda^3 f(0). \tag{3.24}$$

This implies that the point is stable if $f(0) > 0$, which concludes the stability analysis of point $\mathcal{B}$.

3.2.2 Discussion of Critical Points

Points $\mathcal{O}, \mathcal{C}$ and $\mathcal{D}$ all exist independently of our choice of potential, and represent saddle points in the phase space that the universe will eventually evolve away from. They essentially represent the same point, as λ_a in point $\mathcal{C}$ will include arbitrary values of λ. Points $\mathcal{A}_\pm$ and $\mathcal{E}_\pm$ represent decelerating solutions that are dominated by the scalar field, but whose stability implies that they cannot be late-time attractors. Point $\mathcal{B}$ is an accelerated solution that mirrors canonical dark energy with $w_\phi = -1$. It is either stable or a saddle point, depending on whether $f(0) > 0$. This is significant, as evolution towards this point can be achieved for a wide range of potentials. We can also see that without specifying a potential, we have recovered some of the behaviour required of our model universe; that is, evolution away from the fluid-dominated point $\mathcal{C}$ towards the accelerated point $\mathcal{B}$. Also of significant interest are the points $\mathcal{F}$ and $\mathcal{G}$. $\mathcal{G}$ is the so-called *scaling solution*, where the EoS parameter w_ϕ exactly mimics that of the fluid. The *nucleosynthesis bound*, derived in [24] and [57], provides an upper limit on the energy density of the field at the time of nucleosynthesis: $\Omega_\phi < 0.2$. This bound is satisfied for a wide range of potentials and initial conditions in the scaling regime, which is our first indication that the scalar field model can partially alleviate the fine-tuning issues of canonical ΛCDM.

If we assume that the scaling solution is relevant at the time of nucleosynthesis, we can derive a constraint on λ_*. Firstly, we write Ω_ϕ in terms of x and y:

$$\Omega_\phi = \frac{\kappa^2 \rho_\phi}{3H^2} = x^2 + y^2, \tag{3.25}$$

which we can evaluate at point $\mathcal{G}$: $\Omega_\phi|_\mathcal{G} = \frac{3(w+1)}{\lambda_*^2}$. Then, we assume that radiation is the dominant fluid at nucleosynthesis, for which $w = \frac{1}{3}$. This yields $\Omega_\phi|_\mathcal{G} = \frac{4}{\lambda_*^2}$, and the constraint on λ_* is thus $\lambda_*^2 > 20$.

The scaling regime of point $\mathcal{G}$, while an important feature of the model, cannot provide acceleration at late times for physically acceptable values of w. This naturally leads us to ask; how might the universe transition from the decelerating scaling regime to the current epoch of acceleration? In [207], such a mechanism of transition is proposed, whereby the field's value changes suddenly via spontaneous symmetry breaking and a transition to a de Sitter universe is achieved. This will be discussed briefly in Section 3.3.

The final critical point is point $\mathcal{F}$, which represents a scalar field-dominated solution similar to $\mathcal{B}$. Note that the point does not exist for every model, and that it cannot be stable while $\mathcal{B}$ is stable. The universe will be accelerated at $\mathcal{F}$ only if $\lambda_*^2 < 2$, which is clearly at odds with the nucleosynthesis constraint. This is the first indication that a quintessence model with a single scalar field and simple potential may need to be extended, in order to include viable scaling and accelerated solutions.

3.3 Specifying a Potential

We have already briefly mentioned the specific potentials $V \propto e^{-\alpha\kappa\phi}$ and $V \propto \phi^{-a}$, both of which provide interesting dynamical results.

The exponential potential [56] reduces (3.10)-(3.12) to a two-dimensional system; its five critical points are all copies of the points in Table 3.1, while λ_* becomes a parameter which must be provided in the exact form of the potential. Both the scaling and scalar field-dominated solutions are present in this model, which is partially why the exponential potential is prominent in the literature.

The natural extension of the simple exponential potential is a potential such as [22]

$$V(\phi) = M^4(e^{\alpha\kappa\phi} + e^{\beta\kappa\phi}), \tag{3.26}$$

where M is some mass scale. This form for $V(\phi)$, despite being an almost trivial extension of the exponential potential, remarkably provides the transition from the scaling regime to the accelerated regime at late times. The details of this result, found in [22], also demonstrate that this transitionary behaviour is exhibited for a large range of initial conditions, while the parameters of the theory are quite naturally within observational bounds. We should then ask: instead of choosing a potential as a starting point, can we 'reverse engineer' the model such that we derive $V(\phi)$ by imposing the transitionary behaviour as a requirement?

This question is precisely what motivates the use of the generic approach developed throughout this chapter, as in Table 3.1 we have found generic critical points that are independent of the model in question. In [207], this method is used along with a choice of Γ

$$\Gamma = 1 + \frac{1}{\beta} + \frac{\alpha}{\lambda}, \tag{3.27}$$

to provide a concrete mechanism that allows the universe to transition to de Sitter expansion at late times. This is achieved by considering a region of the phase space known as a *basin of attraction*. Each attractor has such a region, and trajectories within the basin will all converge to the point in question. When point $\mathcal{B}$ is stable, and either $\mathcal{F}$ or $\mathcal{G}$ are also stable, a jump from one basin of attraction to another can occur when a field is introduced whose value changes a sufficient amount in a short time. This is achieved via spontaneous symmetry breaking of the field. This is perhaps a more natural and less contrived transitionary mechanism than the double exponential potential, as symmetry breaking is of course an important phenomenon in particle physics.

There are a whole host of other potentials that have been investigated in the literature; we list some of the more interesting models below:

- $V(\phi) = V_0(\cosh(\lambda\phi - 1))^p$ [173]. This model features scaling solutions, and can extended to include a unified description of dark matter and the quintessence field, given by

$$V(\phi, \psi) = V_\phi(\cosh(\lambda_\phi\phi - 1))^{p_\phi} + V_\psi(\cosh(\lambda_\psi\psi - 1))^{p_\psi}, \tag{3.28}$$

  where ψ is the field representing dark matter;

- $V(\phi) = V_0 e^{1/\phi}$ [73];

- $V(\phi) = V_0 e^{-c\phi}(1 + \alpha\phi)$ [54]. This potential has been used to explore *thawing models*, where the quintessence field EoS parameter is "frozen" at $w_\phi = -1$ until the field's energy density becomes significant. Only after this point does the EoS become dynamical. The authors of [54] use recent observational data to constrain the thawing model;

- $V(\phi) = V_0 e^{\alpha e^{\beta\kappa\phi}}$ [142].

A common feature of almost all the quintessence models we have discussed so far is the relative ease with which the models agree with observational data. We have seen that the coincidence problem can readily be solved by the existence of tracking solutions, and that late-time acceleration is an intrinsic aspect of quintessence. Perhaps most importantly, we have shown that scalar field models are remarkably easy to study using dynamical systems techniques. Quintessence models are thus phenomenologically significant, as their predictions for the dynamics of the universe are relatively easy to extract, and the parameters of the theory can be easily tested against observational data.

3.4 The Non-Quintessential Scalar Field

In the previous section, our starting point for the scalar field model was the familiar scalar field action (3.1) where ϕ is weakly coupled to gravity. Before exploring non-scalar field models of dark energy, we will first investigate some of the non-quintessential frameworks for describing dark energy that have been proposed in the literature.

3.4.1 The Tachyon Field

Tachyonic fields appear naturally in string theory, and the cosmological consequences of the tachyon field have been extensively studied in the literature [28, 91, 148, 178]. The tachyon has been considered explicitly as the source of early-universe inflation [3, 29, 102, 132]. It has also been shown that purely tachyonic inflation can be problematic [117] without a *hybrid* mechanism, whereby the tachyon only becomes relevant in a later epoch, while the early inflation is driven by some other field.

We will of course focus on tachyonic models of dark energy and late-time behaviour [21, 185]. A treatment of tachyonic dark energy using dynamical systems techniques has also been performed in [58], [99] and [124]. Interestingly, the tachyon is coupled to dark matter in [99] and [124]. We will instead pursue the simple approach of a tachyon coupled minimally to gravity, and demonstrate some of the basic features of tachyonic dark energy using this simple model.

We begin with the action describing the tachyon, which is given by the usual Einstein-Hilbert term, the background fluid term, in addition to the *Born-Infield action*:

$$S = S_{EH} + S_B - \int d^4x V(\phi)\sqrt{-det(g_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi)}. \tag{3.29}$$

The field equations in an FLRW background are then given by

$$3H^2 = \kappa^2 \left(\frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}} + \rho_m + \rho_r \right), \tag{3.30}$$

$$\dot{H} = -\frac{\kappa^2}{2}\left(\frac{\dot{\phi}^2 V(\phi)}{\sqrt{1-\dot{\phi}^2}} + \rho_m + \frac{4}{3}\rho_r\right), \tag{3.31}$$

where we have included both a matter and radiation contribution to the background fluid. The equation of motion for the tachyon is then

$$\frac{\ddot{\phi}}{1-\dot{\phi}^2} + 3H\dot{\phi} + \frac{V_{,\phi}}{V(\phi)} = 0. \tag{3.32}$$

We introduce the following dimensionless variables

$$x = \dot{\phi}, \qquad y = \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \qquad z = \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H}, \qquad \lambda = -\frac{V_{,\phi}}{\kappa V^{\frac{3}{2}}}, \qquad \Gamma = \frac{VV_{,\phi\phi}}{V_{,\phi}^2}, \tag{3.33}$$

such that we can recast the field equations as the following dynamical system:

$$x' = (x^2-1)\left(3x - \sqrt{3}\lambda y\right), \tag{3.34}$$

$$y' = \frac{1}{2}\left(-\sqrt{3}\lambda x y^2 + 3y\left(\frac{y^2(x^2-1)}{\sqrt{1-x^2}} + 1 + \frac{z^2}{3}\right)\right), \tag{3.35}$$

$$z' = -2z + \frac{3}{2}z\left(\frac{y^2(x^2-1)}{\sqrt{1-x^2}} + 1 + \frac{z^2}{3}\right), \tag{3.36}$$

$$\lambda' = \sqrt{3}\lambda^2 xy\left(\frac{3}{2} - \Gamma\right). \tag{3.37}$$

We see immediately that (3.37) implies that for $\Gamma = \frac{3}{2}$, λ is a constant. This corresponds to a potential such that

$$V(\phi) \propto \phi^{-2}. \tag{3.38}$$

The cases where $\lambda' \neq 0$ are explored in [45] and [58], but we will focus on the simple case, corresponding to the inverse power-law potential above. The critical points with $\Gamma = \frac{3}{2}$ are given in Table 3.3, along with the values of w_ϕ and Ω_ϕ given by

$$w_\phi = -1 + x^2, \tag{3.39}$$

$$\Omega_\phi = \frac{y^2}{\sqrt{1-x^2}}. \tag{3.40}$$

The points with $x_c = \pm 1$, despite being singular points of the system (3.34)-(3.37), still act as effective critical points. Also, the limiting value of Ω_ϕ as $x \to \pm 1$ must be calculated in order to fully characterise the singular points; this analysis is performed in [188].

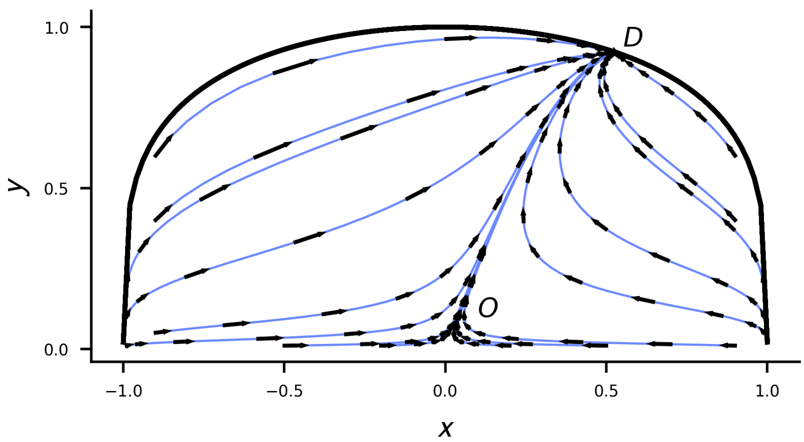
We see that point O is the familiar matter-dominated solution, with the tachyon contribution to the energy density vanishing, with a constant EoS parameter. The point is a saddle and thus

TABLE 3.3: Critical points and stability of the system (3.34)-(3.37) with $\Gamma = \frac{3}{2}$

Label	(x_c, y_c, z_c)	w_ϕ	Ω_ϕ	Stability
O	$(0,0,0)$	-1	0	Saddle
$A_\pm$	$(\pm 1, 0, 0)$	0	undefined	Saddle
$B_\pm$	$(\pm 1, 0, 1)$	0	undefined	Unstable
C	$(0,0,1)$	-1	0	Saddle
D	$\left(\frac{\lambda}{\sqrt{3}}\left(\frac{\sqrt{\lambda^4+36}-\lambda^2}{6}\right)^{\frac{1}{2}}, \left(\frac{\sqrt{\lambda^4+36}-\lambda^2}{6}\right)^{\frac{1}{2}}, 0\right)$	$-1+\frac{\lambda^2}{3}\frac{\sqrt{\lambda^4+36}-\lambda^2}{6}$	1	Stable

repels trajectories along the y-axis. The points $A_\pm$ are also saddle points, and represent matter-dominated solutions. Points $B_\pm$ are the unstable radiation-dominated solutions and represent past attractors. Point C is also radiation dominated but with $w_\phi = -1$, and so the tachyon field mimics a cosmological constant around this point. Finally, point D is the tachyon-dominated solution that always represents the late-time attractor of the system. Acceleration occurs at this point if $\lambda^2 < 2\sqrt{3}$.

FIGURE 3.1: Projection of (3.34)-(3.37) onto the (x, y) plane with $\lambda = 1$. The solid black line represents the Friedmann constraint (3.30) projected onto the (x, y) plane.



The numerical solutions to (3.34)-(3.37) are plotted in Figure 3.1, where we have reduced the system to three dimensions by taking $\lambda = 1$, then projected onto the (x, y) plane. The numerical analysis reveals much of what we have already seen from the linear stability analysis; evolution away from radiation-dominated points and towards either the matter-dominated saddle point or the dark energy point. So, we conclude that there is some sensitivity to initial conditions in the tachyon model, since a matter-dominated epoch is not guaranteed if the universe begins far enough away from the $y = 0$ line.

3.4.2 k-Essence

The origin of k-*essence* was a scalar field model of inflation with a non-canonical action, that allowed for a graceful exit from the inflationary epoch [19]. Naturally, the action of k-*inflation* was applied to the epoch of late-time acceleration in [20] and [53], which resolved the fine-tuning and coincidence problems similarly to quintessence.

The k-essence action is

$$S = S_{EH} + S_B + \int d^4x\sqrt{-g}\,(p(\phi, X))\,, \tag{3.41}$$

where $X = -\frac{1}{2}(\nabla\phi)^2$. This obviously includes quintessence models, but also allows for accelerated expansion and scaling solutions even in the absence of a potential term. Thus we will consider a scalar field action with only kinetic terms, hence the 'k' in k-essence. Such actions are natural in the low-energy effective string action, for which the form of $p(\phi, X)$ is

$$p(\phi, X) = K(\phi)X + L(\phi)X^2, \tag{3.42}$$

where both $K(\phi)$ and $L(\phi)$ are given in terms of the coupling functions in the string action. The field equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2\left(T^B_{\mu\nu} + \frac{\partial p(\phi, X)}{\partial X}\nabla_\mu\phi\nabla_\nu\phi + p(\phi, X)g_{\mu\nu}\right), \tag{3.43}$$

from which we can identify $p(\phi, X)$ as the pressure of the scalar field [19], while its energy density is

$$\rho_\phi = 2X\frac{\partial p}{\partial X} - p(\phi, X). \tag{3.44}$$

Following [53], we redefine the field such that

$$\phi_{new} = \int^{\phi_{old}} d\phi\sqrt{\frac{L(\phi)}{|K(\phi)|}}, \tag{3.45}$$

which allows us to write

$$p(\phi, X) = f(\phi)(-X + X^2), \tag{3.46}$$

where $\phi \equiv \phi_{new}$, $X \equiv X_{new} = \frac{L}{K}X_{old}$ and $f(\phi) \equiv \frac{K^2(\phi_{old})}{L(\phi_{old})}$. Using the redefined field, the equation of state parameter is

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{X-1}{3X-1}, \tag{3.47}$$

while using the field equations we obtain

$$\dot{\rho}_\phi = -\frac{2(1+w_\phi)}{(1+w_B)(t-t_0)}\rho_\phi. \tag{3.48}$$

We should then recall that a scaling solution is such that the equation of state parameter of the field mimics that of the background fluid and remains constant. Under this condition, (3.47) informs us that X will also be constant, and we can then deduce from (3.48) that

$$f(\phi) \propto (\phi - \phi_0)^{-2(1+w_\phi)/(1+w_B)} \tag{3.49}$$

for any scaling solution. Note that the assumption $\rho_B \gg \rho_\phi$ makes the conclusion from (3.49) invalid at late times, when the k-essence field should absolutely be dominating the energy density. We can also state (3.49) conversely; if $f(\phi) \propto \phi^{-\alpha}$, then

$$w_\phi = \frac{(1+w_B)\alpha}{2} - 1. \tag{3.50}$$

To show that the scaling solution is an attractor, we can adopt the dynamical systems methods used to study the canonical scalar field in largely the same manner. The equation of motion for the field during the fluid-dominated epoch ($\rho_B \gg \rho_\phi$) is

$$\ddot{\phi}(1-3\dot{\phi}^2) + \frac{2}{t(1+w_B)}(1-\dot{\phi}^2)\dot{\phi} + \frac{f'}{4f}(2-3\dot{\phi})\dot{\phi}^2 = 0, \tag{3.51}$$

where $f' = \frac{\partial f}{\partial \phi}$. If we make a change of variables [53]

$$\tau \equiv \ln t \tag{3.52}$$

$$u \equiv \frac{\phi}{\phi_s}, \tag{3.53}$$

where ϕ_s is the scaling solution

$$\phi_s = \sqrt{\frac{2(1-w_\phi)}{1-3w_\phi}}\,t \equiv \epsilon_s t, \tag{3.54}$$

we can transform (3.51) into a system of two first-order equations in τ:

$$u' = v \tag{3.55}$$

$$v' = -v + \frac{1}{1-\epsilon_s^2(v+u)^2}\left[\frac{2}{1+w_B}\left(-(v+u) + \epsilon_s^2(v+u)^3\right) + \frac{\alpha}{4u}(v+u)^2\left(2-\epsilon_s^2(v+u)^2\right)\right]. \tag{3.56}$$

The fixed points of the above system in the (u, v) phase space are $(0,0)$, $(\pm 1, 0)$, which we can use to evaluate the Jacobian to determine each point's stability. $(1,0)$ is of course the scaling solution, and evaluating the Jacobian at $(1,0)$ indicates that it is always a stable point. The same is true for the trivial point $(0,0)$, which corresponds to $X = 0$.

Thus, we have seen that a large class of Lagrangians, defined by (3.42) and motivated by the low-energy effective string action, can provide scaling solutions without needing to provide a potential. The numerical analysis of [53] also shows that this occurs for a range of initial conditions, potentially alleviating the coincidence problem. We should note, however, that in [136] the basis of attraction for the scaling solution is found to be much smaller than that of the k-essence-dominated solution. This implies that the model in fact has some sensitivity to initial conditions, at least more so than the quintessence models, and that quintessence is perhaps a preferable solution to the coincidence problem.

3.5 Coupled Scalar Fields and Scaling Solutions

We have so far considered situations in which the scalar field is minimally coupled to gravity, but not coupled to the background fluid at all. In fact, there is every reason to believe that the field couples to matter non-trivially. The strength of this coupling is of course limited, as such an interaction would have been detected on cosmological scales. There are two possibilities for avoiding this problem; firstly, it is possible that the field couples to standard matter differently than to non-standard dark matter [63]. Secondly, the coupling to standard matter could be a small, residual coupling [16] that remains below the level of detection. The second case is the one we will focus on.

Why is the coupled field a worthwhile avenue of investigation? Recall that when we discussed the critical points of Section 3.2, we discovered that the scaling solution (point $\mathcal{G}$ in Table 3.1) cannot provide acceleration for any physically reasonable values of w. However, we will now see that if a non-trivial coupling is introduced, such a scaling solution can in fact produce accelerated dynamics. We should also note that in [16], only a quintessential model with a given potential is investigated; we will instead utilise the general approach of [189], from which we can derive the necessary conditions for the presence of scaling solutions, while allowing for non-canonical scalar fields such as k-essence.

