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The Chern–Simons Action & Quantum Hall Effect: Effective Theory, Anomalies, and Dualities of a Topological Quantum Fluid

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Abstract

This dissertation will examine the use of quantum field theory in the paradigmatic setting of the quantum Hall effect. This work aims to provide a comprehensive review of the use of non-perturbative field theory methods in the context of the Chern–Simons effective action of this system. Particular emphasis is placed on the role of anomalies in 2+1 dimensions and their responsibility in producing gapless, chiral edge excitations. The canonical quantisation of the theory allows for the full description of the edge modes of Laughlin states, and its spectrum is described using conformal field theory techniques. This is extended by the work of Moore, Read, and Witten to detail the duality between the topological Chern–Simons action and its conformal boundary — a process which allows for the calculation of the bulk wavefunction. The emergence of anyons with fractionalised charge and statistics is scrutinised, and particular attention is paid to the contradiction in the literature arising from the flux quantisation condition of the statistical gauge field describing these particles. Bosonisation is discussed throughout: first exactly by examining 1+1 dimensional edge excitations, and then by following a recent field of work which provides a bosonisation duality in 2+1 dimensions that relates particle and vortex excitations. This is finally applied to the half-filled lowest Landau level to motivate the pioneering Dirac composite fermion theory of Son. Throughout this work, attention is paid to the validity of dualities and effective theories away from the zero-magnetic field limit, an often neglected regime which is of the utmost physical relevance in the quantum Hall setting.

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The Quantum Hall Effect 1

"It is fortunate that solid-state and many-body theorists have so far been spared the plagues of quantum field theory"

Mattis and Lieb (1965)

1.1 Introduction

In much of theoretical physics, the study of spaces which do not have one temporal and three spatial dimensions amounts to an abstract act of curiosity. Indeed one finds a rich catalogue physical phenomena which only become possible outside of 3+1 dimensional spacetime, but Nature does not allow us to probe these behaviours quite so simply.

Far from being abstract, the physics of electrons in a humble semiconductor provides a revolutionary theoretical playground for the behaviour of fermions in lower dimensions: the quantum Hall effect (QHE). In a process well established since the 1980s, the boundary between semiconducting silicon and its insulating oxide allows for the confinement of electrons in a two-dimensional 'inversion layer'.

The application of a strong magnetic field adds further complexity by splitting the spectrum of the two-dimensional electrons into Landau levels separated by large energy gaps. The well-known classical result of passing a current through such a system is to produce a 'Hall voltage' perpendicular to the current. However at high magnetic field, the behaviour of this system fundamentally changes as quantum mechanics dominate.

Experimentally this was first noticed when the measured Hall conductivity of the system showed noticeable plateaus at precise multiples of a quantum of conductivity, as shown in Fig. 1.1 (von Klitzing et al., 1980). The plateaus of Hall conductivity are interpreted as occurring at exact filling of ν Landau levels, and at these points the longitudinal conductivity drops to zero because there are no partially-filled levels which are able to conduct. The fractional quantum Hall effect was also observed within two years of this initial observation (Tsui et al., 1982); this phase was first identified by a $\nu = \frac{1}{3}$ plateau of the Hall conductivity. Strong interactions between the electrons of this system stabilise the system when a third of the lowest Landau level is filled, which introduces a small gap in energy to the filling of additional states. This gap leads to a plateau in the conductivity for a small range of filling fractions around a third.

Nowadays the quantum Hall system is a ubiquitous and all-important tool for metrology as it provides an unparalleled measurement of the fine structure constant,

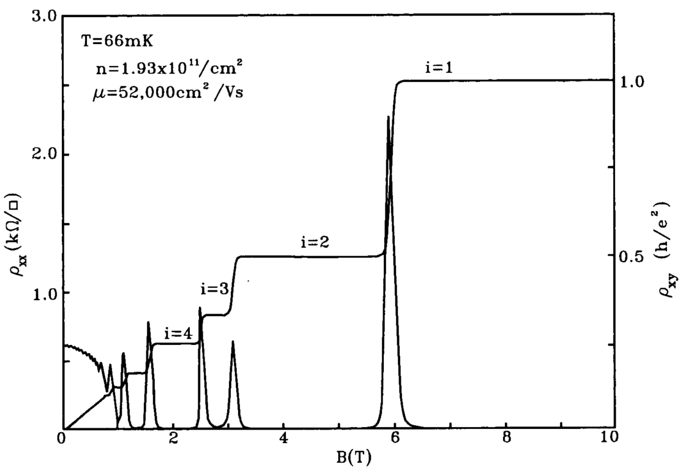

Figure 1.1: Hall resistivity ρ_{xy} plotted as the filling fraction is varied (through experimentally varying the magnetic field B). The visible plateaus are the hallmarks of the quantum Hall effect. Reproduced from the Nobel prize lecture by Tsui (1998).

and its theoretical study has won two Nobel prizes and spurred countless fields of study in the condensed matter community (von Klitzing, 2017).

This dissertation concerns the description of this quantum regime of the Hall effect using field theory methods. Quantum field theory is a stunning framework which has been applied to understanding fundamental physics with astounding precision. In the field of condensed matter physics, field theory is equally powerful but perhaps less intuitive since one does not get to use the familiar inventory of fields and symmetries which describe Nature. We will aim to describe the quantum Hall effect with an *effective theory* of the background electromagnetic (EM) gauge field A in the sample. The interactions of the electrons and the magnetic field will be encoded in the action of this effective theory. The beauty of physics in these strongly interacting materials, however, is that there may arise *emergent* degrees of freedom which one may describe with quantum fields which provide an even larger inventory of possible theories. In the fractional quantum Hall system, the emergent degree of freedom is a dynamical gauge field, denoted a, which couples to the applied background EM field A in a way which gives it fractional charge.

What is the nature of the order that defines the quantum Hall phase? Rather surprisingly there are no local operators which can be used to diagnose the phase as being distinct from an ordinary gapped electron system. Though there do exist unforeseen non-local signifiers of the phase which can clearly distinguish it from so-called *trivial* phases: the degeneracy of the ground state is quantity which only depends on topological character of the manifold, and this quantity cannot be lifted by disorder. This interesting feature defines a *topological* phase of matter (this name arising from a deep connection with topology which will be investigated).

Such topological phases are characterised by some fascinating novel features: most interestingly gapless chiral fermion edge excitations and bulk excitations with fractional charge.

The edge excitations are the only gapless excitations in the theory, and in fact these are the only degrees of freedom which can contribute to the Hall conductance. The chiral nature of these fermions is fundamentally due to the magnetic field causing a precession of the electrons of the bulk in only one direction. However in our attempts to describe the system with field theory, the consideration of the one-dimensional edge theory in isolation is inconsistent with the fact that a local 1+1 dimensional theory can never be chiral. The edge theory is therefore intrinsically connected to the bulk theory, and cannot be described independently if we want our model to be local.

The bulk excitations of a with fractional charge are another inherent feature of topological phases. These fascinating emergent particles are neither bosons nor fermions, but instead under exchange they can pick up any phase which is not ± 1 — for this reason they were dubbed 'anyons'. The study of quantum Hall systems spans four decades: from the first discovery in 1980 to just this year when anyons were first measured. Another direction of research which propels forward the field of quantum Hall physics is the project of realising a quantum computer using anyonic excitations. The robustness of observables to local noise makes topological phases (such as the QHE) particularly promising platforms for this purpose. In this realisation, the exotic braiding properties of anyons — namely their fractional and even possibly non-Abelian exchange properties — provide a platform for manipulating quantum bits of information.

This work seeks to discuss the role of *topological quantum field theories* (TQFTs) in the description of the quantum Hall effect. Specifically we will ask how the Chern–Simons action describes the low-energy behaviour of electrons in two dimensions and how the quantised conductivity of the QHE emerges in such a theory.

How can the general emergence of chiral edge modes be described in this framework? The tools of field theory will show us that their behaviour can be predicted through understanding the variation of the Chern–Simons action under gauge transformations, and links this to the 'chiral anomaly' of edge fermions. We will then ask about the implications of the boundary fermion possessing a conformal symmetry, and attempt to develop a duality framework which will predict the wavefunction of the quantum Hall electrons simply from correlation functions of the boundary fermions.

What special behaviours do topological quantum field theories have, and what exactly is their relation to topology? A jumping-off point for this question is a calculation of the ground state degeneracy of this system which shows how the topology of the background manifold encodes the dependence upon geometry. Going even further, we will derive general and exact expressions for the correlation functions and winding properties of anyons through canonically quantising the topological action.

There exist particularly interesting fractional quantum Hall phases which are gapless and have excitations which look like fermions bound to quanta of magnetic flux. We will finally attempt to develop a duality framework which allows for the description of such phases using a relativistic field theory. In order to develop such a framework, we must develop a deep understanding of the notion of 'flux attachment' which we generalise to relativistic theories.

Through this work the discussion of the quantum Hall effect will be at the forefront. Many of the field theoretical tools we use, such as particle-vortex dualities and effective theories, will require care to be taken in this context due to the background magnetic field.

Field theoretical tools in condensed matter are nowadays a tool of utmost importance. A number of the canonical condensed matter models first proposed in the '60s and '70s, such as the Luttinger liquid of electrons and the Hubbard model of strongly interacting spins, are now best understood with the use of field theory. Indeed, in the case of the Luttinger system of interacting electrons in one spatial dimension which are filled up to the crossing of Dirac cones, the model was at first solved incorrectly due to the technicalities involved in quantising a continuum theory (Mattis and Lieb, 1965). Julian Schwinger even commented a few years earlier on the "paradoxical contradictions" inherent in quantum field theory which foreshadowed this mistake (Schwinger, 1959). However once one understands the subtleties of this calculation, and by following the work of Wen (1990b) this result can be incorporated into the description of the quantum Hall effect and will be identified as the cause of chiral edge modes which are now seen as a fundamental and inseparable feature of the quantum Hall system.

Moreover, the much more recent phenomenology of topologically ordered phases (not limited to just the QHE but including phases such as s-wave superconductors and quantum spin liquids) provides a new landscape in which to apply field theory. Making connection again with high energy physics, the topological bulk theory will be seen to be dual to a conformally invariant boundary theory. This relation — which is sometimes dubbed holography — may allow for the prediction of novel quantum Hall phases through postulating boundary theories then dualising. In fact, extending this process to other topologically ordered phases may allow the description of completely new phases of matter altogether. This is not the same form of holography being investigated in string theory, but is an entirely real duality between tractable theories.

Since the turn of the century there has been striking interest in the theoretical description of, and experimental realisation of, anyons in the (particularly non-Abelian) quantum Hall effect. These anyons with their exotic braiding behaviour are lauded as a potential platform for the next generation of quantum computing, which is set apart by its resilience to noise (Das Sarma et al., 2005). A major step towards control over these particles was taken this year when a team first reported direct measurement of the fractionalised statistics of anyons in the quantum Hall effect (Nakamura et al., 2020).

In the quantum Hall effect, the tools of field theory are indispensable. Topological QFTs underpin our understanding of behaviours of this system and novel tools such as particle-vortex duality provide a new set of non-perturbative tools to better understand the behaviour of quantum matter in two spatial dimensions. This dissertation presents a thorough review of modern aspects of topological quantum field theory and their nuances, focusing on the application of the Abelian Chern–Simons action to the quantum Hall effect for the most part. We will build towards present-day research concerning FQHE states and the Dirac composite fermion theory of the $\nu = \frac{1}{2}$ system.

In an interesting case of apparently universal behaviours arising in widely different model systems, the specific discussion of particle-vortex dualities applied to the half-filled Landau level will also draw a fascinating connection with the surface states of topological insulators. We will conclude by discussing the connection of this duality to the bodies of literature covering holography and supersymmetric dualities, and will highlight the strengths of this new approach in forming developing new bosonisation tools in higher dimensions.

In the remainder of Chapter 1 the quantum Hall effect will be discussed in the more traditional many-body perspective. This will develop some intuition about the behaviour of these interesting phases before we begin applying tools of quantum field theory to the same system.

1.2 Integer QHE

1.2.1 Landau Levels

In order to introduce the quantum Hall effect, we will first show how electrons in a magnetic field split into 'Landau levels'. Consider a physical system of electrons moving in two spatial dimensions (x, y) and a magnetic field with magnitude B passing through the surface in the z-direction. Choosing the Landau gauge, the vector potential can be written

$$A_y = Bx. \tag{1.1}$$

Written in the notation of differential forms, this implies

$$A = x\,\mathrm{d}y \qquad \Longrightarrow \qquad \mathrm{d}A = B\,\mathrm{d}x \wedge \mathrm{d}y\,. \tag{1.2}$$

The magnetic field adjusts the free dispersion of the electron as follows

$$H = \frac{1}{2m}\left(p + eA\right)^2, \tag{1.3}$$

which is more naturally written in terms of the new conjugate momentum

$$\pi = m\dot{x} = p + eA. \tag{1.4}$$

This implies the peculiar canonical commutator

$$\left[\pi_x, \pi_y\right] = -ie\hbar B \tag{1.5}$$

which implies the different spatial components of the momentum are conjugate variables. Following this identification, we define the conjugate variables in terms of the cyclotron frequency $\omega_B = eB/m$

$$a = \frac{1}{\sqrt{2\hbar B}}\left(\pi_x - i\pi_y\right) \qquad \Longrightarrow \qquad H = \frac{1}{2m}\pi^2 = \hbar\omega_B(a^\dagger a + 1/2). \tag{1.6}$$

This result resembles a harmonic oscillator in 0 dimensions, with evenly spaced Landau levels labelled by an integer n which are each macroscopically degenerate.

Let us now calculate eigenstates of this Hamiltonian in the Landau gauge; in terms of the different spatial components it reads

$$H = \frac{1}{2m}\left[p_x^2 + \left(p_y + Bx\right)^2\right] \tag{1.7}$$

because of the y-independence we may search for eigenstates ψ_k with the plane wave ansatz $\psi_k(x,y) = f_k(x)\exp(iky)$. Substituting this into the Hamiltonian gives a momentum-space Hamiltonian $H_k\phi_{n,k} = \omega_n\phi_{n,k}$ with

$$H_k = \frac{p_x}{2m} + \frac{m\omega_B^2}{2}(x + kl_B^2)^2, \tag{1.8}$$

where $l_B^2 = \hbar/eB$ is the magnetic length. This once again makes clear that the spectrum is the same as a harmonic oscillator, with levels labelled by n and the wavefunctions as follows

$$\omega_n = \hbar\omega_B\,(n + 1/2)\,, \quad \phi_{n,k}(x,y) \sim e^{iky}H_n(x + kl_B^2)\,e^{-(x+kl_B^2)^2/2l_B^2}, \tag{1.9}$$

which are Gaussian-localised wavepackets located at $x_n = -kl_B^2$. Let us calculate the number of states in each Landau level by regulating the system to a finite size and calculating how many wavepackets fit in the system:

- Regulate L_y: The system is periodic in y, so $L_y/2\pi$ states fit in this direction.
- Regulate L_x: States are localised around kl_B^2 with width l_B, so the approximate number of states allowed is L_x/l_B^2.

Together the number of states per level is

$$g = \frac{L_x L_y}{2\pi l_B^2} = \frac{B \times \text{Area}}{\Phi_0}, \tag{1.10}$$

in terms of flux quantum $\Phi_0 = h/e = 2\pi\hbar/e$. The Landau levels specify flat bands of electrons, in each of which a macroscopic number of charges must exist. We will henceforth set the magnetic length to equal $l_B = 1$ with a choice of units, but retain factors of $\hbar$ and e for now.

Now we can derive the integer quantum Hall effect by applying an electric field and observing the current

$$\mathbf{J} = \sigma\mathbf{E} \tag{1.11}$$

when $E_x = E$ is the applied electric field strength. The Hamiltonian of the Hall system with an applied electric field has an extra term $-eEx$, so the spectrum is linearly dispersive (to first order in E/B)

$$\omega(k) = \hbar\omega_B\,(n + 1/2) - eEk \tag{1.12}$$

$$\phi = \phi_{n,k}(x - mE/eB^2, y). \tag{1.13}$$

The group velocity calculated from this linear dispersion is $v_g = E/B$ in the y direction, and the conductivity receives a contribution from each of ν filled Landau levels. This result shows the off-diagonal conductivity is $\sigma_{xy} = \nu g/2\pi$, which is quantised when ν Landau levels are filled.

In order to construct the full Hilbert space of states it is easier to work in the symmetric gauge where

$$A = \frac{B}{2}(x\,\mathrm{d}y - y\,\mathrm{d}x). \tag{1.14}$$

Define another type of conjugate momentum $\tilde{\pi} = p - eA$, which differs from (1.4) by a sign. This results in an additional non-zero commutator which differs from (1.5) by a sign:

$$\left[\tilde{\pi}_x, \tilde{\pi}_y\right] = +ie\hbar B, \quad \left[\pi_i, \tilde{\pi}_j\right] = 0. \tag{1.15}$$

This shows π and $\tilde{\pi}$ are simultaneously diagonalisable, and the two ladder operators defined from each 'momentum' will together span the full Hilbert space. Define this new ladder operator as follows

$$b = \frac{1}{\sqrt{2\hbar B}} \left(\tilde{\pi}_x - i\tilde{\pi}_y\right). \tag{1.16}$$

The action of b and $b^\dagger$ is to move between different degenerate states within the same Landau level. Moving to a position representation of these operators, we may gain intuition about the meaning of the b-ladder. In holomorphic coordinates where $z = x + iy$ and $\bar{z}$ is its complex conjugate, the ladder operators take the form

$$a \sim \pi_x + i\pi_y \sim -i\left(\bar{\partial} - \frac{z}{4}\right) \tag{1.17}$$

$$b \sim \tilde{\pi}_x + i\tilde{\pi}_y \sim -i\left(\partial + \frac{\bar{z}}{4}\right). \tag{1.18}$$

Note that the derivatives act like $\partial = \partial_x - i\partial_y$. We may define the operator

$$J = (b^\dagger b - a^\dagger a), \tag{1.19}$$

which has eigenvalue m. In these holomorphic coordinates this operator is explicitly the angular momentum operator

$$J = z\partial - \bar{\partial}\bar{z}. \tag{1.20}$$

A general state is $|n, m\rangle$ and is constructed from the ground state $|0, 0\rangle$ by acting with $a^\dagger, b^\dagger$. In the holomorphic representation, the wavefunction of the ground state can be evaluated by using the fact that it is annihilated by a, b

$$\langle z|0, 0\rangle = e^{-|z|^2/4}. \tag{1.21}$$

Acting on the ground state with $(b^\dagger)^m$ gives the level-m angular momentum state in the lowest Landau level

$$\langle z|0, m\rangle = z^m e^{-|z|^2/4}. \tag{1.22}$$

This eigenstate with angular momentum m is peaked at a radius $r = \sqrt{2m}$; comparing this radius to a sample area $A = \pi R^2$ again yields the same number of states in each Landau level as the Landau gauge calculation.

