Imperial College London

Department of Physics

Gravitational Waves and their Primordial Power Spectrum from Inflation

Author:

Nikola Kolev (CID: 01055052)

Date: September 28, 2020

Abstract

With the construction of LIGO and VERGO, there has been a plethora of new research in to what gravitational waves may tell us about the universe. At the moment, the only information we have about the primordial gravitational waves comes from observing the CMB. Although unlikely to be seen by the first generation of observatories, it is important to make predictions of the primordial gravitational wave spectrum now before it has been measured. We look at the possible primordial gravitational waves produced by a non-specific theory of inflation. We start with a review of some of the theory for linearised General Relativity, first with a massless graviton and then also for the massive case. The linearised action is derived in each case for Minkowski and FRW spacetimes. We also delve a little into bimetric gravity's prediction for the primordial spectrum in a low energy limit using the dRGT Lagrangian.

Acknowledgements

My most important thanks has to go out to my parents who not only made this year possible, but my whole schooling up to this point. They have given me the chance to follow my desires at university and to do so without many of the stresses commonplace in students, whether that be from material or moral support.

It was a strange experience to write a dissertation while not being able to talk face-to-face with my supervisor and with a blackboard at hand. For her help in making this easier, for her rapid email responses, and the zoom conversations we had which helped my comprehension of the topic, my supervisor, Professor Claudia de Rham, also has my gratitude.

Lastly, a quick shout out to my course mates for making the year's load a little lighter.

Contents

Conventions

$\eta_{\mu\nu} = diag(-1, 1, 1, 1)$

Spatial indices are raised and lowered using the delta function.
$\delta_{ij}A^j = A_i \qquad \delta^{ij}B_j = B^i$

Therefore, we also define $\nabla^2 = \delta^{ij}\partial_i\partial_j$.

Differentiation with respect to conformal time η is denoted with a prime (a'(η)), while differention with respect to physical time t is given by a dot above the function ($\dot{a}$(t)).

We use units with $\hbar = c = 1$.

We use the reduced Planck mass defined as $M_{PL}^2 = \frac{1}{8\pi G}$.

1 Introduction

And in the beginning, God said “let there be metric perturbations”, and there were gravitational waves. They rippled and they rushed, filling the cosmos with the echoes of creation. Almighty beings may work in mysterious ways, but humanity has a knack at deciphering the Universe’s puzzles by speculation and observation. And now, a new era of astronomy is being ushered in, that of gravitational waves. First theorized more than 100 years ago by Einstein, it was not until 2015 that they were empirically confirmed. The coalescing of two black holes radiated the equivalent of three solar masses worth of energy [1]. While just this month, a paper was published detailing a merger that emitted the energy of seven and a half solar masses [2], the biggest explosion observed since the Big Bang. It is highly suspected that these mergers do not emit light of any form, so these events would have otherwise gone unnoticed. Luckily, we have new high precision instruments that allow us to observe the invisible and will continue to do so. Humanity is like an old man having just found his glasses, and his two lenses are named LIGO and VERGO.

An exciting prospect emerges when considering the very early stages of the Universe. It is well known that the earliest light visible today, the CMB, was not emitted until some 300,000 years after the Big Bang. Direct observation of the universe prior to this moment of recombination would be highly beneficial. There is a possibility that a CNB (cosmic neutrino background) may be observed, but here we focus on those very first gravitational waves (GWs) and what they may be able to tell us about our origin. More precisely, we will be interested in those remnants predicted by a theory of inflation. Unfortunately, current detectors do not have the capability to detect this background radiation, because they are too weak in the frequency band at which LIGO and VERGO operate. Although it is stronger at larger frequencies, ground based detectors would still have trouble detecting them since they then experience interference from other gravitational phenomenon such as the Coriolis effect. The lowest frequency GWs can be detected using the CMB since they generate E and B mode polarisations (E mode being parallel or perpendicular to the plane wave’s oscillation, and B mode being at a 45 degree angle to it) as the photons in the CMB travel from the surface of last scattering to us. Scalar perturbations can be detected from the CMB too as they cause temperature anisotropies in the CMB and E mode polarisations. Thus, B mode is a phenomenon unique to gravitational waves and any detected would be due to them alone. It is the power spectrum of the perturbations which is what gives us a physical prediction from the theory, as well as the tilt and tensor-to-scalar ratio. The power spectrum is a function of the momentum, and essentially gives the amplitude of the signal at a given frequency. It is given by the 2-point correlation function of the field, whether it be scalar, vector or tensor perturbations. In practice, this is done by quantising the field and finding the vev, and it is only the scalar and tensor that give a measurable contribution.

Inflation was first conceptualised in the late 70s, and it is now well-known as a possible solution to numerous problems in the current ΛCDM model of cosmology. The scale factor dictating the size of the universe was not only increasing during inflation, but the expansion was accelerating, pushed on by the negative pressure of a scalar field, called the inflaton. The increase was so fast in fact that what were small quantum fluctuations grew in to the seeds of structure that eventually became galaxies. We can use what is known about the cosmological constant to work out what properties the inflaton field must have in order to cause an approximately exponential expansion, namely $\rho = -p$. The inflaton is given a potential, which can be chosen to satisfy the condition needed for a negative pressure. Inflation was first devised to solve the magnetic monopole problem, which arose from Grand Unified Theories. It was predicted that in the time when the four fundamental forces coalesced in to one, stable yet heavy magnetic monopoles must have been made in abundance. If this occurred before inflation, then the density of these monopoles would have rapidly decreased and today be undetectable. Since then, it has also been shown to solve both the horizon and flatness problem which will be detailed later on in our discussion.

In spite of its enticing promises to solve so many problems, and some experimental evidence to back it, many still remain skeptical, as a good scientist always must. In part, this is because "inflation" as a theory is so malleable and hundreds of different variants exist, so finding one that fits to experiment is perhaps not so difficult. Nevertheless, we will explore the predictions of inflation without narrowing ourselves too much to one specific branch, as well as how these primordial gravitational waves were produced.

One can never have too many experimental predictions to test a theory with. To confirm inflation would have an astounding impact, not only in the world of physics. If inflation is true, then most likely so is eternal inflation – our early universe was just but a small part of the wider whole and outside our horizon, inflation is still occurring, creating more universes but all of them causally disconnected from us. An infinite number of universes will no doubt make waves in the fields of philosophy and theology. It even offers a relatively nice solution to the Fermi paradox.

The very first predictions were made by just perturbing the flat space metric, demanding the metric to be of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ kept small. In this model, it can be useful to think of $h_{\mu\nu}$ as a spin-2 field living in a flat, static and unrelated background. The expansion of Einstein's field equations to first order in the perturbations then reveal a wave equation for $h_{\mu\nu}$. However, to get there one must take advantage of General Relativity's diffeomorphism invariance, which at the linearised level takes the form of a gauge invariance, somewhat analogous to that found in electromagnetism. The symmetric tensor $h_{\mu\nu}$ has ten degrees of freedom (d.o.f), using this gauge invariance we will show that in fact there is only two. These are the two polarisations of gravity and are most apparent in what is called the transverse traceless gauge. Concepts such as polarization, mass and energy are harder to define if not working in a maximally symmetric spacetime such as Minkowski, but luckily since we are focusing on cosmology and more importantly inflation (which is approximately a de Sitter metric), we need not worry about this.

Another convenient way to break down the perturbations is using a helicity decomposition whereby we write the components of the tensor as irreducible representations of the SO(2) group. This includes scalars, transverse vectors and transverse traceless rank-2 tensors, which are all invariant under rotations around a fixed axis. The momentum of the gravitational wave is taken to be in the direction of this axis, which follows naturally from the wave equation obeyed by $h_{\mu\nu}$. A similar decomposition can be performed for the stress energy tensor which simplifies the final equations of motion for our gravitational waves. The vector perturbations are not discussed much in literature because they decay quickly and have little effect on the physics.

Since they are transverse and traceless, they only couple to the transverse and traceless tensor component of the stress-energy tensor, which is called the anisotropic stress. During inflation, the inflaton stress-energy has no anisotropic stress, meaning that there is no source for the GWs if we ignore the metric's self-interaction. At second order, the scalar metric perturbations can give rise to GWs, although the primordial GWs even at linear order are thought to be weaker than other sources, that the second order corrections are practically irrelevant for now.

Although LIGO gave the first direct observations of GW, in the eighties two neutron stars orbiting each other were observed to be slowing down due to the energy they were emitting as GWs [37]. With this we see the first flaw of the flat space toy model (apart from the obvious missing matter). With GWs themselves carrying energy, then they should bend the spacetime they travel through. To define the stress-energy tensor of gravity is rather ambiguous for somewhat similar reasons to electromagnetic radiation, although extra complications exist. An average must be taken over several wavelengths – no local energy density can be given since the curvature is changing rapidly. Perhaps a bigger problem, there is no curvature if we change to a local inertial frame at that point. It is evident that gravity is self-coupling then, and this is one of the reasons why it has proven so hard to quantise.

Although first predicted by GR, GWs may lead us to a new theory beyond. At the very least, they will rule out many of the competing theories which agree with GR in many other areas. Despite GR's undoubted success, we know it is not the final answer. However, at low energies at least, with each new theory rebuked, it seems ever more likely it is the correct theory. It may then seem somewhat pointless to try and correct it in the low energy regime, but it is an essential aspect of the scientific process. Without it, we would not be able to say with such conviction that it is **the** theory of gravity. Nevertheless, there are areas we would like to explore that are definitively **not** in the low energy regime, such as black hole singularities, and for these a new theory, likely quantum, is definitely needed.

An alternative that has gained a renewed lease of life recently is massive gravity and through it bi or even multi metric gravity. The mass of the graviton provides a Yukawa potential and so a decrease in strength over long distances, somewhat alluring in the world of cosmology where the source of "dark energy" is still badly understood. A massive graviton, with a minute mass, would have much the same properties of GR at short distance where the mass is unnoticeable. It is only at large cosmological lengthscales its effect will be seen. Knowing how to add a mass term to the linearised theory of GR has been known about for a long time now, as well as some of its main consequences: gauge invariance is broken much like in Proca theory; the d.o.f increase to 5 with a new vector mode and one new scalar mode. Beyond this, all attempts to add terms at any higher order were plagued with ghost instabilities (these are d.of that appear and have a negative kinetic energy).

A carefully constructed Lagrangian given by de Rham, Gabadadze, and Tolley [14][15], was able to work around these ghosts by introducing a second metric, usually dubbed $f_{\mu\nu}$, to a theory now known as dRGT gravity. In the original publications, the reference metric was just taken to be that of flat Minkowski space. However, if we choose to make it dynamical, even just introducing perturbations to the flat space metric, we are forced to add a kinetic term for it too. We already know what this needs to be, the Einstein-Hilbert action has been shown to be the unique non-linear kinetic term for a spin-2 field. In fact, in FRW spacetime, it is likely that the reference metric cannot be the Minkowski one, as it has been shown to lead to instabilities. The dRGT Lagrangian can also lead naturally to a cosmological constant term. Both metrics have their own “Planck mass”, M_{PL} for our original $g_{\mu\nu}$ and M_f for the reference metric. These dictate the strength with which the respective metric couples to matter and how the theory is interpreted. If $M_{PL} >> M_f$ then what we have is essentially the usual theory of GR, with a new as yet undiscovered spin-2 field. If we have the reverse, $M_{PL} << M_f$, then we find something very similar to massive gravity models with a single metric. When talking about bi-gravity theories, the two constants are usually taken to be roughly the same, $M_{PL} \approx M_f$.

We will begin with an introduction to gravitational waves, both massive and massless, in flat Minkowski. This is meant to outline some basic concepts, some of which are covered near the end of many introductory GR courses. FRW is then briefly looked over, since inflation does occur in an FRW spacetime. A large part is dedicated to inflation: the background and main ideas mentioned in the introduction are expanded upon, before concentrating on the scalar and tensor perturbations’ power spectrum. The limiting behaviours of the massive gravity Power Spectrum are then also derived. Lastly, we use the ADM formalism in FRW and linearise and then diagonalise the action for dRGT, resulting in a massive and massless mode. The power spectra derived in the previous section then allow us to quickly find the power spectra for the linear dRGT theory.

2 Gravitational waves in Minkowski

2.1 Deriving the action

The first work on gravitational waves was done by taking dynamic perturbions on flat Minkowski space. I will give a quick overview of them here, as well as introducing some concepts that will be useful for gravitational waves in a general spacetime. Our first assumption is that the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1}$$

Where $h_{\mu\nu}$ are the perturbation. Since we will be using the action to derive the equation of motion, which we desire to be of first order, we need the action to be of second order. With this in mind, we also give the inverse metric, to first order.

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{2}$$

Where we define $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$ and $h = \eta^{\mu\nu}h_{\mu\nu}$. This may seem like an unrealistic toy model at first, but once we are sufficiently far away from the source (say colliding black holes) then spacetime will be very close to flat, especially when considering the the curvature with respect to the size of existing measuring devices, such as LIGO. We begin with linearising the well-known and loved Einstein-Hilbert action.

$$S = \frac{M_{PL}^2}{2}\int d^4x\,\sqrt{-g}R \tag{3}$$

Where M_{PL} is the Planck mass. Appendix A contains the explicit working out for linearising the action in Minkowski spacetime, below we outline the steps and give the final answer in (5). We begin by finding the Christoffel symbols and from this the Riemann tensor. To help keep the workload tidier and so easier to follow, we first find the first order contributions to the Riemann tensor (and then curvature), before finding the second order contributions. The volume element must also be linearised, but only to first order, since we have no zeroth order contribution from the Riemann curvature. The resulting action is of the form

$$S = \frac{M_{PL}^2}{2} \int d^4x (R^{(1)} + \frac{1}{2} h R^{(1)} + R^{(2)}) \tag{4}$$

$R^{(1)}$ is actually a total derivative, so it goes to zero. We would also like to add a matter part to this action, which is done simply enough by adding a coupling term.

$$S = \frac{M_{PL}^2}{8} \int d^4x [\partial_\mu h_{\alpha\beta} \partial^\mu h^{\alpha\beta} - \partial_\mu h \partial^\mu h + 2\partial_\mu h^{\mu\nu} \partial_\nu h - 2\partial_\mu h^{\mu\nu} \partial^\rho h_{\rho\nu}] + \frac{1}{2} \int d^4x h_{\mu\nu} T^{\mu\nu} \tag{5}$$

2.2 Diffeomorphism invariance

Now that we have the action, it is simple enough to vary it and get the equation of motion. This action (after a rescaling of the field) is also known as the Fierz-Pauli action [24] and was shown to be the unique linear action for a massless, spin-2 field. It is important to note what happens with the diffeomorphism invariance of general relativity. We take an infinitesimal coordinate transformation, of the same order as the perturbation and use the transformation law for a tensor.

$$\begin{aligned} x'^\mu &= x^\mu + \xi^\mu \quad \text{with} \quad \mathcal{O}(\xi) \approx \mathcal{O}(h) \\ g'_{\mu\nu}(x') &= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \end{aligned}$$

For now we will work with a general background metric so that $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$, and switch to Minkowski at the end. We absorb any changes in the metric due to the coordinate transformation in to the perturbation. In the context of perturbation theory, it is common to call this infinitesimal diffeomorphism the gauge symmetry of the theory because of the clear similarities to the gauge transformation of electromagnetism. We linearise the diffeomorphism first, and then right it in the form of a gauge transformation (i.e. all terms in the equation are functions of the same coordinates).

$$\begin{aligned} g'_{\mu\nu}(x') &= \left(\delta^\rho_\mu - \frac{\partial \xi^\rho}{\partial x'^\mu}\right)\left(\delta^\sigma_\nu - \frac{\partial \xi^\sigma}{\partial x'^\nu}\right) \tilde{g}_{\rho\sigma}(x) + \left(\delta^\rho_\mu - \frac{\partial \xi^\rho}{\partial x'^\mu}\right)\left(\delta^\sigma_\nu - \frac{\partial \xi^\sigma}{\partial x'^\nu}\right) h_{\rho\sigma} + \mathcal{O}(h^2) \\ &= g_{\mu\nu}(x') - \xi^\alpha \partial_\alpha g_{\mu\nu}(x') - g_{\rho\nu}(x') \partial_\mu \xi^\rho - g_{\mu\sigma}(x') \partial_\nu \xi^\sigma + h_{\mu\nu} + \mathcal{O}(h^2) \\ &= g_{\mu\nu}(x') + h_{\mu\nu} - \tilde{\nabla}_\mu \xi_\nu - \tilde{\nabla}_\mu \xi_\mu + \mathcal{O}(h^2) \end{aligned} \tag{6}$$

Where the tilde on the covariant derivative signifies it is with respect to the background metric. Therefore, at linear order the perturbation transforms as

$$h_{\mu\nu}(x) \longrightarrow h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \tilde{\nabla}_\mu \xi_\nu - \tilde{\nabla}_\nu \xi_\mu \tag{7}$$

In Minkowski space, the covariant derivatives simply change to partial. It is simple enough to see that this is in fact a symmetry that leaves the action unchanged. By looking at the effect of a general Lorentz transformation, we see that we can treat the perturbation as a rank-2 Lorentz tensor living in flat Minkowski space (since the flat metric is obviously left unchanged while the perturbation changes). This can be useful when trying to define the energy momentum tensor of gravitational waves since it allows us to use field theoretical tools such as Noether's theorem. The treatment of gravitational waves can in many ways be done analogously to electromagnetic radiation. For instance, we can split it in to multipole radiation, since any source will be very far away from our detectors here on earth. We do not get any dipole radiation like we do for electromagnetism, there is only quadrupole and above. This can be seen to come from the graviton being a spin-2 particle, as we will motivate later on, and the photon spin-1.

2.3 Transverse traceless gauge in Minkowski

Now we turn to finding the equation of motion, we vary the action to find

$$-\Box h_{\mu\nu} + \eta_{\mu\nu}\Box h - \partial_\mu \partial_\nu h - \eta_{\mu\nu}\partial_\alpha \partial_\beta h^{\alpha\beta} + 2\partial_\mu \partial_\alpha h^\alpha{}_\nu = \frac{1}{M_{PL}^2} T_{\mu\nu} \tag{8}$$

This equation is somewhat complicated, so it would be useful to simplify. Luckily we have the gauge freedom to do so, and by doing so we can also find how many degrees of freedom the perturbations contain. We go down the standard route and go to the de Donder gauge, which is the equivalent of the Lorentz gauge.

$$\partial_\mu \bar{h}^\mu{}_\nu - \frac{1}{2}\partial_\nu h = 0 \tag{9}$$

Which can also be written using a redefined field

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{10}$$

as $\partial^\mu \bar{h}_{\mu\nu} = 0$. The redefined field transforms as $\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \partial^\rho \xi_\rho$. So if before the transformation we had $\partial^\mu \bar{h}_{\mu\nu} = f_\nu(x)$ then we need the ξ_μ to satisfy $\Box \xi_\mu = f_\mu(x)$. The D'Alembertian is of course an invertible operator and so we know that it is always possible to choose the de Donder gauge. The equation of motion now reads

$$\Box \bar{h}_{\mu\nu} = \frac{1}{M_{PL}^2} T_{\mu\nu} \tag{11}$$

From this we can finally see why we refer to these perturbartions as gravitational waves. When in a vacuum, (11) just becomes a wave equation. Whatsmore, the propagation speed is the same as that of light, so we conclude that the graviton is massless, at least in Einstein's general relativity. There is a residual gauge symmetry left over that we have not utilised yet. We just need to choose ξ_μ so that our de Donder condition is not affected. If we would like the de Donder condition to be kept, then the ξ_μ functions need to satisfy $\Box \xi_\mu = 0$. This is most useful outside the source since we can set to zero some of the components of the perturbation, as well as it's trace (thus setting $\bar{h}_{\mu\nu} = h_{\mu\nu}$). To do this, we need to solve 4 coupled differential equations.

$$\bar{h} + 2\partial^\mu \xi_\mu = 0 \tag{12}$$

$$\bar{h}_{0i} - \partial_0 \xi_i - \partial_i \xi_0 = 0 \tag{13}$$

$$(13) \implies \xi_i = \int dt (\bar{h}_{0i} - \partial_i \xi_0)$$

Substitute this in to (12)

$$\implies \bar{h} + 2\partial_0 \xi^0 + 2\int dt (\partial^i \bar{h}_{0i} - \partial_i \partial^i \xi_0) = 0$$

We can then use the de Donder gauge to make the substitutions $\partial_i \partial^i \xi^0 = \partial_0 \partial^0 \xi^0$ and $\partial^i \bar{h}_{0i} = -\partial_0 \bar{h}_{00}$.

$$\bar{h} - 2\bar{h}_{00} + 4\partial_0 \xi^0 = 0$$

$$\implies \xi^0 = -\tfrac{1}{4} \int dt (\bar{h} - 2\bar{h}^{00})$$

Showing that it is always possible to make this gauge choice as long as we are outside the source. Lastly, using the de Donder condtion we show that the h_{00} is not time-dependent.

$$\partial^\mu h_{\mu 0} = -\partial^0 h_{00} + \partial^i h_{i0} = \partial^0 h_{00} = 0$$

Gravitational waves are by definition taken to be time-dependent, so we ignore the (00) component of the tensor. What we are left with is called the **tranverse traceless gauge**,

(h_{ij}^{TT}). We have a free wave-equation that is non-trivial only in the spatial components. The solution of which is

$$h_{ij}^{TT} = A_{ij} e^{ik.x} \tag{14}$$

Where A_{ij} are polarisation tensors, and k^μ is the momentum of the gravitational wave. From the transverse condition we know that the polarisation tensors are non-zero only in the directions perpendicular to the momentum. We must also only take the real part of (14) just as we would in electromagnetism. Without loss of generality, we can take the propagation direction to be $\hat{z}$ so that

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} cos(k.x) \tag{15}$$

We see that there are two physical polarisations and we define their polarisation tensors as

$$e_{ij}^{+} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_{ij}^{\times} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

Which we can give a more general definition (i.e for gravitational waves travelling in any direction), using the unit vectors $\hat{u}$ and $\hat{v}$ which are perpendicular to each other and to the direction of propagation.

$$e_{ij}^{+} = \hat{u}_i \hat{u}_j - \hat{v}_i \hat{v}_j \quad e_{ij}^{\times} = \hat{u}_i \hat{v}_j + \hat{v}_i \hat{u}_j \tag{17}$$

These have been chosen so that they have the normalisation $e_{ij}^{A} e^{A',ij} = 2\delta^{AA'}$.

