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Information paradox under the scope of AdS/CFT correspondence

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Abstract

This thesis project is submitted for the Master of Science of Imperial College London Msc Quantum Fields and Fundamental Forces and it a review about Hawking imformation paradox and some recent developments concerning it from the perspective of AdS/CFT correspondence. The aim of the project is to present what the information paradox is and where it comes from, also how a black hole can be seen as a quantum mechanical object evolving in a unitary way. Then our interest will focus on a recent development which offers a way to recover the Page curve of the entropy of Hawking radiation which is an important tool to determine if a black hole as a quantum system evolves in a unitary manner, by considering a black hole system in 2-dimensions living in an asymptotically AdS_2 spacetime.

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Contents

Introduction

Black holes are some of the most interesting macroscopic objects we have ever detected in the universe and also their very existence has given a lot of food for thought in modern theoretical physics, in many different perspectives. During the 60s and 70s there was a productive and thorough theoretical study of black holes from classical perspective, where it had been realized that their mechanics obey some laws and behave like thermodynamic objects [1, 2] with temperature $T = \frac{\hbar\kappa}{2\pi k_B}$ (Hawking temperature), where κ is the surface gravity of the black hole, and entropy given by Bekenstein-Hawking expression $S_{BH} = \frac{Area}{4G_N\hbar}$ which is proportional to the area of the event horizon.

Those quantities took actual physical meaning when in 1974 Hawking [3] proved that considering a quantum field theory in a dynamical curved spacetime of a spherically symmetric black hole, an observer at causal infinity $\mathscr{J}^+$ detects a constant flow of particles emitted by the black hole. He also proved that the spectrum of this radiation follows the Planck distribution of black body's spectrum with temperature equal to the Hawking temperature above. This phenomenon now is called Hawking radiation and is the first endeavour of unifying a theory of dynamical spacetime and quantum field theory in a semiclassical approximation where spacetime itself is a classical geometric entity of General Relativity but is also supplied with quantum fields. Hawking radiation itself is a result of the fact that in a curved spacetime there is not a preferred way to define positive or negative modes in a field theory and this implies, in terms of multiparticle states, that the vacuum of a region next to black hole's event horizon does not seem empty at all, by an observer at infinity but full of multipartilcle states instead. Also microscopically it can be seen as the creation of entangled particle pairs where one is outgoing and escapes at infinity whereas its partner falls into the black hole, so the full Hilbert space consists of the subspace related to the outgoing states and the subsystem concerning the interior and can be written as $\mathcal{H} = \mathcal{H}_{int} \otimes \mathcal{H}_{rad}$.

Then the main implication of Hawking radiation was Information paradox [4] which is perhaps one of the most astonishing and subtle problems of theoretical physics come forth during the 20th century. Information paradox emerged soon after Hawking first published his famous paper about Hawking radiation when he argued that due to the radiation a black hole of mass M_0 loses mass and eventually shrinks up to the point that after time $t_{evap} \sim M_0^3$ completely evaporates leaving behind only thermal radiation. Therefore, all information concerning the initial quantum state of the black hole is lost forever. In other words, if we start with some collapsing matter to form a black hole described by a pure quantum state $|\psi\rangle_{BH}$ then after evaporation is completed we will have a thermal state of Hawking radiation. But the later does not contain any information about former black hole's degrees of freedom and moreover, it is **not** a pure state anymore. This means that from a pure quantum state we end up with a mixed thermal state, something which cannot be acquired by a unitary time evolution operator whose action on a pure state should give a pure state. This seems to violate one of key assumptions of quantum mechanics and formulates the paradox.

Of course the paradox is a paradox only under the assumption of the black hole being an ordinary quantum system with $\frac{Area}{4G_N\hbar}$ degrees of freedom, dependent on the area of the horizon. During the 90s works by Preskill and Suskind [5, 6] gave a new perspective to the black hole realization by introducing the so-called holographic principle which reduces the study of a black hole as a d-dimensional gravitational system to d-1-dimensional holographic quantum theory. Then in 1997 the discovery of AdS/CFT correspondence became the best studied example of a holographic duality. The hypothesis mentioned above is widely accepted today especially from the perspective of string theory and is considered as a main basis for studying the information paradox. Subsequently, whatever the solution for the paradox may be it must respect the unitarity of the black hole as an ordinary quantum system.

An implication of unitarity is that the time evolution of the entanglement or von Neumann entropy of Hawking radiation $S_R = -Tr_{BH}(\rho_R \log \rho_R)$ (where we trace out the black hole degrees of freedom and ρ_R the density operator concerning the state of radiation) should follow the *Page curve* [7, 8, 9]. Initially, just after black hole's formation, the entanglement entropy of radiation is very small compared to Bekenstein entropy (practically zero) and should increase, until it reaches a maximum value equal to Bekenstein entropy (where the degrees of freedom of radiation are equal to black hole's ones) at Page time $t \sim 0.6 t_{evap}$.

Then it starts decreasing until the final evaporation where it takes zero value. The usefulness of Page curve is determined by the fact that its successful derivation by certain gravity theory would indicate that this theory would respect unitarity and therefore, information paradox seems to be resolved. However, we have to emphasize the that Page curve on its own can tell us nothing sufficient about the final quantum state after evaporation is completed. Notice that Hawking himself strongly disagreed with unitarity and argued that the entanglement entropy entropy of Hawking radiation should increase monotonically from whence the black hole is created and settles down until it finally evaporates [4].

A major difficulty in order to recover the Page curve is that we do not have yet a full, consistent theory of quantum gravity which means that it is not known what the exact state of the system radiation-black may be. Therefore, it is not easy to calculate the entropy by direct calculation using its definition. Nevertheless, in the last 15 years there has been significant development in this field in terms of AdS/CFT correspondence. It turns out that the entanglement entropy of a gravitational system coupled to a conformal field theory living in d-dimensions is associated with finding a minimal (or rather extremal) surface in $d+1$-dimensional dual bulk theory in AdS_{d+1} spacetime. The first formula was proposed by Ryu and Takyanagi [10] and then it has been extended by various authors [11, 12, 13]. The most recent realization is that the entanglement entropy of a gravitational system of an evaporating black hole is given by a minimal quantum extremal surface [13]. This is a surface that extremizes the generalized entropy (Bekenstein+von Neumann entropy of quantum fields in black hole region) and if there are more than one such surfaces the minimal one determines the entropy. The benefit of Ryu-Takyanagi prescription is that the problem of entropy becomes a geometrical problem which in general is easier to be solved.

Recent works [14, 15] show that for an evaporating black hole at early times the extremal surface is the trivial and thus the initial increase in entropy of radiation happens due to the quantum fields contribution, as the Hawking radiation starts to escape to infinity. If the early state of the black is pure then the whole interior region belongs to the entanglement region or the *entanglement wedge* of the black hole. Past the Page time there is a new quantum extremal surface which is located behind the event horizon and at Page time there is a phase transition where the surface behind the horizon becomes minimal. Then most of black holes's interior is now excluded from the entanglement wedge of the black hole and the as the black hole continues to evaporate and shrink the corresponding area shrinks too and thus the entanglement entropy of the black hole.

A new proposal in 2019 [16] is that the Page curve of Hawking radiation can be recovered by considering a gravity theory in 2-dimensions with holographic matter, i.e. a conformal field theory which has a dual bulk theory in 3-dimensions coupled to a 2-dimensional holographic bath. The main innovative idea is that finding appropriate quantum extremal surfaces in this case is equivalent with Ryu-Takyanagi (or rather its cavariant) prescription and it follows that the extremal surfaces of radiation and black hole coincide and thus it follows the same rise before the Page time and then the same decrease recovering the full Page curve for radiation entropy. An important point is that this prescription gives rise to a disconnected entanglement wedge of radiation which consists of two region: one would represent Hawking result from the radiation escaped at infinity but also there is a second region which covers most of the interior at late times beneath the quantum extremal surface. This region is called quantum extremal island and it is a region in the gravitational theory where matter is entangled with the external quantum system. Another point which is worth noticing, is that this island, which in 2-dimensional realization is disconnected with the rest of the entanglement wedge, in the dual 3-dimensional description is actually connected via the extra dimension.

The aim of this thesis project is to discuss the above topics. The thesis is organized in four chapters. We start our discussion with a review of black hole thermodynamics and also a thorough presentation of Hawking radiation considering that unfortunately, due to the COVID19 pandemic was not covered in QFFF class. The next chapter is about the entanglement entropy in quantum mechanical systems and we analyze information paradox. Then follows a discussion about what AdS/CFT correspondence is and some of its main conclusions especially its usefulness in calculating entanglement entropies with Ryu/Takyanagi prescription. Finally, in the last chapter we are going to review the paper [16] and see how the Page curve can be obtained by holography. The standard conventions are $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ and in the standard natural units $c = \hbar = k_B = G_N = 1$, although we mostly write down the Newton's constant.

Chapter 1

Hawking radiation

1.1 Review of black hole Thermodynamics

In this chapter we will try to review some topics concerning the black hole mechanics and thermodynamics and then we will see that black holes are actually thermal bodies with non zero temperature and therefore they emit Hawking radiation. This chapter is mainly based on the original papers from 70s' of Hawking and Beckenstein [1, 3] as well as lecture notes by F. Dowker [17], J. Gauntlett [18] (Imperial College London),H. Reall [19] P.K. Townsend [20] (University of Cambridge). Many details for black hole thermodynamics are omitted for the sake of simplicity. The goal is to give a main idea that the black holes are thermodynamic objects and where the Hawking radiation comes from.

Uniqueness theorems

Black holes are some of the most remarkable results provided by General Relativity (GR). The basic way of the formation of a black hole is by the stellar collapse of a supermassive star during a supernovae explosion. Soon after its formation it is classically thought the black hole settle down to a "time independent"[1], stationary state.

By stationary state we mean a state where the black hole spacetime is asymptotically flat [17] and it admits a Killing vector k which is timelike near causal infinity. If a black hole with mass M is rotating with angular momentum $J = Ma$ (in that context a is the angular momentum per unit mass) and also has electric charge Q then this black hole is described by Kerr-Neuman solution (1965) [21] of the Einstein's equations in the vacuum

[1] In reality this happens very fast

of Einstein-Maxwell theory[2] with metric in Boyer-Lindquist coordinates (with $G_N = 1$)

$$ds^2 = -\Big(\frac{\Delta - a^2 sin^2\theta}{\Sigma}\Big)dt^2 + \frac{\Sigma}{\Delta}dr^2 - 2\frac{asin^2\theta}{\Sigma}(r^2 + a^2 - \Delta)dtd\phi$$

$$+\Sigma d\theta^2 + \Big(\frac{(r^2 + a^2)^2 - \Delta a^2 sin^2\theta}{\Sigma}\Big)sin^2\theta d\phi^2 \tag{1.1}$$

where $\Sigma = r^2 + a^2 cos^2\theta$ and $\Delta = r^2 - 2Mr + Q^2 + a^2$ with a corresponding electromagnetic potential expressed as

$$A = (A_t, A_r, A_\theta, A_\phi) = \Big(\frac{Qr}{\Sigma}, 0, 0, -\frac{Qarsin\theta}{\Sigma}\Big) \tag{1.2}$$

This is the most general solution for charged rotating black holes. Moreover, for $a = 0$, $Q = 0$ we recover the spherically symmetric solution i.e. the usual Schwarzschild metric whereas if only $Q = 0$ then the metric (1.1) reduces to the Kerr solution.

The Kerr-Neumann solution is axisymmetric which means that it admits another Killing vector m that is spacelike near causal infinity and all orbits of m are closed [20]. In local coordinates this Killing vector can be expressed as $m = \partial_\phi$.

Between 1967 and 1975 were proved some very powerful uniqueness theorems by Israel (1967)[22], Carter (1971) [23], Hawking (1973) [24] and Robinson (1975) [25] which lead to the conclusion that the unique stationary asymptotically flat black hole solution of the vacuum Einstein-Maxwell theory is the Kerr-Neumann with three parameter family M, J, Q. Those theorems indicate that no matter how the black hole was initially formed, the equilibrium state will be Kerr-Neumann and the information about its formation has been lost either by falling into the black hole or by radiation.

The conclusions of those theorems is the so-called no-hair theorem "the black holes have no hair" first quoted by J. Wheeler. The no hair theorem seems presumably correct at least in the classical description of a black hole. In other words one black hole can be fully described by a family of just three classical parameters mass, angular momentum and electric charge.

We carry on by giving a few definitions and theorems, for more details we refer the

[2] The vacuum of gravity theory coupled with Electromagnetism in classical regime.

reader to Black holes Lecture Notes by Dr. P.K. Townsend [20]. Let $S(x)$ be a smooth function of some spacetime coordinates x^μ. Then we can obtain a family of hypersurfaces with normal vector field to those hypersurfaces

$$l = f(x)(g^{\mu\nu}\partial_\nu S)\frac{\partial}{\partial x^\mu} \tag{1.3}$$

where f an arbitrary non zero function.

Definition: A hypersurface $\mathcal{N}$ for which $l^2 = 0$ is called null hypersurface.

A null hypersurface has the property that since for a normal on it $l^2 = l \cdot l = 0$ any normal on that surface is also tangent vector.

Definition: A null hypersurface $\mathcal{N}$ is called Killing horizon if there is a Killing vector field ξ which is normal to the null hypersurface $\mathcal{N}$.

There is also an interesting theorem by Hawking (1972) [26]

Theorem: For an analytic, asymptotically flat vacuum (or Einstein-Maxwell vacuum) spacetime of a black hole the future event horizon $\mathcal{H}$ is also a Killing horizon. By the definition of the Killing horizon we conclude that

$$\xi = fl \tag{1.4}$$

Subsequently we have for ξ that

$$\xi^\mu \nabla_\mu \xi^\nu = fl^\mu \nabla_\mu (fl^\nu) = fl^\nu l^\mu \nabla_\mu f = \kappa \xi^\nu \tag{1.5}$$

where

$$\kappa = \xi^\mu \partial_\mu ln|f| \tag{1.6}$$

this quantity is called surface gravity and it is called so because is constant on the horizon[3] and its physical meaning is just the force which must be exerted at infinity in order to maintain a unit mass on the horizon.

Now we have not said anything about the stress energy tensor of Einstein's equations. We want to consider only physical matter which respects the causal structure of spacetime. Therefore, there is a motivation to define some energy conditions the stress energy tensor must respect.

Dominant energy condition: The stress energy tensor $T_{\mu\nu}$ satisfies the dominant en-

[3] This is indeed the zeroth law of black hole thermodynamics

ergy condition if for all future-directed timelike vector fields v, the vector field

$$j(v) = -v^{\mu}T^{\nu}_{\mu}\partial_{\nu} \tag{1.7}$$

called "energy-momentum current" is future directed causal.
Weak energy condition: It is a less restrictive condition which states that

$$T_{\mu\nu}v^{\mu}v^{\nu} \geq 0 \tag{1.8}$$

for any timelike vector v. Note that the weak energy condition is actually implied by the dominant which is the physically important one.

Black hole thermodynamics

We are now proceeding to the laws of black hole thermodynamics. In 1973 a paper by Bardeen, Carter and Hawking [1] was published concerning the "The laws of black hole thermodynamics". The idea of this article was that as in classical thermodynamics there are four fundamental laws which determine the state of a classical thermodynamic system, for the black holes also exist four laws which correspond to the thermodynamic ones.
We now present those laws.

Oth Law

The surface gravity κ is constant on the future event horizon $\mathcal{H}^+$ of a stationary black hole spacetime obeying the dominant energy condition.
This law transcends the 0th law of thermodynamics which states that the temperature T throughout a physical thermodynamic system in thermal equilibrium is uniform across the system.

1st Law

The 1st Law of black hole thermodynamics implies the conservation of energy by expressing how the three fundamental classical quantities M, J and Q change if a black hole initially characterized by those three quantities is perturbed to a new stationary state characterized by $M + dM$, $J + dJ$, $Q + dQ$. The mathematical expression of the 1st Law is

$$dM = \frac{1}{8\pi}\kappa dA + \Omega_H dJ + \Phi_H dQ \tag{1.9}$$

where here Ω_H is the angular velocity of the black hole, Φ_H is the surface electric potential and A is the Area of the horizon. The thermodynamic analog is the usual first law of thermodynamics which is expressed as $dE = TdS + \sum_i \mu_i dN_i$.
Here again the first law has a similar structure with the thermodynamic analog. Again the quantity $\frac{\kappa}{8\pi}$ is like the temperature and the area A is the analog of entropy.

2nd Law (Area Law)

This law demands that the Area of the event horizon can never decrease so

$$\delta A \geq 0 \tag{1.10}$$

That implies that if we start for example with two stationary black holes which eventually merge and form a new black hole, the Area of the new black hole must be greater than the sum of the surfaces of the initial ones.
Again it is evident the analog of entropy from the second law of thermodynamics which states that $\delta S \geq 0$ is the surface A. But the second law of black hole thermodynamics is slightly stronger than the corresponding traditional second thermodynamic law. Classically we can transfer entropy from one system to another, under the requirement that the total entropy of the universe does not decrease. However in the case of black holes it is not possible to do this with the area, from one black hole to another since black holes cannot bifurcate. Hence, the second law requires that the area of each black hole should not decrease individually.

3rd Law

It is impossible to reduce surface gravity κ to zero by a physical process. The corresponding 3rd thermodynamic law states that we cannot achieve zero temperature by a physical process.

Bear in mind that initially the authors of this paper [1] perceived those quantities as ones which just correspond to the original thermodynamic but they do not have exact physical correspondence but rather purely mathematical. The primary idea had been that the black holes have zero temperature, since nothing can escape from them, and therefore do not radiate nor possess physical entropy. This idea of course changed swiftly after Hawking published his famous paper on black hole radiation [3].
Nontheless, a year later in 1974 Bekenstein [2] argued that the second law of thermodynamics would be violated if black holes had no entropy, since one could throw an arbitrary

large amount of matter into a black hole and thus reduce the total entropy of the universe according to an observer who stands outside of the hole horizon, but this is a disaster because we demand that the classical thermodynamic laws are still valid. Thus, he proposed a generalized second Law which affirms that the total entropy of the entire universe must not decrease and by total entropy or generalized entropy otherwise we mean the sum of the entropy of the external universe and the entropy of the black hole which should be proportional to the area A. In other words defining $S_{gen} = S_{ext} + S_{BH}$

$$\delta S_{gen} = \delta(S_{ext} + S_{BH}) \geq 0 \tag{1.11}$$

Then in the same year Hawking derived the result that black holes in reality are thermal bodies and they radiate with black body temperature (expressed with all constants $G_N, \hbar, k_B$)

$$T = \frac{\hbar\kappa}{2\pi k_B} \tag{1.12}$$

and the black hole entropy is

$$S_{BH} = \frac{A}{4G_N\hbar} \tag{1.13}$$

Which is called Bekenstein-Hawking entropy[4]. Meanwhile the quantity $G\hbar$ has dimension $[l]^2$ and so we can define the Planck length $l_P = \sqrt{G\hbar}$ to express thw entropy as $S_{BH} = \frac{A}{4l_P^2}$. Also, if we work in natural units $\hbar = k_B = 1$ then we have the term in first law of black hole thermodynamics analogous with the classical thermodynamic temperature. Thus we realize that this analogy was not just a bare coincidence but a one to one correspondence or rather a physical unification [17].

1.2 Quantum Field Theory in curved spacetime

Our goal now is to show and elaborate the result of Hawking (1974) [3]. We begin by generalizing the quantum field theory from the flat Minckowski space to an arbitrary curved spacetime with metric $g_{\mu\nu}$. We focus on the real, scalar field ϕ with Lagrangian

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 \tag{1.14}$$

[4]We will simply call it from now and on Bekenstein entropy or simply just black hole coarse grained entropy see chapter 2. Note also that Bekenstein did not have in his mind that the entropy of the black hole had the expression named after him but rather a more general one.

