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Pati-Salam Model Grand Unification and Violation of Lepton Universality

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Abstract

In recent years, evidence from a variety of experiments at CERN has indicated that the universality of leptons, as predicted by the standard model of particle of physics, might be violated. If true, these results would point to the first false experimental predictions made by the standard model- an exciting invitation to explore physics beyond the standard model. In this thesis, I explore in detail a Grand Unified Theory known as the Pati-Salam model, in which violation of lepton universality arises naturally through new interactions of the fermions with 'leptoquarks'- the gauge bosons associated to the $SU(4)$ subgroup of Pati-Salam. Moreover, although the explorations are of Pati-Salam in particular, a universal approach to understand the theory is emphasised. Thus, the analysis carries through to Grand Unified Theories more generally.

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Contents

1 Introduction

1.1 Unification Theories

The most ambitious goal of theoretical physics is, for many people, to discover a theory of everything. Unpacking that phrase, a theory of everything means a single, unified physical theory from which one is able to explain (at least in principle) all observed phenomena. In fact, physicists have been on the lookout for unifying theories for some centuries. Although it is now taken for granted, phenomena that we casually relate to one another were likely observed by humans to be completely sporadic and unrelated until recently. The motion of the stars and planets seemed completely distinct to the behaviour of massive objects on earth. The idea that the immense force of electricity in thunderstorms and the peculiar attraction between magnetite rocks were two consequences of the same underlying theory would have surely seemed ridiculous until only a few centuries ago.

It is by no means obvious that phenomena may be unified in such a way, and there is no guarantee that we will be able to continue this process ad infinitum. However, the appeal (both philosophical, and practical) is too great. Therefore, theorists have continued chasing unification to this day.

Perhaps the greatest unification theory to date is unenthusiastically dubbed the standard model of particle physics. In the standard model, 3 out of 4 of the fundamental forces are described. All of the particles we can currently observe, as well as their interactions with one another can all be seen to arise from a single physical theory. Thus, excluding the effects of gravity, the standard model [1] has been able to explain every single phenomenon, whether it involves the strong and weak forces of nuclear physics, or the interactions between light and matter. A more intuitive view of the process of unification can be obtained by considering Maxwell and Faraday's work in the 19^{th} century, in which the electric and magnetic forces were revealed to be different facets of the unified, electromagnetic theory [2]. In fact, the standard model predicts that the electromagnetic force unifies with the weak nuclear force above certain energies, thereby forming the single electroweak force. In a similar manner to the unification of electricity and magnetism, GUT's aim to present all of nature's most fundamental forces as the various parts of a single whole, unifying phenomena which appear to be different and independent of one another. Grand Unified Theories have the ambition to further unify the electroweak force with the strong nuclear force, meaning that above certain energies, all three forces are in fact described by a single one, with a single coupling constant. The exact details of the standard model, including precise predictions of this unification theory will be discussed in more detail later on. It is unfortunate that the theory which constitutes our most successful realisation as of yet of the beautiful idea that is the unification of physical phenomena should be called a standard model, and this has no doubt meant that the theory has been broadly underappreciated by the general public.

1.2 Symmetries, Lagrangians and The Principle of Least Action

The most successful and readily generalised principle of modern theoretical physics arguably has to be the principle of least action via the Lagrangian formalism. Having successfully reproduced the equations of motion for Newtonian Mechanics, General Relativity and Quantum Mechanics, the Lagrangian formalism and the use of this principle also proved to be extremely useful in Quantum Field Theories. Given the stunning precision of certain measurements in QED [3], it appears unarguable that the principle of least action permeates physical laws at all scales of reality. It just so happens that there is inherent mathematical beauty to the formulation of physical theories using a Lagrangian (density). For example, it makes manifest the existence of symmetries in physical laws, which in turn, give rise to conserved currents and charges [4] which may be tested for in experiments. Conversely, prior knowledge of a conserved quantity (and thus a continuous symmetry) can guide us in our search for the Lagrangian which more correctly describes our physical universe. One finds that (internal) symmetries of the Lagrangian come in the form of compact Lie groups, and may act on fields in different representations. The classical equations of motion arise very naturally in this formalism, as well as the existence of conserved Noether currents and charges. Thus, this principle also invokes the use of sophisticated mathematics, namely the representation theory of lie groups and algebras.

1.3 Gauge Groups and Gauge Bosons

The main distinctive factor between theories that use the Lagrangian formalism (including GUT's) is the choice of gauge group- the internal symmetry transformations that keep the Lagrangian invariant. In this section, I would like to emphasise the unique part gauge groups and consequently gauge bosons play in the formation of a potential Unified Theory, like the standard model. Many different fields may be added to or subtracted from a physical theory, often with the foresight endowed by knowing which particles you want your theory to predict. However, given a gauge group there is an inherent set of fields that are always intrinsic to a physical theory which rely on that particular gauge group, assuming local symmetry is enforced. To any simple Lie group one is able to find the Lie algebra by considering elements infinitesimally close to the identity. It turns out that the gauge fields are elements of the Lie algebra associated to the gauge group, and thus the gauge boson particle spectrum is intimately connected to the choice of gauge group. Unlike the existence of Higgs particle(s) [5] and additional fermions, gauging a Lagrangian with a given symmetry group comes hand in hand with the associated gauge bosons. This then offers us a convenient route for discovering GUT's, via some additional impressive mathematics, namely Goldstone's theorem [6] and the Higgs mechanism.

1.4 Goldstone Theorem and The Higgs Mechanism- The Standard Model

Given a Lagrangian with some gauge fields (associated to a gauge group), a scalar field and possibly some fermion matter fields, what might one be able to say about the particle spectrum of the theory? Moreover, how might physicists use this information when looking for extensions to the standard model gauge group? The answers lie in Goldstone's theorem and the Higgs mechanism.
Let us use this opportunity to discuss the standard model Lagrangian. Therefore, suppose our theory has the symmetry group $SU(3)_{strong} \times SU(2)_{weak} \times U(1)_Y$, and is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}(\mathrm{Tr}[F_{3\mu\nu}F_3^{\mu\nu} + F_{2\mu\nu}F_2^{\mu\nu}] + F_{\mu\nu}F^{\mu\nu}) + (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \frac{\lambda}{2}(\phi^\dagger\phi)^2 + \mathcal{L}^f_{kinetic} + \mathcal{L}^f_{Yukawa}, \quad (1)$$

with $D^\mu = (\partial^\mu + ig_2 A^\mu + ig_1 Y B^\mu)$ and m^2 some real parameter. The field strength tensors are defined here in the usual way from the gauge fields A^μ, A_2^μ and A_3^μ belonging to $U(1)$, $SU(2)$ and $SU(3)$ respectively, namely

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (2)$$

,

$$F_2^{\mu\nu} = \partial^\mu A_2^\nu - \partial^\nu A_2^\mu + ig_2[A_2^\mu, A_2^\nu], \tag{3}$$

and

$$F_3^{\mu\nu} = \partial^\mu A_3^\nu - \partial^\nu A_3^\mu + ig_3[A_3^\mu, A_3^\nu]. \tag{4}$$

The Higgs field ϕ consists of two complex components and is in the fundamental representation of $SU(2)$ and in the trivial representation of $SU(3)$, so that it transforms as $\phi_i \mapsto M_{ij}\phi_j$, for some element M of $SU(2)$.

We delay the discussion of the kinetic and Yukawa sector associated to the fermions to the next chapter. The reason is that the notion of lepton universality will arise from careful examination of these terms. Since our aim later on will be to explore the possible violation of lepton universality within the Pati-Salam model, we prefer to discuss these terms in more detail by themselves,

We would like to be able to say something about the particle spectrum of the above theory. As we will see, the spectrum will depend on whether the parameter m^2 is positive or negative. Supposing that m^2 is positive, our Lagrangian (1) can be expanded to second order to find the particle spectrum (we do this in more detail for Pati-Salam later on, so here it will suffice to summarise what we would find). Doing so, one would find that our current particle spectrum consists of 12 massless gauge bosons and 4 massive scalar bosons. Considering that massless gauge bosons may oscillate in two perpendicular directions, and that a complex 2 component Higgs field contains four real degrees of freedom, our total number of degrees of freedom is 28.

Now let us suppose that m^2 is negative. We would then find a continuous vacuum manifold of physically identical, non-zero vevs, out of which we pick

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ v \end{bmatrix}. \tag{5}$$

It is a simple calculation (which we will do in detail for Pati-Salam) to see that due to this vev degeneracy, our potential term in the Lagrangian would read off an entirely different particle spectrum: 9 massless and 3 massive gauge bosons, whilst retaining only a single massive Higgs particle [7].

If we were to again consider symmetry transformations which keep our potential invariant, we would find that it is only invariant under the *residual symmetry group* $SU(3) \times U(1)_Q$. The reason that our residual symmetry group is smaller is because the 3 missing generators are broken, in that they do not leave the vev invariant. Counting the degrees of freedom again (recalling that massive gauge bosons, being spin one particles, have three) we indeed find that we still have 28 degrees of freedom overall. The standard model Lagrangian (for our typical energy scales) supposes that m^2 is indeed negative- thus we have summarised that our gauge and scalar particle spectrum in the standard model corresponds to a massive Higgs particle, 9 massless gauge bosons (the gluons, the photon) and 3 massive gauge bosons (W^+, W^- and Z).

These results are a consequence of Goldstone's theorem alongside the Higgs mechanism [8], and may be stated as follows: **When gauging a symmetry for a Lagrangian with a scalar field ϕ and a non zero vev, the particle spectrum D *massless gauge bosons and* K *massive Higgs particles* turns into the spectrum $D - N$ *massless gauge bosons,* N *massive gauge bosons and K-N massive Higgs bosons*, where N is the number of broken generators of our lie algebra**. We shall use this statement repeatedly when demonstrating the particle content associated to a scalar potential, before and after symmetry breaking. We recall that the fermion particle spectrum will be discussed later on, but let us mention that the Higgs mechanism is also responsible for the fermion masses.

So, as a theoretician aspiring to find an appropriate GUT, one must make sure that whatever gauge group is chosen, the number of massive and massless gauge bosons (as well as their relative masses to one another) reduces to that of the standard model, at least for the energies probed at our current particle accelerators. Having seen the structure of the standard model, let us now discuss Grand Unified Theories and lepton universality.

2 Grand Unified Theories and Lepton Universality

2.1 Background and Motivation

A GUT usually refers to a model in which all known forces (except gravity) may be unified so that they all arise from a single force, originating from a simple Lie group, called the gauge group of the theory. If the Lie group is semisimple, like Pati-Salam, we will still sometimes refer to it as a GUT (Pati-Salam may be embedded in a simple Lie group). A gauge group, in turn, refers to the group of transformations on the fields of a given Lagrangian that leave it invariant.

GUT's differ primarily in the choice of gauge group, and these in turn heavily constrain the particle spectra predicted by the theory, as well as the various interactions made possible. Thus, one may use GUT's to predict the existence of particles and interactions, and conversely one may use experimental data as evidence for or against a given GUT.

Broadly speak, two main motivating reasons may be given to the search for a GUT. The first and more practical one is that the standard model appears incomplete. Examples include the issue of neutrino masses and the inability to incorporate gravity into the theory [2]. The second reason is regarding the elegance or simplicity of the theory- it appears like many of the 'coincidences' in the standard model should have a more comprehensible origin. For example, the similar structure of (left handed) quarks and leptons under $SU(2)$ appears as a strange coincidence in the standard model, but as we shall see, can be neatly explained by Pati-Salam. It also seems strange that there exist 3 copies of fermions (different only by mass), dubbed the 3 fermion families, with no apparent explanation to the origin of these additional families [2].

It is for these reasons that various GUT's have been suggested over the last 40 years to replace the standard model, with varying degrees of success at reproducing and correctly extending the standard model [2]. As a motivating reason for this project, recent data from CERN has suggested that lepton universality might be violated in experiments [9]. In brief, lepton universality states that the three generations of leptons should have identical interactions with the other particles in the standard model, and thus no process should favor the production of one lepton over another [10]. More details on lepton universality will be given later on in this chapter. If the data suggesting violation of lepton universality is correct, this would be exciting news, since the standard model cannot account for this violation. This would then suggest that the standard model gauge group isn't the complete theory, and that another GUT might be the source of this violation. As it turns out, the Pati-Salam model might offer an explanation for the origin of this violation [11]. Let us briefly note that the Pati-Salam gauge group by itself is a 'partial unification model', but since it may be embedded in $SO(10)$ it can be viewed as a GUT. From now on we shall ignore this difference.

2.2 Suggested GUT Models

In 1974, Georgi and Glashow came up with the first GUT extension of the standard model [12] based on the gauge group $SU(5)$. Shortly after in the same year, Pati and Salam came up with a model based on the gauge group $SU(4) \times SU(2)_L \times SU(2)_R$ [13]. Both of these models unify quarks and leptons into a single irreducible representation, though this occurs differently in both models. In $SU(5)$, combining leptons and quarks in such a way leads to interactions which may violate conservation of baryon number- this means that there exist processes which can cause proton decay. Earlier $SU(5)$ models have therefore been rejected since they predict proton decay rates that have been found in experiments not to be true. Extensions of $SU(5)$ that avoid that issue of proton decay have since been suggested (e.g. flipped $SU(5)$, given by $SU(5) \times U(1)$). Most Pati-Salam models, on the other hand, although also unifying quarks and lepton, appear to be free of the issue of proton decay since the gauge sector leads to conservation of both baryon and lepton numbers.
In 1975, Georgi [14] and (independently) Fritzsch and Minkowski [15] suggested a GUT model based on the gauge group $SO(10)$ which includes both $SU(5)$ and $SU(4) \times SU(2)_L \times SU(2)_R$ as subgroups, and in fact there have been models suggesting the symmetry breaking pattern through one or the other as intermediate steps, as well as directly into the standard model gauge group. Another model which has been suggested as a GUT is based on the exceptional Lie algebra E_6 [16]. Much more information regarding the specific predictions of each model, as well as the specific manifestations of the standard model subgroup is given in Ref. [17]. Our main interest will be the Pati-Salam model, since recent experimental data suggests that lepton universality might be violated, and Pati-Salam is able to explain this. For this reason, let us discuss the origin of lepton universality in more detail.

2.3 Lepton Universality in the Standard Model

To be able to discuss lepton universality in the standard model, we first have to understand the fermion sector a little better. However, our purpose here is not to extensively discuss the fermion sector of the standard model- we will do this for Pati-Salam. We will also not be very careful with definitions at this stage- this will be done in detail for the real model we are considering in this paper. Therefore, some prior knowledge of standard notation and symbols used to describe the standard model are assumed in this chapter.

Our reason for introducing the fermion sector in this context is not as much to give an idea of how quarks and leptons appear in the standard model Lagrangian, but rather to discuss why leptons appear to be 'universal' whilst quarks do not. This will be very important later on, since recent experimental data from CERN suggests lepton universality might be violated. Having the prerequisite background to explain why lepton universality arises in the first place in the standard model, will help us explain how it might be violated in a model like Pati-Salam.

To that end, let us consider some of the terms that were suppressed as $\mathcal{L}^f_{kinetic}$ and $\mathcal{L}^f_{Yukawa}$ in Lagrangian (1) from before, that capture the essence of what we motivated above.

