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Monopole Formula and Ungauging Schemes of Coulomb branches of 3d $\mathcal{N} = 4$ Gauge Theories

Author:

Keerati Keeratikarn

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Supervisor:

Professor Amihay Hanany

Imperial College London

South Kensington Campus

London SW7 2AZ, UK

Keeratikarn, Keerati:

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Contents

Abstract

1 Introduction and Motivation

1.1 Motivation 1

2 Preliminary

2.1 3d $\mathcal{N}$ =4 Supersymmetry 5

2.2 3d $\mathcal{N}$=2 to 3d $\mathcal{N}$=4 6

2.2.1 Decompsition of 3d $\mathcal{N}$=4 Vector Multiplet 6

2.2.2 Decompsition of 3d $\mathcal{N}$=4 Hyperumltiplet 7

2.2.3 Spaces of Supersymmetric Vacua 7

2.3 Moduli Spaces of Vacua 9

2.3.1 Moduli Spaces: $\mathcal{M}$ 9

2.3.2 Chiral Operators 10

2.3.3 Simplest Example of $\mathcal{M}$ 11

2.4 Hilbert series: HS 13

2.4.1 Example of HS 14

2.4.2 Refined Hilbert Series 18

2.5 Plethystic Programmes 21

2.6 Highest Weight Generating Function: HWG 23

2.7 Quiver Diagrams in 3d $\mathcal{N}$=4 24

3 Coulomb Branches

3.1 Monopole Operators 27

3.2 Balanced quivers 30

3.3 $U(1)$ with N Flavours 31

3.4 Minimal Nilpotent Orbit of A_2 34

3.5 Minimal Nilpotent Orbit of A_3 37

3.6 Supra Minimal Nilpotent Orbit of A_3 39

3.7 Minimal Nilpotent Orbit of A_4 42

3.8 Supra Minimal Nilpotent Orbit of A_4 44

4 Ungauging Scheme

4.1 Modified Monopole Formula 47

4.2 Choice of Ungauging Scheme 48

4.3 Affine $\tilde{G}_2$ 50

4.3.1 Ungauging 50

4.3.2 Re-expression 52

5 Conclusions and Future Work

5.1 Conclusions 55

A Detailed Calculation

A.1 Supra Minimal Orbit of A_4 57

A.2 G_2 in $SU(3)$ Expression 61

B Bibliography

Abstract

This dissertation focus on three dimensional $\mathcal{N} = 4$ quiver gauge theories. We study these quiver theories by counting holomorphic functions on the moduli space of vacua via computing Hilbert series. To obtain the Coulomb branch classically, we introduce the monopole formula to compute the corresponding Hilbert series for A-series quivers corresponding to some minimal nilpotent orbits, i.e. the minimal nilpotent orbit of A_1, A_2, A_3, A_4 and the supra minimal nilpotent orbit of A_3, A_4. We then try to distinct the moduli space of the A_3 and A_4 quivers with the different orbits. Consequently, the moduli space of the quivers corresponding to the supra minimal orbit is big comparing to the minimal orbit quivers.

We also provide a technique of ungauging scheme applied to the monopole formula for a non-simply laced quivers without any flavour group, i.e. affine $\tilde{G}_2$. choices of the ungauging scheme are on the long side of the non-simply laced edges. We find that all the choices are equivalent and provide the same moduli space.

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1

Introduction and Motivation

1.1 Motivation

Supersymmetric (SUSY) gauge field theories have played a crucial role in quantum field theory. A model of the theory can be written as SUSY lagrangian which contain irreducible representation (irreps) of the Super-Poincaré algebra. These irreps are called supermultiplets. In this dissertation, we study three dimensional $\mathcal{N} = 4$ gauge theories so the corresponding supermultiplets are vector multiplets and hypermultiplets corresponding to the gauge groups and the matter fields in the theory. There is also a superpotential, being included in the lagrangian, specifying the interactions in the theories. One can obtain these 3d $\mathcal{N} = 4$ theories by performing dimensional reduction from 4d $\mathcal{N} = 2$ theories. More precisely, the $\mathcal{N} = 4$ vector multiplets are constructed from the $\mathcal{N} = 2$ vector and the $\mathcal{N} = 2$ chiral multiplets; while $\mathcal{N} = 4$ vector multiplets are formed by combining the $\mathcal{N} = 2$ chiral and anti-chiral multiplets. However, as shown in [1], the lagrangian of the theories is very complicated and difficult to study. Therefore, the challenge for us is to find a succinct way to encode the matter contents of the theories, the gauge group which they all transform under and also the superpotential.

Quiver diagrams was first introduced to physics in [2] by Douglas and Moore. In 3d $\mathcal{N} = 4$, these diagrams encode the gauge groups and matter contents of the theories. 3d $\mathcal{N} = 4$ SUSY gauge theories, that are described by the quiver diagrams, are known as 3d $\mathcal{N} = 4$ *Quiver Theories.*

These theories have a moduli space of vacua, including a Higgs branch and a Coulomb branch. The moduli space is defined as the space of vacua parameterised by VEVs of scalars in the theory. We may see the VEVs of *Gauge invariant Chiral Operators* (GIO) as holomophic functions on the moduli space. To understand the moduli space, we can count GIO in the chiral ring of the theories [3]. This is mathematically equivalent to enumerating holomorphic functions in the moduli space [4] and yields a generating function [3], called a *Hilbert series*. This perspective allows us to study the moduli space as algebraic varieties. This aspect will be discussed in chapter 2.

Generally, we are fascinated by the 3d $\mathcal{N} = 4$ Higgs and the 3d $\mathcal{N} = 4$ Coulomb branch, repectively parameterised by VEVs of in the hypermultiplet and the vector multiplet in the theory. Since the Higgs branch is an object in classical level, it does not receive any quantum correction [AH-mono]. Moreover, its Hilbert series can be determined by *Molien-Weyl integrals* as illustrated in [5; 6; 7]. On the other hand, the Coulomb branch is a quantum object which is difficult to compute its Hilbert series due to quantum corrections. In the past few decades, there has been the method, introduced in [8], to achieve the Coulomb branch by computing its mirror Higgs branch or understanding the metric of the Coulomb branch at one-loop correction. These methods are exhausted and time-consuming when we face a big quiver with a large number of gauge groups, which means that they work efficiently if the gauge group are sufficiently small. Therefore, we wish to find an approach to compute the Coulomb branch classically.

In the recently years, there has been an efficient formula, called the *monopole formula* [9]. It allows us to compute the Coulomb branch Hilbert series via counting dressed monopole operators which are the classical objects. It can be thought of multiplying by the classical factors corresponding to the gauge groups in our theory. We will investigate more computing details by using the formula or , particularly, calculate the Coulomb branch Hilbert series of A-series quivers corresponding some minimal nilpotent orbits in chapter 3.

In addition to the computation of the Coulomb branch as the moduli space of dressed monopole operators, for a simply (single) laced quiver with purely gauge nodes, one has to ungauge a residual center-of-mass $U(1)$ symmetry. We can make any choice to ungauge the $U(1)$ symmetry, leading to the equivalence in the unique Coulomb branch [10]. This is non-trivial for a flavorless non-simply laced quiver with edge multiplicity $\lambda = 2,3$, since the monopole formula need to be modified in order to achieve its Coulomb branch Hilbert series. More importantly, all choices of *ungauging scheme* are inequivalent, introduced in [10] in this April. More precisely, this technique, interestingly, yields a valid or non-valid Coulomb branch depending on the choices of the ungauging scheme. In chapter 4, we will only take the analysis of the ungauging scheme on the long side of the non-simply laced edge quiver resulting in the same valid Coulomb branch. For any choice on the short side, one may look further in [10].

2
Preliminary

In this chapter we would like to introduce briefly the space of vavua, also called the moduli space, in 3d $\mathcal{N}$=4 gauge theory through physical and mathematical aspects. We also serve mathematical machinery to find the key features of moduli spaces, including their *dimensions*, their *generators* and their *relations*, in the following sections.

2.1 3d $\mathcal{N}$ =4 Supersymmetry

In this paper we interest moduli spaces of 3d $\mathcal{N}$=4 gauge theory. This theory carries 8 supercharges, which it can be obtained from dimensional reduction of 5+1d $\mathcal{N}$=(1,0) theory. In 5+1d $\mathcal{N}$=(1,0) $SU(2)_R$ is the R-symmetry group rotating the supercharges [11]. In the representations of $SU(2)_R$ ×SO(4)$_{\text{littile}}$, there is the 5+1d vector multiplet consisting of a right-handed chiral spinor and a 5+1d gauge field. We can decompose the 5+1d vector V_ν , $\nu = 0,1,...,5$, into a 2+1d vector A_μ, $\mu = 0,1,2$, and three scalar fields ϕ^i, i= 1, 2, 3. These scalar fields lie in the defining representation of $SO(3)_S$ which corresponds to a rotation in three reduced dimensions (V_ν, $\nu = 3,4,5$).

Since 3d $\mathcal{N}$=4 vector multiplet contains a 2+1d vector field, three scalars and a 4 components fermionic field transforming in (1,0,0), (0,1,0) and $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ spin representation respectively under the $SO(2,1) \times SO(3)_S \times SU(2)_R$ [12], we also have to take the double cover $SU(2)_S$ of $SO(3)_S$ to include other fermions. Thus the R-symmetry of a 3d $\mathcal{N}$=4 gauge theory is

$SO(4) \cong SU(2)_S \times SU(2)_R$, and the scalar fields in 3d $\mathcal{N}$=4 vector multiplet lie in the adjoint representation of $SU(2)_S$.

In the next subsection we will state how to construct a 3d $\mathcal{N}$=4 gauge theory, namely embodying 3d $\mathcal{N}$=2 multiplets as the building blocks for the $\mathcal{N}$=4 theory.

2.2 3d $\mathcal{N}$=2 to 3d $\mathcal{N}$=4

Due to high constrains in $\mathcal{N} = 4$ Super Yang-Mills (SYM) theories, i.e. 3d $\mathcal{N}$=4 gauge theories, there is an easy way to build up the 3d $\mathcal{N}$=4 SUSY by working on 3d $\mathcal{N}$=2 supersymmetric theories which have a half of supercharge numbers. By doing this, it might be easy to write the superpotential of the theories in order to consider their moduli spaces. To construct $\mathcal{N}$=4 from $\mathcal{N}$=2 in three dimensions, one can decompose the $\mathcal{N}$=4 vector multiplet into the $\mathcal{N}$=2 vector multiplet V and the chiral multiplet Φ; and the $\mathcal{N}$=4 hyper multiplet into the $\mathcal{N}$=2 chiral multiplet C and the anti-chiral multiplet $\bar{C}$ [9].

2.2.1 Decompsition of 3d $\mathcal{N}$=4 Vector Multiplet

We will follow [13] to help us discuss the decompositions of the $\mathcal{N}$=4 vector multiplet in more details. The $\mathcal{N}$=2 vector multiplet V contains a vector field A_μ, a Dirac spinor ζ_α and a real scalar φ; while the chiral Φ includes a complex scalar field χ and a Dirac spinor λ_α. These fields are in the adjoint representation of the gauge group. In the terms of the global symmetry $SU(2)_S \times \mathrm{SU}(2)_R$, both Dirac spinors ($\zeta_\alpha$,$\lambda_\alpha$) transform in $\left(\frac{1}{2}, \frac{1}{2}\right)$ spin representaion as doublets of the global symmetry. This is equivalent to a vector transforming under the global symmetry $SO(4)$. To analyse the scalars' transformation, we can take the scalars ϕ_1, ϕ_2, ϕ_3 from the vector of the reduced dimensions in the previous section 2.1. So let us set $\varphi \equiv \phi_3$ and $\chi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ so that one can map $(\phi_1, \phi_2, \phi_3) \mapsto (\varphi, Re[\chi], Im[\chi])$ being in the triplet of $SU(2)_S$, or (1,0) under the global.

2.2.2 Decompsition of 3d $\mathcal{N}$=4 Hyperumltiplet

Now analysing the $\mathcal{N}$=4 hypermultiplet under the same global symmetry $SU(2)_S$ ×SU$(2)_R$, the hyper multiplets is built by combining the $\mathcal{N}$=2 chiral C and anti-chiral $\bar{C}$ [13]. These two chirals C and $\bar{C}$ contain the same types of fermionic fields which are ψ_α, ξ_α Dirac spinors and A, B complex scalars. Those two dirac spinors transform in $(\frac{1}{2},\frac{1}{2})$ as doublets of the global, while the real and imaginary parts of those two complex scalar fields can be arranged as $(A, B^\dagger)$ and $(A^\dagger, B)$ transforming in $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ respectively under the global. In the terms of the gauge group, one could have the the $\mathcal{N}$=2 chiral transforms in the fundamental representation R of the gauge group and the $\mathcal{N}$=2 anti-chiral in the conjugate representation R^* of the chiral.

2.2.3 Spaces of Supersymmetric Vacua

To understand the moduli spaces of the theory, let us consider a holomorphic superpotential (2.1) restricted by $\mathcal{N}$=4 SUSY. [13]

$$S_{sp} = -i\sqrt{2}\int d^3xd^2\theta \sum_{matter}(\bar{C}\Phi C) + c.c. \tag{2.1}$$

where the summation is running over matters, which is charged under the gauge group, connected with the adjoint $\mathcal{N}$=2 chiral Φ. Moreover, this sum has a couple of the tensor product of $R^* \otimes Adj \otimes R$ corresponding to the cubic terms , i.e. $\bar{C}\Phi C$, in the superpotential. This leads to an addition of a trace to the superpotential and one obtains

$$W = Tr(\bar{C}_{ij}\Phi_{jk}C_{kl}) \tag{2.2}$$

We can vary the superpotential W with respect to the superfields, i.e. $\bar{C}, \Phi, C$; subsequently, and set each of them to zero. By doing this we will obtain the set of F-term equations of the theory, as

$$\frac{\partial W}{\partial \bar{C}_{ij}} = (\Phi C)_{ji} = 0, \quad \frac{\partial W}{\partial C_{ij}} = (\bar{C}\Phi)_{ji} = 0, \quad \frac{\partial W}{\partial \Phi_{ij}} = (C\bar{C})_{ji} = 0 \tag{2.3}$$

Let us analyse a case where the scalar vacuum expectation value(VEV), $\langle\Phi\rangle$, is non-zero. To satisfy the first two terms in (2.3), C and $\bar{C}$ have to vanish. Since the moduli spaces parameterised by Φ which are in the $\mathcal{N}$=4 vector multiplet, this space of vacua correspond to the Coulomb branch, $\mathcal{M}_C$. However, one cannot calculate $\mathcal{M}_C$ through the F-term due to the requirement of quantum corrections, discussing in [14].