The equation of motion is the Klein-Gordon equation in curved spacetime expressed as

$$\nabla_\mu \nabla^\mu \phi - m^2 \phi = 0 \tag{1.15}$$

where $\phi = \phi(t, \vec{x})$ and the conjugate momentum defined as

$$\pi = \frac{\partial \mathcal{L}}{\partial(\nabla_0 \phi)} = \sqrt{-g}\dot{\phi} \tag{1.16}$$

As always for the quantization of the system we consider the usual equal time commutation relations:

$$[\hat{\phi}(t, \vec{x}), \hat{\phi}(t, \vec{y})] = 0 \tag{1.17}$$

$$[\hat{\pi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = 0 \tag{1.18}$$

$$[\hat{\phi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = \frac{i}{\sqrt{-g}}\delta^3(\vec{x} - \vec{y}) \tag{1.19}$$

We assume a globally hyperbolic spacetime i.e. one which has a *Cauchy surface*. **Definition:** A partial Cauchy surface Σ is a hypersurface where no past and future inextendible causal curve intersects more than once. when a manifold admits a Cauchy surface, it can be foliated by a family of Cauchy surfaces.

The significance of Cauchy surfaces lies in the fact that once a Cauchy surface has been chosen along with some initial data on it, the corresponding solution to the equations of motion is completely determined on the entire spacetime [17]. Consequently, everything can be determined from a set of initial data.

We will define a new inner product as between two (generally complex) functions

$$(f, g) = i \int_\Sigma d\Sigma \sqrt{-g}(f_1 \nabla_\mu g^* - g \nabla_\mu f^*) \tag{1.20}$$

and for two real scalar fields ϕ_1, ϕ_2 will be

$$(\phi_1, \phi_2) = i \int_\Sigma d\Sigma \sqrt{-g}(\phi_1 \nabla_\mu \phi_2 - \phi_2 \nabla_\mu \phi_1)$$

For Klein-Gordon solutions this inner product does not depend on the choice of the Cauchy slice. To show that let's take two arbitrary Cauchy slices Σ, Σ' then

$$(\phi_1, \phi_2)_{\Sigma'} - (\phi_1, \phi_2)_\Sigma = i \int_{\Sigma'} d\Sigma \sqrt{-g}(\phi_1 \nabla_\mu \phi_2 - \phi_2 \nabla_\mu \phi_1) - i \int_\Sigma d\Sigma \sqrt{-g}(\phi_1 \nabla_\mu \phi_2 - \phi_2 \nabla_\mu \phi_1)$$

$$= i \int_{\partial S} dS \sqrt{-g} (\phi_1 \nabla_\mu \phi_2 - \phi_2 \nabla_\mu \phi_1) = i \int_S \nabla^\mu (\phi_1 \nabla_\mu \phi_2 - \phi_2 \nabla_\mu \phi_1)$$

where in the last step we used Gauss ' theorem. Now note that

$$\nabla^\mu (\phi_1 \nabla_\mu \phi_2 - \phi_2 \nabla_\mu \phi_1) = \nabla^\mu \phi_1 \nabla_\mu \phi_2 + \phi_1 \nabla^2 \phi_2 - \nabla^2 \phi_1 \phi_2 - \nabla_\mu \phi_1 \nabla^\mu \phi_2 = \phi_1 m^2 \phi_2 - m^2 \phi_1 \phi_2 = 0$$

and therefore

$$(\phi_1, \phi_2)_{\Sigma'} = (\phi_1, \phi_2)_\Sigma$$

When we do the quantization of the scalar field in Minckowski spacetime we expand the field in basis of positive and negative frequencies $e^{\pm ipx}$ which is Lorentz invariant and so respects the symmetries of Minckowski space. Nonetheless, in a curved spacetime there is not such a thing as the Lorentz invariance. Thus, it is not so easy to define a basis of positive and negative frequencies whatsoever, because a using a different basis the new observer would not necessarily see the same positive or negative frequencies. In fact, even worse, there is not a preferred choice of basis.
So first of all we consider an orthonormal basis $\{\psi_i\}$ [18] such that

$$\begin{aligned} (\psi_i, \psi_j) &= \delta_{ij} \\ (\psi_i, \psi_j^*) &= 0 \\ (\psi_i^*, \psi_j^*) &= -\delta_{ij} \end{aligned} \tag{1.21}$$

where here the indices i, j have similar role as the momentum in the flat Minckowski space and they are continuous indices but we emphasize they do not represent the physical momentum because on Minkowski space we assume the on-shell condition whereas here not. Now the quantized field can be expressed by expanding in that basis as

$$\phi(x) = \sum_i a_i \psi_i(x) + a_i^\dagger \psi_i^*(x) \tag{1.22}$$

and we require that the creation and annihilation operators obey the usual commutation relations of quantum field theory $[a_i, a_j^\dagger] = \delta_{ij}$. The basis of the Hilbert space is the vacuum $|0\rangle$ with $a_i |0\rangle = 0$ and the multiparticle states constructed by the action of creation operator on the vacuum. Nevertheless, the above condition does not fix the space of all solutions. In a region of an asymptotically flat spacetime the basis functions ψ_i should contain only positive frequencies with respect to the time coordinate and would remain positive for all observers whereas for a general curved spacetime there is nothing which demands this anymore. So if we start, for instance, with our initial basis which can be seen as an initial flat region followed by a region of curved spacetime and then again by a third flat region, the vacuum of the final region will not be the same as the one of the initial one because the basis of the first region will not be identical to the one in third region while both contain positive frequencies. Subsequently, the action of annihilation operator of the first region will not generally give zero if it acts on the vacuum of the third. This rather obscure statement can be interpreted as a particle creation of the scalar field by the gravitational field itself.

Now if we take another basis $\{\psi_i'\}$ we can expand ψ' with respect to the previous basis as

$$\psi_i' = \sum_i A_{ij}\psi_j + B_{ij}\psi_j^* \tag{1.23}$$

The new basis satisfies the orthonormality relations if

$$AA^\dagger - BB^\dagger = I \tag{1.24}$$

$$AB^T - BA^T = 0 \tag{1.25}$$

The A,B coefficients satisfying the above conditions are called *Bogoliugov coefficients*.

Now if we want to invert (1.23) we would have

$$\psi_j = \sum_k A'_{jk}\psi_k' + B'_{jk}\psi_k'^* \tag{1.26}$$

with

$$A' = A^\dagger, B' = -B^T \tag{1.27}$$

and the requirement that A', B' obey the same conditions as A, B do gives that

$$A^\dagger A - B^T B^* = I \tag{1.28}$$

$$A^\dagger B - B^T A^* = 0 \tag{1.29}$$

1.3 Particle creation in curved spacetime

Now in order to elaborate this result of particle creation we consider a globally hyperbolic spacetime which is time dependent and is stationary at early times, then becomes non-stationary and at late times becomes again stationary [19]. We can denote this spacetime as $M = M^{-} \cup M_0 \cup M^{+}$ where -,0 and $+$ for early, intermediate and late times respectively. So we have for that spacetime

- Region 1: M^{-} which corresponds to the spacetime at very early times before the formation of the black hole and it is asymptotically flat. We take this region at $\mathscr{J}^{-}$.
- Region 2: M^{0} which is the curved non-stationary spacetime around the black hole after its formation.
- Region 3: M^{+} at late times after gravitational collapse and very far from the black hole. Hence this spacetime is asymptotically flat. We then take that region in $\mathscr{J}^{+}$.

At early and late times as we said the spacetime is stationary and there is indeed a preferred basis of positive frequencies (which are well defined) subspace. Label this preferred choice of positive frequency eigenfunctions as u_i with respect to a stationary future-directed time translation Killing vector field k [18]. We also choose them to obey the standard orthonormality relations. Since this Killing vector generates a symmetry, the Lie derivative of the field ϕ with respect to k, $\mathcal{L}_k\phi$ must be a solution of Klein-Gordon equation provided ϕ is a solution itself and therefore, we deduce that k maps solutions of Klein-Gordon to some other solutions. Furthermore, the Lie derivative $\mathcal{L}_k$ is an antihermitian operator, a fact which indicates that we can choose a basis of positive frequencies with imaginary eigenvalues. $\mathcal{L}_k u_i = -i\omega_i u_i$ with $\omega_i > 0$

If there is a positive mode in the first region passing through the curved specetime region of the black hole, then an observer in the third region may see a superposition of positive and negative modes. This can be viewed as the fact that the gravitational field in region 2 is a new potential V which can alter the original quantum state as a perturbation and therefore, in region 3 a superposition of positive and negative frequencies may be revealed.

Having considered the above we are now able to expand the field with respect to the preferred positive frequency eigenfunctions, $u_i^{\pm}$ which are solutions of Klein-Gordon in $M^{\pm}$. Notice also that for the intermediate region the field cannot be expressed in such a basis since the spacetime is not stationary.
Hence we have in $M^{\pm}$

$$\phi(x) = \sum_i a_i^{\pm} u_i^{\pm}(x) + a_i^{\pm\dagger} u_i^{*\pm}(x)$$

We also saw that since the inner product defined by (1.20) does not depend on the choice of the Cauchy surface hence the matrix product of A, B will be same. So, expressing the field as

$$\phi = \sum_i a_i^- u_i^- + a_i^{-\dagger} u_i^{*-}$$

$$= \sum_i \left(a_i^- \sum_j (A_{ij} u_j^+ + B_{ij} u_j^{*+}) + a_i^{-\dagger} \sum_j (A_{ij}^* u_j^{*+} + B_{ij} u_j^+) \right)$$

$$= \sum_{i,j} (a_i^- A_{ij} + a_i^{-\dagger} B_{ij}^*) u_i^+ + h.c. = \sum_i (a_i^+ u_i^+ + a_i^{+\dagger} u_i^{*+})$$

we conclude that the Bogoliugov transformation of the creation-annihilation operators are

$$a_i^+ = \sum_j (a_i^- A_{ij} + a_i^{-\dagger} B_{ij}^*) \tag{1.30}$$

We define now the number operator of the field in regions $M^\pm$ respectively as

$$N^\pm = a_i^{\pm\dagger} a_i^\pm \tag{1.31}$$

What would one see if they act with number operator N_i^+ on the vacuum of region M^- denoted as $|0^-\rangle$?
Assuming the corresponding vacuum expectation value one can notice that

$$\langle 0^- | N_i^+ | 0^- \rangle = \langle 0^- | a_i^{+\dagger} a_i^+ | 0^- \rangle = \sum_{j,k} \langle 0^- | (a_k^- B_{ki})(a_j^{-\dagger} B_{ji}^*) | 0^- \rangle$$

$$= \sum_{j,k} \langle 0^- | a_k^- a_j^{-\dagger} | 0^- \rangle B_{ki} B_{ij}^\dagger = (B^\dagger B)_{ii} = Tr(B^\dagger B)$$

So the expected number of particles is

$$\langle N_i^+ \rangle = Tr(B^\dagger B) \tag{1.32}$$

[5] and since $B^\dagger B$ is positive by its definition it turns out that this expectation value vanishes if and only if B vanishes. Non zero B implies particle creation which escapes from the black hole at infinity. Our aim for now and on is to discover the form of this Bogoliugov coefficient B_{ij}. We will eventually see that this radiation turns out to be thermal. Let's have a look over that amazing result calculated by Hawking in 1973 which eventually was named after him, Hawking radiation and it's the first essential result of coupling

[5] here there is no summation over the i index

between the gravity of GR and the quantum field theory.

For the purpose of the calculation we are going to consider a spherically symmetric solution of the Einstein equation, the Schwarzschild metric, which illustrates a non rotating uncharged black hole, and a massless scalar field ϕ. The Schwarzschild metric in 4-dimensions is

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2 = \Big(1 - \frac{2M}{r}\Big)(-dt^2 + dr_*^2) + r^2 d\Omega_2^2 \tag{1.33}$$

using tortoise coordinate r_* for which

$$\frac{dr^2}{1 - \frac{2M}{r}} = \Big(1 - \frac{2M}{r}\Big)dr_*^2 \Rightarrow r_* = r + 2M \log\Big|\frac{r}{2M} - 1\Big|$$

and it can be also expressed by the advanced and retarded coordinates $u = t - r_*$ and $v = t + r_*$ as

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)(-du dv) + r^2 d\Omega_2^2 \tag{1.34}$$

the tortoise coordinate r_* for large r, $r_* \approx r$ and so we can use it in the regions 1 and 3 instead of r.

In the spherically symmetric metric the Klein-Gordon equation of (1.15) can be rewritten using the formula from differential geometry

$$\nabla^\mu \nabla_\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \tag{1.35}$$

and writing $\phi = \phi(t, r_*, \theta, \phi)$. Expanding in spherical harmonics $\phi(t, r_*, \theta, \phi) = e^{-i\omega t} R_l(r_*) Y_{lm}(\theta, \phi)$ we have for the radial part $R_l(r_*)$ that it solves the equation

$$(\partial_{r_*}^2 + \omega^2 - V_l(r_*)) R_l(r_*) \tag{1.36}$$

where

$$V_l(r_*) = \Big(1 - \frac{2M}{r}\Big)\Big[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\Big] \tag{1.37}$$

This is the potential barrier we mentioned before.
Near $\mathscr{J}^{\pm}$ the solutions are just plane waves. Specifically on $\mathscr{J}^{-}$

$$f_{lm\omega'(out)} = \frac{1}{(2\pi\omega')^1/2} e^{-i\omega' u}\frac{Y_{lm}}{r} \qquad \text{(outgoing)} \tag{1.38}$$

$$f_{lm\omega'(in)} = \frac{1}{(2\pi\omega')^1/2} e^{-i\omega' v}\frac{Y_{lm}}{r} \qquad \text{(ingoing)} \tag{1.39}$$

and on $\mathscr{J}^{+}$

$$g_{lm\omega(out)} = \frac{1}{(2\pi\omega)^1/2} e^{-i\omega u}\frac{Y_{lm}}{r} \qquad \text{(outgoing)} \tag{1.40}$$

$$g_{lm\omega(in)} = \frac{1}{(2\pi\omega)^1/2} e^{-i\omega v}\frac{Y_{lm}}{r} \qquad \text{(ingoing)} \tag{1.41}$$

A positive mode g_ω at $\mathscr{J}^{+}$ is given by (1.40) and we would like to investigate its past, in other words how it is related with the ingoing mode $f_{\omega'}$ at $\mathscr{J}^{-}$ given by (1.38). Note that in reality plane waves such as g_ω at $\mathscr{J}^{+}$ are completely delocalized [18], but despite that, we still can construct wave packets on $\mathscr{J}^{+}$ which are localized around an ω_0 and u_0.
The mode g_ω can be expressed as an integral over the incoming frequencies ω' at $\mathscr{J}^{-}$ as

$$g_\omega = \int_0^\infty d\omega' (A_{\omega\omega'} f_{\omega'} + B_{\omega\omega'} f^*_{\omega'}) \tag{1.42}$$

We assume that g_ω, following a null cosmic line γ, arrives at $\mathscr{J}^{+}$ in infinite time. So going backwards in time from $\mathscr{J}^{+}$ the wave g_ω at some point was close to the event horizon $\mathcal{H}^{+}$ and eventually meets the dynamical potential barrier. Then part of the wave g^R_ω will be reflected by the barrier to $\mathscr{J}^{-}$ and will not experience the curved geometry at all. Therefore, it ends up to $\mathscr{J}^{-}$. The other part g^T_ω of the wave transmits through the barrier and thus enters the time dependent geometry and ends up as a mixture of positive and negative modes on $\mathscr{J}^{-}$.

In fact, since we are interested in a wave packet localized and peaked at late times where $u_0 >> 1$ and finite frequency ω_0, the wave packet will be peaked at a very high frequency as it enters the collapsing matter due to the gravitational blueshift [17]. Because of that we are allowed to use *geometric optics approximation* [27] where $g_\omega = A(x)e^{iS(x)}$ and $A(x)$ is has very small variation in comparison with $S(x)$. Subsequently, the Klein-Gordon gives $\nabla^\mu S \nabla_\mu S = 0$ which implies that surfaces of constant phase are null. These surfaces seem to accumulate close to the horizon.

Consider now a null geodesic congruence which contains those hypersurfaces and also the event horizon which is at $S = \infty$. Take also its tangent vector l and also a future directed null vector n, generator of the Killing horizon $\mathcal{H}^+$ at a point x which is directed inwards and has $nl = -1$. This is a connecting vector because $-\epsilon n$ $(\epsilon > 0)$ connects a point on the event horizon with a nearby null hypersurface of constant phase. The spherical symmetry of the problem allows us to take the vector n with vanishing angular components.

The mode g_ω going backwards following the null geodesic γ starts at some point with coordinate $u = u_0$ at $\mathcal{J}^+$ and hits $\mathcal{J}^-$ at some $v = v_0$. The generator of $\mathcal{H}^+$ can be extended to the past so that it hits $\mathcal{J}^-$ at some point with coordinate, without loss of generality, $v = 0$ [6]. Then $v_0 < 0$. Near the horizon the Kruskal coordinates ($U = -e^{-\kappa u}$, $V = e^{\kappa v}$) define an affine distance along n and we can use them in order to measure the distance between γ and γ_H. Then outside the horizon the null geodesic is located at $U = -c\epsilon$. By the definition of U itself we deduce that $u = -\frac{1}{\kappa}\log(-U)$ and so at late times γ will have coordinate $u = -\frac{1}{\kappa}\log(c\epsilon)$ with c positive constant.
Furthermore, since γ is outgoing null geodesic with phase $g_\omega \sim e^{i\omega u}$ we take [20]

$$-i\omega u = \frac{i\omega}{\kappa}\log(c\epsilon) \tag{1.43}$$

with $c > 0$ constant and then at $\mathcal{J}^-$ l, n can be expressed in u, v coordinates as $l \sim \partial_u$, $n \sim D^{-1}\partial_v$ where D again a positive constant. Thus on $\mathcal{J}^-$ the proper distance between γ and γ_H is $-D^{-1}\epsilon$ and the phase $\frac{i\omega}{\kappa}\log(-cDv)$. Therefore, the transmitted wave on $\mathcal{J}^-$ ignoring the nornalization and a constant phase is given by

$$g^T_\omega \sim e^{\frac{i\omega}{\kappa}\log(-v)} \quad v > 0 \tag{1.44}$$

and it vanishes for $v \leq 0$.
Now in order to find the coefficients $A_{\omega\omega'}$ and $B_{\omega\omega'}$ of (1.42) consider the Fourier transformation of the g^T_ω

$$\tilde{g}_\omega(-\omega') = \int_{-\infty}^{+\infty} dv e^{i\omega' v} g^T_\omega(v) = \int_{-\infty}^{0} dv e^{i\omega' v + \frac{i\omega}{\kappa}\log(-v)} \tag{1.45}$$

[6]since the spacetime is invariant in translations

Now for $g^T_\omega(-\omega')$ it can be proven [17, 20] that

$$\tilde{g}^T_\omega(-\omega') = -e^{\frac{\pi\omega}{\kappa}}\tilde{g}^T_\omega(\omega'), \quad \omega' > 0 \tag{1.46}$$

Having this result now we take the inverse Fourier transformation

$$g^T_\omega(v) = \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}e^{-i\omega' v}\tilde{g}^T_\omega(\omega') = \int_0^{+\infty}\frac{d\omega'}{2\pi}e^{-i\omega' v}\tilde{g}^T_\omega(\omega') + \int_0^{+\infty}\frac{d\omega'}{2\pi}e^{+i\omega' v}\tilde{g}^T_\omega(-\omega')$$

$$= \int_0^{\infty} d\omega' N_{\omega'} f_{\omega'}(v)\tilde{g_\omega}^T(\omega') + \int_0^{\infty} d\omega' N^*_{\omega'} f^*_{\omega'}(v)\tilde{g_\omega}^T(-\omega')$$

where $N_{\omega'}$, $N^*_{\omega'}$ are normalization factors. Hence we have for positive ω, ω'

$$A_{\omega\omega'} = N_{\omega'}\tilde{g}^T_\omega(\omega)$$
$$B_{\omega\omega'} = N^*_{\omega'}\tilde{g}^T_\omega(-\omega)$$

and thereby using the result of (1.46) we finally find

$$|B_{\omega\omega'}| = e^{-\frac{\pi\omega}{\kappa}}|A_{\omega\omega'}| \tag{1.47}$$

Now finally the normalization of the transmitted wave is given by the defined inner product

$$\Gamma_\omega = (g^T_\omega, g^T_\omega) = \int_{\omega'}\int_{\omega''}(A_{\omega\omega'}f_{\omega'} + B_{\omega\omega'}f^*_{\omega'}, A_{\omega\omega''}f_{\omega''} + B^*_{\omega\omega''}) = \int_{\omega'}(|A_{\omega\omega'}|^2 - |B_{\omega\omega'}|^2)$$

$$= (e^{\frac{2\pi\omega}{\kappa}} - 1)\int_{\omega'}|B_{\omega\omega'}|^2 = (e^{\frac{2\pi\omega}{\kappa}} - 1)(BB^\dagger)_{\omega\omega}$$

and thus finally

$$\langle N^+_\omega\rangle = Tr(BB^\dagger) = \frac{\Gamma_\omega}{e^{\frac{2\pi\omega}{\kappa}} - 1} \tag{1.48}$$

But this is exactly a black body radiation with a factor Γ_ω which can be seen as an absorption cross section and most importantly with temprtature $T = \frac{\kappa}{2\pi}$ which is the Hawking temperature. The result is remarkable, it shows that expectation value of the number of particles counted at $\mathcal{J}^+$ follows a thermal radiation distribution and it is a continuous flow of thermal particles whose temperature depends only on the surface gravity. For Schwarzschild black hole $\kappa = 4M$ so using the thermodynamic definition of temperature $\frac{dS}{dE} = \frac{1}{T}$, identifying $E = M$ and taking the condition $S(E = 0) = 0$ we recover the Beckenstein result of entropy $S = \frac{A}{4}$ (ignoring some constants).