To begin, we generalise (3.41) to include a field-matter coupling

$$S = S_{EH} + S_B[\phi, \Psi_i, g_{\mu\nu}] + \int d^4x\sqrt{-g}\left(p(\phi, X)\right), \tag{3.57}$$

where $X = -\frac{1}{2}(\nabla\phi)^2$ as before, and S_B is now a functional of both the scalar field and matter fields Ψ_i. Variation of the above action with respect to ϕ yields

$$\ddot{\phi}\left(\frac{\partial p}{\partial X} + \dot{\phi}^2\frac{\partial^2 p}{\partial X^2}\right) + 3H\frac{\partial p}{\partial X}\dot{\phi} + \frac{\partial^2 p}{\partial\phi\partial X} - \frac{\partial p}{\partial\phi} = -\sigma, \tag{3.58}$$

where $\sigma = -\frac{1}{\sqrt{g}}\frac{\delta S_B}{\delta\phi}$, characterising the coupling between the field and the background fluid. Using (3.44), we can rewrite (3.58) as

$$\frac{d\rho_\phi}{d\eta} + 3(1+w_\phi)\rho_\phi = -Q\rho_B\frac{d\phi}{d\eta}, \tag{3.59}$$

where $Q = \frac{\sigma}{\rho_B}$. A similar equation holds for the energy density of the background fluid:

$$\frac{d\rho_B}{d\eta} + 3(1+w_B)\rho_B = Q\rho_B\frac{d\phi}{d\eta}. \tag{3.60}$$

As in [189], we consider an effective Friedmann equation given by

$$H^2 = \beta_n^2\rho_T^n, \tag{3.61}$$

where ρ_T is the total energy density of the universe and β and n are constants. Then we recall that a scaling solution is such that the energy density of the scalar field is proportional to that of the background fluid, i.e $\frac{\rho_\phi}{\rho_B} = const$. In terms of the parameter η, this implies

$$\frac{d\log\rho_\phi}{d\eta} = \frac{d\log\rho_B}{d\eta}, \tag{3.62}$$

which we can use along with equations (3.59) and (3.60) to obtain

$$\frac{d\phi}{d\eta} = \frac{3\Omega_\phi}{Q}(w_B - w_\phi). \tag{3.63}$$

Using (3.63), we can then express the scaling condition (3.62) as

$$\frac{d\log\rho_\phi}{d\eta} = \frac{d\log\rho_B}{d\eta} = -3(1+w_e), \tag{3.64}$$

where w_e is the effective equation of state parameter, defined by

$$w_e = w_B + \Omega_\phi(w_\phi - w_B). \tag{3.65}$$

Using the effective Friedmann equation (3.61) and the definition of X, we can write

$$2X = H^2\left(\frac{d\phi}{d\eta}\right)^2 \propto H^2 \propto \rho_T^n. \tag{3.66}$$

Crucially, this implies that the scaling behaviour of ρ_B and ρ_ϕ translates to the same scaling behaviour for X, up to a factor of n, i.e

$$\frac{d\log X}{d\eta} = -3n(1+w_e), \tag{3.67}$$

and since $p_\phi = w_\phi \rho_\phi$, we also have $\frac{d\log p}{d\eta} = -3(1+w_e)$. Combining these results, we obtain a key relation that characterizes scaling solutions:

$$n\frac{\partial \log p}{\partial \log X} - \frac{\Omega_\phi(w_B - w_\phi)}{Q(1+w_e)}\frac{\partial \log p}{\partial \phi} = 1, \tag{3.68}$$

where, as in Section 3.4.2 we have identified the Lagrangian $p(\phi, X)$ as the pressure p_ϕ of the scalar field. If we define a parameter λ such that

$$\lambda = \frac{Q(1+w_e)}{\Omega_\phi(w_B - w_\phi)}, \tag{3.69}$$

then (3.68) can be used to constrain the form of $p(\phi, X)$:

$$p(\phi, X) = X^{\frac{1}{n}} g(Xe^{n\lambda\phi}). \tag{3.70}$$

Also of importance is the deceleration parameter $q = -\frac{\ddot{a}a}{\dot{a}^2}$, which in the scaling regime yields

$$q = \frac{3n}{2}\frac{(1+w_B)\lambda}{\lambda + Q} - 1. \tag{3.71}$$

We see that in a general background, with a general scalar field Lagrangian, there are many possibilities for scaling solutions that are also accelerated. Contrast this with the models we have discussed so far, where the EoS parameter for the field is tightly constrained by requiring acceleration. For example, consider a GR background ($n = 1$) and a pressureless fluid with $w_B = 0$. In this case the acceleration condition $q < 0$ translates into a constraint on the coupling given by

$$Q > \frac{\lambda}{2}. \tag{3.72}$$

The relation (3.70) is the main motivation for closely following [189], as our expression for $p(\phi, X)$ encapsulates both canonical and non-canonical scalar fields, as well as general cosmological backgrounds beyond General Relativity. It is therefore possible to study generically the scaling behaviour of quintessence, tachyon, and k-essence dark energy in the context of the Lagrangian (3.70). In [189], the interesting non-GR case corresponding to $n \neq 1$ is also considered for canonical and non-canonical fields, which highlights the strength of the generic approach outlined throughout this section.

Chapter 4

Non-Scalar Fields

We have focused up to this point on the scalar field as a model of dark energy. This is because models involving the scalar field are relatively easy to construct, while scalars themselves are quite natural in particle physics and string theory. Further, the observational indication that dark energy is currently dominating, but that this hasn't always been the case, is readily achieved by an array of scalar field models.

Of course, the scalar of Chapter 3 does not necessarily fit neatly into any models of particle physics that we know of. We should thus have no reservations about exploring non-scalar models, since they are equally as speculative as the scalar dark energy that is so prevalent in the literature.

Examples of non-scalars include spinor, vector and tensor fields, along with higher-degree differential forms such as the three-form. Of course, one (not necessarily strict) requirement of a non-scalar model is that it has some degree of observational distinction from that of the scalar field, which makes sense if we are looking to eventually reference the predictions of a model against data. Where possible, we will thus pay close attention to the predictive features of each model that make it distinct from the scalar field.

4.1 Vector Cosmology

The vector field has been extensively studied as a potential candidate for inflation [82, 92] and dark energy [35, 115, 122, 141]. It is worthwhile to examine vector dark energy models using the same dynamical systems methods we employed for the scalar, and search for features that distinguish vector dark energy from scalar dark energy.

Two approaches have emerged in the study of vector dark energy. The first [18, 125] aims to construct an isotropic cosmology using the so-called *cosmic triad*: a set of three equal length, mutually orthogonal vectors. Should we wish to consider an FLRW background, the triad is necessary due to the inherent anisotropies generated by the vector field. The second approach [122] is to do away with FLRW cosmology, and instead focus on a Bianchi I universe with a small amount of anisotropy. The second approach is, in fact, in agreement with anomalous CMB anisotropies [123], and is thus a meaningful area of inquiry.

For simplicity, we consider the triad model [18], for which the action is

$$S_A = -\sum_{a=1}^{3} \int d^4x\sqrt{-g}\left(\frac{1}{4}F^2 + V(A^{a2})\right), \tag{4.1}$$

where A^a_μ represents a member of the triad, $F^a_{\mu\nu} = \partial_\mu A^a \nu - \partial_\nu A^a_\mu$, and $F^2 = F^{a\mu\nu}F^a_{\mu\nu}$.

The field equations derived from the action (4.1) are then

$$\partial_\mu\left(\sqrt{-g}F^{a\mu\nu}\right) = 2\sqrt{-g}V_{,A}\, A^{a\nu}, \tag{4.2}$$

and the energy-momentum tensor for the vector is

$$T^A_{\mu\nu} = \sum_{a=1}^{3} T^a_{\mu\nu} = \sum_{a=1}^{3}\left(F^a_{\mu\sigma}F^{a\sigma}_\nu + 2V_{,A}\, A^a_\mu A^a_\nu - g_{\mu\nu}\left(\frac{1}{4}F^a_{\rho\sigma}F^{a\rho\sigma} + V(A^{a2})\right)\right), \tag{4.3}$$

and $V_{,A} = \frac{dV}{dA^{a2}}$. As in [125], we choose an ansatz for the spatial components of the triad that is compatible with the FLRW symmetries:

$$A^a_i = \delta^a_i A(t), \tag{4.4}$$

and a simple choice of potential

$$V = V_0 e^{-\frac{3\lambda A^2}{a^2}}, \tag{4.5}$$

where now $A^2 = A(t)^2$, and a is the scale factor. Then, we include a phenomenological coupling of the form $Q = \frac{3q\rho_B\dot{A}}{a}$, where q is some positive constant, such that the continuity equations are [1]

$$\dot{\rho}_A + 3H(\rho_A + p_A) = -Q, \tag{4.6}$$

$$\dot{\rho}_B + 3H(\rho_B + p_B) = Q, \tag{4.7}$$

and the equation of motion for the field becomes

$$\ddot{A} + H\dot{A} + 2V_{,A}\, A = qa\rho_B, \tag{4.8}$$

[1]This should remind the reader of the coupling in Section 3.5, which was shown to easily provide a solution to the coincidence problem.

where $V_{,A}$ is now defined such that $V_{,A} = \frac{dV}{d\left(\frac{3A(t)^2}{a^2}\right)}$. Lastly, we derive the Friedmann equations from the total action $S = S_{EH} + S_B[A, \Psi_i, g_{\mu\nu}] + S_A$, which in an FLRW background yields [2]

$$H^2 = \frac{1}{3}\left(\frac{3\dot{A}^2}{2a^2} + 3V + \rho_B\right), \tag{4.9}$$

$$\dot{H} = -\frac{1}{2}\left(\frac{2\dot{A}^2}{a^2} + \frac{2V_{,A}A^2}{a^2} + (1+w_B)\rho_B\right). \tag{4.10}$$

We now define dimensionless variables, as in [125]:

$$x = \frac{\dot{A}}{\sqrt{2}Ha}, \qquad y = \frac{\sqrt{V}}{H}, \qquad z = \frac{A}{a}, \qquad \lambda = -\frac{V_{,A}}{V}, \tag{4.11}$$

such that that the Friedmann constraint (4.9) can be written as $\Omega_A + \Omega_B = 1$, where $\Omega_{A,B} = \frac{\rho_{A,B}}{3H^2}$. As usual, we then take the derivative of each variable with respect to $\eta = \ln a$, and the dynamical system is [3]

$$x' = -2x + \sqrt{2}\lambda y^2 z - \frac{3q}{\sqrt{2}}(1 - x^2 - y^2) - x\left(\lambda y^2 z^2 - 2x^2 - \frac{3(1+w_B)}{2}(1 - x^2 - y^2)\right), \tag{4.12}$$

$$y' = -3\lambda yz(\sqrt{2}x - z) - y\left(\lambda y^2 z^2 - 2x^2 - \frac{3(1+w_B)}{2}(1 - x^2 - y^2)\right), \tag{4.13}$$

$$z' = \sqrt{2}x - z. \tag{4.14}$$

The critical points of the system (4.12)-(4.14) are given in Table 4.1. We also calculate the EoS paramters w_A and w_e at each point, defined by [4]

$$w_A = \frac{p_A}{\rho_A} = \frac{x^2 - 3y^2 - 2\lambda y^2 z^2}{3(x^2 + y^2)}, \tag{4.15}$$

$$w_e = \frac{p_{tot}}{\rho_{tot}} = w_B + x^2\left(\frac{1}{3} - w_B\right) - y^2\left(1 + w_B + \frac{2\lambda z^2}{3}\right). \tag{4.16}$$

As usual, we also calculate the Jacobian eigenvalues at each point; these are given in Table 4.2 We see that point $A^\dagger$, corresponding to vector dark energy domination, is accelerated and stable, and we recover the desired dark energy point. Points $B^\dagger_\pm$ are again vector dominated. However, the universe is decelerating in this case, and the equation of state is equal to that of radiation: $w_e = \frac{1}{3}$. Lastly, the saddle point $C^\dagger$ is only valid in the case $w_B \neq \frac{1}{3}$. For a small enough value of q, the vector energy density can be sub-dominant, with the background matter fluid

[2] Here and throughout the rest of this thesis, we set $\kappa^2 = 1$, which aligns much more closely with the literature.

[3] Using the variables defined in [125], the correct form for (4.14) is the one we have given here. The corresponding equation given in [125] appears to be incorrect. This only marginally affects the stability analysis.

[4] It appears that [125] includes a mistake when calculating w_A, as the factor of y^2 is missing in the term $2\lambda y^2 z^2$. Again this does not drastically affect the conclusions we draw from the stability analysis, though it should still be noted.

TABLE 4.1: Critical points, w_A, w_e, and Ω_A for the system (4.12)-(4.14)

Label	(x_c, y_c, z_c)	w_A	w_e	Ω_A
$A^\dagger$	$(0,1,0)$	-1	-1	1
$B^\dagger_\pm$	$(\pm 1, 0, \pm\sqrt{2})$	$\frac{1}{3}$	$\frac{1}{3}$	1
$C^\dagger$	$(\frac{2\sqrt{3}q}{3w_B-1}, 0, \frac{6q}{3w_B-1})$	$\frac{1}{3}$	$w_B - \frac{6q^2}{3w_B-1}$	$\frac{18q^2}{(3w_B-1)^2}$

dominating instead. The saddle nature of this point also means that the universe will naturally evolve from matter to vector domination, as is required at late times, while the absence of a radiation-dominated point implies that the triad model is really only viable for modelling the late-time, asymptotic behaviour.

TABLE 4.2: Jacobian eigenvalues and stability for the system (4.12)-(4.14)

Label	Jacobian eigenvalues	Stability
$A^\dagger$	$\left\{-3(w_B+1), \pm\frac{\sqrt{8\lambda+1}}{2} - \frac{3}{2}\right\}$	Stable if $-\frac{1}{8} < \lambda < 1$, stable spiral for $\lambda < -\frac{1}{8}$
$B^\dagger_\pm$	$\{-1, 2, \pm 3\sqrt{2}q - 3w_B + 1\}$	Saddle
$C^\dagger$	$\left\{-1, -\frac{3(6q^2-3w_B^2-2w_B+1)}{2(3w_B-1)}, -\frac{18q^2-9w_B^2+6w_B-1}{2(3w_B-1)}\right\}$	Saddle

4.1.1 Massive Vectors

Another model for vector-like dark energy involves coupling the vector(s) non-minimally to the gravitational terms in the action, for which the action reads

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2}\left(1-\omega A^2\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2A^2 - \eta A^\mu A^\nu R_{\mu\nu}\right) + S_B, \tag{4.17}$$

where the parameters ω and η define the coupling strength between the field and the Ricci scalar and Ricci tensor respecively, and m defines the mass of the vector. In [35], the authors derive the late-time behaviour of the scale factor and find that it corresponds to de Sitter expansion with a cosmological constant generated by the mass of the vector field. The authors also find that the mass of the vector is given by

$$m \approx 1.67 \times \sqrt{6(4\omega+\eta)} \times 10^{-63}\ \text{g}, \tag{4.18}$$

with constraints on the coupling parameters derived from solar system observations.

4.2 Two-Forms

Models of inflation, driven by a two-form non-minimally coupled to the gravitational terms in the action, have been studied in [89], [112], [121] and [149]. More recently, the authors of [25] have considered a dark energy scenario whereby a two-form is coupled to a canonical scalar field, and given a full dynamical systems analysis of the model. The origin of the scalar field coupling lies in the inflationary scenario studied in [149], where the scalar plays the role of the inflaton, and anisotropic inflation arises due to the presence of the two-form. It is therefore worthwhile to investigate the late-time behaviour of the model, and search for similar anisotropies in the dark energy epoch.

To begin, we note that the field strength tensor for a two form $C_{\mu\nu}$ is given by $H_{\mu\nu\sigma} = 3\partial_{[\mu}C_{\nu\sigma]}$, where square brackets indicate antisymmetrisation on the indices. The action is then given by

$$S = S_{EH} - \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi) + \frac{1}{12}f(\phi)H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} - P_B(Z) \right), \tag{4.19}$$

where ϕ is the familiar canonical scalar field, $V(\phi)$ is its potential, and $Z = -\frac{1}{2}\partial_\mu\chi\partial^\mu\chi$. Note, χ is the field describing the background fluid of the model, e.g. dark matter [90]; it serves the same purpose as the $\mathcal{L}_\mathcal{B}$ term that we are familiar with. $P_B(Z)$ is therefore a purely k-essence Lagrangian. As in the Bianchi I case, we will consider an anisotropic metric given by

$$ds^2 = -N(t)^2dt^2 + a(t)^2\left(e^{-4\sigma(t)}dx^2 + e^{2\sigma(t)}(dy^2 + dz^2)\right), \tag{4.20}$$

where $N(t)$ is the ADM lapse function [95], $a(t)$ is the average scale factor, and $\sigma(t)$ is the shear. Note that the (y, z) rotational symmetry originates from the assumption that

$$C_{\mu\nu}dx^\mu \wedge dx^\nu = 2v(t)dy \wedge dz, \tag{4.21}$$

where $v(t)$ is some function of the cosmic time t.

Rather than varying the action (4.19) with respect to each field and finding the field equations, we can use the metric (4.20) to rewrite the action as

$$S = \int d^4x \left(\frac{3e^{3\alpha}}{N}(\dot{\sigma}^2 - \dot{\alpha}^2) + e^{3\alpha}\left(\frac{\dot{\phi}^2}{2N} - NV(\phi)\right) + \frac{f(\phi)}{2N}e^{-\alpha-4\sigma}\dot{v}^2 + Ne^{3\alpha}P_B(Z) \right), \tag{4.22}$$

from which we can derive the following field equations (with $N = 1$)

$$3H^2(1 - \Sigma^2) = \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_C + \rho_B\right), \tag{4.23}$$

$$\dot{H} + 3H^2\Sigma^2 = -\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{3}\rho_C + \frac{1}{2}(\rho_B + P_B)\right), \tag{4.24}$$

$$H\dot{\Sigma} + (\dot{H} + 3H^2)\Sigma = \frac{2\rho_C}{3}, \tag{4.25}$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} - \frac{f_{,\phi}}{f}\rho_C = 0, \tag{4.26}$$

$$\dot{\rho}_B + 3H(\rho_B + P_B) = 0, \tag{4.27}$$

where $H = \frac{\dot{a}}{a}$ as usual, $\Sigma = \frac{\dot{\sigma}}{H}$, and $f_{,\phi} = \frac{df}{d\phi}$. The energy densities ρ_C and ρ_B are, as usual, defined by identifying terms in the Lagrangian with components of the energy-momentum tensor for both fields respecively, which yields

$$\rho_C = \frac{f(\phi)}{2} e^{-4\alpha - 4\sigma} \dot{v}^2, \qquad \rho_B = \dot{\chi}^2 P_{B,Z} - P_B. \tag{4.28}$$

Note that ρ_B includes a contribution from both matter and radiation, i.e. we can simply write $\rho_B = \rho_m + \rho_r$, as is familiar.

The dimensionless parameters we will use to construct the dynamical system are

$$x_1 = \frac{\dot{\phi}}{\sqrt{6}H}, \qquad x_2 = \frac{\sqrt{V}}{\sqrt{3}H}, \qquad \Omega_C = \frac{\rho_C}{3H^2}, \qquad \Omega_{r,m} = \frac{\rho_{r,m}}{3H^2}, \tag{4.29}$$

which allows us to write the Friedmann constraint (4.23) as

$$\Omega_m + \Omega_r = 1 - x_1^2 - x_2^2 - \Sigma^2 - \Omega_C. \tag{4.30}$$

We will choose the form of both $V(\phi)$ and $f(\phi)$ to be exponential, i.e.

$$V(\phi) \propto e^{-\lambda\phi}, \qquad f(\phi) \propto e^{-\mu\phi}, \tag{4.31}$$

and we can recast (4.23)-(4.27) as the following dynamical system

$$x_1' = \frac{\sqrt{6}}{2}(\lambda x_2^2 - \mu\Omega_C) + \frac{3}{2}x_1(x_1^2 - x_2^2 - 1 + \Sigma^2 - \frac{1}{3}\Omega_C + \frac{1}{3}\Omega_r), \tag{4.32}$$

$$x_2' = \frac{3}{2}x_2(x_1^2 - x_2^2 + \Sigma^2 + 1 - \frac{\sqrt{6}}{3}\lambda x_1 - \frac{1}{3}\Omega_C + \frac{1}{3}\Omega_r), \tag{4.33}$$

$$\Sigma' = \frac{3}{2}\Sigma(x_1^2 - x_2^2 + \Sigma^2 - 1 - \frac{1}{3}\Omega_C + \frac{1}{3}\Omega_r) - 2\Omega_C, \tag{4.34}$$

$$\Omega_C' = 3\Omega_C(x_1^2 - x_2^2 + \Sigma^2 + \frac{4}{3}\Sigma + \frac{1}{3} + \frac{\sqrt{6}\mu x_1}{3} - \frac{1}{3}\Omega_C + \frac{1}{3}\Omega_r), \tag{4.35}$$

$$\Omega_r' = 3\Omega_r(x_1^2 - x_2^2 + \Sigma^2 - \frac{1}{3} - \frac{1}{3}\Omega_C + \frac{1}{3}\Omega_r), \tag{4.36}$$

where a prime denotes the derivative with respect to $\eta = \ln a$. Note that we have eliminated Ω_m from the system by using the Friedmann constraint (4.30). We search for the critical points of the system above by obtaining simultaneous solutions to $x_1' = x_2' = \Sigma' = \Omega_C' = \Omega_r' = 0$. The relevant points are given in Table 4.3, along with the EoS parameters w_{DE} and w_e, and the matter energy density at each point. We should note that in this model, the dark sector consists of both the quintessence-like scalar field ϕ and the two-form $C_{\mu\nu}$, so we define a total dark sector energy density and pressure, denoted ρ_{DE} and p_{DE} respectively:

$$\rho_{DE} = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_C + 3H^2\Sigma^2, \tag{4.37}$$

$$p_{DE} = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{\rho_C}{3} + 3H^2\Sigma^2. \tag{4.38}$$