In the standard model, left handed leptons and quarks appear in the fundamental representation of $SU(2)$, whereas right handed leptons and quarks transform trivially under $SU(2)$. There are three 'copies' of each pair (singlet) of left (right) handed leptons and quarks. Let us denote

$$l^1_L = \begin{bmatrix} \nu_{eL} \\ e_L \end{bmatrix}, \tag{6}$$

$$l^2_L = \begin{bmatrix} \nu_{\mu L} \\ \mu_L \end{bmatrix}, \tag{7}$$

and

$$l^3_L = \begin{bmatrix} \nu_{\tau L} \\ \tau_L \end{bmatrix}. \tag{8}$$

Similarly, the three left handed quark families as

$$q_L^1 = \begin{bmatrix} u_L \\ d_L \end{bmatrix}, \tag{9}$$

$$q_L^2 = \begin{bmatrix} c_L \\ s_L \end{bmatrix}, \tag{10}$$

and finally

$$q_L^3 = \begin{bmatrix} t_L \\ b_L \end{bmatrix}. \tag{11}$$

The right handed copy of all of (the components of) the fields above appear as $SU(2)$ singlets, except for right handed neutrinos, which do not appear in the standard model. Given the above, quarks and leptons seem to be very similar. So why would leptons be 'universal' while quarks are not? It turns out that it is to do with the Yukawa terms responsible for the mass of the fermions (through the Higgs mechanism). Specifically, the difference between a quark doublet versus a lepton doublet is that the neutrino counterpart to the charged lepton in the standard model is **massless**. For example, via the Higgs mechanism, the Yukawa term for l_L^1 evaluated on the Higgs vev is given by

$$\mathcal{L}_{Yukawa} = ... + Y(\bar{l}_L^1 \phi_0 l_R^1 + \bar{l}_R^1 \phi_0^\dagger l_L^1). \tag{12}$$

Substituting in our definitions for l^1 above, and our Higgs vev from before, we find

$$\mathcal{L}_{Yukawa} = ... + \frac{Yv}{\sqrt{2}}(\bar{e}_L e_R + \bar{e}_R e_L). \tag{13}$$

Note that we therefore do not have a mass term for the corresponding neutrino in this l_L^1 doublet- only the "electron-like" term obtains mass.

A similar treatment to the electron mass above is given for the other quarks and leptons in basic treatments of the standard model, and we therefore won't be carefully deriving it here. Instead, let us state the end result for the left handed up quark and down quark (where we acknowledge that similar terms arise for up-like and down-like quarks):

$$\frac{Y'v}{\sqrt{2}}(\bar{d}_L d_R + \bar{d}_R d_L) \tag{14}$$

and

$$\frac{Y''v}{\sqrt{2}}(\bar{u}_L u_R + \bar{u}_R u_L). \tag{15}$$

The reason for writing the above, was so as to draw the distinction between the lepton doublet and the quark doublet. As mentioned, the neutrino counterparts do not obtain mass in this way. However, both "up-like" and "down-like" quarks obtain mass in precisely **the same way** as the electron did above. Although it is not yet clear why, it is this reason that gives rise to quark non universality. Also, it is the fact that the neutrino masses are zero via the mechanism given above, that give rise to lepton universality.

To see why, let us note that the most general Yukawa terms are given by a 3×3 complex Yukawa matrix, and can in fact mix generations of quarks and fermions. Let us denote "up-like", "down-like" and "electron-like" particles as $\tilde{u}$, $\tilde{d}$ and $\tilde{e}$ (i.e. by specifying a family index superscript, we denote which particle we actual refer to, so for example $\tilde{e}^3 = \tau$). Then, writing the most general second order Yukawa term evaluated on our Higgs vev, we obtain mass terms of the form

$$\begin{aligned} \mathcal{L}_{Yukawa} = ... &+ \frac{v}{\sqrt{2}}(Y_l^{fg}\bar{\tilde{e}}_L^f \tilde{e}_R^g + (Y_l^{fg})^* \bar{\tilde{e}}_R^g \tilde{e}_L^f) \\ &+ \frac{v}{\sqrt{2}}(Y_{\tilde{d}}^{fg}\bar{\tilde{d}}_L^f \tilde{d}_R^g + (Y_{\tilde{d}}^{fg})^* \bar{\tilde{d}}_R^g \tilde{d}_L^f) \\ &+ \frac{v}{\sqrt{2}}(Y_{\tilde{u}}^{fg}\bar{\tilde{u}}_L^f \tilde{u}_R^g + (Y_{\tilde{d}}^{fg})^* \bar{\tilde{u}}_R^g \tilde{u}_L^f), \end{aligned} \tag{16}$$

where the sum over family indices f, g is implied. The subscripts on the matrices Y simply denote which type of particle they are related to.

Looking at the above, it seems strange that we obtain mixing Yukawa mass terms between different generations of particles, given that we know they all have different masses. In fact, this issue may be resolved by redefining the quark and lepton fields above in such a way that Y_l, $Y_{\tilde{d}}$ and $Y_{\tilde{u}}$ all become 3×3 diagonal, real matrices. Recalling that our "particle-like" fields are in fact three component vectors for the three families, one finds that each of the fields must be multiplied by a unitary matrix to diagonalise the Yukawa matrices, so that our new fields are given by

$$V_{lL} \cdot \tilde{e}_L, V_{lR} \cdot \tilde{e}_R, V_{\tilde{d}R} \cdot \tilde{d}_R, V_{\tilde{d}L} \cdot \tilde{d}_L, V_{\tilde{u}R} \cdot \tilde{u}_R, V_{\tilde{u}L} \cdot \tilde{u}_L. \tag{17}$$

We also redefine the left handed neutrinos, using the same unitary matrix V_{lL} we used to redefine their "electron-like" counterparts, as we shall see this will be useful very soon.
We are at the verge of being to explain (non) universality in (quarks) leptons. The only missing terms are found in the kinetic part, $\mathcal{L}_{kinetic}$. Within those, we find the coupling between gauge fields and fermions, for example

$$\mathcal{L}_{kinetic} = ... + ig_2 \bar{l}^f_L \gamma^\mu A_{2\mu} l^f_L + ig_2 \bar{q}^f_L \gamma^\mu A_{2\mu} q^f_L, \tag{18}$$

again summing over f, and recalling our definitions for l^f and q^f above as well that $A_{2\mu}$ is the gauge field associated to $SU(2)$. Consider the W^+, W^- gauge fields associated to A^μ_2 above, which occupy the off diagonal terms of A^μ_2. Then, multiplying out the expression above we find (again summing over f)

$$\mathcal{L}_{kinetic} = ... + ig_2 \bar{\tilde{e}}^f_L \gamma^\mu W^+_\mu \nu^f_L + ig_2 \bar{\tilde{u}}^f_L \gamma^\mu W^+_\mu \tilde{d}^f_L + h.c.. \tag{19}$$

Upon the rotations described above, we see that because $\tilde{e}^f$ and ν^f are rotated via the same unitary matrix V_{lL} , the first term of $\mathcal{L}_{kinetic}$ in fact remains invariant after this redefinition (as it involves a neutrino and anti electron, and V_{lL} is unitary). However, in the second term, different unitary matrices are used for $\tilde{u}^f$ and $\tilde{d}^f$, namely $V_{\tilde{u}L}$ and $V_{\tilde{d}L}$. Therefore, the second term involving the quarks gives us interactions between "up-like" and "down-like" quarks of **different generations of families**, unlike the case for leptons.
The analysis above shows two things- firstly, that quarks have interactions which may change flavour. These effects are described by the Cabbibo-Koboyashi-Maskawa matrix (CKM) [18], given by $V^\dagger_{\tilde{u}L} V_{\tilde{d}L}$ Secondly, even considering same family interaction terms, the redefinition of our fields means that the gauge couplings to the W gauge field differ for quarks of different families. For example, $\bar{u}_L$ and d_L do not couple to W the same way $\bar{c}_L$ and s_L do.

Unlike the above, leptons have neither flavour violation, nor different gauge couplings to the W gauge boson. Therefore, we say that they are universal, while quarks are not.
In the final chapter of this thesis, we shall say in more detail what are the implications of such results. In particular, we will discuss experiments that have tested out lepton universality, describing what they expected to find, and what discrepancies seem to be arising recently. We will also show that the Pati-Salam model has the capability of explaining this violation of lepton universality.
Therefore, without further ado, let us introduce the Pati-Salam model.

3 The Pati-Salam Model

3.1 The Complete Lagrangian

There are two main Pati-Salam models which we will consider. However, both models start off the same way: Consider the Pati-Salam gauge group $SU(4) \times SU(2)_L \times SU(2)_R$. The first thing we wish to discuss is the symmetry breaking pattern of the above gauge group into the standard model. In fact, there are two stages to this:

$$SU(4) \times SU(2)_L \times SU(2)_R \mapsto SU(3) \times SU(2)_L \times U(1)_Y, \tag{20}$$

due to a Higgs field ϕ, followed by

$$SU(3) \times SU(2)_L \times U(1)_Y \mapsto SU(3) \times U(1)_Q, \tag{21}$$

due to a Higgs field ψ or χ for models 1 or 2 respectively. Let us mention that Model 1 and model 2 differ in the choice of Higgs representation for the second symmetry breaking pattern,

$$SU(3) \times SU(2)_L \times U(1)_Y \mapsto SU(3) \times U(1)_Q, \tag{22}$$

and we shall discuss their differences later on. Therefore, all of the analysis relating to the first symmetry breaking pattern is in fact true for both models 1 and 2.

Since our main purpose currently is to highlight the fundamental properties of a GUT, in the following sections and chapters we will be adding complexity only as it is needed to understand the fundamental structure of our theory. However, to make it clear where we are heading, let us first state the result we are trying to build up to. Let us therefore denote ϕ as our first Higgs field responsible for the Pati-Salam symmetry breaking pattern (the same in both models), and ψ or χ our second Higgs fields responsible for electroweak symmetry breaking in model one or two. Also take ψ_L and ψ_R as the fermion fields, and let $G_{\mu\nu}$, $F_{L\mu\nu}$ and $F_{R\mu\nu}$ correspond to the field strength tensors of $SU(4)$, $SU(2)_L$ and $SU(2)_R$ respectively. Then, the full Lagrangian for the first Pati-Salam model we will be looking at is given by

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}\,\mathrm{Tr}\big[G_{\mu\nu}G^{\mu\nu} + F_{L\mu\nu}F_L^{\mu\nu} + F_{R\mu\nu}F_R^{\mu\nu}\big] + D_\mu\phi_{i\alpha}D^\mu\phi^{i\alpha} + D_\mu\psi_{i\alpha}D^\mu\psi^{i\alpha} \\
&+ i\bar{\psi}_L\gamma^\mu D_\mu\psi_L + i\bar{\psi}_R\gamma^\mu D_\mu\psi_R + \mathcal{L}_{Yukawa} - [-2\alpha_R^2\phi_{i\alpha}\phi^{i\alpha} + \beta_{R1}(\phi_{i\alpha}\phi^{i\alpha})^2 \\
&+ \beta_{R2}\phi_{i\alpha}\phi^{j\alpha}\phi_{j\beta}\phi^{i\beta} - 2\alpha_L^2\psi_{i\alpha}\psi^{i\alpha} + \beta_{L1}(\psi_{i\alpha}\psi^{i\alpha})^2 + \beta_{L2}\psi_{i\alpha}\psi^{j\alpha}\psi_{j\beta}\psi^{i\beta} \\
&+ \alpha_{LR1}\psi_{i\alpha}\psi^{i\alpha}\phi_{j\beta}\phi^{j\beta} + \alpha_{LR2}\psi_{i\alpha}\phi^{j\alpha}\psi^{i\beta}\phi_{j\beta} + \alpha_{LR3}(\psi_{i\alpha}\phi^{j\alpha}\psi^i_\beta\phi^\beta_j + h.c.)],
\end{aligned}
\tag{23}
$$

where we delay the explicit treatment of the fermion masses and the Yukawa sector to our discussion of model 2, to a later chapter. Let us comment that the potential terms above arise by considering the most general invariant potential given the Higgs fields representations used above (which we will define later on), as is demonstrated in Ref. [19].

Precise definitions of the fields and operators in the Lagrangian (23) will be given in due course. In the coming sections, we will break apart the above Lagrangian and consider its different sectors. For completeness purposes, let us also present the Lagrangian for model 2, although our scalar potential analysis will be done in detail only for the Lagrangian (23) above. In any case, the Lagrangian for model 2 is given by

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\,\mathrm{Tr}\big[G_{\mu\nu}G^{\mu\nu} + F_{L\mu\nu}F_L^{\mu\nu} + F_{R\mu\nu}F_R^{\mu\nu}\big] + D_\mu\chi_i^I D^\mu\chi_I^i + D_\mu\phi_{i\alpha}D^\mu\phi^{i\alpha} \\ &+ i\bar{\psi}_L\gamma^\mu D_\mu\psi_L + i\bar{\psi}_R\gamma^\mu D_\mu\psi_R + \lambda_1(\psi_{LI\alpha}\chi_i^I\psi_R^{i\alpha} + \psi_L^{I\alpha}\chi_I^i\psi_{Ri\alpha}) \\ &+ \lambda_2(\psi_{LI\alpha}\tilde{\chi}_i^I\psi_R^{i\alpha} + \psi_L^{I\alpha}\tilde{\chi}_I^i\psi_{Ri\alpha}) - [-2\alpha_R^2\phi_{i\alpha}\phi^{i\alpha} + \beta_{R1}(\phi_{i\alpha}\phi^{i\alpha})^2 \\ &+ \beta_{R2}\phi_{i\alpha}\phi^{j\alpha}\phi_{j\beta}\phi^{i\beta} + V(\phi,\chi)],\end{aligned} \tag{24}$$

where we again delegate the job of carefully defining the objects above to the sections where we discuss model 2 in more detail.

Let us just clarify the square bracketed term above involving $V(\phi,\chi)$ - we chose to write the scalar potential of model 2 in a way that emphasises that the non mixing terms for ϕ are indeed the same for both models. In other words, $V(\phi,\chi)$ above contains only the **mixing** terms for ϕ and χ, whereas the terms involving ϕ are the same for both models, and are responsible for the first symmetry breaking pattern. As such, let us demonstrate this symmetry breaking in more detail.

3.2 The First Symmetry Breaking Pattern

We wish to begin by demonstrating the symmetry breaking pattern $SU(4)\times SU(2)_L\times SU(2)_R \mapsto SU(3)\times SU(2)_L\times U(1)_Y$. As we shall see, this need only involve the Higgs field ϕ and the gauge fields (and thus applies to both models 1 and 2). Then, let us suppose our theory currently consists only of those fields. Looking at the Lagrangian (23) or equivalently (24), we therefore choose to ignore the fermion fields, the Yukawa terms, the Higgs field ψ (χ) **and** the terms that couple the Higgs field ϕ to ψ (χ). Therefore, this leaves us with a Lagrangian only consisting of

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\,\mathrm{Tr}\big[G_{\mu\nu}G^{\mu\nu} + F_{L\mu\nu}F_L^{\mu\nu} + F_{R\mu\nu}F_R^{\mu\nu}\big] + D_\mu\phi_{i\alpha}D^\mu\phi^{i\alpha} \\ &- 2\alpha_R^2\phi_{i\alpha}\phi^{i\alpha} + \beta_{R1}(\phi_{i\alpha}\phi^{i\alpha})^2 + \beta_{R2}\phi_{i\alpha}\phi^{j\alpha}\phi_{j\beta}\phi^{i\beta}.\end{aligned} \tag{25}$$

Now, let us define the various objects more carefully. Firstly, let us let us view ϕ as an 8-tuplet, $\phi^{i\alpha}$, with $i=1,2$ and $\alpha=1,2,3,4$ indices for $SU(2)_R$ and $SU(4)$ respectively. The first line above concerns the gauge boson sector, and the gauge boson-Higgs boson coupling. The second line contains the potential terms associated to the first Higgs field only.

One may write ϕ as

$$\phi = \begin{bmatrix} \phi_{11} \\ \phi_{12} \\ \phi_{13} \\ \phi_{14} \\ \phi_{21} \\ \phi_{22} \\ \phi_{23} \\ \phi_{24} \end{bmatrix}. \tag{26}$$

We suppose that ϕ transforms in the fundamental representation of $SU(4)$ and $SU(2)_R$ but trivially under $SU(2)_L$ (i.e. it is in the representation (4,1,2)). By this we mean that

$$\phi^{i\alpha} \mapsto M_{ij} N_{\alpha\beta} \phi^{j\beta}, \tag{27}$$

where M, N are the elements of $SU(2)_R$ and $SU(4)$ respectively. Similarly, let us take $\phi_{i\alpha}$ to be the **transpose conjugate** of $\phi^{i\alpha}$. We will adopt this notation of a field and its transpose conjugate **for the rest of this paper**. Note, we choose to view ϕ as a vector rather than as a 2×4 matrix since this form will be easier to work with when constructing the symmetry breaking matrices later on.