But what exactly is that rather peculiar Hawking radiation?

Well it is a result coming from quantum field theory in a gravitational field of a black hole. A good intuitive but of course not accurate picture is the following. The vacuum of the quantum field theory is characterized by constant activity which means spontaneous creation of pairs of particle-antiparticle which very quickly annihilate in order to maintain the total energy of the vacuum zero. However in the environment of the black hole close to the event horizon it is possible that a pair be created close to the horizon and a particle escapes at $\mathscr{J}^+$ while its partner falls into the black hole. In order the total energy of the vacuum is maintained ,since the escaping particle has positive energy, the infalling particle must have negative energy. That means this is a virtual particle which falls into the black hole and reduces its mass. In other words the Hawking radiation makes the black hole to evaporate. This process is a great milestone of modern physics as it brings forth a difficult problem *The information paradox* for which we are going to discuss in the next chapter.

1.4 Black hole evaporation

The result of Hawking is that at infinity we have a constant flow of particles. This implies that energy flows from the black hole to infinity and thus this energy must be taken away from its mass. This process is called *Black hole evaporation* and it leads to the eventual disappearance of the black hole. Considering a black hole as black body at Hawking temperature we can calculate the power of that radiation classically from Steffan's law

$$\frac{dE}{dt} = -\sigma A T_H^4 \tag{1.49}$$

and since $E = M$, $A \sim M$ and also $T_H \sim \frac{1}{M}$ for Schwarzschild black hole, we deduce that $\frac{dM}{dt} = -\frac{a}{M^2}$, where a is a constant and therefore the evaporation time $t_{evap} = \gamma M_0^3$. Here $\gamma = \frac{1}{3a}$ and M_0 is the initial mass of the black hole after gravitational collapse. The lifetime of a solar mass black hole is then estimated at 10^{64} years, compared to the universe age around 10^{13} years!!!

We need to emphasize here that this calculation is purely classical and thus contains no information about the geometry and its backreaction because of the evaporation process but still remains an accurate approximation for the loss of energy while it is radiated away in a slow rate i.e. $\frac{dM}{dt} << 1$. Nevertheless, this picture does not remain equally accurate during the late stages of evaporation because as its mass becomes smaller and smaller the emission rate increases drastically and when black hole's mass becomes comparable to Planck mass this rate become enormous. In order to be able to conclude more accurate results for those late times we need a theory of quantum gravity.

Chapter 2

Black Hole Information

2.1 Pure and mixed states in Quantum Mechanics

In Quantum mechanics any quantum system is described by a state $|\psi\rangle$ which is a vector that lives on a Hilbert space $\mathcal{H}$. Every ideal meaurement on the quantum system is related to a projection operator Π_i in the way that the probability of a measurement to give an i result on the quantum system, can be found by the action of the projection operator $\hat{\Pi}_i$ on the state $|\psi\rangle$ [28]

$$Pr(i) = |\Pi_i |\psi\rangle |^2 \langle\psi| \Pi_i^\dagger \Pi_i |\psi\rangle \tag{2.1}$$

A projection operator is one that $\Pi_i^2 = \Pi_i$. We are going to define now a new object called the *desnity operator* ρ (or density matrix) defined as $\rho = |\psi\rangle \langle\psi|$.
Using the above definition of density operator and expanding in an appropriate basis we can write the probability of a measurement as

$$Pr(i) = \sum_n \langle\psi|n\rangle \langle n| \Pi_i^\dagger \Pi_i |\psi\rangle = \sum_n \langle n| \Pi_i^\dagger \Pi_i |\psi\rangle \langle\psi| |n\rangle \tag{2.2}$$

$$= \sum_n \langle n| \Pi_i^\dagger \Pi_i \rho |n\rangle = Tr(\Pi_i^\dagger \Pi_i \rho) \tag{2.3}$$

and similarly if we want to compute an expectation value of an operator A we find that

$$\langle\psi| A |\psi\rangle = Tr(A\rho) \tag{2.4}$$

Consequently, we are deduce that the density operator is a useful and important object in quantum mechanics and within it, all the information of the quantum system is included.

A *pure state* is defined as one where the density operator can be written in the form $\rho = |\psi\rangle\langle\psi|$. By the definition of ρ we easily see that $Tr\rho = \langle\psi|\psi\rangle = 1$ and also $Tr\rho^2 = \langle\psi|\psi\rangle^2 = 1$. On the other hand, a *mixed* state is one that has a density operator $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ and more than one of p_i is non zero, this corresponds to a state which is written as $|\psi\rangle = \sum_i \sqrt{p_i}\,|\psi_i\rangle$ where $|\psi_i\rangle$ is an orthonormal basis and $p_i \geq 0$. In this case again if we take the trace of ρ we have $Tr\rho = \sum_i p_i = 1$ but on the other hand if we take the $Tr\rho^2$ we will have $Tr\rho^2 \neq 1$ because $\rho^2 = \sum_i p_i^2 |\psi_i\rangle\langle\psi_I|$ which means

$$Tr\rho^2 = \sum_i (p_i(p_i - 1) + p_i) = \sum_i p_i(p_i - 1) + 1 < 1$$

[28, 29] and therefore the way to check whether a state is mixed is to take the trace of ρ^2 and see if it is 1 or less. The density operator also has the good properties that it is hermitian and also positive definite.

2.2 Entanglement and Entropy

If we consider now a bipartite quantum system with a density matrix ρ living on a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, this system consists of two subsystems A and B with a complete and orthonormal basis say $|n\rangle_A$, $|m\rangle_B$ respectively and also density matrix $\rho = \rho_{AB}$ [28, 29]. An observer who has access only to one system, without loss of generality assume A, what they can measure comes from a density matrix regarding the subsystem A, ρ_A. We can define the *reduced density operators* for each subsystem as $\rho_A = Tr_B\rho$ and $\rho_B = Tr_A\rho$. For such a bipartite system a state will be pure if it is factorizable and thus if it can be written as a tensor product of two states of each subsystem i.e $|\psi\rangle = |n\rangle_A \otimes |m\rangle_B \equiv |n, m\rangle_{AB}$. On the other hand, if a state cannot be expressed as such a tensor product the state is mixed and we say that the two subsystems are entangled which means they are associated to one another.

If we have a factorizable state of a bipartite system then the reduced density matrices describe pure states as well. For example, if we take a state $|\psi\rangle_{AB} = |0, 0\rangle_{AB}$ then $\rho = \rho_{AB} = |0, 0\rangle\langle 0, 0|$ and so

$$\rho_A = Tr_B\rho = |0\rangle_A \langle 0|_A$$

whisch again describes a pure state. So any factorizable states are not entangled.

However, if we have a non factorizable state in general, the reduced density matrices will *not* describe pure states.

As an example we consider a Bell state [28, 30] which obviously is not factorizable and thus

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0,1\rangle + |1,0\rangle) \tag{2.5}$$

Here the density matrix has a form

$$\rho = |\psi\rangle_{AB}\langle\psi|_{AB} = \frac{1}{2}(|0,1\rangle + |1,0\rangle)(\langle 0,1| + \langle 1,0|)$$

$$= \frac{1}{2}\Big((|0\rangle\langle 0|)_A(|0\rangle\langle 0|)_B + (|0\rangle\langle 1|)_A(|0\rangle\langle 1|)_B + (|1\rangle\langle 0|)_A(|1\rangle\langle 0|)_B + (|1\rangle\langle 1|)_A(|1\rangle\langle 1|)_B\Big)$$

which means that if we trace out with respect to system B we are going to acquire

$$\rho_A = Tr_B\rho = \frac{1}{2}\Big(\,|0\rangle_A\langle 0|_A + |1\rangle_A\langle 1|_A\Big) \equiv \frac{1}{2}1_A \tag{2.6}$$

This Bell state is the maximally mixed or maximally entangled state which means that while the state described by the full density matrix ρ_{AB} is pure, the state of the subsystem A is described by ρ_A density matrix which is proportional to the identity. The 2 in the denominator indicates the dimensionality of the state (indeed a bipartite system).

2.2.1 Entanglement entropy

We define now a new quantity called the *von Neumann entropy* which is a very important quantity in order to quantify the information of a quantum system [31]. It is defined as

$$S_{vN} = -Tr(\rho\log\rho) = -\sum_i p_i \log p_i \tag{2.7}$$

where p_i are the eigenvalues of ρ. When we have a pure state, there is only one non zero eigenvalue with value equal to 1 and thus the von Neumann entropy vanishes. The von Neumann entropy or sometimes called *entanglement entropy* or *fine grained entropy* or just *quantum entropy* essentially is a way to measure how much information a quantum system possesses or rather how much it misses and also a measurement of the entanglement between subsystems. Specifically it is a measure of how mixed is a state since a mixed state will have non zero entropy while for a pure state is strictly zero. For a pure state we possess all available knowledge we are able to about the system.

In order to investigate this further, assume the same bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$. If

we take the subsystem A with $\rho_A = Tr_B \rho_{AB}$ this defines an entanglement entropy for the subsystem A $S_A = -Tr_A \rho_A \log \rho_A$. Also it is worth noticing that for an entangled bipartite system like the Bell state of (2.5) the von Neumann entropy of the entire system, which itself is in pure state, is zero while its subsystems have non zero entropy. So, in the bipartite system when the original pure state contains some entanglement between the two subsytems A, B although $S_{vN(A \cup B)} = 0$, S_A, S_B are non zero.
The von Neumann entropy has the following properties [29, 32, 33]

- $S(\rho) \geq 0$ with equal only for a pure state.

- $S(U^\dagger \rho U) = S(\rho)$ for U a unitary evolution operator. That means that if ρ evolves unitarily in time as $\rho(0) \to U^\dagger(t)\rho(0)U(t)$ then $S = constant$. Consequently, if a state is initially pure then it will remain pure in the future and similarly a mixed state will remain mixed. This is a very important property which just reflects the unitarity of time evolution of a quantum system. It means we are able to know about the past of the system by just going backwards in time under the action of a unitary operator

- $S(\rho) \leq \log d$ where d is the dimensionality of the Hilbert space where the system lives, where the equality is about a maximally entangled state.

- The von Neumann entropy of a spatial region of spacetime Σ is determined by a density matrix ρ_Σ and so $S_{vN}(\Sigma) = S_{vN}(\rho_\Sigma)$

- Strong subadditivity relation. For three quantum subsystems A,B,C $S_{A+B+C} + S_B \leq S_{A+B} = S_{B+C}$ and $S_A + S_C \leq S_{A+B} + S_{B+C}$.

- For a pure state with density matrix ρ at zero temperature the von Neumann entropy of a subsystem A is equal to the entropy of its complemment. This manifestly shows that the entanglement entropy is not an extensive quantity. This equality is violated at finite temperature. [10]

2.2.2 Coarse-grained entropy

Furthermore, we give a second notion of entropy, the so-called *coarse-grained entropy* which will be proven quite valuable in the following discussion. The coarse-grained entropy is constructed [32] by considering the same density matrix ρ of a quantum system along with a subset of macroscpic observables $\langle O_i \rangle = Tr(\rho O_i)$ followed from the action of some Hermitian operators $\hat{O}_i$. Then we need to consider all possible density matrices $\tilde{\rho}$ which give the same observable expectation values $\langle O_i \rangle$ as ρ and finally take the von Neumann entropy (2.7) of $\tilde{\rho}$ and maximaze it over all possible density matrices $\tilde{\rho}$. The definition of coarse-grained entropy embodies the usual thermodynamic entropy which increases under the action of a unitary time evolution (unlike fine-grained entropy which remains constant) and thus obeys the second thermodynamic law. In order to take the coarse-grained entropy we have to choose some observable quantities which for thermodynamics can be an approximate energy and volume. The thermodynamic entropy is thus computed by maximizing the fine-grained entropy among all possible states with the same energy and volume.

Note that by the definition of the coarse grained entropy we have that

$$S_{vN} \leq S_{coarse}$$

and the equality is for $\tilde{\rho} = \rho$ the density matrix of the state of the system and essentially it is an upper bound of how many degrees of freedom can be available to the system.

2.3 Seeing a black hole as a quantum system

In the previous chapter we had discussed about how a black hole can be viewed as a thermodynamic system obeying the usual thermodynamic laws. We also saw that the entropy of a black hole which classically was given by Bekenstein and depends only on the area of the horizon. Although we have an expression for the entropy so far we actually have no idea what might be the nature of this quantity. From the expression $S = \frac{A}{4G\hbar}$ we conclude that the entropy of a black hole even of a proton size would be $S \sim 10^{40}$ a number which is huge. So if we associate this entropy with its classical statistical definition which is the number of available microstates of a classical system, then we have a system with extraordinarily large (but finite) number of degrees of freedom. But that contradicts the non-hair theorem we saw in the previous chapter which quotes that black holes are fully described by only three degrees of freedom mass, angular momentum and electric charge and hence there are no microstates which also naively indicates zero statistical entropy. The problem of large number of microstates remains open and we refer the reader to [34]. However, in the context of string theory the result of that number of degrees of freedom has been already confirmed by working on extremal black holes in five dimensions by counting the degeneracy of BPS soliton bound states [35]. Introducing quantum fields does not fully solve the problem of entropy's large number yet but still indicates that we have to regard a black hole as an ordinary quantum system.

So our main assumption from now and on will be that *as seen from an observer in the exterior, a black hole can be described as a quantum system with* $\frac{A}{4G\hbar}$ *degrees of freedom and the system evolves in a unitary manner*. In their review Almheiri, Hartman, Maldacena and Tajidini [32] characterize this hypothesis as "central dogma" a name borrowed from molecular biology where the central dogma shows the flow of genetic information from DNA to RNA and eventually to proteins. In this case it corresponds to quantum information. Thus by considering a cutoff surface or "brickwall" around the black hole at Planck length from the horizon we can define the black hole region which resembles the quantum system defined.

Here we need to note some details related to the central dogma hypothesis. First of all, it does not concern the interior of the black hole at all, in reality it does not provide any information about what is happening behind the event horizon. In addition, the degrees of freedom indicated by the entropy do not come nor manifest in the gravity picture of the black hole. The same argument holds for the unitary evolution of the system. The later means that the Hamiltonian of the system does not manifest on gravity description. It is interesting, finally, to mention that Hawking himself was opponent of this hypothesis [32].

2.4 Hawking Information Paradox

After this short discussion about the entanglement in a bipartite system and the various definitions of quantum mechanical entropy we are ready to move on and discuss one of the most important and subtle problems of modern physics, the *information paradox*.

First of all we need to have a look back to the evaporation process. As we had seen a black hole emits Hawking radiation as a black body of temperature equal to the Hawking temperature $T = \frac{\kappa}{4\pi}$ and it radiates away with faster rate as it shrinks. Remarkably the radiation is influenced only by the surface gravity which subsequently depends utterly on black hole's mass for Schwarzschild solution. In terms of the quantum state of outgoing thermal particles, we can describe the evaporation process in the following way. Let's consider an initial configuration of matter in a pure state which collapses gravitationally to form a black hole. The newly formed black hole will settle down to a stationary state, as it seems from the outside, however the gravitational collapse of the matter will continue in the interior and the geometry also continues to elongate in one direction whereas it shrinks to zero size in angular direction and becomes a singularity [32]. If we consider that the central dogma holds for the black hole system we can think of the evaporating black hole as a bipartite quantum system where the two subsystems are the black hole itself and the radiation. As the black hole settles down there starts the spontaneous creation of particle pairs which constitute the Hawking radiation and actually they are entangled pairs where one particle remains trapped behind the event horizon and the other escapes to infinity as thermal radiation. This process as we saw carries on for an extremely long time and takes away mass from the black hole and consequently makes it shrink until it finally fully evaporates leaving a smooth, flat spacetime that contains only this thermal radiation from Hawking's process.

Information paradox arises directly from Hawking radiation. When the initial configuration collapses to form a black hole, after the black hole settles down to a stationary pure state we have lost forever every information concerning its formation and the matter before the gravitational collapse [32, 29]. On its own this does not create any issue because we accept the existence of the horizon which is the causal boundary of an external observer, the fact that we have no access to the information from the outside does not mean that it has been lost. Instead, it fell through the event horizon towards singularity. However a problem arises when the black hole finally evaporates its full mass to Hawking radiation. Then if nothing remains in the position of the entirely evaporated black hole, all the information about its past has been lost forever. But then what is the difference between a burning piece of coal and an evaporating black hole? Theoretically one could restore all the information initially contained in the piece of coal if carefully collect the ashes and the radiation produced from burning. Therefore, if we consider the coal is in a pure state, it will remain so albeit it has changed its shape and properties. So its information is not lost [18]. In contrast, this does not hold for Hawking radiation.

As mentioned before it is accepted that the black hole as a quantum system has to evolve unitarily. That means that a pure state has to remain a pure state forever. Nonetheless, the radiation is thermal and if the black hole starts as a pure state, after the end of the evaporation the remaining Hawking radiation will be in a mixed state. But this is impossible if the evolution is unitary and thus hawking radiation seems to violate the unitarity of quantum mechanical description.

