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An Introduction To Shape Dynamics

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Abstract

In this paper we provide a review of Shape Dynamics, a new theory of gravity which overlaps with General Relativity in places but is built from fewer and more fundamental principles. Shape Dynamics is based on a different symmetry group to General Relativity and uses this to implement spatial and temporal relationalism as well as successfully satisfying the Mach-Poincaré Principle.

Contents

1 Introduction

Einstein's theory of General Relativity (GR) is currently the most successful theory of gravity that we have and has provided a wealth of experimentally verifiable results within cosmology and astrophysics ref. Nonetheless it is also at the centre of arguably the largest problem in contemporary physics, that of finding a quantum field theory of gravity consistent with the standard model ref.

One of many approaches to improving our understanding of gravity is that of Shape Dynamics (SD). Like General Relativity, Shape Dynamics is a gauge theory of gravity that is based on a set of physical principles that are taken to be fundamental axioms of nature ref. SD differs from GR in that we require fewer and somewhat more fundamental principles to produce the same physics, although that is not to say that GR and SD align in all cases. Of course what is meant by "more fundamental" will be expanded upon later. As we will see there are indeed solutions of SD which are not solutions of GR and likewise, solutions of GR that are not solutions of SD.

To understand what is different about Shape Dynamics, it is first useful to recap some of the basic conceptual structure of General Relativity.

In GR we assume the existence of a 4-dimensional pseudo-Riemannian manifold $\mathcal{M}$ which we call spacetime, with a dynamical metric $g_{\mu\nu}$. Shape Dynamics however does not assume the existence of a spacetime, instead we work with a collection of 3-geometries who may fit together to act as a four dimensional spacetime. Whilst GR is a theory concerned with diffeomorphism invariance - the ability to leave physics invariant under general active coordinate transformations - Shape dynamics is based on the spatial conformal symmetry of Weyl transformations. A Weyl transformation is a rescaling of the metric at every point ref.

$$g_{\mu\nu} \to \phi^2(x) g_{\mu\nu} \tag{1}$$

.

The 3-geometries are in fact conformal geometries, meaning that the associated metrics form an equivalence class under Weyl transformations [1].

$$\{g_{ij} \sim g'_{ij} \mid g'_{ij} = \phi^4(x) g_{ij},\ \phi(x) > 0\ \forall x\} \tag{2}$$

Attempts have previously been made to use conformal methods in the formulation of modern theories of gravity, but so far SD is the only, in fact this goes as far back as 1918 with a conformal theory proposed by Weyl [1], only a few years after Einstein began to publish his work on General Relativity. In fact Shape Dynamics itself was motivated by the work of Dirac on a particular gauge fixing of the ADM description of GR [2]. The ADM description will be covered in section 2. This conformal structure is of fundamental importance to the conceptual and technical difference between Shape Dynamics and General Relativity. The Weyl transformations facilitate the idea that in Shape Dynamics, lengths are not physical, only the angle-determining part of the metric is considered physical [3]. Shape Dynamics is formulated in such a way that it is able to describe physics from relational principles, specifically temporal and spatial relationalism. On top of this Shape Dynamics is a successful implementation of the Mach-Poincaré principle. Einstein originally sought to implement Mach's principle in General Relativity but could not find a practical way to do it ref, and instead ended up with the current formulation of GR that we understand today. The fundamental physical principle of Shape Dynamics can be summarised as the following [4].

- Spatial Relationism: The positions and sizes of objects are determined relative to each other.
- Temporal Relationalism: The flow of time is due to physical changes.
- Mach-Poincaré Principle: A point and a direction in the configuration is space uniquely specify a solution.

The motivation for reformulating mechanics based on these principles come from investigating the canonical Hamiltonian formulation of GR. One would want to write General Relativity in the Hamiltonian language as this would form the first steps canonical quantisation. Section 2 is concerned with formulating this Hamiltonian description and highlighting the relevant issues associated with quantising it through the Wheeler-deWitt equation.

[1] In SD literature it is common to use ϕ^4 rather than ϕ^2 as it simplifies expressions later on.

2 The ADM Formulation of General Relativity

2.1 The Hamiltonian Formulation of GR

The lagrangian formulation provides an invaluable useful tool for investigating field theories in physics. Across any area of physics we may start with a lagrangian (density) that is a function of the canonical variables $\mathcal{L}\left(q_i, \dot{q}_i\right)$, such that the action

$$S = \int d^4x\, \mathcal{L}\left(q_i, \dot{q}_i\right) \tag{3}$$

exhibits the symmetries we desire in the theory. In General Relativity, we usually start with the Einstein-Hilbert action

$$S_{EH} = \int d^4x \sqrt{g} R \tag{4}$$

the equations of motion this produces are simply the Einstein field equations without matter sources $G_{\mu\nu} = 0$ [5]. This action provides a jumping off point for less trivial extensions, such as the inclusion of matter terms or alternate theories that involve modifications of the Einstein-Hilbert actions, such as the coupling to a scalar field in Brans-Dicke Theory [6]

$$S_{BD} = \int d^4x \sqrt{g} \left[\frac{\phi}{16\pi} R - \frac{\lambda}{16\pi\phi} (\partial\phi)^2 \right] \tag{5}$$

or higher-dimensional actions

$$S = \int d^{d+4}X \sqrt{G} \left(\frac{1}{16\pi G_{d+4}} R[G_{ab}] + \mathcal{L}_{matter} \right) \tag{6}$$

. the cosmology of these models is well studied [7–10] and considering such modifications to the Einstein-Hilbert action continues to provide motivation for further research. That being said there is still much to be learnt from the Lagrangian formulation of Einsteins General Relativity. For example one may posit the existence of gravitational waves simply by considering the Einstein-Hilbert action with respect to first-order perturbations of the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The ability to detect gravitational waves in recent times has opened up a new way of probing cosmology and astrophysics, and is a topic of great excitement within those research areas [11, 12].

Whilst the Lagrangian formulation is extremely useful, it is sometimes more appropriate to think in terms of the Hamiltonian formulation, particularly when paying attention to the constraints of a system, which we shall see is important in Shape Dynamics.

The canonical momenta is extracted from the Lagrangian as

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{7}$$

and the Hamiltonian density by performing a Legendre transformation

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}\left(q_i, \dot{q}_i\right) \tag{8}$$

This allows one to define a Hamiltonian that is the generator of time translations $H(t) = \int d^3x \mathcal{H}$ and obeys the standard Poisson bracket formalism

$$\dot{q}_i = \{q_i, H\} \qquad \dot{p}_i = \{p_i, H\} \tag{9}$$

The ADM formulation [13–16] casts General Relativity in this Hamiltonian language and we will outline the procedure here. This will lead to the Wheeler-DeWitt equation [17] which shall help shed some light on the issues associated with quantising GR that Shape Dynamics provides a solution to.

To work in the Hamiltonian framework, we must choose a preferred time variable. This is done by foliating the spacetime into timelike hypersurfaces. The spacetime is split up into a family of thin sheets Σ_t embedded in the 4-dimensional spacetime and parameterised by t. Because of the separation of spatial and temporal variables, this is sometimes referred to as the "3+1" formulation of GR. On each hypersurface the 4-metric $g_{\mu\nu}$ is then decomposed into a spatial 3-metric γ_{ij}, a lapse function N and the shift N_i

$$\begin{aligned} &g_{ij} = \gamma_{ij} && g_{0i} = N_i \\ &g_{00} = -N^2 + \gamma^{ij} N_i N_j && \end{aligned} \tag{10}$$

where γ^{ij} is the inverse spatial metric defined by $\gamma^{ik}\gamma_{kj} = \delta^i_j$. With this decomposition we can write the 4-metric in matrix form as

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \gamma^{ij} N_i N_j & N_i \\ N_j & \gamma_{ij} \end{pmatrix} \tag{11}$$

from this it is easy to read off the decomposition of the inverse metric

$$\begin{aligned} g^{ij} &= \gamma^{ij} - \frac{N^i N^j}{N^2} \quad g^{0i} = \frac{N^i}{N^2} \\ g^{00} &= -\frac{1}{N^2} \end{aligned} \tag{12}$$

where the shift vector is inverted using the spatial 3-metric $N^i = \gamma^{ij} N_j$. The determinant of the 3-metric γ, is related to the determinant of the 4-metric g by $g = -N^2 \gamma$.

The lapse function is a measure of the rate of change of proper time τ w.r.t. the time coordinate t as one moves in the direction of the unit normal vector to Σ_t

$$n^\mu = \left(1/N, -N^i/N\right) \tag{13}$$

and the shift vector is the amount by which the spatial coordinate system *shifts* when moving along n^μ. Now that the decomposition of the metric is understood we may now apply it to the Einstein-Hilbert action (including a cosmological constant).

$$S_{EH} = \int d^4x \sqrt{-g} \left(R_4 - 2\Lambda\right) \tag{14}$$

where R_4 is the 4-dimensional scalar curvature. To express this action in terms of the spatial metric, lapse and shift we make use of the Gauss-Codazzi equations [18] which allow us to write the 4-dimensional Ricci scalar in terms of the 3-dimensional intrinsic curvature of the foliations R and their extrinsic curvature due to the embedding in spacetime

$$K_{ij} = \frac{1}{2N} \left(L_{N_i} \gamma_{ij} - \frac{d}{dt} \gamma_{ij} \right) = \left(\nabla_i N_j + \nabla_j N_i - \frac{d}{dt} \gamma_{ij} \right) \tag{15}$$

Here, L_{N_i} is the Lie derivative w.r.t. the shift vector N_i and covariant derivatives are taken w.r.t. the spatial metric

$$\nabla_i A^j = \partial_i A^j + \gamma^j_{ik} A^k \tag{16}$$

using connection coefficients $\gamma^j_{ik} = \frac{1}{2} \gamma^{jl} \left(\partial_i \gamma_{kl} + \partial_k \gamma_{li} - \partial_l \gamma_{ik}\right)$ The Gauss-Codazzi equations state that the 4-dimensional Ricci scalar decomposes as

$$R_4 = R + K_{ij} K^{ij} - K^2 - 2\nabla_\mu \left(K n^\mu\right) - \frac{2}{N} \nabla_i \nabla^i N \tag{17}$$

The Einstein-Hilbert action can then be written as

$$S_{EH} = \int d^4x\, N\sqrt{\gamma}\left[R + K_{ij}K^{ij} - K^2 - 2\nabla_\mu\left(Kn^\mu\right) - \frac{2}{N}\nabla_i\nabla^i N - 2\Lambda\right] \tag{18}$$

Two of the terms in the integrand can be simplified, firstly if we inspect the middle term we find that it is the integral of a total derivative and thus vanishes assmuing that the current J^μ is zero at infinity.

$$\int d^4xN\sqrt{\gamma}\,\nabla_\mu\left(Kn^\mu\right) = \int d^4x\sqrt{-g}\nabla_\mu J^\mu = 0 \tag{19}$$

Likewise the second to last term can also be shown to vanish due to being a 3-dimensional total derivative

$$\int d^4xN\sqrt{\gamma}\left(\frac{1}{N}\nabla_i\nabla^i N\right) = \int d^4x\sqrt{\gamma}\nabla_i\left(\nabla^i N\right) = \int dt\int d^3x\sqrt{\gamma}\nabla_i j^i = 0 \tag{20}$$

The simplified expression for the Einstein-Hilbert action is then

$$S_{EH} = \int d^4xN\sqrt{\gamma}\,\left[R - 2\Lambda + K_{ij}K^{ij} - K^2\right] \tag{21}$$

With this expression for the action we are now ready to write GR in Hamiltonian form, using the spatial metric γ_{ij}, lapse N and shift N_i canonical variables. The lapse and shift do not enter the theory dynamically, the ADM lagrangian $\mathcal{L}_{ADM} = N\sqrt{\gamma}\left[R - 2\Lambda + K_{ij}K^{ij} - K^2\right]$ only contains time derivatives of the spatial metric γ_{ij} with the lapse and shift acting as Lagrange multipliers with vanishing conjugate momenta. This helps make clear the perspective of GR as a constrained Hamiltonian system. The *primary* [2] constraints associated with the Lagrange multipliers are derived by taking the variation of the action with respect to N & N_i.

$$\mathcal{H}^i = -2\nabla_j p^{ij} \approx 0 \tag{22}$$

$$\mathcal{H} = \frac{1}{\sqrt{\gamma}}\left(p_{ij}p^{ij} - \frac{1}{2}p^2\right) \approx 0 \tag{23}$$

[2]Primary constraints are those satisfied by the momenta due to their definition and not due to extremising the action [19]

where p^{ij} is the conjugate momenta to the spatial 3-metric.

$$p^{ij} = \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{g_{ij}}} = \sqrt{\gamma}\left(K\gamma^{ij} - K^{ij}\right). \tag{24}$$

We use the notation $\approx$ to mean that the expressions should hold on solutions to the theory. We refer to H as the Hamiltonian constraint and $\mathcal{H}^i$ as the diffeomorphism constraint as it is the generator of diffeomorphism transformations. The Hamiltonian constraint however does not admit a simple geometrical interpretation. Performing the Legendre transformation 8 we obtain the ADM Hamiltonian.

$$\mathcal{H}_{ADM} = \int d^3x \left(NH + N_i H^i\right) \tag{25}$$

This clearly vanishes on-shell since it is proportional to the constraints. Furthermore, the Hamiltonian constraint is quadratic in the momenta, we will see in deriving the Wheeler-deWitt equation that this is one of the issues when it comes to quantising the theory. Defining the Poisson bracket as

$$\{F(g_{ij}, p^{ij}), G(g_{ij}, p^{ij})\} = \int d^3x \left(\frac{\delta F}{\delta g_{ij}}\frac{\delta G}{\delta p^{ij}} - \frac{\delta F}{\delta p^{ij}}\frac{\delta G}{\delta g_{ij}}\right) \tag{26}$$

the ADM formulation admits a Poisson bracket structure

$$\begin{aligned}
&\{N(\boldsymbol{x}), \pi_N(\boldsymbol{x}')\} = \delta^3(\boldsymbol{x} - \boldsymbol{x}') \\
&\{N^i(\boldsymbol{x}), \pi^N_j(\boldsymbol{x}')\} = \delta^i_j \delta^3(\boldsymbol{x} = \boldsymbol{x}') \\
&\{\gamma_{ij}, p^{kl}\} = \delta^k_{(i}\delta^l_{l)}\delta^3(\boldsymbol{x} - \boldsymbol{x}')
\end{aligned} \tag{27}$$

2.2 The Wheeler-deWitt Equation

The Wheeler-deWitt equation derives from an attempt to apply the canonical quantisation procedure to the ADM formulation. We introduce a quantum state Ψ and following deWitt's original work [17], we work with states in the *metric representation*, in which Ψ is a functional of the 4-metric components.

$$\Psi = \Psi\left[\gamma_{ij}, N, N_i\right] \tag{28}$$

In the canonical quantisation procedure, the momenta becomes the functional derivative operator $p = -i\hbar\delta/\delta q$. The primary constraints 22 & 100 involve only the spatial 3-metric, thus the wave functional

$$\Psi\left[\gamma_{ij}\right] : \mathbf{Riem} \to \mathbb{C} \tag{29}$$

only depends on the spatial 3-metric, where **Riem** is the space of Riemann 3-geomtries.