Secondly, given gauge fields G^μ and A^μ_R associated to $SU(4)$ and $SU(2)_R$ respectively, we define

$$\hat{G}^a_\mu = \begin{cases} G^a_\mu & \text{for } a = 1, 2, ..., 15 \\ A^{a-15}_{R\mu} & \text{for } a = 16, 17, 18. \end{cases} \tag{28}$$

The precise form of the corresponding Lie algebra generators T^a will be discussed in more detail next subsection. Our covariant derivative is defined as

$$D^\mu = (\partial^\mu + ig_4 \hat{G}^\mu), \tag{29}$$

Moreover, the field strength tensors are defined as

$$G^{\mu\nu} = \partial^{\mu} G^{\nu} - \partial^{\nu} G^{\mu} + ig_4 [G^{\mu}, G^{\nu}], \tag{30}$$

$$F_L^{\mu\nu} = \partial^{\mu} A_L^{\nu} - \partial^{\nu} A_L^{\mu} + ig_L \left[A_L^{\mu}, A_L^{\nu}\right], \tag{31}$$

and

$$F_R^{\mu\nu} = \partial^{\mu} A_R^{\nu} - \partial^{\nu} A_R^{\mu} + ig_R \left[A_R^{\mu}, A_R^{\nu}\right]. \tag{32}$$

Having defined the objects used in our current Lagrangian (25), let us discuss the particle spectrum before and after symmetry breaking. Starting with the former, let us suppose that our vev ϕ_0 is zero. Then, considering only the second order terms in our fields we find that

$$\begin{aligned}\mathcal{L} \approx -\frac{1}{4}[(\partial^{\mu} G^{\nu k} - \partial^{\nu} G^{\mu k})(\partial_{\mu} G_{\nu}^{k} - \partial_{\nu} G_{\mu}^{k}) + (\partial^{\mu} A_L^{\nu a} - \partial^{\nu} A_L^{\mu a})(\partial_{\mu} A_{L\nu}^{a} - \partial_{\nu} A_{L\mu}^{a}) \\ + (\partial^{\mu} A_R^{\nu a} - \partial^{\nu} A_R^{\mu a})(\partial_{\mu} A_{R\nu}^{a} - \partial_{\nu} A_{R\mu}^{a})] + \partial_{\mu} \phi_{i\alpha} \partial^{\mu} \phi^{i\alpha} - 2\alpha_R^2 \phi_{i\alpha} \phi^{i\alpha}.\end{aligned} \tag{33}$$

From this, we may deduce that before spontaneous symmetry breaking, our theory describes 21 massless gauge bosons, fifteen for the components k associated to the gauge field G^{μ}, and the remaining six associated to the two fields A_L^{μ} and A_R^{μ}. There are two degrees of freedom associated to each massless gauge boson above, altogether giving 42 degrees of freedom. Moreover, since our Higgs field contains 8 free (complex) parameters, we expect 16 massive Higgs particles. Overall, we have 58 degrees of freedom.

A natural next step would now be to consider the above Lagrangian for a non zero vev. Recall that our potential for the simplified Lagrangians (25), (24) is given by

$$V(\phi) = -2\alpha_R^2 \phi_{i\alpha} \phi^{i\alpha} + \beta_{R1} (\phi_{i\alpha} \phi^{i\alpha})^2 + \beta_{R2} \phi_{i\alpha} \phi^{j\alpha} \phi_{j\beta} \phi^{i\beta}. \tag{34}$$

Detailed analysis of the potential above is given in Ref. [19]. The author demonstrates that the absolute minimum for our potential is given by a vev ϕ_0 of the form

$$\phi_{0i\alpha} = \begin{pmatrix} 0 & 0 & 0 & \frac{v_R}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{35}$$

Let us see what conditions are imposed on this vev by requiring that it is a minimum of the potential. We find that

$$\frac{\partial V(\phi)}{\partial \phi_{k\gamma}} = -2\alpha_R^2 \phi^{k\gamma} + 2\beta_{R1}\phi^{k\gamma}\phi_{i\alpha}\phi^{i\alpha} + \beta_{R2}(\phi^{j\gamma}\phi_{j\beta}\phi^{k\beta} + \phi_{i\alpha}\phi^{k\alpha}\phi^{i\gamma}). \tag{36}$$

Evaluating this at $\phi = \phi_0$, and equating to zero, we note that we only have one non trivial component. Indeed, for $k = 1$, $\gamma = 4$ we obtain the vev condition

$$\phi_0^{i\alpha}\phi_{0i\alpha} = \frac{{v_R}^2}{2} = \frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}}, \tag{37}$$

used in Ref. [19].

We are now ready to demonstrate the symmetry breaking pattern. One expects that the initial symmetry breaking, from the unbroken gauge group $SU(4) \times SU(2)_L \times SU(2)_R$ to the unbroken standard model gauge group, $SU(3) \times SU(2) \times U(1)_Y$, would result in 9 massive gauge bosons (corresponding to 9 broken generators). This means that those new degrees of freedom (from Goldstone's Theorem alongside the Higgs Mechanism) had to come from the 16 massive Higgs particles of the unbroken theory.

Thus, upon expansion of our potential (34) around our non zero vev (35), we should find that the particle spectrum of our Higgs field has changed appropriately, to reflect the fact that only 7 massive Higgs particles should now be present. Indeed, as the explicit calculation in equation (114) of Appendix A shows in detail, we find that

$$V(\phi) = V(\varphi + \phi_0) \approx 2\alpha_R^2\left(\left(\frac{\beta_{R1}}{\beta_{R1} + \beta_{R2}} - 1\right)(|\varphi_{21}|^2 + |\varphi_{22}|^2 + |\varphi_{23}|^2) + 2Re(\varphi_{14})^2\right), \tag{38}$$

which shows us that only 7 massive Higgs bosons (7 degrees of freedom) remain. Using Goldstone's theorem, we conclude that there must now be 9 massive (27 degrees of freedom) and 12 massless (24 degrees of freedom) gauge bosons. This is none other than the spontaneous symmetry breaking of Pati-Salam into the unbroken standard model gauge group, but we have not explicitly shown that yet. The reason being that although the number of unbroken generators implied by the potential above agrees the number of unbroken generators in $SU(4) \times SU(2)_L \times SU(2)_R \mapsto SU(3) \times SU(2) \times U(1)_Y$, we have not shown explicitly that the remaining generators give rise to the residual symmetry group $SU(3) \times SU(2) \times U(1)_Y$, rather than some other subgroup. We therefore introduce a different method to demonstrate symmetry breaking patterns, which can explicitly produce the residual symmetry group.

3.3 The First Symmetry Breaking Matrix

Recall that a broken generator T of a Lie algebra associated to a gauge group is one such that $T\phi_0 \neq 0$. We are interested in finding the largest set of unbroken generators for a given gauge group, as those will form the residual symmetry group and contain the information regarding our gauge boson particle spectrum. To that end, let us build the following matrix: given the gauge group $SU(4) \times SU(2)_L \times SU(2)_R$, we construct the *symmetry breaking matrix*

$$S^{ab} = \phi_0^\dagger \Big\{ T^a, T^b \Big\} \phi_0. \tag{39}$$

We will then use this in the following way to find the largest set of unbroken generators: Suppose that there exists some linear combination of generators $\lambda^a T^a = \bar{T}^a$ such that $\bar{T}^a \phi_0 = 0$. Then, it is a very simple calculation to verify that the vector λ^a is an eigenvector of S^{ab} with eigenvalue 0. Thus, an easy way to find the largest set of unbroken generators is simply to construct the matrix S^{ab} and diagonlise it to find the number of zero eigenvalues- each such eigenvalue corresponds to an unbroken generator and conversely every non zero eigenvalue is a broken generator. We can then read off the symmetry breaking pattern and find our residual symmetry group simply by noting which are the broken and unbroken generators. Not only that, but the gauge bosons particle spectrum is almost equal to the eigenvalues of our matrix S^{ab}, more specifically the mass of each gauge boson equals the square root of the eigenvalue of S^{ab} multiplied by the coupling constant (g_4 in our case).

To construct this matrix, we first have to correctly embed the generators of $SU(4)$ and $SU(2)_R$ into an 8×8 matrix representation, which we will do shortly. We note that using this method to derive the particle spectrum, whilst one still needs to 'guess' the form of our non zero vev ϕ_0, does not require us to have a pre-established potential term. This means that one may try out gauge groups as potential candidates for a Grand Unified Theory without having to go through the process of finding a potential term in the Lagrangian which correctly reproduces a given vev. In other words, starting only with a gauge group and a possible vev ϕ_0, one may already discover whether their symmetry breaking pattern has the possibility of reproducing the standard model, which allows us to dispose of hopeless candidates without having to explicitly write down any potentials.

Let us construct the symmetry breaking matrix S^{ab} described above. Recall that $S^{ab} = \phi_0^\dagger \{T^a, T^b\}\phi_0$ and that ϕ_0 is an 8-tuplet. Then, let us embded $SU(4)$ and $SU(2)_R$ into 8 by 8 matrices. We proceed in the following manner: Suppose that M_{ij} in $SU(2)_R$ equals $\exp\left\{i\theta^a t^a_{ij}\right\}$ and $N_{\alpha\beta}$ in $SU(4)$ equals $\exp\left\{i\eta^A T'^A_{\alpha\beta}\right\}$ (we use the $SU(4)$ generators given in Appendix A of [19], which we will explicitly write out soon, and the Pauli matrices for $SU(2)$). Then, since ϕ transforms in the fundamental representation of both, we may observe that to first order,

$$\phi^{i\alpha} \mapsto \phi^{i\alpha} + i\theta^a t^a_{ij}\delta_{\alpha\beta}\phi^{i\beta} + i\delta_{ij}\eta^A T'^A_{\alpha\beta}\phi^{j\beta}, \tag{40}$$

which suggests that our generators could be defined in the following way:

For

$$\text{a=1,2,3...,15, } T^a = \begin{pmatrix} \delta_{11}T^a & \delta_{12}T^a \\ \delta_{21}T^a & \delta_{22}T^a \end{pmatrix} = \begin{pmatrix} T^a & 0 \\ 0 & T^a \end{pmatrix} \tag{41}$$

and for

$$\text{a=16,17,18, } T^a = \frac{g_R}{g_4}\begin{pmatrix} t^{a-15}_{11}\mathbb{1}_{4\times4} & t^{a-15}_{12}\mathbb{1}_{4\times4} \\ t^{a-15}_{21}\mathbb{1}_{4\times4} & t^{a-15}_{22}\mathbb{1}_{4\times4} \end{pmatrix} \tag{42}$$

.

Writing these generators explicitly would take up a lot of space since there are 18 of them and each matrix is 8 by 8, however for the sake of clarity (and due to their simple form) I will write down T^{15}

and T^{18}.

Starting with

$$T'^{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{43}$$

and

$$t^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{44}$$

we end up with

$$T^{15} = \mathrm{Diag}(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{3}{2}}) \tag{45}$$

and

$$T^{18} = \frac{g_R}{g_4} \mathrm{Diag}(1, 1, 1, 1, -1, -1, -1, -1). \tag{46}$$

Now we have our generators correctly embedded, we may easily compute S^{ab}. Let us start by only stating the diagonlised matrix, given by (letting $\frac{v_R}{\sqrt{2}} = v$ for simplicity)

$$\mathrm{Diag}(0, 0, 0, 0, 0, 0, 0, 0, 0, v^2, v^2, v^2, v^2, v^2, v^2, \frac{g_R^2 v^2}{g_4^2}, \frac{g_R^2 v^2}{g_4^2}, \frac{v^2 \left(3g_4^2 + 2g_R^2\right)}{2g_4^2}), \tag{47}$$

since it is this object that tells us our particle spectrum, and the masses of our gauge bosons. Note that we have now essentially showed the same result as our potential (38) shows- namely that we have 9 broken generators (since there are 9 non zero eigenvalues in the diagonlised matrix above). However, we had hoped to show more than this, in particular we wanted to explicitly demonstrate that the residual symmetry group is $SU(3) \times SU(2) \times U(1)_Y$. To do this, we need to look more carefully at the generators of $SU(4)$ used in the construction above and the full symmetry breaking matrix. In particular, the $SU(4)$ generators are given by

$$T^{'1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T^{'2} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T^{'3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T^{'4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$T^{'5} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T^{'6} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T^{'7} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T^{'8} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$T^{'9} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, T^{'10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, T^{'11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, T^{'12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$
$$T^{'13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, T^{'14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, T^{'15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \tag{48}$$

The reason that we go through the effort of explicitly writing these is that one may notice that the first 8 generators are precisely the Gell-Mann matrices, i.e. the generators of $SU(3)$, embedded in a

4×4 matrix representation. Therefore, upon construction of the matrix S^{ab},

$$S = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3v^2}{2} & 0 & 0 & -\frac{\sqrt{\frac{3}{2}}\mathrm{g}_R v^2}{\mathrm{g}_4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{g}_R^2 v^2}{\mathrm{g}_4^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{g}_R^2 v^2}{\mathrm{g}_4^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{\frac{3}{2}}\mathrm{g}_R v^2}{\mathrm{g}_4} & 0 & 0 & \frac{\mathrm{g}_R^2 v^2}{\mathrm{g}_4^2}
\end{pmatrix}, \tag{49}$$

we see that the 9 unbroken $SU(4)$ generators are the 8 Gell-Mann matrices, and a linear combination of the diagonal elements of $SU(4)$ and $SU(2)_R$ (T^{15} and T^{18}, which we explicitly wrote above in (45), (46)). Therefore, as claimed, the residual symmetry group generated by these is precisely $SU(3) \times SU(2) \times U(1)_Y$.

We can now conclusively read off that, due to our Higgs field ϕ having a non-zero vev, the Pati-Salam gauge group spontaneously broke to the unbroken standard model gauge group, and that in addition to the 12 massless gauge bosons (9 from the symmetry breaking and 3 from $SU(2)_L$), we expect to find, at some higher energies, 9 additional massive gauge bosons (for our current theory- as we shall see later on, electroweak symmetry breaking gives a further 3 broken generators). Moreover, the masses of our gauge bosons are given by

$$(0,0,0,0,0,0,0,0,0,g_4v,g_4v,g_4v,g_4v,g_4v,g_4v,g_Rv,g_Rv,\frac{v}{\sqrt{2}}\sqrt{3g_4^2+2g_R^2}). \tag{50}$$

Finally, as shown in the previous section, we also expect to have 7 massive Higgs particles. To conclude this chapter and end our discussion regarding the first symmetry breaking pattern of Pati-Salam, let us make some comparisons between the Pati-Salam generators and the ones of the unbroken standard model. We note from (49) that the $U(1)_Y$ subgroup, belonging to the unbroken standard model gauge group, is generated by a linear combination of the diagonal element T^{18} of $SU(2)_R$ and the diagonal element T^{15} of $SU(4)$. Let us relate these generators to hypercharge Y directly: Given a $U(1)$ gauge group, the hypercharge Y appears in the covariant derivative as follows: $D_\mu = \partial_\mu + ... + ig_y Y B_\mu$. On the other hand, diagonalising (49) we find that our new $U(1)_Y$ generator is given by $\bar{T}^{15} = \frac{g_R}{g_4}(\sqrt{\frac{2}{3}}T^{15}+\frac{g_4}{g_R}T^{18})$. Then, writing out the part of the covariant derivative belonging to this generator (in some new field $\bar{G}^{15}$), we find $D_\mu = \partial_\mu + ... + ig_4\bar{T}^{15}\bar{G}^{15} = \partial_\mu + ... + ig_4\frac{g_R}{g_4}(\sqrt{\frac{2}{3}}T^{15} + \frac{g_4}{g_R}T^{18})\bar{G}^{15}$. Comparing the two expressions, we conclude that $g_R = g_Y$ and that $Y = \sqrt{\frac{2}{3}}T^{15} + \frac{g_4}{g_R}T^{18}$ is the $U(1)_Y$ generator (in agreement with the literature [19]). Having concluded the demonstration of the first symmetry breaking pattern, let us finish this chapter by discussing the relation between the vev structure and the symmetry breaking patterns.