Let's try now to give a bit more light to this mixed state arising from the evaporation. We will carefully follow the arguments of S.D. Marthur [36, 37, 38]. Our goal is to find the relation between the vacuum of an observer close to the horizon with the vacuum of an observer far away. Suppose that the observer close to the horizon has a vacuum state $|0\rangle_A$ while the further observer has $|0\rangle_B$. In section 1.3 we had shown the relation between field configurations for different observers in a black hole background. Starting from the observer A, there is a field configuration suppose $\phi = \sum_i a_i f_i + a_i^\dagger f_i^*$ and for B say $\phi = \sum_i b_i g_i + b_i^\dagger g_i^*$ and as it was shown

$$a_i = \sum_j A_{ij} b_j + B_{ij} b_j^\dagger \tag{2.8}$$

Therefore, by the definition of the vacuum state for A we derive that

$$a_i |0\rangle_A = \Big(\sum_j A_{ij} b_j + B_{ij} b_j^\dagger\Big) |0\rangle_A = 0 \tag{2.9}$$

For the simple case of only one mode we can write

$$(b + \gamma b^\dagger) |0\rangle_A = 0 \tag{2.10}$$

This equation has a solution which can be written as

$$|0\rangle_A = Ce^{\zeta b^\dagger b^\dagger} |0\rangle_B = C \sum_n \frac{\zeta^n}{n!} (b^\dagger b^\dagger)^n |0\rangle_B = \sum_n \zeta^n |2n\rangle_B \tag{2.11}$$

where $|2n\rangle = (b^\dagger b^\dagger)^n |0\rangle$ are multipartlicle states which contain n particle pairs, also C is a normalization factor and ζ a complex number. The normalization factor is calculated as

$$1 = \langle 0|0\rangle_A = C^* C \sum_n \sum_m \zeta^{*n} \zeta^{,} \langle 2n|2m\rangle = |C|^2 \sum_m |\zeta|^{2n} = \frac{|C|^2}{1 - |\zeta|^2}$$

which means that $|C| = \sqrt{1 - |\zeta|^2}$. Now we need to compute ζ. Starting from the basic commutation relation $[b, b^\dagger] = 1$ one can show that

$$b(b^\dagger b^\dagger)^n = ((b^\dagger b^\dagger)^n + 2nb^\dagger (b^\dagger b^\dagger)^{n-1} \tag{2.12}$$

from this expression we obtain

$$be^{\zeta b^\dagger b^\dagger} |0\rangle_B = 2\zeta b^\dagger e^{\zeta b^\dagger b^\dagger} |0\rangle_B \tag{2.13}$$

and comparing this result with the expression (2.10) we finally take that $\zeta = \frac{\gamma}{2}$ and thus

$$|0\rangle_A = \sqrt{1 - \frac{|\gamma|^2}{4}} e^{\frac{\gamma}{2} b^\dagger b^\dagger} |0\rangle_B \tag{2.14}$$

This relation shows the result we had already seen in chapter (1.3), the vacuum of A seems to contain particles from the point of view of B since the vacuum of A can be written as $|0\rangle_A = C_0 |0\rangle_B + C_2 b^\dagger b^\dagger |0\rangle_B + C_4 (b^\dagger b^\dagger)^2 |0\rangle_B + ...$
That means that the vacuum of A seems to be a superposition of multiparicle states of B. The existence of two creation operators imply that the particles are always produced in pairs.
Now the general solution for all modes in (2.9) should be then

$$|0\rangle_A = Ce^{-\frac{1}{2}\sum_{n,m} b^\dagger_m \gamma_{mn} b^\dagger_n} |0\rangle_B \tag{2.15}$$

and here γ will be a symmetric matrix given by $\gamma = \frac{1}{2}(A^{-1}B + (A^{-1}B)^T)$. Because the black hole spacetime is stationary we require the conservation of momentum during the process of particle pair creation. That means that the operators $b^\dagger_n$, $b^\dagger_m$ which act on the vacuum of B must represent the same momentum state for each particle. Thus, γ must have zero off-diagonal elements $\gamma_{mn} = 0$ for $m \neq n$ and for simplicity without loss of generality set $\gamma_{nn} = \gamma$.

Apart from the conservation of momentum we also have the creation of pairs where one particle falls into the black hole and the other escapes to infinity. We can perceive the creation of infalling particles happens in the interior of the horizon while the creation of outgoing ones outside. So we can split the Hilbert space of B and write the creation operator of the first group as $b^\dagger_n$ and relabel the ones of the second group as $c^\dagger_n$. Hence we are able to rewrite the state of (2.15) absorbing the $-1/2$ factor of the exponential in the constant C as

$$|0\rangle_A \equiv |\psi\rangle \, Ce^{\sum_n b^\dagger_n \gamma c^\dagger_n} |0\rangle_B = Ce^{b^\dagger_1 \gamma c^\dagger_1} e^{b^\dagger_2 \gamma c^\dagger_2} ... |0_{b_n}, 0_{c_n}\rangle \tag{2.16}$$

This state is called Hawking-Hartle vacuum [39] and it is the vacuum state of the quantum fields in a Schwarzschild black hole background. Now if we denote

$$|\psi\rangle_n \sim e^{b_n^\dagger \gamma c_n^\dagger} |0\rangle_B$$

and thus

$$|\psi\rangle = |\psi\rangle_1 \otimes |\psi\rangle_2 \otimes ... \tag{2.17}$$

Note that this full state is a tensor product and therefore, is a pure state and consequently there is no entanglement. Also as far as the creation operators $b_n^\dagger$ and $c_n^\dagger$ may concern, their commutator must vanish for different n, that implies that the creation of a pair of particles does not influence the creation of another. Furthermore for the same n their commutator vanishes again if the two operators concern different time, this also means that the particles created in a different moment do not interact among themselves and so there is no interaction among the various $|\psi\rangle_n$. This fact can be used in order to simplify the problem and investigate only the state $|\psi\rangle_1 = C(|0\rangle_{b_1} |0\rangle_{c_1} + \gamma |1\rangle_{b_1} |1\rangle_{c_1} + \gamma^2 |2\rangle_{b_1} |2\rangle_{c_1} + ...)$ this form indicates that the two subsystems are entangled.

Suppose now a black hole at early times before the emission of the first pair of particles. Then there exists only the quantum state of the black hole which is a pure state and it can be denoted by $|\psi\rangle_{BH}$. Hence the von Neumann entropy is initially zero as we discussed. Nevertheless, very soon after the black hole settles down we have the first emission ,and considering that this does not affect black hole's mass something that would subsequently alter its quantum state, the new quantum state of the system black hole-radiation will be

$$|\psi\rangle = |\psi\rangle_{BH} \otimes |\psi\rangle_1 = |\psi\rangle_{BH} \otimes C(|0\rangle_{b_1} |0\rangle_{c_1} + \gamma |1\rangle_{b_1} |1\rangle_{c_1} + \gamma^2 |2\rangle_{b_1} |2\rangle_{c_1} + ...) \tag{2.18}$$

Apparently this state consists of the system black hole-radiation and the von Neuman entropy for the particle created by $b_1^\dagger$ is clearly non zero anymore. This process carries on with the creation of new pairs and after the creation of N pairs the state of the full system will be

$$|\psi\rangle = |\psi\rangle_{BH} \prod_{n=1}^{N} \otimes |\psi\rangle_n \tag{2.19}$$

This so far does not seem to create any significant issue. The creation of the pairs creates a state consisting of tensor products and therefore the total state remains pure. However, when the black hole is at its very final stages of evaporation we cannot consider anymore that its state remains $|\psi\rangle_{BH}$.

Considering that the previous configuration remains accurate up to the point that black hole's mass becomes Planckian then there are mainly three possible scenarios

1. When black hole's mass becomes Planckian the Hawking radiation stops and what remains behind is a *remnant* of Planck length size. The advantage of such consideration is that the state remains pure and thus the problem of unitartity is solved. Nontheless, this remnant should be entangled with every pair of particle created by Hawking process. But this seems absurd since it implies that the remnant has extraordinary amount of entropy greater than Beckenstein entropy and hence a huge number of degrees of freedom. Because of that the existence of remnants is not widely accepted as a physical solution.
2. The black hole continues to radiate until it fully evaporates and when finally fully obliterates the remaining state consists of only Hawking-Harte vacuum and the state is not pure. Thus indeed we have started from a pure state an we end up with a mixed one and this as mentioned above violates the fundamental principle of unitarity in quantum mechanics.
3. The black hole evolution is indeed unitary and the information in some way is contained in the radiation and so there is no information loss. Therefore, there must be something wrong in the assumptions leading to the thermal state at late times so that the state is actually pure but so far to some extent this remains unclear. From the point of view of string theory and gauge/gravity duality as we will see that indeed this is what happens.

In terms of the previous discussion about the central dogma, information paradox can be viewed actually as a paradox only provided that we accept that central dogma is valid [32]. In other case it should be a characteristic of a theory of quantum gravity. The second scenario is often characterized as information loss. Hawking himself [4] argued that there cannot be such a unitary evolution from the initial state of a collapsing mass forming a black hole up to the end of the evaporation. Hawking stated [4, 32] that for a theory of quantum gravity, if quantum gravity effects are confined in a certain scale of Planck length and the vacuum of the theory is unique, then there will be information loss.

Another possible proposal about the final state of radiation is that in reality does not come out in a mixed state but rather a pure state which is formed by very complex correlations between photons (and gravitons) of Hawking radiation. Then, any small subsystem looks thermal validating the picture of information loss. However, this consideration essentially questions the validity of quantum field theory in curved spacetime itself and so this violation should have been detectable in other ways but we will not discuss about it any further here. [29]

For the last 45 years information paradox has remained one of the unsolved problems in modern physics and it is one of the indicatives of the incompatibility of quantum mechanics and the classical gravity at least at the semiclassical domain we have considered so far where spacetime is classical supplied with quantum fields living in. At the moment there are several proposed solutions but all of them have their pros and cons and therefore, there is none globally satisfying. The reader is also strongly suggested to have a look over Preskill's review for more conceptual understanding [40].

2.5 Unitary evolution and the Page curve

From the discussion of the previous sections we need to keep in mind two main points which are crucial. First of all the black hole as a gravitational system has entropy and specifically in the first chapter we had seen that it is the generalized entropy which obeys the second law

$$S_{gen} = \frac{Area}{4G_N} + S_{outside} \tag{2.20}$$

In recent years however there is a more precise version of the entropy [12] where quantum fields are considered in semiclassical geometry (including gravitons) and their von Neuman entropy out of the horizon is included and thus.

$$S_{gen} = \frac{Area}{4G_N} + S_{vN} + ... \tag{2.21}$$

where the dots denote some extra terms related to Wald terms [41] and counterterms. This expression is correct in order G_N^0 and also it is the entropy where the second thermodynamic law holds [42].

The second point is the acceptance of the central dogma and therefore the acceptance of unitary evolution of a black hole as quantum mechanical system. Since the whole spacetime of a black whole is asymptotically flat, the unitarity implies that the entire process from the formation of the black hole, from infalling collapsing matter, up to the final evaporation can be seen as a scattering process characterized by an S-matrix which is the unitary linear map of the process [29]. The "scattering" describes infalling *massive* particles during the gravitational collapse and finally after very long time (in terms of quantum field theory infinite) we only have outgoing *massless* particles. This interpretation has an advantage since the spacetime is asymptotically flat we can assume that the ingoing/outgoing particles at $\pm\infty$ have scattered so much from one another so that they do not interact and hence can be considered independent from one another. Thus we can consider them as well defined quantum states and hence those states consist a physical basis of those 1-particle ingoing and outgoing states, where the S-matrix can act upon [29]. An example of such an S-matrix is coming from the BFSS model [43].

At this stage we want to examine how the various entropies of the black hole system evolve in the passing of time. As we had seen in section 1.4, the mass of the black hole decreases as $\frac{dM}{dt} = \frac{a}{M^2}$. Solving this ordinary differential equation we get that

$$M(t) = M_0\Big(1 - \frac{t}{t_{evap}}\Big)^{1/3} \tag{2.22}$$

In the semiclassical approximation for Hawking radiation [44, 45] we take time dependent Bekenstein entropy which is coarse-grained for the black hole (recall $t_{evap} = \gamma M_0^3$ which is

$$S_{BH}^{coarse} \approx 4\pi M_0^2\Big(1 - \frac{t}{\gamma M_0^3}\Big)^{2/3} \tag{2.23}$$

and for the radiation we assume most of the Hawking radiation is emitted into photons in the lowest modes l, since Schwarzschild solution has zero angular momentum. That can approximately be described it as a 1+1 dimensional photon gas

$$S_R^{coarse} \approx 4\pi\beta M_0^2\Big[1 - \Big(1 - \frac{t}{\gamma M_0^3}\Big)^{2/3}\Big] \tag{2.24}$$

where $\beta \approx 1.48$ [9]. Note that those entropies are obeying the 2nd thermodynamic law and thus they are coarse grained entropies and hence $S_{BH} \leq S_{BH}^{coarse}$ where S_{BH} is the von Neuman or fine-grained of black hole and so the story is similar for the radiation. The Hilbert space related to the outgoing states can be described as a bipartite system which is decomposed into the subspace concerning the black hole and the subspace concerning the radiation $\mathcal{H}_{outgoing} = \mathcal{H}_{BH} \otimes \mathcal{H}_R$ and we are interested in calculating the fine-grained entropy of radiation subsystem as a function of time $S_R(t)$. Of course the system black hole-radiation is a pure state so its full von Neumann entropy will be zero.

The plot of $S_R(t)$ was first given by Don Page and it is called [8, 9] *The Page curve of Hawking radiation.* The form of the curve we can expect is based on some relatively simple arguments. At early times after gravitational collapse there is no Hawking radiation yet and the black hole is in a pure state, so we can expect that $S_R(0) = 0$. As the time passes the black hole radiates and S_R starts to increase. However the Bekenstein entropy of the black hole will decrease monotonically as the area shrinks and there will be a time (called Page time) where it will be equal to the fine-grained entropy of radiation $S_R = S_{BH}^{coarse}$. According to Hawking [4] the non-unitarity would require that the entropy of radiation continues to surge, while the thermodynamic entropy of black hole decreases, up to the end of the evaporation where the first has a maximum value and the later vanishes.

However, the more radiation particles are created the more degrees of freedom of the black hole are entangled with them. Because of that when $S_R = S_{BH}^{coarse}$, S_R cannot increase anymore because there are no other available degrees of freedom where radiation particles can be entangled with. Moreover, the statement of unitary evolution demands that $S_R = 0$ at the end of evaporation so that the final state remains pure.

Even more, D. Page has stated a *theorem* [7] which says that for a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with dimensionalities $|A|$, $|B|$ such that $|A| << |B|$ the von Neumann entropy of subsystem A

$$S_A = \log|A| - \frac{1}{2}\frac{|A|}{|B|} + ... \tag{2.25}$$

Therefore it is justified by Page's theorem that at early times $S_{BH}^{coarse} >> S_R$ since $S_R^{coarse} = \log|R|$ and $S_{BH}^{coarse} = \log|BH|$ where $|BH| >> |R|$ initially. In addition from exressions (2.34) and (2.35) we have that there is a time where the two coarse entropies are equal, this is also the time when the they are equl to the fine-grained entropy of radiation. This is the Page [9] time and it is equal to

$$t_{Page} = \Big(1 - \Big(\frac{\beta}{\beta+1}\Big)^{3/2}\Big)t_{evap} \approx 0.54 t_{evap} \tag{2.26}$$

and at this point the fine-grained entropy will be

$$S_R = 4\pi M_0^2\Big(\frac{\beta}{\beta+1}\Big) = S_0\Big(\frac{\beta}{\beta+1}\Big) \approx 0.6 S_0 \tag{2.27}$$

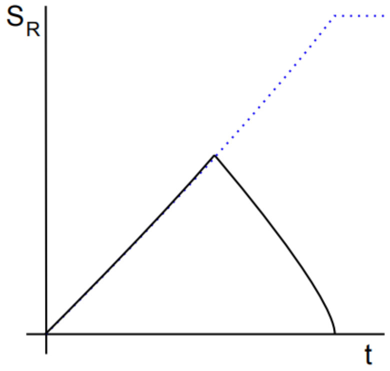

Figure 2.1: Heuristic plot of Page curve of Hawking radiation, the dashed lines show Hawking idea. [29]

The entanglement entropy of radiation should be by definition the minimum of the coarse grained expressions and in a good approximation it follows the increase of coarse-grained entropy of radiation until Page time and after it follows the decrease of the Bekenstein entropy. So we can write

$$S_R \approx \pi\beta M_0^2\Big[1-\Big(1-\frac{t}{\gamma M_0^3}\Big)^{2/3}\Big]\theta(t_{Page}-t)+4\pi M_0^2\Big(1-\frac{t}{\gamma M_0^3}\Big)^{2/3}\theta(t-t_{Page}) \quad (2.28)$$

where θ is the usual step-function.
Thus, at very early times the S_R increases approximately in a linear way with respect to time, when $t << t_{Page}$, then at Page time the entropy takes a maximal value and finally starts to drop as $S_R \sim S_0\Big(1-\frac{t}{t_{evap}}\Big)$.

With all those assumptions and results in mind we can give a picture of the Page curve for the 4-dimensional Schwarzschild black hole (see figure 2.1). Of course if we investigate more carefully the details of the evaporation such as greybody factors and the number and helicities of the available massless particles [29] we can have a better quantitative idea of when exactly is the Page time and the value of S_R at this point.

A subtle point we need to note here is that in all this discussion about the Page curve in reality what we compute is the renormalized entanglement entropy of radiation because we need to take into account the UV divergences arising from quantum fields [29]. So actually we have absorbed in the renormalized radiation entropy the fine-grained entropy of the vacuum which normally should have taken into account as well but for simplicity when we write S_R we mean a renormalized quantity. The Page curve is a very power tool in black hole information problem because it shows qualitatively and quantitatively how the entanglement entropy of radiation should behave as time passes in order to preserve unitarity. So a gravity theory which is able to reproduce the Page curve is also a good candidate to solve the information paradox altogether. During the last years there has been a significant progress on this topic in the scope of AdS/CFT, which we attempt to present in the following chapters.

Chapter 3

AdS/CFT correspondence

In this chapter we are going to introduce the holographic AdS/CFT correspondence or also well known as gauge/gravity duality [46, 47, 48, 49]. The AdS/CFT correspondence is a conjecture which emerged from string theory in the late 90s and has stormed the realm of modern theoretical physics since then in several fields. The conjecture states that there is a duality between a d-dimensional conformal field theory (CFT) and a specific gravity theory i.e. a theory of dynamical spacetime that lives in d+1-dimensions in Anti de-Sitter spacetime. The gravitational theory should be a supersymmetric string theory that can reduce to a usual theory of gravity and matter. Furthermore, the dual description of gravity is manifestly background-independent and depends only on the boundary conditions of AdS [50, 48]. Note that that the duality is additionally, a duality between a strong coupled theory with many degrees of freedom and a weakly coupled theory. Therefore , when a CFT gauge theory is strongly coupled, and thus cannot be studied in the context of perturbation theory as it has been done in weakly coupled QFT, we can use the dual gravitational theory (string theory) to draw conclusions about CFT. On the other hand, if CFT is weakly coupled the dual theory is strongly coupled. Moreover, the mapping of the two theories obey the holgraphic principle first proposed by t'Hooft [5] and Suskind [6] (see also section 3.4) and because of the many degrees of freedom of CFT the higher dimensional "bulk" physics can be packaged into a lower dimensional field theory [48].