We need only concern ourselves then, with the momenta associated with $\hat{\gamma}_{ij}$ (where the metric and momenta have been promoted to quantum operators).

$$\hat{p}^{ij} = -i\frac{\delta}{\delta\gamma_{ij}} \tag{30}$$

The constraints are imposed as operator equations of the states

$$\hat{H}\Psi = 0 \tag{31}$$

$$\hat{H}^i\Psi = 0 \tag{32}$$

The diffeomorphism constraint simply reads

$$\hat{H}^i = \nabla_j \frac{\delta}{\delta\gamma_{ij}} \tag{33}$$

while the Hamiltonian constraint requires a small amount of manipulation.

$$\begin{aligned}\hat{H} &= \frac{1}{\sqrt{\gamma}}\left(\hat{p}_{ij}\hat{p}^{ij} - \frac{1}{2}\hat{p}^2\right) - \sqrt{\gamma}\left(2\Lambda - \hat{R}\right) \\ &= \frac{1}{\sqrt{\gamma}}\left(-\frac{\delta}{\delta\gamma_{ij}}\frac{\delta}{\delta\gamma^{ij}} + \frac{1}{2}\hat{\gamma}_{ij}\hat{\gamma}_{kl}\frac{\delta}{\delta\gamma_{ij}}\frac{\delta}{\delta\gamma_{kl}}\right) + \sqrt{\gamma}\left(2\Lambda - \hat{R}\right) \\ &= \frac{1}{\sqrt{\gamma}}\frac{\delta}{\delta\gamma_{ij}}\left(-\frac{\delta}{\delta\gamma^{ij}} + \frac{1}{2}\hat{\gamma}_{ij}\hat{\gamma}_{kl}\frac{\delta}{\delta\gamma_{kl}}\right) + \sqrt{\gamma}\left(2\Lambda - \hat{R}\right)\end{aligned} \tag{34}$$

We need only lower the indices on one of the derivative operators which will introduce two factors of the 3-metric

$$\frac{\delta}{\delta\gamma^{ij}} = \frac{\delta}{\delta\gamma_{kl}}\hat{\gamma}_{ik}\hat{\gamma}_{jl} \tag{35}$$

The final result is the Wheeler-deWitt equation $\hat{H}\Psi = 0$ with $\hat{H}$ defined as

$$\hat{H} = \frac{1}{\sqrt{\gamma}}\left(\hat{\gamma}_{ik}\hat{\gamma}_{jl} - \frac{1}{2}\hat{\gamma}_{ij}\hat{\gamma}_{kl}\right)\hat{p}^{ij}\hat{p}_{kl} + \sqrt{\gamma}\left(2\Lambda - \hat{R}\right) \tag{36}$$

The issue with the Wheeler-deWitt equation is that it's solutions are stationary wave-functionals, reflecting the fact that the Hamiltonian does not generate any dynamics. This is referred to as the "problem of time". The

Hamiltonian constraint is quadratic in the momenta so one would also have to deal with the ordering of the operators which has been ignored here.

The problem of time can be attributed to the fact that GR does not offer an external time parameter, Shape Dynamics tries to solve this issue by identifying an internal variable as a clock, we will see in later sections that this is the "York time" τ. By formulating Shape Dynamics to satisfy the Mach-Poincaré, it is also able to address the related issue of "many-fingered time". General Relativity does not satisfy the Mach-Poincaré principle on it's configuration space, called superspace $\mathbf{W} = \mathbf{Riem}/\mathbf{Diff}$, due to GR's refoliation invariance. Each choice of lapse function N corresponds to a different configuration in $\mathbf{W}$, however they represent the same spacetime, only with a different foliation. Shape Dynamics does away with this by replacing the Hamiltonian constraints with local constrains on conformal 3-geometries and the volume-preserving conformal constraint [20]. This reduces the configuration space to $\mathbf{C} \times \mathbb{R}^+$, where $\mathbf{C}$ is conformal superspace - the space of conformal 3-geomtries which are equivalence classes of 3-geomtries related by a conformal transformation [21] - $\mathbb{R}^+$ represents the volume of a compact 3-geometry $V = \int d^4x\sqrt{g}$ which is invariant under the transformations of which the volume-preserving conformal constraint is a generator (hence the name!). On this configuration space Shape Dynamics is able to realise the Mach-Poincaré principle through York's solution to the initial value problem in GR. This was laid out across a series of papers [22–25] and the relevant results will be covered in section 4.2.

3 Relational Dynamics

3.1 Best-Matching

The present a formal derivation of Shape Dynamics we first have to establish the mathematical framework that allows us to reformulate mechanics from a relational perspective. This is implemented through a technique developed by Barbour and Bertotti [26], an excellent introduction to this is given by Mercetti [4] which we will try to follow where possible.

The idea behind the best-matching approach is to find an *intrinsic* measure of change of a systems configuration. This leads naturally to the idea of *shape space* through successive quotienting of the configuration space.

To develop this idea we first start with the smaller problem of 3 particles interacting through Newtonian gravity. Say the 3 particles lie on the same plane in 3D space, and you are given two different "snapshots" of the particle configurations without any information on the orientation of the frame in each snapshot. What we require is an intrinsic measure of change between the two configurations. The only raw data we have access to are the relative separations of the particles $r_{12}(t), r_{13}(t)$, $t_{23}(t)$ at the two instants in time t_i, t_f. Each particle at any instant is represented by a vector in $\mathbb{R}^3$ and the collection of these 3 vectors makes up the Cartesian representation of the configuration.

$$\mathbf{r}_a(t) = (x_a, y_a, z_a) \in \mathbb{R}^3 \tag{37}$$

where a labels the particles. However all of this information can be equivalently represented by a single vector in $\mathbb{R}^9$. Thus we can write the initial and final configurations q^i & q^f as two vectors in $\mathbb{R}^9$.

$$q = \oplus_a \mathbf{r}_a = (\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}) = (x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) \tag{38}$$

These are the elements of the configuration space Q^N. The configurations define triangles in $\mathbb{R}^3$. To measure the intrinsic change between the two triangles we are able to use to Euclidean metric that $\mathbb{R}^9$ comes equipped with. The issue is that the Euclidean metric also contains dependence on the relative orientations of the two configurations. This can be removed by quotienting the configuration space by the set of transformations that leave the relative separations r_{ab} unchanged. This is the Euclidean group $ISO(3)$ of translations and rotations. This forms the relative configuration space $Q_N^R = Q_N/ISO(3)$. Shape Dynamics insists that only angles and ratios are physical, so we want to further quotient the configuration space by configurations related by a scale transformation. The result of the quotienting process is shape space $S^N = Q^N/Sim(3)$, where $Sim(3)$ is the similarity group (The group of Euclidean and scale transformations). The Euclidean transformations act on the coordinate vectors as

$$\mathbf{r}_a \rightarrow \Omega \mathbf{r}_a + \boldsymbol{\theta} \tag{39}$$

where $\boldsymbol{\theta} \in \mathbb{R}^3$ is a translation vector and $\Omega \in SO(3)$ is a 3x3 rotation matrix. On the configuration space elements, the Euclidean transformations act individually on each coordinate vector

$$q \rightarrow T[q] = \oplus_a (\Omega \mathbf{r}_a + \boldsymbol{\theta}) \tag{40}$$

Quotienting the configuration space is done by minimising the Euclidean metric

$$d\left(q^{i}, q^{f}\right)=\left[\sum_{a}\left|\mathbf{r}_{a}^{i}-\mathbf{r}_{a}^{f}\right|^{2}\right]^{1/2} \tag{41}$$

with respect to Euclidean transformations of the final configuration q^f. This results in what's called the *best-matched* distance

$$d_{BM}\left(q^{i}, q^{f}\right)=\inf_{\Omega,\theta}\left(\sum_{a}\left|\mathbf{r}_{a}^{i}-\Omega\mathbf{r}_{a}^{f}-\boldsymbol{\theta}\right|^{2}\right)^{1/2} \tag{42}$$

We now perform a constrained variation of the parameters θ & Ω to find the best-matched configuration q^{BM} defined by

$$d\left(q^{i}, q^{BM}\right)=\inf_{q} d\left(q^{i}, q\right) \tag{43}$$

. It's also sufficient to vary the distance-squared since it's a monotonic function. Starting with the translations we consider $\boldsymbol{\theta}\rightarrow\boldsymbol{\theta}+\delta\boldsymbol{\theta}$.

$$\begin{aligned}
d^{2}\left(q^{i}, T[q^{f}]\right) &\rightarrow \sum_{a}\left|\boldsymbol{r}_{a}^{i}-\Omega\boldsymbol{r}_{a}^{f}-\boldsymbol{\theta}-\delta\boldsymbol{\theta}\right| \\
&=\sum_{a}\left(\boldsymbol{r}_{a}^{i}-\Omega\boldsymbol{r}_{a}^{f}-\boldsymbol{\theta}-\delta\boldsymbol{\theta}\right)\cdot\left(\boldsymbol{r}_{a}^{i}-\Omega\boldsymbol{r}_{a}^{f}-\boldsymbol{\theta}-\delta\boldsymbol{\theta}\right) \\
&=d^{2}+\left(2\sum_{a}\left(\boldsymbol{r}_{a}^{i}-\Omega\boldsymbol{r}_{a}^{f}\right)-2\boldsymbol{\theta}\sum_{a}\right)\cdot\delta\boldsymbol{\theta} \\
&=d^{2}+\left(2\sum_{a}\left(\boldsymbol{r}_{a}^{i}-\Omega\boldsymbol{r}_{a}^{f}\right)-6\boldsymbol{\theta}\right)\cdot\delta\boldsymbol{\theta}
\end{aligned} \tag{44}$$

So we have the variation

$$\frac{\delta d^{2}\left(q^{i}, T[q^{f}]\right)}{\delta\boldsymbol{\theta}}=2\sum_{a}\left(\boldsymbol{r}_{a}^{i}-\Omega\boldsymbol{r}_{a}^{f}\right)-6\boldsymbol{\theta}=0 \tag{45}$$

The solution to this equation gives the best-matching condition

$$\boldsymbol{\theta}^{BM}=\frac{1}{3}\sum_{a}\left(\boldsymbol{r}_{\boldsymbol{a}}^{i}-\Omega\boldsymbol{r}_{a}^{f}\right) \tag{46}$$

Geometrically what this variation does, is to align the barycentres of the triangles. We can anticipate that the variation for Ω will rotate the triangles so that they are as congruent as possible. Putting $\boldsymbol{\theta}^{BM}$ back into the best-matched distance gives

$$d_{BM}\left(q^i, q^f\right) = \inf_{\Omega}\left(\sum_a |\Delta \boldsymbol{r}_a^i - \Omega \Delta_{r}{}_a^f|^2\right)^{1/2} \tag{47}$$

where the coordinate vectors are now modified by $\boldsymbol{\theta}^{BM}$ to $\Delta \boldsymbol{r}_a = \boldsymbol{r}_a - \frac{1}{3}\sum_a \boldsymbol{r}_a$.

Secondly we come to the variations with respect to the rotation matrix Ω. Each component of the matrix cannot be varied independently as with $\boldsymbol{\theta}$, as the resulting matrix $\Omega + \delta\Omega$ must remain an element of the rotation group $SO(3)$. Elements of $SO(3)$ are those 3x3 matrices with unit determinant that satisfy $\Omega\Omega^T = 1$ It is standard procedure to find the variation that imposes this condition.

$$\begin{aligned} (\Omega - \delta\Omega)(\Omega + \delta\Omega)^T &= 1 \\ \Omega\Omega^T + \Omega\delta\Omega^T + \delta\Omega\Omega^T + \mathcal{O}(\delta^2) &= 1 \\ \delta\Omega\Omega^T &= -\left(\delta\Omega\Omega^T\right)^T \end{aligned} \tag{48}$$

This is the requirement that the $\delta\Omega\Omega^T$ is an anti-symmetric matrix. To isolate $\delta\Omega$, we make use of the fact that in three dimensions any anti-symmetric matrix can be written in terms of the cross product operator as such

$$\delta\Omega\Omega^T = \delta\boldsymbol{\omega}\times \tag{49}$$

where $\delta\boldsymbol{\omega}$ is an infinitesimal vector and $\delta\boldsymbol{\omega}\times$ is a matrix with components $\varepsilon_{ijk}\delta\omega_k$. Right multiplying by Ω we obtain

$$\delta\Omega = \delta\boldsymbol{\omega} \times \Omega \tag{50}$$

Substituting this into 47 and performing the same variation procedure we obtain

$$\delta\boldsymbol{\omega} \cdot \sum_a \left(\Omega \Delta \boldsymbol{r}_a^f\right) \times \Delta \boldsymbol{r}_a^i = 0 \tag{51}$$

Without loss of generality we can take (modified) coordinate vectors to be in the x-y plane $\Delta \boldsymbol{r}_a = (\Delta x_a, \Delta y_a, 0)$ and take Ω as a rotation in this plane.