The dark sector equation of state is then given by

$$w_{DE} = \frac{p_{DE}}{\rho_{DE}} = \frac{3(x_1^2 - x_2^2 + \Sigma^2) - \Omega_C}{3(x_1^2 + x_2^2 + \Sigma^2 + \Omega_C)}, \tag{4.39}$$

and its energy density parameter is

$$\Omega_{DE} = x_1^2 + x_2^2 + \Sigma^2 + \Omega_C. \tag{4.40}$$

The effective EoS parameter is given by

$$w_e = x_1^2 - x_2^2 + \Sigma^2 - \frac{1}{3}\Omega_C + \frac{1}{3}\Omega_r. \tag{4.41}$$

TABLE 4.3: Critical points, w_{DE}, w_e, and Ω_m for the system (4.32)-(4.36)

Label	$(x_{1,c}, x_{2,c}, \Sigma_c, \Omega_{C,c}, \Omega_{r,c})$	w_{DE}	w_e	Ω_m
$\mathcal{A}^*$	$(0,0,0,0,0)$	undefined	0	1
$\mathcal{B}^*$	$(0,0,0,0,1)$	undefined	$\frac{1}{3}$	0
$\mathcal{C}^*$	$(\frac{2\sqrt{6}}{3\lambda}, \frac{2\sqrt{3}}{3\lambda}, 0, 0, 1-\frac{4}{\lambda^2})$	$\frac{1}{3}$	$\frac{1}{3}$	0
$\mathcal{D}^*$	$(\frac{\sqrt{6}}{2\lambda}, \frac{\sqrt{6}}{2\lambda}, 0, 0, 0)$	0	0	$1-\frac{3}{\lambda^2}$
$\mathcal{E}^*$	$(\frac{\lambda}{\sqrt{6}}, \sqrt{1-\frac{\lambda^2}{6}}, 0, 0, 0)$	$-1+\frac{\lambda^2}{3}$	$-1+\frac{\lambda^2}{3}$	0
$\mathcal{F}^*$	$(-\frac{\sqrt{6}\mu}{3\mu^2+8}, 0, -\frac{4}{3\mu^2+8}, \frac{2}{3\mu^2+8}, \frac{3\mu^2+4}{3\mu^2+8})$	$\frac{1}{3}$	$\frac{1}{3}$	0
$\mathcal{G}^*$	$(-\frac{\sqrt{6}\mu}{2(3\mu^2+8)}, 0, -\frac{2}{3\mu^2+8}, \frac{3}{2(3\mu^2+8)}, 0)$	0	0	$\frac{3\mu^2+6}{3\mu^2+8}$
$\mathcal{H}^*$	((4.42), (4.42), (4.42), (4.42), 0)	(4.42)	(4.42)	0

The equations defining the point $\mathcal{H}^*$ are as follows:

$$
\begin{aligned}
x_{1,c} &= \frac{(2\lambda+\mu)\sqrt{6}}{\lambda^2+5\lambda\mu+3\mu^2+8},\\
x_{2,c} &= \frac{\sqrt{3(\lambda\mu+\mu^2+4)(3\mu^2+4\lambda\mu+8)}}{2\lambda^2+5\lambda\mu+3\mu^2+8},\\
\Sigma_c &= -\frac{2(\lambda^2+\lambda\mu-2)}{2\lambda^2+5\lambda\mu+3\mu^2+8},\\
\Omega_{C,c} &= \frac{3(3\mu^2+\lambda\mu+8)(\lambda^2+\lambda\mu-2)}{(2\lambda^2+5\lambda\mu+3\mu^2+8)^2},\\
w_{DE} = w_e &= -1+\frac{2\lambda(\lambda+\mu)}{2\lambda^2+5\lambda\mu+3\mu^2+8}.
\end{aligned}
\tag{4.42}
$$

4.2.1 Stability of Critical Points

As with the pure quintessence model of Chapter 3, we can calculate the Jacobian matrix and evaluate its eigenvalues at each critical point to determine the point's stability. The results of this are given in Table 4.4

TABLE 4.4: Jacobian eigenvalues and stability for the system (4.32)-(4.36)

Label	Jacobian eigenvalues	Stability
$\mathcal{A}^*$	$\{-\frac{3}{2},-\frac{3}{2},-1,1,\frac{3}{2}\}$	Saddle
$\mathcal{B}^*$	$\{-1,-1,1,2,2\}$	Saddle
$\mathcal{C}^*$	$\{-1,-\frac{1}{2\lambda}(\lambda+\sqrt{64-15\lambda^2}),-\frac{1}{2\lambda}(\lambda-\sqrt{64-15\lambda^2}),\frac{2(\lambda+2\mu)}{\lambda},1\}$	Saddle
$\mathcal{D}^*$	$\{-\frac{3}{2},-1,\frac{\lambda+3\mu}{\lambda},-\frac{3}{4\lambda}(\lambda-\sqrt{-7\lambda^2+24}),-\frac{3}{4\lambda}(\lambda+\sqrt{-7\lambda^2+24})\}$	Saddle for BBN constraint
$\mathcal{E}^*$	$\{\frac{\lambda^2}{2}-3,\frac{\lambda^2}{2}-3,\lambda^2+\lambda\mu-2,\lambda^2-4,\lambda^2-3\}$	Stable if $\lambda^2<2$ and $\lambda^2+\lambda\mu-2<0$
$\mathcal{F}^*$	$\{-\frac{1}{2}\pm\sqrt{\frac{-3(7\mu^2+8)}{3\mu^2+8}},-1,\frac{6\mu^2+3\lambda\mu+16}{3\mu^2+8},1\}$	Saddle
$\mathcal{G}^*$	$\{-\frac{3}{4}\pm\sqrt{\frac{-(5\mu^2+8)}{3\mu^2+8}},-\frac{3}{2},-1,\frac{3(3\mu^2+\lambda\mu+8)}{2(3\mu^2+8)}\}$	Saddle
$\mathcal{H}^*$	(4.43)	Stable if $\lambda^2+\lambda\mu-2<0$ and $4\lambda^2-3\mu^2-2\lambda\mu-8<0$

The Jacobian eigenvalues for point $\mathcal{H}^*$ are

$$
\Big\{-\frac{3(3\mu^2+4\lambda\mu+8)}{2\lambda^2+5\lambda\mu+3\mu^2+8},\quad \frac{3(2\lambda^2-3\mu^2-3\lambda\mu-8)}{2\lambda^2+3\mu^2+5\lambda\mu+8},\quad \frac{2(2\lambda^2-6\mu^2-7\lambda\mu-16)}{2\lambda^2+3\mu^2+5\lambda\mu+8},\quad -\frac{3(3\mu^2\pm A+8)}{2(2\lambda^2+3\mu^2+5\lambda\mu+8)}\Big\},
\tag{4.43}
$$

where

$$
A=\sqrt{-44\lambda^3\mu^3-12\lambda\mu^5-(40\lambda^2-33)\mu^4-16\mu^2(\lambda^4+8\lambda^2-13)-128\lambda^2-32\mu(3\lambda^3-4\lambda)+320}.
\tag{4.44}
$$

We can divide all of the critical points into two categories: anisotropic and isotropic. The isotropic points are those for which the shear term Σ vanishes; these are the points $\mathcal{A}^*$ through $\mathcal{E}^*$, while $\mathcal{F}^*$ through $\mathcal{H}^*$ are the anisotropic points.

The point $\mathcal{A}^*$ is a matter-dominated saddle point. $\mathcal{B}^*$ is the corresponding radiation dominated saddle point and likely to only be relevant in the early stages of the universe. Point $\mathcal{C}^*$ is a familiar scaling solution; in this case the dark sector equation of state mimics that of radiation. Note that Ω_{DE} in the radiation scaling regime is constrained, as in the case of the standard quintessence model, by the nucleosynthesis constraint [24, 57]. The authors of [24] give the tighter constraint of $\Omega_{DE} < 0.045$, from which we can derive a lower limit on λ: $\lambda > 9.4$. This signifies a rather steep potential for the field, and is an issue which plagues almost all quintessential models with an exponential potential, as we saw in Chapter 3. Crucially, the two-form dynamical system allows for a separate radiation scaling solution with a less strict lower bound on the coupling parameter μ, as we shall see. The matter scaling solution, point $\mathcal{D}^*$, is also a saddle point if we assume the BBN constraint $\lambda > 9.4$. The dark energy-dominated point $\mathcal{E}^*$ is accelerated if $\lambda^2 < 2$, in which case the point is stable if $\lambda^2 + \lambda\mu - 2 < 0$. The two anisotropic points $\mathcal{F}^*$ and $\mathcal{G}^*$, corresponding to radiation and matter eras respectively, are also saddle points. If we apply the BBN constraint to point $\mathcal{F}^*$, we obtain $\mu > 5.2$, independently of the value of λ; a weaker constraint than that imposed on λ. Lastly, we have the condition $\Omega_C > 0$, which for point $\mathcal{H}^*$ implies that $\lambda^2 + \lambda\mu - 2 > 0$. This directly contradicts the stability condition of the other dark energy solution, point $\mathcal{E}^*$, and implies that if the anisotropic dark energy point $\mathcal{H}^*$ exists, $\mathcal{E}^*$ is a saddle rather than stable.

The natural question to then ask is: what is the likely evolutionary track of the universe given the phase space structure outlined above? This question is analysed in detail in [25], and it is worth outlining the key evolutionary features obtained from the numerical calculations. It is shown that the likely evolution of the coupled two-form model is ($\mathcal{B}^* \to \mathcal{F}^* \to \mathcal{G}^* \to \mathcal{H}^*$), with some dependence on initial conditions. Importantly, it is also demonstrated that w_{DE} oscillates around the value -1 before settling; this is a key feature that distinguishes the two-form model from both ΛCDM and quintessence. Further, the shear term σ is shown to have interesting effects on the CMB quadrupole, well within the current CMB observational bounds.

Thus, the two-form model of dark energy can readily produce the desired evolutionary epochs of the universe. The intrinsic anisotropy of two-form cosmology can also be used to distinguish the model from others using observational data.

4.3 Three-Forms

We conclude our discussion on n-forms by considering the highest degree form that can still be dynamical in $(1+3)$ dimensions; the three-form. Interestingly, three-form cosmology can easily be applied in a standard FLRW background, since the components of the form, which we will denote $A_{\alpha\beta\gamma}$, can be chosen to respect the spatial symmetry of the FLRW metric. As with the lower-degree forms we have discussed, the minimally coupled three-form can be used as a model of both inflation and dark energy, as shown by the authors of [118], [119], and [121]. More recently, a coupling term between the three-form and dark matter has been considered in [120], [143], and [201]. We will perform a dynamical systems analysis of the simplest three-form model, where the form is minimally coupled, before briefly investigating the cosmology of the coupled three-form model.

As usual, we consider a universe comprised of some combination of matter and radiation perfect fluids, as well as the three-form field. The action is then

$$S = S_{EH} + S_B - \int d^4x \left(\frac{1}{48} F^2 + V(A^2) \right), \tag{4.45}$$

where the components of the field strength are given by $F_{\alpha\beta\mu\nu} = 4\partial_{[\alpha} A_{\beta\mu\nu]}$. The field equations are then given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^A_{\mu\nu} + T^B_{\mu\nu}, \tag{4.46}$$

where $T^B_{\mu\nu}$ is the usual energy-momentum tensor for the background fluid, and

$$T^A_{\mu\nu} = \frac{1}{6} F_{\mu\alpha\beta\sigma} F_{\nu}^{\alpha\beta\sigma} + 6V_{,A}\, A_{\mu\alpha\beta} A_{\nu}^{\alpha\beta} - g_{\mu\nu} \left(\frac{1}{48} F^2 + V(A^2) \right), \tag{4.47}$$

where $V_{,A} = \frac{dV}{d(A^2)}$. The three-form can be chosen such that $A_{0\mu\nu} = 0$, with isotropic spatial components which allow the spatial symmetry of the FLRW metric to be respected. This corresponds to choosing $A = a(t)^3 X(t) dx \wedge dy \wedge dz$, where $X(t)$ is a comoving field such that $A^2 = 6X^2$. Thus, the Friedmann equations are

$$H^2 = \frac{1}{3} \left(\frac{1}{2} (\dot{X} + 3HX)^2 + V(X) + \rho_m + \rho_r \right), \tag{4.48}$$

$$\dot{H} = -\frac{1}{2} \left(\frac{dV}{dX} X + \rho_m + \frac{4}{3} \rho_r \right), \tag{4.49}$$

where ρ_r and ρ_m correspond to the energy densities of radiation and baryonic matter respectively.

We can also derive the equation of motion for the form:

$$\nabla_\mu F^{\mu\nu\alpha\beta} - 12 V_{,A}\, A^{\nu\alpha\beta} = 0, \tag{4.50}$$

which we can rewrite in terms of the comoving field X:

$$\ddot{X} + 3(H\dot{X} + \dot{H}X) + \frac{dV}{dX} = 0. \tag{4.51}$$

The energy density and pressure of the field are given by

$$\rho_X = \frac{1}{2}\left(\dot{X} + 3HX\right)^2 + V(X), \tag{4.52}$$

$$p_X = -\frac{1}{2}\left(\dot{X} + 3HX\right)^2 - V(X) + \frac{dV}{dX}X, \tag{4.53}$$

such that the equation of state parameter is

$$w_X = \frac{p_X}{\rho_X} = \frac{dV}{dX}\frac{X}{\rho_X} - 1. \tag{4.54}$$

We now define our dimensionless variables as in [120] and [143]:

$$x = X, \qquad y = \frac{1}{\sqrt{6}}(X' + 3X), \qquad z = \frac{\sqrt{V}}{\sqrt{3}H}, \qquad \Omega_{r,m} = \frac{\rho_{r,m}}{3H^2}, \qquad \lambda = -\frac{V_{,X}}{V}, \tag{4.55}$$

where a prime denotes the derivative with respect to $\eta = \ln a$ and $V_{,X} = \frac{dV}{dX}$. Using these variables, the Friedmann constraint (4.48) becomes

$$1 = y^2 + z^2 + \Omega_r + \Omega_m. \tag{4.56}$$

We can then write the dynamical system as

$$x' = \sqrt{6}y - 3x, \tag{4.57}$$

$$y' = \frac{3}{2}\left(\lambda z^2\left(\sqrt{\frac{2}{3}} - xy\right) + y\left(1 - y^2 - z^2 + \frac{1}{3}\Omega_r\right)\right) \tag{4.58}$$

$$z' = -\frac{1}{2}z\left(\lambda(\sqrt{6}y - 3x) + 3(\lambda z^2 x + y^2 + z^2 - 1 - \frac{1}{3}\Omega_r)\right), \tag{4.59}$$

$$\Omega_r' = 3\Omega_r(\frac{1}{3}\Omega_r - \lambda z^2 x - y^2 - z^2 - \frac{1}{3}), \tag{4.60}$$

$$\lambda' = (3x - \sqrt{6}y)\lambda^2(\Gamma - 1), \tag{4.61}$$

where we have eliminated Ω_m using (4.56), and defined the familiar Γ such that

$$\Gamma = \frac{VV_{,XX}}{V_{,X}^2}. \tag{4.62}$$

We then apply the standard procedure for finding the critical points of the system; these points are given in Table 4.5, along with the corresponding value of Ω_m, and the effective EoS parameter defined by

$$w_e = \frac{1}{3}\Omega_r - y^2 - z^2 - \lambda z^2 x. \tag{4.63}$$

TABLE 4.5: Critical points, w_e, and existence of critical points for the system (4.57)-(4.61)

Label	$(x_c, y_c, z_c, \Omega_{r,c}, \lambda_c)$	w_e	Ω_m
A	$(0, 0, 0, 0, \lambda_*)$	0	1
B	$(0, 0, 0, 1, \lambda_*)$	$\frac{1}{3}$	0
C	$(0, 0, 1, 0, 0)$	$-\frac{2}{3}$	0
$\mathbf{D}_\pm$	$(\pm\sqrt{\frac{2}{3}}, \pm 1, 0, 0, \lambda_*)$	-1	0
$\mathbf{E}_\pm$	$(x_*, \frac{\sqrt{6}x_*}{2}, \pm\sqrt{1-\frac{3x_*^2}{2}}, 0, 0)$	-1	0

Note that x_* and λ_* denote arbitrary values of x and λ. Interestingly, three of the fixed points, **C** and $\mathbf{E}_\pm$, do not exist in the case where $\lambda \neq 0$. The familiar exponential potential, $V(X) = V_0 e^{-\alpha X}$, is one such case.

The points corresponding to accelerated expansion are **C**, $\mathbf{D}_\pm$, and $\mathbf{E}_\pm$, while **A** and **B** are the matter and radiation-dominated points respectively. However, the absence of scaling solutions is worrying; the simple three-form model has no mechanism by which the field can track the EoS of the background fluid, and the coincidence problem is not alleviated.

4.3.1 Stability of Critical Points

The eigenvalues of the stability matrix for the system (4.57)-(4.61) are given in Table 4.6

TABLE 4.6: Jacobian eigenvalues and stability properties for the system (4.57)-(4.61)

Label	Jacobian eigenvalues	Stability
A	$\{-3, -1, 0, \frac{3}{2}, \frac{3}{2}\}$	Saddle
B	$\{-3, 0, 1, 2, 2\}$	Saddle
C	$\{-4, -3, -3, 0, 0\}$	Non-hyperbolic
$\mathbf{D}_\pm$	$\{-4, -3, -3, 0, 0\}$	Non-hyperbolic
$\mathbf{E}_\pm$	$\{-4, -3, -3, 0, 0\}$	Non-hyperbolic

We see that many of the fixed points for the three-form system are non-hyperbolic, with their non-zero eigenvalues being negative. The use of centre manifold techniques to analyse the stability of these points does not seem to have been performed in the literature; this should be performed in future work so that the stability of the three-form system can be fully characterised.

If we assume that the three-form dominated points $\mathbf{D}_\pm$ are stable, we recover the correct de Sitter expansion at late times, following the early-time matter and radiation-dominated epochs. The simple three-form model is therefore a partially viable model of dark energy. However, it does not solve the coincidence problem, and must be ruled out in favour of the more attractive coupled model which we will briefly consider in Section 4.3.3.

4.3.2 The Little Sibling Big Rip

It is well known that phantom dark energy can lead to the so-called *Big Rip* scenario [46], in which the phantom energy density becomes infinite in a finite time, and all structure in the Universe is catastrophically torn apart. The *Little Sibling Big Rip* (LSBR) [38] is a slightly less severe scenario; at an infinite time in the future, the Hubble rate approaches infinity while its derivative does not. In fact, while the singularity occurs in the infinite future, bound structures will still be torn apart at some finite time, hence its designation as the *little sibling* of the Big Rip.

The LSBR is relevant to our discussion on three-forms, since the authors of [139] have shown that an LSBR scenario is a likely feature of the minimally coupled three-form model. We reproduce some of the results of [139] and briefly discuss their implications.

Consider a late-time epoch in which the energy densities of radiation and matter are negligible. We can use the second Friedmann equation (4.49) to rewrite the equation of motion for X (4.51) as

$$\ddot{X} + 3H\dot{X} + \left(1 - \frac{3x^2}{2}\right) V_{,X} = 0. \tag{4.64}$$

Importantly, one static solution of this equation is given by $X_* = \sqrt{\frac{2}{3}}$. This allows us to rewrite the first Friedmann equation (4.48) as

$$H = \frac{1}{3}\frac{1}{X_*^2 - X^2}\left(X\dot{X} \pm |X\dot{X}|\sqrt{1 + (X_*^2 - X^2)\frac{\dot{X}^2 + 2V}{(X\dot{X})^2}}\right), \tag{4.65}$$

where we recall that $\rho_m = \rho_r = 0$. In the limit $X \to X_*$, (4.65) yeilds

$$H \stackrel{(X \to X_*)}{=} \frac{1}{6}\frac{\dot{X}^2 + 2V}{|X_*\dot{X}|}. \tag{4.66}$$

If V is positive, we see that the Hubble parameter diverges as $X \to X_*$. This is our first indication of an LSBR scenario, and the only constraint we have imposed so far is that $V > 0$. The second Friedmann equation (4.49) yields

$$\dot{H} \stackrel{(X \to X_*)}{=} \mp\frac{1}{2}X_* V_{,X}\,|_{\pm X_*} = -\frac{2}{3}X_* V_{,XX}\,|_{\pm X_*}. \tag{4.67}$$

If we assume that the Hubble parameter diverges at X_*, its derivative must therefore be positive at that point. Using the above version of the second Friedmann equation, this implies that an LSBR scenario is possible if $X_* V_{,XX}\,|_{\pm X_*} < 0$.

The authors of [139] show that the LSBR event occurs at an infinite cosmic time for any potentials satisfying $X_* V_{,XX}\,|_{\pm X_*} < 0$, and that an attractive way of avoiding the LSBR is to introduce a dark sector coupling. We will consider such a coupling in the next section.

4.3.3 Coupling to Dark Matter

To conclude the discussion on n-forms, we will consider a basic model whereby the three-form is coupled to dark matter, as in [143]. We have already seen some coupled models, such as the quintessence-matter coupling of Section 3.5. We saw that coupled quintessence models could readily alleviate the coincidence problem, since they provide scaling solutions for a wide class of Lagrangians. Naturally, we should look to couple the three-form in a similar manner to dark matter. The coupling essentially allows for a decay from dark energy into dark matter and vice-versa, so we can quite intuitively understand how the relative energy densities could be similar in the present epoch.