The conjecture was proposed by J. Maldacena in November of 1997 [46] and back then it concerned a CFT which is $\mathcal{N} = 4$ YM, a 3+1-dimensional, maximally supersymmetric $U(N)$ Yang-Mills theory. According the conjecture, this theory has a holographic dual IIB string theory in a 10-dimensional asymptotically $AdS_5 \times \mathbb{S}^5$ spacetime. The correspondence was discovered by working on D3-branes in low energy limit of string theory which reveals a duality between open and closed superstrings. [47, 48] The duality can be perceived by the fact that a U(N) gauge field theory can be obtained by open strings which end on an N D-brane and also in terms of closed strings where the D-branes at large N produce a non trivial gravitional theory in 10 dimensional $AdS_5 \times \mathbb{S}^5$ spacetime. Here the gauge theory has a coupling constant which for large enough N can be written as $\lambda = Ng^2_{YM}$ while the dual gravitational description has a weak coupling $g \sim \frac{1}{N^2}$ [47]. At the time being, there is no direct, rigorous mathematical derivation of the conjecture and the case studied by Maldacena is the best understood so far [50].

Remarkably, wherever is has been possible to test with exact calculation in both sides of the correspondence, such as in the maximally supersymmetric case where we can use the tools of integrability [51, 50], the conjecture has proven to be valid. However a common critic against AdS/CFT correspondence concerns its very origin itself: the whole conjecture has been developed in the context of string theory but if string theory is incomplete or even wrong as a quantum theory of gravity that would indicate that the conjecture is relied on a non strong theoretical basis and so it would not be trustworthy [50].

The aim of the following sections is not to provide a very deep and thorough investigation of this remarkable conjecture but rather the general idea of what is a conformal field theory, the AdS spacetime, the duality and most importantly how the black hole entropy arises in AdS.

3.1 Conformal field theories (CFTs)

3.1.1 Conformal group

A conformal field theory is a relativistic quantum field theory which is invariant under Poincare group of translations and Lorentz transformations but it is also inavariant under one additional *scale symmetry*

$$x^\mu \to \lambda x^\mu \tag{3.1}$$

and if the theory lives in $d > 2$ dimensions, also under the *special conformal transformations*

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2x^\mu a_\mu + a^2 x^2} \tag{3.2}$$

The Poincare transformations plus those new transformations constitute a larger group of transformations named the *conformal group* which is isomorphic to $SO(2,d)$. Abstractly, the conformal group is defined as the set of transformations of Minkowski which preserve angles but not necessarily lengths [29]. We also need to note that normally the comformal symmetry is broken in quantum physics because of renormalization procedure [52]. As it is known the renormalized coupling constant depends on a scale μ which is introduced at the regularization of the theory and in general destroys the scale symmetry. A quantum field theory is explicitly conformal if and only if the beta function $\beta(\mu) = 0$.

As usual, the translations of the Poinacare group have generator P_μ which by acting on scalar f gives

$$P_\mu f(x) = i\partial_\mu f \tag{3.3}$$

and the Lorentz transformations are generated by $M_{\mu\nu}$ with

$$M_{\mu\nu} f = i(x_\mu \partial_\nu - x_\nu \partial_\mu) f \tag{3.4}$$

Similarly the scale symmetry has as a generator the *dilaton scaling operator* D which on a scalar acts as

$$Df(x) = ix^\mu \partial_\mu f \tag{3.5}$$

and finally the special conformal transformations have generator K_μ with

$$K_\mu f = i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu) f \tag{3.6}$$

So the entire conformal group is generated by those four generators P_μ, $M_{\mu\nu}$, D and K_μ. Those generators satisfy the conformal Lie algebra

$$\begin{aligned}
[P_\mu, P_\nu] = [K_\mu, K_\nu] = [M_{\mu\nu}, D] &= 0 \\
[M_{\mu\nu}, P_a] &= -i(\eta_{a\mu} P_\nu - \eta_{a\nu} P_\mu) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\nu\rho} M_{\rho\sigma}) \\
[D, P_\mu] &= -iP_\mu \\
[D, K_\mu] &= iK_\mu \\
[M_{\mu\nu}, K_\rho] &= -i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \\
[P_\mu, K_\nu] &= 2i(M_{\mu\nu} - \eta_{\mu\nu} D)
\end{aligned} \tag{3.7}$$

where the first three relations manifest Lie algebra of the Poincare group while the rest provide the full conformal algebra. The whole conformal group has an algebra isomorphic to $so(2,d)$ In order to reproduce the algebra of $so(2,d)$ if we define the generators J_{ab} with antisymmetric indices $a, b = 0, 1..., d+1$

$$\begin{aligned} J_{\mu\nu} &= M_{\mu\nu} \\ J_{\mu d} &= \frac{1}{2}(K_\mu - P_\mu) \\ J_{\mu(d+1)} &= \frac{1}{2}(K_\mu + P_\mu) \\ J_{(d+1)d} &= D \end{aligned} \tag{3.8}$$

Thus we can write J_{ab} as an antisymmetric matrix

$$J_{ab} = \begin{pmatrix} J_{\mu\nu} & J_{\mu d} & J_{\mu(d+1)} \\ -J_{\mu d} & 0 & D \\ -J_{\mu(d+1)} & -D & 0 \end{pmatrix} \tag{3.9}$$

and if we define $G_{ab} = diag(-1, -1, 1, 1, 1)$ then the commutator

$$[J_{ab}, J_{cd}] = i(G_{bc}J_{ad} - G_{ac}J_{bd} - G_{bd}J_{ac} + G_{ad}J_{bc}) \tag{3.10}$$

in the same way as $\eta_{\mu\nu}$ does in the commutator $[M_{\mu\nu}, M_{\rho\sigma}]$ reproducing the algebra $so(1,d)$ [48].

3.1.2 Correlators

Consider now the transformations of the field ϕ under a represention of Poincare and dilaton transformation [48, 49]. Because we consider a scalar field which is a function under the action of Poincare group, $x \to x' = \Lambda x + a$ where $\Lambda \in SO(1, d-1)$, it transforms $\phi(x) \to \phi'(x')$. Nontheless, if we include the scale transformation $x \to \lambda x$ the field transforms also up to a scale

$$\phi(x) \to \phi'(x') = \lambda^{-\Delta}\phi(\lambda x) \tag{3.11}$$

here Δ is called scaling dimension of the field. Then the dilaton acts on the field as

$$[D, \phi(x)] = i(\Delta + x^\mu \partial_\mu)\phi(x) \tag{3.12}$$

and also the action of K_μ is

$$[K_\mu, \phi(x)] = i(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu - 2x_\mu \Delta)\phi(x) \tag{3.13}$$

The generators P_μ and K_μ can be seen as creation and annihilation operators upon D correspondingly with eigenvalue $-i\Delta$ by considering

$$[D, \phi(0)] = -i\Delta\phi(0) \tag{3.14}$$

and thus build representations of the conformal group by considering fields, eigenfunctions of D with eigenvalue $-i\Delta$ under D. We are interested in representations which are bound from below something which is guaranteed by the fact that

$$[K_\mu, \phi(0)] = 0 \tag{3.15}$$

Here $\phi(0)$ is the field operator at the origin of Minkowski space and has conformal dimension Δ. The generators P_μ and K_μ acting upon $\phi(0)$ increase or lowers the scale dimension by 1 respectively, because

$$\begin{aligned} DP_\mu\phi(0) &= [D, P_\mu] + P_\mu D\phi(0) = -i(\Delta + 1)P_\mu\phi(0) \\ DK_\mu\phi(0) &= [D, K_\mu] + K_\mu D\phi(0) = -i(\Delta - 1)P_\mu\phi(0) \end{aligned}$$

Therefore the field operator $\phi(0)$ has the lowest scale dimension and so it is called *primary field*. In general, in CFT any local operator O which transforms under dilaton transformation as $O \to \lambda^{-\Delta}O$ is called *primary operator* of conformal dimension Δ. For a unitary CFT Δ is real and positive and if O is a scalar then $\Delta \geq \frac{d-2}{2}$. The derivatives of a primary operator $\partial^n O$ are called *descendants* and in general are not primary operators but have conformal dimension $\Delta + n$ and so under scale transformation they transform as $\lambda^{-\Delta-n}$. Furthermore another interesting feature of the primary operators in CFT, is that at any spacetime point x there is a natural bijection with a complete basis of the Hilbert space of the theory if we take the quantization on a cylinder $\mathbb{R} \times \mathbb{S}^{d-1}$ which is conformally equivalent to Euclidean $\mathbb{R}^d$ [29].

With the use of primary scalar operators we are able to build correlation functions which have to be constrained by conformal invariance and obey the following property

$$\langle 0|\,\phi_i(x_1)\phi(x_2)...\,|0\rangle \rightarrow \lambda^{\Delta_1+\Delta_2+...}\,\langle 0|\,\phi_i(x_1)\phi(x_2)...\,|0\rangle \tag{3.16}$$

Thus the 2-point function $< \phi_i(x)\phi_j(y) >\sim f((x-y)^2)$ and specifically it turns out

$$< \phi_i(x)\phi_j(y) >= \frac{c_{ij}}{(x-y)^{2\Delta}} \tag{3.17}$$

for fields with the same conformal dimension whereas if they do have different ones, the correlation function vanishes [48, 49]. thus $c_{ij} = \delta_{ij}$ and also we can define higher-points correlation functions. Note also that the vacuum concerning those correlation functions is a conformally invariant state meaning that $D\,|0\rangle = 0$.

3.1.3 Stress energy tensor and central charge

Before we move on to the other side of the duality and discuss about the AdS spacetime it would be fruitful to have a look over the stress energy tensor of a CFT. The stress energy tensor $T_{\mu\nu}$ has been derived in any relativistic QFT as a Noether current generated by the translation invariance. It tuns also that it exists for every CFT and as usual it is symmetric and starfishes $\partial_\mu T^{\mu\nu} = 0$. Also because of the conformal symmetry has to be traceless. Also the stress energy tensor has conformal dimension $\Delta = d$ the number of dimensions the theory lives in. Because of that, the 2-point correlalation function has to be $< T_{\mu\nu}(x)T_{\rho\sigma}(y) >\sim \frac{C}{(x-y)^{2d}}$ where $C \sim N$ is a quantity proportional to the degrees of freedom of the theory and it is called effective central charge.

3.2 Anti de Sitter spacetime

Let's now move on to Anti de Sitter space which is the spacetime where the IIB string theory of the AdS/CFT correspondence lives in, for the original study [46]. The *AdS* spacetime is a maximally symmetric solution to Einstein's equations with negative cosmological constant λ. A d-dimensional *AdS* spacetime, AdS_d can be represented as a hypersurface of hyperboloid with radius L embedded in d+1 dimensions of Minkowski spacetime with coordinates $(X_0, X_1, ...X_d)$. In those coordinates the metric of spacetime is expressed as

$$ds^2 = -dX_0^2 + \sum_{i=1}^{d-1} X_i^2 - dX_d^2 \tag{3.18}$$

and thus the hyperboloid representing the AdS_d is given by the expression

$$X_0^2 + \sum_{i=1}^{d-1} X_i^2 + X_d^2 = L^2 \tag{3.19}$$

where the radius L of the hyperboloid is related to the cosmological constant Λ $L^2 \sim \frac{1}{\Lambda}$. Note that since AdS_d is a maximally symmetric spacetime, it has $\frac{d(d+1)}{2}$ linearly independent Killing vectors which generate the isometry group $SO(2, d-1)$ which is the conformal group of Minkowski spacetime in $d-1$ dimensions. In addition, another characteristic of AdS spacetime emerged from the form of the hyperboloid embedding is that there are closed timelike curves defined by $X_i = constant$.

Now, the hyperboloid of (3.19) can be parametrized in several coordiante systems which construct defferent foliations of AdS_d. Here we present the two most important and useful.
The first possible parametrization is to use *global coordinates* (τ,ρ,Ω_i) and write

$$\begin{aligned} X_0 &= L\cosh\rho\cos\tau \\ X_i &= L\sinh\rho\Omega_i \\ X_d &= L\cosh\rho\sin\tau \end{aligned} \tag{3.20}$$

where Ω_i $(i = 1, ...d-1)$ give a parametrization for the metric of unit sphere in $d-2$ dimensions with $\sum_i \Omega_i^2 = 1$ and $0 \leq \rho < \infty$ while $0 \leq \tau \leq 2\pi$. Using those global coordinates the metric (3.18) can be rewritten as

$$ds^2 = L^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}) \tag{3.21}$$

The fact that the time direction coordinate τ has values like an angle, within a closed interval manifests the existence of the closed timelike curves we mentioned above. In order to tackle this issue we have to take the universal cover of AdS_d where τ coordinate gets unwrapped and thus extends from minus to plus infinity and thus does not include closed timelike curves [48].

Furthermore, the metric (3.21) can be expressed in an equivalent way by substituting $\tan\theta = \sinh\rho$ with $\theta \in [0, \frac{\pi}{2})$ and using that $\cosh^2\rho - \sinh^2\rho = 1$ we get that $\cosh\rho = \sqrt{1+\sinh^2\rho} = \sqrt{1+\tan^2\theta} = \frac{1}{\cos\theta}$. Thus, the metric metric of (3.21) is expressed in a clearly hyperbolic way as

$$ds^2 = \frac{L^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-2}^2) \tag{3.22}$$

which apparently diverges for $\theta \to \frac{\pi}{2}$. However the divergence exist due to the $\cos\theta$ in denominator, so we are able to perform a conformal compactification transformation similar to one we perform in order to construct Carter-Penrose diagrams of an arbitrary spacetime. We achieve that by taking a rescaling of the metric with the factor $\frac{L^2}{\cos^2\theta}$ which cancels the existing factor and leaves a metric

$$ds^2 = -d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-2}^2 \tag{3.23}$$

and now we are able to include the $\theta = \frac{\pi}{2}$ as a conformal boundary of AdS_d. The Penrose diagram of this configuration has a topology of $\mathbb{R} \times \mathbb{S}^{d-1}$ representing the half of Einstein static universe[1][53] . The conformal boundary itself has $\theta = \frac{\pi}{2}$ and subsequently, a topology $\mathbb{R} \times \mathbb{S}^{d-2}$. Another point is that using global coordinates we do not have the full $SO(2, d-1)$ symmetry group of AdS_d anymore but a residual isometry group isomorphic to $SO(2) \times SO(d-1)$ which give rotations of time dimension τ in addition to rotations on $\mathbb{S}^{d-2}$ (or for $SO(2)$ translations in time if we take the universal cover where $\tau \in \mathbb{R}$).

We continue our discussion with another foliation of AdS the *Poincare patch* with use of local coordinates (t, z, x_i), $i = 1, ..., d-2$ which have values $t, x_i \in \mathbb{R}$ and $z \in (0, \infty)$ [54]. The parametrization of Poicare patch is given as follows

$$\begin{aligned} X_0 &= \frac{z}{2}(1 + \frac{1}{z^2}(L^2 + \sum_{i=1}^{d-2} x_i - t)) \\ X_i &= \frac{Lx_i}{z} \\ X_{d-1} &= \frac{z}{2}(1 - \frac{1}{z^2}(L^2 - \sum_{i=1}^{d-2} x_i^2 + t^2)) \\ X_d &= \frac{Lt}{z} \end{aligned} \tag{3.24}$$

[1]Which can be produced by the conformal compactification of d-dimensional Minkowski spacetime (Dowker) to $\mathbb{R} \times \mathbb{S}^{d-1}$, but the difference is that 12850$[0, \pi]$ while in AdS $\theta \in [0, \frac{\pi}{2}]$ meaning that it represents half of the Einstein spacetime.

The set of those coordinates cover only the half of the hyperboloid as $z > 0$ and by substituting the above relations in (3.18) we get the metric of the Poincare patch

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + dz^2 + \sum_{i=1}^{d-2} dx_i^2) \tag{3.25}$$

This metric has a coordinate singularity at $z = 0$. However, by performing a conformal compactification as before we can include point $z = 0$ in the patch and the metric there becomes

$$ds^2 = -dt^2 + \sum_{i=1}^{d-2} dx_i^2 \tag{3.26}$$

The conclusion is rather remarkable. By the choice of those local coordinates we have a part of AdS_d with $d-1$-dimensional conformal boundary at $z = 0$, which is Minkowski spacetime!!!

The form of the metric (3.25) indicates Poincare invariance in (t, x_i) coordinates. Also it shows a scale symmetry where $(t, z, x_i) \to (\lambda t, \lambda z, \lambda x_i)$ so the residual symmetry group is $ISO(1, d-2) \times SO(1,1)$ [49].

Another interesting conclusion that we are able to exert from the Poincare patch metric is that a massless particle following a null line from the center of AdS can reach the conformal boundary in a finite time. That can be seen if we make the substitution $\frac{z}{L} = e^{-y}$ (so that the boundary is at $y \to \infty$ which leaves the metric (3.25) as

$$ds^2 = e^{2y}(-dt^2 + \sum_{i=1}^{d-2} dx_i) + L^2 dy^2 \tag{3.27}$$

then $ds^2 = 0$ and taking $x_i = 0$ gives that $dt = Le^{-y}dy$ and the time obtained by integrationg this expression is indeed finite .

To sum up we can think of AdS as a box with a conformal boundary and specifically if we choose reflective boundary counditions then a massless particle started from the center gets all the way out to the boundary and back in a finite proper time π, as seen by somebody at the center. By the way this holds for massive particles as well. Conisder one is floating in the center of AdS and throws a ball away. It will go out some finite distance (unlike a massless particle it will not manage to make all the way to the boundary), but eventually it will turn around and return to its starting point after a time of order one in AdS units. These observations are formalized in the statement that the boundary is timelike [29].

3.3 AdS/CFT dictionary

The modern statement of AdS/CFT correspondence states that a CFT_d living on $\mathbb{R}\times\mathbb{S}^{d-1}$ is dual i.e is equivalent to a theory of quantum gravity which lives in an asymptotically $AdS_{d+1} \times M$ spacetime where M is a non trivial compact manifold [29]. The CFT itself lives on the conformal boundary of the asymptotic AdS spacetime. For now we do not possess any precise theory of quantum gravity in asymptotically AdS spacetime but we can use a certain CFT which is known and well defined and thus extract some conclusions about the bulk gravity theory with a semiclassical limit in AdS.

We would like to extract conclusions upon how the observable quantities in one theory are translated in its holographic dual. So we need a map, called the dictionary, between the two sides of the correspondence. Another issue is that because of the lack of a consistent quantum gravity theory we must determine when a CFT is indeed dual to a semiclassical gravity. It turns that as in the original paper [46], if the number of degrees of freedom in the CFT N are sufficiently large then the gravitational theory is weakly coupled and has a limit to classical gravity.

As far as the dictionary may concern, first of all the duality establishes that the Hilbert spaces of the two dual theories are identical which means that there is a map between the states of CFT and the gravitational theory in the bulk and also the Hamiltonian H determines time evolution is also identical for the two dual theories. Moreover the symmetry generators of CFT symmetry group $SO(2,d)$ have corresponding bulk symmetry generators in AdS space.