Computing the variation explicitly we have

$$\sum_a \left[\Delta x_a^i \left(\cos\phi \Delta y_a^f + \sin\phi \Delta x_a^f\right) - \Delta y_a^i \left(\cos\phi \Delta x_a^f - \sin\phi \Delta y_a^f\right)\right] = 0$$
$$\sum_a |\Delta \boldsymbol{r}_a^i \times \Delta \boldsymbol{r}_a^f| \cos\phi + \sum_a \Delta \boldsymbol{r}_a^i \cdot \Delta \boldsymbol{r}_a^f \sin\phi = 0 \tag{52}$$
$$\tan\phi = \frac{\sum_a |\Delta \boldsymbol{r}_a^f \times \Delta \boldsymbol{r}_a^i|}{\sum_a \Delta \boldsymbol{r}_a^i \cdot \Delta \boldsymbol{r}_a^f}$$

The best-matches rotation can now be written in terms of a closed-form solution

$$\tilde{\phi} = \arctan\left(\frac{\sum_a |\Delta \boldsymbol{r}_a^f \times \Delta \boldsymbol{r}_a^i|}{\sum_a \Delta \boldsymbol{r}_a^i \cdot \Delta \boldsymbol{r}_a^f}\right) \tag{53}$$

The resulting best-matched distance allows us to define a measure of intrinsic change.

$$d_{BM}\left(q^i, q^f\right) = \left(\sum_a |\Delta \boldsymbol{r}_a^i - \Omega(\tilde{\phi}) \Delta \boldsymbol{r}_a^f|^2\right)^{1/2} \tag{54}$$

The best-matched distance we have derived here depends only on the relative positions of the particles but not on their masses or the potential they interact through, this can be corrected for by weighting the contributions to Euclidean metric in the following manner

$$d\left(q^i, q^f\right) = \left(U\left(r_{bc}\right) \sum_a m_a |\boldsymbol{r}_a^i - \boldsymbol{r}_a^f|^2\right)^{1/2} \tag{55}$$

Configurations that are related by a global rotation or translation transformation should be weighted the same so we require $U(r_{bc})$ to be invariant under such transformations. We can now take the limit of a discretised system to see how the best matching procedure will work for continuous curves $q(s)$ in Q^N. We want to minimise the distance between configurations separated by an infinitesimal change in the continuous parameter $s \to s + ds$.

$$d^2\left(T_k[q^k], T_{k+1}[q^{k+1}]\right) = U(r_{bc}) \sum_a m_a |\Omega d\boldsymbol{r}_a + d\Omega \boldsymbol{r}_a + d\boldsymbol{\theta}|^2 \tag{56}$$

The process for varying with respect to translations and rotations as above can be repeated here and gives

$$d_{BM}\mathcal{L}^2 = \inf_{d\boldsymbol{\omega} d\boldsymbol{\theta}} U(r_{bc}) \sum_a m_a |d\boldsymbol{r}_a + d\boldsymbol{\omega} \times \boldsymbol{r}_a + d\boldsymbol{\theta}|^2 \tag{57}$$

Integrating over a path parameterised by s allows us to define a notion of intrinsic change of the configuration along that path.

$$\int d_{BM}\mathcal{L} = \inf_{d\boldsymbol{\omega} d\boldsymbol{\theta}} \int ds \left(U(r_{bc}) \sum_a m_a | \frac{d\boldsymbol{r}_a}{ds} + \frac{d\boldsymbol{\omega}}{ds} \times \boldsymbol{r}_a + \frac{d\boldsymbol{\theta}}{ds} |^2 \right)^{1/2} \tag{58}$$

3.2 Free End Point Variation

We want to formulate a variational procedure that implements best-matching, to do this it is important that we consider the G-bundle structure inherited by Q^N from the group action upon it, as the G-bundle allows us to define a natural "horizontal" direction in Q^N along which actual physical change occurs, and a "vertical" direction where the change to the configuration is not physical. A principle G-bundle [27] P is a smooth manifold with a map $G \times P \rightarrow P$, and is such that G acts on P smoothly and transitively, meaning that all points in P are left invariant by only the identity element of G

$$G_p = \{g \in G | \; gp = p\} = \{e\} \tag{59}$$

The base space of the G-bundle is the quotient space $B = P/G$, in the case of the configuration space, the base space is the relative configuration space as it is formed by $Q_R^N = Q^N/G$. The base space is considered the space of physically distinct configurations. The G-bundle contains a subspace - $V_p \subset T_pP$ - the space of tangent vectors to P, parallel to the group orbits of G. The group orbits move through configurations which are physically indistinct, this forms our notion of a vertical direction. The horizontal direction is simply the complement to this, such that $T_pP = V_p \oplus H_p$. H_p however is not necessarily unique unless one has a metric on the tangent space, in this case H_p can be defined as the unique orthogonal complement to V_p. It is not always the case that we have a metric on the tangent space, so we need to introduce a connection on the G-bundle, this will provide a way to parallel transport vectors in the tangent space so that we can form a smooth choice of "horizontal" subspaces in a neighbourhood of P.

We require trajectories on Q^N to be horizontal according to a connection on the configuration space. In the case that there does exist a metric on the tangent space, it can be used to define a connection as long as the metric is $G-invariant$, meaning the scalar product of tangent vectors of two curves that intersect at a point is invariant under the group action on the curves.

For a G-invariant metric we recover the familiar case from GR, that the physical, "horizontal", curves are just geodesics of the metric. The variational procedure that implements best matching makes use of a theorem by Bertotti Barbour

Theorem. *Let P be a G-bundle with a base space B and a G-invariant metric. Given a sheet lifted above a curve in B, the horizontal curves on that sheet minimise the free endpoint length between the initial and final group orbits. The G-invariant metric assigns the same length to all horizontal curves on the sheet.*

The variational procedure that implements best-matching, two-stage variation, is slightly different from the one we are used to in standard Lagrangian mechanics. There are two stages. In Lagrangian mechanics we vary the action with fixed boundary conditions which allows us to throw away the boundary terms when performing this variation. This is not the case for two-stage variation, we allow the end-points of the curves the trial curves that we are varying to be free as we first vary in the G-bundle space P where there are a family of curves that all correspond to the same physical configuration in the base space P/G, this allows us to find an action associated with a family of curves in the G-bundle. Then we perform the physical variation in P/G keeping the end points fixed to find the physical extremal curve.

Stage I: Free End Point Variation

1. We take a trial curve in P which has free end points on two group orbits (fibres of the G-bundle). Acting on this curve with the group generates a family of curves in P that correspond to the same physical curve in P/B, with end points on the same orbits. We call this family of curves a "sheet" in P. The different end points (that all lie along the same orbit) correspond to the same physical configuration in P/G so there is no reason to keep them fixed.

2. If P is equipped with a G-invariant metric, the horizontal curves on the sheet all minimise the distance between the two group orbits and will be assigned the same length by the metric.

3. Define the value of the action of the physical curve in P/G as the length of the curves on the sheet in P, in this way we have an action that is invariant under the group action.

Stage II: Physical Variation

1. There will be infinitely many paths in the base space P/G joining the two group orbits, the action defined above can be used to assign a value to each of them.

2. We then minimise this action as we do in standard Lagrangian analysis to find a unique physical curve in the base space.

Consider the following example to see how this works in practise. Say we have a G-invariant metric on the tangent space T_pP

$$(d\boldsymbol{r}_a, d\boldsymbol{r}_b) = \sum_{a,b} M^{ab} d\boldsymbol{r}_a \cdot d\boldsymbol{r}_b \tag{60}$$

To implement the first stage of the variational procedure, take a general path in P, $\boldsymbol{r}_a(s) : [s_1, s_2] \to P$. The sheet of curves is then formed by acting on $\boldsymbol{r}_a(s)$ with the group action

$$\boldsymbol{r}_a(s) \to O(s)\boldsymbol{r}_a(s) \tag{61}$$

The G-invariant metric defines a length for the curves in sheet which we take as the action

$$S_{bare} = \int d\mathcal{L}_{bare} = \int |d\boldsymbol{r}_a(s)| \tag{62}$$

Here we follow the ideas layed out by Anderson in his seminal work [28], that in a framework that implements temporal relationism (which we will discuss shortly in section ..), there is no meaningful external notion of time, and thus we cannot form velocities of configuration variables as we usually would. Instead we are left only with the notion of change of one configuration variable with respect to another, so velocities are replaced simply with differentials of configuration variables. Actions become integrals over an arc-element corresponding to a metric as we have alluded to above in the discussion of the two-stage variational procedure. For this reason we work with the lagrangian as $d\mathcal{L}$ to make clear that it is effectively a differential of arc-length to be integrated over.

Since the paths $\boldsymbol{r}_a(s)$ already minimise the bare action due to the metric being G-invariant, we only need to vary with respect to the group elements whilst keeping the end points free to obtain the best-matched action.

$$S_{BM} = \inf_{O(s)} \int d\mathcal{L} = \inf_{O(s)} \int |d\left[O(s)\boldsymbol{r}_a(s)\right]| \tag{63}$$

The action is G-invariant which allows us to act on the integrand by O^{-1} which puts it in a more useful form

$$\begin{aligned} S_{BM} &= \inf_{O(s)} \int |O^{-1}\left(O(s)d\boldsymbol{r}_a + \left(dO(s)\right)\boldsymbol{r}_a(s)\right)| \\ &= \inf_{O(s)} \int |d\boldsymbol{r}_a(s) + O^{-1}\left(dO(s)\right)\boldsymbol{r}_a| \end{aligned} \tag{64}$$

Assuming the $O(s)$ is a matrix representation of a Lie algebra $O(s) = e^{\varepsilon(s)}$, we are able to simplify the $O^{-1}\left(dO(s)\right)$ term.

$$\begin{aligned} e^{\varepsilon(s)} &= O(s) \\ e^{\varepsilon(s)}d\varepsilon(s) &= dO(s) \\ O(s)d\varepsilon(s) &= dO(s) \\ d\varepsilon(s) &= O^{-1}dO(s) \end{aligned} \tag{65}$$

The integrand then becomes just the "best-matching differential"

$$\mathcal{D}\boldsymbol{r}_a(s) = d\boldsymbol{r}_a(s) + d\varepsilon(s)\boldsymbol{r}_a(s) \tag{66}$$

$$S_{BM} = \inf_{\varepsilon(s)} \int |\mathcal{D}\boldsymbol{r}_a(s)| \tag{67}$$

Now we have an action

$$S = \int d\mathcal{L}\left(\boldsymbol{r}_a, d\boldsymbol{r}_a, \varepsilon, d\varepsilon\right) \tag{68}$$

that we can perform the free end point variational procedure on. This action is to be extremised given fixed boundary conditions on the canonical coordinates $\boldsymbol{r}_a(s_1)$ & $\boldsymbol{r}_a(s_2)$, but keeping the end points of the coordinate $\varepsilon(s)$ that move along the gauge orbits free. Taking the variation of 68 with respect to ε and $d\varepsilon$

$$\delta S = \int \left[\frac{\delta d\mathcal{L}}{\delta\varepsilon} - d\left(\frac{\delta d\mathcal{L}}{\delta d\varepsilon}\right)\right]\delta\varepsilon + \frac{\mathcal{L}}{\delta d\varepsilon}\delta\varepsilon\Bigg|_{s=s_1}^{s=s_2} \tag{69}$$

The action must be stationary under all variations from the extremising solution $\tilde{\varepsilon}$, so this includes fixed end point variations for which $\delta\varepsilon(s_1) = \delta\varepsilon(s_2) = 0$ which would produce the standard Euler-Lagrange equations, as well as the free end point variations. For both of these statements to be true simultaneously we require the Euler-Lagrange term to vanish as well as the boundary terms

$$\frac{\delta d\mathcal{L}}{\delta\varepsilon} - d\left(\frac{\delta d\mathcal{L}}{\delta d\varepsilon}\right) = 0 \tag{70}$$

$$\left.\frac{\delta d\mathcal{L}}{\delta d\varepsilon}\right|_{s=s_1} = \left.\frac{\delta d\mathcal{L}}{\delta d\varepsilon}\right|_{s=s_2} = 0 \tag{71}$$

However writing explicitly the form of the lagrangian $d\mathcal{L} = |d\boldsymbol{r}_a + d\varepsilon\boldsymbol{r}_a|$, it is clear that the coordinate ε is cyclic, so the Euler-Lagrange equation reduces to $d\left(\delta d\mathcal{L}/\delta d\varepsilon\right) = 0$. Along with the conditions on the boundary terms 71 we can establish the equations of motion simply as

$$\frac{\delta d\mathcal{L}}{\delta d\varepsilon} = 0 \tag{72}$$

Although we have given a formalised procedure here that we could follow every time we wish to find the equations of motion that implement best-matching, in practise it's not necessary to go through the entire process of acting on the bare lagrangian with the group elements to then take the free end point variations. As we will show now, it is sufficient to simply minimise the bare action and take 72 as an initial condition on the solution.

The bare actions that we started with 62 are G-invariant, so global group transformations $\boldsymbol{r}_a \rightarrow e^{\varepsilon}\boldsymbol{r}_a$ are symmetries of the action. According to Noether's theorem there should then be an associated conserved current

$$d\left(\left.\frac{\delta d\mathcal{L}_{bare}\left(\varepsilon\boldsymbol{r}_a\right)}{\delta\varepsilon}\right|_{\varepsilon=0}\right) = 0 \tag{73}$$

This implies that solutions to the Euler-Lagrange equations for the bare action will remain "horizontal" in the sense defined by the G-bundle as long as they start horizontal, meaning $\delta d\mathcal{L}/\delta d\varepsilon|_{s_1} = 0$ since the Noether current makes the quantity $\delta d\mathcal{L}/\delta d\varepsilon$ constant along the solutions. In this way we can treat the best-matching condition 72 as an initial condition and work with solutions that minimise the bare action.

3.3 N-body Problem

To see how these methods work in practise we can apply them to the problem of N bodies interacting through Newtonian gravity. This will also lead us to the concept of ephemeris time and a discussion of how temporal relationism is implemented as the material covered thus far pertains only to spatial relationism.