The coupling can be implemented as a term in the action, as we have already seen, or we can simply modify the continuity equation (2.13) for both the comoving field X and the dark matter fluid such that

$$\dot{\rho}_d = -3H\rho_d - Q, \qquad \dot{\rho}_X = -3H\rho_X(1+w_X) + Q, \tag{4.68}$$

where ρ_d is the energy density of dark matter, and Q denotes the flow of energy between the two dark sector components. For simplicity, we consider dark matter to be the only background contribution to the total energy denisty, neglecting both baryonic matter and radiation. The system (4.57)-(4.61) then reduces to

$$x' = \sqrt{6}y - 3x, \tag{4.69}$$

$$y' = q + \frac{3}{2}\left(\lambda z^2\left(\sqrt{\frac{2}{3}} - xy\right) + y\left(1 - y^2 - z^2\right)\right), \tag{4.70}$$

$$z' = -\frac{1}{2}z\left(\lambda(\sqrt{6}y - 3x) + 3(\lambda z^2 x + y^2 + z^2 - 1)\right), \tag{4.71}$$

$$\lambda' = (3x - \sqrt{6})\lambda^2(\Gamma - 1), \tag{4.72}$$

where $\Omega_d = \frac{\rho_d}{3H^2}$, and

$$q = \frac{Q}{\sqrt{6}(\dot{X} + 3HX)H^2}. \tag{4.73}$$

Where we have eliminated the equation for Ω_d using the Friedmann constraint

$$1 = y^2 + z^2 + \Omega_d. \tag{4.74}$$

The task of choosing a form for the coupling Q is complicated. We can begin by noting that in the context of quintessence, the most commonly studied form for Q is $Q \propto \rho\dot{\phi}$, with ρ sometimes being replaced by the Hubble rate H. This coupling model is conformally equivalent to a class of Brans-Dicke Lagrangians [16] and thus quite well motivated, so it is worth exploring whether a similar coupling can be useful for the three-form.

For the three-form, we consider a covariant coupling of the form [143]

$$Q^\mu = \sqrt{\frac{2}{3}}\frac{\beta T_d}{24a^3}(\epsilon \circ F)u^\mu, \tag{4.75}$$

where the circle $\circ$ denotes contracting over all four indices in order, T_d is the trace of the dark matter energy-momentum tensor, β is a constant, and u^μ is the four-velocity of the dark matter fluid. Note that the zero component of the coupling four-vector is precisely the scalar coupling: $Q = Q_0$. This yields

$$Q = \sqrt{\frac{2}{3}}\beta\rho_d(\dot{X} + 3HX), \tag{4.76}$$

which implies that

$$q = \beta\Omega_d. \tag{4.77}$$

If we consider an exponential potential such that $V(X) = V_0 e^{-\alpha X}$ and $\lambda = \alpha$, [5] with the coupling $q = \beta\Omega_d$ the system (4.69)-(4.72) becomes

$$x' = \sqrt{6}y - 3x, \tag{4.78}$$

$$y' = \beta(1 - y^2 - z^2) + \frac{3}{2}\left(\lambda z^2\left(\sqrt{\frac{2}{3}} - xy\right) + y\left(1 - y^2 - z^2\right)\right), \tag{4.79}$$

$$z' = -\frac{1}{2}z\left(\lambda(\sqrt{6}y - 3x) + 3(\lambda z^2 x + y^2 + z^2 - 1)\right). \tag{4.80}$$

The critical points of the above system are listed in Table 4.7, along with the EoS parameter w_e and the stability properties of each point.

Table 4.7: Critical points, w_e, and stability for the coupled three-form system

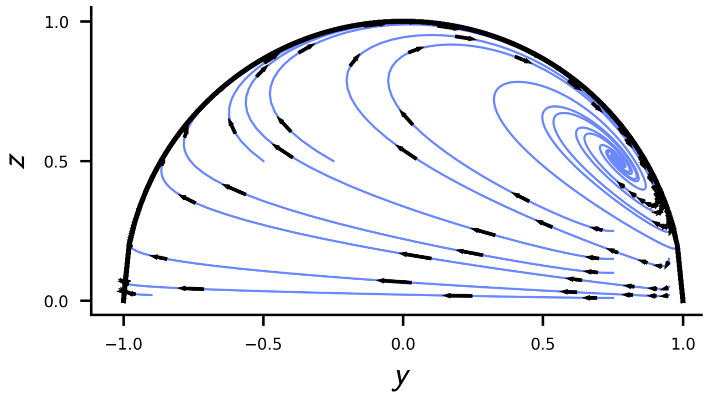
Label	(x_c, y_c, z_c)	w_e	Ω_d	Stability
$\mathbf{A}^*{}_\pm$	$(\pm\sqrt{\frac{2}{3}}, \pm 1, 0)$	-1	0	Non-hyperbolic if $\lvert\beta\rvert > \frac{3}{2}$
$\mathbf{B}^*$	$(-\frac{2\beta}{3}\sqrt{\frac{2}{3}}, -\frac{2\beta}{3}, 0)$	$-\frac{4\beta^2}{9}$	$1 - \frac{4\beta^2}{9}$	Stable if $\lvert\beta\rvert > \frac{3}{2}$
$\mathbf{C}^*$	$\left(-\frac{1}{\beta}\sqrt{\frac{3}{2}}, -\frac{3}{2\beta}, \sqrt{\frac{\frac{9}{2}-2\beta^2}{\sqrt{6}\lambda\beta - 2\beta^2}}\right)$	-1	$-\sqrt{\frac{3}{2}}\frac{\lambda(4\beta^2-9)}{2\beta^2(2\beta-\sqrt{6}\lambda)}$	Figure 4.1

We see that points $\mathbf{A}^*{}_\pm$ correspond to dark energy domination. In the case where $\lvert\beta\rvert > \frac{3}{2}$, the stability of these points cannot be determined by linear stability analysis, and centre manifold theory must be employed. Such an investigation is carried out in [34], although with different variables defining the phase space. Point $\mathbf{B}^*$ is potentially a scaling solution, since the dark energy and dark matter energy densities are of the same order. However, if we impose $\Omega_d < 1$, we obtain the constraint $\lvert\beta\rvert < \frac{3}{2}$, which directly contradicts the stability condition. So, if $\lvert\beta\rvert < \frac{3}{2}$

[5] Gaussian and power-law potentials are discussed in [143].

and point $\mathbf{B}^*$ becomes a saddle, we have a partial solution to the coincidence problem, since the saddle will still attract trajectories along a single axis of the phase space. Lastly, $\mathbf{C}^*$ also represents an accelerated scaling solution. The Jacobian eigenvalues are highly complicated, and a numerical approach is likely to be more fruitful in finding the stability of $\mathbf{C}^*$. In Figure 4.1, we see the strong spiral stability of point $\mathbf{C}^*$ when $\beta = -1.95$. The more detailed analysis of [143] reveals that $\mathbf{C}^*$ is a scaling solution only for $|\beta| > \sqrt{\frac{45}{15}}$, which represents a rather strong coupling in the dark sector. The numerical analysis reveals that under this condition, there is no matter-dominated epoch, which is a troubling conclusion that rules out our simple model.

FIGURE 4.1: Phase space of the coupled three-form system (4.78)-(4.80), where we have projected onto the (y, z) plane and set $\beta = -1.95$, $\lambda = 1$.



Thus, we see that with the coupling (4.75), the three-form has some attractive features such as an accelerated scaling regime. However, the numerical computations of [143] highlight irreconcilable problems that should motivate us to extend the model by, for example, considering different forms of coupling.

4.3.4 Other Coupling Models

We have seen that the coupling (4.75) can yield critical points corresponding to late-time acceleration, but that the scaling critical points are unstable. Our first step in finding a scaling solution could be to try a different potential, as we did for quintessence, or to try different forms for the covariant coupling. The latter option is explored in [143], and the coupling $Q^\mu = -\sqrt{6}\gamma\frac{1}{24a^3}(\epsilon \circ F)u^\mu$ introduces stable solutions where $w_X = w_d$.

Also of note is the recent work of [201], in which the authors consider a generalised Lagrangian, allowing for both a non-canonical three-form and a general coupling function. The action reads

$$S = S_{EH} + \int d^4x \left(-\frac{1}{48}F^2 N(A^2) - I(A^2)\rho_d + \alpha_1(g_{\mu\nu}u^\mu u^\nu + 1) + \alpha_2 \nabla_\mu(\rho_d u^\mu) \right), \qquad (4.81)$$

where $N(A^2)$ is the function that allows for a non-canonical three-form, $I(A^2)$ is the coupling function, and $\alpha_{1,2}$ are constant parameters. The model allows for separate scaling and accelerated solutions, and the authors successfully constrain the model parameters using recent CMB and supernova data. Significantly, using the constrained parameters to calculate the equation of state parameter for X, w_X, results in w_X crossing the phantom boundary at a redshift of $z \approx 0.2$.

The authors of [34] introduce new parameters that compactify the phase space (4.69)-(4.72), and use centre manifold techniques to find the stability of the non-hyperbolic critical points. A simple coupling $Q = \alpha\rho_d H$ is employed.

Lastly, recall that we briefly discussed the LSBR scenario in the previous section, and noted that in [139], such a catastrophe can be avoided by introducing a dark sector coupling. The authors perform a thorough dynamical systems analysis of the coupled model, using a Gaussian potential and including different coupling forms defined by

$$Q = 3H(\rho_d + \rho_X) \sum_{i,j=0} \lambda_{ij} \left(\frac{\rho_d}{\rho_d + \rho_X}\right)^i \left(\frac{\rho_X}{\rho_d + \rho_X}\right)^j, \tag{4.82}$$

where λ_{ij} are the coupling parameters. It is shown that if the coupling has no dependence on the dark matter energy density, ρ_d, the LSBR event can be avoided. When the critical points corresponding to the LSBR are removed, the remaining late-time attractors are the de Sitter attractors that we are familiar with. Further, it is shown that two interesting coupling models, $Q \propto \rho_X$ and $Q \propto \rho_X^2$, can produce very distinct observational signatures. The *statefinder diagnostic* [8, 172] is used to concisely parametrise the disparities between the two models, with the models becoming increasingly divergent at low redshifts.

4.4 Other Non-Scalar Models

4.4.1 Spinor Dark Energy

We have considered in some detail the natural extension to scalar dark energy; n-form cosmologies with $n > 1$. Another equally natural model is that of a spinor field driving the expansion of the Universe. It is likely that spinor dark energy has received far less attention than quintessence or vector models due to the relative difficulty of dealing with spinors in the context of cosmology. This difficulty is partially alleviated by utilising the dynamical systems methods we are now familiar with, and the usual process of stability analysis can, in theory, determine the viability of spinor dark energy models.

A basic spinor cosmology has been investigated in [165]; the authors show that a spinor source is able to drive both the inflationary and dark energy epochs. Spinorial dark matter is considered in [97] and [98], and it is shown that such a matter source can exhibit an equation of state with $w \approx -1$, providing a natural description of cosmic acceleration.

The spinors considered in [97], [98] and [165] are the familiar Dirac spinors. Another class of fermionic field with applications to cosmology is the so-called *ELKO* spinor. First formulated in [4] and expanded upon in [5], [6], [7], and [62] the ELKO (*Eigenspinoren des Ladungskonjugationsoperators*) spinor is an eigenspinor of the charge conjugation operator, and can be written as

$$\lambda = \begin{pmatrix} \pm\sigma_2\phi_L^* \\ \phi_L^* \end{pmatrix}, \tag{4.83}$$

where ϕ_L is the left-handed part of a Dirac spinor in the Weyl representation, σ_2 is the second Pauli matrix, and ϕ_L^* is the complex conjugate of ϕ_L. ELKO spinors also satisfy $(CPT)^2 = -\mathbb{I}$. [6]

ELKO fields have been studied as dark matter candidates [1, 4], as models of inflation [31, 36, 157], and as a dark energy field [32, 37]. There is also an abundance of dynamical systems literature concerned with ELKO cosmology, the earliest example of which appears to be [192]. The author of [192] places emphasis on the existence of scaling solutions, and finds that simple ELKO models with a coupling between the spinor and background matter do not present a solution to the coincidence problem. A similar conclusion is reached by the author of [171] by considering a general self-interaction potential for the ELKO field. In [23], the authors redefine the variables used to describe the ELKO dynamical system, and viable late-time accelerated solutions are found. An inconsistency in [23], regarding a missing factor of 2 in the dynamical system equations, is highlighted by the authors of [158]. They also introduce a promising method of parameterising the potential, finding both scaling solutions and a late-time accelerated phase when the ELKO field is coupled to the background. This analysis is significantly extended in [175]; the authors consider various coupling forms and find that the coincidence problem can indeed be solved within the ELKO framework.

4.4.2 Yang-Mills and Higgs Fields

The final model we will consider is Yang-Mills (YM) dark energy. Of all the possible forms of dark energy we have considered, the YM field is the most closely related to the Standard Model, and thus perhaps the most well-motivated from a particle physics perspective. YM cosmology consists of a YM field, the gauge field of some non-Abelian gauge group, coupled minimally to gravity and in some cases coupled to the background matter.

The coupled model is considered in [202]. The authors show that an $SU(2)$ YM condensate, described by a one-loop effective action, can drive the late-time acceleration of the universe. The coupling also naturally solves the coincidence problem for a wide range of initial conditions, although some fine-tuning is required regarding the energy scale in the effective action. This analysis is extended in [205] and [206], with particular focus on the existence of scaling solutions and the crossing of the phantom barrier $w = -1$. The full dynamical systems treatment of the

[6] Compare this with the usual relation for Dirac spinors: $(CPT)^2 = \mathbb{I}$.

coupled YM model is given in [204], where it is also shown that the scaling attractor corresponds to an effective EoS parameter of $w_e = -1$.

Somewhat related to the pure YM dark energy of [202] is the so-called *Einstein Yang-Mills Higgs* (EYMH) model of [168]. The Higgs field, an $SU(2)$ doublet, along with the correpsonding $SU(2)$ YM field, is considered as a unified model of inflation and dark energy. The acceleration originates from the $SU(2)$ symmetry of the Higgs, which carries an associated charge, which in turn appears in the Friedmann equations to yield acceleration. The stability of the critical points is analysed numerically, with the accelerated critical point being a stable late-time attractor. A very similar analysis is carried out in [167] for the gauge group $SO(3)$, with the numerical solutions again confirming the existence of a late-time accelerated phase, following periods of radiation and matter domination. More recently, the authors of [14] have taken issue with the conclusions of [168]. The complication lies in the interaction term between the Higgs and the gauge field, namely the following section of the EYMH action:

$$S_{EYMH} \subset -\int d^4x\sqrt{-g}(D^\mu\Phi)^\dagger(D_\mu\Phi), \tag{4.84}$$

where D_μ is the gauge covariant derivative and Φ is the Higgs doublet. Such an interaction leads to off-diagonal stress terms in the energy-momentum tensor, and, should we wish to impose isotropy, greatly restricts the form of the Higgs doublet. Despite this, the authors use dynamical systems methods and numerical computations to show that the model is cosmologically viable. Once the Higgs field is gauge fixed, it is shown that radiation, matter, and dark energy-dominated epochs exist, along with an early-time epoch dominated by the Higgs kinetic energy. [7] The authors of [14] then considered the $SO(3)$ gauge group in an anisotropic Bianchi I background in [15]. The critical point corresponding to dark energy is shown to be isotropic, although an observably small amount of shear, well within observational bounds, is still possible.

[7] This should remind the reader of the well-studied purely Higgs inflationary scenarios [170].

Chapter 5

Modified Gravity

We have, up to this point, considered Einstein's General Relativity as the correct theory of gravity on cosmological scales. In Einstein gravity, we have almost exclusively explained the acceleration of the Universe by introducing some content, such as the scalar field, into the Einstein field equations (2.2). GR is, of course, well tested at Solar System scales [147, 163], but at cosmological scales a natural question arises: is the presence of dark energy a symptom of the breakdown of GR? This certainly seems plausible; Newtonian gravity was ousted as the prevailing theory once observation proved GR to be correct, and the same fate could conceivably await GR. Further, we could interpret the numerous pitfalls of the ΛCDM-GR cosmological model, such as the flatness problem, coincidence and fine-tuning problems, and the unexplained dark sector, as an indication that a more capable theory is needed. GR is also famously non-renormalisable. It is therefore insufficient as a quantum theory of gravity, and, much like the cosmological arguments above, signifies the breakdown of GR at both very large and very small scales.

Modified theories of gravity have existed for almost as long as GR itself. Kaluza [113] and Klein [116] developed a five-dimensional theory that unified electromagnetism with the familiar four-dimensional GR. Brans and Dicke [41] developed a theory of gravity in four dimensions whereby the gravitational constant G is replaced by a dynamical scalar field. A whole host of other theories have since been developed, most commonly utilising higher dimensions, extra field content in the gravitational action, non-locality, or terms involving higher-order powers of geometric quantities such as R. We give a (non-comprehensive) list of modifications to GR in Table 5.1.

We will investigate a small number of modified gravity (MG) theories using the dynamical systems toolbox, and demonstrate some of the features that distinguish them from the case of standard GR. In all cases, the simplest formulation of each model is considered for the sake of brevity; this will allow us to cover multiple MG models in sufficient depth. For a detailed consideration of the numerous MG theories in the literature, see [43, 55, 144].

Table 5.1: Theories of modified gravity

Theory	References	Type
Scalar-tensor and Brans-Dicke	[40, 85]	Added field content
TeVeS	[181]	
Massive Gravity	[164]	
Chern-Simons	[12]	
Fab Four	[59]	
$f(R)$	[66, 184]	Higher-order
Hořava-Lifschitz	[153, 183]	
Gauss-Bonnet	[145, 146]	
Kaluza-Klein	[151]	Higher-dimensional
Randall-Sundrum	[152, 160, 161]	

5.1 Brans-Dicke Theory

Brans-Dicke (BD) gravity [41] was originally formulated as a gravitational theory more closely in line with *Mach's Principle* than GR. Mach's Principle, roughly speaking, postulates that the metric tensor is fully determined by the matter distribution of the Universe. To this end, the Brans-Dicke action includes a scalar field ϕ that replaces the gravitational constant:

$$S_{BD} = \int d^4x\sqrt{-g}\left(\frac{\phi R}{2} - \frac{\omega}{2\phi}\partial^\mu\phi\partial_\mu\phi + \mathcal{L}_B\right), \tag{5.1}$$

where ω is the Brans-Dicke parameter. Note that we can view the Brans-Dicke action as a generalisation of GR, and that the gravitational sector is now described by a combination of geometry (R) and a scalar interaction.

We can also include a self-interaction potential for the scalar such that

$$S_{BD} = \int d^4x\sqrt{-g}\left(\frac{\phi R}{2} - \frac{\omega}{2\phi}\partial^\mu\phi\partial_\mu\phi - V(\phi) + \mathcal{L}_m\right). \tag{5.2}$$

Importantly, we have stated the action of BD gravity in the so-called *Jordan frame*, in which the resulting field equations cannot be written in the same form as (2.2). Correspondingly, we could state the BD action in the *Einstein frame*, which is related to the Jordan frame via a conformal transformation: [77]

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = \phi g_{\mu\nu}, \tag{5.3}$$

and a redefinition of the field:

$$\phi \rightarrow \bar{\phi} = \int \frac{(2\omega+3)^{\frac{1}{2}}}{\phi} d\phi. \tag{5.4}$$

We are technically free to choose the frame in which we formulate the theory [81], but it should be noted that the physical equivalence of the two frames is contested [27, 77]. It is worthwhile to continue in the Jordan frame, since it is possible that the phase space of the Einstein frame dynamical system constitutes only a subspace of that of the Jordan frame [13].

The field equations derived from the action (5.2) are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{T_{\mu\nu}}{\phi} + \frac{\omega}{\phi^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}V(\phi) - g_{\mu\nu}\Box\phi\right), \quad (5.5)$$

and

$$\Box\phi = \frac{1}{3+2\omega}(2\phi V_{,\phi}\, 2T), \quad (5.6)$$

where $(\partial\phi)^2 = \partial^\mu\phi\partial_\mu\phi$, $\Box\phi = \nabla^\sigma\nabla_\sigma\phi$, and $T = T^\mu_\mu$. Note that (5.6) is derived by taking the trace of (5.5) to obtain the following expression for R:

$$R = -\frac{1}{\phi}\left(T - \frac{\omega}{\phi}(\partial\phi)^2 - 4V(\phi) - 3\Box\phi\right), \quad (5.7)$$

and thus eliminating R from the equation of motion for the field.

Using the FLRW metric (2.4), the field equations become

$$3H^2\phi + 3H\dot{\phi} = \rho + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi} + V, \quad (5.8)$$

$$2\dot{H}\phi - H\dot{\phi} + \ddot{\phi} + \omega\frac{\dot{\phi}^2}{\phi} = -(1+w)\rho, \quad (5.9)$$

$$\ddot{\phi} + 3H\dot{\phi} = \frac{1}{3+2\omega}(4V - 2\phi V_{,\phi} + \rho(1-3w)), \quad (5.10)$$

where w is the usual EoS parameter for the background fluid. The continuity equation (2.13) is unchanged in BD gravity, since it is derived from imposing covariant conservation of the fluid energy-momentum tensor.