The correspondence states that every bulk scalar field $\phi = \phi(\tau, z, \Omega)$ in the AdS is related to a primary operator O of the CFT on the boundary of AdS (in reality it turns out that this is true in general for any kind of field). So we can determine the boundary conditions of the conformal boundary of CFT by studying a local bulk matter field in AdS close to the boundary. It turns out that the bulk field can be expressed in terms of a sum free fields. and actually there is only one fields' dominant contribution, let it be ϕ_0. If we take the generating functional for the two theories i.e. the CFT and the theory in AdS which is gravity with some matter bulk field ϕ and fixed boundary condition, the statement of the duality indicates that

$$Z_{CFT}[J] = Z_{AdS}[J]. \tag{3.28}$$

where here J is a usual source in terms of CFT generating functional but in terms of the AdS dual, J is a bit more subtle story. It has to do with the configuration of a bulk field on AdS boundary. Specifically, a bulk field corresponds to a dual primary operator as we explained and it has to have appropriate boundary conditions on AdS boundary such that the field fall off to the function J on the boundary. In other words in the context of AdS $J = \phi_0$. We demand that the conformal boundary satisfy certain boundary conditions because a null geodesic line can reach it as we saw in finite time and also massive modes get near it in finite time and thus we are interested in seeing bulk field's behaviour close to the boundary.

3.4 Holographic entanglement entropy

3.4.1 The Ryu-Takayanagi formula

The holographic principle had been realized by t'Hooft and Suskind way before the emergence of gauge/gravity duality claiming that the degrees of freedom in a $d+1$-dimensional theory of quantum gravity should have a correspondence with a quantum system of many bodies in d. This manifests in the Beckenstein thermodynamic entropy of a black hole which as we saw is proportional to the area of the horizon. Nowadays, after the first proposition of AdS/CFT correspondence conjecture we have more concrete examples of holographic dual gravity theories in AdS with certain CFTs living on the boundary of AdS [50]. We discussed also in chapter 2 about the fine-grained entanglement entropy of the black hole which is smaller than the coarse-grained thermodynamic entropy and also that the fine-grained entropy of a spatial slice Σ is $S_\Sigma(\rho) = S(\rho_\Sigma)$ where ρ_Σ is the reduced density matrix which concerns the subsystem of the slice Σ. Our aim now is to present a useful tool which will give the ability to compute fine-grained entropies of quantum systems in context of AdS/CFT correspondence.

Suppose that we have a quantum system A which lives on a spacelike submanifold A on a spatial slice Σ with boundary ∂A. It turns out that the entanglement entropy for a d-dimensional free scalar field theory is always divergent but it can be regularized by a

UV cut-off ϵ and thus it has a leading divergent term in the UV limit

$$S_A = \lim_{\epsilon \to 0} \gamma \frac{Area(\partial A)}{\epsilon^{d-1}} + ... \tag{3.29}$$

where here the coefficient γ does only depend on the quantum system but not on A. It is obvious that this expression has a similarity with the black hole entropy [55]. In the context of central dogma for the black holes we are interested in the calculation of entanglement entropy of radiation which must follow the Page curve in order to respect the unitary evolution.

In 2006 Shinsei Ryu and Tadashi Takayanagi [10] came up with a new formula for the entanglement entropy S_A of a subsystem A in d-dimensional CFT on $\mathbb{R} \times \mathbb{S}^{d-1}$ with boundary $\partial A \in S^{d-1}$. Using the AdS/CFT correspondence their proposal was that

$$S_A = \frac{Area(\gamma_A)}{4G_N^{d+1}} \tag{3.30}$$

where γ_A is the $d-1$-dimensional minimal static surface which has a $d-2$-dimensional boundary $\partial\gamma_A = \partial A$ and G_N^{d+1} the Newton's constant of the gravity theory in AdS_{d+1}. The above expression is known as Ryu-Takayanagi (RT) formula and appears similar to the definition of Beckenstein entropy.

3.4.2 Heuristic proof of RT formula

In order to prove the RT formula we first need to define a new object called Rényi entropy [56] which is defined in terms of the moments of ρ_A

$$S_A^{(n)} = \frac{1}{1-n} \log Tr_A(\rho_A^n) \tag{3.31}$$

with n a positive integer but we can perform analytic continuation so that $n \in \mathbb{R}^+$. The entanglement entropy is equal with

$$S_A = \lim_{n \to 1} S_A^{(n)} \tag{3.32}$$

This way of calculation of S_A is called *replica trick*. We have to calculate the entropy from QFT and thus it will be useful to construct a path integral that computes ρ_A. Then we have to compute the Renyi entropies by taking a functional integral on a 'branched cover' geometry in the following way given by [55, 33, 57]: seperate the scalar field of CFT as $\phi(x) = (\phi_A(x), \phi_{A^c}(x))$ and impose boundary conditions for field in A

$$\phi_A|_{t=0^{\pm}} = \phi_{\pm} \tag{3.33}$$

where conventionally we took $t = 0$ the time of the constant time slice where A was defined. Thus this is equivalent to cutting open the path integral in a restricted domain of space A at time $t = 0^{\pm}$, and projecting the result onto definite field values $\phi_{\pm}$. Therefore, we can write ρ_A as

$$(\rho_A)_{-+} = \int D\phi e^{-S(\phi)}\delta_E(\phi_{\pm}) \tag{3.34}$$

with $\delta_E(\phi_{\pm}) = \delta(\phi_A(0^-) - \phi_-)\delta(\phi_A(0^+) - \phi_+)$. then

$$(\rho_A)^n_{-+} = \int \prod_{j=1}^{n-1} d\phi_+^{(j)}\delta(\phi_+^{(j)} - \phi_-^{(j+1)}) \times \int \prod_{k=1}^{n} D\phi^{(k)}e^{-\sum_{k=1}^{n} S(\phi^{(k)})}\delta(\phi_{\pm A}) \tag{3.35}$$

We can now look at each copy of ρ_A as being computed on a copy of the background spacetime B on the CFT boundary. Thus, the path integral can be computed by integration over the fields of the background B_n, where the later is made by n copies of B, also B_n is characterized by a quantity called deficit angle $\delta = 2\pi(1-n)$. Hence we define $Z_n[A] = Tr_A(\rho_A^n) \equiv Z[B_n]$. Therefore the Rényi entropy is expressed as

$$S_A^{(n)} = \frac{1}{1-n}\log\Big(\frac{Z_n[A]}{Z_1[A]^n}\Big) = \frac{1}{1-n}\log\Big(\frac{Z[B_n]}{Z[B]^n}\Big) \tag{3.36}$$

Now we could try to calculate directly those CFT_d generating functions but this turns out to be a very difficult task instead recall we have from the dictionary of AdS/CFT that $Z[B_n] = Z_{CFT} = Z_{AdS}^{(n)} = e^{-iS_{grav}}$ and it is easier to calculate the Z_{AdS} by realizing that the back reacted geometry related to S_n is given by a n-sheeted AdS_{d+1}, which has the deficit angle δ localized on a codimension two[2] surface γ_A. Thus, the Ricci scalar of gravity+fields in the bulk can be expressed as $R = 4\pi(1-n)\delta(\gamma_A) + R^{(0)}$, where the $R^{(0)}$ is the Ricci scalar of the pure gravity. Therefore, if we substitute this Ricci scalar to the gravitational action of the AdS we have that

$$\log Z_{AdS}^{(n)} = -\frac{1}{16\pi G_N^{d+1}}\int d^{d+1}x\sqrt{-g}(R+\Lambda) + ...$$

[2]codimension two says that the surface has two dimensions less than spacetime dimensions.

$$= \frac{4\pi(1-n)Area(\gamma_A)}{16\pi G_N^{d+1}} - \frac{1}{16\pi G_N^{d+1}} \int d^{d+1}x\sqrt{-g}(R^{(0)} + \Lambda) + ...$$

and so $\log Z[B_n] = \frac{(1-n)Area(\gamma_A)}{16\pi G_N^{d+1}} + O(n^0)$. Finally by (3.36) we obtain the final result

$$S_A^{(n)} = \frac{Area(\gamma_A)}{4G_N^{d+1}}$$

which in turn gives the entanglement entropy in the limit $n \to 1$.

3.4.3 Proof of Strong Subadditivity

The RT formula is well defined since it respects the properies that $S_A = S_{A^c}$ where A^c is the complement of A and the strong subadditivity. The first is geometrically obvious since the minimal surface is minimal for both A and A^c. For the second consider three regions A, B, C on a slice of constant time such that there are no overlaps between them. According to AdS/CFT duality we can extend the boundary setup towards the bulk and consider the entropies S_{A+B}, S_{B+C} and according to RT formula they are qual to the minimal area surfaces γ_{A+B} and γ_{B+C} respectively, also for them $\partial\gamma_{A+B} = \partial(A+B)$ and $\partial\gamma_{B+C} = \partial(B+C)$. Then we can devide these two minimal surfaces into four pieces and recombine either into two surfaces γ'_B, γ_{A+B+C} or two surfaces γ'_A, γ_C'. Each γ'_X is surface which satisfies $\partial\gamma'_X = \partial X$ and which in general are not minimal so$Area(\gamma_X) \geq Area(\gamma_X)$. Therefore, we directly see that [33]

$$Area(\gamma_{A+B}) + Area(\gamma_{B+C}) = Area(\gamma'_B) + Area(\gamma'_{A+B+C}) \geq Area(\gamma_B) + Area(\gamma_{A+B+C})$$
$$Area(\gamma_{A+B}) + Area(\gamma_{B+C}) = Area(\gamma'_A) + Area(\gamma'_C) \geq Area(\gamma_A) + Area(\gamma_C)$$

And thus by dividing with $4G_N^{d+1}$ we complete the proof.

3.4.4 The Hubeny-Rangamani-Takayanagi formula

The idea of RT formula is brilliant because it makes the calculation of entanglement entropy of a quantum mechanical system, which is characetrized by a density matrix and a potentially hard calculation to a purely geometrical problem which usually is easier to solve. However the weakness of RT formula is that it is referred strictly to static time-independent systems. But when it comes to a quantum system that evolves in time we would like to study how its entropy evolves as well. Thus we need a new covariant prescription which generalizes the RT formula for an arbitrary time dependent system. In

2007 Hubeny, Rangamani and Takayanagi [11] (HRT) generalized the RT result arguing that the entanglement entropy of a quantum system associated with a region on the CFT boundary is given by the area of a codimension two bulk surface, which instead of a constant time slice is a Cauchy surface. The HRT proposal for the entanglement entropy of a system A living on a region of the CFT_d is that

$$S_A = min \frac{Area(\mathcal{X}_{ext})}{4G_N^{d+1}} \tag{3.37}$$

where $\mathcal{X}$ a codimension 2 extremal (which has zero null geodesic expansion) surface in the bulk B. [3] and if there is more than one such surface we chose the minimal one. We require also that $\partial\mathcal{X} = \partial A$.

Let's discuss briefly about the motivation of HRT proposal. Consider a time dependent version of AdS/CFT correspondence where the system A of the CFT on the boundary is in a time dependent state on a fixed background ∂M. Then the bulk theory living on M is also explicitly time dependent. Because the metric on the boundary is not dynamical we are free to select a foliation of equal time slices for the boundary and write $\partial M = \partial N_t \times \mathbb{R}$. Taking now a region of A at fixed time t, the slice at this fixed time is $A_t \in \partial$ we can calculate the entanglement entropy by using the path integral we used for the proof of RT formula. In the bulk we should expect a kind of minimal hypersurface as in the static case. However now the conformal boundary ∂M is actually a Lorentzian manifold and and its equal time foliation does not generically to a foliation in the bulk M which is motivated by a certain symmetry but for now assume that we select such a natural foliation. By picking a spacelike (not spatial as before) slice N_t of the bulk M and this slice is an extension of the slice on the boundary ∂M a minimal surface can be well defined and can be found by using holography. The minimal surface (labelled from now and on) $\mathbb{S}$ will be such that $\partial S|_{\partial M} = \partial A$.

This indicates according to the original paper [11] that we look for a covariantly defined spacelike slice of the bulk, N_t , anchored at ∂N_t , which reduces to the constant-t slice for static bulk .

[3] By expansion of an orthogonal null geodesic congruence to the surface we mean that the trace of null extrinsic curvature is zero which is the maximal extremized value it can take. See section 3.2 of HRT paper [11].

3.5 Examples of calculation of entanglement entropy

Keeping the discussion about the RT formula in mind, let's now give some examples where holography can be used in order to calculate those RT surfaces. Those examples are provided directly from the original paper of Ryu and Takyanagi and the later reviews [10, 55, 33, 57].

3.5.1 Holographic dual of CFT_2 on an interval $\mathbb{R}^{1,1}$

Consider a CFT_2 theory, in zero temperature, which lives on an a manifold equivalent to an interval in $\mathbb{R}$, $A = \{x|x \in (a, a)\}$ and at time fixed $t = 0$ and we aim to calculate the entaglement entropy with holgraphy in order to pass the problem to its AdS_3 dual description. We should find the geodesic between two points $A = (x_1, z) = (-a, \epsilon)$ and $B = (x_2, z) = (a, \epsilon)$ in the Poincare patch with metric

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + dx^2 + dz^2) = \frac{L^2}{z^2}(dx^2 + dz^2)$$

Then the geodesic action is expressed as

$$I = L \int d\xi \frac{\sqrt{x'(\xi)^2 + z'(\xi)^2}}{z} \tag{3.38}$$

the equations of motions for this action reveal that the geodesic is a half circle in xz plane $(x, z) = a(cos\xi, sin\xi)$ with $-\frac{\epsilon}{a} \leq \xi \leq \frac{\epsilon}{a}$. This is the minimal "surface" γ_A in terms of RT formula. It's "area" is actually length as it is a 1-dimensional object and it can be calculated as

$$Length(\gamma_A) = -2L \log \frac{\epsilon}{2a} = 2L \log \frac{2a}{\epsilon} \tag{3.39}$$

thus the entanglement entropy is

$$S_A = \frac{Length(\gamma_A)}{4G_N^{(3)}} = 2L \int_{\epsilon/a}^{\pi/2} \frac{d\xi}{\sin \xi} = \frac{L}{2G_N^{(3)}} \log \frac{2a}{\epsilon} = \frac{c}{3} \log \frac{2a}{\epsilon} \tag{3.40}$$

[4] and of course it diverges in the UV limit of $\epsilon \to 0$ as we expected.

Note that if instead of a single interval we had an A consisted of many disconnected intervals $A = \cup_i A_i$ where $A_i = \{x \in \mathbb{R}|x \in (r_i, s_i)\}$ then each interval on its own has a minimal surface which is calculated in the way we saw. The total entanglement entropy

[4] The constant $c = \frac{3L}{2G_N^{(3)}}$ is in reality equal to the central charge of CFT_2

of the system is nothing more than the contribution of all minimal surfaces of all intervals but the difference here is that we have to consider geodesics that connect the left endpoint of one-interval, say A_i , with the right endpoint of any other A_j, including itself. So the length of those geodesics are proprtional to $2\log\frac{|s_i-r_j|}{\epsilon}$ and the total entropy will be

$$S_A = min\Big(\frac{c}{3}\sum_{i,j}\log\frac{|s_i-r_j|}{\epsilon}\Big) \tag{3.41}$$

3.5.2 Holographic dual of CFT_2 on $\mathbb{R}\times\mathbb{S}^1$

The next example is again a system A in CFT_2 on $\mathbb{R}\times\mathbb{S}^1$ with zero temperature. This has an AdS_3 dual with a corresponding metric in global coordinates (τ,ρ,θ)

$$ds^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\theta^2) \tag{3.42}$$

where R is the radius of AdS and with the conformal boundary at $\rho\to\infty$ so we have to use a cut-off $\rho\le\rho_0$ and suppose that the conformal boundary lies at this radius. Here the CFT_2 lives on a spacetime characterized by coordinates (τ,θ) at the boundary of AdS_3, so it is equivalent topologically to a cylinder. The subsystem A that we are interested in, lives in a region of the cylinder $0\le\theta\le\frac{2\pi l}{L}$ where l and L are the lengths of systems A and A^c respectively. The cut-off $\rho=\rho_0$ itself should satisfy the approximate relation. $e^{\rho_0}\sim\frac{L}{\epsilon}>>1$. Then for fixed time τ the geodesic that conects the points with $\theta=0$ and $\theta=\frac{2\pi l}{L}$ determines the minimal surface γ_A. Again here the $Area(\gamma_A)$ of the minimal surface is equal to the geodesic length which is determined by the equations of motion of the geodesic action of (3.42) for the time fixed metric (i.e $d\tau=0$).
The geodesic action is given by

$$I = R\int d\xi\sqrt{\Big(\frac{d\rho}{d\xi}\Big)^2+\sinh^2\rho\Big(\frac{d\theta}{d\xi}\Big)^2} \tag{3.43}$$

with the constraint that $\rho\le\rho_0$. That leads to an equation of motion for $Area(\gamma_A)$

$$\cosh\Big(\frac{Area(\gamma_A)}{R}\Big) = 1+2\sinh^2\rho_0\sin^2\frac{\pi l}{L} \tag{3.44}$$

and thus in the cut-off limit

$$Area(\gamma_A) = R\log\Big(e^{2\rho_0}\sin^2\frac{\pi l}{L}\Big) \tag{3.45}$$

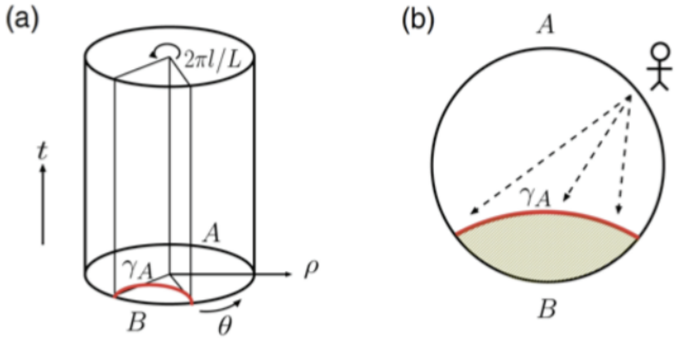

Figure 3.1: Minimal surface for CFT_2 on $\mathbb{R}\times\mathbb{S}^1$ [10]

which in turn gives the entanglement entropy

$$S_A = \frac{R}{4G_N^{(3)}}\log\left(e^{2\rho_0}\sin^2\frac{\pi l}{L}\right) = \frac{c}{3}\log\left(e^{\rho_0}\sin\frac{\pi l}{L}\right) \tag{3.46}$$

3.5.3 Entanglement entropy of CFT_2 at finite temperature

Consider now the same setup of the previous example. This example however, is a bit different than the previous one due to the fact that it is characterized by non zero temperature $\beta = \frac{1}{T}$. We assume that the system has spatial length L which is infinitely long such that $\frac{\beta}{L} << 1$. At high temperature limit the CFT has a dual gravitational theory in AdS_3 which is the Euclidean BTZ black black hole [58]. The Euclidean BTZ metric is given in the set of global coordinates (τ,ρ,ϕ) as

$$ds^2 = (\rho^2-\rho_+^2)d\tau^2 + \frac{R^2}{\rho^2-\rho_+^2}d\rho^2 + \rho^2 d\phi^2 \tag{3.47}$$

and the system A is $0\le\phi\le\frac{2\pi l}{L}$ as before. Here the euclidean time is compactified so that τ $\tau\sim\tau=\frac{2\pi R}{r_+}$ and also $\phi\sim\phi+2\pi$. Those conditions give a smooth geometry. As usual the boundary is taken at $\rho=\rho_0\to\infty$ and by this we find that the relation between the CFT and the BTZ black is given by

$$\frac{\beta}{L} = \frac{R}{r_+} << 1 \tag{3.48}$$

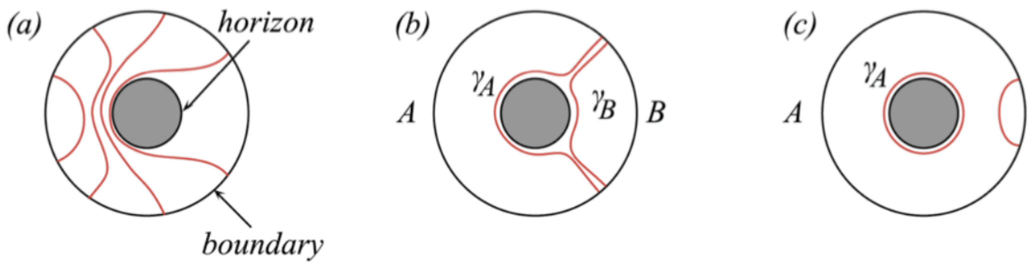

Figure 3.2: Minimal surfaces in the BTZ black hole for various sizes of A. (b) γ_A and γ_B wrap the different parts of the horizon. (c) When ∂A becomes larger, γ_A bifurcates into two parts: one wrapped on the horizon and the other localized near the boundary. [33]

The geodesic length determining the area of the minimal surface can be found similarly as in the other cases and obeys the equation

$$\cosh\left(\frac{Area(\gamma_A)}{R}\right) = 1 + \frac{2\rho_0^2}{\rho_+^2}\sinh^2\left(\frac{\pi l}{\beta}\right) \tag{3.49}$$

where the UV cut-off $e^{\rho_0} \sim \frac{\beta}{\epsilon}$. Then the RT formula gives

$$S_A(\beta) = \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\sinh\left(\frac{\pi l}{\beta}\right)\right) \tag{3.50}$$

This is the black hole entropy for BTZ system and since the dual CFT is a thermal field theory it is equal to the thermal entropy of CFT. In a geometrical point of view (see fig. 3.2), when A has small length, it behaves as the ordinary AdS_3. Nevertheless, when it becomes large in size, the turning point of the geodesic line approaches the horizon and eventually the geodesic line covers a part of horizon. This explains why the thermal behavior is apparent when $\frac{l}{\beta} >> 1$. Recall also that the entanglement entropy of A is equal to the one of the complement A^c when the temperature is zero, however here for non zero temperature this is not true in general.