The configuration space Q^N for the N-body problem with total energy E, is equipped with a G-invariant metric

$$d\mathcal{L}^2 = 4(E-V)\sum_a \frac{1}{2} m_a d\boldsymbol{r}_a \cdot d\boldsymbol{r}_a \tag{74}$$

where G is the Euclidean group $Eucl(3)$ and the Newtonian potential is

$$V = -\sum_{a<b} \frac{m_a m_b}{|\boldsymbol{r}_a - \boldsymbol{r}_b|} \tag{75}$$

The best-matched action is formed by promoting the ordinary differentials to the best-matching differential given by the action of $Eucl(3)$ on the canonical coordinates $\mathcal{D}\boldsymbol{r}_a = d\boldsymbol{r}_a + d\boldsymbol{\omega} \times d\boldsymbol{r}_a + d\boldsymbol{\theta}$

$$S_{BM} = \int d\mathcal{L} = \int 2\left((E-V)\sum_a \frac{1}{2} m_a |\mathcal{D}\boldsymbol{r}_a|\right)^{1/2} \tag{76}$$

The quantities represented by $\delta d\mathcal{L}/\delta d\varepsilon$ in section 3.2 are the free end point variations taken with respect to $d\boldsymbol{\omega}$ and $d\boldsymbol{\theta}$. Starting with the $d\boldsymbol{\omega} \to d\boldsymbol{\omega} + \delta d\boldsymbol{\omega}$ variation

$$\begin{aligned} \delta d\mathcal{L} &= \left((E-V)\sum_a \frac{1}{2} m_a |\mathcal{D}\boldsymbol{r}_a|^2\right)^{-1/2} \delta\left((E-V)\sum_a \frac{1}{2} m_a \mathcal{D}\boldsymbol{r}_a \cdot \mathcal{D}\boldsymbol{r}_a\right) \\ &= (E-V)^{1/2}\left(\sum_a \frac{1}{2} m_a |\mathcal{D}\boldsymbol{r}_a|^2\right)^{-1/2} \sum_a m_a \mathcal{D}\boldsymbol{r}_a \cdot \delta\mathcal{D}\boldsymbol{r}_a \end{aligned} \tag{77}$$

From which we arrive at

$$\frac{\delta d\mathcal{L}}{\delta d\boldsymbol{\omega}} = -\frac{1}{d\chi}\sum_a m_a \mathcal{D}\boldsymbol{r}_a \times \boldsymbol{r}_a = 0 \tag{78}$$

where $d\chi$ is the "differential of the instant" defined as

$$d\chi = (E-V)^{-1/2}\left(\sum_a \frac{1}{2}m_a|\mathcal{D}\boldsymbol{r}_a|^2\right)^{1/2} \tag{79}$$

Similarly the variation with respect to $d\boldsymbol{\theta}$ yields

$$\frac{\delta d\mathcal{L}}{d\delta\boldsymbol{\theta}} = \frac{1}{d\chi}\sum_a m_a\mathcal{D}\boldsymbol{r}_a = 0 \tag{80}$$

Defining the canonical momentum as

$$\boldsymbol{p}^a = \frac{\delta d\mathcal{L}}{\delta d\boldsymbol{r}_a} = m_a\frac{\mathcal{D}\boldsymbol{r}_a}{d\chi} \tag{81}$$

simplifies the Euler-Lagrange equations 70 which read

$$d\boldsymbol{p}^a = -d\chi\frac{\partial V}{\partial\boldsymbol{r}_a} - d\boldsymbol{\omega}\times\boldsymbol{p}^a \tag{82}$$

Furthermore, the $d\boldsymbol{\omega}\times\boldsymbol{p}^a$ term appears in the best-matched differential of the momentum

$$\mathcal{D}\boldsymbol{p}^a = d\boldsymbol{p}^a + d\boldsymbol{\omega}\times\boldsymbol{p}^a \tag{83}$$

The $d\boldsymbol{\theta}$ term does not appear in the best-matched differential hear because the linear momentum is translation invariant so it is not acted on by the rotation subgroup of $Eucl(3)$. The equations of motion can then be written as

$$\frac{\mathcal{D}\boldsymbol{p}^a}{d\chi} = -\frac{\partial V}{\partial\boldsymbol{r}_a} \tag{84}$$

This is analogous to Newtons second law $\dot{\boldsymbol{p}}_a = -\partial V/\partial\boldsymbol{r}$ with the differential of the instant acting as a time variable. The ephemeris time is formed by solving the best-matching conditions, calling the solutions $d\boldsymbol{\omega}_{BM}$ and $d\boldsymbol{\theta}_{BM}$. The ephemeris time

$$dt_{eph} = (E-V)^{-1/2}\left(\sum_a m_a|d\boldsymbol{r}_a + d\boldsymbol{\omega}_{BM}\boldsymbol{r}_a + d\boldsymbol{\theta}_{BM}|^2\right)^{1/2} \tag{85}$$

is a natural, intrinsic parameterisation that measures the duration from one instant to another. In terms of the canonical momentum $\boldsymbol{p}^a$, the best-matching conditions take the form of the momentum conservation constraints

$$\boldsymbol{P} = \sum_a \boldsymbol{p}^a = 0 \tag{86}$$

$$\boldsymbol{L} = \sum_s \boldsymbol{r}_a \times \boldsymbol{p}^a = 0 \tag{87}$$

As was mentioned in section 3.2, we also could have arrived at this by simply taking variations of the bare action and imposing the best-matching equations 78 & 80 as initial conditions.

3.4 Temporal Relationalism

So far we have focused almost entirely on developing a framework that implements spatial relationalism and have only briefly mentioned the temporal aspect. What is meant by temporal relationalism is that the system contains no meaningful notion of time, as we have seen with the ephemeris time, one may emerge naturally, but the system does not contain such a notion as a priori. This concept is implemented in practise by using actions that are reparameterisation invariant, and are defined without the use of an external time variable.

The mathematical machinery for this was originally developed by Jacobi, although not in an attempt to implement temporal relationalism. Consider the action

$$S_J = \int_{s_i}^{s_f} ds \sqrt{(E-V)T_k} \tag{88}$$

for a system with a fixed value of energy E. T_k is the kinetic energy of the system defined by

$$T_k = \sum_a \frac{1}{2} m_a \frac{d\boldsymbol{r}_a}{ds} \cdot \frac{d\boldsymbol{r}_a}{ds} \tag{89}$$

This action is reparameterisation invariant due to the local square root factor. Reparameterising the action to a new variable $s = s(s')$ we have

$$ds = \frac{\partial s}{\partial s'} ds' \tag{90}$$

$$T_k = \sum_a \frac{1}{2} m_a \left(\frac{\partial s'}{\partial s}\right)^2 \frac{d\boldsymbol{r}_a}{ds'} \cdot \frac{d\boldsymbol{r}_a}{ds'} \tag{91}$$

the action then takes the form

$$
\begin{aligned}
S_J &= \int_{s'_i}^{s'_f} ds' \frac{\partial s}{\partial s'} \sqrt{\left(\frac{\partial s'}{\partial s}\right)^2 (E-V) \sum_a \frac{1}{2} m_a \frac{d\boldsymbol{r}_a}{ds'} \cdot \frac{d\boldsymbol{r}_a}{ds'}} \\
&= \int_{s'_i}^{s'_f} ds' \sqrt{(E-V) T_k}
\end{aligned}
\tag{92}
$$

Provided the end points are not changed $s'_i(s_i) = s_i$ & $s'_f(s_f) = s_f$ the action is reparameterisation invariant. The ephemeris time emerges naturally by considering the equation if motion of this action

$$
m_a \frac{d}{ds}\left[\left(\frac{E-V}{T_k}\right)^{1/2} \frac{d\boldsymbol{r}_a}{ds}\right] = -\sqrt{\frac{T_k}{E-V}} \frac{\partial V}{\partial \boldsymbol{r}_a} \tag{93}
$$

which simplifies to the familiar Newtonian form

$$
m_a \frac{d^2 \boldsymbol{r}_a}{dt_{eph}^2} = -\frac{\partial V}{\partial \boldsymbol{r}_a} \tag{94}
$$

by defining the ephemeris time increment as

$$
dt_{eph} = \left(\frac{T_k}{E-V}\right)^{1/2} ds \tag{95}
$$

This gives us an emergent notion of change from one instant to another, that is not defined externally as an a priori of the theory.

4 Shape Dynamics

4.1 Deriving Local Lorentz Invariance

Before we can present a full derivation of Shape Dynamics, we first need to understand how one may tie together the best matching framework which implements spatial relationalism with local square-root (& reparameterisation invariant) actions which provide the temporal relationalism to reproduce the familiar dynamics of special relativity and general relativity. This was mainly done through the work of Mercati, Anderson, Barbour, Foster & Ó

Murchadha in a series of papers [29–33]. In particular, much attention is paid to the non-holonomic constraints of the theory and the formalism developed by Dirac to analyse them [34]. General Relativity after all can be viewed as a constrained gauge theory [35].

In the work published by Barbour, Foster & Ó Murchadha [32], it is shown how, using the relational dynamics established in section 3, it possible to derive the same physics as General Relativity without the presumption of a spacetime from the BSW action.

$$S_{BSW} = \int d\lambda \int d^3x \sqrt{g}\sqrt{R - 2\Lambda}\sqrt{\left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\left(\frac{\partial g_{ij}}{\partial \lambda} - L_\xi g_{ij}\right)\left(\frac{\partial g_{kl}}{\partial \lambda} - L_\xi g_{kl}\right)} \tag{96}$$

where λ is a monotonic parameter and $(K\xi)_{ij}$ is the Lie derivative of g_{ij} w.r.t. ξ^i. We may first see how local Lorentz invariance follows naturally on the basis of 3 fundamental assumptions.

- Assume the existence of a spatial 3-geometry with metric g_{ij} and let the action be a function of only metric and it's first derivatives.
- The action must be of the Jacobi type considered in section 3.4
- The theory must be free from redundancies in the field descriptions, facilitated by best-matching.

To demonstrate how to apply the best-matching principles, consider first the following lagrangian. this is the simplest, although not most general, lagrangian that satisfies the above assumptions.

$$d\mathcal{L} = \int d^3x \sqrt{g}\sqrt{R - 2\lambda}\sqrt{\left(g^{ik}g^{jl} - g^{ij}g^{kl}\right) dg_{ij}dg_{kl}} \tag{97}$$

The canonical momenta conjugate to the metric is give by

$$p^{ij} = \frac{\delta d\mathcal{L}}{\delta dg_{ij}} = \frac{\sqrt{g}}{2d\chi}\left(g^{ik}g^{jl} - g^{ij}g^{kl}\right) dg_{kl} \tag{98}$$

in this case the differential of the instant $d\chi$ is defined as

$$d\chi = \frac{1}{2}\sqrt{\frac{\left(g^{ik}g^{jl} - g^{ij}g^{kl}\right) dg_{ij}dg_{kl}}{R - 2\Lambda}} \tag{99}$$

Purely from the definition of the momentum we are able to derive the Hamiltonian constraint

$$\mathcal{H} = \frac{1}{\sqrt{g}}\left(p^{ij}p_{ij} - \frac{1}{2}p^2\right) - \sqrt{g}(R - 2\Lambda) \approx 0 \tag{100}$$

$\mathcal{H}$ is referred to as "weakly vanishing" denoted by $\approx$ since it is equal to zero on the constraint surface. Equation 100 is a primary constraint, meaning that it follows directly from the definition of the momenta. Applying the Euler-Lagrange equations to 97 yields the equations of motion for the system

$$\begin{aligned} dp^{ij} &= \sqrt{g}\left(Rg^{ij} - R^{ij} + \nabla^i\nabla^j - g^{ij}\triangle\right)d\chi - 2\Lambda\sqrt{g}g^{ij}d\chi \\ &\quad - 2\frac{d\chi}{\sqrt{g}}\left(p^{ik}p^j_k - \frac{1}{2}pp^{ij}\right) \end{aligned} \tag{101}$$

The equations of motion do not necessarily uphold the primary constraint 100, if the constraints were not propagated by the equations of motion then the theory would be internally inconsistent. The differential $d\mathcal{H}$ is a depends on dp^{ij}, we may test whether the constraint is propagated by substituting in the expression for the equation of motion.

$$d\mathcal{H} = \frac{2}{d\chi}\nabla_i\left(d\chi^2\nabla_j p^{ij}\right) \tag{102}$$

This expression can be forced to vanish weakly by imposing a secondary constraint $\mathcal{H}^i = -2\nabla_j p^{ij} \approx 0$. Again it must be checked that this is propagated by the equations of motion or else we would require the implementation of a tertiary constraint and so on until any inconsistencies can be removed. Thankfully this constraint is naturally propagated by the equations of motion $d\mathcal{H}^i = -\nabla^i d\chi\mathcal{H} \approx 0$. The constraints $\mathcal{H} \approx 0$ & $\mathcal{H}^i \approx 0$ are the familiar Hamiltonian and diffeomorphism constraints from the ADM formulation in section 2.1. As mentioned previously, the diffeomorphism constraint is named as such because it generated diffeomorphism transformations of the metric. This can be seen through taking the Poisson bracket

$$\left\{g_{ij}|(\xi_k|\mathcal{H}^k)\right\} = \nabla_i\xi_j + \nabla_j\xi_i = L_\xi g_{ij} \tag{103}$$

where $L_\xi g_{ij}$ is the Lie derivative of g_{ij}, with the Poisson bracket and scalar product defined respectively as

$$\{F(q,p), G(q,p)\} = \int d^3x \left(\frac{\delta F}{\delta q}\frac{\delta G}{\delta p} - \frac{\delta F}{\delta p}\frac{\delta G}{\delta q}\right) \tag{104}$$

$$(f(x)|g(x)) = \int d^3x\sqrt{g}f(x)g(x) \tag{105}$$

The best matched action is formed by replacing the derivatives in the bare action 97 by best matched derivatives $\mathcal{D}g_{ij} = dg_{ij} + L_{d\xi}g_{ij}$.

$$d\mathcal{L} = \int d^3x\sqrt{g}\sqrt{R-2\lambda}\sqrt{\left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\mathcal{D}g_{ij}\mathcal{D}g_{kl}} \tag{106}$$

The canonical momenta is the same as in 4.1 but with the differentials replaced by best-matched differentials

$$p^{ij} = \frac{\sqrt{g}}{2d\chi}\left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\mathcal{D}g_{kl} \tag{107}$$

with the differential of the instant defined similarly.

$$d\chi = \frac{1}{2}\sqrt{\frac{\left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\mathcal{D}g_{ij}\mathcal{D}g_{kl}}{R-2\Lambda}} \tag{108}$$

The best-matched differential produces a new term in the equation of motion which is the Lie derivative of the momentum $L_{d\xi}p^{ij} = d\xi^k\nabla_k p^{ij} + \nabla_k d\xi^k p^{ij} - \nabla_k d\xi^i p^{kj} - \nabla_k d\xi^j p^{ik}$.