Another case is $\langle\Phi\rangle = 0$, then C and $\bar{C}$ must not be zero. This gives us a non-trivial third term in (2.3). In this case the space of vacua is parameterised by the complex scalars found in the $\mathcal{N}$=4 hypermultiplet, leading to the Higgs branch moduli space, $\mathcal{M}_H$. It can be computed by using the F-term equations, since $\mathcal{M}_H$ does not receive quantum corrections. In [15], the relations are derived by using the F-terms in order to define $\mathcal{M}_H$ as an algebraic variety. More interestingly, one can use 3d mirror symmetry to obtain a corresponding algebraic variety of $\mathcal{M}_C$ [14; 16].

I the following section, we would provide an understanding and algebraic description of the moduli space of supersymmetric vacua via a scalar potential of the theory.

2.3 Moduli Spaces of Vacua

2.3.1 Moduli Spaces: $\mathcal{M}$

We firstly look at the physical aspect of the space of vacua solutions by extracting the 3d $\mathcal{N}$=2 scalar potential. Since the 3d $\mathcal{N}$=2 multiplets can be obtained by dimensional reduction of 4d $\mathcal{N}$=1 vector and chiral multiplet [4]. Classically[1], one can write down the 3d $\mathcal{N}$=2 scalar potential as in 4d $\mathcal{N}$=1 theories.

Under the gauge group G with representation R, let us consider the 3d $\mathcal{N}$=2 scalar potential V with Φ_i being the i^{th} complex chiral multiplet, $i = 1,2,...,dim(R)$ [4].

$$V = \sum_i |F_i|^2 + \frac{g^2}{2}\sum_a (D^a)^2 \tag{2.4}$$

where $F_i = \partial W/\partial\Phi^i$ are the F-term equations, $D^a = \sum_{ij}\Phi_i^\dagger (T^a)^i_j \Phi^j$ are the D-terms and a gauge coupling constant g.

Physically, the moduli space ($\mathcal{M}$) or the space of vacua determines the lowest energy behaviour of the supersymmetric QFT under the gauge group. Mathematically, the **moduli space** ($\mathcal{M}$) is defined as the set of constant field configurations minimising the potential (2.4) with gauge invariance considered.

$$\mathcal{M} = \frac{\{(\Phi,\Phi^\dagger)|F_i = 0, \forall i \quad D^a = 0, \forall a\}}{G} \tag{2.5}$$

It provides a connection between supersymmetric field theory and algebraic geometry. So we can always see $\mathcal{M}$ as the space of supersymmetric vacua solutions.

[1] When we reduce 4d $\mathcal{N}$=1 to 3d $\mathcal{N}$=2, there are similarities between those two theories at classical level ($\mathcal{M}_H$). However, they behave differently quantum-mechanically ($\mathcal{M}_C$) as it receives quantum loop corrections. Surprisingly, we can have a new tool, called monopole operators [4], help us compute the coulomb branch. We will see it in the following chapters.

2.3.2 Chiral Operators

Let us move to crucial physical objects that can bridge between physical and mathematical perspective of the moduli space of 3d $\mathcal{N} = 2$ theory. First are **chiral** operators, $\mathcal{O}_i$, which break one half of $\mathcal{N}$=4 SUSY. There is also a commutative ring, called **chiral ring** $\mathcal{R}_C$, of which are vacuum expectation values of chiral operators. The VEVs $\langle \mathcal{O}_i \rangle$ of gauge invariant chiral operators are holomophic functions on the moduli space $\mathcal{M}$. With relations taken into account, we obtain a one-to-one correspondence between $\langle \mathcal{O}_i \rangle$ and holomophic functions on $\mathcal{M}$ [4].

One may consider a coordinate ring $\mathbb{C}[\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_n]$ of $\mathcal{M}$, with generators $\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_n$ (being a surrogate for VEVs of gauge invariant chiral operators). This ring consists of all feasible polynomial functions, formed from the generators, on $\mathcal{M}$. By quotienting the coordinate ring with its ideal $\mathcal{I}$ (generated by the relations that the chiral operators obey) [17], there exist a chiral ring

$$\mathcal{R}_C[\mathcal{M}] = \frac{\mathbb{C}[\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_n]}{\mathcal{I}} \tag{2.6}$$

This picture help us have a nice view of the moduli space as an algebraic variety characterised by the generators and relations in the theory. Thus we can equivalently study the moduli space, turning this into a geometry problem.

We will provide an example that might help one have a nice picture, in the next subsection.

2.3.3 Simplest Example of $\mathcal{M}$

As we have in the previous subscetion, 3d $\mathcal{N}=2$ supersymmetry implies that $\mathcal{M}$ is a **Kähler**[2] Manifold with possible sigularities.

This subsection provides the simplest example of a two complex dimensional orbifold. It is constructed by taking a complex plane $\mathbb{C}^2$ and using its quotient by the central symmetry, i.e. $\mathbb{Z}_2$, of the origin. The corresponding orbifold is therefore $\mathbb{C}^2/\mathbb{Z}_2$. Note that this space is homeomorphic to a cone $\mathbb{CP}^1$. To identify the algebraic variety of $\mathbb{C}^2/\mathbb{Z}_2$, we use x_1 and x_2 be $\mathbb{C}^2$ coordinates and take the action of a group $\mathbb{Z}_2$. Under this parity action, one can have

$$(x_1, x_2) \leftrightarrow (-x_1, -x_2)$$

Then let us find all of the possible monomial functions $f(x_1, x_2) = x_1^a x_2^b$ being invariant under the $\mathbb{Z}_2$ action. We need to consider all powers that satisfy

$$a - b = 0 \quad mod \quad 2 \tag{2.7}$$

So that we obtain $\mathbb{Z}_2$ three invariant monomial functions of degree two as

$$\begin{aligned} X &\equiv x_1^2; \quad a = 2, \quad b = 0 \\ Y &\equiv x_2^2; \quad a = 0, \quad b = 2 \\ Z &\equiv x_1 x_2; \quad a = b = 1 \end{aligned}$$

where X, Y, Z are complex variables describing the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ algebraically. These variables are called *generators* of $\mathbb{Z}_2$ invariant polynomial functions and also have a *relation* as the constraint to each other, as following

[2] In fact, the moduli spaces of 3d $\mathcal{N}$=4 gauge theory are a special class of manifolds, called **HyperKähler**. The paper [18] provide a great introduction to the HyperKähler

$$XY = Z^2 \tag{2.8}$$

This constraint is such an algebraic curve in $\mathbb{C}^3$. To think of X, Y, Z as real coordinates, (2.8) is a simple cone equation so that $\mathbb{C}^2/\mathbb{Z}_2$ is usually called a complex cone with the conical singularity at the origin.

Furthermore, all of these descriptions can be written algebraically as

$$\mathbb{C}^2/\mathbb{Z}_2 = \{X, Y, Z \in \mathbb{C}^3 | XY = Z^2\} \tag{2.9}$$

We will show the way to find the supesymmetric quantum field theory that has these *generators* and their *relation*, i.e. providing the same $\mathcal{M}_C$, in section 3.3.

We might see these mathematical objects, i.e. chiral rings and moduli space, can be used to describe the 3d $\mathcal{N}$=2 moduli space. However, this tool works very well classically, since the observables (formed by chiral operators) in the theory are protected from quantum corrections. There is thus great understand of the Higgs branch [6; 9; 14; 19]. Regarding the Coulomb branch, we have to concern quantum corrections and the study is not quite simple, Therefore we require new tools and operators, i.e. monopole operators, for better pictures and calculations.

2.4 Hilbert series: HS

We are capable of describing the moduli space $\mathcal{M}$ of vacua by using the idea of the chiral ring and then extracting the algebraic variety of the supersymmeric gauge theories. Although this idea helps us to exploit as much as feasible the symmetry of the theory; generally, determining generators and relations of the chiral ring is very tough. There is a very convenient and useful mathematical tool, called **Hilbert series (HS)** [3; 4; 20], that is used for counting scalar gauge invariant chiral operators (GIO).

The Hilbert series is generating functions of the polynomial ring, relevant to the chiral ring, in terms of fugacity[3] t for $\mathcal{M}$.

One may have the form of the Hilbert series [22; 23] as

$$H(t;\mathcal{M}) = \sum_{i=0}^{\infty} m_i t^i \tag{2.10}$$

where m_i is the number of linearly independent combinations of monomials of degree i. To expand all the series, we experience each combination is graded by the degree of the polynomial i [24]. We also normally write the HS (2.10) in terms of a rational function, looking like

$$H(t;\mathcal{M}) = \frac{Q(t)}{(1-t)^{dim\mathcal{M}}} \tag{2.11}$$

Where $Q(t)$ is a polynomial in terms of the fugacity t [25]. There is also the **theorem** stating the order of the pole of the Hilbert Series at t = 1 is the dimension of $\mathcal{M}$.

[3] t is a fugacity that is relevant to the chemical potential in a statistical mechanics perspective [21].

2.4.1 Example of HS

We will try to determine the Hilbert series of the moduli space $\mathbb{C}^2/\mathbb{Z}_2$ to relate them to (2.9).

Let us firstly consider the simplest moduli space $\mathbb{C}$ with a coordinate ring of a holomorphic function $\mathbb{C}[x]$, where x is a complex variable. The corresponding number of linearly independent (LI) monomials is given in table 2.4.1

i	No.	LI Monomial(s)
0	1	1
1	1	x
2	1	x^2
⋮	⋮	⋮

Table 2.4.1 shows LI monomials which live in the coordinate ring $\mathbb{C}[x]$ and the number of them at each order.

It is apparently clear that there is only LI monomials of each degree and the generator x. Therefore, $m_i = 1$ and the Hilbert series is taken as

$$\begin{aligned} H(t;\mathbb{C}) &= \sum_{i=0}^{\infty} (1)\, t^i \\ &= \frac{1}{1-t} \end{aligned} \tag{2.12}$$

Where $|t| < 1$[4]

The next example of the moduli space is $\mathbb{C}^2$ with two generators x_1 and x_2. One will have

[4] We will always take an assumption of t being small in this thesis.

i	No.	LI Monomial(s)
0	1	1
1	2	$x_1 \quad x_2$
2	3	$x_1^2 \quad x_1x_2 \quad x_2^2$
$\vdots$	$\vdots$	$\vdots$
d	$d+1$	$x_1^d \quad \dots \quad x_2^d$
$\vdots$	$\vdots$	$\vdots$

Table 2.4.2 shows LI monomials which live $\mathbb{C}^2[x_1, x_2]$ and the number of them at each order.

Thus the Hilbert Series follow

$$\begin{aligned} H(t;\mathbb{C}^2) &= \sum_{i=0}^{\infty}(i+1)\,t^i \\ &= \frac{1}{(1-t)^2} \end{aligned} \tag{2.13}$$

Subsequently, we now consider $\mathbb{C}/\mathbb{Z}_2$. The parity action is taken into account on the generator x as $(x) \to (-x)$. We obtain

i	No.	LI Monomial(s)
0	1	1
1	0	-
2	1	x^2
3	0	-
4	1	x^4
$\vdots$	$\vdots$	$\vdots$
2d	1	x^{2d}
$\vdots$	$\vdots$	$\vdots$

Table 2.4.3 shows LI monomials which is in the ring $\mathbb{C}/\mathbb{Z}_2[x]$ and the number of them at each order.

There is only generator x^2 at degree 2 and the Hilbert series are

$$\begin{aligned} H(t;\mathbb{C}/\mathbb{Z}_2) &= \sum_{i=0}^{\infty}(1)\,t^{2i} \\ &= \frac{1}{(1-t^2)} \end{aligned} \tag{2.14}$$

The final example is $\mathbb{C}^2/\mathbb{Z}_2$ and the corresponding LI monomials is

i	No.	LI Monomial(s)
0	1	1
1	0	-
2	3	$x_1^2 \quad x_1x_2 \quad x_2^2$
3	0	-
4	5	$x_1^4 \quad x_1^3x_2 \quad x_1^2x_2^2 \quad x_1x_2^3 \quad x_2^4$
$\vdots$	$\vdots$	$\vdots$
$2d$	$2d+1$	$x_1^{2d} \quad \ldots \quad x_2^{2d}$
$\vdots$	$\vdots$	$\vdots$

Table 2.4.4 shows LI monomials which live $\mathbb{C}^2[x_1,x_2]/\mathbb{Z}_2$ and the number of them at each order.

This identifies the generators x_1^2, x_1x_2, x_2^2. So the Hilbert series take form

$$\begin{aligned} H(t;\mathbb{C}^2/\mathbb{Z}_2) &= \sum_{i=0}^{\infty}(2i+1)\,t^{2i} \\ &= \frac{1+t^2}{(1-t^2)^2} \\ &= \frac{1-t^4}{(1-t^2)^3} \end{aligned} \tag{2.15}$$

Simply, we could read off 3 generators (corresponding to $X = x_1^2$, $Z = x_1x_2$, $Y = x_2^2$ in 2.3.3) at degree 2 from the denominator $(1-t^2)^3$ and 1 relation (corresponding to (2.8)) at degree 4 from the numerator $(1-t^4)^1$.

One may find that the first and the second examples do not have relations, since there is no quotient. On the other hand, the third and the fourth experience LI monomials at some degree disappear due to the parity action.