3.4 Vev Dependence of Symmetry Breaking Matrices

We have demonstrated above the symmetry breaking pattern of Pati-Salam into the unbroken standard model. Doing so, we have seen that by choosing an appropriate form for the vev (35), correctly embedding our generators and constructing the symmetry breaking matrix, one obtains a rather neat, nearly diagonal matrix for Pati-Salam, which gives residual symmetry groups precisely matching the standard model gauge group. However, this is by no means the 'typical' case. In other words, by choosing different forms for our vevs, we may end up with a completely different residual symmetry group in each case, with the symmetry breaking matrix being anything but 'neat', and certainly incapable of reproducing the standard model gauge group. Suffices to say, if one chooses a random form for the vev ϕ_0 for example, the number of broken generators does not match the ones we obtained earlier in this paper, and consequently cannot reproduce the correct symmetry braking pattern. Not only that, but it would be an understatement to say that the symmetry breaking matrix and eigenvalues are extremely complicated. Possibly the neatest symmetry breaking pattern that deviates from our required result (looking at the initial symmetry breaking pattern), is given by the vev

$$\phi_0 = \begin{pmatrix} 0 & 0 & 0 & u & u & 0 & 0 & 0 \end{pmatrix}. \tag{51}$$

From this, we obtain the diagonal matrix

$$\mathrm{Diag}(0,0,0,0,0,0,u^2,u^2,u^2,u^2,u^2,u^2,u^2,u^2,u^2,\frac{2u^2\left(g_4^2+g_R^2\right)}{g_4^2},\frac{2u^2\left(g_4^2+g_R^2\right)}{g_4^2},\frac{2u^2\left(g_4^2+g_R^2\right)}{g_4^2}), \tag{52}$$

and although we do not recommend letting u $\mapsto$ v in the 5th component of ϕ_0, this will certainly demonstrate how horrific the system of eigenvalues can become.

Considering the symmetry breaking pattern above, and noting that it does not give the standard model as a residual symmetry group, one might then like to answer the question **what is the most general form of the vevs which reproduces the correct symmetry breaking pattern into the unbroken standard model gauge group?**

It turns out that choosing a vev of the form

$$\phi_0 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_1 & v_2 & v_3 & v_4 \end{pmatrix}, \tag{53}$$

one still obtains the symmetry breaking pattern required. Not only that, one even precisely the same structure of eigenvalues as for the true Pati-Salam vev. In particular, having had

$$\left(0,0,0,0,0,0,0,0,0,v^2,v^2,v^2,v^2,v^2,v^2,\frac{\mathrm{g}_R^2 v^2}{\mathrm{g}_4^2},\frac{\mathrm{g}_R^2 v^2}{\mathrm{g}_4^2},\frac{v^2\left(3\mathrm{g}_4^2+2\mathrm{g}_R^2\right)}{2\mathrm{g}_4^2}\right) \tag{54}$$

as the set of eigenvalues for our original vev, we now find

$$\begin{aligned}
&\mathrm{Diag}(0,0,0,0,0,0,0,0,0,2\left(v_1^2+v_2^2+v_3^2+v_4^2\right),2\left(v_1^2+v_2^2+v_3^2+v_4^2\right)\\
&,2\left(v_1^2+v_2^2+v_3^2+v_4^2\right),2\left(v_1^2+v_2^2+v_3^2+v_4^2\right),2\left(v_1^2+v_2^2+v_3^2+v_4^2\right),\\
&2\left(v_1^2+v_2^2+v_3^2+v_4^2\right),\frac{2g_R^2(v_1^2+v_2^2+v_3^2+v_4^2)}{g_4^2},\frac{2\mathrm{g}_R^2(v_1^2+v_2^2+v_3^2+\mathrm{v}_4^2)}{\mathrm{g}_4^2},\\
&\frac{\left(3\mathrm{g}_4^2+2g_R^2\right)\left(v_1^2+v_2^2+v_3^2+v_4^2\right)}{g_4^2}).
\end{aligned} \tag{55}$$

From the above, we see that both sets of eigenvalues are proportional to one another, and the difference only arises due to the norm of the vevs in each case.

This analysis of the vev structure was done in detail above for the symmetry breaking pattern of the first Higgs field into what should be the unbroken standard model gauge group. It was demonstrated in detail that even small changes to the vev above may produce extremely messy symmetry breaking patterns, unable to reproduce the standard model. It is interesting however to note that an expression as general as (53) above still reproduces the symmetry breaking pattern required, with effectively identical eigenvalues (and thus gauge boson masses). Finally, it is noted that the same analysis may be done for the other symmetry breaking patterns investigated in this paper, and similar results arise.

3.5 The Second Symmetry Breaking Pattern (Model 1)

Having successfully demonstrated the symmetry breaking pattern $SU(4)\times SU(2)_L\times SU(2)_R\mapsto SU(3)\times SU(2)_L\times U(1)_Y$ using the simplified Lagrangian (25), let us discuss the second spontaneous symmetry breaking which occurs in the Pati-Salam model 1, $SU(3)\times SU(2)_L\times U(1)_Y\mapsto SU(3)\times U(1)_Q$. To demonstrate this, we will need to use more terms in our complete Lagrangian (23). In particular, we now consider that Lagrangian, ignoring only the fermions and Yukawa terms. Also, the parameter α_{LR3} does not play a part in the vev condition, and so to simplify our calculations we shall ignore it. Therefore, our current Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\,\mathrm{Tr}\big[G_{\mu\nu}G^{\mu\nu} + F_{L\mu\nu}F_L^{\mu\nu} + F_{R\mu\nu}F_R^{\mu\nu}\big] + D_\mu\phi_{i\alpha}D^\mu\phi^{i\alpha} + D_\mu\psi_{i\alpha}D^\mu\psi^{i\alpha} \\ &- 2\alpha_R^2\phi_{i\alpha}\phi^{i\alpha} + \beta_{R1}(\phi_{i\alpha}\phi^{i\alpha})^2 + \beta_{R2}\phi_{i\alpha}\phi^{j\alpha}\phi_{j\beta}\phi^{i\beta} - 2\alpha_L^2\psi_{i\alpha}\psi^{i\alpha} + \beta_{L1}(\psi_{i\alpha}\psi^{i\alpha})^2 \\ &+ \beta_{L2}\psi_{i\alpha}\psi^{j\alpha}\psi_{j\beta}\psi^{i\beta} + \alpha_{LR1}\psi_{i\alpha}\psi^{i\alpha}\phi_{j\beta}\phi^{j\beta} + \alpha_{LR2}\psi_{i\alpha}\phi^{j\alpha}\psi^{i\beta}\phi_{j\beta}.\end{aligned} \tag{56}$$

The only new object above that needs to be defined is the second Higgs field ψ. Similarly to ϕ, let us let us view ψ as an 8-tuplet, $\psi^{i\alpha}$, with $i = 1,2$ and $\alpha = 1,2,3,4$ indices for $SU(2)_L$ and $SU(4)$ respectively.

One may therefore write ψ as

$$\psi = \begin{bmatrix}\psi_{11}\\ \psi_{12}\\ \psi_{13}\\ \psi_{14}\\ \psi_{21}\\ \psi_{22}\\ \psi_{23}\\ \psi_{24}\end{bmatrix}, \tag{57}$$

where ψ transforms in the fundamental representation of $SU(4)$ and $SU(2)_L$ but trivially under $SU(2)_L$. Again, $\psi_{i\alpha}$ is the **transpose conjugate** of $\psi^{i\alpha}$.

We would like to analyze this Lagrangian before and symmetry symmetry breaking, using the scalar potential and the symmetry breaking matrix. From the above, we see that our current potential is defined as

$$\begin{aligned}V(\phi,\psi) = &-2\alpha_R^2\phi_{i\alpha}\phi^{i\alpha} + \beta_{R1}(\phi_{i\alpha}\phi^{i\alpha})^2 + \beta_{R2}\phi_{i\alpha}\phi^{j\alpha}\phi_{j\beta}\phi^{i\beta} \\ &- 2\alpha_L^2\psi_{i\alpha}\psi^{i\alpha} + \beta_{L1}(\psi_{i\alpha}\psi^{i\alpha})^2 + \beta_{L2}\psi_{i\alpha}\psi^{j\alpha}\psi_{j\beta}\psi^{i\beta} \\ &+ \alpha_{LR1}\psi_{i\alpha}\psi^{i\alpha}\phi_{j\beta}\phi^{j\beta} + \alpha_{LR2}\psi_{i\alpha}\phi^{j\alpha}\psi^{i\beta}\phi_{j\beta}.\end{aligned} \tag{58}$$

Note that the first line we associate to the initial symmetry breaking pattern $SU(4) \times SU(2)_L \times SU(2)_R \mapsto SU(3) \times SU(2) \times U(1)_Y$. The second and third line give us the second symmetry breaking pattern. The unbroken theory with vevs ϕ_0 and ψ_0 both equal to zero is nearly identical to the theory we looked at before (25) (with the exception of new massive Higgs particles associated with ψ) and therefore we ignore this case. Let us now consider the case where both Higgs fields have non zero vevs. Again, in Ref [19] the author shows that the form of the vevs which give an absolute minimum to the potential above is given by

$$\phi_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{v_R}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{59}$$

and

$$\psi_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{v_L}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{60}$$

Let us consider the potential (58) and see what vev conditions we obtain. We find that

$$\begin{aligned} \frac{\partial V(\phi,\psi)}{\partial \phi_{k\gamma}} &= -2\alpha_R^2 \phi^{k\gamma} + 2\beta_{R1}\phi^{k\gamma}\phi_{i\alpha}\phi^{i\alpha} + \beta_{R2}(\phi^{j\gamma}\phi_{j\beta}\phi^{k\beta} + \phi_{i\alpha}\phi^{k\alpha}\phi^{i\gamma}) \\ &- \alpha_{LR1}\psi_{i\alpha}\psi^{i\alpha}\phi^{k\gamma} + \alpha_{LR2}\psi_{i\alpha}\phi^{k\alpha}\psi^{i\gamma}, \end{aligned} \tag{61}$$

and

$$\begin{aligned} \frac{\partial V(\phi,\psi)}{\partial \psi_{k\gamma}} &= -2\alpha_L^2 \psi^{k\gamma} + 2\beta_{L1}\psi^{k\gamma}\psi_{i\alpha}\psi^{i\alpha} + \beta_{L2}(\psi^{j\gamma}\psi_{j\beta}\psi^{k\beta} + \psi_{i\alpha}\psi^{k\alpha}\psi^{i\gamma}) \\ &- \alpha_{LR1}\phi_{i\alpha}\phi^{i\alpha}\psi^{k\gamma} + \alpha_{LR2}\phi_{i\alpha}\psi^{k\alpha}\phi^{i\gamma}. \end{aligned} \tag{62}$$

Substituting in our vevs ϕ_0 and ψ_0, we indeed find the vev conditions (for the case $k = 1$, $\gamma = 4$)

$$-2\alpha_L^2 + (\beta_{L1} + \beta_{L2})v_L^2 + \frac{1}{2}(\alpha_{LR1} + \alpha_{LR2})v_R^2 = 0 \tag{63}$$

and

$$-2\alpha_R^2 + (\beta_{R1} + \beta_{R2})v_R^2 + \frac{1}{2}(\alpha_{LR1} + \alpha_{LR2})v_L^2 = 0 \tag{64}$$

presented in Ref. [19]. We are now ready to demonstrate the symmetry breaking pattern, by considering our potential (58) centred around its non zero vevs (59) and (60).

Let us note that the vev condition $-2\alpha_L^2 + (\beta_{L1} + \beta_{L2})v_L^2 + \frac{1}{2}(\alpha_{LR1} + \alpha_{LR2})v_R^2 = 0$ can be reduced to our vev condition $\phi_0^{i\alpha}\phi_{0i\alpha} = \frac{{v_R}^2}{2} = \frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}}$ on ϕ_0 for the first symmetry breaking pattern, by letting $\alpha_{LR1} \mapsto -\alpha_{LR2}$. This therefore seems like an appealing simplification to make, as it decouples the two symmetry breaking patterns- note that the norm of each vev ϕ_0, ψ_0 does not depend on the Higgs mixing parameters α_{LR1} and α_{LR2} when this simplification is made. Before making this assumption, we show in Appendix B in full detail that the potential centred around the non zero vevs is given by

$$
\begin{aligned}
V(\varphi + \phi_0, \Psi + \psi_0) \approx\ & (-2\alpha_R^2 + v_R^2\beta_{R1})(|\varphi_{21}|^2 + |\varphi_{22}|^2 + |\varphi_{23}|^2) + (-2\alpha_R^2 + 3\beta_{R1}v_R^2 + 3\beta_{R2}v_R^2)Re(\varphi_{14})^2 \\
& + (-2\alpha_R^2 + v_R^2\beta_{R1} + v_R^2\beta_{R2})(|\varphi_{11}|^2 + |\varphi_{12}|^2 + |\varphi_{13}|^2 + Im(\varphi_{14})^2) + |\varphi_{24}|^2) \\
& + (-2\alpha_L^2 + v_L^2\beta_{L1})(|\Psi_{21}|^2 + |\Psi_{22}|^2 + |\Psi_{23}|^2) + (-2\alpha_L^2 + 3\beta_{L1}v_L^2 + 3\beta_{L2}v_L^2)Re(\Psi_{14})^2 \\
& + (-2\alpha_L^2 + v_L^2\beta_{L1} + v_L^2\beta_{L2})(|\Psi_{11}|^2 + |\Psi_{12}|^2 + |\Psi_{13}|^2 + Im(\Psi_{14})^2) + |\Psi_{24}|^2) \\
& + \alpha_{LR1}((|\Psi_{11}|^2 + |\Psi_{12}|^2 + |\Psi_{13}|^2 + |\Psi_{14}|^2 + |\Psi_{21}|^2 + |\Psi_{22}|^2 + |\Psi_{23}|^2 + |\Psi_{24}|^2)\frac{v_R^2}{2} \\
& + (Re(\varphi_{14})^2 + |\varphi_{21}|^2 + |\varphi_{22}|^2 + |\varphi_{23}|^2)\frac{v_L^2}{2} + 2v_Lv_R Re(\varphi_{14})Re(\Psi_{14})) \\
& + \alpha_{LR2}(\frac{v_L^2}{2}(Re(\varphi_{14})^2) + \frac{v_R^2}{2}(|\Psi_{14}|^2 + |\Psi_{24}|^2) + 2v_Lv_R Re(\varphi_{14})Re(\Psi_{14})).
\end{aligned}
\tag{65}
$$