$$\begin{aligned} dp^{ij} = {} & \sqrt{g}\left(Rg^{ij} - R^{ij} + \nabla^i\nabla^j - g^{ij}\triangle\right)d\chi - 2\Lambda\sqrt{g}g^{ij}d\chi \\ & - 2\frac{d\chi}{\sqrt{g}}\left(p^{ik}p_k^j - \frac{1}{2}pp^{ij}\right) + L_{d\xi}p^{ij} \end{aligned} \tag{109}$$

The Lie derivative can be absorbed into the best-matched differential, slightly simplifying the equation to

$$\begin{aligned} \mathcal{D}p^{ij} = {} & \sqrt{g}\left(Rg^{ij} - R^{ij} + \nabla^i\nabla^j - g^{ij}\triangle\right)d\chi - 2\Lambda\sqrt{g}g^{ij}d\chi \\ & - 2\frac{d\chi}{\sqrt{g}}\left(p^{ik}p_k^j - \frac{1}{2}pp^{ij}\right) \end{aligned} \tag{110}$$

The diffeomorphism constraint is derived through the best matching condition $\delta d\mathcal{L}/\delta d\xi_i \approx 0$ which reads

$$\frac{\delta d\mathcal{L}}{\delta d\xi_i} = -2\nabla_j\left(\frac{\sqrt{g}}{2d\chi}(g^{ik}g^{jl} - g^{ij}g^{kl})\mathcal{D}g_{kl}\right) = -2\nabla_j p^{ij} \approx 0 \tag{111}$$

We can now consider coupling a scalar field to the metric by adding appropriate potential and kinetic terms to the lagrangian

$$d\mathcal{L}_\phi = \int d^3x\sqrt{g}\sqrt{R - 2\Lambda + \lambda g^{ij}\nabla_i\phi\nabla_j\phi + U(\phi)}\sqrt{G^{ijkl}\mathcal{D}g_{ij}\mathcal{D}g_{kl} + (\mathcal{D}\phi)^2} \tag{112}$$

Where $G^{ijkl} = g^{ik}g^{jl} - g^{ij}g^{kl}$ is used in this case to simplify the expressions, and the best-matched derivative of the scalar field ϕ is $\mathcal{D}\phi = d\phi + L_{d\xi}\phi = d\phi + d\xi^i\nabla_i\phi$. Following the same procedure as before we find the two momenta associated with the metric and scalar field respectively

$$p^{ij} = \frac{\sqrt{g}}{2d\chi}G^{ijkl}\mathcal{D}g_{kl} \tag{113}$$

$$\pi = \frac{\delta d\mathcal{L}_\phi}{\delta d\phi} = \frac{\sqrt{g}}{2d\chi}\mathcal{D}\phi \tag{114}$$

In this case, the Hamiltonian constraint, formed by virtue of the momenta definitions, involves both the metric momentum and the scalar momentum.

$$\mathcal{H} = \sqrt{g}\left(R - 2\Lambda + \lambda g^{ij}\nabla_i\phi\nabla_j\phi + U(\phi)\right) - \frac{1}{\sqrt{g}}(p^{ij}p_{ij} - \frac{1}{2}p^2 + \pi^2) \approx 0 \tag{115}$$

The best matching condition produces an extra term in the diffeomorphism constrain compared to the previous example

$$\frac{\delta d\mathcal{L}_\phi}{\delta d\xi_i} = -2\nabla_j p^{ij} + \pi\nabla^i\phi = \mathcal{H}^i \approx 0 \tag{116}$$

Just as before, we must check that the Hamiltonian constraint is propagated by the equations of motion, so that the theory is not inconsistent. The equations of motion are

$$\begin{aligned}\mathcal{D}p^{ij} =& \sqrt{g}\left(\frac{1}{2}Rg^{ij} - R^{ij} + \nabla^i\nabla^j - g^{ij}\triangle\right)d\chi \\ &- \frac{2d\chi}{\sqrt{g}}\left(p^{ik}p^j_k - \frac{1}{2}pp^{ij}\right) + \lambda\sqrt{d}d\chi\nabla^i\phi\nabla^j\phi \\ &+ \sqrt{g}g^{ij}d\chi\left(\lambda\nabla^k\phi\nabla_k\phi + U(\phi) - 2\Lambda\right)\end{aligned} \tag{117}$$

$$\mathcal{D}\pi = -2\lambda\sqrt{g}\nabla_i\left(\nabla^i\chi\right) + \frac{\delta U}{\delta\phi}d\chi \tag{118}$$

Attempting to propagate the Hamiltonian constraint we find

$$\mathcal{DH} = \frac{2}{d\chi}\sqrt{g}\nabla_i(d\chi^2\nabla_j p^{ij}) + (4\lambda+1)\frac{\sqrt{g}}{d\chi}\nabla^i(\pi\nabla_i\chi^2) \tag{119}$$

Substituting in the diffeomorphism constraint 116 $2\nabla_j p^{ij} \approx \pi\nabla^i\phi$ gives a weak expression

$$\mathcal{DH} \approx (4\lambda+1)\frac{\sqrt{g}}{d\chi}\nabla^i(\pi\nabla_i\phi d\chi^2) \tag{120}$$

There are two possible options by which we may propagate the constraint, i.e. make $\mathcal{DH} \approx 0$. The first is to introduce a further constraint $\pi\nabla_i\phi \approx 0$. However in this case we are not permitted to do so. Every constraint will remove a degree of freedom from the system, with $\pi\nabla_i\phi \approx 0$ being a vector equation it actually kills six degrees of freedom. From standard classical field theory we know that a scalar field introduces carries with it two degrees of freedom, so adding this further constraint would remove more degrees of freedom than are available in the system. The only option left to propagate the constraint is to choose $\lambda = -1/4$.

To see how this affects the dynamics, we may take small perturbations of metric around a static background $g_{ij} = \delta_{ij} + h_{ij}$ and expand the equation of motion of ϕ to first order. We fix the coordinate frame by setting $d\xi = 0$ which makes the best-matched differentials into ordinary differentials. The equation of motion then reads

$$\frac{d^2\phi}{d\chi^2} + 4\lambda\partial_i\partial^i\phi = \frac{\partial(2U(\phi))}{\partial\phi} \tag{121}$$

We have already identified the requirement $= -1/4$ and with this we have the familiar wave equation of a scalar field propagating at the speed of light through a potential $\tilde{U} = 2U$

$$\Box\phi = \frac{\partial\tilde{U}(\phi)}{\partial\phi} \tag{122}$$

The significance of this result is that propagation of the constraints has required the scalar field to satisfy the same light cone as the metric for small regions of space and time, thus enforcing local lorentz invariance. The mechanism that forces matter minimally coupled to the metric to respect the same light cone is in fact shown to be universal in [32]. This is one of the ways in which Shape Dynamics can be considered more fundamental than General Relativity, since GR assumes local lorentz invariance among it's fundamental principles, where as SD is able to derive it from a smaller starting set of principles.

4.2 Satisfying The Mach-Poincaré Principle

We mentioned in section 2.2, General Relativity does not satisfy the Mach-Poincaré Principle and so far neither does the relational dynamics framework. Due to GR's refoliation invariance, there will be multiple curves between two fixed points in superspace that minimise the Einstein-Hilbert action given some initial data. Thus we cannot specify a unique solution given initial data on the configuration space. This redundancy in the foliation description can also be seen from considering the degrees of freedom. To arrive at superspace, the physical configuration space of GR, one first starts with **Riem**, the space of suitably smooth Riemann geometries and then quotients to the space of geometries that are physically distinct under diffeomorphisms to obtain superspace. **Riem** has six degrees of freedom (per space point), quotienting by diffeomorphims reduces this to three degrees of freedom. However the Hamiltonian constraint further removes one degree of freedom so the metric actually end up with only 2 degrees of freedom, while the configuration space is left with one redundant degree of freedom.

Formally, this is issue is captured by the initial value problem: To construct initial data g_{ij} & p^{ij} that satisfy the Hamiltonian and diffeomorphism constraints. York provided the solution to this problem, building on previous work by Lichnerowicz [36]. Lichnerowicz was able to provide a satisfactory solution in the case of Yamabe positive metrics. These are the class of metrics whose Yamabe constant

$$y\left[\Sigma, g\right] = \inf_{\phi}\left\{\frac{\int d^3x\sqrt{g}(\phi^2 R - 8\phi\Delta\phi)}{\int d^3x\sqrt{g}\phi^6}\right\} \tag{123}$$

is positive. Lichnerowicz's approach exploits the fact that the diffeomorphism constraint is invariant under conformal transformations of the metric in the case where the Cauchy Hyper surface that g_{ij} and p^{ij} lie on has an extrinsic curvature with vanishing trace $K = 0$. The Hamiltonian constraint however is not invariant, and so this can be used to decouple the two equations.

In terms of the extrinsic curvature K^{ij} and the scalar curvature R the

Hamiltonian and diffeomorphism constraint respectively are

$$K^{ij}K_{ij} - R = 0 \tag{124}$$

$$\nabla_j K^{ij} = 0 \tag{125}$$

Lichnerowicz's method is to start with an arbitrary metric and extrinsic curvature $\bar{g}_{ij}$ & $\bar{K}^{ij}$ that do not satisfy the constraint equations. We then perform a conformal transformation

$$g_{ij} = \phi(x)^4 \bar{g}_{ij} \qquad K^{ij} = \phi(x)^{-10} \bar{K}^{ij} \tag{126}$$

where $\phi(x)$ is a smooth, positive function. Under this transformation the diffemorphism constraint transforms as

$$\nabla_i K^{ij} = \phi^{-10} \bar{\nabla}_j \bar{K}^{ij} + \phi^{10} \partial^i (\phi^{-10} \bar{K}) = 0 \tag{127}$$

However the trace of the transformed curvature also vanishes $TrK = \phi^{-10} Tr\bar{K} = 0$, So we find that the constraint is invariant under the conformal transformation.

$$\bar{\nabla}_j \bar{K}^{ij} = 0 \tag{128}$$

The Hamiltonian constraint transforms as

$$\bar{K}^{ij} \bar{K}_{ij} - \phi^8 \bar{R} + 8\phi^7 \bar{\Delta}\phi = 0 \tag{129}$$

which is not invariant under the conformal transformation. In this way the two constraints are decoupled and the problem has reduced to simply finding an appropriate ϕ that allows us to go from an arbitrary metric and curvature, to a pair that satisfy the constraint equations. The transformed Hamiltonian constraint is an elliptic, quasiliniear equation in $\phi(x)$ for which there are existence and uniqueness theorems [36]. So from equation 129 we can always find a unique ϕ that will give us a curvature satisfying $R = K^{ij}K_{ij}$. The issue here is that $K^{ij}K_{ij}$ is strictly positive by construction so this demands that the scalar curvature R must also be positive which in turn restricts the geometries this method can be applied to. As was mentioned previously, this is specifically those geometries with a positive Yamabe constant 123.

York and Ó Murchadha extended this solution to a wider class of geometries. In their approach, the extrinsic curvature is decomposed into a transverse-traceless (TT) part and a part that is proportional to the metric by a (spatial) constant τ [3]

$$K^{ij} = K_{TT}^{ij} + \frac{1}{3}\tau g^{ij} \tag{130}$$

and as in Lichnerowicz's solution, we start with an initial, arbitrary pair $\bar{g}_{ij}$ & $\bar{K}^{ij}$ then make a conformal transformation fo the metric $g_{ij} = \phi^4 \bar{g}_{ij}$ and of the TT part of the curvature $K_{TT}^{ij} = \phi^{-10}\bar{K}_{TT}^{ij}$ so that K^{ij} transforms as

$$K^{ij} = \phi^{-10}\bar{K}_{TT}^{ij} + \frac{1}{3}\phi^{-4}\tau\bar{g}^{ij} \tag{131}$$

Then taking the diffeormorphism constraint

$$\begin{aligned} \bar{\nabla}_j \bar{K}^{ij} &= \bar{\nabla}_j \bar{K}_{TT}^{ij} + \frac{1}{3}\bar{\nabla}_j(\tau\bar{g}^{ij}) \\ &= \frac{1}{3}\left[\nabla_j(\tau\bar{g}^{ij}) + \tau\bar{g}^{kj}\Delta\Gamma_{jk}^{i} + \tau\bar{g}^{ik}\Delta\Gamma_{jk}^{j}\right] = 0 \end{aligned} \tag{132}$$

where $\Delta\Gamma_{jk}^{i} = 2\left(\delta_j^i\partial_k \ln\phi + \delta_k^i\partial_j \ln\phi - g_{jk}g^{il}\partial_l \ln\phi\right)$ is the change in the Christoffel symbol under the conformal transformation. Just like in Lichnerowicz's solution we are able to decouple the diffeomorphism constraint and the Hamiltonian constrain which transforms to another quasilinear elliptic equation

$$\phi^{-12}\bar{g}_{ik}\bar{g}_{jl}\bar{K}_{TT}^{ij}\bar{K}_{TT}^{kl} - \frac{2}{3}\tau^2 - \phi^{-4}\bar{R} + 8\phi^{-5}\bar{\Delta}\phi = 0 \tag{133}$$

This is the York-Lichnerowicz equation, for arbitrary compact or asymptotically flat manifolds there exists a unique solution to this equation. This can be shown be considering the polynomial

$$f(z) = \frac{2}{3}\tau^2 z^3 + Rz^2 - KK \tag{134}$$

where $z = \phi^4$ & $KK = \bar{g}_{ik}\bar{g}_{jl}\bar{K}_{TT}^{ij}\bar{K}_{TT}^{kl} > 0$. The York-Lichnerowicz equation will have a unique solution if and only if the polynomial has a single positive root at every point in space. The turning points of this function are at $z = 0$

[3] In general any symmetric 2-tensor can be decomposed into $X^{ij} = X_{TT}^{ij} + \frac{1}{3}Xg^{ij} + (LY)^{ij}$ where $(LY)^{ij}$ is the conformal killing form of a vector Y^i, in the case of the extrinsic curvature this term vanishes [22]

and $z = -R/\tau^2$ with the nature of the turning points being given by the sign of the second derivatives at those points $f''(0) = 2R$ & $f''(-R/\tau^2) = -2R$. The analysis breaks down into three cases $R > 0$, $R < 0$ and $R = 0$. In the case $R > 0$ then $z = 0$ is a local minima and $z = -R/\tau^2 < 0$ is a local maxima. Since $f(0) = -KK < 0$ then $f(z)$ will always have one root $z_* > 0$ such that $f(z_*) = 0$ for any value of KK. For the case $R < 0$, $z = 0$ is a local maxima and $z = -R/\tau^2 > 0$ is a local minima. Likewise, in this case there will always be a single positive root $z_* > 0$. For $R = 0$ there is a single turning point which is a point of inflection and we can solve exactly for the solution $z_* = (2KK/3\tau^2)^{1/3}$

So we see that York's solution works for those spacetimes that can be foliated into geometries with constant mean extrinsic curvature (CMC) $\tau = const$. This is a much larger class of geometries than the ones Lichnerowicz's solution is restricted to. The spatial constant τ is referred to as York time and as we will see, it can be used to define a universal internal time and parameterise solutions in Shape Dynamics.