All of these examples are a special class of algebraic varieties called *complete intersections* which are, generally, given by

$$H(t;\mathcal{M}) = \frac{\prod_j \left(1-t^{b_j}\right)^{r_j}}{\prod_i (1-t^{a_i})^{g_i}} \tag{2.16}$$

(2.16) illustrates the coordinate ring which has generator numbers $\sum_i g_i$. The g_i generators live in degree a_i. Moreover, all the generators obey $\sum_j r_j$ relations where r_j relations are of degree b_j[5].

Note that the Hilbert series are not always appear as (2.16). Even though we are able to calculate the Hilbert series in the rational form, its numerator might not be factorisable[6]. This is what we call the *syzygy problem.* The problem has not only the generators obeying relations, but the relations obeying their own relations.

[5] In the example 1 and 2, their algebraic varieties are trivially a complete intersection. We call them a freely generated variety.

[6] There is an theorem [26] state that an affine variety is *Calabi-Yau*, if and only if the numerator of the rational form Hilbert Series is *palindromic* or *antipalindromic.*
Additionally, **complete intersection** variety have palindromic or antipalindromic numerators and, hence, are Calabi-Yau.

2.4.2 Refined Hilbert Series

One may write the refined form of Hilbert Series by accounting for the global symmetry of the algebraic variety. Let us consider the example of the moduli space $\mathbb{C}^2$ with complex variables x_1 and x_2. A natural $U(2)$ global symmetry acts on $\mathbb{C}^2$. There is maximal torus $U(1)^2$ as the *Cartan subgroup* which the first $U(1)$ acts on x_1 and another acts on x_2. We may introduce fugacities t_1 and t_2 for each $U(1)$, which they are used to count x_1 and x_2 respectively. With no group of symmetry taken into account, the Hilbert series[7] is given by

$$\begin{aligned} H(t_1, t_2; \mathbb{C}^2) &= \sum_{i_1, i_2 = 0}^{\infty} t_1^{i_1} t_2^{i_2} \\ &= 1 + (t_1 + t_2) + (t_1^2 + t_1 t_2 + t_2^2) + \dots \\ &= \frac{1}{(1-t_1)} \frac{1}{(1-t_2)} \end{aligned} \tag{2.17}$$

Let us perform the fugacity map $t_1 \to xt$ and $t_2 \to t/x$, so we obtain

$$\begin{aligned} H(t, x; \mathbb{C}^2) &= 1 + \left(x + \frac{1}{x}\right) t + \left(x^2 + 1 + \frac{1}{x^2}\right) t^2 + O(t^3) \\ &= \frac{1}{(1-xt)} \frac{1}{\left(1 - \frac{t}{x}\right)} \\ &= \sum_{k=0}^{\infty} \chi([k]_{SU(2)}) t^k \end{aligned} \tag{2.18}$$

where $[k]_{SU(2)}$ is a Dynkin label, corresponding to the characters $\chi([k]_{SU(2)})$ of $SU(2)$. If we take the fugacity map $t_1, t_2 \to t$, i.e. setting $x = 1$, of (2.17), we will get the same Hilbert series as (2.13). So far, we always call (2.18)

[7] One may consider $\mathbb{C}^n$ as the general case. There are, similarly, fugacities $t_1, \dots, t_n$ for each $U(1)$, counting $x_1, \dots, x_n$ respectively. Therefore the Hilbert series $H(t_1, \dots, t_n; \mathbb{C}^n) = \prod_{i=1}^{\infty} \frac{1}{(1-t_i)}$

as the *refined Hilberts series* of the moduli space $\mathbb{C}^2$, and obviously (2.13) is the *unrefined Hilberts series* of $\mathbb{C}^2$ and take the form

$$H(t;\mathbb{C}^2) = \sum_{k=0}^{\infty} dim(\chi([k]_{SU(2)}))\, t^k \tag{2.19}$$

We now, again, analyse the moduli space $\mathbb{C}^2/\mathbb{Z}_2$. The natural $U(2)$ acts on the space $\mathbb{C}^2$, with Cartan subalgebra $U(1)^2$. So we can choose the same fugacity t_1, t_2 as in the $\mathbb{C}^2$ case, but the summation of Hilbert series need to obey the parity action, i.e. $i_1 - i_2 = 0 mod 2$. So the Hilbert series is (2.20).

$$\begin{aligned} H(t_1,t_2;\mathbb{C}^2/\mathbb{Z}_2) &= \sum_{i_1,i_2=0}^{\infty} t_1^{i_1} t_2^{i_2} \qquad ; i_1 - i_2 = 0 mod 2 \\ &= 1 + (t_1^2 + t_1 t_2 + t_2^2) \\ &\quad + (t_1^4 + t_1^3 t_2 + t_1^2 t_2^2 + t_1 t_2^3 + t_2^4) + \dots \\ &= \frac{(1 - t_1^2 t_2^2)}{(1 - t_1^2)(1 - t_1 t_2)(1 - t_2^2)} \end{aligned} \tag{2.20}$$

To get a character expression, we then follow the same map as for $\mathbb{C}^2$.

$$\begin{aligned} H(t,x;\mathbb{C}^2/\mathbb{Z}_2) &= 1 + \left(x^2 + 1 + \frac{1}{x^2}\right) t^2 \\ &\quad + \left(x^4 + x^2 + 1 + \frac{1}{x^2} + \frac{1}{x^4}\right) t^4 + O(t^6) \\ &= \frac{(1 - t^4)}{(1 - x^2 t^2)(1 - t^2)(1 - \frac{t^2}{x^2})} \\ &= \sum_{k=0}^{\infty} \chi([2k]_{SU(2)})\, t^{2k} \end{aligned} \tag{2.21}$$

where $[2k]_{SU(2)}$ is also a Dynkin label of $SU(2)$. One may see that the unrefined Hilbert series of the case is given as (2.15) if we perform the fugacity map $t_1, t_2 \to t$ of (2.20).

We obtain

$$H(t;\mathbb{C}^2/\mathbb{Z}_2) = \sum_{k=0}^{\infty} dim(\chi([2k]_{SU(2)}))\, t^{2k} \tag{2.22}$$

Moreover, (2.21) is thus the refined form of the Hilbert series for the moduli space $\mathbb{C}^2/\mathbb{Z}_2$.

To summarise, we are capable to describe algebraic varieties and what representation of a global symmetry their monomials in each order transform in, so the *refined* and the *unrefined* Hilbert series, generally, are

$$H(t,x;\mathcal{M}) = \sum_{k=0}^{\infty} \chi_k[(x_1,...,x_r)_G]\, t^k \tag{2.23}$$

$$H(t;\mathcal{M}) = \sum_{k=0}^{\infty} dim(\chi_k[(x_1,...,x_r)_G])\, t^k \tag{2.24}$$

where $\chi_k[(x_1,...,x_r)_G]$ is the character of a representation of a group G rank r, where the LI monomials of a certain degree transform in.

Note that we will work in a basis z_i of root fugacities instead of in a basis x_i of character fugacities. However, we can always map between these two basis by using

$$z_i = \prod_{j=1}^{\infty} (x_j)^{M_{ij}} \tag{2.25}$$

where M_{ij} is an element of the Cartan matrix M of the group G.

In the next section we will introduce the useful mathematical tools with help us identify the number of generators, the relations among the generators and the syzygies; due to the fact that the algebraic varieties are not always complete intersections and the Hilbert series is not (2.16) which is difficult to be extracted those properties.

2.5 Plethystic Programmes

To enumerate chiral operators in the moduli space, this section talk about mathematical tools for extracting the number of *generators*, the *relations* (between the generators) and even the *syzygies* (relations of the relations). The plethystic programme contains these tool [23]. One of them is called **Plethystic Exponential (**PE**)** of a multivariable function $f(t_1,...,t_n)$ with variables $t_1,...,t_n$. It is defined as

$$PE[f(t_1,...,t_n)] = \exp\left(\sum_{r=1}^{\infty} \frac{f(t_1^r,...,t_n^r)}{r}\right) \tag{2.26}$$

Due to being used for symmetric products of any multivariable function, it is very vital in the calculation of Higgs branch [5; 6; 14; 23; 27]. In this dissertation we interest in computing the Coulomb branch Hilbert series, so we would not talk about the PE in the further chapter.

The relevant tool, which plays a crucial role in the Coulomb branch, is the inverse form of the PE. It is **Plethystic Logarithm (**PL**)** given as

$$PL[f(t_1,...,t_n)] = \sum_{r=1}^{\infty} \frac{\mu(r)}{r} \log(f(t_1^r,...,t_n^r)) \tag{2.27}$$

where $\mu(r)$ is the *Möbius function* defined as

$$\mu(r) = \begin{cases} 0 & \text{if } r \text{ has a repeated prime factor} \\ 1 & \text{if } r = 1 \\ (-1)^n & \text{if } r \text{ is a product of } n \text{ distinct primes} \end{cases} \tag{2.28}$$

For instant, we can evaluate the PL of (2.13) and (2.18) for the moduli space $\mathbb{C}^2$. So we obtain

$$PL[H_{\mathbb{C}^2}(t);t] = 2t \tag{2.29}$$

$$PL[H_{\mathbb{C}^2}(t,x);t,x] = [1]t \tag{2.30}$$

These tell us there are 2 generators at order 1, which transform as the fundamental representation of $SU(2)$.

Another example is computing PL of (2.15) and (2.21) for the moduli space $\mathbb{C}^2/\mathbb{Z}_2)$. The PL are

$$PL[H_{\mathbb{C}^2/\mathbb{Z}_2}(t);t] = 3t^2 - 1t^4 \tag{2.31}$$

$$PL[H_{\mathbb{C}^2/\mathbb{Z}_2}(t,x);t,x] = [2]\,t^2 - [0]\,t^4 \tag{2.32}$$

Hence these tell us there are 3 generators at order 2, which transform as the adjoint representation of $SU(2)$; and there is 1 relation at order 4, transforming as the trivial representation of $SU(2)$.

It is very easy to read off the generators and relations in terms of finite PL if the algebraic variety is a complete intersection. Generally, the PL of (2.16) is given by

$$PL[H_{\mathcal{M}}(t);t] = \sum_i g_i t^{a_i} - \sum_j r_j t^{b_j} \tag{2.33}$$

This is the same content as we have mentioned in the subsection 2.4.1. We can also take the PL of the refined Hilbert Series of a complete intersection. Consequently, we will see the representation characters of the global symmetry that the generators and relations transform in. [22; 28; 29]

In the syzygy problem which the variety is **NOT** a complete intersection, the PL of the Hilbert Series is infinite. We can identify the first few positive terms are the generators, the first few negative terms are the relations, and the following terms are syzygies. We will discuss them more in chapter 3.

2.6 Highest Weight Generating Function: HWG

The last, but not least, mathematical tool is a generating function encapsulating the refined Hilbert series with a new compact rational form. It is called *highest weight generating function (HWG)* [29] counting the *highest weight monomials*[8] $\mu_1^{n_1}...\mu_r^{n_r}$ of the irreps, that the coordinate ring monomials of the algebraic set transform in.

We specify an irrep of a group G with a rank r as

$$\begin{aligned}[n_1,...,n_r]_G \leftrightarrow & \prod_{i=1}^{r} \mu_i^{n_i} \\ & = \mu_1^{n_1}...\mu_r^{n_r}\end{aligned} \tag{2.34}$$

$[n_1,...,n_r]_G$ is the *Dynkin label* for the group G.

To clarify, we take an example with $G = SU(5)$ [30], so we will get the Dynkin fugacity map

$$[n_1,n_2,n_3,n_4]_G \leftrightarrow \mu_1^{n_1}\mu_2^{n_2}\mu_3^{n_3}\mu_4^{n_4} \tag{2.35}$$

For the adjoint representation of $SU(5)$ $[1,0,0,1]$, the highest weight monomial take the form $\mu_1^1\mu_2^0\mu_3^0\mu_4^1 = \mu_1\mu_4$.

Another example is mapping the Hilbert series in (2.21) to a highest weight generating function. The map is given by

$$\begin{aligned} H(t,x) &= \sum_{k=0}^{\infty} \chi([2k]_{SU(2)})\, t^{2k} \\ &= 1 + [2]\,t^2 + [4]\,t^4 + [6]\,t^6 + O(t^8) \\ \leftrightarrow &\quad 1 + \mu^2 t^2 + \mu^4 t^4 + \mu^6 t^6 + O(t^8) \\ &= \frac{1}{1-\mu^2 t^2} = HGW(t,\mu) \end{aligned} \tag{2.36}$$

[8] It is probably called a *Dynkin label fugacity*.

It is obviously compact comparing to its Hilbert series. Furthermore, one can calculate its PL for extracting the generators of the HWG.

So, in this case, we get

$$PL[HWG(t,\mu)] = \mu^2 t^2 \tag{2.37}$$

This clearly shows that there is only generator μ^2 at degree 2.

We will combine all these mentioned concepts and these mathematical approaches to understand the 3d $\mathcal{N}=4$ Coulomb branch in chapter 3 and 4. In the next section we will, additionally, introduce one of the 3d $\mathcal{N}=4$ supersymmetric gauge theories in order to encode its gauge groups and its matter contents into a diagram. Analysing this diagram provides the Coulomb branch of the theory which will be discussed in chapter 3 and 4.

2.7 Quiver Diagrams in 3d $\mathcal{N}$=4

In the supersymmetric gauge theory, a quiver gauge theory is one of the Super Yang-Mill(SYM) theories whose gauge groups and matter contents can be encoded into a diagram. This diagram is also known as a quiver diagram. Each diagram can be differed by its dimensions and number of supercharges. We would like to introduce how to construct a quiver diagram in the theory in the following subsection.

In a 3d $\mathcal{N}$=4 Quiver diagram there are simple components; representing a gauge group, a matter content and a flavour group [30].

□ : Square node

It denotes the flavour group the theory. In this paper we will focus only a $SU(N)$ flavour group.

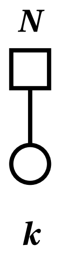

Figure 2.7: A simple example of a quiver diagram in 3d $\mathcal{N} = 4$ gauge theory. In the diagram, the **square node** and the **circular node** represent a $SU(N)$ flavour group and a $U(k)$ gauge group respectively.

O : **Circular node**

It represents a gauge group G in the quiver theory. Vector multiplets in the theory transform under the adjoint representation of the gauge group. In this paper we will focus only a $U(k)$ gauge group.