Then, using our simplifying condition $\alpha_{LR1} \mapsto \alpha_{LR2}$, which also implies $\phi_0^{i\alpha}\phi_{0i\alpha} = \frac{{v_R}^2}{2} = \frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}}$ and $\psi_0^{i\alpha}\psi_{0i\alpha} = \frac{{v_L}^2}{2} = \frac{\alpha_L^2}{\beta_{L1} + \beta_{L2}}$, we find that

$$
\begin{aligned}
V(\varphi + \phi_0, \Psi + \psi_0) \approx\ & 2\alpha_R^2((\frac{\beta_{R1}}{\beta_{R1} + \beta_{R2}} - \frac{1}{2\alpha_R^2}\frac{\alpha_L^2}{\beta_{L1} + \beta_{L2}} - 1)(|\varphi_{21}|^2 + |\varphi_{22}|^2 + |\varphi_{23}|^2) + 2Re(\varphi_{14})^2)) \\
& + 2\alpha_L^2((\frac{\beta_{L1}}{\beta_{L1} + \beta_{L2}} - 1)(|\Psi_{21}|^2 + |\Psi_{22}|^2 + |\Psi_{23}|^2) + 2Re(\Psi_{14})^2)) \\
& + \alpha_{LR1}((|\Psi_{11}|^2 + |\Psi_{12}|^2 + |\Psi_{13}|^2 + |\Psi_{21}|^2 + |\Psi_{22}|^2 + |\Psi_{23}|^2)\frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}},
\end{aligned}
\tag{66}
$$

which shows that as expected, we have 7 non zero components for φ and 13 non zero components for Ψ, implying 9 unbroken generators (as would be required for $SU(3)\times SU(2)_L\times U(1)_Y \mapsto SU(3)\times U(1)_Q$). Let us now consider the case where α_{LR1} and α_{LR2} are free parameters. So as not to have to write out our potential in full again, let us note that the only off diagonal terms in (65) are the ones involving $Re(\varphi_{14})Re(\Psi_{14})$, therefore it is only these terms (as well as $Re(\varphi_{14})^2$, $Re(\Psi_{14})^2$) that need to be diagonalised in the mass matrix. Hence, keeping only such terms, we find

$$\begin{aligned} V(\varphi+\phi_0,\Psi+\psi_0) \approx ... &+ (-2\alpha_R + 3v_R^2(\beta_{R1}+\beta_{R2}) + \frac{v_L^2}{2}(\alpha_{LR1}+\alpha_{LR2}))Re(\varphi_{14})^2 \\ &+ (-2\alpha_L + 3v_L^2(\beta_{L1}+\beta_{L2}) + \frac{v_R^2}{2}(\alpha_{LR1}+\alpha_{LR2}))Re(\Psi_{14})^2 + \\ &+ 2v_Lv_R(\alpha_{LR1}+\alpha_{LR2})Re(\varphi_{14})Re(\Psi_{14}) \\ &+ .... \end{aligned} \tag{67}$$

Again, seeing as we can already read off the spectrum for all of the other components, our remaining mass matrix is

$$\begin{pmatrix} -2\alpha_R + 3v_R^2(\beta_{R1}+\beta_{R2}) + \frac{v_L^2}{2}(\alpha_{LR1}+\alpha_{LR2}) & v_Lv_R(\alpha_{LR1}+\alpha_{LR2}) \\ v_Lv_R(\alpha_{LR1}+\alpha_{LR2}) & -2\alpha_L + 3v_L^2(\beta_{L1}+\beta_{L2}) + \frac{v_R^2}{2}(\alpha_{LR1}+\alpha_{LR2}) \end{pmatrix}. \tag{68}$$

Upon diagonalisation, the eigenvalues of this matrix are pretty horrendous. However, using

$$-2\alpha_L^2 + (\beta_{L1}+\beta_{L2})v_L^2 + \frac{1}{2}(\alpha_{LR1}+\alpha_{LR2})v_R^2 = 0 \tag{69}$$

and

$$-2\alpha_R^2 + (\beta_{R1}+\beta_{R2})v_R^2 + \frac{1}{2}(\alpha_{LR1}+\alpha_{LR2})v_L^2 = 0, \tag{70}$$

we may then simplify the mass matrix to the form

$$\begin{pmatrix} 2v_R^2(\beta_{R1}+\beta_{R2}) & v_Lv_R(\alpha_{LR1}+\alpha_{LR2}) \\ v_Lv_R(\alpha_{LR1}+\alpha_{LR2}) & 2v_L^2(\beta_{L1}+\beta_{L2}) \end{pmatrix}. \tag{71}$$

Solving for the eigenvalues of this system, we find something interesting. The masses of the fields $Re(\varphi_{14})$ and $Re(\Psi_{14})$ are given by

$$m^2_{Re(\varphi_{14})} = 2v_R^2(\beta_{R1} + \beta_{R2}) + 2v_L^2(\beta_{L1} + \beta_{L2}) - F(\alpha, \beta, v) \tag{72}$$

and

$$m^2_{Re(\Psi_{14})} = 2v_R^2(\beta_{R1} + \beta_{R2}) + 2v_L^2(\beta_{L1} + \beta_{L2}) + F(\alpha, \beta, v), \tag{73}$$

with

$$F(\alpha, \beta, v) = 4[(2v_L^2(\beta_{L1} + \beta_{L2}) - 2v_R^2(\beta_{R1} + \beta_{R2}))^2 + 4v_L^2 v_R^2(\alpha_{LR1}^2 + 2\alpha_{LR1}\alpha_{LR2} + \alpha_{LR2}^2)]^{\frac{1}{2}} \tag{74}$$

a rather messy looking function of the coefficients in the potential and the vevs. Note that apart from the term involving $F(\alpha, \beta, v)$, the masses of each of the fields are the same. More importantly, the masses of each of the fields depend on the vevs and coefficients of the other field. Now, allowing $\alpha_{LR1} \mapsto -\alpha_{LR2}$, we find that $F(\alpha, \beta, v) \mapsto 2v_L^2(\beta_{L1} + \beta_{L2}) - 2v_R^2(\beta_{R1} + \beta_{R2})$. Indeed, when $\alpha_{LR1} = -\alpha_{LR2}$ the masses of the fields reduce to

$$m^2_{Re(\varphi_{14})} = 4v_R^2(\beta_{R1} + \beta_{R2}) = 8\alpha_R^2 \tag{75}$$

and

$$m^2_{Re(\Psi_{14})} = 4v_L^2(\beta_{L1} + \beta_{L2}) = 8\alpha_L^2 \tag{76}$$

(Note that this can be seen more readily before solving the eigensystem - simply plugging in $\alpha_{LR1} = -\alpha_{LR2}$ into the mass matrix (68) one immediately obtains the diagonalised matrix).

It is interesting to note that in this case, the masses of one field no longer depend on the vev and coefficients of another. Moreover, given that in the limit $\alpha_{LR1} \mapsto -\alpha_{LR2}$ the vevs ϕ_0 and ψ_0 are 'decoupled', we will later find that the associated gauge boson masses are then also decoupled for the two symmetry breaking patterns, giving them a possible mass hierarchy. That is why we consider this case as a decoupling limit.

In any case, we have now demonstrated that given the Lagrangian (56) and Higgs fields ϕ and ψ with non zero vevs of the form (59) and (60), one may read off 7 massive Higgs particles associated to ϕ and a further 13 associated to ψ. Counting degrees of freedom, we see that this implies 12 broken generators. The first 9, we associate to the initial symmetry breaking pattern $SU(4) \times SU(2)_L \times SU(2)_R \mapsto SU(3) \times SU(2)_L \times U(1)_Y$, as we haved showed earlier in this chapter. The final 3, we would like to associate with $SU(3) \times SU(2)_L \times U(1)_Y \mapsto SU(3) \times U(1)_Q$, though we have not yet explicitly shown this.

3.6 The Second Symmetry Breaking Matrix (Model 1)

Let us conclusively show the second symmetry breaking pattern discussed above, by constructing a symmetry breaking matrix. Note that we can do this with no prior knowledge of a potential term. Firstly, let us realise that our starting point is now the residual symmetry group from the previous symmetry breaking pattern. Therefore, besides the 3 generators belonging to $SU(2)_L$, we need only include the 9 unbroken generators from the previous case, namely the first 8 generators of $SU(4)$ (which turn out to precisely be the $SU(3)$ generators as embedded in $SU(4)$), and the weak hypercharge, namely the linear combination of the diagonal generators of $SU(2)_R$ and $SU(4)$. However, since this second Higgs transforms trivially under $SU(2)_R$, the diagonal generator of $SU(2)_R$ will not play a role here. Then, the generators we are after are the following:

For

$$\text{a=1,2,3...,8, } \hat{T}^a = T^a \tag{77}$$

and for

$$\text{a=9, } \hat{T}^9 = \frac{g_R}{g_4}\sqrt{\frac{2}{3}}T^{15} \tag{78}$$

(T^{18} does not appear in the linear combination of $\hat{T}^9$ since this Higgs field is in the trivial representation of $SU(2)_R$)

and for

$$\text{a=10,11,12, } \hat{T}^a = \frac{g_L}{g_4}\begin{pmatrix} t_{11}^{a-9}\mathbb{1}_{4\times4} & t_{12}^{a-9}\mathbb{1}_{4\times4} \\ t_{21}^{a-9}\mathbb{1}_{4\times4} & t_{22}^{a-9}\mathbb{1}_{4\times4} \end{pmatrix} \tag{79}$$

.

So, constructing the symmetry breaking matrix $S'^{ab} = \psi_0^\dagger\Big\{\hat{T}^a, \hat{T}^b\Big\}\psi_0$ using the generators $\hat{T}^a$ defined in above (letting $\frac{v_L}{\sqrt{2}} = u$ for simplicity), we find that

$$S^{'} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\frac{g_R^2u^2}{g_4^2}&0&0&-\frac{g_Lg_Ru^2}{g_4^2}\\
0&0&0&0&0&0&0&0&0&\frac{g_L^2u^2}{g_4^2}&0&0\\
0&0&0&0&0&0&0&0&0&0&\frac{g_L^2u^2}{g_4^2}&0\\
0&0&0&0&0&0&0&0&-\frac{g_Lg_Ru^2}{g_4^2}&0&0&\frac{g_L^2u^2}{g_4^2}
\end{pmatrix} \tag{80}$$

which, when diagonalised, gives us the matrix

$$\text{Diag}(0,0,0,0,0,0,0,0,0,\frac{g_L^2u^2}{g_4^2},\frac{g_L^2u^2}{g_4^2},\frac{u^2(g_L^2+g_R^2)}{g_4^2}). \tag{81}$$

Again, we may note from these that the final zero eigenvalue comes from a linear combination of the diagonal generator of the previously unbroken $U(1)_Y$ group and the diagonal generator of $SU(2)_L$. This linear combination is indeed the charge generator Q which finally gives the unbroken $U(1)_Q$

subgroup of the standard model. We find that the electric charge generator is

$$Q = \sqrt{\frac{2}{3}}T^{15} + \frac{g_4}{g_R}T^{18} + \frac{g_4}{g_L}\hat{T}^{12} \tag{82}$$

and noting that T^{18} and $\hat{T}^{12}$ are defined with a factor of the inverse form of the couplings appearing here, we see that conserved electromagnetic charge Q takes a rather simple form in Pati-Salam (in agreement with [19]). Finally, we may read off of this matrix the correct particle spectrum: starting with all massless gauge bosons, coming from 8 unbroken $SU(3)$ generators, one unbroken $U(1)_Y$ generator and 3 unbroken $SU(2)_L$ generators, we now have 9 massless gauge bosons, coming from $SU(3) \times U(1)_Q$, and 3 massive gauge bosons from the broken $SU(2)_L \times U(1)_Y$ subgroup. Moreover, we note that the gauge boson masses (closely related to the eigenvalues above) even predict that two of them have equal mass and the third one is heavier, as required from electroweak symmetry breaking. Thus, we now conclude our demonstration of the second symmetry breaking pattern, having shown explicitly that the remaining unbroken generators are precisely those that would give rise to the residual symmetry group $SU(3) \times U(1)_Q$.

3.7 The Fermion Sector (Model 1)

In this section we would have included the final missing terms for our 'complete Lagrangian' for model 1 (23) that we were building up to throughout this chapter. However, we postpone the discussion of the fermion sector of Pati-Salam to a model 2. The reason is that the model we are considering at the moment isn't the exact one that we shall use to understand fermions. In particular, it appears that in the model where electroweak symmetry breaking occurs due to a left handed Higgs in the representation used above, fermions do not attain mass from tree level diagrams, but from radiative loop corrections [19]. The simpler and more phenomenologically sound version of Pati-Salam undergoes electroweak breaking via a Higgs bidoublet. This theory does have tree level masses for fermions [19], and we shall use that framework to understand them.

3.8 Fermion Sector and The Second Symmetry Breaking Matrix (Model 2)

The only parts of our Lagrangian for model 1 (23) that we missed out in the previous chapter were the terms involving fermions. In this section, we will look at another model of Pati-Salam and we will consider the fermion sector. Let us start again by taking the gauge group $SU(4) \times SU(2)_L \times SU(2)_R$, and supposing we have two Higgs fields. The first one, ϕ, transforms in the representation $(4, 1, 2)$ (exactly as before) and is responsible for the symmetry breaking into the unbroken standard model. The second one, χ, transforms trivially under $SU(4)$ and in the fundamental representation of both $SU(2)_L$ and $SU(2)_R$ and is responsible for electroweak symmetry breaking. The gauge sector is identical to model 1 (as should be expected), besides the obvious change in coupling between Higgs and gauge fields.

The difference between the models then begins to arise when one considers the potential term and the fermion sector. It turns out that the most general potential term for this model takes a rather complicated form [19], and it will not aid our subsequent explorations to write this potential explicitly. Since we have already made a detailed and explicit calculation demonstrating the symmetry breaking pattern directly from a potential in model 1, we will not repeat that process for the even more complicated potential given in model 2. Let us again begin by recalling the Lagrangian for model 2,

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}\,\mathrm{Tr}\big[G_{\mu\nu}G^{\mu\nu} + F_{L\mu\nu}F_L^{\mu\nu} + F_{R\mu\nu}F_R^{\mu\nu}\big] + D_\mu\chi_i^I D^\mu\chi_I^i + D_\mu\phi_{i\alpha}D^\mu\phi^{i\alpha} \\
&+ i\bar{\psi}_L\gamma^\mu D_\mu\psi_L + i\bar{\psi}_R\gamma^\mu D_\mu\psi_R + \lambda_1(\psi_{LI\alpha}\chi_i^I\psi_R^{i\alpha} + \psi_L^{I\alpha}\chi_I^i\psi_{Ri\alpha}) \\
&+ \lambda_2(\psi_{LI\alpha}\tilde{\chi}_i^I\psi_R^{i\alpha} + \psi_L^{I\alpha}\tilde{\chi}_I^i\psi_{Ri\alpha}) - [-2\alpha_R^2\phi_{i\alpha}\phi^{i\alpha} + \beta_{R1}(\phi_{i\alpha}\phi^{i\alpha})^2 \\
&+ \beta_{R2}\phi_{i\alpha}\phi^{j\alpha}\phi_{j\beta}\phi^{i\beta} + V(\phi,\chi)],
\end{aligned}
\tag{83}
$$

with λ_1 and λ_2 real. Some definitions are now in order for this Lagrangian. Firstly, as mentioned the Higgs field χ is in the representation (1,2,2) (with complex components) and is given by

$$
\chi = \begin{pmatrix} \chi_1^1 & \chi_2^1 & \chi_1^2 & \chi_2^2 \end{pmatrix}. \tag{84}
$$

Note that (following notation used in [19]) χ_i^I has indices i and I above, corresponding to $SU(2)_R$ and $SU(2)_L$ respectively. Again, we use the notation where superscript and subcscript indices are **transpose conjugates of one another**, meaning here for example that χ_i^I is the transpose conjugate of χ_I^i. We also need to form another bidoublet $\tilde{\chi}_i^I$ with the same transformation law, namely

$$\tilde{\chi} = \left(\begin{array}{cccc} (\chi_2^2)^* & -(\chi_1^2)^* & -(\chi_2^1)^* & (\chi_1^1)^* \end{array} \right). \tag{85}$$

We also define the covariant derivative for χ above as $D^\mu = (\partial^\mu + ig_L A^{\mu a} t'^a)$, for $a = 1,2,3,4,5,6$, where the first three components correspond to $SU(2)_L$ and the final three to $SU(2)_R$ (with the usual order for the Pauli matrices). This implies that our gauge fields are defined as

$$A^{a\mu} = \begin{cases} A_L^{a\mu} & \text{for } a = 1{,}2{,}3, \\ A_R^{(a-3)\mu} & \text{for } a = 4{,}5{,}6\ . \end{cases} \tag{86}$$

Also, we then define our generators as

$$\text{a=1,2,3, } t'^a = \begin{pmatrix} \delta_{11} t^a & \delta_{12} t^a \\ \delta_{21} t^a & \delta_{22} t^a \end{pmatrix} = \begin{pmatrix} t^a & 0 \\ 0 & t^a \end{pmatrix} \tag{87}$$

and for

$$\text{a=4,5,6, } t'^a = \frac{g_R}{g_L} \begin{pmatrix} t^a & 0 \\ 0 & t^a \end{pmatrix}. \tag{88}$$

Finally, the covariant derivative associated to ϕ is the same as in model 1. Let us now define the fields that constitute the fermion sector. These are denoted above as ψ_R and ψ_L, and we suppose that they transform in the representations $(4,1,2)$ and $(4,2,1)$ of our gauge group. Using the gamma matrices we also define above $\bar{\psi}_R \equiv \psi_R^\dagger \gamma^0$ and $\bar{\psi}_L \equiv \psi_L^\dagger \gamma^0$. These objects may seem more complicated than standard model fermions, but in fact they are simpler. As $SU(4)$ vectors, the four components consist of the three colors of the quarks, and the lepton (in that order). For example, the up quark (green, blue, then red) and the electron neutrino. Then, as an $SU(2)$ vector, they are the usual up-down or charged lepton-neutrino pair we are used to from the standard model. The left and right handed fermion representations indeed represent the two chiralities of fermions that exist in the standard model. Thus, as advertised, Pati-Salam unifies quarks and leptons into a single representation, and treats left and right handed fermions equally, thereby simplifying the theory.