The dimensionless variables used to construct the BD dynamical system are [107]

$$x = \frac{\dot{\phi}}{H\phi}, \qquad y = \frac{1}{H}\sqrt{\frac{V}{3\phi}}, \qquad \lambda = -\phi\frac{V_{,\phi}}{V}, \qquad \Gamma = \frac{VV_{,\phi\phi}}{V_{,\phi}^2}, \quad (5.11)$$

such that the Friedmann constraint (5.8) becomes

$$\Omega = 1 + x - \frac{\omega}{6}x^2 - y^2, \quad (5.12)$$

and the second Friedmann equation is

$$\frac{\dot{H}}{H^2} = -\frac{3y^2(2+\lambda)}{3+2\omega} + 2x - \frac{\omega x^2}{2} - 3(1 + x - \frac{\omega x^2}{6} - y^2)\left(\frac{2+\omega(1+w)}{3+2\omega}\right), \quad (5.13)$$

where $\Omega = \frac{\rho}{3H^2\phi}$. Significantly, the constraint (5.12) implies that the positivity of the BD parameter ω fully determines whether the phase space is compact. For $\omega < 0$, the phase space is unbounded and an asymptotic analysis of the system is required [107]. We will consider only the case $\omega > 0$ for the sake of simplicity.

The BD system is given by

$$
\begin{aligned}
x' = & -3x + \frac{6y^2(2+\lambda)}{3+2\omega} - x^2 + 3(1 + x - \frac{\omega x^2}{6} - y^2)\left(\frac{1-3w}{3+2\omega}\right) \\
& - x\left(-\frac{3y^2(2+\lambda)}{3+2\omega} + 2x - \frac{\omega x^2}{2} - 3(1 + x - \frac{\omega x^2}{6} - y^2)\left(\frac{2+\omega(1+w)}{3+2\omega}\right)\right),
\end{aligned} \tag{5.14}
$$

$$
\begin{aligned}
y' = & -y\left(\frac{x}{2}(1+\lambda)\right) \\
& - y\left(-\frac{3y^2(2+\lambda)}{3+2\omega} + 2x - \frac{\omega x^2}{2} - 3(1 + x - \frac{\omega x^2}{6} - y^2)\left(\frac{2+\omega(1+w)}{3+2\omega}\right)\right)
\end{aligned} \tag{5.15}
$$

$$
\lambda' = x\lambda\left(1 - \lambda(\Gamma - 1)\right). \tag{5.16}
$$

Since we are interested in a simple formulation of BD gravity, we will consider the case where $w = 0$ and $\lambda = -2$, such that $V(\phi) = V_0\phi^2$ [107]. For a full consideration of the BD system with different scalar field potentials, the reader is referred to [106], as well as [169] for a similar discussion in the context of generalised BD gravity.

With $V(\phi) = V_0\phi^2$, the system (5.14)-(5.16) reduces to

$$
\begin{aligned}
x' = & -3x\left(1 + x - \frac{\omega x^2}{6} - \left(1 + x - \frac{\omega x^2}{6} - y^2\right)\frac{2+\omega}{3+2\omega}\right) \\
& + 3(1 + x - \frac{\omega x^2}{6} - y^2)\frac{1}{3+2\omega},
\end{aligned} \tag{5.17}
$$

$$
y' = 3y\left(-\frac{x}{2} + \frac{\omega x^2}{6} + \left(1 + x - \frac{\omega x^2}{6} - y^2\right)\frac{2+\omega}{3+2\omega}\right). \tag{5.18}
$$

The critical points of the simple $\lambda = -2$ BD system are given in Table 5.2, along with the effective EoS parameter w_e, defined by

$$
w_e = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}. \tag{5.19}
$$

Table 5.2: Critical points and Jacobian eigenvalues of the system (5.17)-(5.18)

Label	(x_c, y_c)	w_e	Ω	Jacobian Eigenvalues
$\mathcal{H}$	$(0,1)$	-1	0	$\{-3,-3\}$
$\mathcal{I}$	$\left(-\frac{3}{2}, \sqrt{\frac{3\omega+5}{8}}\right)$	$-\frac{1}{2}$	$-\frac{3}{4}(\omega+\frac{3}{2})$	$\{-\frac{3}{8}\left(1 \pm \sqrt{24\omega+41}\right)\}$
$\mathcal{J}$	$\left(\frac{1}{\omega+1}, 0\right)$	$\frac{1}{3(1+\omega)}$	$\frac{(2\omega+3)(3\omega+4)}{6(\omega+1)^2}$	$\{-\frac{3\omega+4}{2(\omega+1)}, \frac{3\omega+5}{2(\omega+1)}\}$
$\mathcal{K}_\pm$	$\left(\pm\frac{\sqrt{6\omega+9}+3}{\omega}, 0\right)$	$1 + \frac{2}{3\omega}(\pm\sqrt{6\omega+9}+3)$	0	$\{\frac{1}{\omega}(3\omega+3+\sqrt{6\omega+9}), 3+\frac{1}{2\omega}(3+\sqrt{6\omega+9})\}$

As we may have expected, the value and stability of the critical points are heavily dependent on the BD parameter ω. A detailed analysis of the phase space and critical points can be found in [107], we will instead briefly summarize the properties of each point.

Firstly, we should note that points $\mathcal{K}_+$ and $\mathcal{J}$ are unable to provide accelerated expansion in the case $\omega > 0$. In principle, all the other points can represent accelerated expansion, with $\mathcal{H}$ being a good candidate for the late-time accelerated attractor due to its stability. Also of note is the saddle behaviour of the point $\mathcal{J}$ for $\omega > 0$. Since $\mathcal{J}$ represents a point with a non-vanishing matter energy density, any trajectory near this point can be used to model a transition between matter domination and a period of late-time acceleration.

Thus, our formulation of BD gravity with a simple power-law potential can be a viable model of cosmic acceleration. Of course, the investigation into more general potentials is worthwhile [106], as is the possibility of $\omega < 0$, which generates a non-compact phase space and requires a more comprehensive analysis.

5.1.1 Scalar-Tensor Theories

Brans-Dicke gravity is in fact a subclass of a wider range of MG theories known as *scalar-tensor* theories. The scalar-tensor action is a straightforward generalisation of the BD action, given by

$$S_{ST} = \int d^4x\sqrt{-g}\left(\frac{F(\phi)R}{2} - \frac{\omega(\phi)}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi) + \mathcal{L}_B\right), \tag{5.20}$$

where F and ω are arbitrary functions of ϕ.

We will not perform a detailed dynamical analysis of scalar-tensor gravity, but instead list some of the scalar-tensor models that have been studied using dynamical systems techniques. In [50], the authors consider $F(\phi) \propto \phi^2$, $V(\phi) \propto \phi^n$, and show that certain forms for $V(\phi)$ allow for late-time solutions that are equivalent to GR. In [69], the entire cosmic history is considered in the context of scalar-tensor gravity; close attention is paid to the canonical evolutionary track of the universe (inflation $\to$ radiation $\to$ matter $\to$ dark energy), in order to derive conditions on the scalar potential and coupling function $F(\phi)$. In [75], it is shown that the dynamics of scalar-tensor cosmology are non-chaotic in both the Einstein and Jordan frame. Lastly, in [52], the scalar is coupled to dark matter and the parameters of the theory are constrained using observational data. For a more general review on the intricacies of scalar-tensor gravity, the reader is referred to [85].

5.2 $f(R)$ Theory

$f(R)$ gravity is another modification of GR that allows for higher-order terms in the Ricci scalar R. Such theories have been extensively studied in the literature, partly due to their simplicity. The introduction of higher-order curvature terms is also well motivated by the low energy string action, as well as the discovery [190] that higher-order terms permit a renormalisable theory of gravity. For a comprehensive review on the subject of $f(R)$ gravity, the reader is referred to [66] and [184].

As usual, we define the theory via the action:

$$S = \int d^4x\sqrt{-g}\,(f(R) + \mathcal{L}_B)\,. \tag{5.21}$$

There is one subtle difference between the $f(R)$ action and the Einstein-Hilbert action. When deriving the field equations from S_{EH}, there is a freedom to choose between the *first order formalism* and *second order formalism*. The latter is the familiar method, where the metric is treated as the important dynamical field while the connection is completely determined by the metric. The *first order formalism* treats the two as independent variables, and the field equations are derived by varying the action with respect to both. In the case of GR, the two methods are equivalent since there is only a linear term in R; this is clearly not the case for $f(R)$.

Due to the inequivalence of the two approaches in $f(R)$ gravity, it is worth carefully deriving the field equations in both cases. We will hereafter refer to the first order formalism as the *metric approach*, and the second order formalism as the *Palatini approach*.

5.2.1 Metric Approach

In order to take the variation of (5.21) with respect to the metric, we should first bring to light a subtle point that has implicitly been ignored until now. Consider the term $\sqrt{-g}R = \sqrt{-g}g^{\mu\nu}R_{\mu\nu}$. When varying this with respect to $g^{\mu\nu}$, we obtain three individual terms such that

$$\delta(\sqrt{-g}R) = \delta(\sqrt{-g})R + \sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}. \tag{5.22}$$

The first two terms yield the familiar Einstein tensor in the equation of motion, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, while the third term may be unfamiliar. We have implicitly ignored the third term in the case of GR, since it can be rewritten as a total derivative in the action and will therefore vanish.

In BD gravity, we cannot ignore the $\delta R_{\mu\nu}$ term due to the presence of the scalar field; this explains the relative complexity of (5.5) compared to (2.2). The same applies to $f(R)$ gravity due to the non-linearity of terms involving R.

With this in mind, we can vary (5.21) with respect to $g^{\mu\nu}$, which yields

$$f_{,R}\,R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f = T_{\mu\nu}, \tag{5.23}$$

where $f_{,R} = \frac{df}{dR}$. This can be rewritten as

$$G_{\mu\nu} = \frac{1}{f_{,R}} \left(T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \left(f - R f_{,R} \right) + \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right) f \right), \tag{5.24}$$

which is reminiscent of the GR field equations (2.2). In an FLRW background (2.4), the field equations are then [184]

$$H^2 = \frac{1}{f_{,R}} \left(\rho + \frac{R f_{,R} - f}{2} - 3H \dot{R} f_{,RR} \right), \tag{5.25}$$

$$2\dot{H} + 3H^2 = -\frac{1}{f_{,R}} \left(p + \dot{R}^2 f_{,RRR} + 2H \dot{R} f_{,RR} + \ddot{R} f_{,RR} + \frac{1}{2} (f - R f_{,R}) \right), \tag{5.26}$$

while the continuity equation (2.13) of the fluid energy density is the same as for GR.

5.2.2 Palatini Approach

The connection is now treated independently from the metric; as such, we should vary (5.21) with respect to both the connection and the metric. To distinguish this case from the metric approach, we denote $\mathcal{R}_{\mu\nu}$ as the Ricci tensor constructed using the independent connection, and $\mathcal{R}$ as the corresponding Ricci scalar.

We have the identity [184]

$$\delta \mathcal{R}_{\mu\nu} = \bar{\nabla}_\sigma \delta \Gamma^\sigma_{\mu\nu} - \bar{\nabla}_\nu \delta \Gamma^\sigma_{\mu\sigma}, \tag{5.27}$$

where $\bar{\nabla}_\sigma$ denotes the covariant derivative with respect to the independent connection. We can use this when varying (5.21) to obtain

$$f'(\mathcal{R}) \mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = T_{\mu\nu}, \tag{5.28}$$

$$- \bar{\nabla}_\sigma \left(\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu} \right) + \bar{\nabla}_\rho \left(\sqrt{-g} f'(\mathcal{R}) g^{\rho(\mu} \right) \delta^{\nu)}_\sigma. \tag{5.29}$$

Taking the trace of (5.29) yields

$$\bar{\nabla}_\rho \left(\sqrt{-g} f'(\mathcal{R}) g^{\rho\mu} \right) = 0, \tag{5.30}$$

which is the second field equation in the Palatini approach, and in the case of $f(\mathcal{R}) = \mathcal{R}$ reduces to the definition of the usual Levi-Civita connection.

5.2.3 Dynamical Analysis

Most work on $f(R)$ gravity in the context of dynamical systems utilises the metric approach, but we should note that the Palatini approach can be equally as insightful when studying the dynamical evolution of the universe, such as in [78] and [150].

An important choice to make in $f(R)$ gravity is of course the form of $f(R)$ itself. A detailed dynamical analysis has been performed for both $f(R) = R^n$ [49] and $f(R) = R + \alpha R^2$ [137]. If instead the form of $f(R)$ is assumed to be more general, as in [17], [128], [177], and [180], it is possible to derive conditions on $f(R)$ that permit a viable cosmological model. For example, a matter-dominated epoch is only present in $f(R)$ theories if the condition

$$\frac{Rf_{,RR}}{f_{,R}} \approx 0 \quad \text{at} \quad \frac{Rf_{,R}}{f} \approx 1, \tag{5.31}$$

is satisfied [17]. There are also observational constraints on the form of $f(R)$, largely obtained from local gravity tests [76, 108, 186].

Equally important is the choice of dynamical system variables. It is shown in [47] that using the variables introduced in [17] for $f(R) = R^p e^{qR}$ yields problematic singularities in the flow of the phase space. Thus, it is worthwhile to investigate whether it is possible to choose variables that allow a generic treatment of $f(R)$ theories, regardless of the exact form of $f(R)$. To this end, the variables used in [47] are [1]

$$\mathbb{R} = \frac{R}{6H^2}, \qquad \Omega = \frac{\rho}{3H^2 f_{,R}}, \qquad \mathbb{J} = \frac{\mathfrak{j}}{3}, \qquad \mathbb{Q} = \frac{3}{2}\mathfrak{q}, \qquad \mathbb{A} = R_0 H^2, \tag{5.32}$$

where

$$\mathfrak{j} = \frac{H''}{H}, \qquad \mathfrak{q} = \frac{H'}{H}, \tag{5.33}$$

and R_0 is a dimensionful parameter such that RR_0 is dimensionless.

The dynamical system is then

$$\mathbb{R}' = 2\mathbb{R}(2-\mathbb{R}) - \frac{4}{Y}(X - \mathbb{R} - \Omega + 1), \tag{5.34}$$

$$\Omega' = \Omega(2 - 3w + X - 3\mathbb{R} - \Omega), \tag{5.35}$$

$$\mathbb{A}' = -2\mathbb{A}(2-\mathbb{R}), \tag{5.36}$$

where

$$X = \frac{f}{6H^2 f_{,R}}, \qquad Y = \frac{24H^2 f_{,RR}}{f_{,R}}, \tag{5.37}$$

and w is the usual EoS parameter for the background fluid.

Without specifying the function $f(R)$, there is some insight to be gained from the quite general system (5.34)-(5.36). Firstly, note that it possesses the invariant submanifolds $\Omega = 0$ and $\mathbb{A} = 0$, signifying that vacuum states will remain vacuum states, and that any point in the phase space for which $R_0 = 0$ will remain that way. $R_0 = 0$ implies that the gravitational part of the action vanishes. There is also the possibility that $Y(\mathbb{A}, \mathbb{R}) \propto \mathbb{R} y(\mathbb{A}, \mathbb{R})$, for some function

[1] We also use the naming conventions of [47] for the variables.

$y(\mathbb{A}, \mathbb{R})$. In this case, $\mathbb{R} = 0$ is also an invariant submanifold that is potentially singular, and so flows in the phase space with $R = 0$ will stay on the $R = 0$ plane.

It is now worth considering a concrete form for $f(R)$; this will elucidate some of the advantages of choosing the variables in (5.32). The simple toy model defined by $f(R) = aR^n$ is a good candidate, as it contains some of the familiar cosmological epochs as solutions. With $f(R) = aR^n$, the quantities X and Y are

$$Y = \frac{\mathbb{R}}{n}, \qquad Y = \frac{4(n-1)}{\mathbb{R}}, \tag{5.38}$$

and the system (5.34)-(5.36) is

$$\mathbb{R}' = \mathbb{R}\left(2(2-\mathbb{R}) - \frac{1}{n-1}\left(\mathbb{R}\left(\frac{1}{n}-1\right) - \Omega + 1\right)\right), \tag{5.39}$$

$$\Omega' = \Omega\left(2 - 3w - \Omega + \mathbb{R}\left(\frac{1}{n} - 3\right)\right). \tag{5.40}$$

Note that (5.36) is decoupled in this case, and can therefore be ignored. The critical points of the system are given in Table 5.3, and the stability analysis is performed for $w = 0$ for simplicity.

Table 5.3: Critical points and stability of the system (5.39)-(5.40)

Label	$(\mathbb{R}_c, \Omega_c)$	Jacobian Eigenvalues	Stability
$\mathfrak{O}$	$(0,0)$	$\{\frac{4n-5}{n-1}, +2\}$	Saddle for $1 < n < \frac{5}{4}$, unstable otherwise
$\mathfrak{A}$	$(0, 2-3w)$	$\{\frac{4n-3}{n-1}, -2\}$	Stable for $\frac{3}{4} < n < 1$
$\mathfrak{B}$	$\left(\frac{n(5-4n)}{2n^2-3n+1}, 0\right)$	$\{-\frac{8n^2-13n+3}{2n^2-3n+1}, -\frac{4n-5}{n-1}\}$	(5.41)
$\mathfrak{C}$	$\left(\frac{-4n+3w+3}{2n}, \frac{8n^2+3w(2n^2-3n+1)-13n+3}{2n^2}\right)$	$\{\frac{9}{2n}, \frac{12n^2-17n+3}{n^2-n}\}$	(5.42)

Point $\mathfrak{B}$ is stable for

$$\begin{aligned} n > \frac{5}{4}, \quad 8n^2 - 13n + 3 > 0 \quad &\text{or} \quad n < \frac{1}{2}, \quad 8n^2 - 13n + 3 > 0 \\ &\text{or} \quad n > \frac{1}{2}, \quad -8n^2 + 13n - 3 > 0, \end{aligned} \tag{5.41}$$

while $\mathfrak{C}$ is stable for

$$n < 0, \qquad -12n^2 + 17n - 3 > 0. \tag{5.42}$$

The question of whether each critical point corresponds to an accelerated solution is explored in [47]. The numerical solutions for the scale factor are calculated for each point, and it is found that both $\mathfrak{B}$ and $\mathfrak{C}$ are possible accelerated attractors.

We thus see that the dynamics of R^n gravity are highly sensitive to the exact value of n, as one would expect. There are stable critical points which can be relevant for the late-time behaviour of the universe, as well as curvature-dominated solutions with a strong dependence on n. We have considered a simple $f(R)$ model to illustrate the use of variables (5.32). In fact, this choice of variables is preferable over those in [49], since the analysis of more complex forms of $f(R)$ is relatively simple using (5.32).

5.3 Braneworld Cosmology

We have already mentioned some cosmological models with foundations in string theory, such as the tachyonic dark energy of Section 3.4.1. *Braneworld cosmology* is a class of MG theories that confines the particles of the Standard Model to a (3+1) dimensional *brane* embedded in a higher-dimensional *bulk* with compactified dimensions. Significantly, gravity is not confined to the brane and can propagate through the extra dimensions. The breadth of research into braneworld cosmology is such we can only cover a small subsection of it in detail, thus the reader is referred to [42], [133], and [152] for a more comprehensive account of the many facets of braneworld theories.

The *DGP*(Dvali-Gabadadze-Porrati) braneworld model [67, 70] considers a bulk consisting of infinite and flat extra dimensions, and recovers Newtonian gravity at short distances. The model is such that at cosmological scales, the modification to gravity negates the need for a cosmological constant, and is thus an attractive and well-studied MG theory. The model has been challenged by CMB and supernovae data [74], but is still a worthwhile avenue of investigation from a theoretical standpoint.

Another important class of braneworld cosmologies are the *Randall-Sundrum* (RS) type I and II models. The RS type I model (RS1) is largely an attempt to solve the Hierarchy Problem [161]. This is achieved via the embedding of two 3-branes in a five-dimensional bulk, with one of the branes containing the Standard Model particles. The RS2 model [160] removes one of the 3-branes, and recovers Newtonian gravity as well as GR as limiting behaviour of the universe. Since the DGP model can produce effects on the late-time evolution of the universe, it is worth investigating whether RS2 models can similarly provide an alternative to dark energy. We will use dynamical systems techniques to study the entire history of the universe with RS2 gravity. In [94] and [131] such an analysis is performed with the inclusion of a scalar field confined to the brane. It is shown that the canonical scalar only affects the early-time behaviour of the universe, and that inflationary critical points exist for a constant scalar potential. In [72], centre manifold theory is used with a wide variety of potentials to study the asymptotic behaviour of RS2 models. We will introduce a scalar described by a general action, as in [68] and [162]; this will allows us to describe phantom and quintessence fields in a unified manner.