2.4 Transverse traceless (TT) with a general background

We show how to find the equivalent of the TT guage in a general spacetime. The full metric is written as $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$. The tilde represent the background metric, this will be FRW for us. We start from the linearised action of gravity in a general spacetime with all covariant derivatives being with respect to the background metric alone.

$$S = \frac{M_{PL}^2}{2} \int d^4x \sqrt{-\tilde{g}} [\nabla_\mu h_{\alpha\beta} \nabla^\mu h^{\alpha\beta} - \nabla_\mu h \nabla^\mu h + 2\nabla_\mu h^{\mu\nu} \nabla_\nu h - 2\nabla_\rho h_{\mu\nu} \nabla^\nu h^{\rho\mu}] \tag{18}$$

Changing once again to $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}h$, where $\tilde{g}_{\mu\nu}$ is the background metric, we get

$$S = \frac{M_{PL}^2}{2}\int d^4x\,\sqrt{-\tilde{g}}[\nabla_\mu \bar{h}_{\alpha\beta}\nabla^\mu \bar{h}^{\alpha\beta}\bar{h} - \frac{1}{2}\nabla_\mu \bar{h}\nabla^\mu \bar{h} - 2\nabla_\mu \bar{h}^{\mu\nu}\nabla_\nu \bar{h} - 2\nabla_\mu \bar{h}^{\mu\nu}\nabla^\rho \bar{h}_{\rho\nu}] \tag{19}$$

Since

$$\begin{aligned} \nabla_\mu h \nabla^\mu h &= \nabla_\mu \bar{h}\nabla^\mu \bar{h} & \nabla_\mu h_{\alpha\beta}\nabla^\mu h^{\alpha\beta} &= \nabla_\mu \bar{h}_{\alpha\beta}\nabla^\mu \bar{h}^{\alpha\beta} \\ \nabla_\rho h_{\mu\nu}\nabla^\nu h^{\rho\mu} &= \nabla_\rho \bar{h}_{\mu\nu}\nabla^\nu \bar{h}^{\rho\mu} + \frac{1}{4}\nabla_\rho h \nabla^\rho h & \nabla_\mu h^{\mu\nu}\nabla_\nu h &= -\nabla_\mu \bar{h}^{\mu\nu}\nabla_\nu \bar{h} + \frac{1}{2}\nabla_\mu \bar{h}\nabla^\mu \bar{h} \end{aligned} \tag{20}$$

Next, we switch to the de Donder gauge, $\nabla_\mu \bar{h}^{\mu\nu} = 0$.

$$\begin{aligned} \bar{h}'_{\mu\nu} &= \bar{h}_{\mu\nu} - 2\nabla_{(\mu}\xi_{\nu)} + \tilde{g}_{\mu\nu}\nabla_\mu \xi^\mu \\ \nabla_\mu \bar{h}'^{\mu\nu} &= 0 \implies 0 = \nabla_\mu \bar{h}^{\mu\nu} - \nabla^\mu \nabla_\mu \xi_\nu - \nabla^\mu \nabla_\nu \xi_\mu + \nabla_\nu \nabla_\mu \xi^\mu \end{aligned} \tag{21}$$

Which must be solved for ξ_μ. We apply this condition to the action and find the equations of motion.

$$\nabla_\mu \nabla^\mu \bar{h}_{\alpha\beta} - 2\nabla^\mu \nabla_{(\alpha}\bar{h}_{\beta)\gamma} - \frac{1}{2}\tilde{g}_{\alpha\beta}\nabla_\mu \nabla^\mu \bar{h} = 0 \tag{22}$$

If we take take the trace, we find that

$$\nabla_\mu \nabla^\mu \bar{h} = 0 \tag{23}$$

So if we impose $\bar{h} = 0$ on some hypersurface, then it remains traceless on all other hypersurfaces if the spacetime is globally hyperbolic [10]. Lastly, the transverse and traceless conditions together allow us to simplify [34] our equations of motion to

$$\nabla_\mu \nabla^\mu h_{\alpha\beta} = 0 \tag{24}$$

2.5 Helicity Decomposition

Our equation of motion, together with the de Donder gauge, implies that we have a conserved energy momentum tensor, although we could also demand this from the theory ourselves. However, this can only be true if the energy and matter responsible for $T_{\mu\nu}$ do not interact with gravity, emitting no gravitational waves. In reality, the conserved quantity would actually be that of the energy and matter, as well as the gravitational field, $T_{\mu\nu} + t^{(2)}_{\mu\nu}$, and so our equation of motion becomes $\Box h_{\mu\nu} = \frac{1}{M^2_{PL}}(T_{\mu\nu} + t^{(2)}_{\mu\nu})$. A new problem now arises that the extra term in the equation of motion requires an extra term in the action. More specifically, $t^{(2)}_{\mu\nu}$ is of second order in h, and so we require a term that is third order in h in the action, implying self interactions between gravitons. This should not be so surprising, seeing as how the full theory of general relativity is known to be highly non-linear. There is now also a third term that appears on the right hand side of the equation of motion. The term that is third order in h in the action then adds another term to the energy-momentum tensor associated with gravity, $t^{(3)}_{\mu\nu}$, through Noether's theorem. One can see this procedure will continue forever but there does exist a resummation algorithm which reproduces Einstein's General Relativity. The extra terms in the action must also change the form of the gauge transformation needed to keep the action invariant (i.e we can no longer cut it off at the linear level). Again, this perhaps should be expected since the full theory has a diffeomorphism invariance, rather than the linear gauge transformation given here.

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) - \frac{1}{M^2_{PL}} \mathcal{O}(h\partial\xi) \tag{25}$$

This complicates things when we try to do a similar gauge transformation to the tranverse traceless gauge in a spacetime such as FRW. One way to work around this is to give a helicity decomposition of the metric perturbations, i.e. split up $h_{\mu\nu}$ in to irreducible representations of SO(3) (for our use only the scalar, vector and tensor are required). Then since we know that gravitational wave are tranverse and traceless in flat spacetime we expect the same in FRW. The helicity decomposition in Minkowski is actually very similar to that of FRW too. The metric is given by (1) again, and we write the perturbation as in (26) [32].

$$\begin{aligned} h_{00} &= 2\psi \\ h_{0i} &= \beta_i + \partial_i \gamma \\ h_{ij} &= -2\phi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\lambda + \frac{1}{2}(\partial_i\epsilon_j + \partial_j\epsilon_i) + h^{(TT)}_{ij} \end{aligned} \tag{26}$$

∇^2 is the flat space Laplacian, and the quantities β_i, ϵ_i and $h^{(TT)}_{ij}$ are transverse. $h^{(TT)}_{ij}$ is also traceless. We have simplified the expression further by splitting the vector in to a longitudinal and transverse part, and have made a similar decomposition for the tensor. So, it seems that we have split the perturbation in to 4 scalars, 2 vectors and 1 tensor. In full, this is actually a decomposition in to irreducible representations of SO(2) (so they are invariant under rotations around some fixed axis **k** which turns out to be the momentum of the field). There are certain boundary conditions we must impose at spatial infinity so that the expressions given in (26) are invertible.

$$\gamma \longrightarrow 0 \ , \quad \lambda \longrightarrow 0 \ , \quad \nabla^2\lambda \longrightarrow 0 \ , \quad \epsilon_i \longrightarrow 0 \tag{27}$$

When finding inverse expressions for the fields in terms of the metric, only ψ and ϕ are local functions of the metric perturbation. The remaining fields involve the inverse Laplacian, which is not a local operator, and also the reason why we require these boundary conditions. We have not yet used our gauge freedom, which will allow us to actually reduce the d.o.f to six. This may seem strange at first after our discussion in 2.3, but the extra four degrees of freedom obey Poisson's equations and so are non-radiative. At a linear level, the gravitational field itself is taken not to be a source of gravitational waves, so the non-radiative d.o.f do not play a role there, but they are still important. For instance, the 2 scalar d.o.f play a major role in inflation to produce the small inhomogeneities which lead to galaxy formation. The gauge functions ξ_μ are also written in the form

$$\xi_0 = A \ , \quad \xi_i = B_i + \partial_i C \tag{28}$$

From this we can find out how the helicity variables transform directly.

$$\begin{gathered} h_{00} = 2\psi \longrightarrow 2\psi - 2\partial_0 A \\ h_{0i} = \beta_i + \partial_i\gamma \longrightarrow (\beta_i - \partial_0 B_i) + \partial_i(\gamma - \partial_0 C - A) \\ h_{ij} = -2\phi\delta_{ij} + (\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2)\lambda + \tfrac{1}{2}(\partial_i\epsilon_j + \partial_j\epsilon_i) + h_{ij}^{(TT)} \longrightarrow \\ -2(\phi - \nabla^2 C)\delta_{ij} + (\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2)(\lambda - 2C) + \partial_{(i}(\epsilon - B)_{j)} + h_{ij}^{(TT)} \end{gathered}$$

Therefore, we get

$$\begin{gathered} \psi' = \psi - \partial_0 A \ , \quad \beta_i' = \beta_i - \partial_0 B_i \ , \quad \gamma' = \gamma - \partial_0 C - A \ , \quad \phi' = \phi - \nabla^2 C \\ \lambda' = \lambda - 2C \ , \quad \epsilon_i' = \epsilon_i - 2B_i \ , \quad h_{ij}^{(TT)\prime} = h_{ij}^{(TT)} \end{gathered} \tag{29}$$

Unsurprisingly, $h_{ij}^{(TT)}$ is completely unchanged by the gauge transformation. This can also be predicted by considering that we have decomposed the gauge functions in to SO(2) irreps, a scalar and a vector. Since $h_{ij}^{(TT)}$ is a tensor, the scalar and vector functions will have no effect on it. From these quantities we may form 3 more gauge invariant quantities that are preferable to work with. There exist two scalar fields (and therefore 2 d.o.f)

$$\Phi = -\phi - \frac{1}{6}\nabla^2\lambda \tag{30}$$

$$\Psi = \psi - \partial_0\gamma + \frac{1}{2}\partial_0\partial_0\lambda \tag{31}$$

and one vector field (therefore 2 d.o.f)

$$\Xi_i = \beta_i - \frac{1}{2}\partial_0\epsilon_i \tag{32}$$

Of course, we also have the transverse, traceless tensor too, which has two d.o.f. So using the four functions ξ_μ, we can eliminate 4 of the 10 degrees of freedom. These gauge invariant quantities are called the Bardeen variables when used in FRW [6].

Another useful aspect about using this helicity decomposition is that the equations of motion can also be split up according to helicity. Only the TT part of the energy-momentum tensor can effect the TT part of the metric perturbation. The decomposition of the energy-momentum tensor is quite similar to $h_{\mu\nu}$, seeing as they are both rank-2 symmetric tensors.

$$T_{00} = \rho \tag{33}$$

$$T_{0i} = S_i + \partial_i S \tag{34}$$

$$T_{ij} = p\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + \partial_{(i}\sigma_{j)} + \sigma_{ij}^{(TT)} \tag{35}$$

Where the equivalent fields obey the same boundary conditions at spatial infinity, and the same transverse conditions. The reason why $h_{ij}^{(TT)}$ is sourced only by the TT part of the energy-momentum is plain to see in this decomposition. In the coupling term, $h_{ij}^{(TT)}$ could only possibly couple to T^{ij}.

$$\begin{aligned} h_{ij}^{(TT)}T^{ij} &= h_{ij}^{(TT)}(p\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + \partial_{(i}\sigma_{j)} + \sigma_{ij}^{(TT)}) \\ &= h_{ij}^{(TT)}\partial^i\partial^j\sigma + h_{ij}^{(TT)}\partial^i\sigma^j + h_{ij}^{(TT)}\sigma^{(TT),ij} \end{aligned} \tag{36}$$

The first two terms go to zero after we integrate by parts since the tensor is transverse. Although this was done in Minkowski, as mentioned earlier, a nearly identical decomposition is possible in FRW with the same result that the tensor part responsible for gravitational waves only couples to the TT part of the stress tensor (the anisotropic stress). For a generic tensor, we can project it on to its TT part using the projector defined as

$$\Lambda_{ij,kl}(\hat{n}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \tag{37}$$

where $P_{ij}(\hat{n}) = \delta_{ij} - n_i n_j$. This definition is transverse and traceless on (i,j) or (k,l) by construction. It also has the property that $\Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}$ and are symmetric upon switching of the two pairs of indices.

2.6 Gravitational waves from Massive Gravity

Later on, we will be investigating the theorised primordial gravitational waves left over from inflation, and will also be comparing these to the predictions from massive gravity. It will be useful to first take a look at massive gravity within Minkowski spacetime. Massive gravity has been studied for a while, with Fierz and Pauli first producing the unique linear action for a spin-2 massive field [24] back in 1939. There are two possible terms one may choose for the mass term in the linear theory without a second "reference" metric [1] being used: h^2 and $h_{\mu\nu}h^{\mu\nu}$. We can have just one of these or a linear combination of the two, but it was shown by Pauli and Fierz that the only combination that does not produce ghosts is $h^{\mu\nu}_{\mu\nu} - h^2$. Ghosts are particles with negative kinetic energy terms in the Lagrangian, which causes instabilities in the theory and especially the vacuum, which can decay in to particles of positive and negative kinetic energy indefinitely. A major consequence of the adding a mass to the theory is that we lose the gauge invariance. At first, it may seem that we have gained 8 degrees of freedom (d.o.f) but thanks to the equations of motion, we can actually still reduce them to

[1] If one allows for a theory that has two metrics then we can use the second one to contract the indices of the perturbation, giving us a few extra options here.

5, in a similar way that one may do with the Proca Lagrangian of a massive spin-1 field.

$$S_{MG} = \frac{M_{PL}^2}{8}\int d^4x\big[\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \partial_\mu h\partial^\mu h + 2\partial_\mu h^{\mu\nu}\partial_\nu h - 2\partial_\mu h^{\mu\nu}\partial^\rho h_{\rho\nu} + m^2(h_{\mu\nu}h^{\mu\nu} - h^2)\big] + \frac{1}{2}\int d^4x h_{\mu\nu}T^{\mu\nu} \tag{38}$$

The coupling to the stress energy tensor could actually have a dimensionless coupling constant, but we set it to one here. One way to find the coupling constant would be to find the weak field limit to massive gravity and match it up to Newton's potential. However, with massive gravity, the weak field limit is broken much easier than in massless and it cannot be used for solar system dynamics. Since the only difference between this action and that of the massless theory is an extra two terms proportional to m^2, it is easy to see that the only difference in the equations motion will also be due to the mass term. We make the same substitution for $\bar{h}$ as before and take the divergence, again demanding that the source is conserved, $\partial^\mu T_{\mu\nu} = 0$. Since any term that was also in the massless equations of motion go to zero, a condition similar to the de Donder condition drops out from the equations.

$$\partial_\mu(h^{\mu\nu} - \eta^{\mu\nu}h) = 0 \tag{39}$$

Which is equivalent to the de Donder if we have a zero trace. If we take the trace instead of the divergence

$$2\partial_\nu\partial_\mu(h^{\mu\nu} - \eta^{\mu\nu}h) = -T - 3m^2 h \implies h = -\frac{1}{3m^2}T \tag{40}$$

Equation (40) along with the condition (39) take away five of the d.o.f leaving the massive graviton with five instead of the two of the massless. The field can be decomposed in to a helicity-2 rank-2 tensor (with two d.o.f), helicity-1 vector (with two d.o.f) and a helicity-0 scalar (with one d.o.f) using the Stuckelberg fields [15]. Once we have introduced the mass term to Gravity, we no longer have Gauge invariance. We can formally bring it back by introducing the Stuckelberg fields A_μ and π. We replace $h_{\mu\nu}$ with:

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu\chi_\nu + \partial_\nu\chi_\mu \tag{41}$$

Where we define $\chi_\mu = \frac{1}{m}A_\mu + \frac{1}{m^2}\partial_\mu \pi$.

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \frac{1}{m}\partial_\mu A_\nu + \frac{1}{m}\partial_\nu A_\mu + \frac{1}{m^2}\partial_\mu\partial_\nu\pi + \frac{1}{m^2}\partial_\nu\partial_\mu\pi \tag{42}$$

To find the new action, we only need to worry about what happens to the mass term since the field redefinition is of the form of the gauge transformation and so clearly leaves the kinetic term (i.e. the massless action) unchanged.

$$\begin{aligned} h_{\mu\nu}h^{\mu\nu} &\longrightarrow h_{\mu\nu}h^{\mu\nu} + \frac{4}{m^2}\partial_{(\mu}A_{\nu)}\partial^{(\mu}A^{\nu)} + \frac{4}{m}h_{\mu\nu}\partial^{(\mu}A^{\nu)} + \frac{4}{m^2}h_{\mu\nu}\partial^{(\mu}\partial^{\nu)}\pi \\ &\quad + \frac{8}{m^3}\partial_{(\mu}A_{\nu)}\partial^{(\mu}\partial^{\nu)}\pi + \frac{4}{m^4}\partial_{(\mu}\partial_{\nu)}\pi\partial^{(\mu}\partial^{\nu)}\pi \\ h^2 &\longrightarrow h^2 + \frac{4}{m}h\partial_\mu A^\mu + \frac{4}{m^2}h\Box\pi + \frac{4}{m^2}(\partial_\mu A^\mu)^2 + \frac{8}{m^3}\partial^\mu A_\mu \Box\pi + \frac{4}{m^4}(\Box\pi)^2 \end{aligned} \tag{43}$$

Adding these together, defining $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ and $S_{m=0}$ as the original linearised massless action (i.e. equation (5) without the $T_{\mu\nu}$ term), then the new action is

$$\begin{aligned} S = S_{m=0} + \frac{M_{PL}^2}{2}\int d^4x\,\sqrt{-g}\big[&-\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + F^{\mu\nu}F_{\mu\nu} - 2m(h_{\mu\nu}\partial^\mu A^\nu - \partial^\mu A_\mu) \\ &- 2(h_{\mu\nu}\partial^\mu\partial_\nu\pi - h\partial^2\pi)\big] \end{aligned} \tag{44}$$

We can see we have brought back the gauge symmetry of the massless action if we also change the vector field at the same time. A gauge transformation of the form $h_{\mu\nu} \longrightarrow h_{\mu\nu} + \frac{2}{m}\partial_{(\mu}\xi_{\nu)}$ will produce a change in the action that is of the same form as the vector terms in (44), so to preserve the action, we must also change $A_\mu \longrightarrow A_\mu - \xi_\mu$ so as to cancel these terms. There is also a gauge symmetry that is of the same form as that of electromagnetism.

$$A_\mu \longrightarrow \tfrac{1}{m}\partial_\mu\lambda \qquad \pi \longrightarrow \pi - \lambda$$

There is one last field redefinition we would like to do so that the kinetic terms of $h_{\mu\nu}$ and π are not mixing.

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \eta_{\mu\nu}\pi \tag{45}$$

This redefinition is no longer of the form of the gauge symmetry of the massless action, so it too will change due to this. The total change in the action (denoted S_Δ) is

$$S_\Delta = \frac{M_{pl}^2}{2}\int d^4x\,\sqrt{-g}[m^2(3h\pi + 6\pi^2) + m\pi\partial^\mu A_\mu - 2\partial_\mu\pi\partial^\mu\pi + 2(h_{\mu\nu}\partial^\mu\partial^\nu\pi - h\partial^2\pi)] \tag{46}$$

The last term cancels the mixing of the kinetic terms of $h_{\mu\nu}$ and π in the old action, which is exactly why we introduced this redefinition. Our final action now reads

$$\begin{aligned} S = S_{m=0} + \frac{M_{PL}^2}{2}\int d^4x\,\sqrt{-g}\big[-\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + F^{\mu\nu}F_{\mu\nu} - 2m(h_{\mu\nu}\partial^\mu A^\nu - \partial^\mu A_\mu) \\ + m^2(3h\pi + 6\pi^2) + m\pi\partial^\mu A_\mu - (\partial_\mu\pi)(\partial^\mu\pi)\big] \end{aligned} \tag{47}$$

and has a gauge symmetry of the form

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \tfrac{2}{m}\partial_{(\mu}\xi_{\nu)} + \eta_{\mu\nu}\lambda \qquad \pi \longrightarrow \pi - \lambda$$
$$A_\mu \longrightarrow A_\mu - \xi_\mu + \tfrac{1}{m}\partial_\mu\lambda$$

We see that we have split the tensor part in to three different fields, and with it we have also split the five propagating d.o.f between the three fields. The tensorial part $h_{\mu\nu}$ (helicity-2 with two degrees of freedom), the vector part A^μ (helicity-1 with two degrees of freedom), and the scalar part π (helicity-0 with one degree of freedom).

From here we can make a gauge transformation so that $h_{0i} = 0$. Remembering that in a vacuum $h = 0$ for the massive theory, and from the equations of motion that the tensor part is transverse.

$$\begin{aligned} h'_{0i} = 0 \implies h_{0i} + \frac{1}{m}\partial_0\xi_i + \frac{1}{m}\partial_i\xi_0 = 0 \\ \xi_i = -\int dt(mh_{0i} + \partial_i\xi_0) \end{aligned} \tag{48}$$

We can simply choose $\xi_0 = 0$ and so there is a single integral left to do. Note that h_{00} can be shown to be time independent once again in the same way as we did for the massless case. The Stuckelberg fields correspond to different polarisations of the graviton. In massless gravity, we only have the two associated with the the tensor, and they squeeze and stretch the spacetime perpendicularly to their direction of travel. Massive gravity actually gets an extra four propagating d.o.f but one of the scalar modes is a Boulware-Deser ghost [8] that is not present if the correct action is used (the Fierz-Pauli at the linear level).