We already know what kind of objects these fermions are, since they appear in precisely the same representations as our Higgs fields from model 1. We will view them as 8-tuplets with the components mentioned above in the same order we had for ϕ earlier in equation (26). To avoid any ambiguity, the first family of left handed fermions, for example, is given by

$$\psi_L = \begin{bmatrix} u_L^g \\ u_L^b \\ u_L^r \\ \nu_{eL} \\ d_L^g \\ d_L^b \\ d_L^r \\ e_L \end{bmatrix} = \begin{bmatrix} \psi_{L11} \\ \psi_{L12} \\ \psi_{L13} \\ \psi_{L14} \\ \psi_{L21} \\ \psi_{L22} \\ \psi_{L23} \\ \psi_{L24} \end{bmatrix}. \tag{89}$$

Let us note that we ignored 2 families of fermions in the expression above for simplicity- of course, all of the terms involving fermions in the Lagrangian above should be repeated for every generation of fermions.

Finally, then, we wish to verify the symmetry breaking pattern and therefore the particle spectrum. It is easy to verify that given the vev

$$\chi_0 = \begin{pmatrix} u_1 & 0 & 0 & u_2 \end{pmatrix} \tag{90}$$

with real parameters u_1 and u_2, one obtains the correct symmetry breaking pattern from $SU(3) \times$

$SU(2)_L \times U(1)_Y$ into the broken standard model gauge group $SU(3) \times U(1)_Q$ upon construction of the symmetry breaking matrix as usual. Let us explicitly demonstrate that.

We recall that the first symmetry breaking occurs in the same way for both models - that is, a right handed Higgs field ϕ in the representation $(4, 1, 2)$ of $SU(4) \times SU(2)_L \times SU(2)_R$. This gave us the symmetry breaking matrix S, with 9 broken generators, leaving the unbroken standard model gauge group as our residual symmetry group. Then, we need to choose which generators to include for this new symmetry breaking matrix, which takes us to the broken standard model gauge group. Since our new Higgs field, χ, is in the representation $(1, 2, 2)$, we would normally expect to have 6 generators altogether, belonging to the two $SU(2)$ groups involved.

However, the group $SU(2)_R$ has already been broken. In fact, only the diagonal element of $SU(2)_R$, when taken together with the diagonal element of $SU(4)$, forms the new unbroken $U(1)_Y$ subgroup. Therefore, it is this generator that will play a part in the symmetry breaking matrix, where we shall ignore the $SU(4)$ generator since χ transforms trivially under it.

Then, we take the remaining four unbroken generators from our definitions above, i.e.

for

$$\text{a=1,2,3, } t'^a = \begin{pmatrix} t^a & 0 \\ 0 & t^a \end{pmatrix} \tag{91}$$

and for

$$\text{a=4, } t'^a = \frac{g_R}{g_L} \begin{pmatrix} t^a & 0 \\ 0 & t^a \end{pmatrix}. \tag{92}$$

.

Then, with the vev defined earlier,

$$\chi_0 = \begin{pmatrix} u_1 & 0 & 0 & u_2 \end{pmatrix}, \tag{93}$$

,

we construct $S''^{ab} = \chi_0^\dagger \Big\{ t'^a, t'^b \Big\} \chi_0$,

from which, we find the symmetry breaking matrix

$$S'' = \begin{pmatrix} \mathrm{u}_1^2 + \mathrm{u}_2^2 & 0 & 0 & 0 \\ 0 & \mathrm{u}_1^2 + \mathrm{u}_2^2 & 0 & 0 \\ 0 & 0 & \mathrm{u}_1^2 + \mathrm{u}_2^2 & \frac{\mathrm{g}_R \mathrm{u}_1^2}{\mathrm{g}_L} + \frac{\mathrm{g}_R \mathrm{u}_2^2}{\mathrm{g}_L} \\ 0 & 0 & \frac{\mathrm{g}_R \mathrm{u}_1^2}{\mathrm{g}_L} + \frac{\mathrm{g}_R \mathrm{u}_2^2}{\mathrm{g}_L} & \frac{\mathrm{g}_R^2 \mathrm{u}_1^2}{\mathrm{g}_L^2} + \frac{\mathrm{g}_R^2 \mathrm{u}_2^2}{\mathrm{g}_L^2} \end{pmatrix}, \tag{94}$$

which, when diagonalised, gives us the matrix

$$\mathrm{Diag}(0, u_1^2 + u_2^2, u_1^2 + u_2^2, \left(g_L^2 + g_R^2\right) \left(\frac{u_1^2}{g_L^2} + \frac{u_2^2}{g_L^2}\right)). \tag{95}$$

Indeed, as is expected, 3 further generators are broken (3 non zero eigenvalues), leaving only a single unbroken generator which corresponds to a linear combination of $U(1)_Y$ and $SU(2)_L$, also known as electric charge generator Q. Moreover, we find three massive gauge bosons, two of equal mass and one slightly heavier, as is required for the weak force. Thus, we have now shown that electroweak symmetry breaking in Pati-Salam has two valid routes- one via a $(4, 2, 1)$ Higgs, and another via the $(1, 2, 2)$ Higgs bidoublet. Given the representations stated above for our Higgs fields, we now know that (alongside the gauge boson spectrum which is the same as model 1) we obtain 7 massive Higgs particles from ϕ and 5 massive Higgs particle from χ. Having defined our objects in the Lagrangian (24) and derived our symmetry breaking pattern, let us talk about fermions in more detail.

3.9 The Fermion Masses (Model 2)

We could have chosen to introduce the fermion sector in model 1, and there is a perfectly valid sector to discuss in that framework [19]. However, since our goal has been to explore the *simplest* formulation of Pati-Salam which reproduces the correct symmetry breaking pattern and is able to qualitatively discuss fermions, we are better off diverting our attention to the Higgs bidoublet model. This is due to the fact that it is impossible (unlike the second model) to write down invariant Yukawa terms that couple a left handed fermion field $\Longleftrightarrow$ left / right handed Higgs fields $\Longleftrightarrow$ right handed fermion field, if the Higgs fields are in the representations used in the first model. In particular, all of the above 3 fields would be in either the fundamental or conjugate representations of $SU(4)$, ie the representation 4 or $\bar{4}$. But this means that the only possible representations of the combined object (as is shown in page 11 of Ref. [19]) are

$$
\begin{aligned}
&4 \otimes 4 \otimes 4 = 20 \oplus 20 \oplus 20 \oplus \bar{4}, \\
&\bar{4} \otimes 4 \otimes 4 = 36 \oplus 4 \oplus 20 \oplus 4, \\
&\bar{4} \otimes \bar{4} \otimes \bar{4} = \bar{36} \oplus \bar{4} \oplus 20 \oplus 4,
\end{aligned}
\tag{96}
$$

none of which allow for singlets (as would be needed for any object in the Lagrangian). This means that the fermion mass terms do not arise in the usual simple way from the Yukawa terms, but rather due to radiative loop corrections [20] that also require the addition of a sterile neutrino into the theory. This added complication does not contribute to our understanding of the fermion sector, nor does it help us to demonstrate where violation of lepton universality might arise from- therefore we choose to set it aside and focus on the second model instead. More information on the fermion sector related to the first model can be found in Ref. [20]. Let us now return to our new Lagrangian (24) and explicitly read off the masses from the Yukawa terms (for one generation, as the rest will be similar). To do this, we recall that the relevant terms are

$$
\lambda_1(\psi_{LI\alpha}\chi_i^I\psi_R^{i\alpha} + \psi_L^{I\alpha}\chi_I^i\psi_{Ri\alpha}) + \lambda_2(\psi_{LI\alpha}\tilde{\chi}_i^I\psi_R^{i\alpha} + \psi_L^{I\alpha}\tilde{\chi}_I^i\psi_{Ri\alpha}). \tag{97}
$$

We also recall the form of our fermions

$$\psi_L = \begin{bmatrix} u_L^g \\ u_L^b \\ u_L^r \\ \nu_{eL} \\ d_L^g \\ d_L^b \\ d_L^r \\ e_L \end{bmatrix}. \tag{98}$$

Finally, we recall our fields

$$\chi = \begin{pmatrix} \chi_1^1 & \chi_2^1 & \chi_1^2 & \chi_2^2 \end{pmatrix} \tag{99}$$

and

$$\tilde{\chi} = \begin{pmatrix} (\chi_2^2)^* & -(\chi_1^2)^* & -(\chi_2^1)^* & (\chi_1^1)^* \end{pmatrix}, \tag{100}$$

and our vev

$$\chi_0 = \begin{pmatrix} u_1 & 0 & 0 & u_2 \end{pmatrix}. \tag{101}$$

Using the above, we substitute the appropriate fields into (97) and evaluate on our Higgs vev χ_0 to find that

$$\begin{aligned} \mathcal{L} &= \lambda_1(\psi_{LI\alpha}\chi_i^I\psi_R^{i\alpha} + \psi_L^{I\alpha}\chi_I^i\psi_{Ri\alpha}) + \lambda_2(\psi_{LI\alpha}\tilde{\chi}_i^I\psi_R^{i\alpha} + \psi_L^{I\alpha}\tilde{\chi}_I^i\psi_{Ri\alpha}) \\ &= \lambda_1(u_1\psi_{L1\alpha}\psi_R^{1\alpha} + u_2\psi_{L2\alpha}\psi_R^{2\alpha}) + \lambda_2(u_2\psi_{L1\alpha}\psi_R^{1\alpha} + u_1\psi_{L2\alpha}\psi_R^{2\alpha}) \\ &= u_1(\lambda_1\psi_{L1\alpha}\psi_R^{1\alpha} + \lambda_2\psi_{L2\alpha}\psi_R^{2\alpha}) + u_2(\lambda_1\psi_{L2\alpha}\psi_R^{2\alpha} + \lambda_2\psi_{La\alpha}\psi_R^{1\alpha}) \\ &= u_1(\lambda_1(\bar{u}_L^c u_R^c + \bar{\nu}_{eL}\nu_{eR}) + \lambda_2(\bar{d}_L^c d_R^c + \bar{e}_L e_R)) + u_2(\lambda_2(\bar{u}_L^c u_R^c + \bar{\nu}_{eL}\nu_{eR}) + \lambda_1(\bar{d}_L^c d_R^c + \bar{e}_L e_R)) \\ &= (u_1\lambda_1 + u_2\lambda_2)(\bar{u}_L^c u_R^c + \bar{\nu}_{eL}\nu_{eR}) + (u_1\lambda_2 + u_2\lambda_1)(\bar{d}_L^c d_R^c + \bar{e}_L e_R)), \end{aligned} \tag{102}$$

where double superscripts c denote a sum over the three colors. We find that the neutrinos have the same mass as the up quarks, and the electrons (or analogous charged leptons for the other two families) have the same mass as the down quarks. We therefore see that our theory now additionally predicts massive fermions, albeit with masses that do not currently match the standard model. In both models, radiative corrections to the masses are required to obtain mass hierarchies for the particle spectra [19]. Nevertheless, for our purposes this is sufficient and now that we understand the fundamental structure of Pati-Salam, we may finally move on to discuss violation of lepton universality and why Pati-Salam might predict this.

4 Lepton Universality and Pati-Salam

4.1 Source of Violation in Pati-Salam

In this final Chapter, we would like to give an idea of how Pati-Salam might predict violation of lepton universality, as well as to mention the results of various experiments over the years which seem to be for or against it. Let us recall the notion of lepton universality.

As shown in chapter 2, in the standard model all charged leptons interact in the same way with the electroweak gauge bosons (the only difference, which can be accounted for, being their masses) [21]. This means that when one calculates the frequency with which standard model processes should produce these leptons as decay products, all three types of charged leptons should be produced at exactly the same rate [21]. In other words, given some standard model processes which produce charged leptons, the ratio of the production of (for example) muons to electrons should converge to one, as the processes are measured many times. However, if there were other interaction terms in the Lagrangian, perhaps mediated via some as of yet undiscovered bosons, which did couple differently to different charged leptons, one would expect these ratios to converge to some other number, different than one. For example, a process currently being investigated experimentally is the decay of B-mesons into a D-mesons, where a new contribution coming from a mediating exotic particle is being considered [11] (much more detail and references will be given in the next subsection). This mediating boson, which decays into a charged lepton and its corresponding neutrino, might couple to these charged leptons differently, and thus the ratio of lepton production for different generations would be distinct in such a process [22].

Indeed, various recent experiments [9], which we will soon discuss, have suggested that these ratios are not quite equal to the predictions made by the standard model, assuming lepton universality. Given the experimental error, these statistical deviations are not individually significant enough to conclude with certainty that lepton universality is truly violated [22]. However when one considers the various experiments for a variety of processes, all of which have noted a slight deviation, there is at the very least cause to inspect other GUT's which may be able to predict such deviations.

With the above in mind, let us see whether our Lagrangians (23), (24) can indeed predict violation of lepton universality, at least in principle. As mentioned above, this would mean that in some way, we should be able to find interaction terms in the above Lagrangian for which the charged leptons differ from one another, so that there will be some kind of deviation in the rate at which each one is produced from certain standard processes.

Looking more closely then at the kinetic terms associated with our left and right handed fermions (recalling that we should have a copy for each generation of fermions), we consider the terms

$$i\bar{\psi}_L\gamma^\mu D_\mu\psi_L + i\bar{\psi}_R\gamma^\mu D_\mu\psi_R. \tag{103}$$

Let us ignore the partial derivative term. Then, ignoring also the generators associated to $SU(2)_L$ and $SU(2)_R$ (since it is our new gauge group $SU(4)$ that we hope can introduce new interaction terms to differentiate the charged leptons), we consider only the $SU(4)$ gauge field interaction terms, with generators as defined in Chapter 3, giving

$$i\bar{\psi}_L\gamma^\mu G^A_\mu T^A\psi_L + i\bar{\psi}_R\gamma^\mu G^A_\mu T^A\psi_R. \tag{104}$$

Moreover, ignoring the generators associated to $SU(3)$ (again, these aren't the ones we expect to give us new interaction terms as they already appear in the standard model), let us for now consider the 6 leptoquarks (see the definition of $\hat{G}^\mu$ on page 2 of Ref. [23] to find the leptoquarks X^-_μ, X^+_μ), i.e. the generators for A= 9,10,...,14. Then, the remaining terms are

$$i\bar{\psi}_L\gamma^\mu X^A_\mu T^A\psi_L + i\bar{\psi}_R\gamma^\mu X^A_\mu T^A\psi_R. \tag{105}$$

Finally, expanding these terms **and including the 3 generations of fermion families** we find interaction terms of the form

$$i\bar{\nu}_e\gamma^\mu X^-_\mu u + i\bar{e}\gamma^\mu X^-_\mu d + i\bar{\nu}_\mu\gamma^\mu X^-_\mu c + i\bar{\mu}\gamma^\mu X^-_\mu s + i\bar{\nu}_\tau\gamma^\mu X^-_\mu t + i\bar{\tau}\gamma^\mu X^-_\mu b + h.c., \tag{106}$$

from which we may finally note that the charged leptons are each coupled to a leptoquark (X^-, with hermitean conjugate X^+) and more importantly a different quark to one another. Therefore, since quarks are non universal in the standard model (see Chapter 2), this would indeed indicate that lepton universality should be broken, when such processes are taken into account.