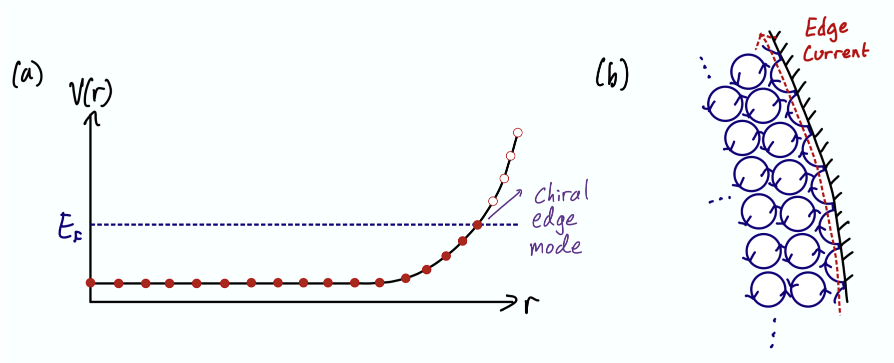

Figure 1.2: (a) The deforming effect of the Landau level in the presence of a confining potential $V(r)$. There are chiral excitations at the Fermi level E_F. (b) Explicitly demonstrating the chiral edge current as a feature of the boundary.

1.2.2 Transport & The IQHE Edge

Confining the system of electrons with a potential $V(|z|^2)$ which is radially symmetric clearly breaks the degeneracy in m, eigenstates of J are roughly shifted by $+V(2m)$. Therefore if $V(|z|^2)$ always increases at larger radii, the system of electrons will fill the Landau levels first in lower angular momentum states, then to higher angular momentum and higher-radius states. This situation is relevant for physical systems where a confining potential deforms the Landau levels to become dispersive. At a finite filling density, the system will form a quantum Hall 'droplet'. There are therefore naturally gapless excitations at the edge, as shown in Fig. 1.2a. These take two forms: firstly there are gapless and neutral excitations which promote a charge to above the Fermi level, and correspond to a deformation in the shape of the edge. There are also excitations which correspond to adding an extra charge to the edge.

In fact all of these edge excitations of the IQHE must be inherently chiral, as the system is comprised of electrons in a magnetic field. At the boundary of the quantum Hall fluid, confining the electrons will lead naturally to a chiral current. The orbits scatter off the boundary, leading to a flow of current in one direction, as shown in Fig. 1.2b. In Section 3.2 we will derive explicitly how these edge excitations propagate, and show that they have a constant velocity.

There exists an argument for the quantised conductivity due to Laughlin (1981) which instead relies upon the idea of spectral flow. Considered again a rotationally symmetric disk-shaped sample of quantum Hall fluid, arranged such that there is a hole in the centre with Φ flux penetrating it. This setup is similar to the previous example, except the additional inside boundary must have its own current with opposite chirality. This annulus-shaped setup is shown in Fig. 1.3

If the flux is varied adiabatically from Φ to $\Phi + \Phi_0$ in time T, there will be an induced voltage $\mathcal{E} = -\Phi_0/T$. Because the flux changes by one quantum, the spectrum before and after this transform must be unchanged, but this process will shift each Landau level n to the former position of the $n + 1$ level. Using the result that in the

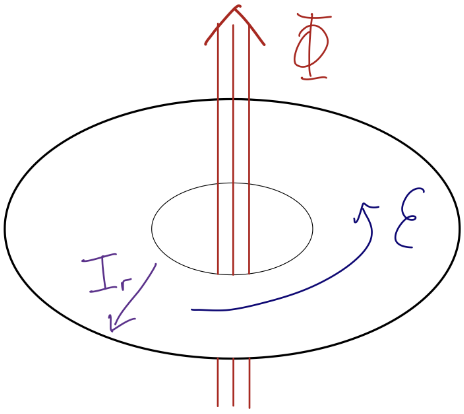

FIGURE 1.3: Annulus geometry of the IQHE, shown with flux Φ penetrating the centre. This induces a voltage $\mathcal{E}$ and a radial Hall current I_r.

radial gauge the wavefunction is peaked at $r_n = \sqrt{2m}$, this spectral flow from $n \to n+1$ involves physically moving electrons from $\sqrt{2m} \to \sqrt{2(m+1)}$. As every electron in-between the innermost and outermost filled state flows outwards, the net result is a shift of one electron from the inside to the outside of the ring. Spectral flow has here led to a contribution to the Hall current.

If ν Landau levels are filled, there will be ν electrons transported. The corresponding Hall conductivity is found in terms of the induced voltage and the radial current

$$\sigma_{xy} = \frac{I_r}{\mathcal{E}} = \frac{e^2}{h}\nu. \tag{1.23}$$

This may be written in units where $e = \hbar = l_B^2 = 1$, which will be the form we prefer in the field theory discussion

$$\sigma_{xy} = \frac{I_r}{\mathcal{E}} = \frac{\nu}{2\pi}. \tag{1.24}$$

In a more thorough quantum mechanical derivation, we may derive this expression for the conductivity by using the Kubo formula in a linear response regime, and show the conductivity is quantised by comparing to the TKNN invariant, of (Thouless et al., 1982). This is an integer that was introduced as a measure of the topology of a series of bands by characterising its Chern class, and its relation to the quantum Hall system will explain the quantised nature of the conductivity.

The Kubo formula for conductivity in terms of quantum Hall ground states $|n=0\rangle$ writes the conductivity as a sum over the commutator of currents

$$\sigma_{xy}(\omega) = \frac{1}{\hbar\omega}\int_0^\infty \mathrm{d}t \, \langle 0| \left[J_y(0), J_x(t)\right] |0\rangle \,. \tag{1.25}$$

In the DC limit $\omega \to 0$, the conductivity is

$$\sigma_{xy}(\omega) \to i\hbar \sum_{n>0} \frac{\langle 0| J_y |n\rangle \langle n| J_x |0\rangle - \langle 0| J_x |n\rangle \langle n| J_y |0\rangle}{(\omega_n - \omega_0)^2}. \tag{1.26}$$

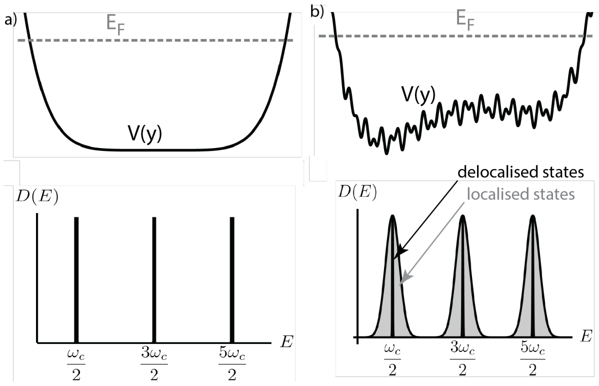

Figure 1.4: (a) The spectrum of a Landau level deformed by a confining spectrum (b) Disorder broadens the density of states of the Landau levels, leaving only some delocalised. Reproduced from (Powell, 2020).

Using the relation (A.8) in Appendix A, we may identify this expression with the Berry curvature

$$\frac{e^2}{\hbar}F_{xy}(\mathbf{k}). \tag{1.27}$$

In this identification, we have considered the Brillouin as the space of parameters and expressed the current operators in position space. To evaluate the total conductivity, integrate the curvature over all momenta (the torus of the Brillouin zone) to give the quantised conductivity

$$\sigma_{xy} = \frac{e^2}{\hbar}\int \frac{\mathrm{d}^2\mathbf{k}}{2\pi}F_{ky}(\mathbf{k}) = \frac{e^2}{2\pi\hbar}C. \tag{1.28}$$

The constant C is the topological invariant Chern number which is naturally quantised. It was calculated by integrating the Berry curvature over the Brillouin zone, which must be integer since the Dirac condition quantises the total Berry flux. This is often referred to as the TKNN invariant due to its similarity to the same calculation on a lattice, but truly this is just the Chern number of the band.

One property is left to explain: how do the plateaus emerge in the quantum Hall effect? Smooth disorder in this potential (due to impurities in the sample away from the inversion layer) is actually necessary for Hall plateaus to be seen experimentally.

This is the key result in reproducing the quantum Hall plateaus in experiments: the small amounts of disorder away from the inversion layer presents a weak and disordered potential for the electrons, broadening the energy width of the Landau level (see Fig. 1.4). Most states now exist in the valleys and on the peaks of the disorder potential; since (semiclassically) the electron orbits drift along equipotentials, all of these states become localised. Only the states which exist in a narrow band around the centre of the energy of the Landau level and exist on an inflection point of the potential are delocalised along the system. These states are the ones which contribute to the spectral flow in the presence of disorder, and thus are the only states which are capable of contributing to the Hall conductivity (Halperin, 1982).

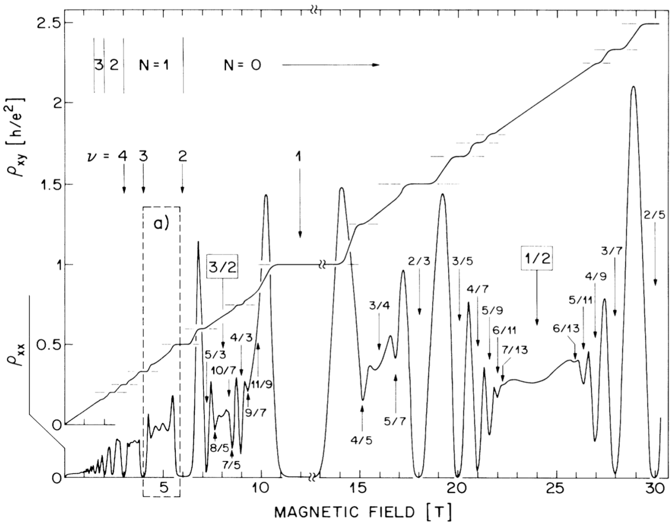

FIGURE 1.5: Hall resistivity ρ_{xy} and diagonal resistivity ρ_{xx} as a function of magnetic field; data and figure reproduced from (Willett et al., 1987).

Therefore, as the Fermi level is adjusted up, when the single extended state is filled there will be an increase in the Hall conductivity because of the contribution of that level through spectral flow. As the Fermi level is continuously increased, there is no further contribution from any other states because they remain localised, leading to the plateau.

1.3 Fractional QHE

1.3.1 Laughlin Wavefunction

The FQHE is an intrinsically many-body effect which eludes a single-particle description. Despite the partial filling of the lowest Landau level, the fluid is rendered incompressible due to strong electron-electron interactions. These interactions open a gap at certain non-integer filling fractions, most notably is the 'Laughlin series' $\nu = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$, and the conjugates $\nu = \frac{2}{3}, \frac{4}{5}, \ldots$.

There exist more exotic states too, shown in Fig. 1.5, including the 'Jain sequence' $\nu = \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots$ and its conjugates which all become narrower as they approach the $\nu = \frac{1}{2}$ state, that is not gapped. There are also plateaus outside of the lowest Landau level, including $\frac{3}{2}, \frac{4}{3}, \frac{5}{2}$, and more.

Now we will attempt to describe the most theoretically well understood series of states using the Laughlin wavefunction. Assuming there is a radially symmetric interparticle potential $V(|z_1 - z_2|)$, we may guess a possible wavefunction of the theory. For example, a class of eigenstates of such a 2-particle potential take the form

$$\psi(z_1, z_2) = (z_1 - z_2)^m e^{-|z_1|^2/4 - |z_2|^2/4}. \tag{1.29}$$

This looks like the general symmetric gauge wavefunctions, labelled by angular momentum m. For a general N-body potential, the general wavefunction is

$$\psi(z_1,\dots,z_N) = f(z_1,\dots,z_N)e^{-\sum_i |z_i|^2/4}, \tag{1.30}$$

where $f(z_1,\dots,z_N)$ is an analytic and totally symmetric prefactor. Laughlin's wavefunction is

$$\psi(z_1,\dots,z_N) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_i |z_i|^2/4}, \tag{1.31}$$

with m odd (Laughlin, 1983). The wavefunction vanishes (with order $(z_1 - z_w)^m$) as any two electrons approach each other, thus instituting the Pauli exclusion principle. One may further use the same wavefunction for even m to describe a new 'bosonic Laughlin' phase, where the wavefunction is even under particle exchange. The filling fraction of such phases is related to the magnetic field B and the electron density ρ by

$$\nu = \frac{2\pi}{B}\rho. \tag{1.32}$$

This trial wavefunction has a strong overlap with numerical studies of small systems, and has been shown to be the exact ground state for the interaction $V(z) = V_2\partial^2\delta(z)$ (Trugman and Kivelson, 1985). Generally it is expected not to be the exact wavefunction of such states, but instead it may live in the same universality class.

There is a close analogy of this wavefunction with superfluid helium, and indeed there are common phonon excitations in both theories. However the FQHE liquid is distinct because its phonon excitations are gapped, rendering the phase incompressible (Girvin et al., 1986). This similarity allows for the construction of 'hydrodynamic' effective theories of the phase (in a similar manner to the superfluid model), which provide a particularly interesting semiclassical model of their dynamics and phase stability (Chen et al., 1989; Abanov, 2013). We will develop the more modern version of such a model in Chapter 5 when we develop the ZHK theory of the FQHE phases.

One may add *quasiholes* to this system, finally actualising the anyons we have been attempting to introduce. These appear in the Laughlin wavefunction as a parameter η in the wavefunction

$$\psi(z_1,\dots,z_N;\eta) = \prod_i (z_i - \eta)\prod_{i<j}(z_i - z_j)^m e^{-\sum_i |z_i|^2/4} \tag{1.33}$$

The quasiholes act — like a fraction of an electron — to repel other charges as z_i approaches η. The order of this pole is lower than for the electron, and correspondingly other charges may approach closer to the quasihole than they may to another electron. Adding additional quasiholes to the wavefunction, and then moving them all to the same point η recovers the order-m pole of an electron — the *fusion* of m quasiholes in this way may generate an electron. This motivates us to identify the charge of a quasihole as e/m.

We can also derive the fractional conductivity of this quantum Hall state by using a familiar flux-threading argument we used on the annulus previously. Recall that threading one unit of flux Φ_0 acted to increase the angular momentum state of the Landau level eigenfunctions by 1. Using the same setup with the Laughlin wavefunction, we may implement this increase by premultiplying the wavefunction by $\prod_i z$, which appears like a quasihole inserted at $z = 0$. This process of inserting flux now only transports e/m of charge from the inner edge of the annulus to the outer, leading to the fractional Hall conductance

$$\sigma_{xy} = \frac{I_r}{\mathcal{E}} = \frac{e^2}{2\pi\hbar}\frac{1}{m}, \tag{1.34}$$

which is described by the $\nu = \frac{1}{m}$ filling factor. There is a 'plasma model' of this Laughlin phase of electrons which provides a robust computational tool. In it the charges are treated as particles in a plasma with charge density $\rho_0 = -1/2\pi B$ and number density $\rho = \nu B/2\pi$, interacting with a background of magnetic field. Importantly, this calculation of the filling fraction can also be verified by calculations in the plasma model.

1.3.2 Fractional Statistics

We may use the Laughlin wavefunction to show that the anyons have fractional statistics: consider looping one anyon around the other and calculate the Berry phase it acquires due to the presence of the other. In a loose sense, consider a static 'quasihole', represented by a unit of magnetic flux. Looping another quasihole around this flux in a loop C picks up the Berry phase

$$i\gamma = i\frac{e}{m}\int_C A = i\frac{e\Phi_0}{m} = \frac{2\pi i}{m}. \tag{1.35}$$

If the operation U acts on two quasiholes to exchange them, then U^2 describes this looping procedure. Therefore the 'statistical phase' acquired on exchange is $\delta = \pi/m$. The full calculation of the Berry phase of the Laughlin wavefunction under adiabatically varying the parameter η is presented by Arovas, Schrieffer, and Wilczek (1984).

Note that in higher dimensions the value of the phase that is acquired under exchange is restricted: looping one particle around another defines the operation U^2, but this path may be lifted out of the plane and smoothly deformed back to a point. This implies that the $U^2 = 1$ operation must be identical to the identity operator, and therefore $U = \pm 1$. In 2+1 dimensions, because paths are fixed on the plane, they cannot be deformed back to a point in this way, and therefore particles may acquire any phase under braiding. The 2+1 dimensional worldlines of such particles which braid in this way will become intrinsically knotted, and we will later show that the braiding phase can be written as a 'knot invariant' of these paths. This is our first example of how two-dimensional theories may have special features not accessible in higher dimensions.

One may also motivate this $\delta = \pi/m$ statistical phase by observing the analytic structure of the Laughlin wavefunction containing two quasiholes. This wavefunction may be written

$$\psi(z_1, \ldots, z_N; \eta_1, \eta_2) = (\eta_1 - \eta_2)^{1/m} \prod_i (z_i - \eta_1)(z_i - \eta_2) \prod_{i<j} (z_i - z_j)^m e^{-\sum_i |z_i|^2/4}, \tag{1.36}$$

which is multi-valued in the parameters η_1, η_2. There is a branch cut in the complex-η_2 plane around the position of the other quasihole η_1. Under rotating η_2 around this branch cut, the wavefunction acquires a phase — called a monodromy — of

$$e^{2i\pi/m}. \tag{1.37}$$

There are m Riemann surfaces of this complex plane, and indeed rotating the quasiholes m times will return the complex coordinate back to itself, with no monodromy phase (Hansson et al., 2017).