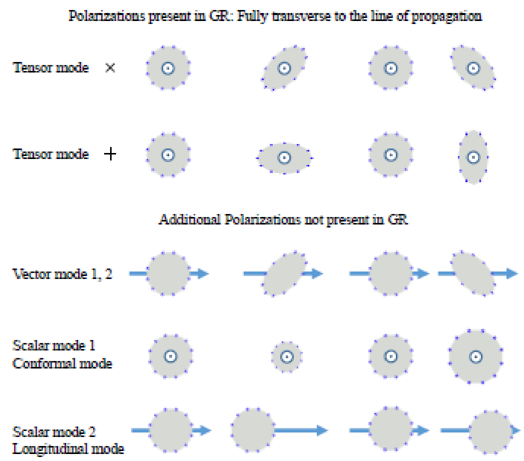

Figure 1: The different polarisations of gravitational waves for massive gravity. The top two are also present in massless gravity and are coming out of the page. The vector modes have the direction of propagation to the right and the scalars are also coming out of the page. The bottom one is actually a ghost and is not present. [13]

2.7 Stress-energy tensor of the gravitational waves

There are a few different ways to derive the stress-energy tensor (SET) for gravitational waves but they all give the same result. When in a general spacetime, techinically we should find the SET by varying the action of the perturbations with respect to the background metric. We can also treat the perturbations as a Lorentz covariant field living within Minkowski spacetime, and take advantage of the field theory methods developed by Noether. For each continuous symmetry of the action, we know there exists a conserved current. We take the derivative of the Lagrangian of $h_{\mu\nu}$ with respect to the coordinates.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\mu} &= \frac{\partial \mathcal{L}}{\partial h_{\nu\rho}} \partial_\mu h_{\nu\rho} + \frac{\partial \mathcal{L}}{\partial(\partial_\sigma h_{\nu\rho})} \partial_\sigma \partial_\mu h_{\nu\rho} \\
\frac{\partial \mathcal{L}}{\partial x^\mu} &= \partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\sigma h_{\nu\rho})} \right) \partial_\mu h_{\nu\rho} + \frac{\partial \mathcal{L}}{\partial(\partial_\sigma h_{\nu\rho})} \partial_\sigma \partial_\mu h_{\nu\rho} \\
\implies \partial_\sigma & \left(\delta^\sigma_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\sigma h_{\nu\rho})} \partial_\mu h_{\nu\rho} \right) = 0
\end{aligned} \tag{49}
$$

It follows that our conserved canonical SET is

$$T^{\sigma}_{\mu} = \delta^{\sigma}_{\mu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_{\sigma} h_{\nu\rho})}\partial_{\mu} h_{\nu\rho} \tag{50}$$

If we are to ask what the energy density is at a specific point, we will get an ambiguous answer. This is integral to the wave-like nature of the perturbations and to be expected. So, a spatial average must be taken over a few wavelengths to get a sensible result. We also apply the transverse traceless gauge to the Lagrangian, although this must be done after the differentiation. In the spatial average we are allowed to integrate by parts, so the the spatial average of the Lagrangian is zero (four terms are zero due to the gauge, and the last one due to integration by parts and use of the equation of motion, $\Box h_{\mu\nu} = 0$. The derivative of the Lagrangian also becomes simpler after noticing that even after differentiation, four of the terms will go to zero due to the fact they are quadratic in terms that are zero in this gauge. This means we only need to worry about a single term in the Lagrangian.

$$\begin{aligned} &\frac{\partial}{\partial(\partial_{\mu} h_{\rho\sigma})}(\kappa \partial_{\alpha} h_{\beta\gamma} \partial^{\alpha} h^{\beta\gamma}) = 2\kappa \partial_{\mu} h_{\rho\sigma} \\ &\implies T^{\sigma}_{\mu} = 2\kappa \langle \partial_{\mu} h_{\rho\nu} \partial^{\sigma} h^{\rho\nu} \rangle \end{aligned} \tag{51}$$

The average over multiple wavelengths is somewhat similar to what is found in electromagnetism for rapidly changing fields. A local energy-density makes no sense. In general relativity, this is even more pronounced since we cannot define a local energy-density even when looking at slowly varying gravitational fields. This is due to the fact that we can always go to normal coordinates at any point so that the spacetime is flat and thus the energy density seems to be zero.

3 FRW

3.1 Single-fluid

Since we are interested in gravitational waves from inflation we will cover a little about FRW spacetime here. More precisely, we will look at the ΛCDM model. This has 3 main eras, each dominated by a different form of matter. The first, when nearly everything was relativistic, was the radiation dominant (RD) era, then the matter dominant (MD) era, and lastly the dark-energy (ΛD) era. Currently, we are in the transition between MD and ΛD. The metric is the well known Friedmann–Robertson–Walker (FRW).

$$ds^2 = -dt^2 + a^2(t)\left(\frac{1}{1-kr^2}dr^2 + r^2 d\Omega_2^2\right) \tag{52}$$

Where k is the curvature and can take values k=-1,0,1 depending on the spatial curvature. From experiment, it seems that k=0 to a very high confidence level [7] so we will assume that from here on for simplicity. Before any perturbations, the energy-momentum tensor is just that of a perfect fluid. This is inevitable, since a homogeneous space requires that the energy and matter is also homogeneous.

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{53}$$

With p being the pressure of the fluid and ρ the energy density. Ignoring the cosmological constant, the Einstein equations then give us

$$\text{for the (0,0) component: } H^2 = \frac{1}{3M_{PL}^2}\rho \tag{54}$$

$$\text{for the (i,j) component: } 2\dot{H} + 3H^2 = -M_{PL}^2 p \tag{55}$$

Where a dot signifies differentiation with respect to the coordinate time, t, and we have defined the Hubble parameter as $H = \frac{\dot{a}}{a}$. If the universe was not spatially flat, then equation (54) would include a curvature term. If the energy density in (54) is then defined as the critical density one would expect for a flat universe, then we have a neat way of measuring the curvature of the universe by comparing the Hubble parameter to the energy density of matter in the universe. To (54) and (55) the conservation of energy equation from the

Bianchi identity $\nabla_\mu T^{\mu\nu} = 0$ can be added.

$$\nu = 0 \implies \dot{\rho} + 3H(\rho + p) = 0 \tag{56}$$

An equation of state connecting p and ρ is also usually assumed so that we have enough information to solve the system of equations.

$$p(t) = w(t)\rho(t) \tag{57}$$

During an era dominated by one form of matter, we can approximate w by the w-value of the dominant form of matter. In inflation we would like a quasi-exponential expansion meaning $a(t) = e^{\alpha t}$ (normalised such that $a(t_0) = 1$) for some constant α.

$$\tfrac{\dot{a}}{a} = H \implies a = e^{Ht}$$

What value does the energy density take? We can look to what we know from the cosmological constant. It too causes an exponential expansion. The equation of state for the cosmological constant is w=-1.

$$\begin{aligned} \dot{\rho} &= -3H(1+w)\rho = 0 \\ &\implies \rho = const. \end{aligned}$$

So, rather strangely, it seems that ”energy” is being created as the universe expands.

3.2 Comoving coordinates and the horizon scale

We define the comoving set of coordinates for FRW. There is not much difference except for the time which now absorbs the expansion factor to give

$$d\eta = \int \frac{1}{a} dt \tag{58}$$

$$\implies ds^2 = a(\eta)^2 \eta_{\mu\nu} dx^\mu dx^\nu \tag{59}$$

with $dx^0 = d\eta$. In these coordinates

$$\mathcal{H} = \frac{a'}{a} = \frac{1}{a} H \tag{60}$$

These coordinates turn out to be useful since an observer that begins as stationary, stays stationary for the rest of time, thus why they are named ”comoving”. Usually, we consider things like galaxies to be stationary within these coordinates, although not exactly true for many, especially the smaller ones.
The horizon scale is a very important concept in cosmology. It dictates what is considered important since it defines the characteristic length scale (or time scale) of the FRW universe.

$$L_H = H^{-1} \tag{61}$$

This helps define two important regimes when looking at gravitational waves, or cosmology in general. The condition for the super-horizon regime (i.e. outside the horizon) is

$$\frac{\lambda}{2\pi} >> \mathcal{H}^{-1} \iff k << \mathcal{H} \tag{62}$$

where k is the momentum in the comoving frame and $\lambda = \frac{2\pi}{k}$. Whereas the sub-horizon limit, when we are firmly inside the horizon

$$\frac{\lambda}{2\pi} << \mathcal{H}^{-1} \iff k >> \mathcal{H} \tag{63}$$

Since $\mathcal{H} = \frac{a'}{a}$, we expect $\mathcal{H} \sim \mathcal{O}(\eta^{-1})$, except of course if $a \propto \eta$, but in that case $\mathcal{H}$ will always be smaller than the absolute value of k. This is useful when we want to use the conditions (62) and (63) in practice. If we want to think about it in the usual coordinates with t instead of η, then we use the ”physical” wavelength $\lambda_{ph} = a(t)\lambda$. The condition is then $\frac{\lambda}{2\pi} >> H^{-1}$.

3.3 Multi-component fluid

In reality, if we want an accurate picture of the universe we need to take in to account all the different forms of matter: radiation(w=$\frac{1}{3}$), cold dark matter (w=0), ”dark energy” (w=-1). We still model all of these as perfect fluids, but now the Friedmann equations are with ρ_{tot} and p_{tot}.

$$\rho_{tot}(t) = \sum_\lambda \rho_\lambda(t) \tag{64}$$

$$p_{tot} = \sum_\lambda w_\lambda(t)\rho_\lambda(t) \tag{65}$$

$$w_{tot} = \frac{\sum_\lambda w_\lambda(t)\rho\lambda(t)}{\sum_\lambda \rho_\lambda \rho_\lambda} \tag{66}$$

So from the Bianchi identity, we the conservation of energy requires $\nabla_\mu T^{\mu\nu}_{tot} = 0$. The $\nu = 0$ component of this gives

$$\dot{\rho}_{tot} + 3H(p_{tot} + \rho_{tot}) = 0 \tag{67}$$

instead of conservation for each species separately. Importantly though, we can take the individual energy-momentum conservation if the interaction rate (Γ) that exchanges energy between species is small compared to H. This is significantly simplifies the problem and also means we can treat some fluids as totally separate apart from their gravitational interaction (e.g. neutrinos after decoupling) which is dictated by the background FRW metric anyway. We also define the energy fractions for the different fluids here

$$\Omega_R = \frac{\rho_R(t)}{\rho_C(t)} \qquad \Omega_M = \frac{\rho_M(t)}{\rho_C(t)} \qquad \Omega_\Lambda = \frac{\rho_\Lambda(t)}{\rho_C(t)} \qquad \rho_C(t) = \frac{3H^2}{8\pi G} \tag{68}$$

The subscripts R, M and Λ correspond to radiation, matter and dark energy respectively whereas ρ_C is the critical density. The Friedmann equations can be used to determine an expression for the energy densities up to some constant of integration. We then use the present day values of the energy fractions, and the present day value of the critical density to give explicit expressions for the energy densities.

$$\rho_\lambda(a) = \rho_0 \Omega_{\lambda,0} a^{-3(1+w_\lambda)} \tag{69}$$

So, for instancee, $\rho_M \propto a^{-3}$. Equation (54) similarly changes to

$$H^2 = \frac{1}{3M_{PL}^2}\rho_{tot} \tag{70}$$

If we decided to include curvature, the equation would gain a term proportional to k (the curvature) which can also be interpreted as an energy density.

$$H^2 = \frac{1}{3M_{PL}^2}(\rho_{tot} + \rho_k) \quad \text{where} \quad \rho_k = -3M_{PL}^2 \frac{k}{a^2(t)} \tag{71}$$

As mentioned earlier, we can use the deviation from the critical density to determine the curvature of the universe. We define the energy fraction for the curvature analogously to those in (68), (i.e. $\Omega_k(t) = \frac{\rho_k(t)}{\rho_C(t)}$). Current measurements give an upper bound of $|\Omega_k(t)| < 0.0094$ at 94% confidence level [7].

The radiation era is still more complex. The primordial soup at the beginning of RD was made up of all the particles in the Standard Model (and any particles we have yet to discover), and all of them were relativistic. As the universe expanded and cooled, different particles left the relativistic phase at different times, with the heavier ones such as top quarks going first. When a particle leaves the relativistic regime, it is referred to as annihilation, since they annihilate with their antiparticles to produce lighter particles (although there obviously was not a perfect symmetry between the number of particles and antiparticles). Since the lighter particles are still relativistic, there is not a big drop in ρ_R seen. Each species begins the transition when the temperature is roughly equal to its mass, but it is not an instantaneous transition.
Even though this era is called ”radiation” dominant, photons only make up a part of the total energy density. With the neutrino experiencing only the Weak force, it decouples almost completely from the other particles at around T=1Mev, producing a cosmic neutrino background akin to the CMB but from a much earlier time. From then on, we treat the neutrinos as a separate fluid since it only weakly interacts with the other. Within the first 10 seconds or so, the only non-relativistic particles left were the photons and neutrinos. In the Standard Model, the neutrino is massless and so would remain relativistic forever. However, experiments in recent years have shown the neutrino probably has a minute but real mass [25][4]. Electron-positron annihilation occurs after the neutrino decoupling, meaning that most of their energy was transferred only to photons. At these energies, photons were still highly-coupled to matter and there was exchange of energy. They are often considered as a single baryon-photon fluid and it remained like this for as long as there were many free electrons. Once the temperature dropped to a point where neutral atoms were forming, the photons were no longer strongly coupled to the matter and their mean free path increased significantly. This is known as recombination and is the origin of the CMB.

3.4 Gravitational Waves

We move on to finding the equation of motion for gravitational waves in FRW. The simplest way to find the action for massless gravitational waves in FRW is to carry out a helicity decomposition, similar to that done in section 2, and then focus solely on the transverse traceless tensor part that we know is responsible for the propagation. The metric is taken to be

$$g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}) \tag{72}$$

and we then decompose $h_{\mu\nu}$ in exactly the same way as in (26). The gauge transformation for curved space (derived in section one) can be used to reduce the variables to five. As in section one, since the gauge transformation functions ξ^μ do not actually have a transverse, traceless part, we already know that $h_{ij}^{(TT)}$ is going to be invariant and so is a real physical quantity.

$$g_{\mu\nu} = a^2(-1, \delta_{ij} + h_{ij}) \quad g^{\mu\nu} = a^{-2}(-1, \delta_{ij} - h_{ij} + h_{ik}h_{kj} + ...) \tag{73}$$

With h_{ij} transverse and traceless. The final action is given by equation (74). The details of the calculation are in the appendix again.

$$S = \frac{M_{PL}^2}{8} \int d^4xa^2(h'_{ij}h'_{ij} - h_{ij,k}h_{ij,k} + h_{ij}\sigma^{(TT),ij}) \tag{74}$$

This leads to an equation of motion

$$h''_{ij} - \partial_k\partial_k h_{ij} + 2\mathcal{H}h'_{ij} = \sigma_{ij}^{(TT)} \tag{75}$$

At the linear level the mass term in a general spacetime is the same Pauli-Fierz term we introduced for Minkowski. Using the metric in (73), the Pauli-Fierz mass term becomes

$$m^2(h_{\mu\nu}h^{\mu\nu} - h^2) = m^2a^{-4}h_{ij}h^{ij} \tag{76}$$

This being the only change in the action, we can see the difference in the equations of motion quite simply by just varying this extra term to get

$$h''_{ij} - \partial_k\partial_k h_{ij} + 2\mathcal{H}h'_{ij} - m^2a^{-2}h_{ij} = \sigma_{ij}^{(TT)} \tag{77}$$

3.5 Stress-energy tensor (SET) in curved space

Since gravitational waves themselves have energy, it follows that they themselves should affect the curvature of the spacetime. To investigate this, the SET is needed. From it we can see how gravity couples to itself, and investigate other problems, such as determining the energy carried away from a source due to gravitational waves. In fact, the first (indirect) observations of gravitational waves were actually made by measuring the decay in orbit of two neutron stars [37]. When thinking about gravitational waves a question of definition arises since they themselves are part of the spacetime. What do we define to be part of the background spacetime, and what is deemed to be a gravitational wave? In Minkowski spacetime it was somewhat simpler since any curvature can be attributed to the waves. For a general metric, $g_{\mu\nu}$, we need an obvious distinction between the background and the fluctuations so that we can write $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$. We define L as the spatial length scale of the background $\tilde{g}_{\mu\nu}$ over which significant changes happen. Then, we can define GW if the reduced wavelength of the fluctuations ($\lambdabar = \frac{\lambda}{2\pi}$) is much smaller than L. Alternatively, but obviously equivalently, we could look at this in terms of frequency. If the largest frequency within the background metric (f_B) is much smaller than the frequency of the fluctuations, a sensible definition can be made. In FRW, the relevant length scale is the Hubble scale. Therefore, to study the SET we assume we are working in such a spacetime where one of

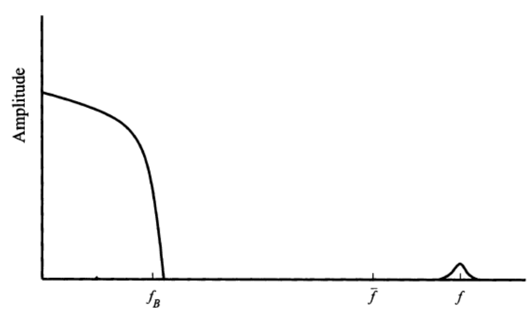

Figure 2: An example of a clear cut situation where we can define gravitational waves in a spacetime. The gravitational waves are at the frequency $f >> f_B$. [31]

the above conditions is satisfied. In curved space, as when we found the canonical SET in section 2, we are required to average over some length l. This coarse-graining procedure is needed not just because the ambiguity of a local energy density definition. By integrating out the shorter wavelengths, we are also getting rid of the high-momentum or high-energy parts of the field. After all, we do not trust our theory above some energy threshold. By getting rid of this finer structure, we can study the slowly-varying d.o.f better. There a few methods that one could use to find SET in general relativity. From the linearised action of the perturbations and then varying the action, followed by the averaging [34].

$$\delta S = \int d^4x \sqrt{-\tilde{g}} t_{\mu\nu} \delta \tilde{g}^{\mu\nu} \tag{78}$$

One could also work with the field equations and expand the Einstein tensor to second order in the perturbations. The zeroth order in the expansion is just due to the background, whereas the first order disappears after the spatial average since it has only single $h_{\mu\nu}$. The second order in the expansion is what we take as the gravitational waves SET (to second order of course). Since it is quadratic in the perturbations, it will have high momenta cancelling out to give a small one that will not be cancelled by the coarse-graining. The answer from all of these methods is actually the same and agrees with the result we obtained in section 2, except with the partial derivatives replaced by covariant derivatives.

$$T_{\sigma\mu} = 2\kappa \langle \nabla_\mu h_{\rho\nu} \nabla_\sigma h^{\rho\nu} \rangle \tag{79}$$

4 Inflation

With General Relativity and the Cosmological Priciple, together with the Standard Model, we can explain much about the universe e.g. the Hubble expansion or the existence of the CMB and its black body spectrum. It does have some downfalls though that can not be addressed such as the horizon problem and the flatness problem.The initial conditions of the universe are chosen so that they fit observations, rather than being explained. Considering how finely-tuned these initial conditions seem to be - up to 60 decimal places if we accept that our current physics models still work all the way back to the Planckian epoch - this is unsettling. Inflation was introduced in order to fix the magnetic monopole problem, but it was soon realised that it could solve other problems too (notably the horizon and flatness problems). It was an era before RD when a scalar field (dubbed the inflaton field) had a very high energy density and drove a roughly exponential expansion.

$$a(t) \approx e^{Ht} \text{ and } H \approx const. \tag{80}$$

Once over, the energy in the inflaton was transferred to the other fields during the reheating phase. From then on, we can revert back to the Big Bang model, with its explanation for events such as nucleosynthesis. One of the advantages of inflation is that is that it has testable predictions, such as the primordial gravitational wave power spectrum it produces (which will be explained further later on). It should also be noted that many inflationary models include more than one inflaton field, but we will only be looking at the single field variants.

4.1 The inflaton and slow-roll conditions

Initially, it was thought that the scalar field needed for inflation was a Higgs field from a grand unified theory but this was soon ruled out. The Higgs field has to be strongly interacting in order for it to induce spontaneous symmetry breaking. This in turn would have made the early universe be highly inhomogeneous due to large quantum fluctuations. The most popular solution was to introduce a different scalar field, dubbed the inflaton (which has so far escaped detection at particle colliders) that is much more weakly interacting. Its action is

$$S_\phi = -\int d^4x\, \sqrt{-g}(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)) \tag{81}$$

and it is taken to be spatially homogeneous to zeroth order. Interestingly, the same effect can be achieved using a modified gravity theory, called the Starobinsky model [35]. It was even the first inflationary model. The Einstein-Hilbert action gains an extra term proportional to R^2 resulting in an extra degree of freedom equivalent to the scalar one in (81). As already discussed in section 2, an exponential expansion of the universe is caused by a constant energy density and implies a negative pressure. This may seem a little weird but one can try and think about it in terms of a piston that closes off a box to illustrate the principles at work in a more familiar setting. Inside the box is a ”vacuum” of constant energy density, i.e. its energy density does not change if the piston moves in or out. Outside the piston is a true vacuum that has a zero energy density. If we move the piston out so that the inside increases by a volume of ΔV, then we must have done work, since the energy inside has increased by $\rho_p \Delta V$ where ρ_p is the energy density inside the piston. We know the work done on the system is equal $W = -p\Delta V$. Therefore, the pressure must be negative. There are some downfalls to this analogy of course, for instance we do not know what is ”outside” our box (which is the universe) or even if such a question makes sense (although there likely is an outside).