– : **Edge or Connection line**

It summarises the matter contents by linking two nodes. In the theory there are hypermultiplets transforming in the bifundamental representation of both the gauge node and the flavour node, or between two the gauge nodes.

In figure 2.7, we provide a simple quiver diagram, consisting of a $U(k)$ gauge group linked to a $SU(N)$ flavour group. The vector multiplets in the theory transforms under $U(k)$ gauge group. Meanwhile, each N hypermultiplets is in the fundamental representation of $U(k)$ so there are kN hypermultiplets in total.

3

Coulomb Branches

3.1 Monopole Operators

Differing from the Higgs branch, the Coulomb branch needs quantum collections to be considered, leading to tough computations [8]. However, there is the concept of *monopole formula* [9] helping us compute the Hilbirt series of Coulomb branch in 3d $\mathcal{N} = 4$. In this section we first review the concept of monopole operators and explain what the terms in the monopole formula represent.

Considering the Coulomb branch, local disorder operators (chiral operators), which are enumerated at position x, are 't Hooft monopole operators $V_m(x)$ [31]. At an insertion point x, they are defined in the infrared conformal field theory (CFT) [32] by specifying the gauge field to have a Dirac monopole singularity [9]. The gauge field one-form, in the northern and the southern hemisphere, is given as

$$A_{N/S} \sim \frac{m}{2}(\pm 1 - \cos\theta)d\varphi \tag{3.1}$$

We can perform the integration of the gauge fields over a two-sphere S^2 covering the singularity at x; consequently, this provides a magnetic charge m [32] being a member of the Lie algebra $\mathfrak{g}$ of the gauge group G.

Moreover, the magnetic charge have to obey a Dirac quantization condition [9]:

$$e^{2\pi i m} = \mathbb{1}_G \tag{3.2}$$

We can see that $m \in \mathbb{Z}$ making sure there is the requirement of m to be in the weight lattice $\Gamma^*_{\hat{G}}$ of the Langland $\hat{G}$ [9; 33][1]. Due to the monopole operators which are specified by m, we use the monopole formula as summing over m. In order to ensure *gauge invariant monopole operators* (GIO) being counted, we require the quotient of the weight lattice by the Weyl group $\mathcal{W}_{\hat{G}}$ to form the quotient space $\Gamma^*_{\hat{G}}/\mathcal{W}_{\hat{G}}$ [33]. Consequently, the magnetic monopoles, which are summed over, are in the Weyl chamber of the weight lattice of $\hat{G}$ [34].

The Hilbert series counts GIO that are graded regarding their dimension and quantum numbers under global symmetry [9; 30]. Our monopole operator are charged under R-symmetry, $U(1)_R$. As mentioned the decomposition of the 3d $\mathcal{N} = 4$ hypermultiplet and vector multiplet in chapter 2, there is the $U(1)_R$, which assigns charge 1/2 to the complex scalars in the chiral and anti-chiral multiplets from decomposing $\mathcal{N} = 4$ hypermultiplets and also charge 1 to the scalars in the adjoint chirals and the gaugino in the vector multiplets from decomposing $\mathcal{N} = 4$ vector multiplet [9]. Thus the quantum number we are looking for is the R-charge, Δ, which is given by

$$\Delta(m) = \Delta_V(m) + \Delta_H(m) = -\sum_{\alpha \in \Delta_+} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{n} \sum_{\rho_i \in \mathcal{R}_i} |\rho_i(m)| \tag{3.3}$$

where the first sum, over **only** positive root $\alpha \in \Delta_+$ in the gauge group G, contributes to the $\mathcal{N} = 4$ vector multiplets. Another term contributes to the $\mathcal{N} = 4$ hypermultiplets, which sum over the weights of the matter field representation $\mathcal{R}_i$ under the same gauge group.

[1] The Langland is also known as GNO dual group of the gauge group G.

We restrict our quivers to the boundary condition in [20]; in the infrared CFT, the R-charge is the conformal dimension if the quivers are *good* ($\Delta(m) \geqslant 1$) or *ugly* ($\Delta(m) \geqslant 1/2$). For $\Delta(m) < 1/2$, the quivers are *bad* which are not the case.

According to the conformal dimension (3.3) stated above, the Hilbert series we will construct just counts *bare* monopole operators; since there are no constant background complex scalars[2] [9]. This background scalar is a contribution of another type of GIO, called *dressed* monopole operators. To include this operators into our Hilbert series, we have to multiply by the classical factor[3] P_G accounting for the residual group H_m of the gauge group G broken by the magnetic charge m [9]. The classical factor is given by

$$P_G(t,m) = \prod_{i=1}^{r} \frac{1}{1 - t^{2d_i(m)}} \tag{3.4}$$

where r is the rank of G and $d_i(m)$ are the degrees of the Casimir invariants of H_m which is left unbroken by the GNO magnetic charge m. [9] provides the expression for this classical factor in Appendix A.

To combine the bare monopole operators and their dressing into the Hilbert series, the monopole[4] is expressed as

$$H(t) = \sum_{m \in \Gamma^*_{\hat{G}}/\mathcal{W}_{\hat{G}}} t^{2\Delta(m)} P_G(t,m) \tag{3.5}$$

This is the unrefined Hilbert series. To refine it, we have to consider the topological symmetry $Z(\hat{G})$ under which the monopole operators are probably charged.

---

2 This scalar field comes from the adjoint chiral multiplet as dicussed in section 2.2.1.

3 This contributes to turning on the constant background complex scalar which is an element of Lie algebra $\mathfrak{h}_m$ of the residual group H_m.

4 The convention of the fugacity t may be different from other papers. In this paper, we use t^2 instead, as a different normalization.

The corresponding charge is called the topological charge $J(m)$ [33]. $Z(\hat{G})$ is a non-trivial topological symmetrySince, when the gauge group G is non-simply connected[5]. let us assign the fugacity z_G, for the gauge group G, which each fugacity carries $J(m)$. Therefore, the refined form of the Hilbert series is

$$H(t,z) = \sum_{m\in\Gamma^*_{\hat{G}}/\mathcal{W}_{\hat{G}}} z_G^{J(m)} t^{2\Delta(m)} P_G(t,m) \tag{3.6}$$

In this paper we only focus on the unitary gauge group in our quivers. One can have a fugacity map which converts the fugacities z_G to the character fugacities of the global symmetry by using (2.25). By doing this, we can promote the topological symmetries of the gauge theory to a global symmetry.

3.2 Balanced quivers

For ADE quivers[6], the balance of each $U(N_i)$ gauge node is given [5]:

$$Balance_{SLE}(U(N_i)) = \sum_{j\in\text{adjacent nodes}} 2N_i - N_j \tag{3.7}$$

where N_j is the rank of the linked $U(N_j)$ gauge node. We call the node i *balanced* if (3.7) is zero, otherwise it *imbalanced* or *excess* [20]. Moreover, we state that quiver is balanced if (3.7) of all its nodes is zero. With an only unbalanced node, the quvier is said to be minimally unbalanced [35].

For BCF and G quivers[7], the long node directly adjoined to the non-simply laced edge (NSLE) obtains *double* and *triple* the contribution from the other side node respected to the NSLE.

[5] $U(N)$ and $SO(N)$ are non-simply connected groups, while $SU(N)$ and $USp(N)$ are simply connected

[6] These types of quivers are simply laced edge (SLE) quivers

[7] These quivers have double and triple laced edge.

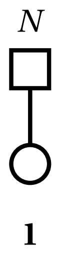

Figure 3.2: the simplest 3d $\mathcal{N} = 4$ quiver diagram, consisting of a $U(1)$ gauge group linked to a $SU(N)$ flavour group.

3.3 $U(1)$ with N Flavours

We now begin with analysing the simplest 3d $\mathcal{N} = 4$ quiver, the gauge theory of $U(1)$ with $SU(N)$ flavour. This quiver is shown in fugure 3.2.

The magnetic charge of the $U(1)$ gauge node is labeled by a. Since this gauge group is abelian, there is no the vector contribution. Therefore, the conformal dimension is only contributed by the matter content and expressed as:

$$\Delta(a) = \frac{N}{2}|a| \tag{3.8}$$

Simply, one can have classical factor for the $U(1)$ gauge group given by

$$P_{U(1)}(t) = \frac{1}{1-t^2} \tag{3.9}$$

For any $a \in \mathbb{Z}$, $U(1)$ cannot be broken to residual groups. Thus the monopole formula (3.5) used to compute the unrefined Hilbert series of the Coulomb branch of the for the quiver is:

$$\begin{aligned} H_{unref}(t) &= \frac{1}{1-t^2} \sum_{a=-\infty}^{\infty} t^{2(\frac{N}{2}|a|)} \\ &= \frac{1-t^{2N}}{(1-t^2)(1-t^N)^2} \end{aligned} \tag{3.10}$$

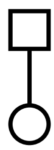

Figure 3.3: the 3d $\mathcal{N} = 4$ quiver diagram, consisting of a $U(1)$ gauge group linked to a $SU(2)$ flavour group.

To balance (3.10)[8], we can choose $N = 2$, so the Hilbert series becomes

$$H_{unref}(t) = \frac{1-t^4}{(1-t^2)^3} = 1 + 3t^2 + 5t^4 + O(t^6) \tag{3.11}$$

One may notice this is the same as (2.15). By seeing at the Taylor series of the Hilbert series, we find that the coefficient[9] of the term t^2 is the dimension of the adjoint representation of $SU(2)$. The global symmetry group is, therefore, $SU(2)$. Interestingly, this balanced quiver given in fig. 3.3 corresponds to the *Dynkin diagram*[10] A_1.

To refine (3.11), we add another fugasity z associating to the gauge node. The refined form and its Taylor expansion are expressed as:

$$\begin{aligned} H_{ref}(t,z) &= \frac{1}{1-t^2}\sum_{a=-\infty}^{\infty} z^a t^{2\left(\frac{N}{2}|a|\right)} \\ &= \frac{1-t^4}{(1-t^2)(1-zt^2)(1-\frac{t^2}{z})} \\ &= 1 + \left(z + 1 + \frac{1}{z}\right)t^2 + \left(z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2}\right)t^4 + O(t^6) \end{aligned} \tag{3.12}$$

[8] Following (3.7), we choose N to obtain $Balance_{SLE}(U(1)) = 0$.
[9] This coefficient always equal to the dimension of the global symmetry group.
[10] There is a **corollary** stating that for a balanced quiver that corresponds to the *Dynkin diagram* of a Lie algebra $\mathfrak{g}$, then the corresponding Lie group G is the global symmetry group.

The Cartan matrix of $SU(2)$ is used to map z to the fundamental weight fugacities x, i.e. $z \to x^2$. We obtain the refined Hilbert series as:

$$\begin{aligned} H_{ref}(t,x) &= 1 + \left(x^2 + 1 + \frac{1}{x^2}\right) t^2 \\ &\quad + \left(x^4 + x^2 + 1 + \frac{1}{x^2} + \frac{1}{x^4}\right) t^4 + O(t^6) \\ &= 1 + [2]\, t^2 + [4]\, t^4 + [6]\, t^6 + O(t^8) \\ &= \sum_{n=0}^{\infty} [2n]\, t^{2n} \end{aligned} \tag{3.13}$$

where $[2n]$ is the *highest weight Dynkin label* for $SU(2)$ irreps. To identify the moduli space of the Coulomb branch, we compute the plethystic logarithm (PL) of (3.13), expessed by

$$PL[H_{ref}](t,x) = [2]\, t^2 - [0]\, t^4 \tag{3.14}$$

There is the positive term at degree 2, corresponding to the 3 generators lying in the adjoint representation of $SU(2)$. There is the following negative term at degree 4, which corresponds to the relation transforming under the trivial representation. One can find that the PL terminates; thus the moduli space is a complete intersection.

Following section 2.6, one can simply turn (3.13) into the highest weight generating function (HWG). The HWG is given by:

$$HWG_{ref}(t,\mu) = \sum_{n=0}^{\infty} \mu^{2n} t^{2n} = \frac{1}{1 - \mu^2 t^2} \tag{3.15}$$

along with its PL:

$$PL[HWG_{ref}](t,\mu) = \mu^2 t^2 \tag{3.16}$$

where μ is the Dynkin label fugacity for $SU(2)$. And we find μ^2 is a generator at t^2 for the HWG.

3.4 Minimal Nilpotent Orbit of A_2

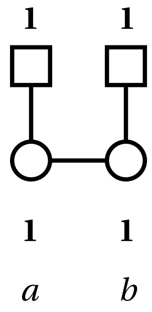

Figure 3.4: The quiver corresponds to the minimal nilpotent orbit of A_2. The magnetic charges a and b associated to the $U(1)$ gauge nodes.

Let us now consider minimal nilpotent orbit[11] of A_2. We label the magnetic charges a, b for both $U(1)$ gauge nodes. Thus the conformal dimension ,associating to three matter contents, reads

$$\Delta(a,b) = \frac{1}{2}(|a| + |a-b| + |b|) \tag{3.17}$$

For the classical factor, the two $U(1)$ provide

$$P_{U(1)^2}(t) = \frac{1}{(1-t^2)^2} \tag{3.18}$$

The unrefined Hilbert series is

$$\begin{aligned} H_{unref}(t) &= \frac{1}{(1-t^2)^2} \sum_{a,b=-\infty}^{\infty} t^{(|a|+|a-b|+|b|)} \\ &= \frac{1+4t^2+t^4}{(1-t^2)^4} \\ &= 1 + 8t^2 + 27t^4 + 64t^6 + O(t^8) \end{aligned} \tag{3.19}$$

[11] See more about quiver gauge theories of classical group nilpotent orbits in [5].