The discussion about the minimal surfaces and the entropy calculation of a quantum system can be generalized for a higher dimensional CFT_d. Again by using the duality conjecture the problem of CFT entropy can be transferred to the dual gravitational theory in AdS_{d+1} and then in general we are able to find suitable minimal surfaces and use the RT formula.

Chapter 4

Holographic derivation of the Page curve

In this chapter we will make a review of some recent developments about the entropy of Hawking radiation in the context of AdS/CFT correspondence. The basic point is that the Page curve can be reproduced by semiclassical geometry approximation.

4.1 The fine-grained entropy of a gravitational system

Initially we will discuss about a generalized expression of the entanglement entropy of a gravitational system like a collapsing mass which forms a black hole. So far, we have defined various kinds of entropy and more importantly we saw that using holography the calculation of the entanglement entropy of a region A corresponding to a quantum system living on the CFT boundary can be realized from an RT/HRT surface in the AdS bulk geometry. In that way we studied the entropy of a BTZ black hole.

We know that the generalized entropy of Hawking and Bekenstein obeys the 2nd thermodynamic law and it should only increase and also the Bekenstein expression for the black hole entropy is a coarse-grained expression. Considering a version of the generalized entropy which contains contributions from the quantum fields it must have the expression of Faukner, Lewkowysk and Maldacena [59] given in the second chapter which proposed that in a semiclassical geometry setting

$$S_{gen} = \frac{Area(X)}{4G_N} + S_{vN} + counterterms \tag{4.1}$$

where X is a suitable surface found by RT/HRT perscription and S_{vN} is the bulk entanglement entropy of across the surface.
The question that now arises is whether we could have an expression for the fine-grained entropy as well. The naive answer seems to be no since we do not exactly possess full knowledge of the quantum gravity effects and also we have no knowledge about the black hole interior. Thus, there is not a known well defined density matrix in order to calculate directly the fine-grained entropy. Remarkably though, It turns out that such a formula exists indeed and it is a generalization of the generalized entropy. The difference is that instead of taking the area of the horizon we have to consider another surface in HRT spirit. There comes the notion from Engelhardt and Wall (2014) [13] that the fine-grained entropy of a CFT with a holographic dual which takes into account quantum corrections, is given at any order of $\hbar$ by the generalized entropy of a quantum extremal surface (QES) $\mathcal{X}_{\mathcal{A}}$ anchored at A and homologous to it. What we call a quantum extremal surface is a surface which extremizes the generalized entropy. Also if there are more than one such surfaces we have to choose the one which minimizes the generalized entropy. So the definition can be expressed as

$$S = min_{\mathcal{X}_A}\Big\{ext_{\mathcal{X}}\Big[\frac{Area(\mathcal{X})}{4G_N} + S_{vN}(\Sigma_{\mathcal{X}})\Big]\Big\} \tag{4.2}$$

where $\Sigma_{\mathcal{X}}$ is the region where we consider the quantum fields of the black hole system and it is bounded by the the QES and the cut-off surface of the black hole as a quantum system [32]. Once again the way that this formula works is that we start with a surface from the outside of the black hole horizon, we move and fix this surface in a way that it extremizes the generalized entropy ans among all the extrema we select the minimum and so we obtain the QES.

For the fine-grained entropy of a black hole, at its very early stages before the Hawking radiation starts escaping the black hole region there are no extremal surfaces emerging by deforming $\mathcal{X}$ inwards up to zero size. So the QES is the trivial surface $\mathcal{X}_0$ and it is called vanishing surface. That means that the area term of the (4.2) vanishes and the fine-grained entropy is just the quantum fields' von Neumann entropy of the black hole region near the cut-off surface. We assume that this contribution is constant in time and hence we can neglect this term provided the initial state of the black hole be pure. So initially the whole interior of the black hole is described by the degrees of freedom of the black hole and not radiation or otherwise the so-called *entanglement wedge* of the black hole system (the region of semiclassical space described by this specific subsystem) includes the interior which means the area whose causal structure is determined by the degrees of freedom of the black hole. As the black hole starts to emit thermal radiation the von Neumann entropy of the black hole region will start to increase as well, due to the existence of entangled pairs of radiation particles and their interior partners. This carries on as more and more entangled interior quanta accumulate [32].

This increase at early stages follows precisely the one that coarse-grained entropy of radiation does and naively it is likely to increase until the evaporation is completed. That is not true all the same, recently it has been shown that shortly after radiation starts a new QES $\mathcal{X}_1$ is formed [14, 15]. The position of this surface depends on the stage of evaporation and thus its position is time dependent but turns out that it lies just behind the horizon for a late stage black hole. Using this QES the generalized entropy has a dominant term which is the area term and subsequently can this can be identified approximately with the area of the horizon. The contribution of the quantum fields in the region of this QES are small enough and thus the corresponding von Neumann entropy can be neglected compared to the area term. Therefore, the form of the fine-grained entropy will be $S = min(S_{\mathcal{X}_0}, S_{\mathcal{X}_1})$ which gives a plot that consists of two sectors the first one corresponds to the trivial QES and describes the initial increase of generalized entropy until the point that it is equal with generalized entropy from the second QES. Then there is a phase transition and it follows the decrease of generalized entropy generated by the second surface. The time evolution of the fine-grained entropy of the black hole seems to have very similar structure to the Page curve indicating unitarity. However what we really look for is the Page curve concerning the entropy of radiation which is related to the information paradox.

4.2 Gravity coupled to holographic matter

In the following sections we will try to give a review of some new developments in calculating the entropy of radiation by assuming a bit different setting. We are going to consider a black hole living in a 2-dimensional gravity theory coupled to a CFT matter. This black hole has very low (non zero) initial temperature. Then this system is coupled to a CFT bath similar to CFT matter theory. We will present mainly the ideas and procedure followed by Almheiri, Mahajan, Maldacena and Zhao paper of 2019 "The Page curve of Hawking radiation from semiclassical geometry" [16]. The result of this discussion is that the entanglement entropy of radiation can be reproduced by a formula similar to (4.2) in [13] spirit and also follows the Page curve.

4.2.1 2-dimensional gravity coupled to CFT_2

In this chapter we will consider a 2-dimensional gravity theory but the results of the discussion can be generalized to higher dimensions.

So, consider now a general 2-dimensional theory of gravity. The classical gravity arises from Einstein-Hilbert action which in four dimensions is a nice theory of a dynamical spacetime but in 2 dimensions is trivial i.e. purely topological and does not contain any local dynamics which are required to describe collapsed matter, black holes, gravitational waves etc. The only contribution to the entropy from this system is then just a constant (equivalent to Euler character).
So we desire a model of interactive gravity in 2-dimensions which describes a dynamical spacetime. The simplest dynamical spacetime in two dimensions can be acquired by coupling of classical gravity to an interaction field which is a scalar dilaton field ϕ equivalent to the description of the evaporation of near extremal black holes[1] in Jackiw-Teitelboim (JT) gravity approximation given in [60, 61]. Note that the same discussion could be done with a spinor or a tensor field but since the quantities we are interested in i.e. entropy and area are purely scalar we prefer the simplest possible choice which clearly is the scalar field. Consider then the general action defined on a 2-dimensional Lorentzian manifold $\mathcal{M}$ with boundary ∂M.

$$I_{grav}[g^{(2)}_{\mu\nu}, \phi] = \frac{1}{16\pi G^{(2)}_N} \int_{\mathcal{M}} d^2x\sqrt{-g}\phi R^{(2)} + U(\phi) \tag{4.3}$$

where here

$$U(\phi) = \frac{1}{16\pi G^{(2)}_N} \int_{\mathcal{M}} 2\phi + \frac{1}{16\pi G^{(2)}_N} \int_{\partial\mathcal{M}} 2\phi K$$

The last bit is called Hawking-Gibbons term, $G^{(2)}_N$ is the renormalized Newton's constant in 2 dimensional theory and $R^{(2)}$ the corresponding Ricci scalar and also $g^{(2)}_{\mu\nu}$ is a 2-dimensional fixed metric where the theory lives in. In this approximation of JT gravity the dilaton field essentially shifts the classical Einstein-Hilbert term by the dilaton field ϕ [16].

[1]Black holes with almost minimal mass given the other two standard three classical degrees of freedom of angular momentum and charge.

Now in order to add matter to the system we need an extra term in the action. The matter is considered as CFT_2 with fields assume χ living on the 2-dimensional boundary of its 3 dimensional holographic dual theory, which is a 3-dimensional gravity theory living in asymptotic AdS_3.

In that case we will have a total action

$$I[g_{\mu\nu}^{(2)}, \phi, \chi] = I_{grav}[g_{\mu\nu}^{(2)}, \phi] + I_{CFT}[g_{\mu\nu}^{(2)}, \chi] \tag{4.4}$$

The entire coupled theory itself we want it to be asymptotically AdS_2. We also require that the central charge c of the matter CFT_2 will satisfy the condition $1 << c << \frac{\phi}{4G_N^{(2)}}$ so that we work in the semiclassical limit and also the radius of the corresponding dual theory is sufficiently big. Also we need a strongly coupled CFT in order to have an Einstein gravity.

Now we want to discover the higher dimensional dual theory which is described by the full action. The main idea is to start from the given 3d geometry add the dilaton field ϕ that lives on the 2d boundary and functionally integrate over ϕ and $g_{\mu\nu}^{(2)}$. Here we note that the integration concerns the 2d fixed background metric and not only the dilaton field. Because of the gravitational degrees of freedom in 3 dimensions the resulting theory behaves locally as AdS_3 with a boundary at finite location which is actually a reason why we take the gravity theory in 2 dimensions. If we had taken it in 3 dimensions then the dual theory would be 4-dimensional and it would contain non trivial degrees of freedom i.e interactions, which would make our study much more complicated.

Therefore, we have a dual theory which has a dynamical conformal boundary which can be seen like the Randall-Sundrum model from string theory [62, 63] which had been introduced as an alternative to dimensional compactification. The dynamical boundary of the dual theory is called *Planck brane*. The CFT_2 lives in spacetime, with metric $g_{\mu\nu}^{(2)}$, which is described by the action $I_{CFT}[g_{\mu\nu}^{(2)}, \chi]$ and has a holographic dual gravity with 2-dimensional dynamical boundary (brane) where the metric obeys the boundary condition that

$$g_{\mu\nu}^{(3)}|_{brane} = \frac{1}{\epsilon^2} g_{\mu\nu}^{(2)} \tag{4.5}$$

and here ϵ is just a short distance cut-off preventing some UV divergences.

The location of the Planck brane in asymptotically AdS_3 can be computed in the spirit of the following arguments. The 2-dimensional coupled theory lives in geometry with a metric asymptotically AdS_2 and in Poincare coordinates is written as $ds^2 = \frac{1}{z^2}(-dt^2+dz^2)$ by change of coordinates $t = \frac{y^+ - y^-}{2}$ $z = \frac{y^+ - y^-}{2}$ can be brought to the form

$$ds^2 = -\frac{dy^+ dy^-}{(y^+ - y^-)^2} = -e^{2\rho(y)} dy^+ dy^- \tag{4.6}$$

and additionally has stress energy tensor $T_{y^\pm y^\pm}(y^\pm)$. The stress energy tensor of the dilaton gravity is explicitly dependent on the Ricci scalar but here we also have the matter theory which is CFT_2. That theory in flat space has as we saw stress energy tensor with zero trace. However, here do not have a flat space and therefore it should not be zero in general, but it still can be expressed in terms of the the curvatures of the spacetime. This is called the conformal anomaly and so the stress energy tensor or rather the part contributes to the trace at least is determined by the Ricci scalar or more specifically by derivates of $\rho(y)$.
Next we introduce new coordinates $w^\pm$ for which the stress energy tensor vanishes. Under a general diffeomorphism the stress energy tensor transforms as

$$\left(\frac{dw}{dy}\right)^2 T_{w^\pm w^\pm} = T_{y^\pm y^\pm} + \frac{c}{24\pi}\{w^\pm(y), y^\pm\} \tag{4.7}$$

where $\{w(y), y\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$ is called Schwarzian derivative. If then we take a conformal transformation of the metric itself and so bring the metric in the form of 2-dimensional Minkowski spacetime $ds^2 = -dw^+ dw^-$ then the stress tensor vanishes universally. So considering the full bulk, locally AdS_3 metric expresseed in $w^\pm$ coordinates

$$ds^2 = \frac{-dw^+ dw^- + dz_w^2}{z_w^2} \tag{4.8}$$

and if we take a slice of $z_w = constant$ in this geometry the induced metric is written as $ds^2 = -\frac{dw^+ dw^-}{z_w^2}$ and obviously on this slice the stress energy tensor vanishes. Because of the condition (4.5) we demand that for 2-dimensional geometry on the boundary i.e the Planck brane

$$-\frac{dw^+ dw^-}{z_w^2} = -\frac{e^{2\rho(y)} dy^+ dy^-}{\epsilon^2}$$

which subsequently gives

$$z_w = \epsilon e^{-\rho(y)} \sqrt{\frac{dw^+}{dy^+} \frac{dw^-}{dy^-}} \tag{4.9}$$

So to resume what we have done: we started with a given 2-dimensional geometry of the coupled theory and then considered that it has a holographic dual in 3 dimensions which is **locally** AdS_3 with a finite localized boundary, the Planck brane, and by imposing appropriate boundary conditions for the geometry and also conditions for the energy tensor we derived an expression that gives the position of this brane in the bulk.

4.2.2 Black hole coupled to holographic bath

On the setup of the 2-dimensional theory described by the action of (4.4) we consider a black hole that lives on this symptotically AdS_2 geometry. Then then we attach the theory to an external, non gravitational CFT_2 bath of zero tempereture [2] same as the CFT_2 of the matter field in (4.4) with appropriate boundary conditions at the point where the two theories are joined such that energy can flow between the gravity+matter theory and the bath.[3] Then the black hole is starts to evaporate into the bath which collects the Hawking radiation. Intuitively we are allowed to do that because the coupled theory has a dual bulk theory which as we mentioned looks locally AdS_3 as well. Therefore, we can consider the full AdS_3 with a conformal CFT_2 living on the AdS_3 boundary which corresponds to the bath and join it with the Planck brane of the dual bulk theory. By definig $\sigma_y = \frac{y^+ - y^-}{2}$ we can conventionally consider that the bath and the coupled theory are joined at $\sigma_y = 0$ and the points where $\sigma_y > 0$ correspond to the bath while the ones with $\sigma_y < 0$ to the coupled gravity+matter.

Also from the central dogma we know that we can see the black hole as a quantum mechanical system and also the 2-dimensional gravity is asymptotically AdS_2 which means that we can use holography again and describe the black hole system as a CFT_1 system which essentially is the usual quantum mechamics with some extra symmetry. Thus an alternative realization of the coupled gravity+matter to the bath is to assume the black hole as a quantum mechanical system dual to the original gravitational, with $\frac{A}{4G_N^{(3)}}$ degrees

[2] or rather it should be called reservoir instead of bath because it remains at a constant zero temperature

[3] Here we have to note that a since the gravity theory is asymptotically AdS_2 we can use very good and special characteristic of AdS spacetimes that emerges: the energy can reach infinity in a finite time which means that the bath can receive energy from the gravitational system in a finite time and so joining the two systems is useful, while in asymptotically flat spacetimes energy needs infinite time to reach spatial infinity.

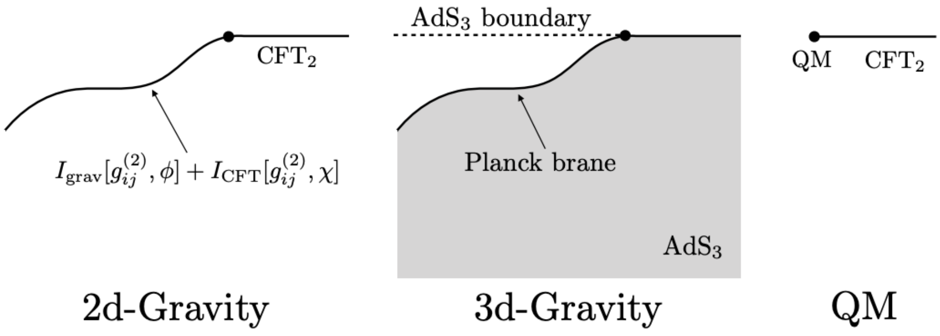

Figure 4.1: Three equivalent considerations of black hole coupling to the bath [16]

of freedom, coupled to the bath. The quantum mechanical system in this case lives at $\sigma_y = 0$ and it is joined with the bath at this point.