Next, we would like to fine tune $V(\phi)$ such that the conditions in (80) are satisfied for some period of time, but do come to an end. The action in (81) leads to an equation of motion $g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = V'(\phi)$. If we work in flat FRW, then

$$\begin{gathered} g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = V'(\phi) \implies g^{\mu\nu}(\delta^\alpha_\mu\partial_\nu - \Gamma^\alpha_{\mu\nu})\partial_\alpha\phi = V'(\phi) \\ g^{00}\partial_0\partial_0\phi - g^{ij}\Gamma^0_{ij}\partial_0\phi = V'(\phi) \\ -\ddot{\phi} - \delta_{ij}H\delta_{ij}\dot{\phi} = V'(\phi) \\ \\ \implies \ddot{\phi} - 3H\dot{\phi} + V'(\phi) = 0 \end{gathered} \tag{82}$$

From the variation of the action (81) with respect to the metric we find the stress-energy tensor of the inflaton (equivalently, it can also be derived using Noether's theorem).

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\big(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\big) \tag{83}$$

Then, by comparing this to the usual fluid $T_{\mu\nu}$ (in the rest frame of the fluid so $u^\mu = (1, \mathbf{0})$).

$$\begin{gathered} T_{00} = \rho \qquad T_{ij} = a^2 p\delta_{ij} \\ \implies \rho = \dot{\phi}^2 - g_{00}(\tfrac{1}{2}g^{00}\dot{\phi}^2 + V) \qquad pa^2 = \partial_i\phi\partial_i\phi - g_{ii}(-\tfrac{1}{2}\dot{\phi}^2 + V) \end{gathered}$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \qquad p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{84}$$

We know already for $a \approx e^{Ht}$ we need an equation of state $p = -\rho$. So, for an inflationary era we need

$$|V(\phi)| >> \frac{1}{2}\dot{\phi}^2 \tag{85}$$

$\implies \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \approx 1$ It should be noted that these conditions for inflation ((85), p=-ρ,$\dot{H} \approx 0, p \approx -V$) classify a certain type of inflation called slow-roll inflation. The first Friedmann equation becomes

$$H^2 \approx \frac{1}{3M_{PL}^2} V(\phi) \tag{86}$$

We only want these to be true for some period of time, or equivalently between $V(\phi_1)$ to $V(\phi_2)$ with the potential "slowly rolling" between the two values of the field. We can get another condition by differentiating (85).

$$\dot{\phi}\ddot{\phi} << V'(\phi)\dot{\phi} \implies |\ddot{\phi}| << |V'(\phi)| \tag{87}$$

Note, the condition for inflation is (85) over some range of ϕ. Then (85) implies (87). We do not need to require (87) to be true. Two slow-roll parameters are usually defined to summarise equations (85) and (87). Firstly, we simplify the equation of motion for the inflaton.

$$\text{From the equation of motion } H \approx -\frac{1}{3\dot{\phi}} V'(\phi) \tag{88}$$

Then, we define the first slow roll parameter as

$$\epsilon_V \equiv \frac{1}{2M_{PL}^2}\left(\frac{V'}{V}\right)^2 = \frac{3}{2}\frac{\dot{\phi}^2}{V} \tag{89}$$

Where we have simplified the expression using (88) and (86). Then we see that $\epsilon_V << 1 \iff (85)$. The second slow-roll parameter is defined differentiating the first one by time.

$$\eta_v \equiv M_{PL}^2 \left|\frac{V''}{V}\right| \tag{90}$$

Often, there are two different slow-roll parameters used that are defined in terms of the Hubble parameter and its derivatives, denoted ϵ_H and η_H.

$$\epsilon_H = -\frac{\dot{H}}{H^2} \qquad \eta_H = \epsilon_H - \frac{\ddot{H}}{2H\dot{H}} \tag{91}$$

These are more useful in general but up to first order they are equal to the parameters defined in (89) and (90). To lighten the notation, I will omit the subscript on the slow-roll parameters from now on unless stated otherwise. Once ϵ is no longer much smaller than one, inflation finishes. V rolls quickly down the potential and oscillates around the minimum. As V rolls down, the energy stored in the inflaton field is transferred to other fields. The conservation of energy equation comes in to play here. During inflation, the decay rate or exchange of energy between the inflaton and other fields was small compared to H (i.e. $\Gamma << H$, Γ being the rate of exchange found in the conservation equation). When inflation finishes, the conservation equation for ϕ now has a Γ term that is no longer negligible

$$\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = \Gamma\rho_\phi \tag{92}$$

This is the process known as reheating and is essential so that inflation fits our current proven models since the inflaton decays into all the known particles increases the temperature rapidly. It does not play out exactly as shown here in all models though. For instance, the inflaton may not even be coupled to conventional matter and so it would have to be created in some roundabout way. The rapid expansion during inflation would have had a big cooling effect on the universe due to dilution of all the other particles, but we know that the temperature at around $10^{-15}s$ must have been higher than 100MeV. Thus the need for the reheating phase.

Although inflation fixes the problem of initial conditions in the ΛCDM model there is a problem of initial conditions for it too, admittedly though it they are much less stringent and actually quite flexible depending on which inflation theory one picks. The minimal requirement is usually taken to be that in the very early universe, before inflation has started, there is some very small region of space that by chance has a high energy density and $\langle\phi\rangle \approx 0$. However, there are other, perhaps more compelling reasons to believe the initial conditions for inflation present little problem, if any. It has been shown that once inflation begins it is unlikely it will ever stop i.e. eternal inflation [36]. This would then lead to many pocket universe being created that would not be in contact with one another, but could also mean evidence of how inflation began may not be found. Then, since it is eternal, as long as the chance of it starting is non-zero then the initial conditions need not be worried about any longer. One may have noticed that while speaking about the highly curved spacetime before inflation we have actually been using the FRW metric with a Euclidean spatial section. This is because we have assumed that the region of space where inflation takes place is small enough so that it can be taken to be spatially flat, or alternatively that we add this to the initial conditions (i.e. we now need a high energy density, a zero v.e.v and a spatially flat region) for the small section of spacetime where it begins.

4.2 Solution to the horizon problem

In ΛCDM, up until recently we lived in a universe with decelerating expansion rate. This means there existed a particle horizon. Two points separated by a particle horizon cannot be in causal contact with one another i.e. no signal could have been sent between them. The issue then comes about when studying the CMB. It is nearly totally homogeneous (with WMAP registering a difference of only 0.0002K). If causality is to remain intact we must then assume one of two things. Either, by some great feat of luck, that the universe is homogeneous by chance, or that despite the apparent particle horizon these points have been in causal contact. An easy way to demonstrate the problem is with a diagram (see figure 3).

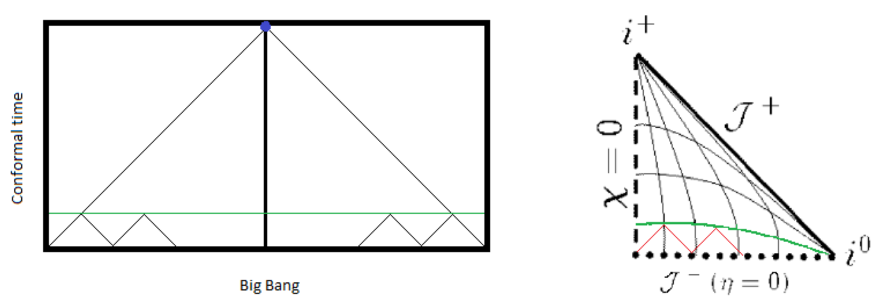

Figure 3: On the left we have a diagram that is easier to understand if one hasn't covered Penrose diagrams. The one on the right is the Penrose diagram for an FRW universe with Euclidean spatial section, $\rho = 0$ and $P > 0$ [17]. χ is the comoving radial distance, which for a Euclidean spatial geometry is usually just denoted as r. In both diagrams the green line is meant to be the surface of last scattering when the CMB was made. The blue dot on the left one is us now.

What is the particle horizon? If we take two photons and set them off in opposite directions we get the expression for it which is

$$d_{PH} = 2\int_{t_1}^{t_2} \frac{dt}{a} = 2\int_{a_1}^{a_2} \frac{d\tilde{a}}{\tilde{a}^2 H(\tilde{a})} \tag{93}$$

This is basically just two times the comoving time η since we set them off. In figure 3, it is the time below the green line. The time from the surface of last scattering to now (i.e. above the green line) would then be given by the integral

$$d_{lss} = 2\int_{a_2}^{a_0} \frac{d\tilde{a}}{\tilde{a}^2 H(\tilde{a})} \tag{94}$$

The problem then arises because $\frac{d_{PH}}{d_{lss}} < 1$. When investigating the CMB, a_2 is fixed from whenever the surface of last scattering was made. A history containing only RD would give an integral over some constant (since $H \propto a^{-2}$ during RD). If instead, a_1 is during an inflationary epoch, we would have a divergent integral as a_1 goes to zero since H is roughly constant. Knowing this, we just need to find out how far back a_1 must be in order for all of the CMB to be in causal contact i.e. we split d_{PH} in two like

$$d_{PH} = \frac{2}{H}\int_{a_1}^{a_3} \frac{d\tilde{a}}{\tilde{a}^2} + 2\int_{a_3}^{a_2} \frac{d\tilde{a}}{\tilde{a}^2 H(\tilde{a})} \tag{95}$$

where a_3 is the value of a from the transition from inflation to RD. Even though this model is somewhat simplified and skips the reheating phase it highlights the key ideas. Roughly speaking, we only really need to take in to account the lower bound of only the first integral (from a_1 to a_3) since all other quantities will be significantly smaller.

$$d_{ph} = \frac{2}{H}\left[\frac{-2}{a}\right]_{a_1}^{a_3} = \frac{4}{a_3 H}\left(e^N - 1\right) \tag{96}$$

Where $N = ln\left(\frac{a_3}{a_1}\right)$, the number of times the universe expanded by a factor of e during inflation. N is usually referred to as the number of e-folds. We take N>> 1, and take the ratio of d_{ph} and d_{lss}.

$$\frac{d_{ph}}{d_{lss}} = \frac{4a_0 H_0}{a_3 H} e^N \tag{97}$$

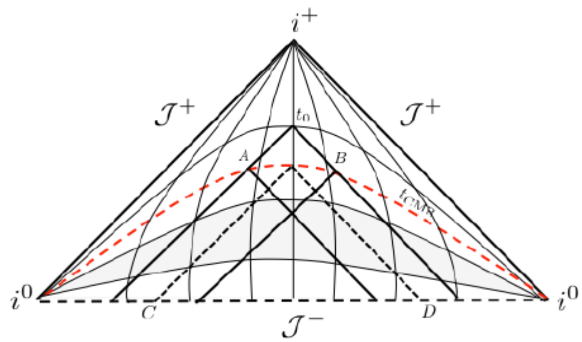

Figure 4: This Penrose diagram includes an inflationary epoch that has been shaded in. The surface of last scattering is the red dashed line. t_0 is an observer today. Points A and B on the surface of last scattering are causally connected if the shaded region lasts long enough. [17]

We can therefore tune N so that the ratio is larger than one and the problem is solved. Equation (106) requires a minimum of sixty e-folds to solve the problem.

4.3 Solution to the flatness problem

As already discussed in section 3, the universe is very flat, with an upper bound on the curvature energy density fraction of $|\Omega_{k,0}| < 5 \times 10^{-3}$. This implies that the earlier universe must have been even flatter since any curvature at the beginning would only have increased as it expands. This simply comes from the fact that matter and radiation get diluted with expansion much quicker than curvature does. Going back to (69), we see that $\rho_R \propto a^{-4}$ and $\rho_M \propto a^{-3}$ whereas $\rho_k \propto a^{-2}$.
We would like to know Ω_k at some time during RD. We only want a rough answer, so we estimate ρ_c by the energy density of the dominant fluid of that era.

$$\text{During MD} \quad \Omega_k(t) \approx \frac{\rho_k}{\rho_M} \approx \frac{a^{-2}}{a^{-3}} = a \ \text{ and during RD} \quad \Omega_k(t) \approx \frac{\rho_k}{\rho_R} \approx \frac{a^{-2}}{a^{-4}} = a^2 \tag{98}$$

Defining a_{eq} as the value of the expansion factor during the RD-MD equilibrium, the explicit expression for $\Omega_k(t_i)$ for any time during RD comes from

$$\frac{\Omega_{k,0}}{\Omega(t_i)} = \frac{\Omega_{k,0}}{\Omega(t_{eq})} \frac{\Omega(t_{eq})}{\Omega(t_i)} = \frac{a_0}{a_{eq}} \frac{a_{eq}^2}{a_i^2} \tag{99}$$

If we turn back the clock to the time of nucleosynthesis, a time when we still highly trust our current description of physics, then we have $|\Omega_k(t_i)| \approx 10^{-18}$. This is quite unnerving to just take as true without cause other than observation. The inflation era before RD explains it rather nicely. In the beginning (or at least, before inflation) the universe was highly curved. Then, the period of rapid expansion in inflation "flattens" out the spacetime. The fact we have an accelerated expansion means $\dot{a}$ is increasing so a^2H^2 is also increasing. Ω_k can in general be written as

$$\Omega_k = \frac{3M_{PL}^2 k}{a^2\rho_c} = \frac{k}{a^2H^2} \tag{100}$$

The period of time with increasing a^2H^2 then leads to a decrease in curvature. We could work out the minimum e-folds needed to solve this problem but it is somewhat similar to the horizon problem and has been done many time in literature.

4.4 Small field and large field inflation

As mentioned, there are many different types of inflation so it is useful to categorise them. Two substantial groups are the small and large field. Although both produce gravitational waves, those from the large field models are significantly bigger and so would be the easiest to detect. The main distinction between the two is their potentials, although this also affects the initial conditions needed.

Large field usually has a power law (some have an exponential) of the form $V(\phi) = \lambda\phi^n, n > 0$ and for some constant λ. The slow-roll condition then gives

$$\epsilon = \frac{n^2M_{PL}^2}{2\phi^2} \tag{101}$$

We need $\epsilon << 1$, therefore, $\phi^2 >> M_{PL}^2$ if n is of order 1. We are also interested in the change of the inflaton's value over the duration of inflation. This can be found from the number of e-folds needed to solve the problems spoken about in sections 4.2 and 4.3, so we take N≈60. The number of e-folds comes from the scale factors growth $a(t_f) = a_{in}e^N = a_{in}e^{\int_{t_i}^{t_f} H(t)dt}$. The Hubble parameter is often approximated as constant during inflation but we keep its time dependence here so that we can use the expression $H \approx \frac{V\phi}{M_{PL}^2 V_\phi}$, which is derived by dividing

(86) by (88).

$$N = -\int_{t_i}^{t_f} \frac{V\dot{\phi}}{M_{PL}^2 V_\phi} = \frac{1}{M_{PL}^2}\int_{\phi_f}^{\phi_i} \frac{V}{V_\phi} d\phi = \frac{1}{nM_{PL}^2}\left(\phi_i^2 - \phi_f^2\right) \approx \frac{\phi_i^2}{nM_{PL}^2} \tag{102}$$

Where we have ignored the final value of the field as it is much smaller than the initial. We find that for large field inflation, the initial value of the inflaton is super-Planckian i.e. $\phi_i \approx \sqrt{2NnM_{PL}^2}$. Below are given two examples for an idea of the size of the field.

N=60 n=2 $\implies \phi_i \approx 15.5M_{PL}$

N=60 n=4 $\implies \phi_i \approx 21.9M_{PL}$

This may seem like it should cause an issue for classical GR, but we actually only need to worry about quantum effects if the field's energy density is super-Planckian i.e. if $M_{PL}^4 \lesssim V(\phi)$. To avoid this we require $\phi << \lambda^{-\frac{1}{n}} M_{PL}^{\frac{4}{n}}$. If we would like to stay in the classical regime (so that we can trust our results better) and still have inflation, we the full requirements are then

$$M_{PL} << \phi << \lambda^{-\frac{1}{n}} M_{PL}^{\frac{4}{n}} \tag{103}$$

in large field inflation. Taking the coupling to be very small ($\lambda << M_{PL}^{4-n}$) provides more leeway with which field values are still considered to be in the classical regime.

Conversely, small field inflation does not require super-Planckian field values. In this branch, the early universe goes through a "phase-change" of sorts. At high temperatures (as is assumed to be the case before inflation) the potential has a minimum at $\phi = 0$. Once the temperature has fallen enough, the potential changes shape to the form usually attributed to small-field models.

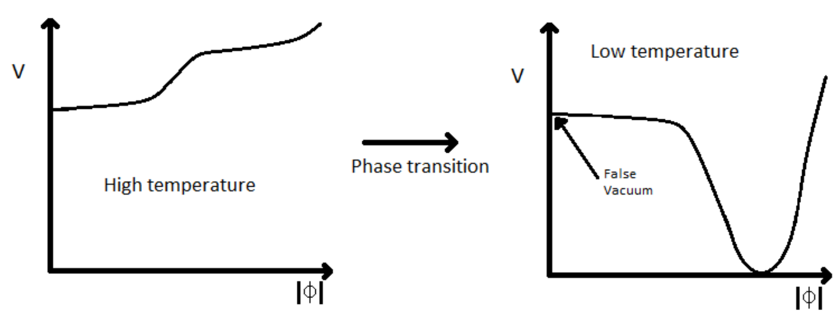

Figure 5: Small field inflation model including the "phase transition". Note, the up turn for higher $|\phi|$ in the low temperature potential requires an extra term not usually included when giving the form of the potential i.e. $V(\phi) = V_0 - g\phi^n + \tilde{g}\phi^m$, (m>n≥3).

After the phase change, ϕ is still at zero, in what is referred to as the ”false vacuum” since it is no longer the state with lowest energy density. Quantum fluctuations cause it to come off the maximum and it begins it slowly roll down the potential, which is usually given as of the form $V(\phi) = V_0 - g\phi^n$, n≥3, at least near the origin. Since ϕ is small during inflation and does not increase to very high value, we can use the first few term of its Taylor expansion accurately and do not need to know the potential's complete form (this is in effect what we are doing when defining $V(\phi) = V_0 - g\phi^n$).

$$\implies \epsilon = \frac{M_{PL}^2}{2}\left(\frac{V'}{V}\right)^2 = \frac{M_{PL}^2}{2}\left(\frac{-gn\phi^{n-1}}{V_0 - g\phi^n}\right)^2 \tag{104}$$

It is evident from (104) that the slow roll condition is satisfied from a small ϕ close to 0.

Now to talk a little about the initial conditions needed for the two branches. Large field is easier to justify, although the discussion earlier on eternal inflation should make the issue of likelihood somewhat of a moot point. It seems reasonable to imagine that the early universe was highly curved, highly non-homogenous, and highly non-isotropic at lengths larger than the Planck length. From there, a small patch may by chance have an extremely high energy density large enough so that the slow-roll conditions are satisfied and sets off inflation. Since the expansion begins out of a chaotic universe, this is also called “chaotic inflation” and was first proposed by Linde [30]. On the other hand, small field inflation may come about from conditions similar to the Hot Big Bang model except that homogeneity and isotropy are not needed. Once the universe has expanded and cooled enough, the potential changes form and we enter the inflationary epoch. Small field inflation was suggested first (after Starobinsky's) with Linde also publishing one of the first papers on it [29]. It should be noted that large field inflation does not necessarily have to adhere to the upper bound, since this only comes from a gap in our knowledge. This an advantage of the small-field inflation. It may be harder to detect the GW signal left behind from it, but the physics can be understood without having to fear that the energies may become super-Planckian.

There is a third type of inflation that is popular called hybrid inflation but it is not single-field. The initial conditions are close to those of large field but do not require the energy density to be super-Planckian. The second scalar field (which should actually be taken to have more than one component) acts similarly to a Higgs field, providing a symmetry breaking mechanism that changes the potential shape once inflation has ended and reheats the universe.

4.5 Tensor perturbations during inflation

What we call primordial gravitational waves are tensor perturbations to the metric that were produced during inflation and are still travelling through the universe. In this section, we will consider only the tensor perturbations and then later we will look at the scalar.
We take the relevant modes as the sub-horizon modes, those with comoving momentum $k >> \mathcal{H}$, which is sometimes restated in terms of the physical wavelength $\frac{\lambda_{ph}}{2\pi} << H$. Inflation's solution to the horizon problem comes in useful again here. Presumably, there are gravitational waves with physical wavelengths that have only recently entered the Horizon. Without inflation, this would cause an issue much akin to that discussed in the horizon problem section, areas of the universe that should not have a correlation have a gravitational wave spanning across them. Assuming inflation did happen, we take it so that all modes that have entered the horizon during RD and MD, had previously been in the horizon during inflation. A graph with both the physical horizon and some of the physical wavelengths can be very useful here.