We can refine (3.19) by taking two additional fugacities z_1, z_2 associating to those two gauge nodes. The refined Hilbert series is

$$\begin{aligned} H_{ref}(t, z_1, z_2) &= \frac{1}{(1-t^2)^2} \sum_{a,b=-\infty}^{\infty} z_1^a z_2^b t^{(|a|+|a-b|+|b|))} \\ &= 1 + (2 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}) t^2 + O(t^4) \end{aligned} \tag{3.20}$$

Since fig. 3.4 is the balanced quiver and the global symmetry is $SU(3)$, we are capable to use the Cartan matrix to map ugacities z_1, z_2 to the character fugacities x_1, x_2, as following

$$z_1 \to \frac{x_1^2}{x_2}, \quad z_2 \to \frac{x_2^2}{x_1} \tag{3.21}$$

The refined Hilbert series is rewritten as

$$\begin{aligned} H_{ref}(t, x_1, x_2) &= 1 + [1,1]\, t^2 + [2,2]\, t^4 + [3,3]\, t^6 + O(t^8) \\ &= \sum_{n=0}^{\infty} [n, n]\, t^{2n} \end{aligned} \tag{3.22}$$

where $[n_1, n_2]$ is the highest weight Dynkin label for $SU(3)$ irreps. We also compute the PL given by

$$PL_{ref}(t, x_1, x_2) = [1,1]\, t^2 - ([1,1] + [0,0])\, t^4 + 2[1,1]\, t^6 + O(t^8) \tag{3.23}$$

This shows that the PL of the Hilbert series is infinite so the moduli space of the theory is not a complete intersection.

Regarding the first positive term at degree 2, we can see the generators of the moduli space transforming under the adjoint irrep of $SU(3)$.

At degree 4, all relations transform in the adjoint and trivial representation. We can construct the order 4 of the Hilbert series by taking

$$Sym^2[1,1] = [2,2] + [1,1] + [0,0] \tag{3.24}$$

Subsequently the relations is used to subtract $[1,1]+[0,0]$ out. At the degree 4, there is $[2,2]$ remained for the moduli space. Moreover, this PL has also provided the syzygies[12] lying in the two adjoints irreps. These syzygies help us find the degree 6 of the Coulomb branch Hilbert series. We take the third symmetric product of [1,1]:

$$Sym^3[1,1] = [3,3] + [2,2] + [1,1] + [0,0] + [3,0] + [0,3] \tag{3.25}$$

And then we subtract with $[1,1]([0,0]+[1,1])$, constructed from degree 2 and 4 of the PL, and add the syzygies to obtain

$$[3,3] = Sym^3[1,1] - [1,1]([0,0]+[1,1]) + 2[1,1] \tag{3.26}$$

One may receive the corresponding HWG as following

$$HWG_{ref}(t,\mu_1,\mu_2) = \sum_{n=0}^{\infty} \mu_1^n \mu_2^n t^{2n} = \frac{1}{1-\mu_1\mu_2 t^2} \tag{3.27}$$

And its PL is given by

$$PL[HWG_{ref}](t,\mu_1,\mu_2) = \mu_1\mu_2 t^2 \tag{3.28}$$

where μ_1,μ_2 correspond to the Dynkin label $[n_1,n_2]$. Therefore, $\mu_1\mu_2$ is a generator at degree 2.

[12] They act as the relations of the relations. We have mentioned about the syzygy problem in section 2.5

3.5 Minimal Nilpotent Orbit of A_3

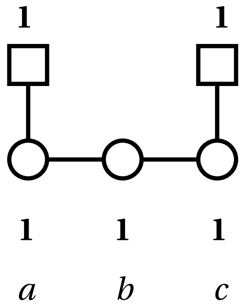

Figure 3.5: The quiver corresponds to the minimal nilpotent orbit of A_3. The charges a, b and c associate to the three $U(1)$ gauge nodes.

Fig. 3.5 illustrates the quiver diagram of the minimal nilpotent orbit of A_3. The conformal dimension of the theory is contributed by four matter fields and expressed as

$$\Delta(a,b,c) = \frac{1}{2}(|a| + |a-b| + |b-c| + |c|) \tag{3.29}$$

Similar to the previous section, there is the classical factor:

$$P_{U^3(1)}(t) = \frac{1}{(1-t^2)^3} \tag{3.30}$$

corresponding to three $U(1)$ gauge nodes. One can construct the unrefined Hilbert series is

$$\begin{aligned} H_{unref}(t) &= \frac{1}{(1-t^2)^3} \sum_{a,b,c=-\infty}^{\infty} t^{2\Delta(a,b,c)} \\ &= \frac{1+9t^2+9t^4+t^6}{(1-t^2)^6} \\ &= 1 + 15t^2 + 84t^4 + 300t^6 + O(t^8) \end{aligned} \tag{3.31}$$

The refined Hilbert series is written as

$$H_{ref}(t, z_1, z_2, z_3) = \frac{1}{(1-t^2)^3} \sum_{a,b=-\infty}^{\infty} z_1^a z_2^b z_3^c t^{2\Delta(a,b,c)} \tag{3.32}$$

where z_1, z_2, z_3 associate to the gauge nodes. We also do the same character map as using the Cartan matrix of the global symmetry group $SU(4)$, since this quiver is balanced:

$$z_1 \to \frac{x_1^2}{x_2}, \quad z_2 \to \frac{x_2^2}{x_1 x_3}, \quad z_3 \to \frac{x_3^2}{x_2} \tag{3.33}$$

(3.32) becomes

$$\begin{aligned} H_{ref}(t, x_1, x_2, x_3) &= 1 + [1,0,1]\,t^2 + [2,0,2]\,t^4 + [3,0,3]\,t^6 + O(t^8) \\ &= \sum_{n=0}^{\infty} [n,0,n]\,t^{2n} \end{aligned} \tag{3.34}$$

where $[n_1, n_2, n_3]$ represents the highest weight Dynkin label for $SU(4)$ irreps. We also compute the PL of the refined Hilbert series:

$$\begin{aligned} PL_{ref}(t, x_1, x_2, x_3) = [1,0,1]\,t^2 &- ([1,0,1] + [0,0,0] + [0,2,0])\,t^4 \\ &+ (2[1,0,1] + 2[0,2,0] + [2,1,0] + [0,1,2])\,t^6 \\ &+ O(t^8) \end{aligned} \tag{3.35}$$

What we can, overall, see that the moduli space is not a complete intersection. At degree 2, the PL tell us there is, similarly, the generators in the adjoint irreps of the global symmetry. The relations at order 4 is used to subtract the second symmetric product of [1,0,1]. Thus the degree 4 of the Hilbert series is given by:

$$[2,0,2] = Sym^2[1,0,1] - ([1,0,1] + [0,0,0] + [0,2,0]) \tag{3.36}$$

and also the syzygies given in (3.35) is used to compute the order 6

$$\begin{aligned} [3,0,3] = Sym^3[1,0,1] &- [1,0,1]([1,0,1] + [0,0,0] + [0,2,0]) \\ &+ (2[1,0,1] + 2[0,2,0] + [2,1,0] + [0,1,2]) \end{aligned} \tag{3.37}$$

For the HWG of the theory, it is very simple to find. We can label μ_1, μ_2, μ_3 corresponding to the Dynkin label of $SU(4)$. We, consequently, obtain HWG and its PL as:

$$HWG_{ref}(t, \mu_1, \mu_3) = \sum_{n=0}^{\infty} \mu_1^n \mu_3^n t^{2n} = \frac{1}{1 - \mu_1 \mu_3 t^2} \tag{3.38}$$

$$PL[HWG_{ref}](t, \mu_1, \mu_3) = \mu_1 \mu_3 t^2 \tag{3.39}$$

This shows that the generator of the HWG is $\mu_1 \mu_3$ at degree 2.

3.6 Supra Minimal Nilpotent Orbit of A_3

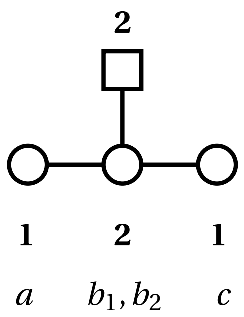

Figure 3.6: The quiver corresponds to the supra minimal nilpotent orbit of A_3. The charges a and c associated to the two $U(1)$ gauge nodes on the left and right, respectively, of the diagram. The magnetic charges b_1 and b_2 are labelled on the middle $U(2)$ gauge node.

The Coulomb branch of the quiver is the supra minimal nilpotent orbit of A_3. There are the magnetic charges of the monopole operators a, b_1, b_2 and c. The conformal dimension is

$$\begin{aligned}\Delta(a, b_1, b_2, c) = &\frac{1}{2}(|a - b_1| + |a - b_2| + 2|b_1| + 2|b_2| + |c - b_1| + |c - b_2|) \\ &- |b_1 - b_2|\end{aligned} \tag{3.40}$$

The first line is the contribution of the matter contents of the theory and the second line associates to the vector multiplet.

The non-abelian $U(2)$ can be broken into the residual symmetry group $U^2(1)$. Therefore we can split the dressing factor into two cases due to whether the $U(2)$ is broken or not. The overall dressing factor is expressed as:

$$P_G(t,b_1,b_2) = \begin{cases} \frac{1}{(1-t^4)(1-t^2)^3} & ; \quad b_1 = b_2 = b \\ \frac{1}{(1-t^2)^4} & ; \quad b_1 < b_2 \end{cases} \tag{3.41}$$

One can sum over the Weyl chamber of the $U(2)$ weight lattice as $\sum_{b_1 \le b_2 \in \mathbb{Z}}$. The unrefined Hilbert series is

$$\begin{aligned} H_{unref}(t) &= \sum_{a,b_1\le b_2,c\in\mathbb{Z}} t^{2\Delta(a,b_1,b_2,c)} P_G(t,b_1,b_2) \\ &= \frac{1}{(1-t^4)(1-t^2)^3} \sum_{a,b,c\in\mathbb{Z}} t^{2\Delta(a,b,c)} \\ &\quad + \frac{1}{(1-t^2)^4} \sum_{a,b_1<b_2,c\in\mathbb{Z}} t^{2\Delta(a,b_1,b_2,c)} \\ &= \frac{1+7t^2+12t^4+7t^6+t^8}{(1-t^2)^8} \\ &= 1+15t^2+104t^4+475t^6+O(t^8) \end{aligned} \tag{3.42}$$

We refine (3.42) by adding z_1, z_2, z_3 with the assigned magnetic charges $(a), (b_1, b_2), (c)$ corresponding to the gauge nodes from the left to the right.

$$H_{ref}(t,z_1,z_2,z_3) = \sum_{a,b_1\le b_2,c\in\mathbb{Z}} z_1^a z_2^{b_1+b_2} z_3^c t^{2\Delta(a,b_1,b_2,c)} P_G(t,b_1,b_2) \tag{3.43}$$

Since the global symmetry group of $SU(4)$, we use the same map stated in 3.33. The refined Hilbert series is rewritten as

$$\begin{aligned} H_{ref}(t,x_1,x_2,x_3) = 1 & \\ &+ t^2([1,0,1]) \\ &+ t^4([2,0,2]+[0,2,0]) \\ &+ t^6([3,0,3]+[1,2,1]) \\ &+ t^8([4,0,4]+[2,2,2]+[0,4,0]) \\ &+ t^{10}([5,0,5]+[3,2,3]+[1,4,1]) \\ &+ O(t^8) \end{aligned} \tag{3.44}$$

The corresponding PL is

$$\begin{aligned} PL[H_{ref}](t,x_1,x_2,x_3) = [1,0,1]\,t^2 - ([1,0,1]+[0,0,0])\,t^4 \\ + ([1,0,1]+[0,2,0])\,t^6 + O(t^8) \end{aligned} \tag{3.45}$$

This PL has less relations comparing to the previous case in section 3.5, which means that we have the smaller subtraction for the degree 4 of the Coulomb branch Hilbert series. Interestingly, it also implies that the moduli space of the quiver in this section is bigger than the moduli space, corresponding to the minimal nilpotent orbit of A_3.

The HWG is

$$\begin{aligned} HWG_{ref}(t,\mu_1,\mu_2,\mu_3) &= (1+\mu_1\mu_3 t^2)\sum_{n=0}^{\infty}(\mu_1^2\mu_3^2 t^4)^n \sum_{m=0}^{n}\left(\frac{\mu_2^2}{\mu_1^2\mu_3^2}\right)^m \\ &= \frac{1}{(1-\mu_1\mu_3 t^2)(1-\mu_2^2 t^4)} \end{aligned} \tag{3.46}$$

along with its PL:

$$PL[HWG_{ref}](t,\mu_1,\mu_2,\mu_3) = \mu_1\mu_3 t^2 + \mu_2^2 t^4 \tag{3.47}$$

It is quite different that this theory has two generators $\mu_1\mu_3$ at t^2 and μ_2^2 at t^4 for the HWG.

3.7 Minimal Nilpotent Orbit of A_4

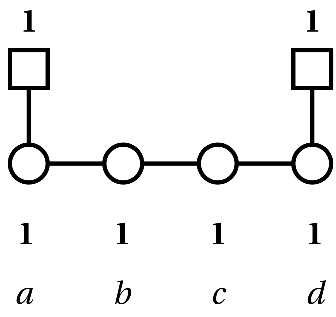

Figure 3.7: The quiver corresponds to the minimal nilpotent orbit of A_4. There are four manetic charges a, b, c and d assigned to the four $U(1)$ gauge nodes.

The analysis for the quiver in fig. 3.7 is very similar to the quivers in section 3.3 and 3.5. It is a balanced quiver so $SU(5)$ is the global symmetry group of the theory.