4.3 Arriving at Shape Dynamics

Hopefully by now, it is clear how one may form a classical theory of mechanics based on the three principles stated in section 1: Spatial Relationalism, Temporal Relationalism and The Mach-Poincaré Principle. To arrive at Shape Dynamics we go back to a generalised version of the action 97 considered in section 4.1

$$d\mathcal{L} = \int d^3x \sqrt{g}\sqrt{aR - 2\Lambda}\sqrt{(\lambda_1 g^{ik}g^{jl} - \lambda_2 g^{ij}g^{kl})dg_{ij}dg_{kl}} \tag{135}$$

where $a, \Lambda, \lambda_1, \lambda_2$ are spatial constants. As usual we can extract the following momentum

$$p^{ij} = \frac{\sqrt{g}}{2d\chi}(\lambda_1 g^{ik}g^{jl} - \lambda_2 g^{ij}g^{kl})dg_{kl} \tag{136}$$

with the differential of the instant defined as

$$d\chi = \frac{1}{2}\sqrt{\frac{(\lambda_1 g^{ik}g^{jl} - \lambda_2 g^{ij}g^{kl})dg_{ij}dg_{kl}}{aR - 2\Lambda}} \tag{137}$$

The momentum as defined above admits the Hamiltonian constraint

$$\mathcal{H} = \sqrt{g}(aR - 2\Lambda) - \frac{1}{\lambda_1\sqrt{g}}\left(p_{ij}p^{ij} - \frac{\lambda_2}{3\lambda_2 - \lambda_1}p^2\right) \approx 0 \tag{138}$$

The equation of motion is

$$
\begin{aligned}
dp^{ij} = a\sqrt{g}\left(\frac{1}{2}Rg^{ij} - R^{ij} + \nabla^i\nabla^j - g^{ij}\Delta\right)d\chi - 2\Lambda\sqrt{g}g^{ij}d\chi \\
- \frac{2d\chi}{\lambda_1\sqrt{g}}\left(p^{ik}p^j_k - \frac{\lambda_2}{3\lambda_2 - \lambda_1}pp^{ij}\right)
\end{aligned}
\tag{139}
$$

Once more, in order for this theory to be consistent and admit sensible solutions, the Hamiltonian constraint must be propagated by the equations of motion. In this case, the constraints are best analysed through Dirac's Poisson bracket formalism [34]. In this formalism, the constraint is propagated if it is "first-class" with respect to itself, meaning that the Poisson bracket of the constraint, integrated against an arbitrary function, with it's self vanishes $\{(df|\mathcal{H}), (dg|\mathcal{H})\} \approx 0$. Computing such a Poisson brakcet, we have

$$
\{(df|\mathcal{H}), (dg|\mathcal{H})\} \approx \frac{a}{\lambda_1}(df\nabla^i dg - dg\nabla^i df| - 2\nabla_j p^j_i + 2\frac{\lambda_2 - \lambda_1}{3\lambda_2 - \lambda_1}\nabla_i p) \tag{140}
$$

There are three choices available that we can use to force this Poisson bracket to vanish. **I**: In the first case we may take $a = 0$. This is the case known as "strong gravity", the choice removes the Ricci scalar from the action. This Theory is essentially the limit of relativity as $c \to 0$ so that all light cones close up into straight lines. This prevents any information being transmitted from one spatial point to another. **II**: In the second case we can take $\lambda_1 \to \infty$. This corresponds to a a theory in which the metric does not enter the equations of motion dynamically and is static. Similar to case **I**, it can be viewed as the limit $c \to \infty$, where all light cones open up so that information can be transmitted instantaneously between spatial points. This is simply Galilean Relativity.

Case **III** is far richer and is what will lead, after a fair amount of analysis, to Shape Dynamics. The last option is to impose a secondary constraint

$$
\mathcal{Z}_i = -2\nabla_j p^j_i + 2\alpha\nabla_i p \approx 0 \tag{141}
$$

where $\alpha = (\lambda_2 - \lambda_1)/(3\lambda_3 - \lambda_1)$. Just like the Hamiltonian constraint, the $\mathcal{Z}_i$ constraint will propagate if it's Poisson bracket with itself integrated against arbitrary functions vanishes.

$$
\{(d\xi^i|\mathcal{Z}_i), (d\sigma^i|\mathcal{Z}_j)\} = ([d\xi, d\sigma]^i|\mathcal{Z}_i + (2\alpha - 6\alpha^2)\nabla^i p) \tag{142}
$$

where $[d\xi, d\sigma]^i$ is the Lie bracket. The Poisson bracket vanishes if $2\alpha - 6\alpha^2 = 0$ which has two solutions at $\alpha = 0$ & $\alpha = 1/3$. Thus we are able to break case **III** into three subcases.

Case **III (a)**: Taking $\alpha = 0$ reduces $\mathcal{Z}_i$ to the familiar diffeomorphism constraint $\mathcal{H}_i = -2\nabla_j p^j_i$ which generates diffeomorphism transformations. This produces the ADM formulation of GR. Having identified the diffeomorphism symmetry, it can be implemented into the theory through the best-matched action

$$d\mathcal{L} = \int d^3x \sqrt{g}\sqrt{aR - 2\Lambda}\sqrt{(g^{ik}g^{jl} - g^{ij}g^{kl})\mathcal{D}g_{ij}\mathcal{D}g_{kl}} \tag{143}$$

$$\mathcal{D}g_{ij} = dg_{ij} + L_{d\xi}g_{ij} \tag{144}$$

Case **III (b)**: The solution $\alpha = 1/3$ corresponds to $\mathcal{S}_i = -2\nabla_j p^j_i + \frac{2}{3}\nabla_i p$. This constraint is the generators of special diffeomorphisms, those diffeomorphisms that leave the volume element $\sqrt{g}$ invariant. The special diffeomorphisms are introduced through the best matched action

$$d\mathcal{L} = \int d^3x \sqrt{g}\sqrt{aR - 2\Lambda}\sqrt{(g^{ik}g^{jl} - \frac{1}{3}g^{ij}g^{kl})\mathcal{D}g_{ij}\mathcal{D}g_{kl}} \tag{145}$$

$$\mathcal{D}g_{ij} = dg_{ij} + L_{d\xi}g_{ij} - \frac{2}{3}g_{ij}\nabla_k d\xi^k \tag{146}$$

Case **III (c)**: The final case covers a generic α in which case we must split $\mathcal{Z}_i$ into the diffeomorphism constraint $\mathcal{H}_i$ and a separate constraint that require $\nabla_i p = 0$, however we are not able to take this as a vector constraint as it would remove more degrees of freedom than are available. Instead we have to take an equivalent scalar constraint $p \propto \sqrt{g}$ that implies $\nabla_i p = 0$. Taking $p = \frac{3}{2}\sqrt{g}$, we see that this constraint has a geometrical interpretation like the diffeomorphism constraint.

The Poisson bracket with the metric is

$$\{(\phi|p), g_{ij}\} = \phi g_{ij} \tag{147}$$

so the constraint generates the volume-preserving conformal transformations, VPCT's, mentioned at the end of section 2.2. Furthermore, assuming that the manifold is compact, τ must be proportional to the spatial average of p

$$\tau = \frac{2}{3}\frac{\int d^3xp}{\int d^3x\sqrt{g}} = \frac{2}{3}\langle p\rangle \tag{148}$$

The constraint can be re-written as $\mathcal{C} = p - \langle p \rangle \sqrt{g} \approx 0$. The VPCT constraint is automatically first-class with respect to itself $\{(d\phi|\mathcal{C}),(d\rho|\mathcal{C})\} = 0$ so we require no additional constraints or fixing or parameters. The configuraton space of this theory is that of superspace quotiented by the VPCT's, which is in fact just the cartesian product $\boldsymbol{C} \times \mathbb{R}^+$ where $\boldsymbol{C}$ is conformal superspace. Making use of York's solution, the theory is able to satisfy the Mach-Poincaré Principle on this configuration space. There are no restrictions on α for this analysis, so in fact **III (a)** & $\mathbf{III}(b)$ are really just special cases of this. The genericα case $\mathbf{III}(c)$ is what corresponds to shape dynamics. The introduction of VPCT's in place of refoliation invariance is usually introduced via a Linking Theory.

4.4 Linking Theory

The idea behind the Linking Theory is to start with a known gauge theory (in this case GR) and go to another gauge theory that possesses a different symmetry. The Linking Theory is a theory that lives in a larger phase space than GR SD, with redundant degrees of freedom. Different gauge fixing the redundant degrees of freedom give produce GR & SD. This approach was developed by Gomes and Koslowski in [37]. Formally they define a linking gauge theory L as a triplet $(T_L, \Sigma_1, \Sigma_2)$ where $T_L = (\Gamma, \{.,.\}, \{\chi_i\}, \{\rho_j\})$ is a previously known gauge theory (in our case GR) with phase space Γ, a Poisson bracket structure $\{.,.\}$, first class constraints χ_i and second class constraints ρ_i.Σ_1 and Σ_2 are sets of gauge fixing conditions such that $\Sigma_1 \cup \Sigma_2$ is a gauge fixing condition for T_L. Additionally it requires that the set of first class constraints can be decomposed into three disjoint subsets $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{X}_0$ such that $\mathcal{X}_1$ and $\mathcal{X}_2$ can be gauge fixed by Σ_1 and Σ_2 respectively and $\mathcal{X}_0$ is not gauge fixed by either.

Practically, one constructs a Linking Theory by extending the phase space of a starting gauge theory with an auxiliary conjugate pair coordinate and momentum. In our case we start with the ADM formulation (with $\Lambda = 0$ to simplify later expressions.)

$$\mathcal{H} = \frac{1}{\sqrt{g}}\left(p_{ij}p^{ij} - \frac{1}{2}p^2\right) - \sqrt{g}R \approx 0 \tag{149}$$

$$\mathcal{H}^i = -2\nabla_j p^{ij} \approx 0 \tag{150}$$

Then we add to the system a scalar field ϕ ad associated momentum π with the first-class constraint

$$\mathcal{Q} = \pi \approx 0 \tag{151}$$

To go to the Linking Theory we perform the canonical transformation

$$(g_{ij}, p^{ij}, \phi, \pi) \to (G_{ij}, P^{ij}, \Phi, \Pi) \tag{152}$$

The transformations are chosen specifically so that they implement the VPCT's.

$$G_{ij} = e^{4\hat{\phi}} g_{ij} \qquad P^{ij} = e^{-4\hat{\phi}} \left[p^{ij} - \frac{1}{3}(1 - e^{6\phi} \langle p \rangle \sqrt{g} g^{ij} \right] \tag{153}$$

$$\Phi = \phi \qquad \Pi = \pi - 4(p - \langle p \rangle \sqrt{g}) \tag{154}$$

where $\hat{\phi} = \phi - \frac{1}{6} \ln \langle \sqrt{g} e^{6\phi} \rangle$. The canonical constraints are then

$$\begin{aligned}
\mathcal{H} &= \frac{e^{-6\hat{\phi}}}{\sqrt{g}} \left(p_{ij} p^{ij} + \frac{1}{3} \sqrt{g} (1 - e^{6\hat{\phi}} \langle p \rangle p - \frac{1}{6} g (1 - e^{6\hat{\phi}})^2 \langle p \rangle^2 - \frac{p^2}{2} \right) \\
&\quad - \sqrt{g} (R e^{2\hat{\phi}} - 8 e^{\hat{\phi}} \Delta e^{\hat{\phi}}) \approx 0 \\
\mathcal{H}^i &= -2 e^{-4\hat{\phi}} [\nabla_j p^{ij} - 2(p - \sqrt{g} \langle p \rangle) \nabla^i \phi] \approx 0 \\
\mathcal{Q} &= \pi - 4(p - \langle p \rangle \sqrt{g}) \approx 0
\end{aligned} \tag{155}$$

As per Dirac's analysis of constrained Hamiltonian systems [34], the total Hamiltonian is a linear combination of the constraints

$$H_{total} = (\mathcal{H}|N) + (\mathcal{H}^i|\xi_i) + (\mathcal{Q}|\rho) \tag{156}$$

This is exactly why the ADM Hamiltonian in 25 is of this form. The two gauge fixing conditions available are $\phi \approx 0$ which is the GR gauge, resulting in the ADM constraints, and the $\pi \approx 0$ gauge which is the Shape Dynamics gauge. After gauge fixing $\pi \approx 0$ there is still a single non-vanishing Poisson bracket

$$\{(\mathcal{H}|N), \pi\} = \frac{\delta(\mathcal{H}|N)}{\delta\hat{\phi}} - \left\langle \frac{\delta(\mathcal{H}|N)}{\delta\hat{\phi}} \right\rangle e^{6\hat{\phi}} \sqrt{g} \tag{157}$$

On the constraint hypersurface one has $\mathcal{H} = 0$ and thus

$$\frac{\delta(\mathcal{H}|N)}{\delta\hat{\phi}} = \left\langle \frac{\delta(\mathcal{H}|N)}{\delta\hat{\phi}} \right\rangle e^{6\hat{\phi}} \sqrt{g} \tag{158}$$

This is "lapse-fixing equation" and can be solved for the scalar lapse function $N(x)$. To calculate the variational derivative consider explicitly $\hat{\phi} \to \hat{\phi} + \delta\hat{\phi}$.