Although the theoretical presence of anyons in the FQHE Laughlin states is well established — a feature which has been bolstered by numerical Monte Carlo studies (Kjønsberg and Leinaas, 1999) — experimental evidence of these fractionalised quasi-particles has been somewhat elusive. There have been works which analyse the 'shot noise' of a FQHE junction, and demonstrate there exists a regime where the charge carriers appear to have fractionalised charge (Hashisaka et al., 2015). Due to the intense technical challenges of controlling anyons it had been until recently impossible to directly measure their braiding statistics. Promising recent work has performed advanced interferometry experiments (Bishara et al., 2009; Bartolomei et al., 2020).

However the strongest evidence was published this year which performed a physical braiding experiment on a sub-micrometer scale (Nakamura et al., 2020); this work claims to provide a direct confirmation of the $\delta = \frac{1}{3}$ exchange statistics of the $\nu = \frac{1}{3}$ Laughlin state. This experiment is performed by running edge currents along one side of a semiconductor heterostructure, which is placed in a FQHE phase using a high magnetic field. The edge current is partly backscattered at two constriction points, where it is directed by the opposing edge to flow backwards, as shown in Fig. 1.6. In-between these two constriction points, there is a relatively large area which hosts thermal anyon excitations. The interference of the currents which are backscattered before and after the bulk anyons provides a measure of the statistical phase acquired by looping the edge currents around these anyons. As magnetic field and gate voltage are varied, the expected number of thermal anyons in the bulk changes, and the authors of this study observe discontinuities in the interference phase — providing a direct measurement of single-anyon's phase contributions.

We have examined the Laughlin state and shown that its excitations are electrons and quasiholes with fractional statistics. Such quasiholes appear in the wavefunction as a depletion of charge at a point η like $\prod_i (z_i - \eta)$. The phase of this expression winds through 2π as any z_i loops around η, which appears like a vortex.

The Laughlin wavefunction is constructed of blocks $(z_i - z_j)^m$, meaning the wavefunction approaches zero order m as the points are brought together. This can be seen as a *composite fermion* (CF) built of an electron $(z_i - z_j)$ and $m-1$ further vortices (Jain, 1989). For Laughlin states, where odd $m = 2p + 1$, the blocks decompose as

$$(z_i - z_j)^m = \underbrace{(z_i - z_j)^{2p}}_{\text{vortices}} \underbrace{(z_i - z_j)}_{\text{electron}}. \tag{1.38}$$

Winding a composite fermion along a closed path in this fluid of CFs, there is a Aharonov–Bohm phase contribution due to the quasiholes attached to the electrons,

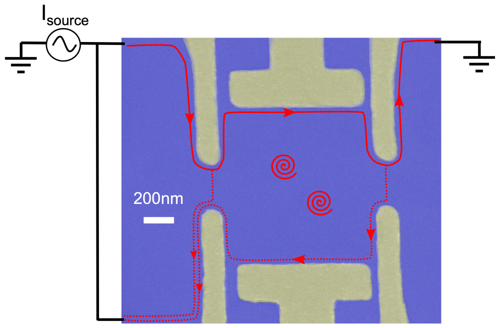

FIGURE 1.6: A schematic of the device used to measure anyon excitations, including the path of edge currents; figure reproduced from (Nakamura et al., 2020).

as well as from the background magnetic field. When the CFs have a density ρ, the accumulated phase per loop-area A in a field B is

$$\gamma = 2\pi\left(\frac{B}{\Phi_0} - 2p\,\rho\right)A. \tag{1.39}$$

since $2p\,\rho$ is the vortex density. This motivates the definition of an effective magnetic field B^* for the CFs, given $B^* = B - 2p\rho\Phi_0$. Recall the definition of the filling fraction of electrons is $\nu = 2\pi\rho/B$ (1.32), we are therefore motivated to define the effective CF filling fraction as

$$\nu^* = \frac{2\pi\rho}{B^*} = \frac{2\pi\rho}{B - 2p\rho\Phi_0} \quad \Longrightarrow \quad \nu = \frac{\nu^*}{1 + 2p\nu^*}. \tag{1.40}$$

Hence the $\nu = \frac{1}{m}$ Laughlin FQHE states are simply lowest Landau level $\nu^* = 1$ IQHE states of the composite fermions (Jain, 2007). Furthermore, we can simply construct the Jain states by taking integer CF Landau level filling $\nu^* = k$ (Blok and Wen, 1990) and fixing $p = 1$ so

$$\nu_{\text{Jain}} = \frac{k}{2k+1}. \tag{1.41}$$

These composite fermions will be the focus of the end of Chapter 5, where a field theoretic method provides a novel understanding of these particles in the particularly interesting $\nu = \frac{1}{2}$ phase.

The Chern–Simons Effective Action 2

2.1 Chern–Simons Theory of the IQHE

2.1.1 The Chern–Simons Action

The Quantum Hall state can be described by the Chern–Simons 2+1 dimensional topological quantum field theory (Frohlich and Zee, 1991). Let us motivate this action by considering which terms we can construct from a 2+1 dimensional vector potential A_μ. We will be using differential form notation in this section, so the unfamiliar reader is directed to Appendix A for a review.

The vector potential 1-form $A = A_\mu \, \mathrm{d}x^\mu$ has a topological current given by

$$J = \frac{1}{2\pi} \star \mathrm{d}A\,, \tag{2.1}$$

such that it is conserved identically in 2+1 dimensions

$$\mathrm{d}^\dagger J = \frac{1}{2\pi} \star \mathrm{d}^2 A = 0, \tag{2.2}$$

and is gauge invariant under $A \to A + \mathrm{d}\lambda$. The factor of $1/2\pi$ in the current is simply a normalisation.

In seeking a description of the integer QHE, we are looking for a low-energy (or *infrared*) description of the gauge field. This means we must construct the action from operators which are *relevant* with respect to the renormalisation group — relevant operators are ones which have a scaling dimension less than $D = 3$, and their couplings increase as distance scales are increased. In our formation of an effective theory of the QHE, we will describe the long-distance physics with a relevant term in the action that encodes the universal properties of the QHE. Higher mass-dimension *irrelevant* operators will describe non-universal microscopic properties of the state and can be ignored at large distances.

Since A has mass dimension 1 and J has mass dimension 2, the lowest-dimension 3-form action we can construct using these two which involves derivatives of A is the *marginal* operator $A_\mu J^\mu = A \wedge \star J$, with mass dimension 3. This is the Chern–Simons action, which when conventionally normalised is written as

$$S_{\mathrm{CS}} = \frac{1}{4\pi} \int A \wedge \mathrm{d}A\,. \tag{2.3}$$

Other operators such as the Maxwell operator $J \wedge \star J$ are irrelevent in the IR and will not affect the universal long-distance physics.

This CS theory has been argued to describe the long-distance physics of a 2+1 dimensional gauge field purely on symmetry and scaling grounds and so one expects that this theory must arise generally as an effective theory in 2+1 dimensions. We will later construct such a theory by integrating out fermions interacting with a gauge field, which is physically motivated by the underlying physics of the quantum Hall effect. In order to derive this effective theory we must only assume that the underlying theory is gapped, and there exists an energy scale below which the dynamics of the heavy fermions can be ignored.

This action is topological because S_{CS} is an integral of a 3-form and is explicitly independent of the metric $g_{\mu\nu}$ when placed on a curved space $\mathcal{M}$. This can be demonstrated by evaluating the wedge product in coordinate basis

$$S_{\text{CS}} = \frac{1}{4\pi} \int A_\mu \partial_\nu A_\rho \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \tag{2.4}$$

$$= \frac{1}{4\pi} \int \mathrm{d}^3x \, \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \tag{2.5}$$

which only depends upon the Levi–Civita symbol ε and not the metric. Consequently we see the stress tensor $T_{\mu\nu}$, defined through $\delta S_{\text{SC}} = \int_{\mathcal{M}} T_{\mu\nu} \delta g^{\mu\nu}$, is identically zero.

In 3+1 dimensions there are a slew of possible relevant actions which form possible low-energy descriptions of quantum materials. Examples include ferromagnetic terms $\mathbf{n} \cdot \mathbf{B}$, (anisotropic) electric polarisation terms $\alpha_{ij} E^i E^j$, etc. These terms (which do not have equivalents in 2+1 dimensions) have in common that they are constructed of manifestly gauge invariant quantities $\mathbf{B}$ and $\mathbf{E}$; in contrast, the Chern–Simons action is constructed from the vector potential A_μ. An important discussion which has so far been deferred is the state of gauge invariance in the Chern–Simons theory. We will later show that the action varies by a boundary term, and is only generally gauge invariant if $S = kS_{\text{CS}}$ with a quantised coupling constant $k \in \mathbb{Z}$.

2.1.2 Gauge Invariance

The variation of the CS action by δA, when on a flat and non-compact manifold $\mathcal{M}$ with no boundaries, is

$$\delta S_{\text{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} [\delta A \wedge \mathrm{d}A + A \wedge \mathrm{d}\delta A] = \frac{k}{2\pi} \int_{\mathcal{M}} \delta A \wedge F = 0. \tag{2.6}$$

The stationary action condition gives the equation of motion $\mathrm{d}A = 0$, which is the on-shell flat-connection condition. Let us investigate the gauge transformation properties of the CS action; the gauge variation in terms of the group element $g \in U(1)$ is

$$\delta A = g^{-1} \, \mathrm{d}g \,. \tag{2.7}$$

Taking the group elements to be $g = \exp(i\lambda)$, then the variation is

$$\delta A = \mathrm{d}\lambda \,. \tag{2.8}$$

(a) Open Manifold (b) Manifold with Boundary (c) Closed Manifold

FIGURE 2.1: Examples of the categories of manifolds $\mathcal{M}$ on which the Chern–Simons gauge-invariance is discussed. (a) shows a non-compact open manifold, (b) shows an manifold with a boundary $\partial\mathcal{M}$, (c) shows a closed manifold.

Now again on a non-compact manifold there are no boundary terms and the CS variation is

$$\delta S_{\mathrm{CS}}[A] = \frac{k}{2\pi}\int_{\mathcal{M}} \mathrm{d}\lambda \wedge \mathrm{d}A = 0. \tag{2.9}$$

This action is therefore totally gauge invariant and well-defined on non-compact *open* spaces without boundaries, as shown in Fig. 2.1a.

If instead we take $\mathcal{M}$ to be a manifold with a boundary $\partial\mathcal{M}$ (Fig. 2.1b), the variation picks up a total-derivative term

$$\delta S_{\mathrm{CS}}[A] = \frac{k}{2\pi}\int_{\mathcal{M}} \delta A \wedge F + \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}(A \wedge \delta A) \tag{2.10}$$

The on-shell condition $F = 0$ removes the first term, and using Stokes' theorem to write the second term as a boundary integral gives

$$\delta S_{\mathrm{CS}}[A] = \frac{k}{4\pi}\int_{\partial\mathcal{M}} A \wedge \mathrm{d}\lambda. \tag{2.11}$$

Therefore, on a manifold with boundaries, the Chern–Simons action is not gauge-invariant. This non-zero variation is called a gauge anomaly of the theory and must be dealt with carefully (Elitzur et al., 1989; Zumino et al., 1984).

However the case of a compact and closed manifold is even more interesting (Fig. 2.1c). Although this case has no boundary, it is still not invariant under so-called 'large gauge transformations', which are not homotopically equivalent to the identity. The gauge function λ along a cycle C on the manifold does not return to itself as it winds around a full loop.

If we consider as an example $\mathcal{M}$ with a compact time dimension and a spatial sphere $\mathcal{M} = S^1 \times S^2$, we must identify the periodic ends of the time coordinate $\tau \sim \tau + \beta$. It can be shown that this periodic-time theory is equivalent to placing the QFT at finite temperature β^{-1} (Cabra et al., 1996). There now exist large gauge transformations which wind N times around the time coordinate

$$g = e^{2\pi i N\tau/\beta}. \tag{2.12}$$

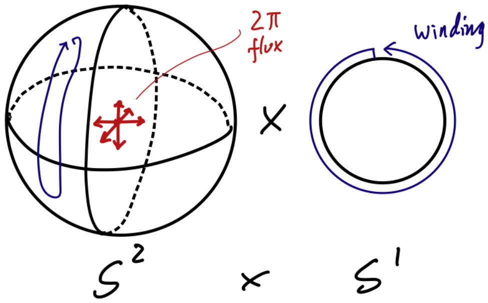

Figure 2.2: A cycle (in blue) on the $\mathcal{M} = S^2 \times S^1$ manifold. The cycle winds around the timelike manifold and is not contractible. Each spacial S^2 manifold has 2π flux penetrating its surface.

This group element is formed by taking $\lambda(\tau, \mathbf{x}) = 2\pi N\tau/\beta$.

Separately, take the spatial volume to be unity and thread 2π of background magnetic flux through the surface of the sphere

$$\int_{S^2} F_{\text{bg}} = 2\pi. \tag{2.13}$$

This configuration is shown in Fig. 2.2. Periodic time means there can exist gauge transformations which are not homotopic to the identity involving the τ coordinate, meaning they have a non-trivial winding number N around the S^1 manifold. Such a transform in this system is

$$A(\tau) \to A(\tau) + \frac{2\pi N}{\beta}\, \mathrm{d}\tau \tag{2.14}$$

which is constant in space; under this the total derivative term is

$$\frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{d}(A \wedge \delta A) = \frac{k}{4\pi} \frac{2\pi N}{\beta} \int_{S^1} \int_{S^2} \mathrm{d}A \wedge \mathrm{d}\tau = 2\pi N k. \tag{2.15}$$

This variation does not leave the action invariant, however the theory is unchanged if the path integral $Z = \exp(iS_{\text{CS}})$ is invariant. Explicitly it changes as

$$e^{iS_{\text{CS}}} \to e^{iS_{\text{CS}}} e^{2\pi N k}; \tag{2.16}$$

the additional phase is 1 for all loops ($N = 0, 1, \dots$) if k is an integer. Therefore on a closed manifold the Chern–Simons action can be made gauge invariant only if $k \in \mathbb{Z}$.

In conclusion, the Chern–Simons action cannot be well defined in isolation on manifolds with a boundary. Instead the theory should be considered with non-compact dimensions, or with periodic (closed) dimensions. We will later investigate the effect of placing a quantum Hall fluid (described by a CS term) on a manifold with a boundary, and the gauge variation on the boundary (2.11) will play a part in the 'anomaly cancellation' story.

2.1.3 Emergence of the QHE

The path integral of the effective theory is written simply as the exponential of the CS action with coupling constant, or 'level', k:

$$Z[A] = e^{ikS_{\mathrm{CS}}[A]}. \tag{2.17}$$

Here, the background gauge field plays the role of an external current which probes the CS action. Expectation values of the ground state current J can be evaluated by taking functional derivatives of the partition function

$$\langle J^\mu \rangle = -i\frac{\delta \log Z}{\delta A_\mu}. \tag{2.18}$$

In contrast to usual descriptions, note that in this context J is the operator and the background field A is its source. Since the gauge field always couples to currents like $A_\mu J^\mu$ we can define the ground state current for the full theory to be

$$\langle J^\mu \rangle = -k\frac{\delta S_{\mathrm{CS}}}{\delta A_\mu(x)} = \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}\partial_\nu A_\rho(x). \tag{2.19}$$

This is proportional to the topological current first introduced in (2.1). Evaluating the components gives

$$\left\langle J^i \right\rangle = \frac{k}{4\pi}2\varepsilon^{ij0}\partial_j A_0 = \frac{k}{2\pi}\varepsilon^{ij}E_j, \tag{2.20}$$

$$\left\langle J^0 \right\rangle = \frac{k}{4\pi}\varepsilon^{0ij}\partial_i A_j = \frac{k}{2\pi}B. \tag{2.21}$$

The first expression (2.20) gives an expression for the Hall current J^2 in terms of the applied electric field E^1; as a result the CS action inherently describes a system with Hall conductivity

$$\sigma_{xy} = \frac{k}{2\pi}. \tag{2.22}$$

The quantised CS coupling (as a result of gauge invariance) therefore means the CS action necessarily describes the integer quantum Hall effect with $\nu = k$ filled Landau levels. There exists a conserved charge associated with the spatial integral of J^0; recall that the number of electrons in each Landau level is

$$g = \int \mathrm{d}^2\mathbf{x}\,\frac{B}{2\pi}, \tag{2.23}$$

and therefore the charge is the number of electrons in k filled Landau levels

$$N_e = k\int \mathrm{d}^2\mathbf{x}\,\frac{B}{2\pi} = kg. \tag{2.24}$$

The parity operator acts on the coordinates through ${P_\mu}^\nu = \mathrm{diag}(1,-1,1)$, and the coordinates transform as $x_\mu \to {P_\mu}^\nu x_\nu$. The gauge field transforms in the same way as the coordinates since it is a covariant vector (it is the components of a one-form). Therefore under this transform the measure is invariant and the action transforms

$$\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \to -\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{2.25}$$

This is affirming, since we expect to use the CS theory to describe a system in a magnetic field which inherently breaks parity. We have now checked that the CS action possesses all of the required features needed for it to describe the integer quantum Hall effect. There seems to be a contradiction at this stage: how can the fractional QHE be described by the CS action when the results obtained thus far imply that we would need a fractional k which would not be gauge invariant? The answer will be delivered later, and comes from the emergence of a new dynamical gauge field a which is itself charged under the background electromagnetic gauge field A.

2.1.4 Chern–Simons Effective Action & Parity Anomaly

The Chern–Simons action can be written down as an *effective action* of a gauge field A interacting with massive Dirac fermions. This section will derive this important result, which will uncover the parity anomaly of fermions in 2+1 dimensions. This result is clearly pertinent to the quantum Hall system, where we expect the gauge field-electron interaction to produce this action.

Consider the path integral of a theory with dynamical degrees of freedom interacting in a background of an EM gauge field: integrating out all the matter fields defines the generating functional in terms of only the gauge field

$$Z[A_\mu] = \int \mathcal{D}(\text{fields})\, e^{iS[\text{fields};A_\mu]} \tag{2.26}$$

and then the Chern–Simons action can be realised as its Wilsonian quantum effective action

$$iS_{\text{eff}}[A_\mu] = \log \frac{Z[A_\mu]}{Z[0]}. \tag{2.27}$$

Since the CS action is parity-odd, we can seek for the CS term arising from the effective theory by evaluating only the parity-odd components.