The conformal dimension is

$$\Delta(a,b,c,d) = \frac{1}{2}(|a| + |a-b| + |b-c| + |c-d| + |d|) \tag{3.48}$$

The classical factor for the residual symmetry group $U^4(1)$ is given by

$$P_{U(1)^4}(t) = \frac{1}{(1-t^2)^4} \tag{3.49}$$

So the unrefined Hilbert series is

$$\begin{aligned} H_{unref}(t) &= \frac{1 + 16t^2 + 36t^4 + 16t^6 + t^8}{(1-t^2)^8} \\ &= 1 + 24t^2 + 200t^4 + 1000t^6 + O(t^8) \end{aligned} \tag{3.50}$$

Equivalent to the minimal orbit cases, one can use the Cartan matrix of $SU(5)$ in order to get the character expression. The map is

$$z_1 \to \frac{x_1^2}{x_2}, \quad z_2 \to \frac{x_2^2}{x_1 x_3}, \quad z_3 \to \frac{x_3^2}{x_2 x_4}, \quad z_4 \to \frac{x_4^2}{x_3} \tag{3.51}$$

Thus the refined Hilbert series is

$$H_{ref}(t,x_1,x_2,x_3,x_4) = \sum_{n=0}^{\infty}[n,0,0,n]\,t^{2n} \tag{3.52}$$

where the highest weight Dynkin label for $SU(5)$ irreps is represented by $[n_1,n_2,n_3,n_4]$. And one may have the refined PL of the Hilbert series given in (3.53)

$$\begin{aligned} PL[H_{ref}](t,x_1,x_2,x_3) = & \; t^2[1,0,0,1] \\ & - t^4([1,0,0,1]+[0,0,0,0]+[0,1,1,0]) \\ & + t^6(2[1,0,0,1]+2[0,1,1,0] \\ & \quad + [2,0,1,0]+[0,1,0,2] \\ & \quad + [1,2,0,0]+[0,0,2,1]) \\ & + O(t^8) \end{aligned} \tag{3.53}$$

Generally, there are a majority of the relations that we have to remove from the moduli space. We will, again, compare this PL with the refined PL (3.60) of the Hilbert seires in the next section.

The highest weight generating function and its PL are expressed as:

$$HWG_{ref}(t,\mu_1,\mu_4) = \sum_{n=0}^{\infty}\mu_1^n\mu_4^n t^{2n} = \frac{1}{1-\mu_1\mu_4 t^2} \tag{3.54}$$

$$PL[HWG_{ref}](t,\mu_1,\mu_4) = \mu_1\mu_4 t^2 \tag{3.55}$$

There is only one generator μ_1,μ_4 at degree 2 for the HWG.

3.8 Supra Minimal Nilpotent Orbit of A_4

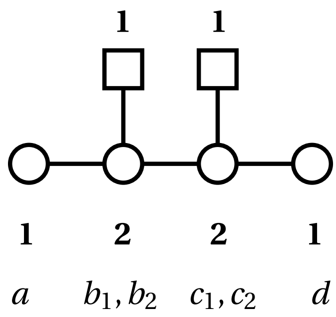

Figure 3.8: The quiver corresponds to the supra minimal nilpotent orbit of A_4. The assigned magnetic charges a and d associated to the leftmost and rightmost $U(1)$ gauge nodes, respectively. The magnetic charges b_1, b_2 and c_1, c_2 are labelled on the two $U(2)$ gauge nodes.

The conformal dimension is

$$\begin{aligned}\Delta(a, b_1, b_2, c_1, c_2, d) = \frac{1}{2}(&|a-b_1| + |a-b_2| + |b_1| + |b_2| \\ &+ |b_1-c_1| + |b_2-c_1| + |b_1-c_2| + |b_2-c_2| \\ &+ |c_1| + |c_2| + |c_1-d| + |c_2-d|) \\ &- (|b_1-b_2| + |c_1-c_2|\end{aligned} \tag{3.56}$$

The first three lines associate to the matter contents of the theory. Since there are two non-abelian gauge groups, $U(2)$; there are the contribution of the two vector multiplets, corresponding to the last line. The analysis of the classical factor can be divided into four cases as following:

- Both $U(2)$ gauge groups are not broken, i.e. $b_1 = b_2, c_1 = c_2$.
- The left $U(2)$ is broken to the residual symmetry groups $U(1)^2$, i.e. $b_1 \neq b_2$.
- The right $U(2)$ is broken to the residual symmetry groups $U(1)^2$, i.e. $c_1 \neq c_2$.
- Both $U(2)$ are broken to the residual groups $U(1)^2$

Those four cases respectively have the corresponding classical factor:

$$P_G(t,b_1,b_2,c_1,c_2) = \begin{cases} \frac{1}{(1-t^2)^4(1-t^4)^2} & ; \quad b_1 = b_2 = b, \quad c_1 = c_2 = c \\ \frac{1}{(1-t^2)^5(1-t^4)} & ; \quad b_1 < b_2, \quad c_1 = c_2 = c \\ \frac{1}{(1-t^2)^5(1-t^4)} & ; \quad b_1 = b_2 = b, \quad c_1 < c_2 \\ \frac{1}{(1-t^2)^6} & ; \quad b_1 < b_2, \quad c_1 < c_2 \end{cases} \tag{3.57}$$

The unrefined Hilbert series[13] is

$$\begin{aligned} H_{unref}(t) &= \sum_{a,b_1\le b_2,c_1\le c_2,d\in\mathbb{Z}} t^{2\Delta(a,b_1,b_2,c_1,c_2,d)} P_G(t,b_1,b_2,c_1,c_2) \\ &= \frac{1+12t^2+53t^4+88t^6+53t^8+12t^{10}+t^{12}}{(1-t^2)^{12}} \\ &= 1+24t^2+275t^4+2024t^6+O(t^8) \end{aligned} \tag{3.58}$$

The refined Hilbert series is

$$H_{ref}(t,z_1,z_2,z_3,z_4) = \sum_{a,b_1\le b_2,c_1\le c_2,d\in\mathbb{Z}} z_1^a z_2^{b_1+b_2} z_3^{c_1+c_2} z_4^d t^{2\Delta(a,\ldots,d)} P_G \tag{3.59}$$

where the fugacities z_1, z_2, z_3, z_4 associate from the gauge nodes leftmost to the rightmost. Due to the global symmetry group $SU(5)$, we use the same character map mentioned in (3.51). The refined Hilbert series is given by:

$$\begin{aligned} H_{ref}(t,x_1,x_2,x_3,x_4) = PE[&t^2[1,0,0,1] - t^4([1,0,0,1]+[0,0,0,0]) \\ &+ t^6[1,0,0,1]] + O(t^8) \end{aligned} \tag{3.60}$$

where $[n_1,n_2,n_3,n_4]$ is the highest weight Dynkin label for $SU(5)$ irreps, as we are familiar. We have written the Hilbert series in term of its PL[14]. This tell us the moduli space of the supra minimal orbit of A quiver is bigger than the moduli space in section 3.7 due to the fact that the rela-

[13] We will provide the calculation in detail and the useful technique compute the exact unrefined Hilbert series in Appendix A

[14] The full character expression is given in (A.3)

tions given in (3.60) is less than in (3.53). This means we have the smaller subtraction for the moduli space of the quiver in fig. 3.8.

One may compute the HWG expressed as:

$$HWG_{ref}(t,\mu_1,\mu_2,\mu_3,\mu_4) = \frac{1}{(1-\mu_1\mu_4 t^2)(1-\mu_2\mu_3 t^4)} \tag{3.61}$$

where μ_1,μ_2,μ_3,μ_4 are labelled corresponding to $[n_1,n_2,n_3,n_4]$. The PL of the HWG is:

$$PL[HWG_{ref}](t,\mu_1,\mu_2,\mu_3,\mu_4) = \mu_1\mu_4 t^2 + \mu_2\mu_3 t^4 \tag{3.62}$$

Hence, there are two HWG generators $\mu_1\mu_4$ at t^4 and $\mu_2\mu_3$ at t^4.

Overall in this chapter, we analyse balanced A series quivers; namely minimal orbit of A_1, A_2, A_3, A_4 and supra minimal orbit of A_3, A_4. We can see that all the unrefined Hilbert series, i.e. (3.11), (3.19), (3.31), (3.42), (3.50) and (3.58), can be written in terms of palindromic form which we have expected. The denominator of these Hilbert series imply the dimension[15] of the Coulomb branch quiver. More interestingly, this dimension can be used to compute the exact unrefined Hilbert series of the complicated quiver from the perturbative approach as be shown in Appendix A.

We also provide generalised A_n Series nilpotent orbit moduli spaces. The following table, which we have picked some parts from [AH,RK-Nilpotent], will show the orbits, the dimensions of the Coulomb branch quiver and the corresponding HWG.

Orbit	Dimension	PL of HWG
Minimal	$2n$	$\mu_1\mu_n t^2$
Supra Minimal ($n \geq 3$)	$4n-4$	$\mu_1\mu_n t^2 + \mu_2\mu_{n-1} t^4$

where n is the number of gauge nodes in the the Coulomb branch quiver.

[15] The dimension of the Coulomb branch quiver is equal to double the sum of the ranks of the gauge nodes [5]

4

Ungauging Scheme

In this chapter, we provide another useful technique used for the computation of the Coulomb branch for a flavourless quiver, i.e. the quiver with purely gauge groups [10]. This technique is called "*ungauging*", since we ungauge or decouple a residual center-of-mass $U(1)$ symmetry.

Regarding a simply laced edge quiver, i.e. ADE series quivers, there are arbitrary choices where this $U(1)$ should be decoupled. Consequently, the Coulomb branch we comepute is invariant due to changing the ungauged choice[1] [10]. For non-simply laced edge quivers, we will start with modify our monopole formula (3.5) to treat NSLE.

4.1 Modified Monopole Formula

What we have to concern about the modification of the formula is just the conformal dimension contributed by the hypers [7; 10]. This part in (3.3) turned into

$$\Delta_H(m) = \frac{1}{2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}|\lambda m_i^{(1)} - m_j^{(2)}| \tag{4.1}$$

where $m^{(1)}$ and $m^{(2)}$ are the assigned magnetic charges associating to $U(N_1)$ and $U(N_2)$ gauge nodes.

[1] One may see the example of ungauging the center-of-mass $U(1)$ symmetry in the affined $\tilde{A}_2$quiver in section 5.2.2 in [14].

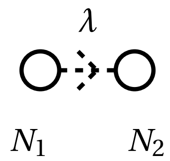

Figure 4.1: the flavourless quiver with two $U(N)$ gauge nodes. The dashed line represents single, double and triple laced edge corresponding to $\lambda = 1, 2, 3$.

(4.1) is the generalisation of the conformal dimension contributed by matter contents: setting $\lambda = 2, 3$ used for a double and a triple laced edge. We can also recover the monopole formula (3.5) by choosing $\lambda = 1$.

4.2 Choice of Ungauging Scheme

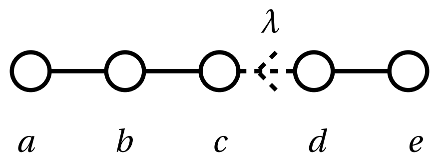

Figure 4.2: the flavourless quiver with five $U(N)$ gauge nodes. The λ dashed line represents single, double and triple laced edge. The magnetic charges a, b, c, d and e are assigned to these nodes.

We provide the flavourless quiver in fig. 4.2 as the example how a choice of a ungauging scheme affects on our monopole calculation. The monopole formula consists of a sum over magnetic charges taking values in the dual lattice as mentioned in section 3.1. We can think of the change from one ungauging scheme to another as a change of the lattice which we sum over (or the shift in the magnetic charges m). Generally, the classical factors $P_G(t, m)$ in (3.5) and also the confromal dimension $\Delta_V(m)$, contributed by the vector multiplets, are invariant under a shift in the charges m [10]. However, the contribution of Hypers, i.e. $\Delta_H(m)$, has a non-trivial change.

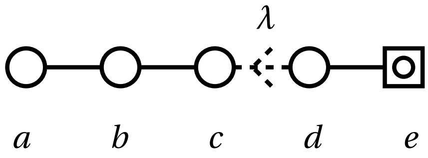

Figure 4.3: this is the quiver in fig. 4.2, ungauged at the e node.

Following Appendix B in [10], now let us consider the choice of the ungauging rightmost gauge node provided in fig. 4.2. The quiver has the multiplicity λ for NSLE and the assigned magnetic charges a, b, c, d and e associate to those five nodes. The conformal dimension contributed by the matters is given by

$$\Delta_H = \sum(|a-b| + |b-c| + |c-\lambda d| + |d-e|) \times \delta(e') \tag{4.2}$$

where $\delta(e')$ ensures that our chosen ungauging scheme needs one of the magnetic charges e, on the (long) rightmost node, set to be zero. We can choose another ungauged d node by shifting the magnetic charges, $e \to e + d'$. The result is

$$\Delta'_H = \sum(|a-b| + |b-c| + |c-\lambda d| + |d-e-d'|) \times \delta(e'+d') \tag{4.3}$$

And we again make a shift $d \to d - e'$, leading to

$$\Delta''_H = \sum(|a-b| + |b-c| + |c-\lambda d| + |d-e|) \times \delta(d') \tag{4.4}$$

where $\delta(d')$ is left in order to guarantee one of the magnetic charges d being zero. (4.4) is the conformal dimension corresponding to the choice of the ungauging scheme on the long d node. Therefore, the Coulomb branch Hilbert series is still the same for both cases. Generally, we can still continue shifting the magnetic charges m_L along the long side of a NSLE quiver, resulting in the same Coulomb branch C_L [10][2].

[2] Regarding the ungauging scheme on the short side of the quiver, we will not analyse it in this thesis. One might see it in [10]

4.3 Affine $\tilde{G}_2$

4.3.1 Ungauging

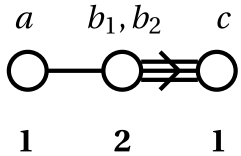

Figure 4.4: The affine $\tilde{G}_2$ quiver. The assigned magnetic charges a, b_1, b_2 and c associated to the gauge nodes from the left to the right.

The quiver given in fig. 4.3.1 is the affine $\tilde{G}_2$ quiver. There are magnetic charges a, b_1, b_2, c labelled for each gauge node from the left to right of the quiver. The conformal dimension is

$$\Delta(a,b_1,b_2,c) = \frac{1}{2}(|a-b_1|+|a-b_2|+|3b_1-c|+|3b_2-c|) - |b_1-b_2| \tag{4.5}$$

To compute the Coulomb branch of the theory, we can make two choices of the ungauging scheme on the long side of the triple laced edge. One is the left $U(1)$ gauge node another is the middle $U(2)$ node. Due to the uncomplicated choice of ungauging the $U(1)$ node, we obtain the ungauged quiver given in fig. 4.3.1 and the conformal dimention express as:

$$\Delta(b_1,b_2,c) = \frac{1}{2}(|b_1|+|b_2|+|3b_1-c|+|3b_2-c|) - |b_1-b_2| \tag{4.6}$$

We can read off the classical factor taking the form

$$P_G(t,b_1,b_2) = \begin{cases} \frac{1}{(1-t^4)(1-t^2)^2} & if \quad b_1 = b_2 = b \\ \frac{1}{(1-t^2)^3} & if \quad b_1 < b_2 \end{cases} \tag{4.7}$$

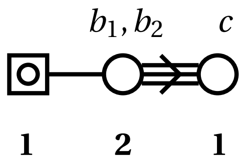

Figure 4.5: The $\tilde{G}_2$ quiver with the ungauging scheme on the left $U(1)$ gauge node.