$$
\begin{aligned}
&(\mathcal{H}|N) + \delta(\mathcal{H}|N) = \\
&\int d^3x \left[\frac{e^{-6\hat{\phi}} e^{-6\delta\hat{\phi}}}{\sqrt{g}} \tilde{p}^{ij}\tilde{p}_{ij} - \frac{1}{6\sqrt{g}} e^{6\hat{\phi}} e^{6\delta\hat{\phi}} \sqrt{g}\langle p\rangle^2 - \sqrt{g}(Re^{2\phi}e^{2\delta\hat{\phi}}) - 8e^{\phi}e^{\delta\hat{\phi}}\Delta(e^{\phi}e^{\delta\hat{\phi}}) \right] N(x) \\
&= \int d^3x \left[\frac{e^{-6\hat{\phi}}}{\sqrt{g}} \tilde{p}^{ij}\tilde{p}_{ij} - \frac{6\delta\hat{\phi}}{\sqrt{g}} e^{-6\hat{\phi}} \tilde{p}^{ij}\tilde{p}_{ij} - \frac{1}{6} e^{6\hat{\phi}} \sqrt{g}\langle p\rangle^2 - \delta\hat{\phi} e^{6\hat{\phi}} \sqrt{g}\langle p\rangle^2 \right. \\
&\left. - \sqrt{g}(Re^{2\hat{\phi}} + 2\delta\hat{\phi} - 8e^{\hat{\phi}}(1 + \delta\hat{\phi})\Delta(e^{\hat{\phi}} + \delta\hat{\phi}e^{\hat{\phi}})) \right] N(x)
\end{aligned} \tag{159}
$$

Where we have defined $\tilde{p}^{ij} = p^{ij} - \frac{1}{3}pg^{ij}$ for simplicity. From 159 one may extract

$$
\begin{aligned}
\delta(\mathcal{H}|N) = \int d^3x &\left[-\frac{6}{\sqrt{g}} \delta\hat{\phi} e^{-6\hat{\phi}} \tilde{p}^{ij}\tilde{p}_{ij} - \delta\hat{\phi} e^{6\hat{\phi}} \sqrt{g}\langle p\rangle^2 \right. \\
&\left. - \sqrt{g}(2R\delta\hat{\phi}Re^{2\hat{\phi}} - 8e^{\hat{\phi}}\Delta(\delta\hat{\phi}e^{\hat{\phi}}) - 8e\hat{\phi}\delta\hat{\phi}\Delta e^{\hat{\phi}}) \right] N(x)
\end{aligned} \tag{160}
$$

Integrating the $e^{\hat{\phi}}\Delta(\delta\hat{\phi}e^{\hat{\phi}})N$ term by parts and recalling the definition of the variational derivative

$$
\delta F[\phi(x)] = \int d^3x \frac{\delta F[\phi(x)]}{\delta\phi} \delta\phi(x) \tag{161}
$$

we have

$$
\frac{\delta(\mathcal{H}|N)}{\delta\hat{\phi}} = e^{\hat{\phi}}\sqrt{g}\left[56Ne^{\hat{\phi}} + 8\Delta(Ne^{\hat{\phi}}) - 2N(4Re^{\hat{\phi}} + e^{5\hat{\phi}}\langle p\rangle^2) \right] \tag{162}
$$

Where we have made use of the constraint condition $\mathcal{H} \approx 0$ to eliminate the term proprtional to $\tilde{p}^{ij}\tilde{p}_{ij}$. The lapse equation is a non-homogeneous, second order linear ODE. The general solution is thus a linear combination $N_* = \lambda_0 N_0 + \lambda_1 N_1 + \lambda_2 N_2$ where N_1 and N_2 are the linearly independent solutions to the homogeneous equation and N_0 is a particular solution to the non-homogeneous equation

$$
\frac{\delta(\mathcal{H}|N)}{\delta\hat{\phi}} =^{6\hat{\phi}} \sqrt{g} \tag{163}
$$

with coefficient

$$\lambda_0 = \left\langle \frac{\delta(\mathcal{H}|N)}{\delta\hat{\phi}} \right\rangle \tag{164}$$

$(\mathcal{H}|N_*) \approx 0$ is a secondary constraint on the system. Explicitly, the Poisson bracket is

$$\{(\mathcal{H}|N_*), \pi\} = \lambda \left[1 - \left\langle e^{6\hat{\phi}}\sqrt{g}\right\rangle\right] e^{6\hat{\phi}}\sqrt{g} \approx 0 \tag{165}$$

The constraint forces $\langle e^{6\hat{\phi}}\sqrt{g}\rangle = 1$ so equivalently we could write it as

$$\mathcal{H}_* = \int d^3x\sqrt{g}\left(e^{6\hat{\phi}} - 1\right) \approx 0 \tag{166}$$

where we regard $\hat{\phi}$ as being fixed in terms of g_{ij} & p^{ij} by the Hamiltonian constraint $\mathcal{H} = 0$. So we arrive at the final constraints defining the Shape Dynamics gauge of the Linking theory

$$\begin{aligned} \mathcal{H}_* &= \int d^3x\sqrt{g}(e^{6\hat{\phi}[g_{ij},p^{kl}]} - 1) \approx 0 \\ \mathcal{H}^i &= -2\nabla_j p^{ij} \approx 0 \\ \mathcal{Q} &= 4(p - \langle p\rangle\sqrt{g}) \approx 0 \end{aligned} \tag{167}$$

The Shape Dynamics structure that has been formulated here allows us to single out the York time τ as a natural internal clock, since it is a spatially constant and monotonic independant variable. Solutions of Shape Dynamics are then curves in conformal superspace parameterised by τ. York time can be used to "deparameterise" the system such that the Shape Dynamics Hamiltonian that generates physical evolution depends on York time as an independent variable. Solutions can then be fully specified by an inital value of τ, a transverse-traceless momenta and a class of conformal geometries. The idea behind the deparameterisation procedure is as follows [4, 38, 39].

Take a reparameterisation-invariant theory with Hamiltonian constraint $\mathcal{H}(q_1, p^1, q_2, p^2, ...) \approx 0$ where we would like to use q_1 as an internal clock for solutions of the theory. The generator of q_1-translations if the conjugate momenta p^1. The Hamiltonian constraint can be solved for p^1 in terms of all other canonical variables. The solution $p^1 = H(q_1, q_2, p^2, ...)$ acts as a Hamiltonian that generates dynamics with respect to q_1.

In the case of Shape Dynamics, the volume $V = \int d^3\sqrt{g}$ is conjugate to τ. We solve $\mathcal{H}_*$ for the volume giving the Shape Dynamics Hamiltonian (also sometimes referred to as the York Hamiltonian [40]).

$$V = H_{SD} = \int d^3x\sqrt{g}e^{6\hat{\phi}[g_{ij},p^{kl};x,\tau)} \tag{168}$$

4.5 Matter Experiences Spacetime

All of the elements to construct a more familiar spacetime structure are now in place. One may start with a metric and momenta $\tilde{g}_{ij}$ and $\tilde{p}^{kl}$. The diffeomorphism constraint can be used to fix $\xi[\tilde{g}_{ij}, \tilde{p}^{kl}; x)$, York's method for solving the initial-value problem allows us to construct a $\phi[\tilde{g}_{ij}, \tilde{p}^{kl}; x, \tau)$ from which we can define the constraint-satisfying 3-metric $g_{ij} = \phi^4\tilde{g}_{ij}$. Finally we solve the Lapse-Fixing Equation with $\hat{\phi}$ regarded as the solution to the Lichnerowicz-York Equation to obtain $N[\phi, \tilde{g}_{ij}, \tilde{p}^{kl}; x, \tau)$. From these elements we can construct a proper Lorentzian spacetime metric

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + g_{ij}\xi^i\xi^j & g_{ij}\xi^j \\ g_{ij}\xi^j & g_{ij} \end{pmatrix} \tag{169}$$

We saw in the toy model studied in section 4.1, that coupling to matter, in that case a scalar field, enforces local Lorentz invariance naturally rather than it having to be postulated. Similarly in this Shape Dynamics theory, one is able to show that the coupling of matter to SD produces natural rods and clocks to measure spacetime geometry with [40] - which is the formalism we are familiar with from relativity. Consider a massive scalar field χ that is weakly coupled to shape dynamics, such that the Hamiltonian constraint becomes $\mathcal{H} - \mathcal{H}_{matter}$ where

$$\mathcal{H}_{matter} = \frac{1}{2}\frac{1}{\sqrt{g}}\pi_\chi^2 + \frac{1}{2}\sqrt{g}(\nabla_i\chi\nabla^i\chi + m^2\chi^2) \tag{170}$$

If we consider perturbations to the conformal factor of the metric $\tilde{\phi} - e^{\hat{\phi}} \to \tilde{\phi} + \delta\tilde{\phi}$, then the first order correction to the Hamiltonian constraint is computed as

$$\Delta_{GF}\delta\tilde{\phi} = \mathcal{H}_{matter} \tag{171}$$

where $\Delta_{GF} = 8\Delta - R - \frac{5}{6}\langle p\rangle^2 - \frac{7}{6}\tilde{p}^{ij}\tilde{p}_{ij}$. The solution can be written in terms of a greens function $\Delta_x G(x, y) = \delta^3(x - y)$

$$\delta\tilde{\phi} = \int d^3y G(x, y)\mathcal{H}_{matter}(y) \tag{172}$$

Recalling that H_{SD} is the generator of the dynamics, we would like to compute the equations of motion of the scalar fluctuations $\chi(x)$ which are given by the Poisson brackets

$$\dot{\chi} = \{\chi, H^1_{SD}\} \tag{173}$$

where H^1_{SD} is the first order perturbation to H_{SD} give

$$\begin{aligned} H^1_{SD} &= 6\int d^3x\sqrt{g(x)}\delta\tilde{\phi}(x) \\ &= 6\int d^3x d^3y\sqrt{g(x)}G(x,y)\mathcal{H}_{matter}(y) \end{aligned} \tag{174}$$

The resulting equation of motion is

$$\ddot{\chi} - N^2_{eff}\Delta\chi + N^2_{eff}m^2\chi + \dot{\chi}\frac{\partial}{\partial\tau}\left(\ln\frac{N_{eff}}{\sqrt{g}}\right) + N_{eff}\nabla_i N_{eff}\nabla^i\chi = 0 \tag{175}$$

where the effective lapse function is $N_{eff} = 6\int d^3x d^3y\sqrt{g(x)}G(x,y)$. Assuming we have plane wave solutions $\chi = \chi_0 e^{ikx}$ then in the high frequency limit, the first two terms will dominate this expression since the second order derivatives produce factors of the frequency squared ω^2 and wave vector squared $|\boldsymbol{k}|^2$. So in this limit we have simply a free wave equation

$$\ddot{\chi} - N^2_{eff}\Delta\chi = 0 \tag{176}$$

This defines the light cone structure experienced by the scalar fluctuations, and thus it sees a spacetime geometry $ds^2 = -d\tau^2 + N^{-2}_{eff}dx^i dx_i$. By Malament's theorem [41], the light cone structure completely determines the topology of the spacetime up to a conformal factor. In [40] Koslowski gives the following argument to derive an appropriate conformal factor, using the idea that a massive field introduces a natural scale m.

The equation of motion, and thus dispersion relation is nonlinear. By observing the interference pattern of a superposition of two waves over a small region of space one defines a time scale $T \sim 1/m$. The dispersion relation defined by the equation of motion 175 is

$$\omega(k)^2 + A\omega(k) + N^2_{eff}k_i k^i + B_i k^i + N^2_{eff}m^2 = 0 \tag{177}$$

where A & B_i are homogeneous coefficients. The dispersion relation implies that the interference pattern of the waves changes as if the mass were $N_{eff}m$.

This implies that we should reparameterise time by a factor of N_{eff}, so the actual spacetime experienced by the scalar field is

$$ds^2 = N_{eff}^2 d\tau^2 - dx_i dx^i \tag{178}$$

In the case of a massless field however one cannot use the same procedure as there is no natural scale introduced by the field.

By this point we have now covered a derivation of Shape Dynamics starting from a motivational discussion of the ADM formulation, and having overseen the development of the relational framework on which it is based. As we have seen, SD really is quite different from GR, basing itself on the symmetry of conformal 3-geometries. One of the most astounding features of Shape Dynamics is that the familiar spacetime geometry emerges as a consequence of SD, rather than an assumed a priori. Of course, as we have just seen above, the spacetime description is not universally applicable, it is limited by simplifying assumptions. This is of course expected as the very nature and role of spacetime in the modern description of gravity is under much scrutiny. Other approaches, such as causal set theory [42, 43] assume far less structure regarding spacetime than SD. The fact that Shape Dynamics does not rely on the existence of a smooth 4-dimensional spacetime geometry makes it well suited to tackle problems where GR fails dues to singularities.