We can explicitly derive this effective action by considering a one-loop expansion of a massive fermion model in 2+1 dimensions (Redlich, 1984b). This calculation must be restricted to $B = 0$ which will allow us to simply calculate the effective action, but still serves to motivate how a CS term can arise by integrating out fermions. However we will later offer more recent results which derives how the CS term appears even at finite magnetic field, which is exactly the case we are interested in when describing the QHE.

The path integral of the massive Dirac fermion is given by

$$Z[A_\mu] = \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi}\, \exp\Big[i \int \mathrm{d}^3x\, \bar{\psi}\, [i\partial\!\!\!/ + A\!\!\!/ + m]\, \psi \Big] \tag{2.28}$$

For each fermion species, Fig. 2.3 shows the dispersion relation. The addition of a mass gaps the spectrum into two spin-polarised bands, and the effective theory will be valid when gauge-field momenta are well below the fermion mass m.

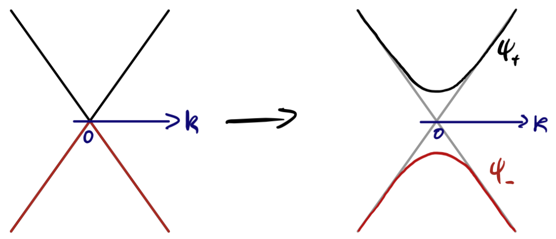

FIGURE 2.3: Adding mass to the fermion splits its spectrum into two bands, a step necessary to integrate out the fermions.

Parity inversion acts as $x^1 \to x^1$ and $x^2 \to -x^2$, and acts on the fermion as $\psi \to \sigma^2\psi$. The mass term $m\bar{\psi}\psi$ breaks this $\mathbb{Z}_2$ parity symmetry, and therefore an effective theory derived from a massive fermion may also break parity symmetry. The Chern–Simons term is one such parity-odd term, and we will indeed find that it arises as a quantum correction to the effective action (Redlich, 1984a). This result, wherein the CS term breaks the classical parity symmetry of the gauge field, is dubbed the 'parity anomaly'.

An anomaly is a symmetry which is conserved in the classical action S, but is broken in the quantum path integral. In our example the tree-level effective action is parity-symmetric but if one tries to quantise the massless theory there are IR divergences which must be regulated with a mass. This mass term immediately breaks parity symmetry (and so do the one-loop vacuum polarisation bubble diagrams which are generated in perturbation theory), but this is an unavoidable step which must be taken to have a regularised quantum theory (Ma, 2018). Therefore it can be shown that the parity symmetry is not a true symmetry of the quantum theory, and the theory is anomalous.

Explicitly integrating out the fermions, the path integral at one-loop is

$$\log Z[A_\mu] = \operatorname{tr}\log(i\partial\!\!\!/ + A\!\!\!/ + m) \tag{2.29}$$

and the second-order expansion of the logarithm in A is

$$iS_{\text{eff}} = \operatorname{tr}\log\left[\frac{1}{i\partial\!\!\!/ + m}A\!\!\!/\right] + \frac{1}{2}\operatorname{tr}\log\left[\frac{1}{i\partial\!\!\!/ + m}A\!\!\!/\frac{1}{i\partial\!\!\!/ + m}A\!\!\!/\right] + \mathcal{O}(A^3). \tag{2.30}$$

These first two terms are the tadpole diagram (which is evaluated to identically zero), and the bubble diagram (shown in Fig. 2.4). This latter term leads to the vacuum polarisation of the gauge field described by the polarisation tensor $\Pi^{\mu\nu}$. The parity-odd element of this term contributes to the parity anomaly (and we therefore suspect it will contain the CS action, based on our RG arguments of Section 2.1.1). Explicitly

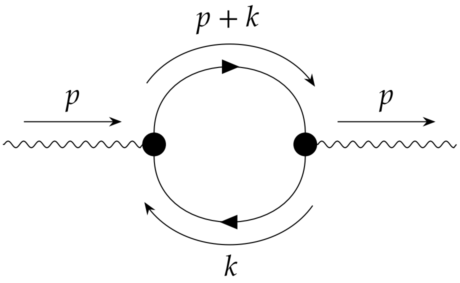

Figure 2.4: One-loop Feynman diagram which contributes the CS term.

evaluating this term

$$iS_{\text{eff}} = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3} A_\mu(p)\,\Pi^{\mu\nu}_{\text{odd}}(p)\,A_\nu(-p), \tag{2.31}$$

$$\Pi^{\mu\nu}_{\text{odd}} = \text{odd}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\text{tr}\left[\gamma^\mu \frac{\not{p}+\not{k}-m}{(p+k)^2-m^2}\gamma^\nu \frac{\not{k}-m}{k^2-m^2}\right]. \tag{2.32}$$

The parity transform on (2.31) takes $p \leftrightarrow -p$ and hence the parity-odd component of $\Pi^{\mu\nu}$ is its antisymmetric part $\Pi^{[\mu\nu]}$. Evaluate this using the only antisymmetric trace contribution $\text{tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = -2\varepsilon^{\mu\nu\rho}$ in 2+1 dimensions. Evaluating the odd part of the polarisation tensor gives

$$\Pi^{\mu\nu}_{\text{odd}}(p) = 2m\int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\varepsilon^{\mu\nu\rho}k_\rho + \varepsilon^{\mu\rho\nu}(p_\rho + k_\rho)}{[(p+k)^2-m^2]\,[k^2-m^2]} \tag{2.33}$$

$$= 2m\,\varepsilon^{\mu\nu\rho}p_\rho \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^3Q}{(2\pi)^3}\frac{1}{\left[Q^2 + D(x,p^2)\right]^2} \tag{2.34}$$

when evaluated using Feynman's integral trick, with $D(x,p^2) = m^2 + p^2x(1-x)$. Evaluate this integral using the Euler gamma function identities, then take the limit $m^2/p^2 \to \infty$, which represents the correct hierarchy of energy scales for the effective theory, to give

$$\Pi^{\mu\nu}_{\text{odd}}(p) = 2m\,\varepsilon^{\mu\nu\rho}p_\rho \int_0^1 \frac{\mathrm{d}x}{8\pi}\frac{1}{\sqrt{D}} \to \frac{\text{sign}\,m}{4\pi}\varepsilon^{\mu\nu\rho}p_\rho. \tag{2.35}$$

This is compatible with the effective field theory limit, where all momenta are much lower than the mass of the fermion being integrated out $p^2 \ll m^2$. Using (2.31) and moving back to real space to recover the form of the Chern–Simons action

$$S_{\text{eff}} = \frac{\text{sign}\,m}{2}\frac{1}{4\pi}\int \mathrm{d}^3x\,\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho = \frac{\text{sign}\,m}{2}\frac{1}{4\pi}\int \mathrm{d}^3x\,A\wedge \mathrm{d}A\,. \tag{2.36}$$

Extending the action to contain k identical copies of the fermion,

$$Z[A_\mu] = \prod_{i=1}^{k}\int \mathcal{D}\psi_i \int \mathcal{D}\bar{\psi}_i\,\exp\!\left[i\int \mathrm{d}^3x\,\bar{\psi}_i\,[i\not{\partial} + \not{A} + m]\,\psi_i\right], \tag{2.37}$$

then integrating them out gives the following CS action

$$S_{\text{eff}} = \frac{\operatorname{sign} m}{2} \frac{k}{4\pi} \int \mathrm{d}^3x\, A \wedge \mathrm{d}A\,. \tag{2.38}$$

This is the level-$k/2$ CS action, which is gauge invariant on an open manifold. If we want to extend the analysis to a closed manifold then we must choose a regularisation scheme — if possible — to preserve the gauge symmetry. Failure to do so would result in a fatal gauge anomaly. Note that this coupling does not get renormalised at higher-loops due to the topological nature of this term (Coleman and Hill, 1985).

The divergences of the theory have not been avoided; the parity-even component of the polarisation tensor (2.32) contains a divergent term

$$\Pi^{\mu\nu}_{\text{div}}(p) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\operatorname{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)k_\rho k_\sigma}{[(p+k)^2 - m^2][k^2 - m^2]} \tag{2.39}$$

$$= 3\int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{2k^\mu k^\nu - k^2\eta^{\mu\nu}}{[(p+k)^2 - m^2][k^2 - m^2]}. \tag{2.40}$$

This scales as $\sim \int \mathrm{d}^3k\,/k^2$, or in polar coordinates $\sim \int \mathrm{d}k$ which means it has a linear UV divergence (Fradkin, 2020a). A Pauli–Villars (PV) regulator must be introduced to cancel this UV divergence, which is done by introducing a heavy fermion with mass $|M| \gg m$ for every fermion of the theory (Turner, 2019, Sec. 9.2). These auxiliary fields are decoupled by taking $|M| \to \infty$ and integrating them out as well as the original fields

$$S_{\text{PV}}[A_\mu] = S_{\text{eff}}[A_\mu] - \lim_{M\to\pm\infty} S_{\text{eff}}[A_\mu, M] \tag{2.41}$$

$$= \frac{\operatorname{sign} m + \operatorname{sign} M}{2} \frac{1}{4\pi} \int \mathrm{d}^3x\, A \wedge \mathrm{d}A\,. \tag{2.42}$$

The PV fields not only cancel the linear UV divergence, but they also contribute a half-quantised CS term when they are integrated out, with a $\pm$ sign depending on m. Taking the limits $m \to +\infty$, $M \to -\infty$ then gives an integer-quantised CS action

$$S_{\text{PV}}[A_\mu] = \frac{k}{4\pi} \int \mathrm{d}^3x\, A \wedge \mathrm{d}A\,. \tag{2.43}$$

This remarkably is gauge invariant on all manifolds, resolving the gauge anomaly at the expense of retaining the parity anomaly. This effective action of the gauge field interacting with infinitely massive fermions has a quantum correction which leads to parity non-invariance, despite the infinitely massive fermions themselves being parity conserving. Alternatively, taking the limits $m \to +\infty$, $M \to +\infty$ means the PV fermions exactly cancel the contribution of the dynamical fermions, removing the CS action from the effective action. In this regularisation scheme the parity symmetry is preserved and there is no anomaly.

2.1.5 Effective Action at $B \neq 0$

The previous Section demonstrated the emergence of the CS action from a gauge field interacting with 2+1 dimensional massive fermions. However there is a problem if we hope to use this result in the context of the quantum Hall effect: the calculation was performed at zero background magnetic field. Recent work has been able to calculate the effect of turning on a background B field, and confirm that a Chern–Simons action indeed appears in this context too.

Let us walk through the previous derivation in order to understand how the assumption of zero magnetic field affected the calculation. The polarisation tensor was defined through the expansion in the perturbation $V = \not{A}$

$$iS_{\text{eff}} = \log\det(1 + G_0 V) \tag{2.44}$$

$$= \operatorname{tr}(G_0 V) - \frac{1}{2}\operatorname{tr}(G_0 V G_0 V) + \cdots, \tag{2.45}$$

where $G_0^{-1} = i\not{\partial} + m$ is the inverse free propagator. The expansion of this second order term (2.31) involved the trace over the whole Hilbert space of the intermediate fermions. When the system is in a magnetic field, the physical states of the fermion are not continuous-momentum representations, but instead are discrete Landau levels in energy. Generally one would expect the propagator amplitude to depend upon the density of states of the virtual particles which appear in loops, and indeed this case is an extreme example where the *only* allowed states are discrete levels.

Now let us aim to perform this calculation with a magnetic field present; consider the non-relativistic fermion as a more simple example (Abanov and Gromov, 2014). This has the action

$$S_{\text{NR}} = \int \mathrm{d}^3x \left[\psi^\dagger (i\partial_0 + A)\psi - \frac{1}{2m}|(\partial_i - iA_i)\psi|^2 \right], \tag{2.46}$$

which has a quadratic dispersion in the free theory. Rather than perturbing around a small gauge field A (as in the previous example), in the presence of a constant (and potentially large) background magnetic field $\overline{B}$ we must expand around this background gauge field $A = \overline{A} + \delta A$ in a small perturbation δA. The presence of a magnetic field in $\overline{A}$ splits the spectrum of fermions into Landau levels at all orders in perturbation theory, and the Hilbert space of the virtual fermions is now the Hilbert space of the harmonic oscillator (generated by ladder operators $a, a^\dagger$). The spectrum of the fermions is split as shown in Fig. 2.5.

This action has inverse free propagator $G_0^{-1} = i\partial_0 - |\partial_\mu - i\overline{A}_\mu|^2/2m$, and perturbation $V = \delta A_0 + |\partial_\mu - i\delta A_\mu|^2/2m$. The calculation presented by Nguyen and Gromov (2016) evaluates the effective action of the gauge field when the fermion is integrated out, and only appears as a virtual particle in loops. The strength of the background magnetic field sets the Fermi level of these virtual fermions, and dictates how many Landau levels are filled; call this number of filled levels N and then the gauge field's one-loop effective action is

$$S_{\text{eff}} = \frac{N}{4\pi} \int \mathrm{d}^3x \, [A \wedge \mathrm{d}A + \cdots] \, . \tag{2.47}$$

This is a level-N Chern–Simons action, and the terms in '$\cdots$' depend only on δE and δB which are not topological. It intuitively makes sense that the CS level is proportional

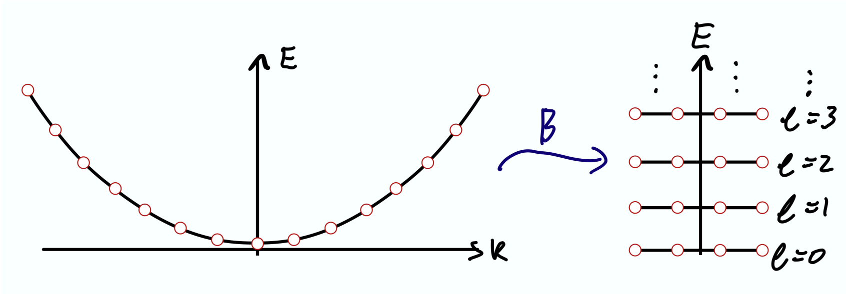

FIGURE 2.5: Non-relativistic dispersion is split into Landau levels in a magnetic field. The vacuum has no states occupied.

to the number of occupied Landau levels of intermediate virtual fermions — if, for example, there were no occupied levels then there could be no fermions propagating around the loop to contribute to the quantum effective action. Moreover, for each level filled there will be a step in the number of electrons which can contribute to the vacuum polarisation, increasing the CS level as if a separate fermion species were added to the UV theory.

Next, we will examine the relativistic *and massless* Dirac theory

$$S_{\text{Dirac}} = \int \mathrm{d}^3x\, \overline{\psi}(i\partial\!\!\!/ + A\!\!\!/)\psi, \tag{2.48}$$

where again $A = \overline{A} + \delta A$, and the background field is nonzero $\overline{B} \neq 0$. The Dirac-cone spectrum of the relativistic virtual fermions are again split into Landau levels by the background field; when filled up to the Dirac point in the zero-field theory, there is a Dirac sea of filled states with negative energy. In the presence of a magnetic field, this sea is replaced by a *Dirac ladder* of occupied negative energy states, and the $\ell = 0$ level is half-filled, as shown in Fig. 2.6. Compared to the effective action derived in Section 2.1.4, this Dirac fermion did not need to have a mass introduced for it to be integrated out. The magnetic field naturally gaps the system, and an effective theory can be derived at energies much below the gap spacing ω_c.

Now by varying the magnetic field, we will also adjust the number of filled Landau levels N above the vacuum which are occupied. The equivalent calculation of the effective action in this Dirac case yields a $N + \frac{1}{2}$ level Chern–Simons term; the difference of a half is due to the $N = 0$ limit corresponding to the half-filled $\ell = 0$ Landau level, which still contributes to the level. Physically this may be seen from Fig. 2.6; because the excitations across the Fermi level cancel, in effect the only contribution of the filled Dirac sea is to contribute an additional $\frac{1}{2}$ to the level.

This result of an additional $\frac{1}{2}$ contribution to the CS level will be important later in the dissertation when we must again consider the Dirac action in a magnetic field. In this future case the Dirac field has a different physical interpretation, but the half-filled zero Landau level of the Dirac ladder vacuum will adjust observables such as the conductivity of the model in the same way.

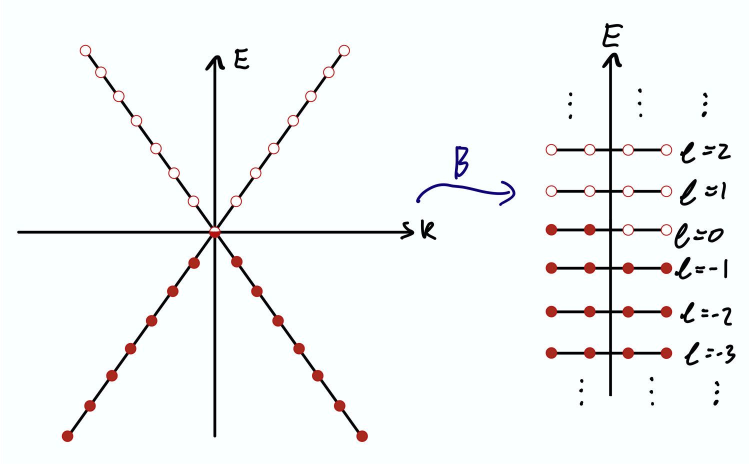

Figure 2.6: Relativistic Dirac dispersion is split into Landau levels in a magnetic field. The vacuum is a 'Dirac ladder' with all negative energy Landau levels filled, as well as half-filling the $\ell = 0$ Landau level.

2.2 Quantum Hall Boundary Anomaly

2.2.1 Relativistic Fermions from Band Theory

In Section 1.2.2 we have motivated the idea that the Hall states should support some sort of chiral fermion at its edges, but we are yet to see how this result fits in with the Chern–Simons approach to the edge. We will find that the presence of such chiral fermions constitutes a 'chiral anomaly', and that it is only resolved through its conjunction with the gauge anomaly of the CS action at a boundary.

Let us now consider a general 1+1 dimensional fermion theory constituted of a number of bands; we will show that at a finite-filling the excitation spectrum looks like a theory of relativistic fermions. Moroever, there exists an important theorem in condensed matter and high-energy physics: the Nielson–Ninomiya theorem which states that chiral fermions are forbidden on a lattice, and so these excitations must be Dirac fermions (Nielsen and Ninomiya, 1981b; Karsten, 1981). To understand this we recall the topological proof provided by Nielsen and Ninomiya (1981a), which begins by considering the spectrum of a generic 1+1 dimensional fermionic Hamiltonian.