The Coulomb branch Hilbert series is

$$H_{unref}(t) = \frac{1+8t^2+8t^4+t^6}{(1-t^2)^6} = 1+14t^2+77t^4+273t^6+O(t^8) \tag{4.8}$$

To refine this Hilbert seires, we add more two fugacities, i.e. z_1 and z_2. The refined Hilbert series takes the form:

$$H_{ref}(t, z_1, z_2) = \sum_{b_1 \le b_2, c \in \mathbb{Z}} z_1^{b_1+b_2} z_2^c t^{2\Delta(b_1,b_2,c)} P_G(t, b_1, b_2) \tag{4.9}$$

The character fugacity map for G_2 is

$$z_1 \to \frac{x_1^2}{x_2^3}, \quad z_2 \to \frac{x_2^2}{x_1}, \tag{4.10}$$

This is used to get its character expression give by

$$H_{ref}(t, x_1, x_2) = \sum_{n=o}^{\infty} [0,n]\, t^{2n} \quad PE[[0,1]\, t^2 - ([2,0]+[0,0])\, t^4 + O(t^6)] \tag{4.11}$$

At degree 2, we have Dynkin label [0,1] which is adjoint representation[3] of G_2. The global symmetry group is G_2. The Coulomb branch is the closure of the minimal nilpotent orbit of $\mathfrak{g}_2$ algebra [10].

[3] In some texts, [1,0] is used to represent the adjoint of G_2.

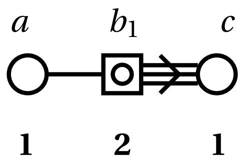

Figure 4.6: The $\tilde{G}_2$ quiver with ungauging a residual center-of-mass $U(1)$ symmetry on the middle $U(2)$ gauge node.

We also can compute the HWG has the form:

$$\begin{aligned} HWG_{ref}(t,\mu_1,\mu_2) &= \frac{1}{(1-\mu_2 t^2)} \\ &= PE[\mu_2 t^2] \end{aligned} \tag{4.12}$$

The HWG has the generator μ_2^2 at degree 2.

We are able to ungauge another residual center-of-mass $U(1)$ symmetry on the middle $U(2)$ gauge node, depicted in fig. 4.3.1. The conformal dimension (4.5) turn into

$$\Delta(a,b_1,c) = \frac{1}{2}(|a-b_1|+|a|+|3b_1-c|+|c|)-|b_1| \tag{4.13}$$

After computing the Hilbert series with this conformal dimension, we will find the refined Hilbert series is also expressed as (4.11)

4.3.2 Re-expression

According to the quiver in fig. 4.3.1, the corresponding Dynkin diagram of the affine $\tilde{G}_2$ contains the Dynkin diagram of A_2, which means that the corresponding Lie group of A_2 is a subgroup of G_2, i.e. $SU(3) \subset G_2$

We can see the character map between the two groups from their Cartan matrix associating to the Dynkin diagram, as shown in fig. 4.3.2:

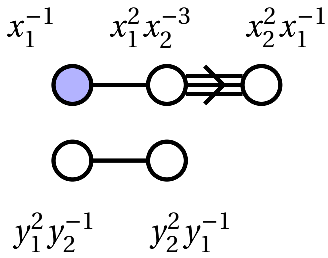

Figure 4.6: The Dynkin diagram of $\tilde{G}_2$ and A_2. Their labelled characters, where x_1, x_2 and y_1, y_2 are the characters for those two group respectively, associate to each node of the Dynkin diagrams.

Therefore the character map between these two groups is:

$$x_1 \to \frac{y_2}{y_1^2}, \quad x_2 \to \frac{1}{y_1}, \tag{4.14}$$

The refined Hilbert series[4] is rewritten in terms of the characters of $SU(3)$ as:

$$\begin{aligned} H_{G_2 \to SU(3)} = PE[&([1,1]+[1,0]+[0,1])\,t^2 \\ &-([1,1]+[2,0]+[0,2]+[1,0]+[0,1]+2[0,0])\,t^4 \\ &+O(t^6) \end{aligned} \tag{4.15}$$

along with the HWG[5] expressed by:

$$HWG_{G_2 \to SU(3)} = \frac{1}{(1-\mu_1 t^2)(1-\mu_2 t^2)(1-\mu_1\mu_2 t^2)} \tag{4.16}$$

Regarding other non-simply laced quivers in [10], one find that any ungauging scheme on the long side of a NSLE provides the Coulomb branch which corresponds to the closure of the minimal nilpotent orbit of Lie algebre $\mathfrak{g}$, corresponding to the global symmetry group G. On the other hand, to ungauge a residual center-of-mass $U(1)$ symmetry of the $U(N_s)$ on the short side of the NSLE gives us different interesting Hilbert series, as described in [10].

[4] We will provide the full character expression in Appendix A.2.
[5] We will guide the way to simplify HWG of this re-expression also in Appendix A.2.

5

Conclusions and Future Work

5.1 Conclusions

In this thesis, we study the Coulomb branch of 3d $\mathcal{N} = 4$ quiver gauge theories throughout the monopole formula. The computation of the monopole formula gives rise to the Coulomb branch Hilbert series which is used to determine the *generators*, the *relations* and even the *syzygies* of the moduli space of vacua in the theories. We also provide the useful mathematical machinery, called *Plethystic Logarithm* or PL, helping us extract those muduli space key properties. Since the refined Hilbert series, sometimes, is very long and complicated to see due to the complexity of the quivers with a majority of gauge nodes; we thus introduce the Highest weight generating function or HWG corresponding to the character expression of the Hilbert seires. Additionally, we also compute the its PL in order to determine the generators of the HWG.

Particularly, the analysis in chapter 3 is calculating the Coulomb branch of the balanced quivers corresponding to A series *minimal* and *supra minimal* nilpotent orbits. As we have found in the computations, the supra minimal orbit of the quivers, which is the next-to minimal nilpotent orbit, has few relations (also syzygies) comparing to the minimal orbit of the same A series quiver. This means that the moduli space of the vacua associating to the supra minimal orbit of the quivers is bigger than the space of the minimal orbit of the same quivers. In terms of HWG, there is an additional generator at order 4 of the quivers corresponding to the supra minimal orbit, confirming their moduli space is bigger.

We have an ungauging scheme for in a computation of a Coulomb branch for a non-simply laced edge (NSLE) quiver with purely gauge groups. The ungauging choice of a residual center-of-mass $U(1)$ symmetry determines the Coulomb branch of the theory. In this paper, we focus on the ungauging schemes on the long side of the NSLE. This results in the same Coulomb branch independent to the choice of ungauging scheme. Accordingly, the global symmetry is the Lie group G which corresponds to the Dynkin diagram of the Lie algebra $\mathfrak{g}$ correlating with the flavourless quiver. Regarding short ungauging schemes, it is pretty different from the long schemes due to the Coulomb branch depending on the ungauging choice. We can see in [10] for more details.

Finally we provide the useful computing technique in Appendix A.1. The technique is "*perturbative approach*" allowed us to obtain the exact Hilbert series by just summing the magnetic charges around the origin of the lattice. This is not a time-consuming computation for the complicated quivers. The yields take palindromic forms of the unrefined Hilbert series which we have expected.

A

Detailed Calculation

A.1 Supra Minimal Orbit of A_4

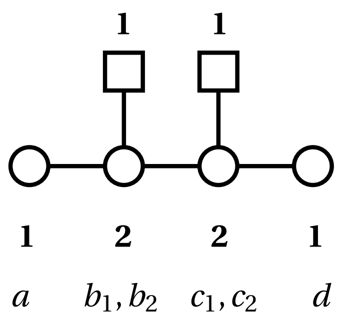

Figure A.1: The quiver corresponds to the supra minimal nilpotent orbit of A_4, mentioned in section 3.7. The assigned magnetic charges are labelled under the quiver diagram.

As we have stated in section 3.7, the gauge groups can be broken and divided into four cases as following:

- Both $U(2)$ gauge groups are not broken, i.e. $b_1 = b_2, c_1 = c_2$.
- The left $U(2)$ is broken to the residual symmetry groups $U(1)^2$, i.e. $b_1 \neq b_2, c_1 = c_2$.
- The right $U(2)$ is broken to the residual symmetry groups $U(1)^2$, i.e. $b_1 = b_2, c_1 \neq c_2$.
- Both $U(2)$ are broken to the residual symmetry groups $U(1)^2$, i.e. $b_1 \neq b_2, c_1 \neq c_2$.

Those four cases have the corresponding conformal dimensions and classical factors, given in:

$$
\begin{aligned}
\Delta_1(a,b,c,d) &= |a-b|+|b|+2|b-c|+|c|+|c-d| \\
\Delta_2(a,b_1,b_2,c,d) &= \frac{1}{2}(|a-b_1|+|a-b_2|+|b_1|+|b_2| \\
&\quad +2|b_1-c|+2|b_2-c|+2|c|+2|c-d|)-|b_1-b_2| \\
\Delta_3(a,b,c_1,c_2,d) &= \frac{1}{2}(2|a-b|+2|b|+2|b-c_1|+2|b-c_2| \\
&\quad +|c_1|+|c_2|+|c_1-d|+|c_2-d|)-|c_1-c_2| \\
\Delta_4(a,b_1,b_2,c_1,c_2,d) &= \frac{1}{2}(|a-b_1|+|a-b_2|+|b_1|+|b_2| \\
&\quad +|b_1-c_1|+|b_2-c_1|+|b_1-c_2|+|b_2-c_2| \\
&\quad +|c_1|+|c_2|+|c_1-d|+|c_2-d|) \\
&\quad -(|b_1-b_2|+|c_1-c_2|
\end{aligned}
\tag{A.1}
$$

$$
P_G(t,b_1,b_2,c_1,c_2)=\begin{cases}
\frac{1}{(1-t^2)^4(1-t^4)^2}=P_{G1} & ; \quad b_1=b_2=b, \quad c_1=c_2=c \\
\frac{1}{(1-t^2)^5(1-t^4)}=P_{G2} & ; \quad b_1<b_2, \quad c_1=c_2=c \\
\frac{1}{(1-t^2)^5(1-t^4)}=P_{G3} & ; \quad b_1=b_2=b, \quad c_1<c_2 \\
\frac{1}{(1-t^2)^6}=P_{G4} & ; \quad b_1<b_2, \quad c_1<c_2
\end{cases}
\tag{A.2}
$$

The unrefined Hilbert series is expressed as:

$$
\begin{aligned}
H_{unref} &= H_1(t)+H_2(t)+H_3(t)+H_4(t) \\
H_1(t) &= \sum_{a,b,c,d\in\mathbb{Z}} t^{2\Delta_1(a,b,c,d)}P_{G1} \\
H_2(t) &= \sum_{a,b_1\le b_2,c,d\in\mathbb{Z}} t^{2\Delta_2(a,b_1,b_2,c,d)}P_{G2} \\
H_3(t) &= \sum_{a,b,c_1\le c_2,d\in\mathbb{Z}} t^{2\Delta_3(a,b,c_1,c_2,d)}P_{G3} \\
H_4(t) &= \sum_{a,b_1\le b_2,c_1\le c_2,d\in\mathbb{Z}} t^{2\Delta_4(a,b_1,b_2,c_1,c_2,d)}P_{G4}
\end{aligned}
\tag{A.3}
$$

One can simplify and rearrange the magnetic charges in the absolute values, but it is very complicated and tough to compute the Hilbert series even using a calculating programme, i.e. Mathematica.

Accordingly, we will provide a useful computing technique, called the perturbative approach. By the process, we only sum the magnetic charge m_i around the origin ($m_i = 0$) of the lattice, i.e. $-10 \leq m_i \leq 10$. After that, we multiply by $(1-t^2)^{12}$ due to the dimension[1] of the Coulomb branch quiver. We will receive the nominator, in the ***palindromic*** form, of the Hilbert series in the quotient form, taking the following form:

$$1+12t^2+53t^4+88t^6+53t^8+12t^{10}+t^{12} \tag{A.4}$$

The exact unrefined HS is

$$H_{unref}(t) = \frac{1+12t^2+53t^4+88t^6+53t^8+12t^{10}+t^{12}}{(1-t^2)^{12}} \tag{A.5}$$

We also refine (A.5) and get

$$\begin{aligned} H_{ref}(t,x_1,x_2,x_3,x_4) = 1 & \\ & + t^2([1,0,0,1]) \\ & + t^4([2,0,0,2]+[0,1,1,0]) \\ & + t^6([3,0,0,3]+[1,1,1,1]) \\ & + t^8([4,0,0,4]+[2,1,1,2]+[0,2,2,0]) \\ & + t^{10}([5,0,0,5]+[3,1,1,3]+[1,2,2,1]) \\ & + t^{12}([6,0,0,6]+[4,1,1,4]+[2,2,2,2]+[0,3,3,0]) \\ & + t^{14}([7,0,0,7]+[5,1,1,5]+[3,2,2,3]+[1,3,3,1]) \\ & + O(t^{16}) \end{aligned} \tag{A.6}$$

[1] This dimension can be obtained by taking double of the sum of the ranks of the gauge nodes [AH,RK-Nilpotent]

and simplify (A.6)

$$\begin{aligned} H_{ref} &= \sum_{n=0}^{\infty} t^{4n}([2n,0,0,2n]+[2n-2,1,1,2n-2]+\cdots+[0,n,n,0]) \\ &\quad + \sum_{n=0}^{\infty} t^{4n+2}([2n+1,0,0,2n+1]+[2n-1,1,1,2n-1]+\cdots+[1,n,n,1]) \end{aligned} \tag{A.7}$$

To obtain HWG, we use the Dynkin label fugacity for $SU(5)$, as mentioned in section 2.6. The easy way to simplify the HWG is dividing into two cases; namely the terms at order t^{4n} and at order t^{4n+2} where $n \geq 0$. Therefore, we get

$$\begin{aligned} HWG_{ref} &= \sum_{n=0}^{\infty} t^{4n}\mu_1^{2n}\mu_4^{2n} \sum_{m=0}^{n} \left(\frac{\mu_2\mu_3}{\mu_1^2\mu_4^2}\right)^m + \sum_{n=0}^{\infty} t^{4n+2}\mu_1^{2n+1}\mu_4^{2n+1} \sum_{m=0}^{n} \left(\frac{\mu_2\mu_3}{\mu_1^2\mu_4^2}\right)^m \\ &= (1+\mu_1\mu_4 t^2) \sum_{n=0}^{\infty} (\mu_1^2\mu_4^2 t^4)^n \sum_{m=0}^{n} \left(\frac{\mu_2\mu_3}{\mu_1^2\mu_4^2}\right)^m \\ &= \frac{1}{(1-\mu_1\mu_4 t^2)(1-\mu_2\mu_3 t^4)} \end{aligned} \tag{A.8}$$

which is equivalent to (3.61).