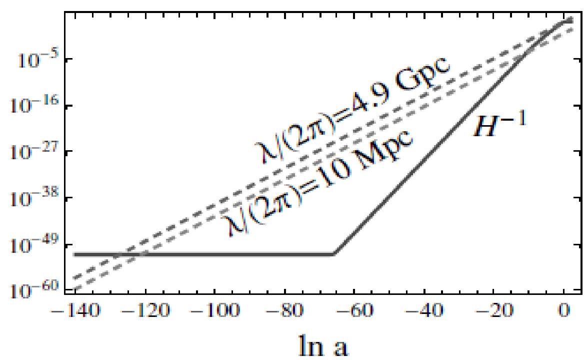

Figure 6: The physical horizon is roughly constant during inflation, but begins to grow during RD and MD (at different rates, although it is not clear in this diagram). It finally tails off back to a constant towards the end since we are thought to be entering another exponential expansion. The physical wavelengths go as straight lines, all with the same gradient since they are just proportional to their comoving wavelengths with a proportionality factor a(t). [32]

We have already derived the action for gravitational waves in flat space FRW back in section 2, as well as their equations of motion. Importantly, during inflation there is no stress that contributes to $\sigma_{ij}^{(TT)}$ so there is no source for the gravitational waves at first order. If we go further, then the perturbations of the metric itself, can act as a source for tensor perturbations. We use the stress-energy tensor from Noether's theorem (i.e. one index up and one down). The metric is defined as before

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu} \tag{105}$$

with $\tilde{g}_{\mu\nu} = a^2\eta_{\mu\nu}$ and since we are worrying about the tensor perturbations responsible for gravitational waves only, $h_{\mu\nu} = (0, a^2 h_{ij}^{(TT)})$, although we will omit the (TT) superscript.

$$\begin{aligned} T^{\mu}_{\ \nu} &= \tilde{g}^{\mu\rho}\partial_\rho\phi\partial_\nu - \delta^{\mu}_{\nu}\left(\frac{1}{2}\tilde{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right) \\ \delta T^{\mu}_{\ \nu} &= \delta g^{\mu\rho}\partial_\rho\phi\partial_\nu\phi + 2\tilde{g}^{\mu\rho}\partial_\rho\phi\partial_\nu\delta\phi \\ &\quad - \delta^{\mu}_{\nu}\left(\frac{1}{2}\delta g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + \tilde{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\delta\phi + V'(\phi)\delta\phi\right) \end{aligned} \tag{106}$$

We insert for $\delta g^{\mu\nu} = (0, -a^{-2}h^{ij})$ and take the (i,j) component of (106), remembering that the inflaton is spatially homogeneous to zeroth order.

$$\delta T^{i}_{j} = -\delta^{i}_{j}\left(\tilde{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\delta\phi + V'(\phi)\delta\phi\right) \tag{107}$$

This obviously does not contribute to the transverse traceless tensor $\sigma_{ij}^{(TT)}$. We remind ourselves that after linearising, the action in comoving coordinates is

$$S[h] = \frac{M_{PL}^2}{8}\int d^3x d\eta a^2(\eta)(\partial_\eta h_{ij}\partial_\eta h_{ij} - \partial_k h_{ij}\partial_k h_{ij}) \tag{108}$$

where we have ignored the energy momentum tensor due the discussion just above. From this, the equation of motion is

$$\partial_\eta^2 h_{ij} - \partial_k^2 h_{ij} + 2\mathcal{H}\partial_\eta h_{ij} = 0 \tag{109}$$

We would like to treat the metric perturbations as a scalar field so we switch to their fourier transform to utilise their decomposition in to polarisation tensors (+ and ×). Then, from the normalisation of the polarisation tensors ($e^{A}_{ij}e^{A'}_{ij} = 2\delta_{AA'}$) we get an action that is like that of two scalar fields in a curved background. By this I mean, we have the usual kinetic term

$\partial_\mu \phi \partial_\nu \phi$, but its contracted with $\tilde{g}^{\mu\nu}$ and of course the volume element includes $\sqrt{-\tilde{g}}$.

$$S[h] = \frac{M_{PL}^2}{4} \sum_A \int d^3x d\eta \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\mu h_A \partial_\nu h_A \tag{110}$$

Whereas the action for canonically normalised scalar fields is

$$S[\phi] = -\frac{1}{2} \sum_A \int d^3x d\eta \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\mu \phi_A \partial_\nu \phi_A \tag{111}$$

So, the gravitational waves only differ from the canonically normalised scalar fields by a proportionality constant (some normalisation).

We would like to get some sort of experimental prediction. What we measure is the amplitude of incoming gravitational waves at different frequencies. For the ground based detectors the frequencies are usually quite high. This is due to the fact that there are many gravitational effects at quadratic order that are not due to the gravitational waves. Luckily, we can still make ground based detectors because these other sources have a lower frequency than many GW sources, but this does mean there are some lower frequency GW that we are not able to detect on the ground. One way around this is to build space based detectors, such as LISA planned by the ESA to be launched in 2034. Another way, for which the infrastructure already exists, is to use accurate CMB readings.

Before introducing the power spectrum, we go over a normalisation convention for the FT used here that just makes the expressions a little lighter on the eyes and easier to follow. We introduce a volume in to the fourier transform since cosmology deals with finite volumes that are measurable. A clear advantage being we do not need to require our functions to decay to zero sufficiently fast.

$$\tilde{f}(\mathbf{k}) = \frac{1}{\sqrt{V}} \int_V d^3x f(\mathbf{x}) e^{-i\mathbf{k}.\mathbf{x}} \iff f(\mathbf{x}) = \sqrt{V} \int_V \frac{d^3k}{2\pi^3} \tilde{f}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}} \tag{112}$$

We have spoken a lot about initial conditions and their origin but so far we have failed to mention that the exact initial conditions cannot be predicted. There is no way that we would be able to explain or predict the value of the field h_{ij} at a certain point in space. We can see this as originating from the quantum nature of reality and the fluctuations that happen on the smallest of scales. Practically, this means we treat the initial conditions at a point as a random variable. We try not to predict $h^{ij}_{initial}$ at a point, but instead its two point correlation function which we will use to define its power spectrum. We assume Gaussian initial conditions from here on since all data from experiments support this. In Gaussian statistics all n-point correlation functions can be found using the 2-point correlation function, so all the information we need is encoded in that. Note that the 3-point correlation function is actually zero. Since the FRW metric is invariant under spatial translations and rotations, we know that the 2 point correlation function is actually only a function of η_{in} and the absolute value of the difference of the two points i.e. $|\mathbf{x}-\mathbf{x}'|$.

$$\langle \tilde{h}_A(\eta_i n, \mathbf{k})\tilde{h}^*_{A'}(\eta, \mathbf{k'})\rangle = \frac{1}{V}\int_V d^3x d^3x' \langle \tilde{h}_A(\eta_{in}, \mathbf{x})\tilde{h}^*_{A'}(\eta_{in}, \mathbf{x'})\rangle e^{-i\mathbf{k}.\mathbf{x}}e^{-i\mathbf{k}'.\mathbf{x}'}$$

We make the substitution $\mathbf{X} = \frac{1}{2}(\mathbf{x}+\mathbf{x}')$ and $\mathbf{y} = \mathbf{x}-\mathbf{x}'$

$$\langle \tilde{h}_A(\eta_i n, \mathbf{k})\tilde{h}^*_{A'}(\eta, \mathbf{k'})\rangle = \frac{1}{V}\int_V d^3X d^3y f(y) e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{X}-\frac{i}{2}(\mathbf{k}+\mathbf{k}').\mathbf{y}} \tag{113}$$

We can do the X integral easily enough resulting in a three dimensional delta function and a factor of $(2\pi)^3$. The remaining integral is packaged in to the definition of the power spectrum P_T, given below in equation (114).

$$\langle \tilde{h}_A(\eta_{in}, \mathbf{k})\tilde{h}^*_{A'}(\eta_{in}, \mathbf{k'})\rangle = \frac{(2\pi)^3}{2V}P_T(\eta_{in};k)\delta^{(3)}(\mathbf{k}-\mathbf{k'})\delta_{AA'} \tag{114}$$

Where V is the finite volume that the perturbations are in. We have also chosen the normalisation convention of $\frac{1}{2}$ because of the polarisation tensors we defined in section 1 that satisfy $e^A_{ij}e^{A'}_{ij} = 2\delta^{AA'}$. It means we have the cleaner expression without the one half normalisation for the 2-point correlation function of the field h_{ij}. From the form given in (112), we can see why we changed normalisation convention for the fourier transform. By including the V in the denominator we avoid the apparent singularity one would get from the delta function if we were to look at $\mathbf{k} = \mathbf{k}'$, i.e $V \equiv (2\pi)^3\delta^{(3)}(\mathbf{0})$. Therefore

$$\langle |\tilde{h}_A(\eta, \mathbf{k})|^2\rangle = \frac{1}{2}P_T(k;\eta) \tag{115}$$

We call P_T the power spectrum of h_A. From this we further define

$$\mathcal{P}_T(k;\eta) = \frac{k^3}{2\pi^2}P_T(k;\eta) \tag{116}$$

Which, confusingly, is also referred to as the power spectrum. When measuring GWs, we usually do so around some reference value of k (called k_*) and within a small band around it. Around this value k_*, we expect a simple form for $\mathcal{P}_T(k;\eta_{in})$ (i.e. the primordial power spectrum). The simplest parametrisation is

$$\mathcal{P}_T(k;\eta_{in}) = A_T(k_*)\left(\frac{k}{k_*}\right)^{n_T(k_*)} \iff n_T(k_*) = \frac{d}{d\,ln(k)} ln(\mathcal{P}_{T,in}(k)) \tag{117}$$

A_T is called the amplitude, and n_T is called the tilt. A flat spectrum is defined as one with zero tilt. However, what we measure is $\mathcal{P}_T(k;\eta_0)$, η_0 being the time today. So, after using making a prediction for $\mathcal{P}_T(k;\eta_{in})$, and evolving it in time (this will depend on the cosmological model used), we can compare our prediction to what is measured in the present day. Going back to the equation (in momentum space this time)

$$\partial_\eta^2 \tilde{h_{ij}} + k^2 \tilde{h_{ij}} + 2\mathcal{H}\partial_\eta \tilde{h_{ij}} = 0 \tag{118}$$

This can be simplified by making a field redefinition $h_A = \frac{1}{a(\eta)M_{PL}}\chi$. Using the chain rule and remembering that $\frac{d\eta}{dt} = \frac{1}{a}$ we get

$$\tilde{\chi}'' + (k^2 - \frac{a''}{a})\tilde{\chi} = 0 \tag{119}$$

For which the general solution is

$$\chi(\eta,\mathbf{x}) = \int \frac{d^3k}{(2\pi^3)}\Big(f(\eta,\mathbf{k})e^{i\mathbf{k}.\mathbf{x}}a(\mathbf{k}) + f^*(\eta,\mathbf{k})e^{-i\mathbf{k}.\mathbf{x}}a^*(\mathbf{k})\Big) \tag{120}$$

To find the two point correlation function we actually find the vacuum expectation value of the quantum field. This means we need to perform a second quantisation in a curved spacetime. There is a difficulty that arises with doing this in general relativity. We are free to change coordinates because of the diffeomorphism invariance, which can cause ambiguities when defining the vacuum. There is no guarantee that the vacuum of one set of coordinates will be the same as the vacuum of another set. This problem also materialises itself when looking at the modes in equation (119). In Minkowski, a Lorentz transformation keeps the positive modes as the positive and the negative modes stay as the negative. This is not necessarily true after a coordinate change in a curved spacetime. Here, we will take the Bunch-Davies vacuum [9] because in the limit $|k\eta| >> 1$ it agrees with the Minkowski vacuum. We therefore have

$$\hat{\chi}(\eta,\mathbf{x}) = \int \frac{d^3k}{(2\pi^3)} \left(f(\eta,\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}} \hat{a}(\mathbf{k}) + f^*(\eta,\mathbf{k}) e^{-i\mathbf{k}.\mathbf{x}} \hat{a}^\dagger(\mathbf{k}) \right) \tag{121}$$

With the $\hat{a}$ obeying the commutation relation $[\hat{a},\hat{a}^\dagger] = (2\pi)^3 \sqrt{V} \delta^3(\mathbf{k}-\mathbf{k}')$, with the rest being zero. The hats will be omitted from now on for ease. The vacuum expectation value is then

$$\begin{aligned}
\langle 0 || \chi(\eta,\mathbf{x})|^2 | 0\rangle = \langle 0| &\int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \left(f(\eta,\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}} a(\mathbf{k}) + f^*(\eta,\mathbf{k}) e^{-i\mathbf{k}.\mathbf{x}} a^\dagger(\mathbf{k}) \right) \\
&\left(f(\eta,\mathbf{k}') e^{i\mathbf{k}'.\mathbf{x}} a(\mathbf{k}') + f^*(\eta,\mathbf{k}') e^{-i\mathbf{k}'.\mathbf{x}} a^\dagger(\mathbf{k}') \right) |0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3} d^3k' f(\mathbf{k}) f(\mathbf{k}') \sqrt{V} e^{i\mathbf{x}(\mathbf{k}-\mathbf{k}')} \delta^3(\mathbf{k}-\mathbf{k}') \\
&= \int \frac{d^3k}{(2\pi)^3} |f(\mathbf{k})|^2 \\
&= \int_0^\infty \frac{dk}{2\pi^2 k} k^3 |f(\mathbf{k})|^2
\end{aligned} \tag{122}$$

From this, we can easily find the power spectrum $\mathcal{P}_T$. We just multiply the two fourier transforms of $h_{ij}(\eta,\mathbf{x})$ and $h_{ij}(\eta,\mathbf{x}')$ (given in (112)) and take the vev. Then substitute in (114), simplify and change to polar coordinates. Now going back to the original field we had for the tensor perturbations, the power spectrum we get is therefore

$$\mathcal{P}_T(k;\eta) = \frac{2}{a^2 \pi^2 M_{PL}^2} k^3 |f_k(\eta)|^2 \tag{123}$$

Before giving the exact solution we will take a look at the two limiting extremes. The full solution is more complicated and harder to extract information from. In the sub-horizon limit ($|k\eta| >> 1$) we get the usual Minkowski wave equation for χ. So the h_A oscillates while its amplitude is decreasing over time as $\frac{1}{a}$, we can see this damping as being a direct result of the expansion of the universe. On the other hand we have the super-horizon limit ($|k\eta| << 1$). Then the two independent solutions are

$$\tilde{\chi}_1 \propto a \text{ and } \tilde{\chi}_2 \propto \frac{1}{a^2} \tag{124}$$

$$\implies \tilde{h}_1 = const. \text{ and } \tilde{h}_2 \propto \frac{1}{a^3} \tag{125}$$

We get a constant mode and one which decays very quickly. We are usually interested in the constant mode since the other will decay to being negligible very soon after η_{in}. We conclude that any mode that exits the horizon during inflation remains "frozen" until it has reentered the horizon.
The approximate exponential expansion means we are in a roughly de Sitter spacetime. In de Sitter, $a = \frac{1}{\mathcal{H}\eta}$. We want to get a more accurate expression for a, and since we are in a slow roll inflation, we can expand $\mathcal{H}$ in terms of ϵ, but only to first order like all the expansions so far.

$$\text{Using} \quad \epsilon = -\frac{\dot{H}}{H^2} \quad \text{and} \quad H = \frac{1}{a}\mathcal{H}$$

$$\begin{aligned}\epsilon = -\frac{a^4}{(a')^4}\frac{d\eta}{dt}\frac{d}{d\eta}(\frac{1}{a}\mathcal{H}) = -\frac{a^2}{(a')^2}\Big(\mathcal{H}' - \frac{a'}{a}\mathcal{H}\Big) = -\mathcal{H}^{-2}\Big(\mathcal{H}' - \frac{a'}{a}\mathcal{H}\Big) \\ \implies \mathcal{H}' = \mathcal{H}^2(1-\epsilon)\end{aligned} \tag{126}$$

We are working only to first order in the slow roll expansion so we can take ϵ to be constant. After integration we get

$$\mathcal{H} = -\frac{(1+\epsilon)}{\eta} \tag{127}$$

By definition we have $\mathcal{H} = \frac{1}{a}\frac{da}{d\eta} = \frac{d(ln(a))}{d\eta}$

$$\implies \int d(ln(a)) = -(1+\epsilon)\int \frac{d\eta}{\eta} \underset{\sim}{\propto} ln(\eta^{-1-\epsilon})$$

$$a \underset{\sim}{\propto} \eta^{-1-\epsilon} \tag{128}$$

Using this expression for $a(\eta)$, we can write

$$\frac{a''}{a} = \frac{2+3\epsilon}{\eta^2} + \mathcal{O}(\epsilon^2) \tag{129}$$

$$\implies \tilde{\chi}'' + (k^2 - \frac{2+3\epsilon+\mathcal{O}(\epsilon^2)}{\eta^2})\tilde{\chi} = 0 \tag{130}$$

$$\implies \tilde{\chi}'' + (k^2 - \frac{\nu^2 - \frac{1}{4}}{\eta^2})\tilde{\chi} = 0 \text{ where } \nu = \frac{3}{2} + \epsilon + \mathcal{O}(\epsilon^2) \tag{131}$$

One can then show that the mode functions take the form

$$f_k(\eta) = -\frac{1}{2}\sqrt{\frac{\pi}{k}}\sqrt{-k\eta}H_\nu^{(1)}(-k\eta) \tag{132}$$

Where $H_\nu^{(1)}(x)$ is the Hankel function of the first kind and we take $\nu \approx \frac{3}{2}$ since ϵ is small.

$$\implies \langle|\chi(\eta_{in}, \mathbf{x})|^2\rangle = \int \frac{d^3k}{(2\pi)^3}|f_k(\eta_{in})|^2 = \int_0^\infty \frac{dk}{k}\frac{k^3}{2\pi^2}|f_k(\eta_{in})|^2$$

$$\implies \mathcal{P}_T(k;\eta_{in}) = \frac{k^3}{2\pi^2 a^2 M_{PL}^2}|f_k(\eta_{in})|^2 \tag{133}$$

$$\mathcal{P}_T(k;\eta_{in}) = \frac{k^2}{2\pi a^2 M_{PL}^2}(-k\eta_{in})|H_\nu^{(1)}(\eta_{in})|^2 \tag{134}$$

We have assumed that the initial condition for the vacuum state is such that $a_{\mathbf{k}}|0\rangle = 0$, which is the Bunch-Davies vacuum as mentioned earlier.

4.6 Massive tensor perturbations during inflation

Later on we will be looking at what happens to the primordial power spectrum were we to couple the usual metric to a second spin-2 field. The coupling term results in a massive mode. Therefore, we will explore the implications of massive gravity on the power spectrum here first. Usually, when a particle ”gains” a mass, its potential gains a decaying exponential i.e. we expect a Yukawa-like potential. This is what happens with photons and the Proca Lagrangian. It is also what happens to the (00) component of $g_{\mu\nu}$, which is the component responsible for the classical Newtonian potential in regular GR. One might expect then that something similar happens to the gravitational waves once they gain mass. In reality, the sub-horizon remain the same while for the super-horizon we get one decaying mode and one increasing mode (at least during inflation). We already derived the equation of motion for massive gravitational waves in section 3.4, equation (77).

$$\implies h_{ij}'' + 2\mathcal{H}h_{ij}' - \partial_k\partial_k h_{ij} - m^2a^2h_{ij} = 0 \tag{135}$$

We then proceed exactly as we did for massless gravity, changing to fourier space and splitting the equation in to the two polarisations before making the substitution $\tilde{h}_A = \frac{M_{PL}}{a}\tilde{\chi}_A$.

$$\tilde{\chi}_A'' + (k^2 - m^2 a^2 - \frac{a''}{a})\tilde{\chi}_A = 0 \tag{136}$$

Since we are in a quasi de Sitter space, $a^2 \approx \frac{1}{H_0^2\eta^2}$. This means that if we look at the sub-horizon modes, all we are left with is the usual wave equations i.e. there is no change to the massive tensor modes. Note that we require the mass to be roughly the same order as H_0 for this to be true. Although current experimental upper bounds place the graviton mass at orders of magnitude bigger than H_0 [18], our requirement is not so stringent or unheard of. Later on, we will be looking at bimetric theories of gravity in FRW, in one paper exploring this a theoretical upper bound for the graviton mass was found to be an order of magnitude less than H_0 [16]. Taking the limit in the other direction, $|k\eta| << 1$, we are left with the equation

$$\tilde{\chi}_A'' - (m^2 a^2 + \frac{a''}{a})\tilde{\chi}_A = 0 \tag{137}$$

Remembering equations (128) and (129), we continue to first order in slow roll. Since $a = H_0^{-1}\eta^{-1-\epsilon}$. We now look to expand $\eta^{-2\epsilon}$ to first order.

$$\begin{aligned} \eta^{-2\epsilon} &= 1 - 2\epsilon ln(\eta) + \mathcal{O}(\epsilon^2) \\ \implies m^2 a^2 &\approx m^2 H_0^{-2}\eta^{-2}(1 - 2\epsilon ln(\eta) + ...) \end{aligned} \tag{138}$$

Then we rely on [16] to ensure the mass is at least one order smaller than H_0, we take $m^2 H_0^2 \epsilon$ as being of the same order as ϵ^2.

$$\implies -m^2 a^2 - \frac{a''}{a} \approx \frac{-m^2 H_0^{-2} - 2 - 3\epsilon}{\eta^2} \tag{139}$$

For ease, we define $b = (m^2 H_0^{-2} + 2 + 3\epsilon)$. The equation is now

$$\tilde{\chi}_A'' - b\eta^{-2}\tilde{\chi}_A = 0 \tag{140}$$

and has two solutions:

$$\tilde{\chi}_{A_\pm} \propto \eta^{r_\pm} \tag{141}$$

With $r_\pm = \frac{1}{2}(1 \pm \sqrt{1+4b})$. We have one decaying mode again, but this time instead of a constant mode we have one that is increasing. However, this does not necessarily mean that $\tilde{h}_A$ is also increasing. We check this next.

$$\tilde{h}_{A_\pm} \propto \eta^{\frac{3}{2}+\epsilon\pm\frac{1}{2}\sqrt{1+4b}} \tag{142}$$

Looking at the power only

$$\begin{aligned}\frac{3}{2}+\epsilon\pm\frac{1}{2}\sqrt{1+4b} &= \frac{3}{2}+\epsilon\pm\frac{1}{2}\sqrt{9+4m^2H_0^{-2}+12\epsilon} \\ &= \frac{3}{2}+\epsilon\pm\frac{1}{2}\sqrt{9+4m^2H_0^{-2}}\pm\frac{3\epsilon}{\sqrt{9+4m^2H_0^2}} \\ &= \frac{3}{2}\pm\frac{1}{2}\sqrt{9+4m^2H_0^{-2}}+\beta_\pm\epsilon\end{aligned} \tag{143}$$

where we have defined $\beta_\pm \equiv 1 \pm \frac{3}{\sqrt{9+4m^2H_0^{-2}}}$

We have also kept the power only to first order in ϵ. It is easy to see that the index of h_{A_+} is positive. However, since $\eta \propto \frac{1}{a}$, this mode actually decays with time. To check the behaviour of the other mode, we expand its index

$$\frac{3}{2}-\frac{3}{2}\sqrt{1+\frac{4m^2H_0^{-2}}{9}}+\epsilon\left(1-\left(1+\frac{4m^2H_0^{-2}}{9}\right)^{-\frac{1}{2}}\right) = -\frac{m^2H_0^{-2}}{3}+\frac{2m^2H_0^{-2}}{9}\epsilon < 0 \tag{144}$$

Therefore, we have a decaying mode and an increasing mode (the power of the increasing mode is of order one or less, according to our assumption earlier $\mathcal{O}(m) \lesssim \mathcal{O}(H_0)$). Since $\beta_+ \longrightarrow 0$ as $m \longrightarrow 0$, when we take the massless limit we get back one decaying mode and one constant mode as expected.