The generic 1+1 dimensional Hamiltonian has no band degeneracies and crosses the Fermi level with a non-zero gradient. Considering in isolation each pair of bands, the theory describing them as a two-level system is described by 2×2 Bloch Hamiltonians. Because in 1+1 dimensions the momentum constitutes one degree of freedom, tuning the eigenvalue these matrices to a specific energy requires fixing three parameters and therefore the bands cannot generically be made to cross. However a subset of bands may intersect the Fermi level (set to $\omega = 0$), and give rise to gapless excitations, which is the behaviour of a metal. The Taylor expansion of the bands about this point are $\omega(p) = v(p - p_i)$ where $v_i = \partial\omega/\partial p|_{p_i}$ defines the velocity of each excitation's propagation. Now, on a lattice, momentum p is periodic and so the bands

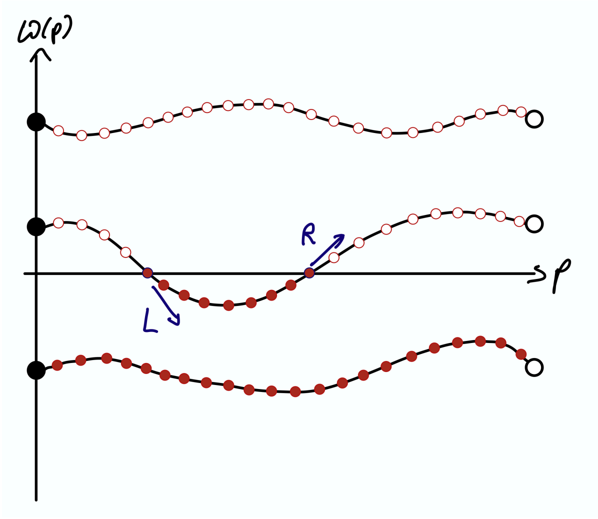

FIGURE 2.7: The spectrum of a generic Hamiltonian filled up to zero energy; bands which cross the zero of energy have massless excitations. The chirality of the excitations is set by their gradient as they cross the Fermi level, and there must be as many left-handed excitations as right-handed ones.

lie in the 'Brillouin zone'. Each band must be periodic over the Brillouin zone and must therefore cross the Fermi level the same number of times upwards as downwards; equivalently, we could say for every $v > 0$ crossing there must be a $v < 0$ crossing as shown in Fig. 2.7.

Importantly we can interpret these left- and right-moving excitations as chiral fermions through inspection of the 1+1 dimensional Dirac action:

$$S = v \int \mathrm{d}^2 x \, \overline{\Psi} \, i \partial\!\!\!/ \, \Psi. \tag{2.49}$$

Its equation of motion is

$$\left(\gamma^0 \partial_t + \gamma^1 \partial_x\right) \Psi = 0 \tag{2.50}$$

and using the Clifford algebra we rearrange to a Hamiltonian expression

$$i\partial_t \Psi = -i\gamma^0\gamma^1 \partial_x \Psi. \tag{2.51}$$

In 1+1 dimensions work in the chiral basis then the chirality operator is $\gamma^5 = \sigma_z$ (see Appendix A). Consequently, the eigenstates of chirality $\Psi_\pm$ are written simply as

$$\Psi_+ = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} 0 \\ \psi_- \end{pmatrix}. \tag{2.52}$$

These have ± 1 eigenvalues $\gamma^5 \Psi_\pm = \pm \Psi_\pm$. The Hamiltonian (2.51) acting on parity eigenstates gives

$$i\partial_t \Psi_\pm(x) = (-i\partial_x)\gamma^5 \Psi_\pm(x) \tag{2.53}$$

$$= \pm(-i\partial_x)\Psi_\pm(x) \tag{2.54}$$

which implies $\omega_p = \pm p$. In the linear dispersion regime this shows the two chiralities are left- and right-movers with positive and negative velocities (Witten, 2015). Together with the Nielson–Ninomiya theorem, this result implies that in 1+1 dimensions on a lattice, there must be as many left-handed fermions as right-handed ones. This is relevant for general 1+1 dimensional condensed matter systems when electrons live on a chain of atoms: the lattice structure provided by this system guarantees that the Brillouin zone is periodic, and therefore that the fermion excitations have a relativistic (and achiral) Dirac nature.

We find the Hamiltonian density of (2.49)

$$\varpi = \frac{\partial \mathcal{L}}{\partial \partial_t \Psi} = iv\Psi^\dagger \qquad \Longrightarrow \qquad \mathcal{H} = v\Psi^\dagger(-i\partial_x)\gamma^1\Psi, \tag{2.55}$$

which, if the state were to contain only one chirality eigenstate $\Psi = \Psi_\pm$ (which would only be possible in a continuous theorem, to evade the NN theorem) then this reduces to

$$\mathcal{H} = \pm v\,\psi_\pm^*(-i\partial_x)\psi_\pm. \tag{2.56}$$

Alternatively, if the state contains both eigenstates $\Psi = \Psi_+ + \Psi_-$ (as must be true on a lattice) then the Hamiltonian is

$$\begin{aligned} \mathcal{H} &= v\Psi^*(-i\partial_x)\sigma_3\Psi & (2.57) \\ &= -v\psi_-^*(-i\partial_x)\psi_- + v\psi_+^*(-i\partial_x)\psi_+, & (2.58) \end{aligned}$$

and the theory decouples into the sum of a chiral and antichiral fermion. This factorisation means that the classical action has two separate $U(1)$ phase symmetries $\psi_\pm \to e^{i\alpha_\pm}\psi_\pm$. In terms of the Dirac fermion Ψ in the chiral basis, the generators of the symmetries are $(1 \pm \sigma_z)/2$, which indeed commute with the Hamiltonian of the theory. This leads to the charges $\rho_\pm = |\psi_\pm|^2$ being individually conserved, which means that although we can excite a right-moving fermion to a higher energy (Fig. 2.8), we cannot turn it into a left-mover (Tong, 2018).

From a different perspective, the classical 1+1 dimensional fermion action (2.49) has a vector symmetry $\Psi \to e^{i\alpha}\Psi$ and axial symmetry $\Psi \to e^{i\alpha\gamma^5}\Psi$. The charges associated with the vector and axial symmetries are defined in terms of the generators of the transformation (1 and σ_z for the vector and axial symmetries, respectively)

$$\begin{aligned} \rho_V &= \Psi^\dagger\, 1\, \Psi = |\psi_+|^2 + |\psi_-|^2 & (2.59) \\ \rho_A &= \Psi^\dagger\, \sigma_z\, \Psi = |\psi_+|^2 - |\psi_-|^2. & (2.60) \end{aligned}$$

Because of the individual conservation of the $\rho_\pm$ charges, both the vector and axial currents are conserved classically.

2.2.2 Chiral Anomaly

Does this necessarily mean the quantum field theory will inherit the same symmetry group? In fact it need not — although the classical action is invariant under axial transformations, the quantum theory may vary. The general process wherein a classical

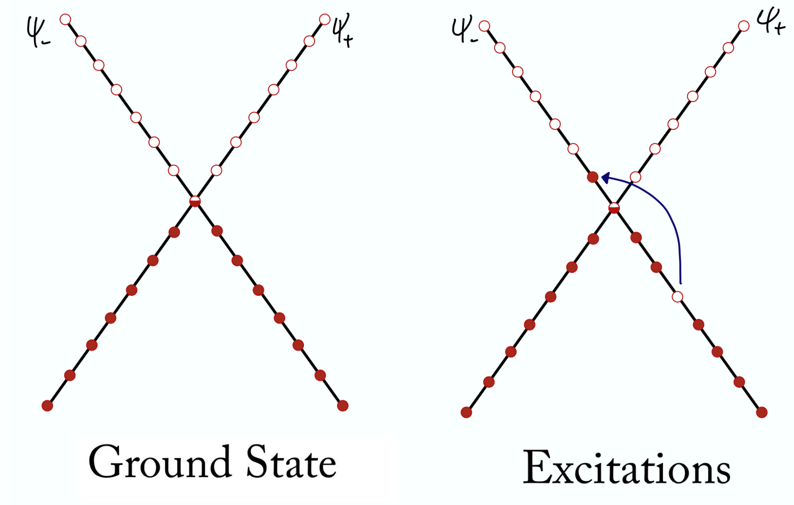

FIGURE 2.8: Excitations of the Dirac spectrum with chiral and antichiral branches preserve the chirality of fermions.

symmetry is broken by quantisation is called an anomaly, and it can be traced to the fact that the path integral measure varies under symmetry transformations.

Working purely in 1-dimension, there is a simple demonstration of this so-called 'axial anomaly' by turning on a (non-dynamical) background gauge field under which the fermions are charged with coupling e

$$S_{\text{gauged chiral}}[A] = v \int d^2x \, \overline{\Psi} \, (i\partial\!\!\!/ + eA\!\!\!/) \, \Psi, \tag{2.61}$$

In the classical theory, the action (2.61) remains invariant under axial transformations. In the quantum theory however, the ground state is a filled Fermi sea of electrons, and we can get an intuitive picture of the anomaly by considering their behaviour under an applied electric field. Adiabatically applying an electric field E causes a shift in momentum of each electron $\Delta p = eE\Delta t$ in time Δt. The density of states of linear bands in 1D is a constant $1/2\pi$, and so the change of density of right/left-moving charge is

$$\dot{\rho}_{\pm} = \pm \frac{eE}{2\pi}. \tag{2.62}$$

This shows that the electric field shifts the right-moving electrons into higher momentum states, and left-moving electrons into lower momentum states. The total number of electrons $\rho_V = \rho_+ + \rho_-$, associated with the vector symmetry is conserved by (2.62). However the difference between left-moving and right-moving chiral fermions $\rho_A = \rho_+ - \rho_-$, which represents the total electric charge, is not conserved

$$\dot{\rho}_A = \frac{eE}{\pi}. \tag{2.63}$$

This violates the rule, due to axial symmetry, that chiral fermions cannot be excited into a state with the opposite chirality. The electric field was able to increase the charge

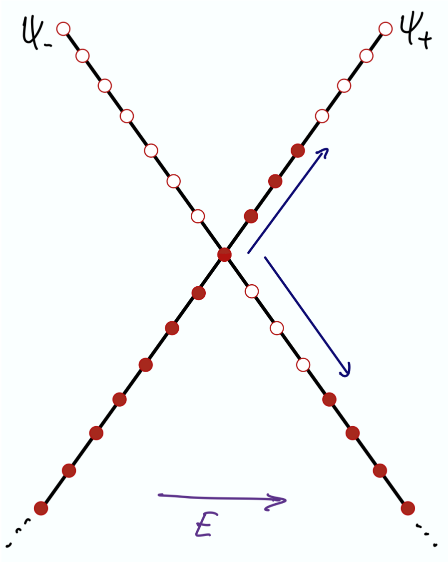

Figure 2.9: Under the application of the electric field there is a chiral anomaly wherein the chirality of fermions is not preserved.

in the right-moving sector because there is an infinite sea of negative-energy states, as shown in Fig. 2.9. This would not be possible in a finite system and it is in this sense that we can say the anomaly exists only in a continuum quantum field theory.

More formally, the quantum theory is defined through its partition function

$$Z[A] = \int \mathcal{D}\Psi\mathcal{D}\overline{\Psi}\, e^{S_{\text{gauged chiral}}[A]}; \tag{2.64}$$

if the classical action $S_{\text{gauged chiral}}[A]$ is invariant under chiral transformations, then the anomaly must be due to the transform of the measure $\mathcal{D}\Psi\mathcal{D}\overline{\Psi}$. Appendix B shows that in the quantum theory the path integral measure transforms under a chiral transform $\delta\Psi = i\lambda(x)\gamma^5\Psi$ goes as

$$\mathcal{D}\Psi\mathcal{D}\overline{\Psi} \to \mathcal{D}\Psi\mathcal{D}\overline{\Psi}\exp\Big(i\int \mathrm{d}^2x\,\lambda(x)\,\partial_\mu J^\mu_A\Big). \tag{2.65}$$

Under the anomaly the axial current, given by

$$J^\mu_A = ie\,\overline{\Psi}\gamma^\mu\gamma^5\Psi, \tag{2.66}$$

is not conserved

$$\partial_\mu J^\mu_A = \frac{eF_{01}}{\pi}. \tag{2.67}$$

Together this allows the definition of the anomalous action

$$S_{\text{anom}} = \frac{e}{\pi}\int \mathrm{d}^2x\,\lambda(x)\,F_{01}. \tag{2.68}$$

This is a 1+1 dimensional equivalent of the Adler–Bell–Jackiw (ABJ, or chiral) anomaly (Adler, 1969; Bell and Jackiw, 1969), which was originally discovered by computing the triangle diagrams of a free fermion in 3+1 dimensions. This massless theory classically has an axial symmetry but the massive counterpart does not — the non-conservation of the global axial symmetry is proportional to the mass m. However ABJ found that there exists an anomaly term due to these triangle diagrams which does not disappear for massless fermions.

One could choose a regularisation scheme which instead preserves the axial symmetry but breaks the vector symmetry, but we choose not to do this because we intend to identify the vector symmetry with the physical $U(1)$ of electromagnetism (by coupling to A_μ, its vector potential). Of course it is therefore possible to gauge the axial symmetry and break the vector symmetry with an anomaly, but this theory is not of interest here since we are seeking a theory that preserves electromagnetism.

In components $F_{01} = E$ and $J^0_A = \rho_A$, therefore (for spatially constant variations) the anomaly (2.63) is reproduced from the non-conservation equation (B.25).

We have reviewed the Nielson–Ninomiya theorem which shows that chiral fermions are not classically allowed in 1D because they must be accompanied by an opposite-chirality partner. However in the presence of an EM field we showed the chiral anomaly leads to non-conservation of the axial current and a buildup of electric charge.

2.2.3 Anomaly Inflow

Previously we found that the Chern–Simons action on a manifold with boundaries has a fatal gauge anomaly, and stated that this forbids the consistent definition of a Chern–Simons action on such manifolds. However this is not necessarily the case, and indeed we will show that the chiral anomaly at the edge acts to cancel the gauge anomaly in the bulk. The missing element of the previous discussion was the behaviour of the edge modes; in this section we will show that there exist massless chiral fermions at the edge of the Chern–Simons fluid which cannot be integrated out. These fermions suffer from the chiral anomaly, and the gauge anomaly arises due to the 'inflow' of this. When the system is considered as a whole, it can be understood consistently.

Recall Section 2.1.4 where we derived the Chern–Simons bulk effective theory from the (UV) action

$$S = \int \mathrm{d}^3x\, \overline{\Psi}(i\partial\!\!\!/ + A\!\!\!/ + m)\Psi. \tag{2.69}$$

We found that integrating out the massive fermions does not decouple the low-energy spectrum (due to the topological Chern–Simons action which emerges at one-loop). With a suitable Pauli–Villars regularisation, the effective action is the level-1 CS action $S_{\text{eff}} = S_{\text{SC}}$. On a manifold with boundaries, and under the gauge transform $\delta A = \mathrm{d}\lambda$ the CS variation (2.10) has the anomaly (2.11)

$$\delta S_{\text{anom}} = \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}\lambda \wedge A. \tag{2.70}$$

Generally the 1+1 dimensional boundary of the theory (2.69) supports gapless chiral fermions which cannot be integrated out — if we do include their behaviour in our theory then the gauge anomaly is cancelled and the theory is healthy.

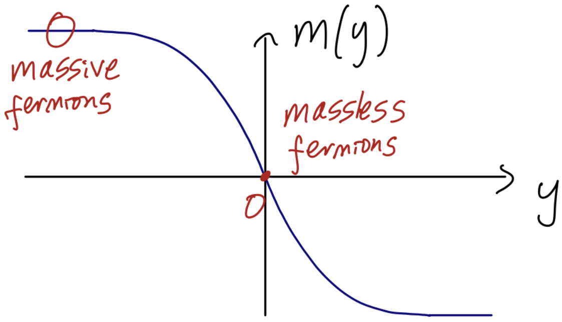

Figure 2.10: Profile of the mass function which crosses zero at $y = 0$. Here there exist massless fermions, but in the bulk the fermion spectrum is gapped, allowing them to be integrated out.

Indeed, it can be generally argued that in order to cancel the gauge anomaly, the system must contain gapless chiral fermions (Witten, 2015). They must be gapless so that they cannot be integrated out, and their chiral nature means that they are indeed anomalous and could not exist in a purely 1+1 dimensional system. The boundary theory hence cannot be consistently written down as a purely local theory on the edge of the sample, and can only exist in conjunction with a Chern–Simons bulk and its gauge anomaly (Maeda, 1996).

We can produce a simple model with this behaviour by localising the fermions in (2.69) around $x^2 = 0$ — this is achieved through tuning the mass to be dependent on $y = x^2$ (Callan and Harvey, 1985). When $m(y)$ changes sign, it can be shown that there must emerge a gapless fermion in 1 spatial dimension, as shown in Fig. 2.10 (Jackiw and Rebbi, 1976). The action written in terms of $x = (t, \sigma)$ and y is

$$S = \int \mathrm{d}^2x \int \mathrm{d}y \, \overline{\Psi}(x,y)(i\partial\!\!\!/ + A\!\!\!/ + i\gamma^5\partial_y + m(y))\Psi(x,y), \tag{2.71}$$

where we have chosen a representation of the 2+1 Clifford algebra with the extra matrix given by the chirality operator $\gamma^5 = \gamma^2$ of the 1+1 dimensional algebra. This has the equation of motion

$$\left[i\partial\!\!\!/ + i\gamma^5\partial_y + m(y)\right]\Psi(x,y) = 0, \tag{2.72}$$

and we may find bound states of Ψ when $E < M$. These states are necessarily gapless and exist at $y = 0$. In the chiral representation $\Psi = (\psi_+, \psi_-)$, the equation of motion implies

$$i\left[\partial_y\psi_+ + (\partial_t + \partial_\sigma)\,\psi_-\right] = -m(y)\psi_+ \tag{2.73}$$

$$i\left[(\partial_t - \partial_\sigma)\,\psi_+ - \partial_y\psi_-\right] = -m(y)\psi_-. \tag{2.74}$$

There is only one gapless normalisable solution at the edge:

$$\Psi(x,y) = \begin{pmatrix} \psi_+(x,y) \\ \psi_-(x,y) \end{pmatrix} = e^{-\int_{-\infty}^{y}\mathrm{d}y' m(y')} \begin{pmatrix} \chi_+(x) \\ 0 \end{pmatrix} \tag{2.75}$$

which is an emergent massless chiral fermion 'zero mode' localised at $m(y) = 0$ which satisfies $(\partial_t - \partial_\sigma)\chi_+(x) = 0$.