The total action of the RS2 model (including a background fluid term and the scalar) is

$$\begin{aligned} S &= S_{RS} + S_\phi + S_B \\ &= \int d^5x\sqrt{-g^{(5)}}\left(2R^{(5)} + \Lambda^{(5)}\right) + \int d^4x\sqrt{-g}\left(\lambda - \frac{1}{2}\mu(\phi)(\nabla\phi)^2 - V(\phi) + \mathcal{L}_B\right), \end{aligned} \tag{5.43}$$

where $R^{(5)}$, $g^{(5)}_{\mu\nu}$ and $\Lambda^{(5)}$ are the bulk Ricci scalar, metric, and cosmological constant respectively. λ is the tension on the 3-brane, $g_{\mu\nu}$ is the 3-brane metric, and $\mu(\phi)$ is the scalar coupling function. Note also the absence of the κ term, this is because the bulk gravitational constant has been chosen such that $(\kappa^{(5)})^2 = 1$. We assume that the brane metric is the usual FLRW metric (2.4), which yields the Friedmann equation [133]:

$$3H^2 = \rho_T \left(1 + \frac{\rho_T}{2\lambda}\right), \tag{5.44}$$

where $\rho_T = \rho_\phi + \rho_B$, and the bulk cosmological constant has been set to zero for simplicity. The second Friedmann equation is

$$2\dot{H} = -\left(1 + \frac{\rho_T}{\lambda}\right)(\mu(\phi)\dot{\phi}^2 + \rho_B), \tag{5.45}$$

and the equation of motion for the scalar is

$$\mu(\phi)\ddot{\phi} + \frac{1}{2}\mu_{,\phi}\,\dot{\phi}^2 + 3H\mu(\phi)\dot{\phi} + V_{,\phi} = 0. \tag{5.46}$$

We utilise the following variables, introduced in [94]:

$$x = \frac{\dot{\phi}}{\sqrt{6}H}, \qquad y = \frac{\sqrt{V}}{\sqrt{3}H}, \qquad z = \frac{\rho_T}{3H^2}, \tag{5.47}$$

which yields

$$\frac{\rho_T}{\lambda} = \frac{2(1-z)}{z}, \tag{5.48}$$

and thus the following constraint on z: $0 \leq z \leq 1$. Note that the low-energy limit, where $\lambda \to \infty$, corresponds to $z = 1$. Brane effects are important in the inverse limit, $\lambda \to 0$, where $z \to 0$.

The Friedmann constraint, written using the variables (5.47), yields

$$\Omega_B = z - x^2 - y^2. \tag{5.49}$$

If we choose the background fluid to be dark matter, such that $w_B = 0$, the dynamical system is

$$x' = -\sqrt{\frac{3}{2\mu}}(\ln V)_{,\phi}\, y^2 - 3x + \frac{3}{2}x\left(z + x^2 - y^2\right)\left(\frac{2-z}{z}\right), \tag{5.50}$$

$$y' = \sqrt{\frac{3}{2\mu}}(\ln V)_{,\phi}\, xy + \frac{3}{2}y\left(z + x^2 - y^2\right)\left(\frac{2-z}{z}\right), \tag{5.51}$$

$$z' = 3(1-z)(z + x^2 - y^2). \tag{5.52}$$

Since we are looking for a simple dark energy model in RS2 gravity, we can choose a potential and coupling function such that $\sqrt{\frac{3}{2\mu}}(\ln V)_{,\phi} = \beta$, where β is a constant. The critical points of the system (5.50)-(5.52) with $\sqrt{\frac{3}{2\mu}}(\ln V)_{,\phi} = \beta$ are given in Table 5.4. We also calculate the scalar field EoS parameter at each point, given by

$$\omega_\phi = \frac{x^2 - y^2}{x^2 + y^2}, \tag{5.53}$$

and the scalar field energy density parameter, $\Omega_\phi = x^2 + y^2$. The deceleration parameter is given by

$$q = -1 - \frac{\dot{H}}{H^2} = -1 + \frac{3}{2}\left(\frac{2-z}{z}\right)(z + x^2 - y^2). \tag{5.54}$$

TABLE 5.4: Critical points, ω_ϕ, Ω_ϕ, q, and stability of the system (5.50)-(5.52)

Label	(x_c, y_c, z_c)	ω_ϕ	Ω_ϕ	Stability	q
$\bar{O}$	$(0,0,0)$	undefined	0	Unstable	2
$\bar{A}_\pm$	$(\pm 1, 0, 1)$	1	1	Saddle	2
$\bar{B}$	$(0,0,1)$	undefined	0	Saddle	$\frac{1}{2}$
$\bar{C}$	$(-\frac{\beta}{3}, \sqrt{1-\frac{\beta^2}{9}}, 1)$	$\frac{2\beta^2}{9} - 1$	1	Stable for $\beta^2 < \frac{9}{2}$	$\frac{\beta^2}{3} - 1$
$\bar{D}$	$(-\frac{3}{2\beta}, \frac{3}{2\beta}, 1)$	0	$\frac{9}{2\beta^2}$	Stable for $\frac{\sqrt{36-7\beta^2}}{\beta} < 1$	$\frac{1}{2}$

The results of the critical point analysis are significant. We see that point $\bar{O}$, the only point for which the brane is important, is decelerated and unstable. The saddle points $\bar{A}_\pm$ are dominated by the scalar field, and again correspond to decelerated expansion. Point $\bar{B}$ corresponds to dark matter domination, and is also decelerated. Point $\bar{D}$ is the matter scaling solution, since the field's EoS parameter mimics that of the background dust ($\omega_\phi = 0$). Since it is stable, we have an elegant solution to the coincidence problem, as the universe will naturally evolve towards a state where the dark energy density is comparable to that of dark matter. The only relevant point for the late-time accelerated phase is therefore point $\bar{C}$. Since $z_c|_{\bar{C}} = 1$, the brane effects are lost and we recover four-dimensional dark energy. Thus, our simple RS2 model, while theoretically interesting, does not appear to present any distinguishing late-time characteristics over the GR-scalar model. The existence of stable scaling and accelerated solutions is, however, an attractive feature that should be investigated further.

The above is a demonstration of the power of dynamical systems techniques; we have seen, with some simple calculations, that the RS2-scalar model cannot be distinguished from GR at late times. Of course, we have considered only a simple formulation of RS2 gravity, and it is possible that some other combination of scalar potentials could yield dynamics that are dependent on the brane. Such an analysis, with exponential and power-law forms of the potential, is performed in [68]. It should also be noted that RS2 can be a viable model for the early inflationary era [162], with the brane having a significant effect at early times.

5.4 Other MG Models

We now briefly discuss a small number of MG theories with applications to dark energy, and give a summary of the dynamical systems literature that exists for each theory.

5.4.1 Massive Gravity

Perhaps less exotic than braneworld cosmology, *Massive Gravity* (MaG) is a theory of gravity whereby the graviton gains a non-zero mass. The intuitive leap towards MaG thus seems almost trivial; there are massive force carriers in the Standard Model, why should gravity be any different? Of course, the process of giving mass to the graviton is rather more involved, and has been investigated in great detail since Fierz and Pauli derived a theory of massive spin-2 particles in 1939 [80]. The massive graviton is an attractive prospect in cosmological contexts, since the decaying strength of gravity at large distances may negate the need for dark energy.

A major difficulty encountered with MaG is the presence of the Boulware–Deser ghost [39], which until recently was thought to be present in any theory of massive gravity. An explicitly ghost-free MaG theory, the dRGT model, was formulated in [164], and has lead to an increased interest in massive gravity in recent years. The cosmological repercussions of massive gravity have been investigated in [60], [61], [65], [93], [100], and [126]. A dynamical systems analysis of a variety of MaG theories has been performed in [86], [101], [129], [155], [174], and [197].

A significant result, derived in [65], is that isotropic and homogeneous solutions in the dRGT model exhibit instabilities. There are two solutions to this difficulty that have been considered in the literature. The first is to study inhomogeneous and/or anisotropic cosmologies that can be closed, open, or flat [96, 101, 100]. This is not necessarily at odds with observation: in the limit of the graviton mass going to zero, it is possible to find solutions corresponding to an FLRW universe, as well as solutions that become increasingly homogeneous and isotropic. The second method of avoiding the FLRW instabilities is to consider the extended dRGT model, outlined in [109], where the graviton mass is determined by a dynamical scalar field. The dynamical analysis [129, 198] reveals a rich phenomenology and a potentially valuable area of investigation. In [129], it is shown that mass-varying MaG exhibits the desired accelerated late-time behaviour, and that the effective dark energy EoS parameter can be either quintessence or phantom-like. The authors also demonstrate that the graviton mass approaches zero asymptotically in the extended dRGT model, and that the model possesses critical points that can be used to solve the coincidence problem.

5.4.2 Teleparallel Gravity

Einstein himself formulated *teleparallel gravity* as a means of unifying gravity and electromagnetism [71], while modern teleparallelism is considered solely as a theory of gravity. More specifically, gravitation is treated as a gauge theory of the translation group [10, 64].

The crucial difference between GR and teleparallelism is the introduction of torsion via the *tetrad field* e^a_μ, where

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu. \tag{5.55}$$

Here the latin indices $a, b...$ relate to the Minkowski tangent space at each point in spacetime, and η_{ab} is the associated Minkowski metric. We then define the *Weitzenböck connection* [64]

$$\Gamma^\sigma{}_{\mu\nu} = e^\sigma_a \partial_\nu e^a_\mu, \tag{5.56}$$

as well as the torsion by

$$T^\sigma{}_{\mu\nu} = e^\sigma_a \left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu\right), \tag{5.57}$$

and the contorsion by

$$K^{\mu\nu}{}_\sigma = -\frac{1}{2}\left(T^{\mu\nu}{}_\sigma - T^{\nu\mu}{}_\sigma - T_\sigma{}^{\mu\nu}\right). \tag{5.58}$$

The torsion scalar T is then defined by

$$\begin{aligned} T &= S_\sigma{}^{\mu\nu} T^\sigma{}_{\mu\nu} \\ &= \frac{1}{2}\left(K^{\mu\nu}{}_\sigma + \delta^\mu_\sigma T^{\rho\nu}{}_\rho - \delta^\nu_\sigma T^{\rho\mu}{}_\rho\right) T^\sigma{}_{\mu\nu}. \end{aligned} \tag{5.59}$$

The scalar T is the important object when constructing teleparallel theories; we can use T itself, functions of T, or couple T to a scalar field as in Section 5.1 in order to write down a teleparallel action. It should be noted that simply replacing the Ricci scalar in the Einstein-Hilbert action with T produces no discernable difference between GR and teleparallelism. Thus, models with $f(T) = T$ are entirely equivalent to GR. Furthermore, it can be shown that adding the quintessence field to an action involving T yields a dark energy model equivalent to the GR quintessence model. We should therefore be cautious in searching for teleparallel models, since there is a possibility of reproducing the well-known results of GR without any new phenomenology.

With this in mind, the worthwhile action to investigate is that of *teleparallel dark energy* [88], given by

$$S_{TDE} = \int d^4x |e| \left(-\frac{1}{2}F(\phi)T - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi) + \mathcal{L}_B\right), \tag{5.60}$$

where $|e| = det(e^a_\mu) = \sqrt{-g}$. The dynamical systems analysis for the above action, with $F(\phi) = 1 + \xi\phi^2$, has been performed in [193] and [200]. In [193], it is shown that there are no scaling solutions for this choice of $F(\phi)$. In [200], it is shown that teleparallel dark energy exhibits an extra late-time attractor over the standard GR quintessence. This attractor corresponds to behaviour similar to the cosmological constant and is independent of the model parameters. Further, the authors find that the EoS parameter for dark energy can dynamically cross the phantom barrier before settling at a constant value of -1. Lastly, the authors of [182] present a framework for studying more general forms of $F(\phi)$, and consider in detail the important differences between the late-time behaviour of GR quintessence and teleparallel dark energy.

As in Section 5.2, where we considered higher order terms of the curvature in the action, we can consider teleparallel actions of the form [26, 79]

$$S = \int d^4x|e|(f(T) + \mathcal{L}_B)\,, \tag{5.61}$$

Such an action has been shown to be useful in describing both the inflationary and dark energy scenarios, and has received much attention in the context of dynamical systems analysis. In [199], the authors find the critical points for the general $f(T)$ model, as well as the power law form $f(T) \propto (-T)^n$. A late-time period of acceleration is found in both cases. In [203], the model $f(T) \propto T \ln\left(\frac{T}{T_0}\right)$ is considered, and it is shown that the fluid-dominated critical points do not exist. The authors of [30] consider a coupling within the dark sector. The stability analysis reveals relevant dynamics for both early and late times, and includes the familiar epoch of dark energy domination. In [138] and [179], the form of $f(T)$ is constrained using dynamical systems methods, and a number of conditions on $f(T)$ are derived. These conditions are then compared with specific forms of $f(T)$. For a more general treatment of $f(T)$ theory and its application to cosmology, the reader is referred to [11] and [44].

5.4.3 Hořava-Lifschitz Gravity

The nonrenormalisability of GR is perhaps the most significant pitfall that burdens Einstein's theory. It signifies that GR breaks down at some scale, and thus that GR is an effective theory with only the lowest order terms in the curvature scalar. An attempt at renormalisation can then be made by introducing higher order curvature terms, and in [187] it is shown that introducing terms quadratic in the curvature results in a renormalisable theory. This comes at the cost of introducing ghosts and thus violating unitarity, as higher-order curvature terms necessarily introduce higher-order time derivatives.

Hořava's proposition [105] was to introduce an anisotropic scaling between space and time at high energies, and thus treat space and time on unequal footing. The theory (hereafter referred to as HL gravity) is power-counting renormalisable, unitary, and most importantly in violation of Lorentz invariance, although this characteristic Lorentz violation may only be relevant at high energies and thus below the current level of detection.

The dynamical variables of HL gravity are the ADM lapse and shift functions [95], labelled N and N_i respectively. The metric is then written using these fields:

$$ds^2 = -N^2dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)\,, \tag{5.62}$$

where g_{ij} is the spatial metric defined on the leaves of the foliation. The HL action is written as

$$S_{HL} = \int dt d^3 x N \sqrt{g} \left(\mathcal{L}_k - \mathcal{L}_p + \mathcal{L}_B \right), \tag{5.63}$$

where $\mathcal{L}_B$ is the familiar background fluid Lagrangian. $\mathcal{L}_k$ is the kinetic Lagrangian, given by

$$\mathcal{L}_k = \alpha \left(K_{ij} K^{ij} - \lambda K^2 \right), \tag{5.64}$$

where α and λ are coupling constants, and K_{ij} is the extrinsic curvature: [2]

$$K_{ij} = \frac{1}{2N} \left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i \right). \tag{5.65}$$

There is some subtlety in dealing with the potential Lagrangian $\mathcal{L}_p$, as noted by Hořava [105]. The *detailed balance condition* restricts the form of the potential, and greatly reduces the number of independent coupling constants of the theory. Under the detailed balance condition, the potential Lagrangian can be written as

$$\mathcal{L}_p = \beta C^{ij} C_{ij} + \gamma \epsilon^{ijk} R_{il} \nabla_j R^l_k + \zeta R_{ij} R^{ij} + \eta R^2 + \delta R + \sigma, \tag{5.66}$$

where C^{ij} is the Cotton tensor defined by

$$C^{ij} = \frac{\epsilon^{ijk}}{\sqrt{g}} \nabla_k \left(R^j_i - \frac{1}{4} R \delta^j_i \right), \tag{5.67}$$

and R_{ij} and R are the three-dimensional Ricci tensor and Ricci scalar respectively. $\beta, \gamma, \zeta, \eta$, and σ are the coupling constants, and ϵ^{ijk} is the three-dimensional Levi-Civita tensor.

We can then carry out the familiar process of deriving the field equations from the action (5.63), and construct a dynamical system using appropriate variables. A generic treatment of Hořava-Lifschitz Gravity can be found in [153] and [183], while the cosmological implications of the theory are investigated in [114] and [140]. In [176], a dark energy model is constructed using Hořava-Lifschitz gravity and two scalar fields.

In [48], dynamical systems methods are employed in the detailed balance case, and it is shown that the dark energy epoch is difficult to obtain in this context. Also of note is that one of the attractors of the system corresponds to oscillatory behaviour. The non-detailed balance case is also considered; it is shown that a dark energy phase can exist, and that HL gravity can produce a viable cosmological model. A similar analysis is performed in [130] with similar conclusions. A highly detailed study of the phase space in HL gravity in a Bianchi IX universe is performed in [134] and [135], with particular focus on the oscillatory nature of the dynamics. A dark energy epoch is found without the need for a cosmological constant, and the model

[2]Note that the covariant derivative ∇_i is defined with respect to the three-dimensional connection.

parameters are constrained using observational data. The reader is also referred to [87] and [127] for lengthy appraisals of the dynamical systems behaviour of HL gravity.

Chapter 6

Conclusion

In this thesis, we have attempted to provide a partial account of the numerous theoretical descriptions of dark energy. We began by outlining the dynamical systems method, which was utilised throughout this work as the main tool for studying dark energy models. Our first foray into cosmology was an elucidation of the canonical cosmological model, the ΛCDM universe. We saw that using dynamical systems techniques, the dynamics of such a universe were incredibly easy to describe, and that an accelerated phase of expansion is intrinsic to the model. We also outlined some issues with ΛCDM, mainly with regards to the cosmological constant Λ.

The next step was to remove Λ entirely, and consider a simple dynamical object, the scalar field. We examined the canonical scalar (quintessence), the tachyon, and the k-essence field, before generalising the Lagrangian further in search of the desirable scaling solutions. The power of the dynamical systems method became evident when discussing these models, as we were able to recast the complicated field equations into a more manageable set of differential equations, before finding the critical points and stability properties of the system. With such an approach, we saw that the entire cosmic history could be studied rather easily and matched against observational data, with close attention paid to the desired evolutionary track of (radiation $\rightarrow$ matter $\rightarrow$ dark energy).

We continued, dynamical systems toolbox in hand, to consider n-form dark energy - a natural extension to the scalar model. We again considered the viability of the n-form as a dark energy candidate, and pitched the predictions of the model against observational constraints. We found that, just like the scalar, a host of n-form models were viable, and pointed out some of the features that distinguish such models from the scalar. We then briefly discussed the other popular forms of non-scalar dark energy, such as the spinor and Yang-Mills fields. Although we considered these in less detail, we summarised some of the key results of the literature, and gave relevant references where necessary.

The final chapter, and perhaps most interesting from a theoretical standpoint, was concerned solely with theories that altered the description of gravity. We saw that modified theories of gravity can, unsurprisingly, also be studied with dynamical systems methods. We performed such an analysis for some of the more prevalent MG theories, including Brans-Dicke, $f(R)$, braneworld and massive gravity. In most cases, we considered the simplest formulation of each theory, opting for a basic evaluation of the theories in the dynamical systems context. In accordance with this aim, we gave a detailed account of the recent literature on the subject, in which the interested reader can find comprehensive studies of dynamical systems applications to modified gravity.

The quest to understand the nature of dark energy is still very much in its infancy, and cosmologists will likely need every tool at their disposal to formulate a satisfactory theoretical description of the acceleration of the Universe. The hope is that this thesis has provided a good argument for dynamical systems techniques to be included in that toolbox, for two reasons. Firstly, we have seen that the cosmological implications of new theories of dark energy can easily be tested using the methods outlined in this work. Secondly, we have demonstrated that constraints on model parameters are relatively easy to obtain using the dynamical systems procedure. We therefore hope that the application of dynamical systems to cosmology continues as a flourishing area of research, in tandem with technological advances and the ever-increasing accuracy of observational constraints.

Bibliography

[1] B. Agarwal et al. “ELKO fermions as dark matter candidates”. In: *Physical Review D* 92.7 (2015). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.92.075027.

[2] N. Aghanim et al. *Planck 2018 results. VI. Cosmological parameters*. 2018. arXiv: 1807.06209 [astro-ph.CO].

[3] J. Aguirregabiria and R. Lazkoz. “Tracking solutions in tachyon cosmology”. In: *Physical Review D* 69.12 (2004). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.69.123502.

[4] D. V. Ahluwalia-Khalilova and D. Grumiller. “Spin-half fermions with mass dimension one: theory, phenomenology, and dark matter”. In: *Journal of Cosmology and Astroparticle Physics* 2005.07 (2005), pp. 012–012. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2005/07/012.

[5] D. V. Ahluwalia and S. P. Horvath. “Very special relativity as relativity of dark matter: the Elko connection”. In: *Journal of High Energy Physics* 2010.11 (2010). ISSN: 1029-8479. URL: http://dx.doi.org/10.1007/JHEP11(2010)078.

[6] D. V. Ahluwalia, C.-Y. Lee, and D. Schritt. “Self-interacting Elko dark matter with an axis of locality”. In: *Physical Review D* 83.6 (2011). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.83.065017.

[7] D. V. Ahluwalia and A. C. Nayak. “Elko and mass dimension one field of spin one-half: Causality and Fermi statistics”. In: *International Journal of Modern Physics D* 23.14 (2014), p. 1430026. ISSN: 1793-6594. URL: http://dx.doi.org/10.1142/S0218271814300262.

[8] U. Alam et al. “Exploring the expanding Universe and dark energy using the statefinder diagnostic”. In: *Monthly Notices of the Royal Astronomical Society* 344.4 (2003), pp. 1057–1074. ISSN: 1365-2966. URL: http://dx.doi.org/10.1046/j.1365-8711.2003.06871.x.

[9] A. Albrecht and P. J. Steinhardt. “Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking”. In: *Physical Review Letters* 48 (1982), pp. 1220–1223. URL: https://link.aps.org/doi/10.1103/PhysRevLett.48.1220.

[10] R. Aldrovandi, J. G. Pereira, and K. H. Vu. “Selected topics in teleparallel gravity”. In: *Brazilian Journal of Physics* 34 (2004), pp. 1374–1380. ISSN: 0103-9733. URL: http://www.scielo.br/scielo.php?script=sci_arttext&pid=S0103-97332004000700014&nrm=iso.

[11] R. Aldrovandi and J. Pereira. *Teleparallel Gravity: An Introduction*. Fundamental Theories of Physics. Springer Netherlands, 2012. ISBN: 9789400751439. URL: https://books.google.co.uk/books?id=2dZ5zuKe%5C_fUC.

[12] S. Alexander and N. Yunes. “Chern–Simons modified general relativity”. In: *Physics Reports* 480.1-2 (2009), pp. 1–55. ISSN: 0370-1573. URL: http://dx.doi.org/10.1016/j.physrep.2009.07.002.