4.7 Scalar perturbations during inflation

Although not dynamical, the scalar perturbations of the metric are of interest when considering how the small inhomogeneities in the universe formed. They are what led to matter accumulation in certain areas and eventually the development of galaxies. If we are exploring gravitational waves at second order or above, then the scalar perturbations can be a source of GWs themselves. Unlike $h_{ij}^{(TT)}$, the scalar perturbations do couple to the inflaton perturbations at the linear level. The quantum fluctuations of the inflaton can be show to be the origin of the small inhomogeneities in the early universe. There are a couple of ways to find the resulting equation of motion. One could go from the perturbed field equations. Alternatively, the action of the inflaton minimally coupled to gravity can be perturbed.

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\left(M_{PL}^2 R - (\partial\phi)^2 - 2V(\phi)\right) \tag{145}$$

Since we are looking at the scalar perturbations, we ignore the tensor and vector perturbations of both the metric and the gauge transformation. In (146) we write the most general FRW metric with scalar perturbations as well as the perturbed inflaton field.

$$\begin{aligned} ds^2 &= -(1+2\zeta)dt^2 + 2a\partial_i\gamma dx^i dt + a^2[(1-2\psi)\delta_{ij} + 2\partial_i\partial_j\lambda]dx^i dx^j \\ \phi(t,\mathbf{x}) &= \phi_0(t) + \delta\phi(t,\mathbf{x}) \end{aligned} \tag{146}$$

The gauge functions ξ^μ are defined as

$$\xi^0 = \alpha \qquad \xi^i = \partial^i\beta \tag{147}$$

Using the curved space gauge transformation $h_{ij} \longrightarrow h_{ij} - 2\nabla_{(\mu}\xi_{\nu)}$ leads to the following transformations

$$\begin{aligned} &\zeta \longrightarrow \zeta - \dot{\alpha} \qquad \gamma \longrightarrow \gamma + a^{-1}\alpha - a\dot{\beta} \quad \lambda \longrightarrow \lambda - \beta \qquad \psi \longrightarrow \psi + H\alpha \\ &\delta\phi \longrightarrow \delta\phi - \dot{\phi}_0\alpha \end{aligned} \tag{148}$$

Note that we are using (t,x,y,z) coordinates in this calculation and of course have $\xi_\mu = \tilde{g}_{\mu\nu}\xi^\nu$ so $\xi_i = \tilde{g}_{ij}\xi^j = a^2\partial_i\beta$. There are four scalar metric d.o.f and one more from the inflaton field so five in total. The gauge invariance removes two and there are constraints from the perturbed field equations which would remove another two. The one physical d.o.f is chosen to be the comoving curvature $\mathcal{R} = \psi + \frac{H}{\partial_t\phi_0}\delta\phi$. On a comoving hypersurface where $\delta\phi = 0$, then $\mathcal{R} = \psi$ and it gives the intrinsic spatial curvature via a Poisson equations dictating ψ ($R^{(3)} = \frac{4}{a^2}\nabla^2\psi$). We can see that $\mathcal{R}$ is gauge invariant. If we go in to the comoving gauge so that $\delta\phi = 0$ and $g_{ij} = a^2[(1-2\mathcal{R})\delta_{ij} + h_{ij}]$ (where h_{ij} is transverse and traceless) then the resulting action is

$$S = \frac{1}{2}\int d^4x\, \sqrt{-g} a^3 \frac{\dot{\phi}}{H^2}[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i \mathcal{R})^2] \tag{149}$$

Next, we make a field redefinition $u = z\mathcal{R}$ where $z^2 = a^2 \frac{(\partial_t \phi)^2}{H^2}$ and change to conformal time. The final action is

$$S = \frac{1}{2}\int d\eta d^3x[(u')^2 - (\partial_i u)^2 + \frac{z''}{z}u^2] \tag{150}$$

Varying this action will give us the Mukhanov-Sasaki (MS) equation.

$$u'' - \nabla^2 u - \frac{z''}{z}u = 0 \qquad \text{or in Fourier space} \qquad \tilde{u}'' + (k^2 - \frac{z''}{z})\tilde{u} = 0 \tag{151}$$

The MS equation turns out to be quite similar in form to the equation we had for the two polarisations of $h_{ij}^{(TT)}$. Both look like simple harmonic motion with a frequency that changes with time. Next, we would like to quantize the MS variable , u, to find its vacuum fluctuations. One may wonder how to go about finding the vacuum fluctuations when the vacuum itself is not unique, as we mentioned earlier. To get around this we choose the same vacuum as before, the Bunch-Davies vacuum, because of its transition to Minkowski at sub-horizon regimes. After quantisation, the mode functions can be written as

$$\hat{\tilde{u}}(\mathbf{k}) = \tilde{u}(\eta,k)\hat{a}(\mathbf{k}) + \tilde{u}^*(\eta,k)\hat{a}^\dagger(-\mathbf{k}) \tag{152}$$

With the $\hat{a}(\mathbf{k})$, $\hat{a}^\dagger(-\mathbf{k})$ the creation and annihilation operators. They are defined using the Wronskian ($W[u,v] = \frac{i}{\hbar}(u^* v' - u^{*\prime} v)$ and satisfy the the commutation relation

$$[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = (2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}) \qquad \text{so that} \qquad W[\tilde{u}(\eta,k), \tilde{u}(\eta,k)] = 1 \tag{153}$$

Differentiating (70) with respect to time (with $\rho_{tot} = \rho_{inf}$), and then using equation (67) (again with the total pressure and energy density taken as that of the inflaton) and (84) (i.e. $\rho_{inf} + p_{inf} = \dot{\phi}^2$), we can show that $\dot{H} = -\frac{M_{PL}^2}{2}\dot{\phi}^2$. Therefore, $z^2 = \frac{M_{PL}^2 a^2}{2}\frac{\dot{H}}{H^2} = \frac{2a^2}{M_{PL}^2}\epsilon$. If we are only working to first order in slow-roll, then we take epsilon as constant meaning that $\frac{z''}{z} = \frac{a''}{a}$ and the equation of motion for the MS variable becomes identical to that of the tensor polarisations except for a normalisation. The power spectrum for the MS variable can then easily be inferred to be

$$\mathcal{P}_u(k;\eta) = \frac{k^2}{2\pi}(-k\eta)\left|H_\nu^{(1)}(\eta)\right| \tag{154}$$

where $\nu = \frac{3}{2} + \epsilon$ again. The power spectrum for $\mathcal{R}$ (at lowest order so we only use the background inflaton for $z = \frac{a\phi_0'}{\mathcal{H}}$) is

$$\mathcal{P}_\mathcal{R}(k;\eta) = \frac{k^2}{2\pi}\left(\frac{\mathcal{H}}{a\phi_0'}\right)^2(-k\eta)\left|H_\nu^{(1)}(\eta)\right| \tag{155}$$

It can be useful to point out that to choose the vacuum, we have to choose appropriate boundary conditions. We take the sub-horizon limit of (151) and find its solution (which are just oscillating modes). The vacuum is then the minimum energy state for this solution. Just as we did with the tensor sector, we can parametrise the scalar perturbations power spectrum around a certain moementum k_* (called the "pivot scale" as before) to get a simpler form. Again, we do this because experiments will only be looking at a relatively small bandwidth of different momenta. We have a tilt again too, although by convention the spectrum is flat this time if the tilt is one rather than zero.

$$\mathcal{P}_{\mathcal{R},in}(k) = A_\mathcal{R}(k_*)\left(\frac{k}{k_*}\right)^{n_s(k_*)-1} \iff n_s(k_*) - 1 = \frac{d}{dln(k)}\mathcal{P}_{\mathcal{R},in}(k) \tag{156}$$

Now that we have both tensor and scalar power spectra, we introduce their ratio. The tensor-to-scalar ratio is usually used when discussing the effects of the metric perturbations on the CMB. It quantifies the strength of the GWs in comparison to the scalar perturbations.

$$r(k) = \frac{\mathcal{P}_{T,in}(k)}{\mathcal{P}_{\mathcal{R},in}(k)} \tag{157}$$

This turns out to be quite small as shown below using (134) and (156).

$$r(k) = \frac{1}{M_{PL}^2}\left(\frac{\phi_0'}{\mathcal{H}}\right)^2 = \frac{1}{M_{PL}^2}\left(\frac{\dot{\phi}_0}{H}\right)^2 \tag{158}$$

Using $\dot{\phi}_0^2 = -\frac{M_{PL}^2}{2}\dot{H}$ and $\epsilon = -\frac{\dot{H}}{H^2}$ gives us

$$r = \frac{1}{2}\epsilon \tag{159}$$

The exact value of the ratio can vary depending on the model of inflation, which is apparent in (159) since the slow-roll parameter does depend on the inflaton's potential (one of the main aspects that defines an inflation model). In general, the r value is around 0.15 to 0.4 for generic models with a minimum number of e-folds. From (159) we can see roughly why large field inflation will have a greater effect on the CMB than small field. We already know what ϵ we get for generic small and large field models. We also need the $\eta = M_{PL}^2 \left|\frac{V''}{V}\right|$.

$$\begin{aligned} \text{Large field} \quad \eta =& M_{PL}^2 \left|\frac{n(n-1)}{\phi^2}\right| \approx \epsilon \\ \text{Small field} \quad \eta =& M_{PL}^2 \left|\frac{gn(n-1)\phi^{n-2}}{V_0}\right| > \epsilon \end{aligned} \tag{160}$$

Inflation ends when either of the slow-roll parameters approaches one. Since they are roughly equal in large field, inflation ends when $\epsilon \approx 1$. In small field, η is larger and so inflation ends before ϵ can reach one, meaning the tensor to scalar ratio will also be smaller. Although this argument is somewhat crude, it gets across the main point. Observations from Planck 2015 give an upper bound on r of 0.11 [3]. This is below most mainstream inflationary models. This is not the only problem with inflation. Inflation has a fine-tuning which is especially unsettling considering it was created to try and fix a previous fine-tuning problem. As mentioned earlier, the coupling parameter in the inflaton potential must be very small in order to produce the small quantum fluctuations that produce the observed amplitude of primordial density fluctuations $\left(\frac{\delta\rho}{\rho} = 10^{-15}\right)$. If we consider [2] $V(\phi) = \lambda\phi^4$, this means we need to have $\lambda \approx \mathcal{O}(10^{-15})$, although not as extreme as what we saw for the flatness problem it should not be taken lightly.

[2]This particular potential has been ruled out by Planck2015 since it r value is too high.

5 Bimetric gravity and its effect on the power spectrum

5.1 Short introduction to dRGT

So far, when considering massive gravity, we have only done so to linear order with the Fierz-Pauli action. Since its first introduction, there have been many attempts to write down the fully non-linear theory and find the complete kinetic term of a massive spin-2 field. Until recently, these attempts could not avoid the Boulware-Deser (BD) ghost that was avoided in the Fierz-Pauli action. There have been theories produced without this ghost but the majority are not Lorentz invariant. The many massive gravity models usually only differ in their predictions once we start probing huge, cosmologically relevant distances. This is somewhat expected, and built in by design since we know that General Relativity is extremely accurate at the level of the solar system. Arguably, it is perfectly healthy at larger scales too, but we do have to introduce dark energy which one may argue points to a hole in the theory rather than a missing "substance". Furthermore, any new theory should agree largely with the predictions of GR for the evolution of the universe, since the ΛCDM model has been thoroughly tested.

The problem of ghosts was solved using the dRGT Lagrangian [15], which is Lorentz invariant and is made up of a two-parameter family of potentials. Since we choose to work with a Lorentz invariant theory, the mass of the graviton (defined using the propagator of the graviton) is given by a single pole. However, the theory must also be invariant under translations, but we do not need to worry about this since we are working with FRW. Although the BD ghost is no longer an issue, there may be other instabilities in the theory, and we will come across some in the bimetric FRW. These are not as problematic as the BD ghost, and well-behaving physical models can be found from dRGT.

To build the dRGT Lagrangian we must define a second reference metric usually denoted as $f_{\mu\nu}$. The full, non-linear Lagrangian of the dRGT theory is then given in (161) (if $f_{\mu\nu}$ is not dynamic).

$$\mathcal{L}_{tot} = \frac{M_{PL}^2}{2}\sqrt{-g}\left(R + m^2\sum_{I=0}^{4}\alpha_I\mathcal{L}_I\right) \tag{161}$$

The α_I are the free parameters, of which only two need to be kept general and the $\mathcal{L}_I$ are

$$\begin{aligned} \mathcal{L}_0 =& 1 \ , \quad \mathcal{L}_1 = [\mathcal{K}] \ , \quad \mathcal{L}_2 = \frac{1}{2}\left([\mathcal{K}]^2 - \left[\mathcal{K}^2\right]\right) \\ \mathcal{L}_3 =& \frac{1}{6}\left([\mathcal{K}]^3 - 3[\mathcal{K}]\left[\mathcal{K}^2\right] + 2\left[\mathcal{K}^3\right]\right) \\ \mathcal{L}_4 =& \frac{1}{24}\left([\mathcal{K}]^4 - 6[\mathcal{K}]^2\left[\mathcal{K}^2\right] + 3\left[\mathcal{K}^2\right]^2 + 8[\mathcal{K}]\left[\mathcal{K}^3\right] - 6\left[\mathcal{K}^4\right]\right) \end{aligned} \tag{162}$$

The square brackets denote the trace operation and

$$\mathcal{K}^\mu{}_\nu \equiv \delta^\mu{}_\nu - \left(\sqrt{g^{-1}f}\right)^\mu{}_\nu \ , \qquad \left(\sqrt{g^{-1}f}\right)^\mu{}_\rho \left(\sqrt{g^{-1}f}\right)^\rho{}_\nu \equiv g^{\mu\rho} f_{\rho\nu} \tag{163}$$

The zeroth term is just the cosmological constant, while the first is to do with the v.e.v of the field. The full non-linear Lagrangian is given here, but when calculating the primordial power spectrum of the gravitational waves, we will only be using the linear Lagrangian (i.e. $\alpha_3 = \alpha_4 = 0$). Often, the second reference metric is just taken as the Minkowski metric. In [20] both the background and the reference were taken to be FRW, although the reference remained without perturbations and varied only with time. This massive gravity theory led to a deviation from the usual Yukawa potential expected by redressing the mass to $\tilde{m}_g^2(H) = m_g^2 \frac{H}{H_0}\left(c_0 + c_1 \frac{H}{H_0} + c_2 \frac{H^2}{H_0^2}\right)$ (the c_i are dimensionless constants that depend on the α_i parameters). This redressing is actually not unique to the FRW model studied in [20] - the mass is commonly seen to change depending on the background spacetime.

If one instead takes $f_{\mu\nu}$ to be a general symmetric rank two tensor that is dynamical, then we would need to include a kinetic term for it too. We already know the form of this, since the Einstein-Hilbert action is the unique action for a massless spin-2 field (at the non-linear level). The sum in (161) is then a coupling term between the two dynamical spin-2 fields and the theory is referred to as bi-gravity.

$$S = \frac{M_{PL}^2}{2}\int d^4x\,\sqrt{-g}R[g] + \frac{M_f^2}{2}\int d^4x\,\sqrt{-f}R[f] + m^2 M_{PL}^2 \int d^4x \sum_{I=0}^{4} \alpha_I \mathcal{L}_I \tag{164}$$

The full action only contains one graviton bare mass. It is essential that we have a massive spin-2 field interacting with a massless in a bimetric theory, since Weinberg's theorem [38] [39] shows that we cannot have two massless spin-2 fields mediating gravity in a Lorentz invariant theory. After diagonalisation , we are left with one massive and one massless tensor mode and an effective mass which is what appears in predictions. There are therefore seven d.o.f in this bimetric theory, two carried by the massless mode and five carried by the massive mode.

5.2 ADM formalism

In this part of we will be closely following [19][22] where the ADM formalism is used. The ADM formalism is a Hamiltonian formulation of General Relativity first introduced in [5]. It takes a generic form of the metric to be

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = (-N^2 + \gamma_{ij}\mathcal{N}^i\mathcal{N}^j)dt^2 + \gamma_{ij}\mathcal{N}^j dx^i dt + \gamma_{ij}dx^i dx^j \tag{165}$$

Where N is the lapse function, $\mathcal{N}^i$ the vector and γ_{ij} the spatial metric. The generalised coordinates are taken to be the spatial metric part of $g_{\mu\nu}$ and the canonical momenta are those of the spatial metric $\left(i.e\ \pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{g}_{ij}}\right)$. This formalism separates time from the three spatial coordinates and spacetime is foliated in to Cauchy surfaces (which can be thought of as spatial surfaces of constant time). In this form, the volume element can be shown to be $\sqrt{-g} = N\sqrt{\gamma}$. In FRW, we have already foliated our spacetime . The general FRW metric with tensor perturbations for gravitational waves is:

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -N^2dt^2 + a(t)^2(\gamma_{ij} + H_{ij}^{(TT)})dx^i dx^j \tag{166}$$

As before, we are taking $\gamma_{ij} = \delta_{ij}$ during inflation. Interestingly, in an FRW universe, the second spin-2 field cannot be the Minkowski metric as it leads to instabilities [16], although this can be worked around by taking approximate FRW solutions instead ([16] drops homogeneity and also provides an upper bound for the graviton mass, while [21] and [27] drop isotropy). We take the reference metric to be of the same form as $g_{\mu\nu}$, but with a different lapse function, as well as a different perturbation to the spatial metric.

$$f_{\mu\nu}dx^{\mu}dx^{\nu} = -n^2dt^2 + b(t)^2(\gamma_{ij} + h_{ij}^{(TT)})dx^i dx^j \tag{167}$$

We have some gauge freedom and can actually set the lapse function of $g_{\mu\nu}$ to be equal to a(t) so that we are using conformal time again. It is not possible to do this for both metrics at the same time though, unless forced by us as a physical condition.

There is a question of how to couple bi-gravity to matter. Coupling both metrics to to the same matter in a covariant way leads to the emergence of a ghost[33], so we eliminate this option. There are a few alternatives. We could couple $T_{\mu\nu}$ to one of the metrics, or couple each metric to its own separate part of $T_{\mu\nu}$. There is also the possibility to couple $T_{\mu\nu}$ to a composite metric, again given in [33], to be of the form $g^{eff}_{\mu\nu} = \alpha^2 g_{\mu\nu} + 2\alpha\beta g_{\mu}\sqrt{g^{-1}f}^{\alpha}{}_{\nu} + \beta^2 f_{\mu\nu}$, where α and β are arbitrary constants. Then the stress energy tensor couples to matter by the action $S_{matter} = \int d^4x\sqrt{-g^{eff}}g^{eff}_{\mu\nu}T^{\mu\nu}$.

5.3 FRW background with two metrics

We now have two metrics, both homogeneous and isotropic (to zeroth order). This results in two sets of ”Friedmann” equations, as well as extra terms in the equations due to the coupling of the two fields to each other. For instance, four of the six background equations are given below and are from [22].

$$3H_g^2 = m^2\rho_{m,g} + \frac{\rho_g}{M_{PL}^2} \qquad 3H_f^2 = \frac{m^2}{\kappa}\rho_{m,f} + \frac{\rho_f}{\kappa M_f^2} \tag{168}$$

$$\begin{aligned} 2\frac{\dot{H}}{N} &= m^2 XJ(\tilde{c}-1) - \frac{\rho_g + P_g}{M_{PL}^2} \\ 2\frac{\dot{H}_f}{n} &= -\frac{m^2}{\kappa X^3\tilde{c}}J(\tilde{c}-1) - \frac{\rho_f + P_f}{\kappa M_{PL}^2} \end{aligned} \tag{169}$$

where ρ_g and ρ_f are the energy densities due to the matter coupled to $g_{\mu\nu}$ and $f_{\mu\nu}$ respectively. We list some definitions for (168) and (169) below, as well as some extra ones we will need later.

$$\begin{aligned} &H_g = \frac{\dot{a}}{aN}\,, \quad H_g = \frac{\dot{a}}{bn}\,, \quad \kappa = \frac{M_f^2}{M_{PL}^2}\,, \quad X = \frac{b}{a}\,, \quad \rho_{m,g} = U(X) - \frac{X}{4}U'(X) \\ &\rho_{m,f} = \frac{1}{4X^3}U'(X)\,, \quad U(X) = -6\alpha_2(X-1)^2 + 4\alpha_3(X-1)^3 - \alpha^4(X-1)^4 \\ &J(X) = \frac{1}{3}[U(X) - \frac{X}{4}U'(X)]'\,, \quad \tilde{c} = \frac{na}{Nb} \end{aligned} \tag{170}$$

We see that we get a Hubble parameter for each metric, and an energy density due to matter for each metric, so unsurprisingly we get a copy of everything in section 2 for the second metric. One could think of $\rho_{m,g}$ or $\rho_{m,f}$ as the energy density due to the coupling of the spin-2 field $f_{\mu\nu}$ to $g_{\mu\nu}$ or vice versa, since $\mathcal{L}_{dRGT}$ is in effect what replaces the coupling via a stress energy tensor and is worked out by variation of the dRGT Lagrangian with respect to the respective metric. From the background equations for $g_{\mu\nu}$ one can find a constraint equation for the theory $J(X)(H_g - XH_F) = 0$ which leads to two "branches" that can be explored. The J=0 branch implies that X is a constant and so $\rho_{m,g}$ is also constant and we get a naturally appearing cosmological constant term. Although this seems appealing at first, at the linear level, the J=0 branch has only four d.o.f instead of the seven that are expected [11] while at the non-linear it has many instabilities akin to the self-accelerating massive gravity FRW model [12]. We therefore focus on the $H_g = XH_f$ branch.