Finally, let us mention that the above exploration gives the 'basic' foundation for lepton universality violation in Pati-Salam, via the existence of a mediating vector leptoquark X. In practice, more complicated variants of Pati-Salam have been used to explain this violation. However, the majority of them do still rely on these leptoquarks which, being gauge fields and thus intrinsic to the choice of gauge group $SU(4) \times SU(2)_L \times SU(2)_R$, appear in almost all of these variants. Therefore, our efforts above are still relevant in that they demonstrate the origin of the vector leptoquarks, which often constitute the fundamental cause for the violation of lepton universality in these models. It is worth mentioning, however, that other Pati-Salam variants involving scalar leptoquarks (for example in Ref. [11]) have also been suggested as an explanation for lepton universality violation.

Having motivated the origin of the violation of lepton universality in Pati-Salam, let us mention some of the experiments over the years that produced evidence for or against it.

4.2 Tests of Lepton Universality

Our aim in this section is not to give a detailed phenomenological breakdown of the experiments that have been performed over the years, testing lepton universality. Indeed, this section is aimed also at theorists and therefore does not presuppose much knowledge of experimental particle physics.

Lepton universality in the standard model was motivated in Chapter 2, with an emphasis on the gauge coupling of leptons from different generations to the W bosons. However, the universality refers also to the couplings of leptons to the photon γ and Z boson. In fact, measurements as far back as 2005 [24] have produced data **in support** of lepton universality of the charged leptons in the so-called 'Neutral Current' interactions, mediated via the Z bosons. These experiments found that, to within a precision of the order of 0.1 percent, the ratio of branching fractions measured experimentally agrees with predictions based on lepton universality. Let us be more explicit about what we mean by branching fraction. For the Neutral Current interactions, the electron branching fraction for the Z boson would be the number of Z particles which decay into electrons, divided the total number of Z particle decays. In other words, if the Z boson decays into electrons 10 percent of the time, the branching fraction would be 0.1.

Then, the 2005 paper [24] shows that the ratios of branching fractions $\mathcal{B}$ for different charged leptons are given by

$$\frac{\mathcal{B}(Z \to \mu\mu)}{\mathcal{B}(Z \to ee)} = 1.0009 \pm 0.0028, \tag{107}$$

and

$$\frac{\mathcal{B}(Z \to \tau\tau)}{\mathcal{B}(Z \to ee)} = 1.0019 \pm 0.0032. \tag{108}$$

Another set of observables, the 'leptonic asymmetry paramters' $\mathcal{A}_l$ (which we will not define here, detailed discussion may be found in Ref. [24]) also yielded results which agree with lepton universality, however to a lower precision of the order of 1 percent. A more in depth summary of the above results is discussed in detail in Ref [25].

Moving on to lepton universality tests via the charged W bosons, in 2013 experiments measured the ratios of the different lepton branching fractions of the W bosons [26]. These results have produced a mix of evidence, some in support of lepton universality and some that hint at its violation. More concretely, the ratio

$$\frac{\mathcal{B}(W \to \mu\nu)}{\mathcal{B}(W \to e\nu)} = 0.993 \pm 0.019, \tag{109}$$

agrees with lepton universality to a precision of about 2 percent, while

$$\frac{\mathcal{B}(W \to \tau\nu)}{\mathcal{B}(W \to l\nu)} = 1.066 \pm 0.025, \tag{110}$$

deviates from the predicted value by 2.6 standard deviations, when lepton universality is assumed. The denominator above represents the average branching fraction of the electron and muon.

Other experiments measuring partial widths (observables which are closely related to branching fractions) seem to also be in support of the universality of electrons and muons, as is discussed in Ref. [25] and shown in detail in the 2018 review [27]. As is mentioned in [9], the larger mass of the τ may make B hadron decays into this heavier charged lepton more sensitive to 'new physics' effects (and thus more readily manifest violation of lepton universality) than its lighter counterparts.

Indeed, the most significant hints of the violation of lepton universality recently come from these 'semitauonic B decays' described above. These experiments [28–37] yielded the results

$$R(D) = \frac{\mathcal{B}(\bar{B}^0 \to \tau^- \bar{\nu}_\tau D^+)}{\mathcal{B}(\bar{B}^0 \to l^- \bar{\nu}_l D^+)} = 0.340 \pm 0.027 \pm 0.013, \tag{111}$$

and

$$R(D^*) = \frac{\mathcal{B}(\bar{B}^0 \to \tau^- \bar{\nu}_\tau D^{*+})}{\mathcal{B}(\bar{B}^0 \to l^- \bar{\nu}_l D^{*+})} = 0.295 \pm 0.011 \pm 0.008, \tag{112}$$

deviating from standard model predictions by 1.4σ and 2.5σ respectively (see [25] for more detail), the combined deviation [38] corresponding to 3.1σ.

As we can see, there is an accumulation of data over the years hinting more and more at the violation of lepton universality. Motivated by such results, Pati-Salam variants of the basic model analysed in this paper have been suggested in recent years [11, 39, 40]. In particular, in Ref. [11] the Pati-Salam variant capable of producing an explanation for the observed values of $R(D^*)$ above requires extending the basic Pati-Salam model to include the existence of a light right-handed neutrino.

5 Conclusions and Caveats

Some natural questions (and from these subtleties) arise from the developments we've made in the previous chapter, regarding violation of lepton universality.

Firstly, one may worry: do these new interactions (106) mediated by the leptoquarks imply the violation of baryon number? If so, how is this reconciled with the standard model? Indeed, at first glance it seems that baryon and lepton numbers could be violated by such interactions. In fact, these leptoquarks themselves carry the lepton and baryon numbers needed to exactly cancel out any discrepancy [41], thus maintaining the conservation of these numbers. However, the conservation of baryon number (at the classical level) is an 'accidental symmetry' [42] of the standard model, so it is unclear what results in baryon number conservation for GUT extensions. Moreover, if this isn't a general requirement on GUT's, why do we still find that Pati-Salam seems to conserve baryon number? It turns out that much like the conservation of B is an accidental symmetry of the standard model, so too is the conservation of B+L an accidental symmetry of the (broken) Pati-Salam group [41], [40]. Combined with the exact B-L symmetry (associated to the diagonal element of $SU(4)$, $T^{'15}$) inherent to Pati-Salam, one indeed finds conservation of baryon number. This, for example, is why the proton is stable in Pati-Salam models [40], unlike certain other GUT's, and thus is still a viable alternative to the standard model.

Secondly, it is interesting to note that the appearance of massive neutrinos in Pati-Salam models mean that the 'usual' reason for lepton flavor universality, shown in chapter 2, is no longer apparent. In fact, it seems like there could now be inter generational mixing associated to the kinetic terms for leptons, i.e. interactions which can even violate lepton flavor universality.

We conclude by reiterating that the main purpose of this paper has been to 1) explore the different paths one should take when trying to understand a GUT and 2) demonstrate where violation of lepton universality might arise in Pati-Salam, which is useful to understand both because current data suggests this might be the case (and if so, would mean the standard model is incorrect) and because it gives us a concrete idea of how different GUT's may predict different phenomenon. Indeed, we explored how one may quickly check whether the symmetry breaking pattern of a GUT can reduce to the standard model via computation of the symmetry breaking matrix, as well as how to obtain the particle spectrum of the theory, before and after symmetry breaking. We demonstrated where new interaction vertices might arise, and how these can give predictions that differ from the standard model. We have not addressed the issue of fermion mass hierarchies, nor have we discussed the running of the Pati-Salam coupling or the phenomenological aspects of our theory, beyond the discussion of leptoquarks and the results of experiments for or against lepton universality. A lot of work has been done on Pati-Salam which addresses some of the more subtle and complex issues mentioned above, and may be found in Ref. [19], and [23]. The goal we had hoped to achieve with this paper is to create some sort of bridge for theoreticians who wish to gain conceptual access to the theoretical investigations of the Pati-Salam Grand Unified Theory. It was written in such a way so as to make the study of Grand Unified Theories seem like a less mysterious and opaque pursuit, for those currently unfamiliar with it beyond knowledge of the standard model of particle physics. In particular, this paper aims to deliver an explicit account (for the first time, to the best of our knowledge) of the fundamentals of the Pati-Salam model.

Given the difficulties theoretical physics has faced over the recent decades, and motivated by the accumulation of experimental data from CERN hinting at new physics, it seems that now is a good time to explore typical GUT's with predictions for beyond the LHC energies to their full extent.

A Symmetry Breaking of Pati-Salam Into Unbroken Standard Model

The following calculation is an explicit verification of the fact that after the first symmetry breaking pattern, $SU(4) \times SU(2)_L \times SU(2)_R \mapsto SU(3) \times SU(2) \times U(1)_Y$, the 16 massive Higgs particles (associated to ϕ) in Lagrangian 25 should reduce to only 7 massive Higgs particles, reflecting the 9 broken generators after symmetry breaking has occured. Let $\phi^{i\alpha}$ be the complex 8-tuplet (with transpose conjugate $\phi_{i\alpha}$), and write $\phi^{i\alpha} = \varphi^{i\alpha} + \phi_0^{i\alpha}$. As mentioned earlier, the vev ϕ_0 is of the form

$$\phi_{0i\alpha} = \begin{pmatrix} 0 & 0 & 0 & \frac{v_R}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{113}$$

where $\phi_0^{i\alpha}\phi_{0i\alpha} = \dfrac{{v_R}^2}{2} = \dfrac{\alpha_R^2}{\beta_{R1}+\beta_{R2}}$. Note, in the following calculation I will only select out the second order terms in the fields (and the equals sign should be understood in this sense) since these reflect the particle spectrum.

Then, expanding our potential around ϕ_0, we have

$$
\begin{aligned}
V(\phi) &= \underline{-2\alpha_R^2\phi_{i\alpha}\phi^{i\alpha}} + \underline{\beta_{R1}(\phi_{i\alpha}\phi^{i\alpha})^2} + \underline{\beta_{R2}\phi_{i\alpha}\phi^{j\alpha}\phi_{j\beta}\phi^{i\beta}} \\
&\approx \underline{-2\alpha_R^2\varphi_{i\alpha}\varphi^{i\alpha}} + \underline{\beta_{R1}(\varphi_{i\alpha}\phi_0^{i\alpha} + \varphi^{i\alpha}\phi_{0i\alpha})(\varphi_{j\beta}\phi_0^{j\beta} + \varphi^{j\beta}\phi_{0j\beta})} + \underline{2\beta_{R1}\varphi_{i\alpha}\varphi^{i\alpha}\phi_{0j\beta}\phi_0^{j\beta}} \\
&+ \underline{\beta_{R2}(\varphi_{i\alpha} + \phi_{0i\alpha})(\varphi^{j\alpha} + \phi_0^{j\alpha})(\varphi_{j\beta} + \phi_{0j\beta})(\varphi^{i\beta} + \phi_0^{i\beta})} \\
&\approx \underline{-2\alpha_R^2(\varphi_{11}^2 + \varphi_{12}^2 + \varphi_{13}^2 + \varphi_{14}^2 + \varphi_{21}^2 + \varphi_{22}^2 + \varphi_{23}^2 + \varphi_{24}^2)} \\
&+ \underline{2\beta_{R1}Re(\varphi_{14})^2 v_R^2} + \underline{{v_R}^2\beta_{R1}(\varphi_{11}^2 + \varphi_{12}^2 + \varphi_{13}^2 + \varphi_{14}^2 + \varphi_{21}^2 + \varphi_{22}^2 + \varphi_{23}^2 + \varphi_{24}^2)} \\
&+ \underline{\beta_{R2}(\varphi_{i\alpha}\varphi^{j\alpha}\phi_{0j\beta}\phi_0^{i\beta} + \varphi_{i\alpha}\varphi_{j\beta}\phi_0^{j\alpha}\phi_0^{i\beta} + \varphi_{i\alpha}\varphi^{i\beta}\phi_0^{j\alpha}\phi_{0j\beta}} \\
&+ \underline{\varphi^{j\alpha}\varphi_{j\beta}\phi_{0i\alpha}\phi_0^{i\beta} + \varphi^{j\alpha}\varphi^{i\beta}\phi_{0i\alpha}\phi_{0j\beta} + \varphi_{j\beta}\varphi^{i\beta}\phi_{0i\alpha}\phi_0^{j\alpha})} \\
&\approx \underline{-2\alpha_R^2(\varphi_{11}^2 + \varphi_{12}^2 + \varphi_{13}^2 + \varphi_{14}^2 + \varphi_{21}^2 + \varphi_{22}^2 + \varphi_{23}^2 + \varphi_{24}^2)} \\
&+ \underline{4\beta_{R1}\frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}}Re(\varphi_{14})^2} + \underline{2\beta_{R1}\frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}}(\varphi_{11}^2 + \varphi_{12}^2 + \varphi_{13}^2 + \varphi_{14}^2 + \varphi_{21}^2 + \varphi_{22}^2 + \varphi_{23}^2 + \varphi_{24}^2)} \\
&+ \underline{\beta_{R2}\frac{{v_R}^2}{2}(\varphi_{1\alpha}\varphi^{1\alpha} + \varphi_{14}\varphi_{14} + \varphi_{i4}\varphi^{i4} + \varphi^{i4}\varphi_{i4} + \varphi^{14}\varphi^{14} + \varphi_{1\alpha}\varphi^{1\alpha})} \\
&\approx \underline{-2\alpha_R^2(\varphi_{11}^2 + \varphi_{12}^2 + \varphi_{13}^2 + \varphi_{14}^2 + \varphi_{21}^2 + \varphi_{22}^2 + \varphi_{23}^2 + \varphi_{24}^2)} \\
&+ \underline{4\beta_{R1}\frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}}Re(\varphi_{14})^2} + \underline{2\beta_{R1}\frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}}(\varphi_{11}^2 + \varphi_{12}^2 + \varphi_{13}^2 + \varphi_{14}^2 + \varphi_{21}^2 + \varphi_{22}^2 + \varphi_{23}^2 + \varphi_{24}^2)} \\
&+ \underline{\beta_{R2}\frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}}(2(\varphi_{11}^2 + \varphi_{12}^2 + \varphi_{13}^2 + \varphi_{14}^2) + 2(\varphi_{14}^2 + \varphi_{24}^2) + 2(Re(\varphi_{14})^2 - Im(\varphi_{14})^2))} \\
&\approx \underline{-2\alpha_R^2(\varphi_{11}^2 + \varphi_{12}^2 + \varphi_{13}^2 + \varphi_{14}^2 + \varphi_{21}^2 + \varphi_{22}^2 + \varphi_{23}^2 + \varphi_{24}^2)} \\
&+ \underline{2\beta_{R1}\frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}}(\varphi_{11}^2 + \varphi_{12}^2 + \varphi_{13}^2 + \varphi_{14}^2 + \varphi_{21}^2 + \varphi_{22}^2 + \varphi_{23}^2 + \varphi_{24}^2)} \\
&+ \underline{2\beta_{R2}\frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}}(\varphi_{11}^2 + \varphi_{12}^2 + \varphi_{13}^2 + \varphi_{14}^2 + \varphi_{24}^2 + 2Re(\varphi_{14})^2)} + \underline{4\beta_{R1}\frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}}Re(\varphi_{14})^2)} \\
&\approx 2\alpha_R^2\left(\left(\frac{\beta_{R1}}{\beta_{R1} + \beta_{R2}} - 1\right)(|\varphi_{21}|^2 + |\varphi_{22}|^2 + |\varphi_{23}|^2) + 2Re(\varphi_{14})^2\right).
\end{aligned}
\tag{114}
$$

The above calculation demonstrates that we have 7 remaining massive Higgs bosons, with the 9 Goldstone bosons no longer present in our Lagrangian (as expected). Indeed, the number of degrees of freedom agrees with our prediction: 7 massive Higgs bosons , 9 massive gauge bosons and 12 massless gauge bosons equals 58 degrees of freedom, in agreement with the number of degrees of freedom before symmetry breaking.