[13] A. Alho, S. Carloni, and C. Uggla. “On dynamical systems approaches and methods in f(R) cosmology”. In: *Journal of Cosmology and Astroparticle Physics* 2016.08 (2016), pp. 064–064. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2016/08/064.

[14] M. Álvarez et al. “Einstein Yang–Mills Higgs dark energy revisited”. In: *Classical and Quantum Gravity* 36.19 (2019), p. 195004. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/1361-6382/ab3775.

[15] M. Álvarez et al. *Anisotropic Einstein Yang-Mills Higgs Dark Energy*.

[16] L. Amendola. “Coupled quintessence”. In: *Physical Review D* 62.4 (2000). ISSN: 1089-4918. URL: http://dx.doi.org/10.1103/PhysRevD.62.043511.

[17] L. Amendola et al. “Conditions for the cosmological viability of f(R) dark energy models”. In: *Physical Review D* 75.8 (2007). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.75.083504.

[18] C. Armendáriz-Picón. “Could dark energy be vector-like?” In: *Journal of Cosmology and Astroparticle Physics* 2004.07 (2004), pp. 007–007. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2004/07/007.

[19] C. Armendáriz-Picón, T. Damour, and V. Mukhanov. “k-Inflation”. In: *Physics Letters B* 458.2-3 (1999), pp. 209–218. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/S0370-2693(99)00603-6.

[20] C. Armendariz-Picon, V. Mukhanov, and P. J. Steinhardt. “Essentials of k-essence”. In: *Physical Review D* 63.10 (2001). ISSN: 1089-4918. URL: http://dx.doi.org/10.1103/PhysRevD.63.103510.

[21] J. S. Bagla, H. K. Jassal, and T. Padmanabhan. “Cosmology with tachyon field as dark energy”. In: *Physical Review D* 67.6 (2003). ISSN: 1089-4918. URL: http://dx.doi.org/10.1103/PhysRevD.67.063504.

[22] T. Barreiro, E. J. Copeland, and N. J. Nunes. “Quintessence arising from exponential potentials”. In: *Physical Review D* 61.12 (2000). ISSN: 1089-4918. URL: http://dx.doi.org/10.1103/PhysRevD.61.127301.

[23] A. Basak et al. “Attractor behaviour in ELKO cosmology”. In: *Journal of Cosmology and Astroparticle Physics* 2013.04 (2013), pp. 025–025. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2013/04/025.

[24] R. Bean, S. H. Hansen, and A. Melchiorri. “Early-universe constraints on dark energy”. In: *Physical Review D* 64 (2001), p. 103508. URL: https://link.aps.org/doi/10.1103/PhysRevD.64.103508.

[25] J. P. Beltrán Almeida et al. “Anisotropic 2-form dark energy”. In: *Physics Letters B* 793 (2019), pp. 396–404. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2019.05.008.

[26] G. R. Bengochea and R. Ferraro. “Dark torsion as the cosmic speed-up”. In: *Physical Review D* 79.12 (2009). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.79.124019.

[27] A. Bhadra et al. “Brans–Dicke Theory: Jordan Versus Einstein Frame”. In: *Modern Physics Letters A* 22.05 (2007), pp. 367–375. ISSN: 1793-6632. URL: http://dx.doi.org/10.1142/S021773230702261X.

[28] N. Bilić, S. Domazet, and G. S. Djordjevic. “Tachyon with an inverse power-law potential in a braneworld cosmology”. In: *Classical and Quantum Gravity* 34.16 (2017), p. 165006. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/1361-6382/aa7e0f.

[29] N. Bilić et al. “Tachyon inflation in the holographic braneworld”. In: *Journal of Cosmology and Astroparticle Physics* 2019.08 (2019), pp. 034–034. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2019/08/034.

[30] S. K. Biswas and S. Chakraborty. “Interacting dark energy in f(T) cosmology: A dynamical system analysis”. In: *International Journal of Modern Physics D* 24.07 (2015), p. 1550046. ISSN: 1793-6594. URL: http://dx.doi.org/10.1142/S0218271815500467.

[31] C. G. Böhmer. “Dark spinor inflation: Theory primer and dynamics”. In: *Physical Review D* 77.12 (2008). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.77.123535.

[32] C. G. Böhmer and J. Burnett. “Dark Energy With Dark Spinors”. In: *Modern Physics Letters A* 25.02 (2010), pp. 101–110. ISSN: 1793-6632. URL: http://dx.doi.org/10.1142/S0217732310032275.

[33] C. G. Böhmer and N. Chan. “Dynamical Systems in Cosmology”. In: *Dynamical and Complex Systems* (2016), pp. 121–156. URL: http://dx.doi.org/10.1142/9781786341044_0004.

[34] C. G. Böhmer, N. Chan, and R. Lazkoz. "Dynamics of dark energy models and centre manifolds". In: *Physics Letters B* 714.1 (2012), pp. 11–17. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2012.06.064.

[35] C. G. Böhmer and T. Harko. "Dark energy as a massive vector field". In: *The European Physical Journal C* 50.2 (2007), pp. 423–429. ISSN: 1434-6052. URL: http://dx.doi.org/10.1140/epjc/s10052-007-0210-1.

[36] C. G. Böhmer and D. F. Mota. "CMB anisotropies and inflation from non-standard spinors". In: *Physics Letters B* 663.3 (2008), pp. 168–171. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2008.04.008.

[37] C. G. Böhmer et al. "Dark spinor models in gravitation and cosmology". In: *Journal of High Energy Physics* 2010.7 (2010). ISSN: 1029-8479. URL: http://dx.doi.org/10.1007/JHEP07(2010)053.

[38] M. Bouhmadi-López et al. "The little sibling of the big rip singularity". In: *International Journal of Modern Physics D* 24.10 (2015), p. 1550078. ISSN: 1793-6594. URL: http://dx.doi.org/10.1142/S0218271815500789.

[39] D. G. Boulware and S. Deser. "Can Gravitation Have a Finite Range?" In: *Physical Review D* 6 (1972), pp. 3368–3382. URL: https://link.aps.org/doi/10.1103/PhysRevD.6.3368.

[40] C. H. Brans. "Jordan-Brans-Dicke Theory". In: *Scholarpedia* 9.4 (2014). revision #151619, p. 31358.

[41] C. Brans and R. H. Dicke. "Mach's Principle and a Relativistic Theory of Gravitation". In: *Physical Review* 124 (1961), pp. 925–935. URL: https://link.aps.org/doi/10.1103/PhysRev.124.925.

[42] P. Brax, C. v. d. Bruck, and A.-C. Davis. "Brane world cosmology". In: *Reports on Progress in Physics* 67.12 (2004), pp. 2183–2231. ISSN: 1361-6633. URL: http://dx.doi.org/10.1088/0034-4885/67/12/R02.

[43] P. Bull et al. "Beyond ΛCDM: Problems, solutions, and the road ahead". In: *Physics of the Dark Universe* 12 (2016), pp. 56–99. ISSN: 2212-6864. URL: http://dx.doi.org/10.1016/j.dark.2016.02.001.

[44] Y.-F. Cai et al. "f(T) teleparallel gravity and cosmology". In: *Reports on Progress in Physics* 79.10 (2016), p. 106901. ISSN: 1361-6633. URL: http://dx.doi.org/10.1088/0034-4885/79/10/106901.

[45] G. Calcagni and A. R. Liddle. "Tachyon dark energy models: Dynamics and constraints". In: *Physical Review D* 74.4 (2006). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.74.043528.

[46] R. R. Caldwell, M. Kamionkowski, and N. N. Weinberg. "Phantom Energy: Dark Energy with w¡-1 Causes a Cosmic Doomsday". In: *Physical Review Letters* 91.7 (2003). ISSN: 1079-7114. URL: http://dx.doi.org/10.1103/PhysRevLett.91.071301.

[47] S. Carloni. "A new approach to the analysis of the phase space of f(R)-gravity". In: *Journal of Cosmology and Astroparticle Physics* 2015.09 (2015), pp. 013–013. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2015/09/013.

[48] S. Carloni, E. Elizalde, and P. J. Silva. "An analysis of the phase space of Hořava–Lifshitz cosmologies". In: *Classical and Quantum Gravity* 27.4 (2010), p. 045004. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/27/4/045004.

[49] S. Carloni et al. "Cosmological dynamics of R^n gravity". In: *Classical and Quantum Gravity* 22.22 (2005), pp. 4839–4868. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/22/22/011.

[50] S. Carloni et al. "Cosmological dynamics of scalar–tensor gravity". In: *Classical and Quantum Gravity* 25.3 (2008), p. 035008. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/25/3/035008.

[51] S. M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press, 2019.

[52] J. L. Cervantes-Cota, R. d. Putter, and E. V. Linder. "Induced gravity and the attractor dynamics of dark energy/dark matter". In: *Journal of Cosmology and Astroparticle Physics* 2010.12 (2010), pp. 019–019. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2010/12/019.

[53] T. Chiba, T. Okabe, and M. Yamaguchi. "Kinetically driven quintessence". In: *Physical Review D* 62.2 (2000). ISSN: 1089-4918. URL: http://dx.doi.org/10.1103/PhysRevD.62.023511.

[54] T. G. Clemson and A. R. Liddle. "Observational constraints on thawing quintessence models". In: *Monthly Notices of the Royal Astronomical Society* 395.3 (2009), pp. 1585–1590. ISSN: 1365-2966. URL: http://dx.doi.org/10.1111/j.1365-2966.2009.14641.x.

[55] T. Clifton et al. "Modified gravity and cosmology". In: *Physics Reports* 513.1-3 (2012), pp. 1–189. ISSN: 0370-1573. URL: http://dx.doi.org/10.1016/j.physrep.2012.01.001.

[56] E. J. Copeland, A. R. Liddle, and D. Wands. "Exponential potentials and cosmological scaling solutions". In: *Physical Review D* 57.8 (1998), pp. 4686–4690. ISSN: 1089-4918. URL: http://dx.doi.org/10.1103/PhysRevD.57.4686.

[57] E. J. Copeland, M. Sami, and S. Tsujikawa. "Dynamics Of Dark Energy". In: *International Journal of Modern Physics D* 15.11 (2006), pp. 1753–1935. ISSN: 1793-6594. URL: http://dx.doi.org/10.1142/S021827180600942X.

[58] E. J. Copeland et al. "What is needed of a tachyon if it is to be the dark energy?" In: *Physical Review D* 71.4 (2005). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.71.043003.

[59] E. J. Copeland, A. Padilla, and P. M. Saffin. "The cosmology of the Fab-Four". In: *Journal of Cosmology and Astroparticle Physics* 2012.12 (2012), pp. 026–026. URL: https://doi.org/10.1088%2F1475-7516%2F2012%2F12%2F026.

[60] G. D'Amico et al. "Massive cosmologies". In: *Physical Review D* 84.12 (2011). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.84.124046.

[61] G. D'Amico. "Cosmology and perturbations in massive gravity". In: *Physical Review D* 86.12 (2012). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.86.124019.

[62] R. Da Rocha and W. A. Rodrigues. "Where Are Elko Spinor Fields In Lounesto Spinor Field Classification?" In: *Modern Physics Letters A* 21.01 (2006), pp. 65–74. ISSN: 1793-6632. URL: http://dx.doi.org/10.1142/S0217732306018482.

[63] T. Damour, G. W. Gibbons, and C. Gundlach. "Dark matter, time-varying G, and a dilaton field". In: *Physical Review Letters* 64 (1990), pp. 123–126. URL: https://link.aps.org/doi/10.1103/PhysRevLett.64.123.

[64] V. C. De Andrade, L. C. T. Guillen, and J. G. Pereira. "Teleparallel Gravity: An Overview". In: *The Ninth Marcel Grossmann Meeting* (2002), pp. 1022–1023. URL: http://dx.doi.org/10.1142/9789812777386_0144.

[65] A. De Felice, A. E. Gümrükçüoğlu, and S. Mukohyama. "Massive Gravity: Nonlinear Instability of a Homogeneous and Isotropic Universe". In: *Physical Review Letters* 109.17 (2012). ISSN: 1079-7114. URL: http://dx.doi.org/10.1103/PhysRevLett.109.171101.

[66] A. De Felice and S. Tsujikawa. "f(R) Theories". In: *Living Reviews in Relativity* 13.1 (2010). ISSN: 1433-8351. URL: http://dx.doi.org/10.12942/lrr-2010-3.

[67] C. Deffayet. "Cosmology on a brane in Minkowski bulk". In: *Physics Letters B* 502.1-4 (2001), pp. 199–208. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/S0370-2693(01)00160-5.

[68] J. Dutta and H. Zonunmawia. "Complete cosmic scenario in the Randall-Sundrum braneworld from the dynamical systems perspective". In: *The European Physical Journal Plus* 130.11 (2015). ISSN: 2190-5444. URL: http://dx.doi.org/10.1140/epjp/i2015-15221-3.

[69] J. Dutta et al. *From inflation to dark energy in scalar-tensor cosmology.* 2020. arXiv: 2007.06601 [gr-qc].

[70] G. Dvali, G. Gabadadze, and M. Porrati. "4D gravity on a brane in 5D Minkowski space". In: *Physics Letters B* 485.1-3 (2000), pp. 208–214. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/S0370-2693(00)00669-9.

[71] A. Einstein. "Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus". In: vol. 1928. 2006, pp. 316–321. ISBN: 9783527608959.

[72] D. Escobar et al. "Phase space analysis of quintessence fields trapped in a Randall–Sundrum braneworld: a refined study". In: *Classical and Quantum Gravity* 29.17 (2012), p. 175005. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/29/17/175005.

[73] W. Fang et al. "Exact analysis of scaling and dominant attractors beyond the exponential potential". In: *Classical and Quantum Gravity* 26.15 (2009), p. 155005. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/26/15/155005.

[74] W. Fang et al. "Challenges to the DGP model from horizon-scale growth and geometry". In: *Physical Review D* 78.10 (2008). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.78.103509.

[75] V. Faraoni, M. N. Jensen, and S. A. Theuerkauf. "Non-chaotic dynamics in general-relativistic and scalar–tensor cosmology". In: *Classical and Quantum Gravity* 23.12 (2006), pp. 4215–4229. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/23/12/016.

[76] V. Faraoni. "Solar system experiments do not yet veto modified gravity models". In: *Physical Review D* 74.2 (2006). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.74.023529.

[77] V. Faraoni and E. Gunzig. "Einstein Frame or Jordan Frame?" In: *International Journal of Theoretical Physics* 38 (1999).

[78] S. Fay, R. Tavakol, and S. Tsujikawa. "f(R) gravity theories in Palatini formalism: Cosmological dynamics and observational constraints". In: *Physical Review D* 75.6 (2007). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.75.063509.

[79] R. Ferraro and F. Fiorini. "Modified teleparallel gravity: Inflation without an inflaton". In: *Physical Review D* 75 (2007), p. 084031. URL: https://link.aps.org/doi/10.1103/PhysRevD.75.084031.

[80] M. Fierz and W. Pauli. "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field". In: *Proc. Roy. Soc. Lond. A* A173 (1939), pp. 211–232.

[81] É. É. Flanagan. "The conformal frame freedom in theories of gravitation". In: *Classical and Quantum Gravity* 21.15 (2004), pp. 3817–3829. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/21/15/N02.

[82] L. H. Ford. "Inflation driven by a vector field". In: *Physical Review D* 40 (1989), pp. 967–972. URL: https://link.aps.org/doi/10.1103/PhysRevD.40.967.

[83] A. Friedmann. "Über die Krümmung des Raumes". In: *Zeitschrift fur Physik* 10 (1922), pp. 377–386.

[84] A. Friedmann. "Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes". In: *Zeitschrift fur Physik* 21.1 (1924), pp. 326–332.

[85] Y. Fujii and K.-i. Maeda. *The Scalar-Tensor Theory of Gravitation*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2003.

[86] R. Gannouji et al. "Quasidilaton nonlinear massive gravity: Investigations of background cosmological dynamics". In: *Physical Review D* 87.12 (2013). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.87.123536.

[87] F. Gao and J. Llibre. *Global dynamics of the Horava-Lifshitz cosmological model in a non-flat universe with non-zero cosmological constant*. 2020. arXiv: 2007.08049 [gr-qc].

[88] C.-Q. Geng et al. ""Teleparallel" dark energy". In: *Physics Letters B* 704.5 (2011), pp. 384–387. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2011.09.082.

[89] C. Germani and A. Kehagias. "P-nflation: generating cosmic Inflation with p-forms". In: *Journal of Cosmology and Astroparticle Physics* 2009.03 (2009), pp. 028–028. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2009/03/028.

[90] D. Giannakis and W. Hu. "Challenges for the kinetic unified dark matter model". In: *Physical Review D* 72.6 (2005). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.72.063502.

[91] G. W. Gibbons. "Thoughts on tachyon cosmology". In: *Classical and Quantum Gravity* 20.12 (2003), S321–S346. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/20/12/301.

[92] A. Golovnev, V. Mukhanov, and V. Vanchurin. "Vector inflation". In: *Journal of Cosmology and Astroparticle Physics* 2008.06 (2008), p. 009. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2008/06/009.

[93] Y.-G. Gong. "Cosmology in Massive Gravity". In: *Communications in Theoretical Physics* 59.3 (2013), pp. 319–323. ISSN: 0253-6102. URL: http://dx.doi.org/10.1088/0253-6102/59/3/13.

[94] T. Gonzalez et al. "Self-interacting Scalar Field Trapped in a Randall-Sundrum Braneworld: The Dynamical Systems Perspective". In: *Physics Letters B* 676 (2009), pp. 161–167. arXiv: 0812.1734 [gr-qc].

[95] E. Gourgoulhon. *3+1 Formalism and Bases of Numerical Relativity*. 2007. arXiv: gr-qc/0703035 [gr-qc].

[96] P. Gratia, W. Hu, and M. Wyman. "Self-accelerating massive gravity: Exact solutions for any isotropic matter distribution". In: *Physical Review D* 86.6 (2012). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.86.061504.

[97] Y.-Q. Gu. "A Cosmological Model With Dark Spinor Source". In: *International Journal of Modern Physics A* 22.25 (2007), pp. 4667–4678. ISSN: 1793-656X. URL: http://dx.doi.org/10.1142/S0217751X07037925.

[98] Y. Gu. "Nonlinear Spinors as the Candidate of Dark Matter". In: *OALib* 04.10 (2017), pp. 1–8. ISSN: 2333-9705. URL: http://dx.doi.org/10.4236/oalib.1103954.

[99] B. Gumjudpai et al. "Coupled dark energy: towards a general description of the dynamics". In: *Journal of Cosmology and Astroparticle Physics* 2005.06 (2005), pp. 007–007. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2005/06/007.

[100] A. E. Gümrükçüoğlu, C. Lin, and S. Mukohyama. "Open FRW universes and self-acceleration from nonlinear massive gravity". In: *Journal of Cosmology and Astroparticle Physics* 2011.11 (2011), pp. 030–030. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2011/11/030.

[101] A. E. Gümrükçüoğlu, C. Lin, and S. Mukohyama. "Anisotropic Friedmann–Robertson–Walker universe from nonlinear massive gravity". In: *Physics Letters B* 717.4-5 (2012), pp. 295–298. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2012.09.049.

[102] Z.-K. Guo and Y.-Z. Zhang. "Cosmological scaling solutions of multiple tachyon fields with inverse square potentials". In: *Journal of Cosmology and Astroparticle Physics* 2004.08 (2004), pp. 010–010. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2004/08/010.

[103] A. H. Guth. "Inflationary universe: A possible solution to the horizon and flatness problems". In: *Physical Review D* 23 (1981), pp. 347–356. URL: https://link.aps.org/doi/10.1103/PhysRevD.23.347.

[104] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1973.

[105] P. Hořava. "Quantum gravity at a Lifshitz point". In: *Physical Review D* 79.8 (2009). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.79.084008.

[106] O. Hrycyna and M. Szydłowski. "Brans-Dicke theory and the emergence of ΛCDMmodel". In: *Physical Review D* 88.6 (2013). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.88.064018.

[107] O. Hrycyna and M. Szydłowski. "Dynamical complexity of the Brans-Dicke cosmology". In: *Journal of Cosmology and Astroparticle Physics* 2013.12 (2013), pp. 016–016. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2013/12/016.

[108] W. Hu and I. Sawicki. "Models of f(R) cosmic acceleration that evade solar system tests". In: *Physical Review D* 76.6 (2007). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.76.064004.

[109] Q.-G. Huang, Y.-S. Piao, and S.-Y. Zhou. "Mass-varying massive gravity". In: *Physical Review D* 86.12 (2012). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.86.124014.

[110] E. Hubble. "A relation between distance and radial velocity among extra-galactic nebulae". In: *Proceedings of the National Academy of Sciences* 15.3 (1929), pp. 168–173. ISSN: 0027-8424. eprint: https://www.pnas.org/content/15/3/168.full.pdf. URL: https://www.pnas.org/content/15/3/168.

[111] D. Huterer and M. S. Turner. "Prospects for probing the dark energy via supernova distance measurements". In: *Physical Review D* 60.8 (1999). ISSN: 1089-4918. URL: http://dx.doi.org/10.1103/PhysRevD.60.081301.

[112] A. Ito and J. Soda. "Designing anisotropic inflation with form fields". In: *Physical Review D* 92.12 (2015). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.92.123533.