If we then multiply the (0,0) equation for $f_{\mu\nu}$ (that is, the right one in (168)) by X^2 and subtract it from the (0,0) component for $g_{\mu\nu}$, we arrive at (171) which we should use to determine X.

$$\rho_m \equiv \rho_{m,g}(X) - \frac{X^2}{\kappa} \qquad \rho_{m,f}(X) \equiv \frac{X^2 \rho_f}{m^2 \kappa M_{PL}^2} - \frac{\rho_f}{m^2 M_{PL}^2} \tag{171}$$

We will find that our effective mass is dependent on $X = \frac{b}{a}$. For convenience and ease in our calculations, it would be better to work in a regime where the effective mass can approximately be taken as constant. One such regime is given by taking $\rho_m << 1$. It has the solution $X = X_c$ =constant, with $\rho_m(X_c) = 0$. Technically, even if $\rho_m(X_c)$ is not zero, it is constant and can be taken as part of the cosmological constant. Equation (171) can be true if and only if

$$\frac{\rho_g}{m^2 M_{PL}^2} << 1 \quad , \quad \frac{X^2 \rho_f}{m^2 \kappa M_{PL}^2} << 1 \tag{172}$$

It is the two conditions in (172) which we refer to as the low energy limit. If we want the low energy condition to agree with the cosmological history provided by GR, then (168) demands that $\rho_{m,g} < \frac{H_g^2}{m^2} << 1$ up until the recent dark energy epoch (i.e. we need the energy density of the matter fields to be dominant). Depending on which matter sector is coupled to which of the two metrics, we may also need corresponding conditions for the second dynamical metric. Another consequence of the low energy limit can be found by differentiating $H_g - XH_f = 0$ with respect to time, and then using equations (169).

$$\dot{X} = \frac{\dot{b}}{a} - \frac{b\dot{a}}{a^2} = \frac{bnH_g}{aX} - \frac{bNH_g}{a} \tag{173}$$

Then just rearrange (169) to get expressions for $\dot{H}_f$ and $\dot{H}_g$. Substitute these all in to $\dot{H}_g - \dot{X}H_f - X\dot{H}_f = 0$ to get (174).

$$2(\tilde{c}-1)\left[\frac{m^2 J(1+\kappa X^2)}{2\kappa X} - 2H_g^2\right] = \frac{\rho_g + P_g}{M_{PL}^2} - \frac{X^2\tilde{c}(\rho_f + P_f)}{\kappa M_{PL}^2} \tag{174}$$

So even though we do not have the gauge freedom to set both a=N and b=n (and therefore $\tilde{c} = 1$), equation (174) ensures that $\tilde{c}$ approaches 1 in the low energy limit, meaning $a \propto b$ and $N \propto n$.

5.4 The tensor perturbations' action

We are interested in how the primordial power spectrum for a bimetric theory of gravity differs from what we found in section 3. What part of the power spectrum is measurable by our instruments very much depends on how the metrics are coupled to matter. The instruments can only detect gravitational waves due to a metric that it is coupled to. We will give the power spectra of both massless and massive eigenstates here, and discuss their coupling later. To find their power spectra we obviously need the action for the two metrics' tensor perturbations. We will only be working to linear order, which means we set $\alpha_3 = \alpha_4 = 0$ from now on.

$$S = \frac{M_{PL}^2}{8}\int d^4x N a^3 \sqrt{\gamma}\left[\frac{\dot{H}^{ij}\dot{H}_{ij}}{N^2} + \frac{H^{ij}}{a^2}\nabla^2 H_{ij} + \kappa\tilde{c}X^4\left(\frac{\dot{h}^{ij}\dot{h}_{ij}}{n^2} + \frac{h^{ij}}{b^2}\nabla^2 h_{ij}\right) \right. \\ \left. - m^2\Gamma(X)\left(H^{ij} - h^{ij}\right)\left(H_{ij} - h_{ij}\right)\right] \tag{175}$$

Where $\Gamma(X) \equiv XJ(X) + \frac{X^2(\tilde{c}-1)}{2}J'(X)$. There a few simplifications we can do from here. Firstly, we can set $\sqrt{\gamma} = 1$. We can also take the low energy limit, letting $X = X_c$ and $\tilde{c} = 1$. This means that $b \approx aX_c$ which then also implies that $n \approx NX_c$. From the interaction term in (175), it is easy to see that the massive mode must be of the form $s(H_{ij} - h_{ij})$. We then require that the other mass eigenstate is of the form $(qH_{ij} + rH_{ij})$, and we have q, r and s to determine. We do so in appendix D, and we also make an attempt at deriving the action (175) in appendix C. We find that the two mass eigenstates are

$$H^{+}_{ij} = \frac{H_{ij} + \kappa X_c^2 h_{ij}}{1 + \kappa X_c^2} \quad , \quad H^{-}_{ij} = H_{ij} - h_{ij} \tag{176}$$

We have chosen the normalisation of each eigenstate so that the new action can easily be separated in to two, each with its own separate "Planck mass", $M_-^2 \equiv \frac{\kappa X_c^2}{1+\kappa X_c^2} M_{PL}^2$ and $M_+^2 \equiv M_{PL}^2(1 + \kappa X_c^2)$. Lastly, we choose the gauge so that $N = a$ and therefore we are in comoving coordinates.

$$\begin{aligned} S = \frac{1}{8}\int d^4x a^2 \Big[& M_+^2 \left(\dot{H}_+^{ij}\dot{H}^+_{ij} + H_+^{ij}\nabla^2 H^+_{ij} \right) \\ & + M_-^2 \left(\dot{H}_-^{ij}\dot{H}^-_{ij} + H_-^{ij}\nabla^2 H^-_{ij} - m_{\text{eff}}^2 H_-^{ij} H^-_{ij} \right) \Big] \end{aligned} \tag{177}$$

$$m_{eff}^2 \equiv \frac{1 + \kappa X_c^2}{\kappa X_c^2} m^2 \Gamma(X_C^2)$$

This is the final form for the action, and makes apparent the fact that we have one massless mode and one massive mode. The power spectra of the two modes are easy enough to read straight off (177), since the fields only differ from those we found in section 4 by a normalisation.

$$\mathcal{P}_{H^+}(k,\eta) = \frac{1}{1+\kappa X_c^2}\mathcal{P}_T(k,\eta) = \frac{k^2}{2\pi^2 a^2 M_{PL}^2 (1 + \kappa X_C^2)}(-k\eta)|H_\nu^{(1)}(\eta)|^2 \tag{178}$$

$$\mathcal{P}_{H^-}(k,\eta) = \frac{1 + \kappa X_C^2}{\kappa X_C^2}\mathcal{P}_{TM}(k,\eta) \tag{179}$$

Where $\mathcal{P}_T$ is the massless power spectrum from single gravity, and $\mathcal{P}_{TM}$ is the massive. It was said in [19] that the massive modes oscillate once outside the horizon, but as shown in section 3, we found one to decay and one to increase. After an email sent to one of the authors, they agreed that they probably made a mistake writing up their results.

Only the massless mode survives, since we know that the massive mode decays once it reaches super-horizon levels. Now we turn to the question of detection. It is obvious that since we are using matter to detect these gravitational waves, that our experiments will only be able to detect perturbations in the metric that the matter is coupled to. In a flat FRW spacetime, using normal linearised GR ($g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu})$), the stress energy tensor for matter is given by equation (180).

$$S_{matter} = \int d^4 x a^2 h_{\mu\nu} T^{\mu\nu} \tag{180}$$

Therefore, the predictions made by bimetric gravity depends on what metric we choose to be coupled to matter. The simplest and most common choice is to take $g_{\mu\nu}$ to be coupled to all matter. Then, $T_{\mu\nu}$ is coupled to a linear combination of massless and massive modes which can be worked out by inverting the equations in (176).

$$S_{matter} = \int d^4 x a^2 \left(H^+_{ij} + \frac{\kappa X_c^2}{1+\kappa X_c^2} H^-_{ij} \right) T^{ij} \tag{181}$$

However, as we know that the majority of the massive modes have decayed, we expect the power spectrum detected to be very similar to that predicted by normal GR except for the normalisation given above in equation (180).
We could take the matter as being coupled to only one of the mass eigenstates. In order to do this, we would need to use a composite metric similar to that mentioned earlier from [33], except that we would have to take the middle term containing $\sqrt{g^{-1}f}^{\mu}{}_{\nu}$ to be zero. I am not sure of the validity of this assumption and have not found literature exploring it either. Nonetheless, coupling $T^{\mu\nu}$ to only one of the mass eigenstates would contradict some of the assumptions made earlier when finding the background equations. Namely, we took each metric to be coupled covariantly to some matter sector (and separately from the other metric) but this can easily be conciled by just taking the energy density of matter to be zero for that metric. If there existed some exotic form of matter that only coupled to $f_{\mu\nu}$, then anyone wishing to make a detector for these different gravitational waves would need to do so out of the exotic matter. If we couple matter to $g_{\mu\nu}$ as given in equation (181), then the power spectrum we would get is that from equation (182).

$$\mathcal{P}_{Tg} = \langle h_{ij} h^{ij} \rangle = \langle H^+_{ij} H^{ij}_+ \rangle + \left(\frac{\kappa X_c^2}{1+\kappa X_c^2} \right)^2 \langle H^-_{ij} H^{ij}_- \rangle = \frac{1}{1+\kappa X_c^2} \mathcal{P}_T(k,\eta) + \left(\frac{\kappa X_c^2}{1+\kappa X_c^2} \right) \mathcal{P}_{TM}(k,\eta) \tag{182}$$

If we take $M_{PL}^2 >> M_f^2 \implies \kappa << 1$, then the power spectrum will be nearly that of GR. Note that we may need to be careful about the details here, since the massive gravity modes actually increase while in the super-horizon region during inflation. If we take the limit the other way, $M_{PL}^2 << M_f^2$ then we would get the power spectrum just for massive gravity. So we see what was mentioned earlier that the strength of the respective Planck masses for the metrics dictates how similar this bigravity theory is to massless or massive gravity. In the low energy limit, we actually require $\kappa << 1$ [19]. If we want to take the predictions from this seriously then we must impose this, meaning we expect little change from current models of normal GR. On one hand, this may be seen as advantageous, considering how successful GR has been, but on the other it means we must have extremely sensitive instruments to find evidence allowing us to distinguish the two theories in reality.

6 Conclusion

This dissertation gave a quick overview of some of the main ideas for gravitational waves, both massless and massive, before going on to explore the predicted signals from a non-specific theory of inflation. In particular, an important point was the Helicity decomposition of the perturbations, which allowed us to significantly simplify our calculations of the action. For massive gravity, we used a Stuckelberg decomposition for Minkowski space, attempts have been made at using a similar method in FRW (or a general spacetime), but no real progress has been made. Once the basics were covered, we moved on to deriving the tensor perturbations for massless and massive gravity in an FRW spacetime. Lastly, we briefly introduced the dRGT Lagrangian which allows for a bimetric theory, which after diagonalisation presents a gravitational force that is communicated by a massive and massless tensor mode. Crucially, we took the low energy limit to be able to get this result, so this significantly narrows the scope of our predictions but ensures we have an approximately constant effective mass. We can still recover most of the universe's history in agreement with the ΛCDM model, and the gravitational waves due to the massive mode are likely many orders of magnitude smaller than the massless (although we did not work out the evolution of the massive modes in RD or MD). Anyway, even if the dRGT predictions for the primordial spectra is wrong, it may in fact be inflation which is the incorrect theory.

A numerical analysis of the differential equation for the massive modes could be used in future discussions in order to get a better picture of the full evolution of these modes without having to result to the super and sub horizon limits. This has been done already for the massless modes and a transfer function has been defined too [32]. Another area we could have explored in more detail is the predictions of different models of inflation for the power spectrum. As mentioned briefly, new data has ruled out many of the "classic" inflation theories. There are new "post-modern" versions which purport to solve the disagreement between theory and experiment. For instance in [23] they give a supergravity model with a scalar field (as well as vector and tensor modes required by supersymmetry) and 3 parameters that can be tuned in any way so that the theory will fit any prediction for the tilt, n_s, and r. This may be unsettling for some as it is not so much a predictive theory, but one that is just fit to the data.

One could also look to alternative theories rather than inflation, such as the ekpyrotic theories that attribute the flatness and smoothness today to a period of slow contraction before the big bang. The horizon problem is solved due to the fact the universe was larger before this big bounce. Predictions from this for the non-gaussianities and r-value have matched the WMAP and Planck2015 data well while sticking to a simple theory without many parameters [28].

References

[1] B.P. Abbott, R. Abbott, T.D. Abbott, M.R. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R.X. Adhikari, and et al. Properties of the binary black hole merger gw150914. *Physical Review Letters*, 116(24), Jun 2016. pages 7

[2] R. Abbott et al. Properties and astrophysical implications of the 150 Msun binary black hole merger GW190521. *Astrophys. J. Lett.*, 900:L13, 2020. pages 7

[3] P. A. R. Ade, N. Aghanim, M. Arnaud, M. Ashdown, J. Aumont, C. Baccigalupi, A. J. Banday, R. B. Barreiro, J. G. Bartlett, and et al. Planck2015 results. *Astronomy Astrophysics*, 594:A13, Sep 2016. pages 58

[4] Q. R. Ahmad, R. C. Allen, T. C. Andersen, J. D.Anglin, J. C. Barton, E. W. Beier, M. Bercovitch, J. Bigu, S. D. Biller, R. A. Black, and et al. Direct evidence for neutrino flavor transformation from neutral-current interactions in the sudbury neutrino observatory. *Physical Review Letters*, 89(1), Jun 2002. pages 31

[5] R. Arnowitt, S. Deser, and C. W. Misner. Dynamical structure and definition of energy in general relativity. *Phys. Rev.*, 116:1322–1330, Dec 1959. pages 61

[6] James M. Bardeen. Gauge-invariant cosmological perturbations. *Phys. Rev. D*, 22:1882–1905, Oct 1980. pages 20

[7] C. L. Bennett, D. Larson, J. L. Weiland, N. Jarosik, G. Hinshaw, N. Odegard, K. M. Smith, R. S. Hill, B. Gold, M. Halpern, and et al. Nine-year wilkinson microwave anisotropy probe (wmap) observations: Final maps and results. *The Astrophysical Journal Supplement Series*, 208(2):20, Sep 2013. pages 27, 31

[8] David G. Boulware and S. Deser. Can gravitation have a finite range? *Phys. Rev. D*, 6:3368–3382, Dec 1972. pages 24

[9] T. S. Bunch and P. C. W. Davies. Quantum field theory in de sitter space: Renormalization by point-splitting. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 360(1700):117–134, 1978. pages 50

[10] K. S. Thorne C. W. Misner and J. A. Wheeler. *Gravitation*. W. H. Freeman, 1973. pages 17

[11] D. Comelli, M. Crisostomi, and L. Pilo. Perturbations in massive gravity cosmology. *Journal of High Energy Physics*, 2012(6), Jun 2012. pages 63

[12] Antonio De Felice, A. Emir Gümrükçüoğlu, and Shinji Mukohyama. Massive gravity: Nonlinear instability of a homogeneous and isotropic universe. *Physical Review Letters*, 109(17), Oct 2012. pages 63

[13] Claudia de Rham. Massive gravity. *Living Reviews in Relativity*, 17(1), Aug 2014. pages 25

[14] Claudia de Rham and Gregory Gabadadze. Generalization of the fierz-pauli action. *Phys. Rev. D*, 82:044020, Aug 2010. pages 10

[15] Claudia de Rham, Gregory Gabadadze, and Andrew J. Tolley. Resummation of massive gravity. *Phys. Rev. Lett.*, 106:231101, Jun 2011. pages 10, 22, 59

[16] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava, and A. J. Tolley. Massive cosmologies. *Physical Review D*, 84(12), Dec 2011. pages 53, 61

[17] George F.R. Ellis and Jean-Philippe Uzan. Causal structures in inflation. *Comptes Rendus Physique*, 16(10):928 – 947, 2015. Cosmic inflation / Inflation cosmique. pages 39, 41

[18] Abbott et al. Tests of general relativity with gw150914. *Phys. Rev. Lett.*, 116:221101, May 2016. pages 53

[19] Matteo Fasiello and Raquel H. Ribeiro. Mild bounds on bigravity from primordial gravitational waves. *Journal of Cosmology and Astroparticle Physics*, 2015(07):027–027, Jul 2015. pages 61, 65, 67, 78

[20] Matteo Fasiello and Andrew J Tolley. Cosmological perturbations in massive gravity and the higuchi bound. *Journal of Cosmology and Astroparticle Physics*, 2012(11):035–035, Nov 2012. pages 60

[21] Antonio De Felice, A. Emir Gümrükçüoğ, Chunshan Lin, and Shinji Mukohyama. Nonlinear stability of cosmological solutions in massive gravity. *Journal of Cosmology and Astroparticle Physics*, 2013(05):035–035, May 2013. pages 61

[22] Antonio De Felice, A. Emir Gümrükçüoğ, Shinji Mukohyama, Norihiro Tanahashi, and Takahiro Tanaka. Viable cosmology in bimetric theory. *Journal of Cosmology and Astroparticle Physics*, 2014(06):037–037, Jun 2014. pages 61, 62

[23] Sergio Ferrara, Renata Kallosh, Andrei Linde, and Massimo Porrati. Minimal supergravity models of inflation. *Phys. Rev. D*, 88:085038, Oct 2013. pages 68

[24] M. Fierz and W. Pauli. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. *Proc. Roy. Soc. Lond. A*, 173:211–232, 1939. pages 13, 21

[25] Y. Fukuda, T. Hayakawa, E. Ichihara, K. Inoue, K. Ishihara, H. Ishino, Y. Itow, T. Kajita, J. Kameda, S. Kasuga, and et al. Measurement of the flux and zenith-angle distribution of upward throughgoing muons by super-kamiokande. *Physical Review Letters*, 82(13):2644–2648, Mar 1999. pages 31

[26] A. Emir Gümrükçüoğ, Chunshan Lin, and Shinji Mukohyama. Cosmological perturbations of self-accelerating universe in nonlinear massive gravity. *Journal of Cosmology and Astroparticle Physics*, 2012(03):006–006, Mar 2012. pages 77

[27] A. Emir Gümrükçüoğlu, Chunshan Lin, and Shinji Mukohyama. Anisotropic friedmann–robertson–walker universe from nonlinear massive gravity. *Physics Letters B*, 717(4-5):295–298, Oct 2012. pages 61

[28] Anna Ijjas and Paul J Steinhardt. Implications of planck2015 for inflationary, ekpyrotic and anamorphic bouncing cosmologies. *Classical and Quantum Gravity*, 33(4):044001, jan 2016. pages 69

[29] A.D. Linde. A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems. *Physics Letters B*, 108(6):389 – 393, 1982. pages 44

[30] A.D. Linde. Chaotic inflation. *Physics Letters B*, 129(3):177 – 181, 1983. pages 44

[31] Michele Maggiore. *Gravitational Waves, Volume 1: Theory and Experiments*. Oxford University Press, 2008. pages 33

[32] Michele Maggiore. *Gravitational Waves, Volume 2: Astrophysics and Cosmology*. Oxford University Press, 2018. pages 18, 45, 68

[33] Claudia de Rham, Lavinia Heisenberg, and Raquel H Ribeiro. On couplings to matter in massive (bi-)gravity. *Classical and Quantum Gravity*, 32(3):035022, Jan 2015. pages 62, 66

[34] Alexander Saffer, Nicolás Yunes, and Kent Yagi. The gravitational wave stress–energy (pseudo)-tensor in modified gravity. *Classical and Quantum Gravity*, 35(5):055011, Feb 2018. pages 17, 34

[35] A.A. Starobinsky. A new type of isotropic cosmological models without singularity. *Physics Letters B*, 91(1):99 – 102, 1980. pages 36

[36] S. T. C. Siklos Stephen Hawking, Gary Gibbons. *The Very Early Universe*, pages 251–266. Cambridge University Press, 1983. pages 38

[37] J. H. Taylor and J. M. Weisberg. A new test of general relativity - Gravitational radiation and the binary pulsar PSR 1913+16. , 253:908–920, February 1982. pages 9, 33

[38] Steven Weinberg. Photons and gravitons in s-matrix theory: Derivation of charge conservation and equality of gravitational and inertial mass. *Phys. Rev.*, 135:B1049–B1056, Aug 1964. pages 60

[39] Steven Weinberg. Photons and gravitons in perturbation theory: Derivation of maxwell's and einstein's equations. *Phys. Rev.*, 138:B988–B1002, May 1965. pages 60

A Linearising the EH action in Minkowski spacetime

This is the linearisation of the EH action in Minkowski space. Therefore, we write the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Firstly, we find the Christoffel symbol to second order

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}(h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta}) - \frac{1}{2}h^{\alpha\beta}(h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta}) \tag{183}$$

Then we work out the Ricci tensor in two, the first part being of first order and the second or second order. This is done solely to make the working out easier to follow as otherwi

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\eta^{\mu\beta}(h_{\beta\sigma,\nu\mu} - h_{\beta\mu,\nu\sigma} - h_{\sigma\nu,\beta\mu} + h_{\nu\mu,\beta\sigma}) \tag{184}$$

$$R^{(1)} = h^{\mu\nu}{}_{\mu\nu} - \Box h \tag{185}$$

$$\begin{aligned} R^{(2)}_{\mu\nu} =& \frac{1}{2}\Big[\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - h^{\alpha\beta}\partial_\nu\partial_\beta h_{\alpha\mu} - h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu} \\ &+ h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial^\beta h^\alpha_\nu \partial_\beta h_{\alpha\mu} - \partial^\beta h^\alpha_\nu\partial_\alpha h_{\beta\mu} - \partial_\beta h^{\alpha\beta}\partial_\nu h_{\alpha\mu} \\ &+ \partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu} - \partial_\beta h^{\alpha\beta}\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial^\alpha h\partial_\alpha h_{\mu\nu} + \frac{1}{2}\partial^\alpha h \partial_\nu h_{\alpha\mu} \\ &+ \frac{1}{2}\partial^\alpha h\partial_\mu h_{\alpha\nu}\Big] \end{aligned} \tag{186}$$

$$\begin{aligned} R^{(2)} =& \frac{1}{2}\big[\frac{3}{2}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + h^{\alpha\beta}\Box h_{\alpha\beta} - 2h^{\alpha\beta}\partial^\mu\partial_\beta h_{\alpha\mu} + h^{\alpha\beta}\partial_\alpha\partial_\beta h \\ &- \partial^\beta h^{\alpha\mu}\partial_\alpha h_{\beta\mu} - 2\partial_\beta h^{\alpha\beta}\partial^\mu h_{\alpha\mu} + \partial_\beta h^{\alpha\beta}\partial_\alpha h - \frac{1}{2}\partial_\alpha h\partial^\alpha h \\ &+ \partial^\alpha h\partial^\mu h_{\alpha\mu} - h^{\nu\sigma}\partial_\nu\partial_\mu h^\mu{}_\sigma + \frac{1}{2}h^{\nu\sigma}\Box h_{\nu\sigma} + \frac{1}{2}h^{\nu\sigma}\partial_\nu\partial_\sigma h\big] \end{aligned} \tag{187}$$