The action for these 1+1 dimensional fermion zero modes has a chiral anomaly given by (2.68), since it inherits the interaction with the gauge field A through its embedding in 2+1 dimensional space. The edge's axial-current nonconservation is given by

$$\partial_\mu J^\mu_A = \frac{eF_{01}}{2\pi}, \tag{2.76}$$

We showed before in Eq. 2.63 that this implies the accumulation of edge charge at a rate $\dot{\rho}_{\text{edge}} = eE/\pi$. Note that the factor of a half arises because χ_+ is a Weyl and not a Dirac fermion. The gauge anomaly (2.11) can be expressed as the fact that the current

$$J^\mu_{\text{gauge}} = \frac{e}{4\pi}\varepsilon^{\mu\nu\rho}F_{\nu\rho} \tag{2.77}$$

is conserved within the bulk but not on the edge (Dunne, 1999). This gauge noninvariance leads to an accumulation of charge at a rate given by the y-component of the current

$$J^2_{\text{gauge}} = \frac{eF_{01}}{2\pi}, \tag{2.78}$$

which explains the source of charge accumulation in the chiral edge fermion sector, as depicted in Fig. 2.11. Indeed in both theories, the nonconservation of a classically-conserved current is due to anomalies, but they cancel through the general process of anomaly cancellation. In the quantum Hall context it not only renders our theory well defined on a finite system with edges (of obvious relevance to experimental realisations of the phase), but the properties of the anomalous edge modes will further produce interesting features of the QH phases.

At the level of actions, the anomalies will also explicitly cancel. The chiral fermion's anomaly is

$$\delta S_{\text{A}} = \frac{1}{2\pi}\int \mathrm{d}^2x\,\lambda(x)F_{01} \tag{2.79}$$

under chiral transformations. Note that the factor of a half arises because χ_+ is a Weyl and not a Dirac fermion. Evaluating the bulk's gauge anomaly (2.70) of the fermion theory which manifests as an integral over this $y = 0$ boundary (Elitzur et al., 1989), and integrating by parts gives

$$\delta S_{\text{gauge}} = -\frac{1}{4\pi}\int \lambda(x)\,\mathrm{d}A = -\frac{1}{4\pi}\int \lambda(x)F_{\mu\nu}\varepsilon^{\mu\nu}\,\mathrm{d}x^0\wedge \mathrm{d}x^1 \tag{2.80}$$

$$= -\frac{1}{2\pi}\int \lambda(x)F_{01}\,\mathrm{d}^2x, \tag{2.81}$$

and hence this term is cancelled by the chiral anomaly of the fermion zero modes

$$\delta S_{\text{gauge}} + \delta S_{\text{A}} = 0. \tag{2.82}$$

Since the failure of gauge invariance is quantised in k and must be equal to the chiral anomaly, then indeed the chiral number $n_+ - n_- = k$ must too be quantised. This leads to the quantised edge conduction that Laughlin predicted.

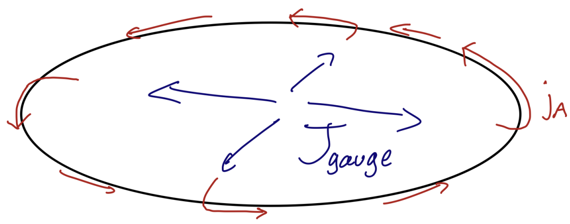

Figure 2.11: Cancellation of the bulk anomaly current and the time-dependent edge current.

2.3 Chern–Simons Theory of the FQHE

2.3.1 Statistical Gauge Field

The Chern–Simons action can also be used to describe the *fractional* quantum Hall effect, and indeed this will be the arena in which the action is most powerful. However in order to describe the FQHE by using the Chern–Simons action we must violate some of the assumptions developed for the IQHE. Specifically the level k must be made non-integer, which naïvely appears to violate the requirement of quantised level imposed by gauge considerations. Surprisingly it is possible to use massive 'topological' fields to change the effective level of the background CS term when they are integrated out (Zhang et al., 1989). The emergent degrees of freedom which have this topological property take the form of a *statistical gauge field*, a, which is a dynamical field of the theory arising from the collective behaviour of the full UV theory of interacting fermions. We will develop the framework which shows that when this field is coupled to matter fields, it attaches flux to them and acts to change their statistics. The effective action which includes the gauge field behaviour is given

$$e^{iS[a,A]} = \int \mathcal{D}(\text{fields})\, e^{iS_{\text{UV}}[\text{fields},a;A]} \tag{2.83}$$

Indeed, it is the presence of *strong interactions* which leads to the emergence of such a gauge field; Lopez and Fradkin (1991) show that the theory of particles interacting under a strong 2-point interaction $V(|\mathbf{x}_i - \mathbf{x}_j|)$ is equivalent to a theory with a new 'statistical' gauge field with a level-m Chern–Simons term

$$m\, S_{\text{CS}}[a] = \frac{m}{4\pi} \int a \wedge \mathrm{d}a\,. \tag{2.84}$$

The second ingredient needed is for the statistical gauge field to be charged under the background $U(1)$ of electromagnetism. The term which achieves this is the 'BF term' (Horowitz, 1989), given

$$S_{\text{BF}}[a,A] = \frac{1}{2\pi} \int a \wedge \mathrm{d}A\,. \tag{2.85}$$

We have now motivated the following action which is proposed to describe the fractional QHE using a dynamical statistical gauge field. This theory is derived through integrating out the UV degrees of freedom of the fermions to leave the effective theory

$$S[a, A] = \int \left[-\frac{m}{4\pi} a \wedge \mathrm{d}a + \frac{1}{2\pi} A \wedge \mathrm{d}a \right]. \tag{2.86}$$

We will analyse this theory and show how the excitations of a are charged under the background EM field and are massive — these will be *anyonic* particles of the FQHE which have fractional exchange statistics.

In order to gain further physical intuition for the a gauge field combine it with a Maxwell term normalised by the dimensionful scale $[\mu] = 1$

$$\int \left[\frac{k}{4\pi} a \wedge f - \frac{1}{\mu} f \wedge \star f \right] \tag{2.87}$$

which is called the Chern–Simons–Maxwell theory. Varying δa, and defining $f = \mathrm{d}a$,

$$\frac{k}{4\pi} \left[\delta a \wedge \mathrm{d}a + a \wedge \mathrm{d}\delta a \right] - \frac{2}{\mu} \delta\, \mathrm{d}a \wedge \star f = 0 \quad \Longrightarrow \quad \frac{k}{2\pi} \delta a \wedge f + \frac{2}{\mu} \delta a \wedge \mathrm{d} \star f = 0 \tag{2.88}$$

and using $\mathrm{d}^\dagger f = \star\, \mathrm{d} \star f$ gives the equation of motion

$$\mathrm{d} \star f + \frac{k\mu}{4\pi} f = 0 \implies \mathrm{d}^\dagger f + \frac{k\mu}{4\pi} \star f = 0 \quad \Longleftrightarrow \quad \partial_\mu f^{\mu\nu} - \frac{k\mu}{4\pi} \varepsilon^{\nu\rho\sigma} f_{\rho\sigma} = 0. \tag{2.89}$$

The inclusion of the Chern–Simons action has modified the usual Maxwell equations by giving the photon a mass. Combine the equations (2.89) for $\mathrm{d} \star f$ and $\star\, \mathrm{d} \star f$ to get

$$\mathrm{d}[\mathrm{d}^\dagger f] = \left(\frac{k\mu}{4\pi} \right)^2 f. \tag{2.90}$$

To see this explicitly, trial the ansatz $f \sim e^{ipx}$ and evaluate the adjoint derivative using the Laplacian $\Delta = \mathrm{d}\,\mathrm{d}^\dagger = -\partial^2$ (when $\mathrm{d}f = 0$) in

$$\mathrm{d}\,\mathrm{d}^\dagger f = \Delta f = p^2 f \tag{2.91}$$

and therefore

$$p^2 f = \left(\frac{k\mu}{4\pi} \right)^2 f \tag{2.92}$$

which implies the spectrum has a gap

$$\omega = k\mu/4\pi. \tag{2.93}$$

The Chern–Simons term has therefore endowed the 2+1 dimensional photon with a 'topological mass' ω through the addition of a topological and gauge invariant term, rather than the typical a_μ^2 term (Deser et al., 1982, 2000).

The limit $\mu \to \infty$ recovers the pure-CS effective action, showing that the topological theory we were considering is the limit of the dynamical CSM theory with infinitely heavy excitations. This limit recovers the pure CS action $a \wedge \mathrm{d}a$ and shows that the mass of the dynamical statistical gauge field is infinite, which will now allow it to be integrated out.

2.3.2 Effective Theory and the FQHE

In order to see how the FQHE is recovered from this theory, consider integrating out the massive statistical gauge field to find an effective theory for just the background field A

$$e^{iS_{\text{eff}}[A]} = \int \mathcal{D}a\, e^{iS[a;A]}. \tag{2.94}$$

Let us calculate the tree-level effective theory in a non-compact space by using the classical equations of motion for a. Vary $S[a;A]$ by δa to get the equation of motion $m\,\mathrm{d}a = \mathrm{d}A$. Defining $f = \mathrm{d}a$ and $F = \mathrm{d}A$ the equation of motion becomes

$$f = \frac{1}{m}F \tag{2.95}$$

which is solved locally by $a = A/m$. Plugging into the action gives the tree-level effective action

$$S_{\text{eff}}[A] = S_{\text{eff}}[A/m;A] = \frac{1}{4\pi}\frac{1}{m}\int A \wedge \mathrm{d}A \tag{2.96}$$

which is a fractional quantum Hall state with filling factor $\nu = \frac{1}{m}$ and Hall conductivity $\nu/2\pi$. Therefore the odd-m theories describe the Laughlin states.

Moreover, states with integer m also describe QHE phases with bosonic Laughlin states, as will be studied in detail later in the case of the $m = 2$ state. Using the composite fermion (CF) picture — where the excitations are electrons bound to flux moving in an effective background field — these integer m states correspond to integer filling of the CF Landau levels. In the CS description, the a gauge field has quantised level m and 'BF coupling' to the background gauge field A, and integrating it out recovers the fractional charge of the anyons. Repeating this procedure, by coupling a to a second gauge field b, and integrating out each one in turn will produce the FQHE behaviour with filling ν in the next level of the hierarchy. Appendix C performs an explicit calculation using the Wen–Zee model, which is a generalisation of this hierarchy and can be used to produce FQHE states of arbitrary filling fraction.

But is this theory a meaningful and gauge invariant description? On non-compact spaces there is no problem, but on compact spaces only the action $S[a;A]$ describes a properly gauge-invariant theory. Although locally this theory gives the correct description (*i.e.* it predicts the fractional Hall current), by integrating out a we have missed some important details of the topological character of the CS fluid when on a compact space. Specifically, integrating out the gauge field using the classical equation of motion is not strictly possible with monopoles present — in this context we should revert to the full action (2.86). Indeed we will later find that the topological nature of the field is important in exactly this environment and even leads to surprising measurable results about the ground state of the theory.

Now we can try to understand what the excitations of the statistical gauge field represent in the quantum Hall context; to (2.86) add an additional source for the emergent gauge field j, and then rewrite the BF term as an EM-field source J

$$S_{\text{eff}}[a_\mu, A_\mu] = \int \left[-\frac{m}{4\pi} a \wedge f + A \wedge \star J + a \wedge \star j \right] \tag{2.97}$$

where the EM-field current is

$$J = \frac{1}{2\pi} \star f. \tag{2.98}$$

Now the Dirac quantisation condition for the emergent field strength requires that the flux integrated over a sphere be quantised. In the language of differential forms, this states

$$Q_{\text{mag}}(S^2) = \int_{S^2} \star J. \tag{2.99}$$

The Hodge dual $\star$ takes the dual of the 1-form J with respect to the full 3-dimensional spacetime, so $\star J$ is a 2-form which can be integrated over the spacelike manifold S^2. Using the current definition (2.98) in terms of the gauge field a, we can recast this charge

$$Q_{\text{mag}}(S^2) = \frac{1}{2\pi} \int_{S^2} f. \tag{2.100}$$

Expressing the field strength 2-form in coordinates on the sphere,

$$f = \frac{1}{2} f_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j = \frac{1}{2} f_{ij} \varepsilon^{ij}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2\,, \tag{2.101}$$

and so the charge in a background 'magnetic field' of the statistical gauge field $f_{12} = b$

$$Q_{\text{mag}}(S^2) = \frac{1}{4\pi} \int_{S^2} f_{ij} \varepsilon^{ij}\, \mathrm{d}^2\mathbf{x} \tag{2.102}$$

$$= \frac{1}{2\pi} \int_{S^2} b\, \mathrm{d}^2\mathbf{x}\,. \tag{2.103}$$

This is the familiar expression for the number of units of magnetic flux (of the statistical gauge field) enclosed by the surface, and so we can write $n_\phi = Q(S^2) \in \mathbb{Z}$.

Variation by δa gives the dynamical gauge field equation of motion, which together with the equation for J relates the two gauge field currents

$$\frac{1}{m} \star j = \frac{1}{2\pi} f \qquad \Longrightarrow \qquad J = \frac{1}{m} j. \tag{2.104}$$

The quantised units of current in the statistical gauge field are therefore fractional currents in the physical gauge field. This important feature is called flux attachment: that the quantised charges of the dynamic statistical gauge field are fractional *magnetic* charges of the physical field. The equation (2.97) still requires that on a compact space, the total physical magnetic charge is an integer, but the individual excitations each carry fractional charge. Physically, this means the fractional excitations must be created in multiples of m.

2.3.3 Flux Quantisation of the Statistical Gauge Field

Before, when working on a non-compact space we related the CS gauge fields using $f = F/m$ in (2.95), which involved integrating out

$$\int \mathcal{D}a\, e^{-S[a,A]} = e^{S_{\text{eff}}[a_{\text{on-shell}}(A), A]}. \tag{2.105}$$

This quadratic integral was taken as being the on-shell classical solution of the equations of motion, however on compact spaces we expect that quantum and topological effects may become important. Indeed, on such spaces the classical solution cannot be globally well defined; consider a background flux which, although it is an integer, is not a multiple of m like

$$\frac{1}{2\pi}\int_{S^2} F = N_\phi \in \mathbb{Z} \tag{2.106}$$

then the flux of the statistical gauge field is not well defined

$$\frac{1}{2\pi}\int_{S^2} f = \frac{N_\phi}{m} = n_\phi \notin \mathbb{Z}. \tag{2.107}$$

Instead, we should explicitly evaluate the path integral over a (2.105), and here we suggest a novel method of doing so. Following the method outlined in (Klebanov et al., 2011), one may attempt to perform this integral by recasting it as a sum over the eigenbasis of the Laplacian

$$\int \mathcal{D}a \to \sum_\lambda a(\lambda), \tag{2.108}$$

where the eigenvalues are of the Laplacian satisfy

$$\Delta a(\lambda) = [\mathrm{d} \star \mathrm{d}\star + \star \mathrm{d} \star \mathrm{d}]a(\lambda) = -\lambda a(\lambda). \tag{2.109}$$

Using a Hodge decomposition for the field a,

$$a = b + \mathrm{d}\phi + h, \tag{2.110}$$

where $\mathrm{d} \star b = 0$ is co-closed, and $\Delta h = 0$ is harmonic.

Using this method, we seek to evaluate partition function on $S^2 \times S^1$, first considering a correctly quantised background satisfying $n_\phi = N_\phi/m \in \mathbb{Z}$. In this case, we may take a simple background like $A_\tau = N_\phi$ and confirm that the path integral recovers the usual $f = F/m$ relation.

The on-shell classical approach has been shown to fail when the flux quantisation condition is broken (*i.e.* when $n_\phi \notin m\mathbb{Z}$). Usual saddle-point approaches to quantum approximations for this path integral will fail, as one may not shift the integration field by the classical solution which does not exist.

Taking an explicit non-trivial background of A_μ with charge that is not a multiple of m will allow for the evaluation of such a non-classical solution, and the resultant path integral will teach us about the quantum behaviour of such a theory. It is an open question what the result of such a calculation will be — whether the path integral will be trivially zero or if it will encode some new behaviours. There is limited scope in this thesis to explore this calculation in more detail, but this constitutes an interesting novel research question which may be explored in the future.

2.4 Anyons & Topological Order

There remain some unanswered questions about the nature of the fractional CS fluid: what are the properties of the anyon excitations, and in what sense is it a topological field theory? In response to the latter question, we may recall that the action was independent of the metric which allowed it to be written as an integral of 3-form fields. The requirements of topological QFTs are explained in depth by Atiyah (1988). There is a much more profound result that observables only depend on the topology of the background manifold which comes as a result of quantising the CS action (Elitzur et al., 1989). In particular we will now work to discover that the ground state degeneracy of the level-m CS theory will be m^g on a genus-g manifold.

Consider a manifold with no boundaries, in particular the spatial torus $\mathcal{M} = T^2 \times \mathbb{R}$. Because there are non-contractible loops in this space, the gauge field a is not solved globally by only local gauge transformations (*i.e.* the connection is not exact). In this case, the physical degrees of freedom are exactly the gauge field a, with action

$$S = \frac{m}{4\pi} \int \mathrm{d}^3x \, \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho. \tag{2.111}$$

Following Tong (2016); Fradkin (2020c), we can now canonically quantise this action to find the following commutation relation

$$[a_1(\mathbf{x}), a_2(\mathbf{x}')] = \frac{2\pi i}{m} \delta^{(2)}(\mathbf{x} - \mathbf{x}'). \tag{2.112}$$

The spatial components of the gauge field along different cycles of the torus are therefore conjugate variables in the quantum theory.