[113] T. Kaluza. "Zum Unitätsproblem der Physik". In: *International Journal of Modern Physics D* 27.14 (2018), p. 1870001. arXiv: 1803.08616 [physics.hist-ph].

[114] E. Kiritsis and G. Kofinas. "Hořava–Lifshitz cosmology". In: *Nuclear Physics B* 821.3 (2009), pp. 467–480. ISSN: 0550-3213. URL: http://dx.doi.org/10.1016/j.nuclphysb.2009.05.005.

[115] V. V. Kiselev. "Vector field as a quintessence partner". In: *Classical and Quantum Gravity* 21.13 (2004), pp. 3323–3335. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/21/13/014.

[116] O. Klein. "Quantentheorie und fünfdimensionale Relativitätstheorie". In: *Zeitschrift fur Physik* 37.12 (1926), pp. 895–906.

[117] L. Kofman and A. Linde. "Problems with Tachyon Inflation". In: *Journal of High Energy Physics* 2002.07 (2002), pp. 004–004. ISSN: 1029-8479. URL: http://dx.doi.org/10.1088/1126-6708/2002/07/004.

[118] T. S. Koivisto and N. J. Nunes. "Inflation and dark energy from three-forms". In: *Physical Review D* 80.10 (2009). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.80.103509.

[119] T. S. Koivisto and N. J. Nunes. "Three-form cosmology". In: *Physics Letters B* 685.2-3 (2010), pp. 105–109. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2010.01.051.

[120] T. S. Koivisto and N. J. Nunes. "Coupled three-form dark energy". In: *Physical Review D* 88.12 (2013). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.88.123512.

[121] T. S. Koivisto, D. F. Mota, and C. Pitrou. "Inflation from N-forms and its stability". In: *Journal of High Energy Physics* 2009.09 (2009), pp. 092–092. ISSN: 1029-8479. URL: http://dx.doi.org/10.1088/1126-6708/2009/09/092.

[122] T. Koivisto and D. F. Mota. "Vector field models of inflation and dark energy". In: *Journal of Cosmology and Astroparticle Physics* 2008.08 (2008), p. 021. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2008/08/021.

[123] K. Land and J. Magueijo. "Examination of Evidence for a Preferred Axis in the Cosmic Radiation Anisotropy". In: *Physical Review Letters* 95.7 (2005). ISSN: 1079-7114. URL: http://dx.doi.org/10.1103/PhysRevLett.95.071301.

[124] R. C. G. Landim. "Coupled tachyonic dark energy: A dynamical analysis". In: *International Journal of Modern Physics D* 24.11 (2015), p. 1550085. ISSN: 1793-6594. URL: http://dx.doi.org/10.1142/S0218271815500856.

[125] R. C. G. Landim. "Dynamical analysis for a vector-like dark energy". In: *The European Physical Journal C* 76.9 (2016). ISSN: 1434-6052. URL: http://dx.doi.org/10.1140/epjc/s10052-016-4328-x.

[126] D. Langlois and A. Naruko. "Cosmological solutions of massive gravity on de Sitter". In: *Classical and Quantum Gravity* 29.20 (2012), p. 202001. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/29/20/202001.

[127] G. Leon and A. Paliathanasis. "Extended phase-space analysis of the Hořava–Lifshitz cosmology". In: *The European Physical Journal C* 79.9 (2019). ISSN: 1434-6052. URL: http://dx.doi.org/10.1140/epjc/s10052-019-7236-z.

[128] G. Leon and A. A. Roque. "Qualitative analysis of Kantowski-Sachs metric in a generic class of f(R) models". In: *Journal of Cosmology and Astroparticle Physics* 2014.05 (2014), pp. 032–032. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2014/05/032.

[129] G. Leon, J. Saavedra, and E. N. Saridakis. "Cosmological behavior in extended nonlinear massive gravity". In: *Classical and Quantum Gravity* 30.13 (2013), p. 135001. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/30/13/135001.

[130] G. Leon and E. N. Saridakis. "Phase-space analysis of Hořava-Lifshitz cosmology". In: *Journal of Cosmology and Astroparticle Physics* 2009.11 (2009), pp. 006–006. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2009/11/006.

[131] Y. Leyva et al. "Dynamics of a self-interacting scalar field trapped in the braneworld for a wide variety of self-interaction potentials". In: *Physical Review D* 80.4 (2009). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.80.044026.

[132] X.-z. Li, D.-j. Liu, and J.-g. Hao. *On the tachyon inflation*. 2002. arXiv: hep-th/0207146 [hep-th].

[133] R. Maartens and K. Koyama. "Brane-World Gravity". In: *Living Reviews in Relativity* 13.1 (2010). ISSN: 1433-8351. URL: http://dx.doi.org/10.12942/lrr-2010-5.

[134] R. Maier and I. D. Soares. "Hořava-Lifshitz bouncing Bianchi IX universes: A dynamical system analysis". In: *Physical Review D* 96.10 (2017). ISSN: 2470-0029. URL: http://dx.doi.org/10.1103/PhysRevD.96.103532.

[135] R. Maier and I. D. Soares. "Curvature late-time acceleration in an eternal universe". In: *Journal of Cosmology and Astroparticle Physics* 2018.12 (2018), pp. 029–029. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2018/12/029.

[136] M. Malquarti, E. J. Copeland, and A. R. Liddle. "k-essence and the coincidence problem". In: *Physical Review D* 68.2 (2003). ISSN: 1089-4918. URL: http://dx.doi.org/10.1103/PhysRevD.68.023512.

[137] J. Miritzis. "Dynamical system approach to FRW models in higher-order gravity theories". In: *Journal of Mathematical Physics* 44.9 (2003), p. 3900. ISSN: 0022-2488. URL: http://dx.doi.org/10.1063/1.1602161.

[138] B. Mirza and F. Oboudiat. "Constraining f(T) gravity by dynamical system analysis". In: *Journal of Cosmology and Astroparticle Physics* 2017.11 (2017), pp. 011–011. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2017/11/011.

[139] J. Morais et al. "Interacting 3-form dark energy models: Distinguishing interactions and avoiding the Little Sibling of the Big Rip". In: *Physics of the Dark Universe* 15 (2017), pp. 7–30. ISSN: 2212-6864. URL: http://dx.doi.org/10.1016/j.dark.2016.11.002.

[140] S. Mukohyama. "Hořava–Lifshitz cosmology: a review". In: *Classical and Quantum Gravity* 27.22 (2010), p. 223101. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/27/22/223101.

[141] S. Nakamura, R. Kase, and S. Tsujikawa. "Coupled vector dark energy". In: *Journal of Cosmology and Astroparticle Physics* 2019.12 (2019), pp. 032–032. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2019/12/032.

[142] S. C. C. Ng, N. J. Nunes, and F. Rosati. "Applications of scalar attractor solutions to cosmology". In: *Physical Review D* 64.8 (2001). ISSN: 1089-4918. URL: http://dx.doi.org/10.1103/PhysRevD.64.083510.

[143] T. Ngampitipan and P. Wongjun. “Dynamics of three-form dark energy with dark matter couplings”. In: *Journal of Cosmology and Astroparticle Physics* 2011.11 (2011), pp. 036–036. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2011/11/036.

[144] S. Nojiri, S. Odintsov, and V. Oikonomou. “Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution”. In: *Physics Reports* 692 (2017), pp. 1–104. ISSN: 0370-1573. URL: http://dx.doi.org/10.1016/j.physrep.2017.06.001.

[145] S. Nojiri and S. D. Odintsov. “Modified Gauss–Bonnet theory as gravitational alternative for dark energy”. In: *Physics Letters B* 631.1-2 (2005), pp. 1–6. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2005.10.010.

[146] S. Nojiri, S. D. Odintsov, and M. Sasaki. “Gauss-Bonnet dark energy”. In: *Physical Review D* 71.12 (2005). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.71.123509.

[147] K. Nordtvedt. “Verification of general relativity: tests in the Solar System”. In: *The Century of Space Science*. Ed. by J. A. M. Bleeker, J. Geiss, and M. C. E. Huber. Dordrecht: Springer Netherlands, 2001, pp. 335–352. ISBN: 978-94-010-0320-9. URL: https://doi.org/10.1007/978-94-010-0320-9_15.

[148] K. Nozari and N. Rashidi. “Some aspects of tachyon field cosmology”. In: *Physical Review D* 88.2 (2013). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.88.023519.

[149] J. Ohashi, J. Soda, and S. Tsujikawa. “Anisotropic non-Gaussianity from a two-form field”. In: *Physical Review D* 87.8 (2013). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.87.083520.

[150] G. J. Olmo. “Palatini Approach To Modified Gravity: F(R) Theories And Beyond”. In: *International Journal of Modern Physics D* 20.04 (2011), pp. 413–462. ISSN: 1793-6594. URL: http://dx.doi.org/10.1142/S0218271811018925.

[151] J. Overduin and P. Wesson. “Kaluza-Klein gravity”. In: *Physics Reports* 283.5-6 (1997), pp. 303–378. ISSN: 0370-1573. URL: http://dx.doi.org/10.1016/S0370-1573(96)00046-4.

[152] A. Padilla. *Braneworld Cosmology and Holography*. 2002. arXiv: hep-th/0210217 [hep-th].

[153] A. Padilla. “The good, the bad and the ugly .... of Hořava gravity”. In: *Journal of Physics: Conference Series* 259 (2010), p. 012033. ISSN: 1742-6596. URL: http://dx.doi.org/10.1088/1742-6596/259/1/012033.

[154] A. Padilla. *Lectures on the Cosmological Constant Problem*. 2015. arXiv: 1502.05296 [hep-th].

[155] L. Parisi, N. Radicella, and G. Vilasi. "Stability of the Einstein static universe in massive gravity". In: *Physical Review D* 86.2 (2012). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.86.024035.

[156] P. Peebles and B. Ratra. "Cosmology with a Time Variable Cosmological Constant". In: *Astrophysical Journal Letters* 325 (1988), p. L17.

[157] S. Pereira and T. Guimarães. "From inflation to recent cosmic acceleration: the fermionic Elko field driving the evolution of the universe". In: *Journal of Cosmology and Astroparticle Physics* 2017.09 (2017), pp. 038–038. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2017/09/038.

[158] S. Pereira, A. P. S.S., and J. H. d. Silva. "Some remarks on the attractor behaviour in ELKO cosmology". In: *Journal of Cosmology and Astroparticle Physics* 2014.08 (2014), pp. 020–020. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2014/08/020.

[159] S. Perlmutter et al. "Measurements of Ω and Λ from 42 High-Redshift Supernovae". In: *The Astrophysical Journal* 517.2 (1999), pp. 565–586. ISSN: 1538-4357. URL: http://dx.doi.org/10.1086/307221.

[160] L. Randall and R. Sundrum. "An Alternative to Compactification". In: *Physical Review Letters* 83.23 (1999), pp. 4690–4693. ISSN: 1079-7114. URL: http://dx.doi.org/10.1103/PhysRevLett.83.4690.

[161] L. Randall and R. Sundrum. "Large Mass Hierarchy from a Small Extra Dimension". In: *Physical Review Letters* 83.17 (1999), pp. 3370–3373. ISSN: 1079-7114. URL: http://dx.doi.org/10.1103/PhysRevLett.83.3370.

[162] A. Ravanpak and G. F. Fadakar. "Dynamical system analysis of Randall-Sundrum model with tachyon field on the brane". In: *Physical Review D* 101.10 (2020). ISSN: 2470-0029. URL: http://dx.doi.org/10.1103/PhysRevD.101.103525.

[163] S. Reynaud and M.-T. Jaekel. *Tests of general relativity in the solar system.* 2008. arXiv: 0801.3407 [gr-qc].

[164] C. de Rham, G. Gabadadze, and A. J. Tolley. "Resummation of Massive Gravity". In: *Physical Review Letters* 106.23 (2011). ISSN: 1079-7114. URL: http://dx.doi.org/10.1103/PhysRevLett.106.231101.

[165] M. O. Ribas, F. P. Devecchi, and G. M. Kremer. "Fermions as sources of accelerated regimes in cosmology". In: *Physical Review D* 72.12 (2005). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.72.123502.

[166] A. G. Riess et al. "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant". In: *The Astronomical Journal* 116.3 (1998), pp. 1009–1038. ISSN: 0004-6256. URL: http://dx.doi.org/10.1086/300499.

[167] M. Rinaldi. "Dark energy as a fixed point of the Einstein Yang-Mills Higgs equations". In: *Journal of Cosmology and Astroparticle Physics* 2015.10 (2015), pp. 023–023. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2015/10/023.

[168] M. Rinaldi. "Higgs dark energy". In: *Classical and Quantum Gravity* 32.4 (2015), p. 045002. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/32/4/045002.

[169] N. Roy and N. Banerjee. "Generalized Brans-Dicke theory: A dynamical systems analysis". In: *Physical Review D* 95.6 (2017). ISSN: 2470-0029. URL: http://dx.doi.org/10.1103/PhysRevD.95.064048.

[170] J. Rubio. "Higgs Inflation". In: *Frontiers in Astronomy and Space Sciences* 5 (2019). ISSN: 2296-987X. URL: http://dx.doi.org/10.3389/fspas.2018.00050.

[171] H. M. Sadjadi. "On coincidence problem in ELKO dark energy model". In: *General Relativity and Gravitation* 44.9 (2012), pp. 2329–2336. ISSN: 1572-9532. URL: http://dx.doi.org/10.1007/s10714-012-1392-x.

[172] V. Sahni et al. "Statefinder—A new geometrical diagnostic of dark energy". In: *Journal of Experimental and Theoretical Physics Letters* 77.5 (2003), pp. 201–206. ISSN: 1090-6487. URL: http://dx.doi.org/10.1134/1.1574831.

[173] V. Sahni and L. Wang. "New cosmological model of quintessence and dark matter". In: *Physical Review D* 62.10 (2000). ISSN: 1089-4918. URL: http://dx.doi.org/10.1103/PhysRevD.62.103517.

[174] M. Sami. "A Note On The Cosmological Dynamics In Finite-Range Gravity". In: *International Journal of Modern Physics D* 12.04 (2003), pp. 689–696. ISSN: 1793-6594. URL: http://dx.doi.org/10.1142/S0218271803003128.

[175] A. P. dos Santos Souza, S. H. Pereira, and J. F. Jesus. "A new approach on the stability analysis in ELKO cosmology". In: *The European Physical Journal C* 75.1 (2015). ISSN: 1434-6052. URL: http://dx.doi.org/10.1140/epjc/s10052-015-3260-9.

[176] E. N. Saridakis. "Hořava–Lifshitz dark energy". In: *The European Physical Journal C* 67.1-2 (2010), pp. 229–235. ISSN: 1434-6052. URL: http://dx.doi.org/10.1140/epjc/s10052-010-1294-6.

[177] I. Sawicki and W. Hu. "Stability of cosmological solutions in f(R) models of gravity". In: *Physical Review D* 75.12 (2007). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.75.127502.

[178] A. Sen. "Rolling Tachyon". In: *Journal of High Energy Physics* 2002.04 (2002), pp. 048–048. ISSN: 1029-8479. URL: http://dx.doi.org/10.1088/1126-6708/2002/04/048.

[179] M. Setare and N. Mohammadipour. "Cosmological viability conditions for f(T) dark energy models". In: *Journal of Cosmology and Astroparticle Physics* 2012.11 (2012), pp. 030–030. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2012/11/030.

[180] P. Shah and G. Samanta. "Stability analysis for cosmological models in f(R) gravity using dynamical system analysis". In: *The European Physical Journal C* 79 (2019).

[181] C. Skordis. "The tensor-vector-scalar theory and its cosmology". In: *Classical and Quantum Gravity* 26.14 (2009), p. 143001. URL: https://doi.org/10.1088%2F0264-9381%2F26%2F14%2F143001.

[182] M. A. Skugoreva, E. N. Saridakis, and A. V. Toporensky. "Dynamical features of scalar-torsion theories". In: *Physical Review D* 91.4 (2015). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.91.044023.

[183] T. P. Sotiriou. "Hořava-Lifshitz gravity: a status report". In: *Journal of Physics: Conference Series* 283 (2011), p. 012034. ISSN: 1742-6596. URL: http://dx.doi.org/10.1088/1742-6596/283/1/012034.

[184] T. P. Sotiriou and V. Faraoni. "f(R) theories of gravity". In: *Reviews of Modern Physics* 82.1 (2010), pp. 451–497. ISSN: 1539-0756. URL: http://dx.doi.org/10.1103/RevModPhys.82.451.

[185] S. K. Srivastava. *Tachyon as a Dark Energy Source*. 2004. arXiv: gr-qc/0409074 [gr-qc].

[186] A. A. Starobinsky. "Disappearing cosmological constant in f(R) gravity". In: *JETP Letters* 86.3 (2007), pp. 157–163. ISSN: 1090-6487. URL: http://dx.doi.org/10.1134/S0021364007150027.

[187] K. S. Stelle. "Renormalization of higher-derivative quantum gravity". In: *Physical Review D* 16 (1977), pp. 953–969. URL: https://link.aps.org/doi/10.1103/PhysRevD.16.953.

[188] N. Tamanini. "Dynamical systems in dark energy models". PhD thesis. 2014.

[189] S. Tsujikawa and M. Sami. "A unified approach to scaling solutions in a general cosmological background". In: *Physics Letters B* 603.3-4 (2004), pp. 113–123. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2004.10.023.

[190] R. Utiyama and B. S. DeWitt. "Renormalization of a Classical Gravitational Field Interacting with Quantized Matter Fields". In: *Journal of Mathematical Physics* 3.4 (1962), pp. 608–618. eprint: https://doi.org/10.1063/1.1724264. URL: https://doi.org/10.1063/1.1724264.

[191] H. E. S. Velten, R. F. vom Marttens, and W. Zimdahl. "Aspects of the cosmological "coincidence problem"". In: *The European Physical Journal C* 74.11 (2014). ISSN: 1434-6052. URL: http://dx.doi.org/10.1140/epjc/s10052-014-3160-4.

[192] H. Wei. "Spinor dark energy and cosmological coincidence problem". In: *Physics Letters B* 695.1-4 (2011), pp. 307–311. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2010.10.053.

[193] H. Wei. "Dynamics of teleparallel dark energy". In: *Physics Letters B* 712.4-5 (2012), pp. 430–436. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2012.05.006.

[194] S. Weinberg. "The cosmological constant problem". In: *Reviews of Modern Physics* 61 (1989), pp. 1–23. URL: https://link.aps.org/doi/10.1103/RevModPhys.61.1.

[195] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. Texts in Applied Mathematics. Springer New York, 2003. ISBN: 9780387001777. URL: https://books.google.co.uk/books?id=RSI4RGdwnU4C.

[196] S. Wiggins. *Ordinary Differential Equations*. 2017. URL: https://figshare.com/articles/dataset/Ordinary_Differential_Equations/5311612/2.

[197] D.-J. Wu. "Cosmological evolutions of F(R) nonlinear massive gravity". In: *Physical Review D* 90.4 (2014). ISSN: 1550-2368. URL: http://dx.doi.org/10.1103/PhysRevD.90.043530.

[198] D.-J. Wu, Y.-S. Piao, and Y. Cai. "Dynamical analysis of the cosmology of mass-varying massive gravity". In: *Physics Letters B* 721.1-3 (2013), pp. 7–12. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2013.02.055.

[199] P. Wu and H. Yu. "The dynamical behavior of f(T) theory". In: *Physics Letters B* 692.3 (2010), pp. 176–179. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2010.07.038.

[200] C. Xu, E. N. Saridakis, and G. Leon. "Phase-space analysis of teleparallel dark energy". In: *Journal of Cosmology and Astroparticle Physics* 2012.07 (2012), pp. 005–005. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2012/07/005.

[201] Y.-H. Yao and X.-H. Meng. "A coupled generalized three-form dark energy model". In: *Physics of the Dark Universe* 27 (2020), p. 100462. ISSN: 2212-6864. URL: http://www.sciencedirect.com/science/article/pii/S2212686419302316.

[202] Y. Zhang, T. Y. Xia, and W. Zhao. "Yang–Mills condensate dark energy coupled with matter and radiation". In: *Classical and Quantum Gravity* 24.13 (2007), pp. 3309–3337. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/24/13/011.

[203] Y. Zhang et al. "Notes on f(T) theories". In: *Journal of Cosmology and Astroparticle Physics* 2011.07 (2011), pp. 015–015. ISSN: 1475-7516. URL: http://dx.doi.org/10.1088/1475-7516/2011/07/015.

[204] W. Zhao. "Attractor Solution In Coupled Yang–mills Field Dark Energy Models". In: *International Journal of Modern Physics D* 18.09 (2009), pp. 1331–1342. ISSN: 1793-6594. URL: http://dx.doi.org/10.1142/S0218271809014947.

[205] W. Zhao and Y. Zhang. "Coincidence problem in YM field dark energy model". In: *Physics Letters B* 640.3 (2006), pp. 69–73. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2006.07.052.

[206] W. Zhao and Y. Zhang. "The state equation of Yang–Mills field dark energy models". In: *Classical and Quantum Gravity* 23.10 (2006), pp. 3405–3417. ISSN: 1361-6382. URL: http://dx.doi.org/10.1088/0264-9381/23/10/011.

[207] S.-Y. Zhou. "A new approach to quintessence and a solution of multiple attractors". In: *Physics Letters B* 660.1-2 (2008), pp. 7–12. ISSN: 0370-2693. URL: http://dx.doi.org/10.1016/j.physletb.2007.12.020.