Lastly, what we need is the determinant to be expanded, but we only need it to first order since the Riemann curvature is at least to first order.

$$\begin{aligned} ln(|det(g)|) &= Tr(ln(g_{\mu\nu})) = Tr(ln(\eta_{\mu\nu} + h_{\mu\nu})) \\ &= ln(det(\eta_{\mu\nu})) + Tr(\eta^{\mu\nu}h_{\mu\nu}) + \mathcal{O}(h^2) \\ &= h \\ \implies |det(g)| &= e^h \\ \implies \sqrt{|g|} &= 1 + \frac{1}{2}h \end{aligned}$$

Putting these all together we would get

$$S = -\frac{c^3}{16\pi G}\int d^4x(R^{(1)} + \frac{1}{2}hR^{(1)} + R^{(2)}) + \frac{1}{2M_{PL}}\int d^4x h_{\mu\nu}T^{\mu\nu} \tag{188}$$

We would also like to add a matter part to this action, which is done simply enough by adding a coupling term. The final action is then given below.

$$S = \frac{M_{PL}^2}{8}\int d^4x\left[\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \partial_\mu h\partial^\mu h + 2\partial_\mu h^{\mu\nu}\partial_\nu h - 2\partial_\mu h^{\mu\nu}\partial^\rho h_{\rho\nu}\right] + \frac{1}{2}\int d^4x h_{\mu\nu}T^{\mu\nu} \tag{189}$$

B Linearising the EH action in FRW spacetime

We begin with the form of the metric. $g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu})$ We only take into account the transverse and traceless peturbations that are the gravitational waves, so $h_{\mu\nu} = h_{ij}^{(TT)}$ (again, will omit the superscript (TT) for ease). The Christoffel symbols to second order are:

$$\begin{aligned}\Gamma^0_{i0} = \Gamma^i_{00} = 0 \quad \Gamma^0_{00} = \mathcal{H} \quad \Gamma^0_{ij} &= \mathcal{H}(\delta_{ij} + h_{ij}) + \frac{1}{2}h'_{ij}\\ \Gamma^i_{j0} &= \mathcal{H}\delta_{ij} + \frac{1}{2}h'_{ij} - \frac{1}{2}h'_{jk}h_{ik}\\ \Gamma^i_{jk} &= \frac{1}{2}(h_{ij,k} + h_{ik,j} - h_{kj,i}) - \frac{1}{2}h_{il}(h_{lj,k} + h_{lk,j} - h_{kj,l})\end{aligned} \tag{190}$$

Then the Ricci tensors are:

$$\begin{aligned}R_{00} &= -3\mathcal{H}' + \frac{1}{2}h''_{ik}h_{ki} + \frac{1}{4}h'_{ij}h'_{ij} + \frac{1}{2}\mathcal{H}h'_{ik}h_{ik}\\ R_{ij} &= \mathcal{H}'(\delta_{ij} + h_{ij}) + \mathcal{H}h'_{ij} + \frac{1}{2}h''_{ij} - \frac{1}{2}h_{ij,kk} - \frac{1}{2}h_{kl}(h_{li,jk} + h_{lj,ik} - h_{ij,lk}) + \frac{1}{2}h_{kl,j}h_{lk,i} + \frac{1}{2}h_{kl,ij}h_{kl}\\ &\quad + 2\mathcal{H}^2\delta_{ij} + 2\mathcal{H}h_{ij} - \frac{1}{2}\mathcal{H}h'_{kl}h_{kl}\delta_{ij} - \frac{1}{2}h'_{kj}h'_{kj} - \frac{1}{4}(h_{lj,k} + h_{lk,j} - h_{kj,l})(h_{ik,l} + h_{kl,i} - h_{il,k})\end{aligned} \tag{191}$$

We then contract the Ricci tensor using the inverse metric.

$$\begin{aligned}R &= g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij}\\ &= -a^{-2}R_{00} + a^{-2}(\delta_{ij} - h_{ij} + h_{ik}h_{kj})R_{ij}\end{aligned} \tag{192}$$

$$
\begin{aligned}
\delta_{ij}R_{ij} &= 3\mathcal{H}' + \frac{1}{2}h_{kl,i}h_{lk,i} + \frac{1}{2}h_{kl}h_{kl,ii} + 6\mathcal{H}^2 - \frac{3}{2}\mathcal{H}h'_{ij}h_{ij} - \frac{1}{2}h'_{ij}h'_{ij} - \frac{1}{2}h_{li,k}h_{kl,i} + \frac{1}{4}h_{il,k}h_{il,k} \\
-h_{ij}R_{ij} &= -\mathcal{H}'h_{ij}h_{ij} - \mathcal{H}h'_{ij}h_{ij} - \frac{1}{2}h''_{ij}h_{ij} + \frac{1}{2}h_{ij,kk} - \frac{7}{2}\mathcal{H}^2h_{ij}h_{ij} \\
h_{ik}h_{kj}R_{ij} &= -\mathcal{H}'h_{ij}h_{ij} + 2\mathcal{H}^2h_{ij}h_{ij}
\end{aligned}
\tag{193}
$$

Add these all together to get the Ricci scalar.

$$
R = a^{-2}(6\mathcal{H}' + 6\mathcal{H}^2 - \frac{1}{4}h'_{ij}h'_{ij} - 3\mathcal{H}h'_{ij}h_{ij} + \frac{3}{4}h_{ij,k}h_{ij,k} + h_{ij,kk}h_{ij} - \frac{1}{2}h_{ij,k}h_{kj,i} - \frac{3}{2}\mathcal{H}^2h_{ij}h_{ij}) \tag{194}
$$

The second last term will go to zero after integration by parts. We also need to take in to account the volume element. We can use the same formula as derived for the Minkowski volume element, but substitute in the FRW metric instead.

$$
\sqrt{-g} = a^4(1 - \frac{1}{4}h_{ij}h_{ij}) \tag{195}
$$

After some integration by parts and ignoring the terms independent of h_{ij}, we get

$$
a^2(-\frac{1}{4}h'_{ij}h'_{ij} + \frac{1}{4}h_{ij,k}h_{ij,k}) \tag{196}
$$

Finally, the action for gravitational waves in comoving FRW coordinates is

$$
S = \frac{M_{PL}^2}{2}\int d^4x\ a^2(h'_{ij}h'_{ij} - h_{ij,k}h_{ij,k}) + \int d^4x\ a^2(h_{ij}\sigma^{(TT),ij}) \tag{197}
$$

This leads to an equation of motion

$$
h''_{ij} - \partial_k\partial_k h_{ij} + 2\mathcal{H}h'_{ij} = \frac{1}{M_{PL}^2}\sigma^{(TT),ij} \tag{198}
$$

C Linearising dRGT Lagrangian

Since we have already linearised the massless action in FRW (albeit, not in the ADM formalism) we will only try to expand the interaction term in the dRGT action to linear order in the tensor perturbations. We will not be using the full non-linear dRGT action, and set

$\alpha_3 = \alpha_4 = 0$, $\alpha_2 = \frac{1}{2}$. The two metrics are assumed to be homogeneous and isotropic to zeroth order (i.e. FRW) and their perturbations are transverse and traceless tensors only.

$$g_{\mu\nu} = (-N^2, a^2\delta_{ij} + a^2 H_{ij}) \implies g^{\mu\nu} = (-N^{-2}, a^{-2}\delta_{ij} - a^{-2}H^{ij} + a^{-2}H^{ik}H^{kj}) \tag{199}$$

$$f_{\mu\nu} = (-n^2, b^2\delta_{ij} + b^2 h_{ij}) \tag{200}$$

and we define

$$X = \frac{b}{a} \qquad \tilde{c} = \frac{na}{Nb} = \frac{n}{NX} \qquad A^{\mu}{}_{\nu} = g^{\mu\rho} f_{\rho\nu} \qquad B^{\mu}{}_{\nu} = \sqrt{A^{\mu}{}_{\nu}} \tag{201}$$

Since we will be taking the low energy limit ($X = X_C$ and $\tilde{c} = 1$) anyway, we decide to implement it now just for ease in the calculation. The square root of a tensor significantly complicates our working. Luckily, since we are only working to second order we only need the square root of the tensor to second order. Below, we use an expansion for a square root of a matrix used in [26].

$$\begin{aligned}
A^{\mu}{}_{\nu} =& \frac{n^2}{N^2}\delta^{\mu 0}\delta_{\nu 0} + X^2\delta^i_j + X_C^2 h^i_j - X_C^2 H^i_j - X_C^2 H^{ik}h_{kj} + X_C^2 H^{ik}H_{kj} \\
\implies& (A^{(0)})^t{}_t = \frac{n^2}{N^2}\,, \quad (A^{(0)})^i{}_t = (A^{(0)})^t{}_j = 0\,, \quad (A^{(0)})^i{}_j = X_C^2\delta^i_j\,, \\
& (A^{(1)})^t{}_t = 0\,, \quad (A^{(1)})^i{}_t = (A^{(1)})^t{}_j = 0\,, \quad (A^{(1)})^i{}_j = X_C^2(h^i{}_j - H^i{}_j)\,, \\
& (A^{(2)})^t{}_t = 0\,, \quad (A^{(2)})^i{}_t = (A^{(2)})^t{}_j = 0\,, \quad (A^{(2)})^i{}_j = X_C^2(H^{ik}H_{kj} - H^{ik}h_{kj})
\end{aligned} \tag{202}$$

$$\begin{aligned}
\implies& (B^{(0)})^t{}_t = \frac{n}{N}\,, \quad (B^{(0)})^i{}_t = (B^{(0)})^t{}_j = 0\,, \quad (B^{(0)})^i{}_j = X_C\delta^i_j\,, \\
& (B^{(1)})^t{}_t = 0\,, \quad (B^{(1)})^i{}_t = (B^{(1)})^t{}_j = 0\,, \quad (B^{(1)})^i{}_j = X_C(h^i{}_j - H^i{}_j)\,, \\
& (B^{(2)})^t{}_t = 0\,, \quad (B^{(2)})^i{}_t = (B^{(2)})^t{}_j = 0\,, \\
& (B^{(2)})^i{}_j = \frac{1}{2}X_C\left(\frac{3}{4}H^{ik}H_{kj} - \frac{1}{4}h^{ik}h_{kj} - \frac{1}{4}H^{ik}h_{kj} - \frac{1}{4}h^{ik}H_{kj}\right)
\end{aligned} \tag{203}$$

We need the potential to second order in h and H. Note that we are working with $\mathcal{K}^\mu{}_\nu = \delta^\mu_\nu - A^\mu_\nu$, not just A^μ_ν.

$$
\begin{aligned}
([\mathcal{K}])^2 &= 16 - 8\frac{n}{N} - 24X + 6\tilde{c}X^2 + \frac{n^2}{N^2} + H^{ij}H_{ij}\left(\frac{3}{4}\tilde{c}X^2 - 3X\right) \\
&\quad + h^{ij}h_{ij}\left(X - \frac{1}{4}\tilde{c}X^2\right) + H^{ij}h_{ij}\left(2X - \frac{1}{4}\tilde{c}X^2\right) \\
[\mathcal{K}^2] &= 4 - 2\frac{n}{N} + \frac{n^2}{N^2} - 6X + 3X^2 + H^{ij}H_{ij}\left(X^2 - \frac{3}{4}X\right) \\
&\quad + \frac{1}{4}Xh^{ij}h_{ij} + H^{ij}h_{ij}\left(\frac{1}{2}X - X^2\right)
\end{aligned}
\tag{204}
$$

$$
\begin{aligned}
([\mathcal{K}])^2 - [\mathcal{K}^2] &= 12 - 6\frac{n}{N} - 18X_C + 6\tilde{c}X_C^2 - 3X_C^2 + H^{ij}H_{ij}X_C\left(X_C\left(1 + \frac{3}{4}\tilde{c}\right) - \frac{15}{4}\right) \\
&\quad + h^{ij}h_{ij}X_C\left(\frac{5}{4} - \frac{1}{4}\tilde{c}X_C\right) + H^{ij}h_{ij}X_C\left(\frac{5}{2} - X_C\left(\frac{1}{4}\tilde{c} + 1\right)\right) \\
&= 12 - 6\frac{n}{N} - 18X_C + 6X_C^2 - 3X_C^2 + H^{ij}H_{ij}X_C\left(\frac{7}{4}X_C - \frac{15}{4}\right) \\
&\quad + h^{ij}h_{ij}X_C\left(\frac{5}{4} - \frac{1}{4}X_C\right) + H^{ij}h_{ij}X_C\left(\frac{5}{2} - \frac{5}{4}X_C\right)
\end{aligned}
\tag{205}
$$

We then need to multiply by $\sqrt{-g} = Na^3(1 - \frac{1}{4}h^{ij}h_{ij})$. The correct expansion is given in [19]. It is given below.

$$
\mathcal{L}_{mass} = -\frac{M_{PL}^2}{8}Na^3m^2\Gamma(X)\left(H_{ij} - h_{ij}\right)\left(H^{ij} - h^{ij}\right)
\tag{206}
$$

Where

$$
\Gamma(X) = X\left(\frac{3}{2} - X\right)
\tag{207}
$$

and the final action is

$$
\begin{aligned}
S = \frac{M_{PL}^2}{8}\int d^4xNa^3\sqrt{\gamma}\Bigg[&\frac{\dot{H}^{ij}\dot{H}_{ij}}{N^2} + \frac{H^{ij}}{a^2}\nabla^2H_{ij} + \kappa\tilde{c}X^4\left(\frac{\dot{h}^{ij}\dot{h}_{ij}}{n^2} + \frac{h^{ij}}{b^2}\nabla^2h_{ij}\right) \\
&-m^2\Gamma(X)\left(H^{ij} - h^{ij}\right)\left(H_{ij} - h_{ij}\right)\Bigg]
\end{aligned}
\tag{208}
$$

D Diagonalising the linearised dRGT Lagrangian

We start from the linearised action from (208). It is clear from the action that one of the mass eigenstates should be of the form $s(H_{ij} - h_{ij})$, where s is some coefficient. The second mass eigenstate can be taken to be $qH_{ij} + rh_{ij}$, again q and r some coefficients yet to be determined. We can bring the time derivatives in the action together and try and match up the coefficients in order to find q, r and s.

$$\begin{aligned}\dot{H}_{ij}\dot{H}^{ij} + \kappa X_c^2 \dot{h}_{ij}\dot{h}^{ij} &= (q\dot{H}_{ij} + r\dot{h}_{ij})(q\dot{H}^{ij} + r\dot{h}^{ij}) + s(\dot{H}_{ij} - \dot{h}_{ij})(\dot{H}^{ij} - \dot{h}^{ij}) \\ &= (q^2 + s)\dot{H}_{ij}\dot{H}^{ij} + 2(qr - s)\dot{H}_{ij}\dot{h}^{ij} + (r^2 + s)\dot{h}_{ij}\dot{h}^{ij} \\ \implies q^2 + s = 1 \ , \quad qr = s \ , \quad r^2 + s = \kappa X_c^2\end{aligned} \tag{209}$$

We substitute the second equation in to the the third one to get q as a function r. This can be used in the first equation (along with $s = \kappa X_c^2 - r^2$) to find r as a function of κ and X_c only. Once we have r, we just work backwards to get q and s.

$$q = \frac{1}{\sqrt{1 + \kappa X_c^2}} \ , \quad r = \frac{\kappa X_c^2}{\sqrt{1 + \kappa X_c^2}} \ , \quad s = \frac{\kappa X_c^2}{1 + \kappa X_c^2} \tag{210}$$

Writing out only the massless eigenstate's action we see that we get

$$\begin{aligned}S_{massless} = \frac{M_g^2}{8}\int d^4xNa^3\Bigg[&\frac{1}{N^2}\left(\frac{\dot{H}_{ij} + \kappa X_c^2\dot{h}_{ij}}{\sqrt{1 + \kappa X_c^2}}\right)\left(\frac{\dot{H}^{ij} + \kappa X_c^2\dot{h}^{ij}}{\sqrt{1 + \kappa X_c^2}}\right) \\ &+ \frac{1}{a^2}\left(\frac{\dot{H}_{ij} + \kappa X_c^2\dot{h}_{ij}}{\sqrt{1 + \kappa X_c^2}}\right)\nabla^2\left(\frac{\dot{H}^{ij} + \kappa X_c^2\dot{h}^{ij}}{\sqrt{1 + \kappa X_c^2}}\right)\Bigg]\end{aligned} \tag{211}$$

Just for cleanliness, we square the denominator of the massless mode and then multiply by an extra factor of $1 + \kappa X_c^2$ (in effect multiplying by one). Finally, we define

$$H^+_{ij} = \frac{H_{ij} + \kappa X_c^2 h_{ij}}{1 + \kappa X_c^2} \tag{212}$$

$$\begin{aligned}\implies S &= \frac{1}{8}\int d^4xNa^3M_g^2(1 + \kappa^2 X_c^2)\left(\frac{\dot{H}^{ij}_+\dot{H}^+_{ij}}{N^2} + \frac{H^{ij}_+}{a^2}\nabla^2 H^+_{ij}\right) \\ &= \frac{1}{8}\int d^4xNa^3M_+^2\left(\frac{\dot{H}^{ij}_+\dot{H}^+_{ij}}{N^2} + \frac{H^{ij}_+}{a^2}\nabla^2 H^+_{ij}\right)\end{aligned} \tag{213}$$

Where we have defined $M_+^2 = M_g^2(1 + \kappa X_c^2)$. For the massive mode, we absorb s in to the coefficient at the front, defining $M_-^2 = \frac{\kappa X_c^2}{1+\kappa X_c^2} M_g^2$. The massless eigenstate is then simply given by $H^{-}_{ij} = H^{ij} - h_{ij}$. A similar procedure for the spatial derivative can also be carried out (or can just be verified). The final action is then

$$S = \frac{1}{8}\int d^4xNa^3\left[M_+^2\left(\frac{\dot{H}_+^{ij}\dot{H}_{ij}^+}{N^2} + \frac{H_+^{ij}}{a^2}\nabla^2 H_{ij}^+\right)\right. \\ \left. + M_-^2\left(\frac{\dot{H}_-^{ij}\dot{H}_{ij}^-}{N^2} + \frac{H_-^{ij}}{a^2}\nabla^2 H_{ij}^- - m_{\text{eff}}^2 H_-^{ij} H_{ij}^-\right)\right] \tag{214}$$

E Showing the Bunch-Davies vacuum agrees with Minkowski at sub-horizon levels

The mode function we found for the scalar field χ was

$$f_k(\eta) = -\frac{1}{2}\sqrt{\frac{\pi}{k}}(-k\eta)^{\frac{1}{2}} H_\nu^{(1)}(-k\eta) \qquad \nu = \frac{3}{2} + \epsilon \tag{215}$$

$H_\nu^{(1)}$ is a Hankel function of the first kind and is defined in (216).

$$H_\nu^{(1)}(x) \equiv J_\nu(x) + iY_\nu(x) \tag{216}$$

$J_{n+\frac{1}{2}}(x) \equiv \sqrt{\frac{2x}{\pi}} j_n(x)$ is Bessel function of the first kind and $Y_{n+\frac{1}{2}}(x) \equiv \sqrt{\frac{2x}{\pi}} y_n(x)$ is the Bessel function of the second kind. $j_n(x)$ and $y_n(x)$ are the spherical Bessel functions. Since $\epsilon << 1$, then $\nu \approx \frac{3}{2}$. So we only need the first spherical Bessel functions. They are given in (**??**).

$$j_1(x) = -\frac{d}{dx}\frac{sin(x)}{x}\,, \quad y_1(x) = \frac{d}{dx}\frac{cos(x)}{x} \tag{217}$$

We write $-k\eta = x$ for ease in the working below.

$$\begin{aligned}
\sqrt{x}H_{1+\frac{1}{2}} &= \sqrt{x}(J_{1+\frac{1}{2}}(x) + iY_{1+\frac{1}{2}}(x)) \\
&= \sqrt{\frac{2}{\pi}}x\,(j_1(x) + iy_1(x)) \\
&= \sqrt{\frac{2}{\pi}}x\left(-\frac{d}{dx}\left(\frac{sin(x)}{x}\right) + i\frac{d}{dx}\frac{cos(x)}{x}\right) \\
&= \sqrt{\frac{2}{\pi x^2}}\,(-xcos(x) + sin(x) - ixsin(x) - icos(x)) \\
&= -\sqrt{\frac{2}{\pi}}\left(e^{ix} + \frac{i}{x}e^{ix}\right)
\end{aligned} \tag{218}$$

For $|x| >> 1$, we get

$$\sqrt{x}H^{(1)}_{1+\frac{1}{2}}(x) \approx -\sqrt{\frac{2}{\pi}}e^{ix} \tag{219}$$

$$\implies f_k(\eta) \approx \sqrt{\frac{1}{2k}}e^{ik\eta} \qquad \text{for} \quad |k\eta| >> 1 \tag{220}$$

Which are the mode functions for Minkowski spacetime.