Now let us try to create a gauge invariant observable: integrating these components of the gauge field around a non-contractible loop Γ_i (shown in Fig. 2.12(a)) defines the operator

$$w_i = \oint_{\Gamma_i} a. \tag{2.113}$$

Due to the canonical commutator of the gauge field components, these non-local operators satisfy the algebra

$$[w_1, w_2] = \frac{2\pi i}{m} \tag{2.114}$$

However this is not gauge invariant under large gauge transformations which have a winding number around the loop, but the exponential of this is (since they are invariant up to a factor of 2π). These operators are the Abelian Wilson loops of the CS theory

$$W_i = \exp\left(\oint_{\Gamma_i} a \right), \tag{2.115}$$

which are now shown to satisfy the algebra

$$W_1 W_2 = e^{2\pi i/m} W_2 W_1. \tag{2.116}$$

We can now calculate the size of the Hilbert spaces of the theory: choose the vacuum so that it is an eigenstate of the Wilson loop operator

$$W_1 \left|0\right\rangle = \left|0\right\rangle, \tag{2.117}$$

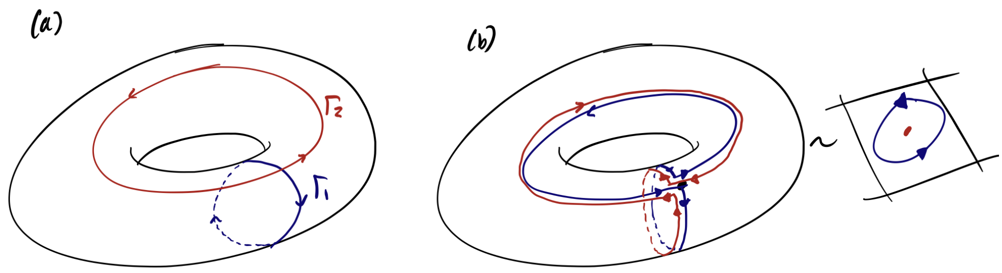

FIGURE 2.12: (a) The two non-contractible loops on a torus. (b) The winding procedure $W_1 W_2 W_1^{-1} W_2^{-1}$ interpreted as the path of anyons on the torus; the result is topologically equivalent to simply looping one anyon around another.

with eigenvalue 1. Next use the algebra to show this implies

$$W_1 W_2 |0\rangle = e^{2i\pi/m} W_2 W_1 |0\rangle = e^{2i\pi/m} W_2 |0\rangle \,, \tag{2.118}$$

or $W_2 |0\rangle$ has eigenvalue $e^{-2i\pi/m}$ under W_1. In fact there are m distinguishable vacuum states given

$$W_1 W_2^k |0\rangle = e^{-2i\pi k/m} W_2 |0\rangle \tag{2.119}$$

with $k = 0, \ldots, m-1$. Because these states all have the same energy, the ground state $|0\rangle$ has an m-fold degeneracy on the torus. Generalising this to a genus-g manifold, one finds that the degeneracy is m^g. The unique ground state on a spatial sphere can be interpreted as being because the space is simply connected.

The CS action has truly led to a topological quantum field theory: we have found an important observable — the ground state degeneracy — is totally determined by the topology of the manifold. This result is independent of paths Γ_i used to define the Wilson loop operators. All that mattered for this derivation was the existence of $g+1$ distinguishable cycles on a genus g manifold, and the canonical quantisation of the CS term immediately told us their commutation relations (Wen, 1989).

This degeneracy is robust against disorder and even particle interactions — in a sense it completely categorises the phase. This naturally introduces the notion of a topological phase: one which is defined not by the symmetry breaking measured by a *local order parameter*, but instead by a *global property* such as ground state degeneracy (Wen, 1990a). More recent work has found that generally such topologically ordered phases are also characterised by an intrinsic 'topological entanglement entropy', which signifies that states arbitrarily far away in the sample are entangled. This produces a constant contribution to the entropy of phase which only depends upon the topology of the background manifold, and does not scale with system or boundary size. It acts as another non-local measure of order which can be used to classify a system as topological (Kitaev and Preskill, 2006; Levin and Wen, 2006).

We can interpret the Wilson lines as physically inserting charged a excitations into the theory and carrying them along the path Γ_i. Now let us consider the action of

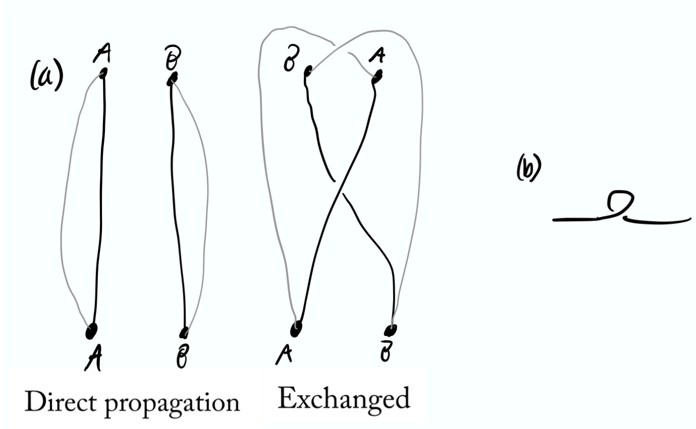

FIGURE 2.13: (a) The comparison of worldlines of direct propagation and exchange of anyons and their antiparticles (b) The self-loop diagram.

generating one particle and taking it along the combined path Γ_2 then Γ_1, which is performed by the operator $W_1 W_2$. Creating a partner particle and taking it on the loop in the inverse direction is generated by the operator $W_1^{-1} W_2^{-1}$. Performing both actions at once gives a braiding operation

$$(W_1 W_2)(W_1^{-1} W_2^{-1}) = e^{2i\pi/m}, \tag{2.120}$$

which was evaluated using (2.116). Unknotting the combined paths from around the torus, the Aharonov–Bohm phase contribution of the braided Wilson lines is identical to simply braiding one line around another. In the interpretation of the Wilson lines as fractionally charged excitations of the gauge field, then we have shown that they have a non-integer statistical phase $\delta = \pi/m$ — this is an anyon (Wilczek, 1982; Wilczek and Zee, 1983).

In fact a much more general statement can be made about the amplitudes of TQFTs which shows how deeply this notion of topology is ingrained in these field theories. The amplitudes of Wilson loops in TQFTs only depend upon the topology of the knotted loops, which in this sense identifies two knots if they can be deformed into each other without passing one line through another. Mathematically such knots are characterised by *knot invariants* which measure the way which loops are intertwined using geometry. These invariants define an algorithm which produces an algebraic expression given a knot such that knots with the same topology produce equivalent invariants. In the interpretation of the Wilson lines as worldlines of anyons, the quantum amplitude of the corresponding process is simply given by a knot invariant (Witten, 1989).

Fig. 2.13(a) shows the worldlines of two particles when they are either exchanged or not. The particles are created in pair-production events, then reconnected with their antiparticle counterpart (which is also exchanged) in order to define closed loops. The exchange event of both particles and antiparticles therefore corresponds to linking the two closed worldlines of the particles in this picture. Alternatively, one may consider looping one particle around another, and receive the same result. Calling one of these loops γ, its amplitude is

$$\langle W[\gamma] \rangle = \exp\left[-\frac{i}{2} \int d^3x \int d^3y \, J^\mu(x) G_{\mu\nu}(x-y) J^\nu(y) \right], \tag{2.121}$$

where $G_{\mu\nu}(x-y) = \left\langle a_\mu(x) a_\nu(y) \right\rangle$ is the correlator of the gauge field. The source $J^\mu(x) = \delta^{(4)}(x^\mu - z^\mu(\lambda))$ where $z^\mu(\lambda)$ is a suitable parameterisation of the loop. Following Witten (1989) and Fradkin (2020c), one can show $J_\mu = \varepsilon_{\mu\nu\rho} \partial_\nu b_\rho(x)$ where $\partial_\mu b^\mu(x) = 0$ in Lorenz gauge. Using Stokes' theorem one can then express the amplitude as

$$\langle W[\gamma] \rangle = \exp\left[-\frac{i\pi}{m} \oint_\gamma dx_\mu \, b^\mu(x) \right] = \exp\left[-\frac{i\pi}{m} \int_\Sigma dS_\mu \, J^\mu(x) \right], \tag{2.122}$$

which simply counts the number of times the Wilson line punctures the surface Σ which is bounded by γ. Call this integer the linking number n_γ, then the amplitude is simply

$$\langle W[\gamma] \rangle = e^{i\pi n_\gamma / m}. \tag{2.123}$$

The linking number measures the number of times the surface is punctured by another Wilson line — it cannot be perturbed by any small amount but can only change when two Wilson loops are linked or unlinked. This accordingly represents a topological invariant, and provides the relation between the two diagrams in Fig. 2.13a

$$W[\text{linked}] = e^{i\pi/m} W[\text{unlinked}], \tag{2.124}$$

which is the calculation of the exchange phase of the two anyons and their antiparticles.

This amazing result shows that all observables of TQFTs are determined solely by the knotting structure, and not on details such as the speed of particles on their respective paths or how close they approach.

One can moreover show that these particles with fractional statistics also have fractional spin, which satisfies the generalised spin-statistics theorem (Dunne et al., 1989). This forms a unitary representation of the Poincaré algebra in 2+1 dimensions (which is not true in higher dimension), and the link between spin and statistics truly holds (Dunne, 1999). In fact the generalised spin-statistics theorem can be calculated using the theory of knot invariants: the quantum amplitude for a particle spinning on its axis is encoded in the diagram of a line with a single loop shape in it, as in Fig. 2.13b.

Bosonisation of the Boundary Excitations 3

3.1 1+1 Dimensional Bosonisation

In this section we will aim to more thoroughly describe the chiral fermions which live on the boundary of the quantum Hall fluid. In the previous chapter, the discussion of the boundary first arose in a rather theoretical analysis of gauge invariance on different manifolds. However when applying the CS action to quantum Hall systems this regime is of utmost importance — indeed all physical quantum Hall samples contain a spacelike boundary, and some of the most interesting observable behaviours of these systems are properties of the edge states.

In order to study these edges we will exploit the chiral anomaly, which when studied in detail will lead to an interesting *bosonisation* duality wherein the fermionic excitations are mapped onto a bosonic particle. The anomalous current-current commutators due to chiral symmetry non-conservation will lead us to identify bosonic fields with the currents, forming the basis of the duality.

At first sight it may seem problematic that there may exist a duality between fermionic and bosonic degrees of freedom, but in 1+1 dimensions there is no real distinction between the statistics of these particles. In fact, in 1 spatial dimension the concept of statistics does not exist, since it is not possible to exchange two particles without moving them over each other. Because of this feature, we may develop a useful procedure called bosonisation which will provide a useful perspective when quantising fermions on the quantum Hall edge, and indeed even become a core part of the discussion of the conformal boundary.

Bosonisation was developed and applied to models introduced by Luttinger (1963) and Tomonaga (1950) which describe strongly-interacting spinless fermions in 1+1 dimensions (see Voit (1995) for a review of the bosonisation of spin-half fermions). The former theory describes a fermionic system filled up to the Fermi level k_F, where the dispersion is linear. This was solved exactly by Mattis and Lieb (1965) who showed that the 1D model's behaviour is given in terms of a bosonic field representing excitations about the Fermi level.

3.1.1 Luttinger Liquid

The Luttinger model is resolved in terms of left- and right-moving $\Psi = (\psi_+, \psi_-)$ fermions on each branch of the Fermi surface (Luttinger, 1963). The free kinetic

Hamiltonian has equal dispersion velocities, as written in (2.57),

$$H_0 = v \int_0^L \mathrm{d}\sigma\, \Psi^\dagger(\sigma)\sigma_z \partial_\sigma \Psi(\sigma) = v \int_0^L \mathrm{d}\sigma \left[\psi_+(\sigma)^* \partial_\sigma \psi_+(\sigma) - \psi_-(\sigma)^* \partial_\sigma \psi_-(\sigma)\right]. \tag{3.1}$$

Moving to momentum space (in a finite system),

$$\psi_\pm(\sigma) = \frac{1}{\sqrt{L}} \sum_k e^{-ik\sigma} a_{\pm,k} \tag{3.2}$$

then the Hamiltonian can be expressed as a momentum-mode sum

$$H_0 = v \sum_k k \left(a^\dagger_{+,k} a_{+,k} - a^\dagger_{-,k} a_{-,k}\right). \tag{3.3}$$

where the chiral currents on each $k > 0$ and $-k < 0$ branch of the Fermi surface are defined as

$$\mathcal{J}_\pm(k) = \sum_p a^\dagger_{\pm,p+k} a_{\pm,k}, \quad \mathcal{J}_\pm(-k) = \sum_p a^\dagger_{\pm,p} a_{\pm,k+p}. \tag{3.4}$$

Now taking the commutator of the free Hamiltonian and the currents gives

$$\left[H_0, \mathcal{J}_\pm(k)\right] = vk\, \mathcal{J}_\pm(k). \tag{3.5}$$

Rewrite the fermionic expression for H_0 in terms of these bosonic currents (Chang, 2003)

$$H_0 = \frac{2v}{L} \sum_{k>0} \left(\mathcal{J}_+(k)\mathcal{J}_+(-k) + \mathcal{J}_-(-k)\mathcal{J}_-(k)\right). \tag{3.6}$$

Now let us clarify the role of these currents in forming the symmetry group of the Luttinger theory. Classically, the model (3.1) has a vector and axial symmetry, generated in the chiral basis by 1 and σ_z respectively. As discussed in Section 2.2.1, the theory separately conserves the vector (2.59) and axial (2.60) charges. In the language of the Luttinger model, these charges are $\mathcal{J}_V = \mathcal{J}_+ + \mathcal{J}_-$ and $\mathcal{J}_A = \mathcal{J}_+ - \mathcal{J}_-$.

We will now show that the chiral currents $\mathcal{J}_\pm(k)$ obey the following bosonic algebra

$$[\mathcal{J}_+(-k), \mathcal{J}_+(k')] = [\mathcal{J}_-(k), \mathcal{J}_-(-k')] = \frac{kL}{2\pi}\delta(k-k'), \tag{3.7}$$

$$[\mathcal{J}_+(k), \mathcal{J}_-(k')] = 0. \tag{3.8}$$

This bosonic algebra — with chiral current operators expressed as sums of pairs of fermionic operators — can construct the full Hilbert space of the Luttinger liquid. The only non-zero commutator is when $k = k'$ in (3.7), which we will now verify explicitly (Schulz, 1995):

$$[\mathcal{J}_+(-k), \mathcal{J}_+(k)] = \sum_p \sum_{p'} \left[a^\dagger_{+,p} a_{+,k+p}, a^\dagger_{+,p+k} a_{+,k}\right] = \sum_p \left(n_{p-k} - n_p\right) \tag{3.9}$$

where the number operator is $n_p = a^\dagger_{+,p} a_{+,p}$. Using the fact that all momentum states below k_F are occupied in the ground state, this sum (3.9) can be evaluated through its vacuum expectation. Regulate this sum with a cutoff function $w(k)$ such that $w(k) = 1$ for $k \leq k_F$ and $w(k) \to 0$ as $k \to \infty$ (Hansson et al., 2017). Then the sum over p converges and the expectation value of Eq. (3.9) in the Dirac vacuum simply counts the number of momentum states between $p - k$ and p:

$$\langle 0| \left[\mathcal{J}_+(-k), \mathcal{J}_+(k) \right] |0\rangle = \frac{Lk}{2\pi}. \tag{3.10}$$

Importantly, the current algebra (3.7) is called the $U(1)$ Kac–Moody algebra, and is a central part of our discussion of the edge theory as a conformal field theory (CFT). This UV-perspective (which explicitly considers a theory of all fermion modes with a given prescribed regulator), shows explicitly how the theory which contains the full *quantum-mechanical* treatment of the Dirac vacuum leads to the anomalous current commutator. Due to the expression for the Hamiltonian in terms of the currents (3.6), the non-trivial commutation of the opposing-chirality currents reveals that the axial charge is not conserved, since $[\mathcal{J}_A, H_0] \neq 0$.

In this sense, the Kac–Moody algebra is said to represent an anomaly of the classical chiral symmetry, and it fundamentally arises because of the Dirac sea vacuum — but indeed this is also what made possible the chiral anomaly in our discussion of the Dirac sea with an applied electric field. Recall in this prior example (Section. 2.2.2) the two branches of the Dirac sea could exchange charge when all momenta changed by $\Delta p = eE\Delta t$, only because each branch had an infinite number of negative energy states.

We have now successfully shown that the fundamental excitations of this fermionic theory are bosonic and rewritten H_0 in this basis. Calling on work by Overhauser (1965) we can state that these bosons construct a complete basis of eigenstates, and hence the Hilbert spaces of the fermionic and bosonic theories are fully equivalent.

An important additional analysis is that of the effect of interactions: consider a inter-particle potential $V(\sigma)$ which scatters left- and right-moving fermions:

$$H_I = 2\lambda \iint_0^L \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \psi_+(\sigma)^\dagger \psi_+(\sigma) V(\sigma - \sigma') \psi_-(\sigma')^\dagger \psi_-(\sigma'). \tag{3.11}$$

The interaction term can be added more simply in momentum-space, noting that V_k is the Fourier transform of the 2-particle potential

$$H_I = 2\lambda \sum_k V_k\, \mathcal{J}_+(k) \mathcal{J}_-(-k). \tag{3.12}$$

Combining these, and ignoring the $k = 0$ modes, we get

$$H = H_0 + H_I = \frac{2}{L} \sum_{k>0} \left[v \left(\mathcal{J}_+(k)\mathcal{J}_+(-k) + \mathcal{J}_-(-k)\mathcal{J}_-(k) \right) + \lambda V_k\, \mathcal{J}_+(k)\mathcal{J}_-(-k) \right]. \tag{3.13}$$

which have a linear dispersion about the Fermi point. A full treatment of the $k = 0$ zero-mode is done by (Haldane, 1979, 1981), but here we follow this derivation while setting $V_0 = 0$.