Hence we argue that there are three equivalent realizations of the gravity+theory coupled to a holographic bath [16].
2-dimensional gravity+holographic matter theory for $\sigma_y < 0$ coupled to a CFT_2 bath with $\sigma_y > 0$
3-dimensional bulk theory in AdS_3 with dynamical boundary (Planck brane) at $sigma_y < 0$attached to a AdS_3 a boundary where the CFT_2 bath lives for $\sigma_y > 0$
QM dual description of the black hole at $\sigma_y = 0$ joined with a 2-dimensional CFT bath on the half line $\sigma_y > 0$
The three considerations are shown in figure 4.1

4.2.3 The entanglement entropy of the 2-dimensional theory

We have already done some discussion about how to calculate the generalized entropy of a system using quantum extremal surfaces. Let's try now to apply this perscription aiming to calculate the fine-grained entropy of the system of 2-dimensional coupled gravity+matter. Cosnider a point y in the 2-dimensional bulk and also the interval $\mathcal{I}_y$ from that point to some region far away where the dilaton field becomes very large and the theory very weakly coupled. For simplicity this ending point can be taken as the 2-dimensional boundary of the asymptotically AdS_2. Then we construct a quantity at the point y similar to the previously defined generalized entropy with the difference that we need to consider the dilaton field ϕ as well as well also the contribution of the matter field

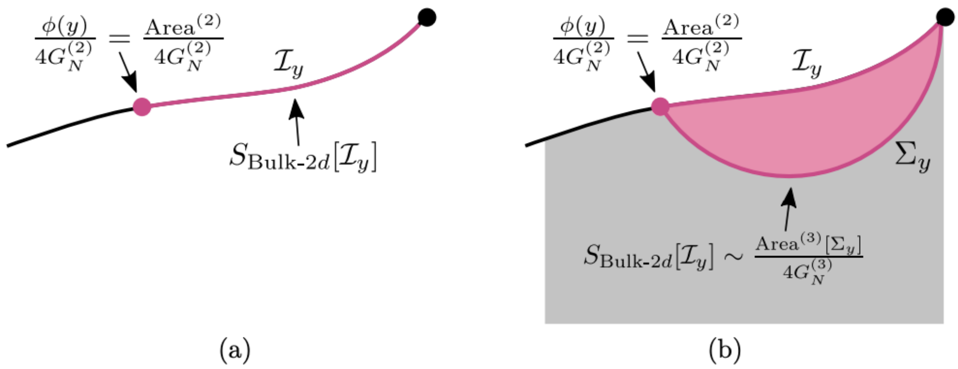

Figure 4.2: (a) surfaces contributing to the generilized entropy in 2d geometry (b) the 3d geometry picture of dual theory in the bulk.[16]

χ from the 2-dimensional bulk. So the constructed gnerealized entropy is

$$S_{gen}(y) = \frac{\phi(y)}{4G_N^{(2)}} + S_{Bulk-2d}[\mathcal{I}_y] \tag{4.10}$$

Because the problem is two dimensional the area of a point is the coefficient of the Ricci scalar term in the action (4.4), the first area term is given by the $\phi(y)$. The second term is the von Neumann bulk entropy $S_{Bulk-2d}$ and includes apart from the dominant contribution of the matter field χ also contributions from quantum fluctuations of the 2-dimensional metric and dilaton field. Now in order to to take the fine-grained entropy we have to find the points y which extremize the expression (4.9) and choose then the point (y_e^+, y_e^-) which makes it minimal and this is the QES[4].

Moreover, the $S_{Bulk-2d}$ can be computed using the holographic 3-dimensional description to leading order using RT/HRT formula. So the aim is to find an extrenal surface Σ_y in the dual AdS_3 geometry starting from the point y on the brane to the endpoint and in this case it is an interval. So, ignoring the quantum fluctuations mentioned above and also higher order terms of 3-dimensional bulk entanglement entropy we have

$$S_{Bulk-2d} \approx \frac{Area^{(3)}(\Sigma_y)}{4G_N^{(3)}} \tag{4.11}$$

and the whole generalized geometry is written as

$$S_{gen}(y) \approx \frac{\phi(y)}{4G_N^{(2)}} + \frac{Area^{(3)}[\Sigma_y]}{4G_N^{(3)}} \tag{4.12}$$

[4]which in the context of 2d theory is just a point.

The extremization of generalized entropy in 2 dimensions is equivalent to the standard RT/HRT area extremization in the 3-dimensional dual with a dynamical boundary the Planck brane. Consequently we look for an area-extremizing surface in 3d with an end-point on the Planck brane. This “area” has a contribution coming from the length of the line Σ_y as well as a contribution from the dilaton field at the Planck brane. In order to find the fine-grained entropy we have to extremize the entire expression (4.11), thus we also need to find the position of the point (y_e^+, y_e^-) on the Planck brane. At leading order the whole problem is equivalent to finding the RT/HRT extremal surface [16].

4.3 Entanglement wedges for an evaporating black hole

In this section we would like to present the extremal surfaces of the holograpic JT gravity presented in [14, 15, 16]. We will try to determine which are the entanglement wedges of the black hole and radiation and the cextremal surfaces of of this model.

4.3.1 Early times

We assume a very low temperature black hole (but non-zero) which is initially decoupled from the bath. As we mentioned before two CFTs of the same dimension can generally be attached by imposing appropriate conditions on their boundary. So in this case the decoupled situation can be described as having the gravity theory where the black hole lives and and the CFT bath joined at the point $\sigma_w = 0$ but initially separated by an intransversible conformal boundary. This conformal boundary does not carry any energy. The holographic dual of this boundary is in AdS bulk and the its conformal boundary condition is reflective in both sides. This boundary is called Cardy brane and seperates the bulk spacetime into two regions as we see in figure. The left side is the locally AdS_3 bulk of gravity+matter theory with the black hole whereas the right one is the AdS_3 bulk of the CFT_2 bath. In this context the two Cardy branes can be described as two straight lines located at $\sigma_w = 0^\pm$ which immerse down into the bulk and they separate the two geometries fully, the Cardy branes also are independent of Ricci scalar and the dilaton field.

Consider now that at time $t = 0$ the two systems are joined and so they can exchange energy. Since the bath initially is actually empty if we join the two systems instantly at $t = 0$ we will get an infinite pulse of energy from the gravity system to the bath. A rather theatrical description of that is to think what would happen if we had a water

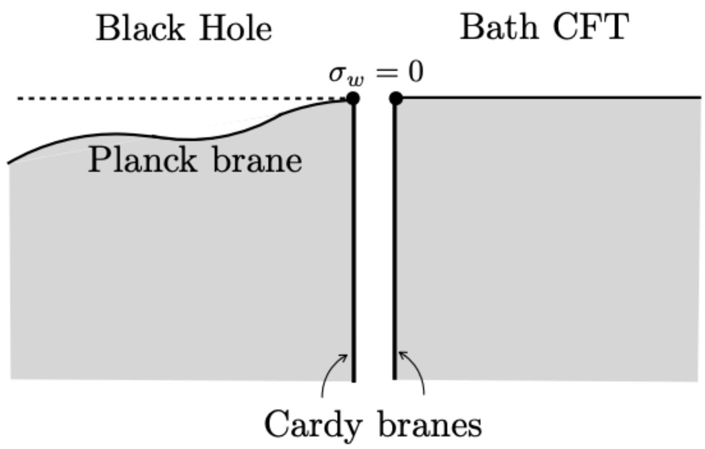

Figure 4.3: Systems black hole-bath before the coupling. [16]

reservoir with a very tall dam and suddenly the dam dissappears...Disaster! This comes from the fact that when the systems becomes suddenly coupled, certain boundary terms called Gibbons-Hawking appearing in the original CFT actions and associated with the initially reflective AdS boundaries change suddenly and become transversable so that the energy can flow. So it is preferred to couple the two systems over some time interval Δt and that gives a pulse of energy $E_1 \sim \frac{c}{\Delta t}$. Then the Cardy brane is removed from the CFT boundary and leaves a Lorentzian geometry of 3 dimensions behind, locally AdS_3, as then and immerses towards the interior of the bulk AdS_3. As the time passes it falls deeper into the bulk further from the physical boundary. Then the pulse of make the black hole to increase its mass and subsequently its temperature. The new temperature of the black hole will be such that

$$E_1 = \frac{\pi\phi}{4G_N^{(2)}} T_i^2 \tag{4.13}$$

As the black hole starts to evaporate this temperature starts to drop and specifically in a way

$$T(t) \approx T_i e^{-\frac{\kappa}{2}t} \tag{4.14}$$

where the constant κ depends on the central charge c and also to the effective gravitianonal coupling of 2-dimensional gravity [16, 14]. Now Hawking radiation is captured by the bath and so the study of entanglement of radiation and the black hole modes, becomes equivalent to studying the entanglement of the black hole and the bath. Initially the entanglement wedge of the black hole occupies the whole black hole region. On the other hand the entanglement entropy of the black hole increases as the Cardy brane falls deep into the bulk and so more and more Hawking particles in captured in the bath

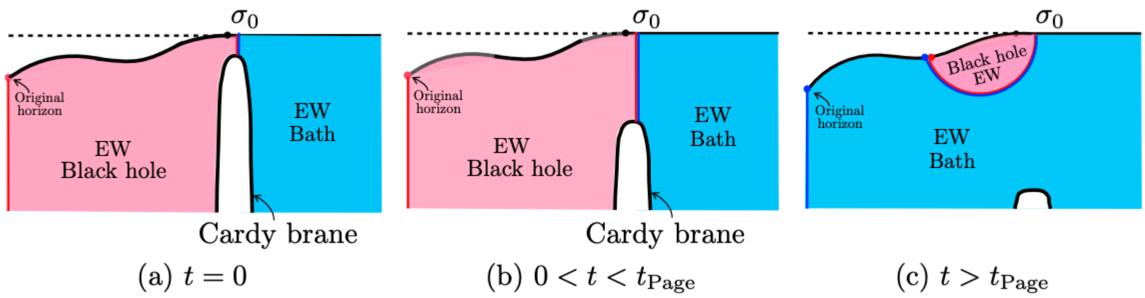

Figure 4.4: Picture of the entanglement wedges of black the hole and the bath.[16]

are entangled with their partners in the black hole region behind the horizon. So the entanglement entropy is also the entropy of radiation and it is approximately

$$S(t) \sim S_{rad}(t) = \frac{\pi c}{6} \int_0^t duT(u) = 2S_{BH}^{coarse}(1 - e^{-\frac{\kappa}{2}t}) \tag{4.15}$$

where S_{BH}^{coarse} is the usual coarse-grained Bekenstein entropy. So this entropy is increasing monotonically in time with a maximal value being $2S_{BH}$ we see. The factor of 2 exists because Hawking radiation is not an adiabatic process [16]. Notice that as far as QES may concern at very early times it is identified as the classical extremal surface, at the bifurcation point of the original black hole horizon, before coupling. In the bulk theory the corresponding surface extends into the bulk.

4.3.2 Late times

After the coupling as we we have the emergence of a new quantum extremal surface which is at the point y_e of the gravitational theory. In the dual quantum mechanical description we have to consider quantum mechanical degress of freedom related to the CFT boundary and this can be done by taking an interval $[0, \sigma_0]$ on the right half line of the coupling between gravity theory with the bath.

Then the quantum extremal surface is at a point (y_e^+, y_e^-) and it can be found behind the horizon by the following argument. We go back along the AdS_2 boundary surface by a time of order $\frac{1}{T(t)} \log S_{BH}$ and we shoot an ingoing light ray. Then the surface is located close to the point where this light ray intersects the horizon. This time scale $\sim \frac{1}{T(t)} S_{BH}$. The time scale $\frac{1}{T(t)} \log S_{BH}$ is called scrambling time [64] and it is very short compared to the evaporation time [32]. A more accurate expression is that its position is determined

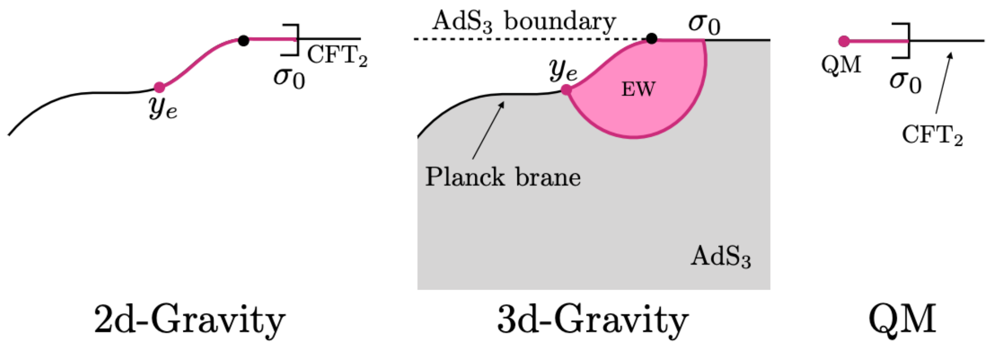

Figure 4.5: The entanglement wedge of the black hole at times which in 2d is a sspatial slice. In 3d gravity picture there is also a RT/HRT surface [16]

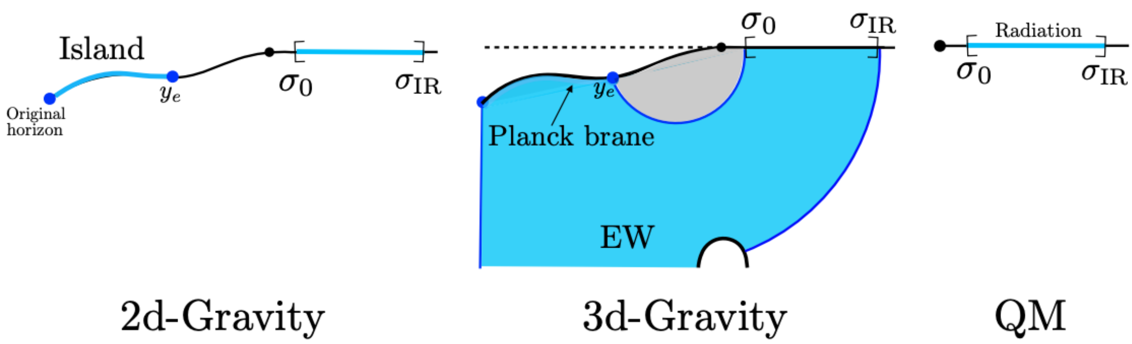

Figure 4.6: The entanglement wedge of the radiation including a contribution from the initial black hole horizon, and a contribution from the IR cutoff in the CFT. [16]

by

$$y_e^+ = t - \frac{1}{2\pi T(t)} \log \frac{S_{BH}(T(t)) - S_0}{c} + ... \tag{4.16}$$

with $S_0 << S_{BH}^{coarse}$ be the extremal entropy associated with the black hole before the coupling with the bath. Therefore the entanglement entropy reduces to an RT/HRT surface from the point y_e to σ_0 which is an interval contribution and the entanglement wedge of the black hole is the causal domain of this interval. The entanglement entropy of the black hole then for this extremal surface is given by

$$S(t) = S_{BH}^{coarse}(T(t)) + O(log) \tag{4.17}$$

so the dominant term is the coarse-grained Bekenstein term and also there are some logarithmic contributions of initial state entropy. The main point of this expression is that it decreases monotonically since the temperature decreases. Now at last let's discuss about radiation which is the most important point in order to derive the form of Page curve. As we saw at early times the contribution from the early time QES gives an increasing entropy of radiation which is approximately equal to the entropy of the black hole. As we mentioned above the entanglement of radiation and the black hole can be viewed as the study of the entanglement of black hole and the CFT bath and so we have to study the entropy of the CFT in the bath, on an interval outside σ_0, $[\sigma_0, \sigma_{IR}] \equiv \Sigma_R$ where $\sigma_{IR} > t_{evap}$ is a cut-off point large enough so that we assume all entropy from radiation emitted from the black hole is approximately contained entirely there. The entanglement wedge in 2-dimensional point of view should be this interval which would reproduce an increasing entropy of radiation and give Hawking result. However at late times the entanglement wedge of the black hole does not contain the whole interior. Thus we assume now that the rest of the interior belongs to the wedge of radiation.

The result is a that apart from the main region of radiation there is also a manifestly disconnected region behind the horizon which is called the "island" which is seperated from the "mainland" which is the interval $[\sigma_0, \sigma_{IR}]$. This seems rather odd, however, when we take the dual picture we see directly that the two regions are actually connected via the extra dimension in the bulk theory. Therefore, considering that at late times the entanglement wedge of radiation is identified by the entanglement wedge of the bath indeed it contains three extremal surfaces. One concerns the initial extremal surface of the of the original black hole horizon, which is located at the leftmost point of the island. The second is an extremal surface found from RT/HRT prescription from σ_0 to the Planck brane. This extremal surface coincides with the surface discussed for black hole and it is the minimal at late times. Finally there is another extremal surface which starts at σ_{IR} and finishes at Cardy brane which at late times is immersed very deep into the bulk giving a very small entropy

$$S \sim \frac{c}{6} \log \sigma_{IR} << S_{BH}^{coarse}$$

. The essential point of this island prescription is that we can have in 2-dimensional consideration, at late times, a disconnected entanglement wedge for radiation which is actually connected via the extra dimension of the dual theory in 3 dimensions. Therfore, the equation (4.2) can be generalized in order to include the island configuration of the entanglement wedge and then the entanglement entropy of radiation is

$$S_{rad} = min\Big\{extr\Big(\frac{Area(\mathcal{X})}{4G_N^{(3)}} + S_{eff}(\Sigma_R \cup \Sigma_{island})\Big)\Big\} \tag{4.18}$$

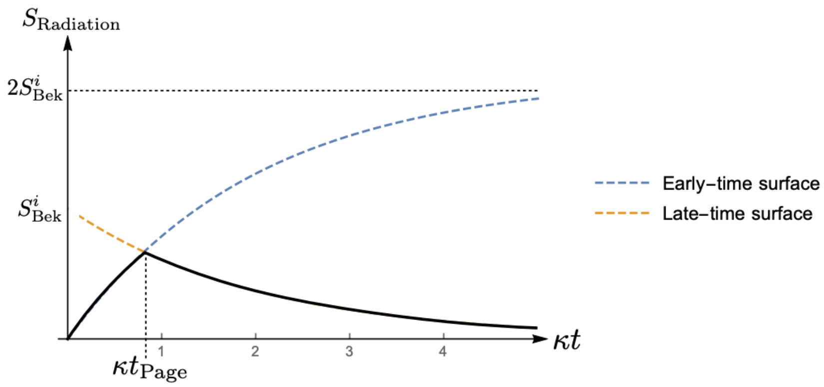

Figure 4.7: Page curve of Hawking radiation of JT gravity model coupling to zero temperature CFT bath.[16]

where S_{eff} is the effective entropy of radiation in semiclassical geometry approximation taking into account the island configuration of the entanglement wedge.

4.3.3 Reproducing the Page curve

So now practically our task is accomplished. The time evolution of the entropy of Hawking radiation will be configured by the above extremal surfaces. Initially the radiation will rise following the equation (4.14), then at Page time its value is equal to Bekenstein entropy which subsequently coincides with the entanglement entropy of the Black hole and at this point, with a minimal extremal surface concerning the original black hole. After this point there is a phase transition in entanglement wedge and minimal extremal surface which gives rise to radiation entropy at late times is the one concerning the black hole at late times and the entropy gradually decreases. Thus we have reproduced the Page curve! Of course details about how smooth is the phase transition or when exactly is have to do with quantum effects (see [15]) but the main point of this model is that it describes a black hole system which illustrates an evaporating black hole and respect unitarity.

Conclusion

So the conclusion of the entire discussion in this thesis is that indeed the Page curve can be derived holographically, by considering a black hole in a 2d gravity theory with holographic matter coupled to a 2d CFT bath. So in terms of AdS/CFT correspondence black holes as quantum systems evolve in unitary fashion. This prescription gives rise to an entanglement wedge which contains a disconnected island region in the interior, in 2-dimensions and it is connected with the exterior mainland region via the third dimension of the holographic dual theory.

The derivation of the Page curve with this holographic prescription is surely a positive step towards finding a consistent solution to information paradox. However, again here we have to emphasize that all this discussion has not given any new detail about the final state and thus about the corresponding density matrix. We actually have discovered a delicate way by using RT/HRT formula in order to calculate entanglement entropies but we do not know precisely the final states which are determined potentially by gravity effects during the final stages and therefore, we do not have a precise result of the elements of density matrix respectively.

This field is still very new and there is surely lot of work to be done. More specifically in 2020 new studies have been published which analyze the same problem but the CFT bath is now a thermal bath of finite temperature [65] and also the case of the 2-dimensional gravity+matter theory in an asympotically flat spacetime instead of asymptotically AdS_2[66]. The whole discussion we have done was about 2-dimensional theory but it can be generalized to d-dimensions, even though, as we had mentioned the coupling of gravity and holographic matter becomes far more complicated.

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