A.2 G_2 in $SU(3)$ Expression

The following $[n_1, n_2]$ is the Dynkin label for $SU(3)$. The full expression of the refined Hilbert series (4.15) is

$$
\begin{aligned}
H_{G_2\to SU(3)} = 1 \\
&+ t^2([11]+[10]+[01]) \\
&+ t^4([22]+[21]+[20]+[12]+[02] \\
&\quad +[11]) \\
&+ t^6([33]+[32]+[31]+[30]+[23]+[13]+[03] \\
&\quad +[22]+[21]+[12]) \\
&+ t^8([44]+[43]+[42]+[41]+[40]+[34]+[24]+[14]+[04] \\
&\quad +[33]+[32]+[31]+[23]+[13]] \\
&\quad +[22]) \\
&+ t^{10}([55]+[54]+[53]+[52]+[51]+[50]+[45]+[35]+[25]+[15]+[05] \\
&\quad +[44]+[43]+[42]+[41]+[34]+[24]+[14] \\
&\quad +[33]+[32]+[23]) \\
&+ t^{12}([66]+[65]+\cdots+[60]+[56]+\cdots+[06] \\
&\quad +[55]+[54]+\cdots+[51]+[45]+\cdots+[15] \\
&\quad +[44]+[43]+[42]+[34]+[24] \\
&\quad +[33]) \\
&+ t^{14}([77]+[76]+\cdots+[70]+[67]+\cdots+[07] \\
&\quad +[66]+[65]+\cdots+[61]+[56]+\cdots+[16] \\
&\quad +[55]+[54]+[53]+[52]+[45]+[35]+[25] \\
&\quad +[44]+[43]+[34]) \\
&+ O(t^{16})
\end{aligned}
\tag{A.9}
$$

After using the Dynkin label fugacity for $SU(3)$, this is very difficult to simplify the HWG. By the way, we will stick the process as splitting into the terms associating to degree t^{4n} and t^{4n+2}.

We first start considering the terms associating to degree t^{4n}. We obtain

$$\sum_{n=0}^{\infty} t^{4n}(\mu_1^{2n}\mu_2^{2n} + \mu_1^{2n-1}\mu_2^{2n-1} + \mu_1^{2n-2}\mu_2^{2n-2} + \cdots + \mu_1^{2n-n}\mu_2^{2n-n}$$
$$+ \mu_1^{2n-0}\mu_2^{2n-1} + \mu_1^{2n-0}\mu_2^{2n-2} + \cdots + \mu_1^{2n-0}\mu_2^{2n-2n}$$
$$+ \mu_1^{2n-1}\mu_2^{2n-2} + \mu_1^{2n-1}\mu_2^{2n-3} + \cdots + \mu_1^{2n-1}\mu_2^{2n-(2n-1)}$$
$$+ \mu_1^{2n-2}\mu_2^{2n-3} + \mu_1^{2n-2}\mu_2^{2n-4} + \cdots + \mu_1^{2n-2}\mu_2^{2n-(2n-2)}$$
$$\vdots$$
$$+ \mu_1^{2n-(n-1)}\mu_2^{2n-n} + \mu_1^{2n-(n-1)}\mu_2^{2n-(n+1)}$$
$$+ \mu_1^{2n-1}\mu_2^{2n-0} + \mu_1^{2n-2}\mu_2^{2n-0} + \cdots + \mu_1^{2n-2n}\mu_2^{2n-0}$$
$$+ \mu_1^{2n-2}\mu_2^{2n-1} + \mu_1^{2n-3}\mu_2^{2n-1} + \cdots + \mu_1^{2n-(2n-1)}\mu_2^{2n-1}$$
$$+ \mu_1^{2n-3}\mu_2^{2n-2} + \mu_1^{2n-4}\mu_2^{2n-2} + \cdots + \mu_1^{2n-(2n-2)}\mu_2^{2n-2}$$
$$\vdots$$
$$+ \mu_1^{2n-(n-1)}\mu_2^{2n-n} + \mu_1^{2n-(n-1)}\mu_2^{2n-(n+1)})$$

$$= \sum_{n=0}^{\infty} (\mu_1^2\mu_2^2 t^4)^n (\sum_{p=0}^{n} \mu_1^{-p}\mu_2^{-p}$$
$$+ \mu_1^{-0}(\sum_{m=1}^{2n-0} \mu_2^{-m}) + \mu_1^{-1}(\sum_{m=1}^{2n-1} \mu_2^{-m}) + \mu_1^{-2}(\sum_{m=1}^{2n-2} \mu_2^{-m}) + \cdots + \mu_1^{-(n-1)}(\sum_{m=1}^{2n-(n-1)} \mu_2^{-m})$$
$$+ (\sum_{m=1}^{2n-0} \mu_1^{-m})\mu_2^{-0} + (\sum_{m=1}^{2n-1} \mu_1^{-m})\mu_2^{-1} + (\sum_{m=1}^{2n-2} \mu_1^{-m})\mu_2^{-2} + \cdots + (\sum_{m=1}^{2n-(n-1)} \mu_1^{-m})\mu_2^{-(n-1)})$$

$$= \sum_{n=0}^{\infty} (\mu_1^2\mu_2^2 t^4)^n \left(\sum_{p=0}^{n} \mu_1^{-p}\mu_2^{-p} + \sum_{p=0}^{n-1} \mu_1^{-p}(\sum_{m=1+p}^{2n-p} \mu_2^{-m}) + \sum_{p=0}^{n-1} \mu_2^{-p}(\sum_{m=1+p}^{2n-p} \mu_1^{-m}) \right) \tag{A.10}$$

Subsequently, we also do the same way, as order t^{4n}, for order t^{4n+2}:

$$\sum_{n=0}^{\infty} t^{4n+2}(\mu_1^{2n+1}\mu_2^{2n+1} + \mu_1^{2n-0}\mu_2^{2n-0} + \mu_1^{2n-1}\mu_2^{2n-1} + \cdots + \mu_1^{2n-n+2}\mu_2^{2n-n+2} + \mu_1^{2n-n+1}\mu_2^{2n-n+1}$$
$$+ \mu_1^{2n+1}\mu_2^{2n-0} + \mu_1^{2n+1}\mu_2^{2n-1} + \cdots + \mu_1^{2n+1}\mu_2^{2n+1-(2n+1)}$$
$$+ \mu_1^{2n-0}\mu_2^{2n-1} + \mu_1^{2n-0}\mu_2^{2n-2} + \cdots + \mu_1^{2n-0}\mu_2^{2n+1-(2n)}$$
$$+ \mu_1^{2n-1}\mu_2^{2n-2} + \mu_1^{2n-1}\mu_2^{2n-3} + \cdots + \mu_1^{2n-1}\mu_2^{2n+1-(2n-1)}$$
$$\vdots$$
$$+ \mu_1^{2n-(n-2)}\mu_2^{2n-(n-1)} + \mu_1^{2n-(n-2)}\mu_2^{2n-n} + \mu_1^{2n-(n-2)}\mu_2^{2n-(n+1)}$$
$$+ \mu_1^{2n-(n-1)}\mu_2^{2n-n}$$
$$+ \mu_1^{2n-0}\mu_2^{2n+1} + \mu_1^{2n-1}\mu_2^{2n+1} + \cdots + \mu_1^{2n+1-(2n+1)}\mu_2^{2n+1}$$
$$+ \mu_1^{2n-1}\mu_2^{2n-0} + \mu_1^{2n-2}\mu_2^{2n-0} + \cdots + \mu_1^{2n+1-(2n)}\mu_2^{2n-0}$$
$$+ \mu_1^{2n-2}\mu_2^{2n-1} + \mu_1^{2n-3}\mu_2^{2n-1} + \cdots + \mu_1^{2n+1-(2n-1)}\mu_2^{2n-1)}$$
$$\vdots$$
$$+ \mu_1^{2n-(n-1)}\mu_2^{2n-(n-2)} + \mu_1^{2n-n}\mu_2^{2n-(n-2)} + \mu_1^{2n-(n+1)}\mu_2^{2n-(n-2)}$$
$$+ \mu_1^{2n-n}\mu_2^{2n-(n-1)})$$

$$= (\mu_1\mu_2 t^2)\sum_{n=0}^{\infty} t^{4n}(\mu_1^{2n}\mu_2^{2n} + \mu_1^{2n-1}\mu_2^{2n-1} + \mu_1^{2n-2}\mu_2^{2n-2} + \cdots + \mu_1^{2n-n+1}\mu_2^{2n-n+1} + \mu_1^{2n-n}\mu_2^{2n-n}$$
$$+ \mu_1^{2n-0}\mu_2^{2n-1} + \mu_1^{2n-0}\mu_2^{2n-2} + \cdots + \mu_1^{2n-0}\mu_2^{2n+1-(2n)}$$
$$+ \mu_1^{2n-1}\mu_2^{2n-2} + \mu_1^{2n-1}\mu_2^{2n-3} + \cdots + \mu_1^{2n-1}\mu_2^{2n+1-(2n-1)}$$
$$+ \mu_1^{2n-2}\mu_2^{2n-3} + \mu_1^{2n-2}\mu_2^{2n-4} + \cdots + \mu_1^{2n-2}\mu_2^{2n+1-(2n-2)}$$
$$\vdots$$
$$+ \mu_1^{2n-(n-1)}\mu_2^{2n-(n)} + \mu_1^{2n-(n-1)}\mu_2^{2n-(n+1)} + \mu_1^{2n-(n-1)}\mu_2^{2n-(n+2)}$$
$$+ \mu_1^{2n-n}\mu_2^{2n-(n+1)}$$
$$+ \mu_1^{2n-1}\mu_2^{2n-0} + \mu_1^{2n-2}\mu_2^{2n-0} + \cdots + \mu_1^{2n-0}\mu_2^{2n+1-(2n)}$$
$$+ \mu_1^{2n-2}\mu_2^{2n-1} + \mu_1^{2n-3}\mu_2^{2n-1} + \cdots + \mu_1^{2n-1}\mu_2^{2n+1-(2n-1)}$$
$$+ \mu_1^{2n-3}\mu_2^{2n-2} + \mu_1^{2n-4}\mu_2^{2n-2} + \cdots + \mu_1^{2n-2}\mu_2^{2n-(2n-2))}$$
$$\vdots$$
$$+ \mu_1^{2n-(n)}\mu_2^{2n-(n-1)} + \mu_1^{2n-(n+1)}\mu_2^{2n-(n-1)} + \mu_1^{2n-(n+2)}\mu_2^{2n-(n-1)}$$
$$+ \mu_1^{2n-(n+1)}\mu_2^{2n-n}) \tag{A.11}$$

$$= (\mu_1\mu_2 t^2) \sum_{n=0}^{\infty} (\mu_1^2\mu_2^2 t^4)^n (\sum_{p=0}^{n} \mu_1^{-p}\mu_2^{-p}$$
$$+ \mu_1^{-0}(\sum_{m=1}^{2n-0} \mu_2^{-m}) + \mu_1^{-1}(\sum_{m=1}^{2n-1} \mu_2^{-m}) + \mu_1^{-2}(\sum_{m=1}^{2n-2} \mu_2^{-m}) + \cdots + \mu_1^{-(n-1)}(\sum_{m=1}^{2n-(n-1)} \mu_2^{-m})$$
$$+ (\sum_{m=1}^{2n-0} \mu_1^{-m})\mu_2^{-0} + (\sum_{m=1}^{2n-1} \mu_1^{-m})\mu_2^{-1} + (\sum_{m=1}^{2n-2} \mu_1^{-m})\mu_2^{-2} + \cdots + (\sum_{m=1}^{2n-(n-1)} \mu_1^{-m})\mu_2^{-(n-1)}$$

$$= (\mu_1\mu_2 t^2) \sum_{n=0}^{\infty} (\mu_1^2\mu_2^2 t^4)^n \left(\sum_{p=0}^{n} \mu_1^{-p}\mu_2^{-p} + \sum_{p=0}^{n-1} \mu_1^{-p}(\sum_{m=1+p}^{2n-p+1} \mu_2^{-m}) + \sum_{p=0}^{n-1} \mu_2^{-p}(\sum_{m=1+p}^{2n-p+1} \mu_1^{-m}) \right) \tag{A.12}$$

Finally we can combine (A.10) and (A.12), resulting in

$$HWG_{G_2 \to SU(3)} = \frac{1}{(1-\mu_1 t^2)(1-\mu_2 t^2)(1-\mu_1\mu_2 t^2)} \tag{A.13}$$

which is also the same as (4.16).

B
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Declaration of Academic Integrity

I hereby declare that I have written the present work myself and did not use any sources or tools other than the ones indicated.

Date: 25 09 2020 ....Keerati...Keeratikarn........

(Signature)