For completeness purposes, let us write down the above potential without assuming the vev condition $\phi_0^{i\alpha}\phi_{0i\alpha} = \frac{{v_R}^2}{2} = \frac{\alpha_R^2}{\beta_{R1}+\beta_{R2}}$, since as showed at an earlier chapter, considering the full Lagrangian (23) we would find a different condition on our vevs. Repeating the above calculation therefore, we find that

$$\begin{aligned} V(\phi) &= (-2\alpha_R^2 + v_R^2\beta_{R1})(|\varphi_{21}|^2 + |\varphi_{22}|^2 + |\varphi_{23}|^2) + (-2\alpha_R^2 + 3\beta_{R1}v_R^2 + 3\beta_{R2}v_R^2)Re(\varphi_{14})^2 \\ &+ (-2\alpha_R^2 + v_R^2\beta_{R1} + v_R^2\beta_{R2})(|\varphi_{11}|^2 + |\varphi_{12}|^2 + |\varphi_{13}|^2 + Im(\varphi_{14})^2) + |\varphi_{24}|^2), \end{aligned} \tag{115}$$

and we note that indeed assuming the vev condition written above, we retrieve our result in (114).

B Model 1 Electroweak Symmetry Breaking Pattern

Adapting the potential given in [19] for model 1, and ignoring only the final term (since we are interested in the simplest possible potential that reproduces the correct symmetry breaking pattern) we have that

$$\begin{aligned} V(\phi,\psi) = & \underline{-2\alpha_R^2\phi_{i\alpha}\phi^{i\alpha}} + \underline{\beta_{R1}(\phi_{i\alpha}\phi^{i\alpha})^2} + \underline{\beta_{R2}\phi_{i\alpha}\phi^{j\alpha}\phi_{j\beta}\phi^{i\beta}} \\ & - 2\alpha_L^2\psi_{i\alpha}\psi^{i\alpha} + \beta_{L1}(\psi_{i\alpha}\psi^{i\alpha})^2 + \beta_{L2}\psi_{i\alpha}\psi^{j\alpha}\psi_{j\beta}\psi^{i\beta} \\ & + \alpha_{LR1}\psi_{i\alpha}\psi^{i\alpha}\phi_{j\beta}\phi^{j\beta} + \alpha_{LR2}\psi_{i\alpha}\phi^{j\alpha}\psi^{i\beta}\phi_{j\beta}, \end{aligned} \tag{116}$$

with vevs ϕ_0 and ψ_0 in the forms given by (59) and (60).

Before going into the rather involved calculation, let us note some simplifying 'facts':

1. The underlined terms are the ones used in the potential (34) are wholly responsible for the initial symmetry breaking pattern. Thus, we shall enforce that the terms involving ϕ in the final potential predict the same particle spectrum for ϕ as the potential in (34) does.

2. To ensure the above, we will use our gauge freedom and work in the unitary gauge, which eliminates 9 components from ϕ, namely the ones corresponding to the Goldstone bosons.

3. Since the second line is in fact identical to the first, with the exception of $\phi \mapsto \psi$ and $R \mapsto L$, and since both vevs ϕ_0 and ψ_0 have the same form, we may skip the calculation for the second line and simply use the final answers from (114) when the condition $\alpha_{LR1} = -\alpha_{LR2}$ is assumed, and (115) otherwise, augmented appropriately.

4. We will denote by T_1 the term involving α_{LR1} and T_2 the term involving α_{LR2}.

5. The following is true for any complex numbers w, z:
   $zw + \bar{z}\bar{w} = 2(Re(z)Re(w) - Im(w)Im(z))$
   $z\bar{w} + \bar{z}w = 2(Re(z)Re(w) + Im(w)Im(z))$

6. Finally, as we've done before, we are only interested in the second order terms of our Higgs fields. Thus, the equal sign should be understood to only take into account those terms.

Then, letting $\psi = \Psi + \psi_0$ and $\phi = \varphi + \phi_0$, we have that

$$
\begin{aligned}
T_1 &= \alpha_{LR1}\psi_{i\alpha}\psi^{i\alpha}\phi_{j\beta}\phi^{j\beta} \\
&= \alpha_{LR1}(\Psi_{i\alpha} + \psi_{0i\alpha})(\Psi^{i\alpha} + \psi^{0i\alpha})(\varphi_{j\beta} + \phi_{0j\beta})(\varphi^{j\beta} + \phi^{0j\beta}) \\
&\approx \alpha_{LR1}((|\Psi_{11}|^2 + |\Psi_{12}|^2 + |\Psi_{13}|^2 + |\Psi_{14}|^2 + |\Psi_{21}|^2 + |\Psi_{22}|^2 + |\Psi_{23}|^2 + |\Psi_{24}|^2)\frac{v_R^2}{2} \\
&+ (|\varphi_{11}|^2 + |\varphi_{12}|^2 + |\varphi_{13}|^2 + |\varphi_{14}|^2 + |\varphi_{21}|^2 + |\varphi_{22}|^2 + |\varphi_{23}|^2 + |\varphi_{24}|^2)\frac{v_L^2}{2} \\
&+ \varphi_{14}\frac{v_L}{\sqrt{2}}(\Psi_{14}\frac{v_R}{\sqrt{2}} + \frac{v_R}{\sqrt{2}}\Psi^{14}) + \frac{v_L}{\sqrt{2}}\varphi^{14}(\Psi_{14}\frac{v_R}{\sqrt{2}} + \Psi^{14}\frac{v_R}{\sqrt{2}})) \\
&= \alpha_{LR1}((|\Psi_{11}|^2 + |\Psi_{12}|^2 + |\Psi_{13}|^2 + |\Psi_{14}|^2 + |\Psi_{21}|^2 + |\Psi_{22}|^2 + |\Psi_{23}|^2 + |\Psi_{24}|^2)\frac{v_R^2}{2} \\
&+ (|\varphi_{11}|^2 + |\varphi_{12}|^2 + |\varphi_{13}|^2 + |\varphi_{14}|^2 + |\varphi_{21}|^2 + |\varphi_{22}|^2 + |\varphi_{23}|^2 + |\varphi_{24}|^2)\frac{v_L^2}{2} \\
&+ \varphi_{14}v_L v_R Re(\Psi_{14}) + \varphi^{14}v_L v_R Re(\Psi_{14})) \\
&= \alpha_{LR1}(|\varphi|^2 + |\Psi|^2 + 2v_L v_R Re(\varphi_{14})Re(\Psi_{14})).
\end{aligned} \tag{117}
$$

Let us now move on to T_2. We find that

$$
\begin{aligned}
T_2 &= \alpha_{LR2}\psi_{i\alpha}\phi^{j\alpha}\psi^{i\beta}\phi_{j\beta} \\
&= \alpha_{LR2}(\Psi_{i\alpha} + \psi_{0i\alpha})(\varphi^{j\alpha} + \phi_0^{j\alpha})(\Psi^{i\beta} + \psi_0^{i\beta})(\varphi_{j\beta} + \phi_{0j\beta}) \\
&\approx \alpha_{LR2}(\frac{v_L^2}{2}(|\varphi_{14}|^2 + |\varphi_{24}|^2) + \frac{v_R^2}{2}(|\Psi_{14}|^2 + |\Psi_{24}|^2) \\
&+ \varphi_{1\alpha}\Psi^{1\alpha}\frac{v_L v_R}{2} + \varphi_{14}\frac{v_L v_R}{2}\Psi_{14} + \frac{v_L v_R}{2}\varphi^{1\alpha}\Psi_{1\alpha} + \frac{v_L v_R}{2}\Psi^{14}\varphi^{14}) \\
&= \alpha_{LR2}(\frac{v_L^2}{2}(|\varphi_{14}|^2 + |\varphi_{24}|^2) + \frac{v_R^2}{2}(|\Psi_{14}|^2 + |\Psi_{24}|^2) + v_L v_R(2Re(\varphi_{14})Re(\Psi_{14})) \\
&+ Re(\varphi_{11})Re(\Psi_{11}) + Im(\varphi_{11})Im(\Psi_{11}) + Re(\varphi_{12})Re(\Psi_{12}) \\
&+ Im(\varphi_{12})Im(\Psi_{12}) + Re(\varphi_{13})Re(\Psi_{13}) \\
&+ Im(\varphi_{13})Im(\Psi_{13})),
\end{aligned} \tag{118}
$$

where we have used condition number five about complex numbers to simplify the last 3 lines in T_2. Recalling that we know the non mixing terms for the Ψ field from condition number three, we now have all of the relevant terms to write down the full potential term (116), as centred around our new non-zero vevs ϕ_0 and ψ_0.

We find that

$$\begin{aligned}
V(\varphi+\phi_0,\Psi+\psi_0) \approx\ & (-2\alpha_R^2+v_R^2\beta_{R1})(|\varphi_{21}|^2+|\varphi_{22}|^2+|\varphi_{23}|^2)+(-2\alpha_R^2+3\beta_{R1}v_R^2+3\beta_{R2}v_R^2)Re(\varphi_{14})^2\\
&+(-2\alpha_R^2+v_R^2\beta_{R1}+v_R^2\beta_{R2})(|\varphi_{11}|^2+|\varphi_{12}|^2+|\varphi_{13}|^2+Im(\varphi_{14})^2)+|\varphi_{24}|^2)\\
&+(-2\alpha_L^2+v_L^2\beta_{L1})(|\Psi_{21}|^2+|\Psi_{22}|^2+|\Psi_{23}|^2)+(-2\alpha_L^2+3\beta_{L1}v_L^2+3\beta_{L2}v_L^2)Re(\Psi_{14})^2\\
&+(-2\alpha_L^2+v_L^2\beta_{L1}+v_L^2\beta_{L2})(|\Psi_{11}|^2+|\Psi_{12}|^2+|\Psi_{13}|^2+Im(\Psi_{14})^2)+|\Psi_{24}|^2)\\
&+\alpha_{LR1}((|\Psi_{11}|^2+|\Psi_{12}|^2+|\Psi_{13}|^2+|\Psi_{14}|^2+|\Psi_{21}|^2+|\Psi_{22}|^2+|\Psi_{23}|^2+|\Psi_{24}|^2)\frac{v_R^2}{2}\\
&+(Re(\varphi_{14})^2+|\varphi_{21}|^2+|\varphi_{22}|^2+|\varphi_{23}|^2)\frac{v_L^2}{2}+2v_Lv_RRe(\varphi_{14})Re(\Psi_{14}))\\
&+\alpha_{LR2}(\frac{v_L^2}{2}(Re(\varphi_{14})^2)+\frac{v_R^2}{2}(|\Psi_{14}|^2+|\Psi_{24}|^2)+2v_Lv_RRe(\varphi_{14})Re(\Psi_{14})\\
&+Re(\varphi_{11})Re(\Psi_{11})+Im(\varphi_{11})Im(\Psi_{11})+Re(\varphi_{12})Re(\Psi_{12})\\
&+Im(\varphi_{12})Im(\Psi_{12})+Re(\varphi_{13})Re(\Psi_{13})+Im(\varphi_{13})Im(\Psi_{13})).
\end{aligned} \tag{119}$$

Now let us use condition number two and implement our unitary gauge, whereby the components $Im(\varphi_{14}), Re(\varphi_{11}), Im(\varphi_{11})$,
$Re(\varphi_{12}), Im(\varphi_{12}), Re(\varphi_{13}), Im(\varphi_{13}), Re(\varphi_{24}), Im(\varphi_{24})$ all equal 0. Then, our potential simplifies to

$$\begin{aligned}
V(\varphi+\phi_0,\Psi+\psi_0) \approx\ & (-2\alpha_R^2+v_R^2\beta_{R1})(|\varphi_{21}|^2+|\varphi_{22}|^2+|\varphi_{23}|^2)+(-2\alpha_R^2+3\beta_{R1}v_R^2+3\beta_{R2}v_R^2)Re(\varphi_{14})^2\\
&+(-2\alpha_R^2+v_R^2\beta_{R1}+v_R^2\beta_{R2})(|\varphi_{11}|^2+|\varphi_{12}|^2+|\varphi_{13}|^2+Im(\varphi_{14})^2)+|\varphi_{24}|^2)\\
&+(-2\alpha_L^2+v_L^2\beta_{L1})(|\Psi_{21}|^2+|\Psi_{22}|^2+|\Psi_{23}|^2)+(-2\alpha_L^2+3\beta_{L1}v_L^2+3\beta_{L2}v_L^2)Re(\Psi_{14})^2\\
&+(-2\alpha_L^2+v_L^2\beta_{L1}+v_L^2\beta_{L2})(|\Psi_{11}|^2+|\Psi_{12}|^2+|\Psi_{13}|^2+Im(\Psi_{14})^2)+|\Psi_{24}|^2)\\
&+\alpha_{LR1}((|\Psi_{11}|^2+|\Psi_{12}|^2+|\Psi_{13}|^2+|\Psi_{14}|^2+|\Psi_{21}|^2+|\Psi_{22}|^2+|\Psi_{23}|^2+|\Psi_{24}|^2)\frac{v_R^2}{2}\\
&+(Re(\varphi_{14})^2+|\varphi_{21}|^2+|\varphi_{22}|^2+|\varphi_{23}|^2)\frac{v_L^2}{2}+2v_Lv_RRe(\varphi_{14})Re(\Psi_{14}))\\
&+\alpha_{LR2}(\frac{v_L^2}{2}(Re(\varphi_{14})^2)+\frac{v_R^2}{2}(|\Psi_{14}|^2+|\Psi_{24}|^2)+2v_Lv_RRe(\varphi_{14})Re(\Psi_{14})).
\end{aligned} \tag{120}$$

Finally, choosing to further simplify by allowing $\alpha_{LR1} = -\alpha_{LR2}$ (which gives us vev norms of the form (59) and (60)), then by collecting like terms, we find that our potential is given by

$$\begin{aligned} V(\varphi + \phi_0, \Psi + \psi_0) \approx 2\alpha_R^2 ((&\frac{\beta_{R1}}{\beta_{R1} + \beta_{R2}} - \frac{1}{2\alpha_R^2} \frac{\alpha_L^2}{\beta_{L1} + \beta_{L2}} - 1)(|\varphi_{21}|^2 + |\varphi_{22}|^2 + |\varphi_{23}|^2) + 2Re(\varphi_{14})^2)) \\ &+ 2\alpha_L^2 ((\frac{\beta_{L1}}{\beta_{L1} + \beta_{L2}} - 1)(|\Psi_{21}|^2 + |\Psi_{22}|^2 + |\Psi_{23}|^2) + 2Re(\Psi_{14})^2)) \\ &+ \alpha_{LR1} ((|\Psi_{11}|^2 + |\Psi_{12}|^2 + |\Psi_{13}|^2 + |\Psi_{21}|^2 + |\Psi_{22}|^2 + |\Psi_{23}|^2) \frac{\alpha_R^2}{\beta_{R1} + \beta_{R2}}, \end{aligned} \tag{121}$$

from which we may read off the correct number of non zero Higgs fields components that would be required to reproduce the broken standard model gauge group.

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