5 Discussion

The Shape Dynamics framework has been applied to investigate many facets of classical and quantum cosmology. Particularly, in [44], the authors investigate a Bianchi type IX cosmological model - one in which the spatial slices of the metric are topologically equivalent to the 3-sphere S^3 [45]

$$g_{ij} = h_{ab}\sigma_i^a \sigma_j^b \tag{179}$$

where σ^a are the translation invariant one-forms which posses the internal $SO(3)$ gauge symmetry. Coupled to a scalar field ϕ with momentum π. The Hamiltonian is a function of the shape space coordinates (q_1, q_2), momenta (p_1, p_2), York time τ and spatial volume V (the conjugate variable to τ).

$$H = p_1^2 + p_2^2 + \frac{\pi^2}{2} - \frac{3}{8}\tau^2 V^2 - V^{4/3} C(q^1, q^2) \approx 0 \tag{180}$$

$C(q^1, q^2)$ is the shape potential given by

$$\begin{aligned} C(q^1, q^2) &= F(2q^2) + F(q^1\sqrt{3} - q^2) + F(-q^1\sqrt{3} - q^2) \\ F(x) &= e^{-x/\sqrt{6}} - \frac{1}{2}e^{2x/\sqrt{6}} \end{aligned} \tag{181}$$

Through a multi-stage process the authors perform a mapping to a new set of variables $\alpha, \beta, \gamma, , \sigma, \xi$ defined by

$$\begin{aligned} & q_1 = |\tan\beta|\cos\alpha \qquad q_2 = |\tan\beta|\sin\alpha \\ & p_1 = sp\cos\left[\alpha + \arcsin\left(\frac{\gamma}{\tan\beta}\right)\right] \qquad p_2 = sp\sin\left[\alpha + \arcsin\left(\frac{\gamma}{\tan\beta}\right)\right] \\ & \tau = s(p\sigma)^{-1/2}e^{\frac{3}{8}\sigma(\sqrt{\tan^2\beta - \gamma^2} - \omega)} \qquad V = (p\sigma)^{3/2}e^{-3/2\sigma(\sqrt{\tan^2\beta-\gamma^2}-\omega)} \\ & \pi = s \end{aligned} \tag{182}$$

where $s = sign(\tan\beta)$. This represents a mapping of the shape plane (q_1, q_2) onto the surface of a sphere S^3. Such a transformation presents the Hamiltonian constraint in way that is scale-invariant. In ordinary GR, there would be a singularity located at the equator $\beta = \pi/2$ where the equations of motion as discontinuous, and thus the description breaks down. This would correspond to a "big-bang" in this cosmological model. The equations of motion in the angular coordinates are

$$\begin{aligned} \frac{d\alpha}{d\beta} &= \frac{\gamma}{\sin^2\beta\sqrt{1 - \frac{\gamma^2}{\tan^2\beta}}} \qquad & \frac{d\gamma}{d\beta} &= \frac{f_\gamma \varepsilon}{\cos^2\beta\sqrt{1 - \frac{\gamma^2}{\tan^2\beta}}} \\ \frac{d\omega}{d\beta} &= \frac{f_\omega \varepsilon}{\cos^2\beta\sqrt{1 - \frac{\gamma^2}{\tan^2\beta}}} \qquad & \frac{d\sigma}{d\beta} &= \frac{f_\sigma \varepsilon}{\cos^2\beta\sqrt{1 - \frac{\gamma^2}{\tan^2\beta}}} \end{aligned} \tag{183}$$

where $\varepsilon = e^{\frac{\sigma}{2}(\omega - \sqrt{\tan^2\beta - \gamma^2})}$ and $f_\gamma, f_\omega, f_\sigma$ are complicated functions whose exact form is not particularly relevant here but is contained in the appendix of [44]. In the Shape Dynamics system, we are able to continue the equations through the equator due to the Pickard-Lindölf theorem which states that such differential equation posesses a unique solution given initial data provided that $f_i\varepsilon/\cos^2\beta\sqrt{1 - \gamma^2/\tan^2\beta}$ is Lipschitz-continuous in the neighbourhood of $\beta = \pi/2$. In crossing the equator, the spatial manifold flips in orientation, the authors note that this could have implications for the discrete symmetries of particle phsyics, such as CPT. There continues to be further work investigating the Bianchi IX model within the context of Shape Dynamics [46], particularly in relation to the fact that shape space is asymmetric and thus provides a natural setting for distinct arrows of time to emerge.

Whilst much of the current literature on Shape Dynamics compares and contrasts the solutions of the theory to those in General Relativity, it is also of interest to study Shape Dynamics solutions in their own right. One of the first questions one may ask about Shape Dynamics is "Does it admit it's own analogue to Birkhoff's Theorem". The initial work on this was done by H.Gomes in [47] with further investigation by F.Mercati [48]. Birkhoffs Theorem [49] asserts that the spherically symmetric vaccum Einstein field equations with asymptotically flat boundary conditions admit a unique solution given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{184}$$

If one imposes the conditions of spherical symmetry and asymptotic flatness on the vacuum equations for Shape Dynamics, we find that the SD Hamiltonian is

$$H_{SD} = -\frac{1}{2\pi} \int d^2y \partial_r \phi (r^2 \sin\theta d\theta d\phi) = M \tag{185}$$

The lapse-fixing equation 158 becomes

$$\partial_i N + 2\partial_r \left(1 + \frac{M}{2r}\right) \partial_r N = 0 \tag{186}$$

The lapse-fixing equation admits a solution $N = 1 - \frac{2C}{M+2r}$ where C is a constant of integration. By examining the equations of motion subject to the asymptotically flat boundary condition one may determine the constant $C = 1$. From this, it is standard Shape Dynamics procedure to construct the 4D-line element

$$ds^2 = -\left(\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 dr^2 + r^2 \left(1 + \frac{M}{2r}\right)^4 d\Omega^2 \tag{187}$$

As the author notes, the solution is not considered as a vacuum spacetime of GR, but what one would obtain if they were to describe a vacuum solution of

SD in language of spacetime. In that way, we can consider it as a background geometry for weakly interacting matter in SD, and perform the same kind of calculations that one would in GR. Lorentz invariance breaks down on the surface $r = m/2$, however this is not an issue in Shape Dynamics as it is formulated without local Lorentz Invariance as a fundamental symmetry. Thus, if one were to draw the Penrose diagram for this "spacetime" to display it's causal structure, it would not contain a singularity at $r = m/2$.

Whilst SD provides a new approach to classical gravity, it has also begun to be applied to problems within quantum gravity. One possible area of application is in Loop Quantum Gravity [50]. Whilst the details of this are beyond the scope of this text, it is sufficient to mention that in Loop Quantum Gravity, there exits the problem of trying to construct a physical Hilbert space for quantum gravity. In Shape Dynamics, this reduces to the problem of quantising the constraints on the conformal 3-geometries. In [51], a novel quantisation procedure is proposed that is based on forming a conformal equivalence class of quantum states, such that the Hilbert space of the quantised Shape Dynamics theory is spanned by the equivalence classes.

References

[1] Weyl, H. Reine Infinitesimalgeometrie. Math Z 2, 384–411 (1918). https://doi.org/10.1007/BF01199420

[2] P. A. M. Dirac, Phys. Rev. **114** (1959), 924-930 doi:10.1103/PhysRev.114.924

[3] H. Weyl, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1918** (1918), 465

[4] F. Mercati, [arXiv:1409.0105 [gr-qc]].

[5] Carroll, S. (2019). Spacetime and Geometry: An Introduction to General Relativity. Cambridge: Cambridge University Press. doi:10.1017/9781108770385

[6] Brans. C. H, Dicke. R. H (November 1, 1961). "Mach's Principle and a Relativistic Theory of Gravitation". Phys. Rev. 124 (3). doi: 10.1103/PhysRev.124.925

[7] A. Galiautdinov and S. M. Kopeikin, Phys. Rev. D **94** (2016) no.4, 044015 doi:10.1103/PhysRevD.94.044015 [arXiv:1606.09139 [gr-qc]].

[8] J. P. Uzan, Living Rev. Rel. **14** (2011), 2 doi:10.12942/lrr-2011-2 [arXiv:1009.5514 [astro-ph.CO]].

[9] Y. Aghababaie, C. P. Burgess, S. L. Parameswaran and F. Quevedo, Nucl. Phys. B **680** (2004), 389-414 doi:10.1016/j.nuclphysb.2003.12.015 [arXiv:hep-th/0304256 [hep-th]].

[10] C. P. Burgess and L. van Nierop, Phys. Dark Univ. **2** (2013), 1-16 doi:10.1016/j.dark.2012.10.001 [arXiv:1108.0345 [hep-th]].

[11] B. P. Abbott *et al.* [LIGO Scientific and Virgo], Phys. Rev. Lett. **119** (2017) no.14, 141101 doi:10.1103/PhysRevLett.119.141101 [arXiv:1709.09660 [gr-qc]].

[12] B. P. Abbott *et al.* [LIGO Scientific and Virgo], Phys. Rev. Lett. **119** (2017) no.16, 161101 doi:10.1103/PhysRevLett.119.161101 [arXiv:1710.05832 [gr-qc]].

[13] R. L. Arnowitt, S. Deser and C. W. Misner, Gen. Rel. Grav. **40** (2008), 1997-2027 doi:10.1007/s10714-008-0661-1 [arXiv:gr-qc/0405109 [gr-qc]].

[14] E. Gourgoulhon, [arXiv:gr-qc/0703035 [gr-qc]].

[15] F. D'Ambrosio, M. Garg, L. Heisenberg and S. Zentarra, [arXiv:2007.03261 [gr-qc]].

[16] G. Schäfer and P. Jaranowski, Living Rev. Rel. **21** (2018) no.1, 7 doi:10.1007/s41114-018-0016-5 [arXiv:1805.07240 [gr-qc]].

[17] DeWitt, B. S. (1967). "Quantum Theory of Gravity. I. The Canonical Theory". Phys. Rev. 160 (5): 1113–1148. doi:10.1103/PhysRev.160.1113

[18] T. Frankel, The geometry of physics: an introduction. Cambridge University Press, 2004.

[19] P. A. M. Dirac, Lectures on Quantum Mechanics. Dover Publications, Yeshivea University, New York, 1964.

[20] J. Barbour, T. Koslowski and F. Mercati, Class. Quant. Grav. **31** (2014), 155001 doi:10.1088/0264-9381/31/15/155001 [arXiv:1302.6264 [gr-qc]].

[21] J. Barbour and N. O. Murchadha, [arXiv:1009.3559 [gr-qc]].

[22] York, J. (1973). "Conformally invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial-value problem of general relativity". Journal of Mathematical Physics, 14(4), 456-464. https://doi.org/10.1063/1.1666338

[23] York, J. "Role of Conformal Three-Geometry in the Dynamics of Gravitation". Phys. Rev. Lett. 1972; 28:1082–1085. https://doi.org/10.1103/PhysRevLett.28.1082

[24] York, J. "Gravitational Degrees of Freedom and the Initial-Value Problem". Phys. Rev. Lett. 1971; 26:1656–1658. https://doi.org/10.1103/PhysRevLett.26.1656

[25] Ó Murchadha, N, York, J. "Initial - value problem of general relativity. I. General formulation and physical interpretation". Phys. Rev. D 1974; 10:428–436. https://doi.org/10.1103/PhysRevD.10.428

[26] Barbour J. B., Bertotti B. and Penrose Roger 1982 Mach's principle and the structure of dynamical theories. Proc. R. Soc. Lond. A382:295–306. http://doi.org/10.1098/rspa.1982.0102

[27] H. de A.Gomes, J. Math. Phys. **52** (2011), 082501 doi:10.1063/1.3603990 [arXiv:0807.4405 [gr-qc]].

[28] E. Anderson, [arXiv:1111.1472 [gr-qc]].

[29] E. Anderson, Gen. Rel. Grav. **36** (2004), 255-276 doi:10.1023/B:GERG.0000010474.63835.2c [arXiv:gr-qc/0205118 [gr-qc]].

[30] E. Anderson, Phys. Rev. D **68** (2003), 104001 doi:10.1103/PhysRevD.68.104001 [arXiv:gr-qc/0302035 [gr-qc]].

[31] E. Anderson and J. Barbour, Class. Quant. Grav. **19** (2002), 3249-3262 doi:10.1088/0264-9381/19/12/309 [arXiv:gr-qc/0201092 [gr-qc]].

[32] J. Barbour, B. Z. Foster and N. O'Murchadha, Class. Quant. Grav. **19** (2002), 3217-3248 doi:10.1088/0264-9381/19/12/308 [arXiv:gr-qc/0012089 [gr-qc]].

[33] E. Anderson and F. Mercati, [arXiv:1311.6541 [gr-qc]].

[34] Dirac Paul Adrien Maurice 1958. Generalized Hamiltonian dynamics. Proc. R. Soc. Lond. A246326–332. https://doi.org/10.1098/rspa.1958.0141

[35] S. Vignolo, R. Cianci and D. Bruno, Int. J. Geom. Meth. Mod. Phys. **3** (2006), 1493-1500 doi:10.1142/S0219887806001818 [arXiv:math-ph/0605059 [math-ph]].

[36] Lichnerowicz, André. On the relativistic equations of gravity. Bulletin of the Mathematical Society of France, Volume 80 (1952), pp. 237-251. doi: 10.24033 / bsmf.1433.

[37] H. Gomes and T. Koslowski, Class. Quant. Grav. **29** (2012), 075009 doi:10.1088/0264-9381/29/7/075009 [arXiv:1101.5974 [gr-qc]].

[38] V. Shyam, [arXiv:1212.0745 [gr-qc]].

[39] N. O. Murchadha, C. Soo and H. L. Yu, Class. Quant. Grav. **30** (2013), 095016 doi:10.1088/0264-9381/30/9/095016 [arXiv:1208.2525 [gr-qc]].

[40] T. Koslowski, Can. J. Phys. **93** (2015) no.9, 956-962 doi:10.1139/cjp-2015-0029 [arXiv:1501.03007 [gr-qc]].

[41] Malament, D.B. (1977). "The Class of continuous timelike curves determines the topology of spacetime". J Math Phys (NY), 18(7), 1399-1404.

[42] S. Surya, Living Rev. Rel. **22** (2019) no.1, 5 doi:10.1007/s41114-019-0023-1 [arXiv:1903.11544 [gr-qc]].

[43] Bombelli L, Lee J, Meyer D, Sorkin RD. Space-time as a causal set. Phys Rev Lett. 1987;59(5):521-524. doi:10.1103/PhysRevLett.59.521

[44] T. A. Koslowski, F. Mercati and D. Sloan, Phys. Lett. B **778** (2018), 339-343 doi:10.1016/j.physletb.2018.01.055 [arXiv:1607.02460 [gr-qc]].

[45] C. Kiefer, N. Kwidzinski and W. Piechocki, Eur. Phys. J. C **78** (2018) no.9, 691 doi:10.1140/epjc/s10052-018-6155-8 [arXiv:1807.06261 [gr-qc]].

[46] J. Barbour, T. Koslowski and F. Mercati, [arXiv:1310.5167 [gr-qc]].

[47] H. Gomes, Class. Quant. Grav. **31** (2014), 085008 doi:10.1088/0264-9381/31/8/085008 [arXiv:1305.0310 [gr-qc]].

[48] F. Mercati, Gen. Rel. Grav. **48** (2016) no.10, 139 doi:10.1007/s10714-016-2134-2 [arXiv:1603.08459 [gr-qc]].

[49] G. D. Birkhoff, "Relativity and modern physics," Cambridge, MA: Harvard University Press, 1923.

[50] T. Thiemann, [arXiv:gr-qc/0110034 [gr-qc]].

[51] T. A. Koslowski, doi:10.1142/9789814623995$_0$381$[arXiv : 1302.7037[gr - qc]]$.