Appendix D explicitly diagonalises the interacting Hamiltonian (3.6) in this 'UV perspective' (meaning the perspective which considers all momentum modes of H and is regulated with a cutoff). The important result which comes from this is that the chiral currents can be represented by a scalar field

$$\mathcal{J}_\pm(\sigma) = \frac{1}{2\pi}\partial_\sigma \varphi_\pm(\sigma). \tag{3.14}$$

3.1.2 Schwinger Terms

We will now develop a more formal field theory method for quantising these fermionic theories which will show that the anomalous commutator is a more general feature of continuum QFTs. These general 'Schwinger terms', which look like non-zero momentum-mode correlators $[\mathcal{J}(p), \mathcal{J}(q)] \sim p\delta(p+q)$ that are proportional to momentum, are the source of the important Kac–Moody algebra in the bosonised excitations. Understanding different representations of this algebra will then allow us to interpret different possible edge excitations.

The Schwinger terms are commutators which are equal to derivatives of the Dirac operator and indicate the presence of an anomaly in the theory (Schwinger, 1959, 1951; Jackiw). In this present case, the current algebra will contain a Schwinger term, which is due to the chiral anomaly. As mentioned, the bosonisation procedure was first investigated by Tomonaga (1950) and later developed by Mattis and Lieb (1965) and Luther and Peschel (1974).

When looking at eigenvalues of the free Luttinger Hamiltonian H_0, given (3.1), the Schwinger terms — and hence the presence of bosonic excitations in representations of the Kac–Moody algebra — arise because the system has a filled Dirac sea (Fradkin, 2013). Consider two different ground states of the theory: the empty vacuum $|0\rangle$, and the filled Dirac sea which is occupied up to the band crossing $|\text{Dirac}\rangle$, shown in Fig. 3.1. The Hamiltonian H_0 normal-ordered with respect to a vacuum $|\text{Dirac}\rangle$ is defined such that the Dirac sea vacuum has zero energy: $:H_0: |\text{Dirac}\rangle = 0$. This permits the decomposition of H_0 into its normal-ordered form and an infinite vacuum energy contribution

$$H_0 = :H_0: + E_F |\text{Dirac}\rangle \langle\text{Dirac}| \tag{3.15}$$

where $H_0 |\text{Dirac}\rangle = E_F |\text{Dirac}\rangle$ is the eigenvalue which depends on the vacuum. The zero of charge is also defined relative to the vacuum, so we expect commutators of current operators will depend on the relative definition we use.

The next step is to develop the current algebra of the Luttinger theory, where we are interested in the charge density and current operators ρ and j. Both have zero eigenvalues in the empty state $|0\rangle$ as they contain no charges and therefore these operators are automatically normal-ordered with respect to this vacuum. To normal-order the operators with respect to the filled Dirac sea will take some careful analysis, and indeed this will be the source of the anomalous Schwinger terms.

These charge and current operators are defined by

$$\rho(x) = :\psi_+^\dagger(x)\psi_+(x) + \psi_-^\dagger(x)\psi_-(x): \tag{3.16}$$

$$j(x) = v\ :\psi_+^\dagger(x)\psi_+(x) - \psi_-^\dagger(x)\psi_-(x):. \tag{3.17}$$

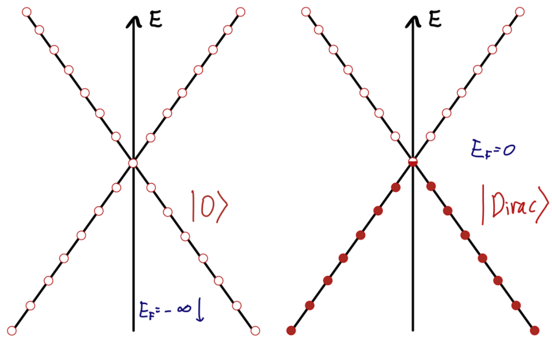

Figure 3.1: Two vacua of the relativistic fermion model: the empty vacuum $|0\rangle$ and the Dirac vacuum filled up to zero energy $|\text{Dirac}\rangle$.

Defining left and right components of the current $\mathcal{J}_\pm(x) = :\psi^\dagger_\pm(x)\psi_\pm(x):$. Therefore we can package into a covariant vector J^μ

$$\begin{pmatrix} J^0(\sigma) \\ J^1(\sigma) \end{pmatrix} = \begin{pmatrix} \rho(\sigma) \\ j(\sigma)/v \end{pmatrix} = \begin{pmatrix} \mathcal{J}_+ + \mathcal{J}_- \\ \mathcal{J}_+ - \mathcal{J}_- \end{pmatrix}. \tag{3.18}$$

This is simpler with a transform into lightcone coordinates $x = (t, \sigma) \to x' = (x^+, x^-)$ defined through

$$x^\pm = vt \pm \sigma. \tag{3.19}$$

In these coordinates, $\partial_\mu J^\mu = 0$ implies $\partial_\pm \mathcal{J}_\pm = 0$. This says that the number of left- and right-movers are separately conserved. Being more careful, the current must be defined through the following point-splitting limit (Affleck, 1986)

$$\mathcal{J}(\sigma)_\pm = \lim_{\epsilon\to 0} :\psi^\dagger_\pm(\sigma-\epsilon)\psi_\pm(\sigma+\epsilon): . \tag{3.20}$$

Now evaluate the current commutator

$$\left[\mathcal{J}(\sigma)_\pm, \mathcal{J}(\sigma')_\pm\right] = \lim_{\epsilon\to 0}\left[:\psi^\dagger_\pm(\sigma-\epsilon)\psi_\pm(\sigma+\epsilon):, :\psi^\dagger_\pm(\sigma'-\epsilon)\psi_\pm(\sigma'+\epsilon):\right] \tag{3.21}$$

$$= \lim_{\epsilon\to 0}\left[\delta(\sigma-\sigma'+\epsilon) - \delta(\sigma'-\sigma+\epsilon)\right]\psi^\dagger_\pm(\sigma-\epsilon)\psi_\pm(\sigma+\epsilon) \tag{3.22}$$

where we used the fermion anticommutator $\{\psi^\dagger_\alpha(\sigma), \psi_{\alpha'}(\sigma')\} = \delta_{\alpha\alpha'}\delta(\sigma-\sigma')$. This expression is non-zero when we normal-order with respect to the filled Dirac sea; split the second term into

$$\psi^\dagger_\pm(\sigma-\epsilon)\psi_\pm(\sigma+\epsilon) = :\psi^\dagger_\pm(\sigma-\epsilon)\psi_\pm(\sigma+\epsilon): + \langle\text{Dirac}|\,\psi^\dagger_\pm(\sigma-\epsilon)\psi_\pm(\sigma+\epsilon)\,|\text{Dirac}\rangle, \tag{3.23}$$

identically in the free theory. When inserted into (3.22), the normal-ordered expectation value vanishes in the $\epsilon \to 0$ limit, and after substituting the momentum space expression (and $p, q \geq 0$ with $\pm$ signs in the exponent to agree with (3.2)) we are left with the following

$$\langle \text{Dirac} | \psi^\dagger_\pm(\sigma - \epsilon)\psi_\pm(\sigma + \epsilon) | \text{Dirac} \rangle = \lim_{\epsilon \to 0} \frac{1}{L} \sum_{p,q \geq 0} \langle \text{Dirac} | e^{\pm ip(\sigma - \epsilon)} a^\dagger_{\pm,p} e^{\mp iq(\sigma+\epsilon)} a_{\pm,q} | \text{Dirac} \rangle \tag{3.24}$$

$$= \lim_{\epsilon \to 0} \frac{1}{L} \sum_{p \geq 0} e^{\mp 2i\epsilon p}. \tag{3.25}$$

Now use $p = 2\pi n/L$ and analytically continue $\epsilon \to \epsilon - i\eta$ to make the geometric series converge

$$\lim_{\epsilon \to 0} \langle \text{Dirac} | \psi^\dagger_\pm(\sigma - \epsilon)\psi_\pm(\sigma + \epsilon) | \text{Dirac} \rangle = \lim_{\eta \to 0^\pm} \frac{1}{L} \sum_{n=0}^{\infty} e^{\mp 2(i\epsilon - \eta)(2\pi n/L)} = \mp \frac{i}{4\pi\epsilon}. \tag{3.26}$$

Combining the above, the current algebra becomes

$$\left[\mathcal{J}(\sigma)_\pm, \mathcal{J}(\sigma')_\pm\right] = \mp \lim_{\epsilon \to 0} \left[\delta(\sigma - \sigma' + \epsilon) - \delta(\sigma' - \sigma + \epsilon)\right] \frac{i}{4\pi\epsilon} \tag{3.27}$$

$$= \mp \frac{i}{2\pi} \partial_\sigma \delta(\sigma - \sigma') \tag{3.28}$$

or in terms of Lorentz components $J(\sigma) = (\rho(\sigma)v, j(\sigma))$, we use (3.18) to get

$$[\mathcal{J}_+, \mathcal{J}_+] = \frac{1}{2}[J^0, J^1] = \frac{1}{2v}[\rho, j] \tag{3.29}$$

and hence

$$[\rho(\sigma), j(\sigma')] = -\frac{iv}{\pi} \partial_\sigma \delta(\sigma - \sigma') \tag{3.30}$$

where other commutators are zero. Hence the equal time density-current commutator becomes non-zero if these currents and densities are normal-ordered with respect to the filled Dirac sea.

The chiral anomaly arises in the Luttinger theory through the same mechanism as in the Dirac model discussed before, where the filled Dirac sea provides a mechanism for the non-conservation of the axial charge (Ambjørn et al., 1983). We have shown how the Schwinger terms arise due to the filled Dirac sea, and indeed the appearance of Schwinger terms in the current algebra is fundamentally due to the presence of the anomaly in the theory. Adam et al. (1993) present a review of the relation between the Schwinger terms and the chiral anomaly in 1+1 dimensions, including a discussion of how the anomalous Schwinger terms come from axial symmetry-violating 'seagull' diagrams (Jackiw; Gross and Jackiw, 1969).

3.1.3 Boson Duality

Now we claim that the above theory is dual to a bosonic theory described by a free scalar boson φ. This process of bosonisation emerges from this Kac–Moody algebra we derived for the 'currents' $\mathcal{J}_{\pm}$. The currents are defined in terms of products of the fermion operators which occupy the system and are therefore bosonic (which should be clear from the form of their commutator above). We will aim to relate the current algebra to the canonical algebra of the free field, given

$$[\varphi(\sigma), \varpi(\sigma')] = i\delta(\sigma - \sigma'). \tag{3.31}$$

To demonstrate the equivalence, let us examine the particle's action

$$S = \int \mathrm{d}^2x \, \frac{1}{2}(\partial\varphi)^2. \tag{3.32}$$

This free theory has a topological current

$$\tilde{J}^{\mu} = \frac{1}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_{\nu}\varphi \tag{3.33}$$

which is conserved $\partial_{\mu}\tilde{J}^{\mu} = 0$ by antisymmetry of ε. Using the expressions (3.18) we can define the current

$$\mathcal{J}_{\pm}(\sigma) = \mp\frac{1}{2\sqrt{\pi}}\partial_{\mp}\varphi = \frac{1}{2}\left[\tilde{J}^0 \pm \tilde{J}^1\right] \tag{3.34}$$

which we now aim to identify with the bosonised current (3.20) of the free fermion theory H_0. Let us demonstrate this in several steps

- Firstly, let us identify the commutators of the boson theory (3.31) with the current Kac–Moody algebra.

  Taking the derivative with respect to σ acts on the first term and gives

  $$[\partial_{\sigma}\varphi(\sigma), \varpi(\sigma')] = i\partial_{\sigma}\delta(\sigma - \sigma'). \tag{3.35}$$

  Identifying these terms with $\tilde{J}^{\mu}$ components using (3.33) and the conjugate momentum $\varpi(\sigma) = \partial_0\varphi$ gives

  $$\pi\left[\tilde{J}^0(\sigma), \tilde{J}^1(\sigma')\right] = -i\partial_{\sigma}\delta(\sigma - \sigma'). \tag{3.36}$$

  Moreover, in terms of $\mathcal{J}_{\pm}(\sigma)$ (by inverting (3.34)) the current algebra reproduces

  $$\left[\mathcal{J}(\sigma)_{\pm}, \mathcal{J}(\sigma')_{\pm}\right] = \mp\frac{i}{2\pi}\partial_{\sigma}\delta(\sigma - \sigma') \tag{3.37}$$

  which is indeed the current algebra (3.28) of the fermionic theory.

- The current is conserved $\partial_{\mu}\tilde{J}^{\mu} = 0$ which implies in lightcone coordinates $\partial_{\pm}\mathcal{J}_{\pm} = 0$ which was the conservation equation for the bosonised H_0 theory.

- There is an additional classically conserved axial current of the fermionic theory $J^{\mu}_A = \overline{\psi}\gamma^{\mu}\gamma^5\psi$. Thus this is dual to the current $\tilde{J}^{\mu}_A = \partial^{\mu}\varphi$ which is conserved on-shell.

- Finally we must identify the stress-energy tensors; the free bosonic action (3.32) has the following Hamiltonian, which can be written in the 'Sugawara form' in terms of chiral currents

$$H = \frac{1}{2}\int d^2x \left[\varpi^2 + (\partial_\sigma \varphi)^2\right] = \frac{\pi}{2}\int d^2x \left[\tilde{J}^0(\sigma)^2 + \tilde{J}^1(\sigma)^2\right] \tag{3.38}$$

$$= \pi \int d^2x \left[\mathcal{J}_+(\sigma)^2 + \mathcal{J}_-(\sigma)^2\right]. \tag{3.39}$$

Now calculating the Hamiltonian of the fermionic theory, we find in lightcone coordinates

$$\mathcal{H} = \frac{1}{2} i : \psi_+^\dagger \partial_+ \psi_+ + \psi_-^\dagger \partial_- \psi_- := \mathcal{T}_+ + \mathcal{T}_-, \tag{3.40}$$

where, using the same point-splitting procedure as before, we may evaluate (Coleman et al., 1969)

$$\mathcal{T}_\pm(\sigma) = \pi \lim_{\epsilon_\pm \to 0} \mathcal{J}_\pm(x_\pm - \epsilon_\pm)\mathcal{J}_\pm(x_\pm + \epsilon_\pm) + \text{const.} \tag{3.41}$$

Thus the Hamiltonian of the fermionic theory (3.1) is in agreement with (3.39) (Haldane, 1981). One can use the same point-splitting method to evaluate the other components of the stress tensor.

A specific treatment of the interacting theory is included in Appendix D, where we find that the only effect of interactions is to renormalise the Fermi velocity.

This procedure has formed a duality between a fermionic theory with a Dirac vacuum and a free bosonic theory. Importantly, this duality preserves all the physical observables of the original theory, and so one can calculate (for example) correlation functions in the free bosonic theory and achieve the correct results for the fermion model. The Dirac theory was the prototypical model of the chiral anomaly to which we have referred several times (Fradkin, 2020a). Now we will reproduce the anomaly in the dual boson theory. First, couple the fermion's charge J^μ to a gauge field

$$\Delta S = J^\mu A_\mu. \tag{3.42}$$

Using the dual current $\tilde{J}^\mu = \frac{1}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_\nu \varphi$ we get the dual action

$$\Delta S = \frac{1}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_\nu \varphi A_\mu \tag{3.43}$$

which modifies the scalar equation of motion to generate the axial anomaly (expressed as non-conservation of the axial current J_A^μ)

$$\partial_\mu j_\mu^5 = \partial^2 \varphi = \frac{1}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_\nu A_\mu. \tag{3.44}$$

Taking the 1+1 dimensional system to have a periodic spatial coordinate θ which ranges from 0 to 2π, we will find a topological argument for the quantisation of charge.

The boson's charge is defined through the zero spacetime component of the dual covariant current

$$\tilde{J}^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu}\partial_\nu\varphi \quad \Longrightarrow \quad \tilde{J} = \begin{pmatrix} \rho \\ j/v \end{pmatrix}. \tag{3.45}$$

The total charge of the field is hence given

$$Q = \int_0^{2\pi} \mathrm{d}\theta\, \rho(t,\theta) = \frac{1}{2\pi}\left[\varphi(2\pi) - \varphi(0)\right] = n \tag{3.46}$$

where n must be an integer because the theory is dual to a fermion theory with an integer number of charges. This implies the periodic boundary conditions for the *compact* boson

$$\varphi(2\pi) - \varphi(0) = 2\pi n. \tag{3.47}$$

The physical observables of a theory with compact bosons are the *vertex operators*, given through the exponentials of the bosonic fields

$$V_n(x) =: e^{in\,\varphi(x)} :, \tag{3.48}$$

where the normal-ordering is needed in the quantum duality. In accordance with the results of Luther and Peschel (1974) (briefly presented in Appendix D), these vertex operators should be identified with the fermions of the original Dirac theory (Mandelstam, 1975). The original works which uncovered this duality in the high-energy physics community showed how the fermion maps onto a 'soliton' in the dual theory: a kink in the field which propagates without changing shape.

One can then construct a 'duality dictionary' which provides a simple form for the fermion mass term $\overline{\psi}\psi \to \frac{1}{2\pi}$ $: \cos(2\sqrt{\pi}\,\varphi) :$ and axial mass term $i\overline{\psi}\gamma^5\psi \to: \sin(2\sqrt{\pi}\,\varphi) :$. The massive Dirac theory therefore maps onto the Sine–Gordon model

$$\overline{\psi}(i\partial\!\!\!/ - m)\psi \to \frac{1}{2}(\partial\varphi)^2 - \frac{m}{2\pi} : \cos(2\sqrt{\pi}\,\varphi) : . \tag{3.49}$$

The duality can be extended to describe an interacting theory, called the massive Thirring (1958) model with interactions given by $(\overline{\psi}\gamma^\mu\psi)^2$, by changing the coefficient of the boson kinetic term (Coleman, 1975).

3.2 Stone Excitations of Edge

Now let us use this bosonisation procedure to describe the chiral excitations on the integer QHE edge. In order to use the bosonisation procedure, we must note that there is a difference between the Luttinger model and the theory of the QHE edge: the edge theory only has one chirality of fermion which maps onto a chiral boson. When studying the edge as a CFT, the nature of the embedding of this chiral boson in a non-chiral theory will be explained, and fortunately the theory happily factorises into two separate chiral sectors.

Now let us develop a theory of the chiral fermions at the QHE edge (Stone, 1991). Consider a QHE sample which is periodic in y, and where the electrons are confined

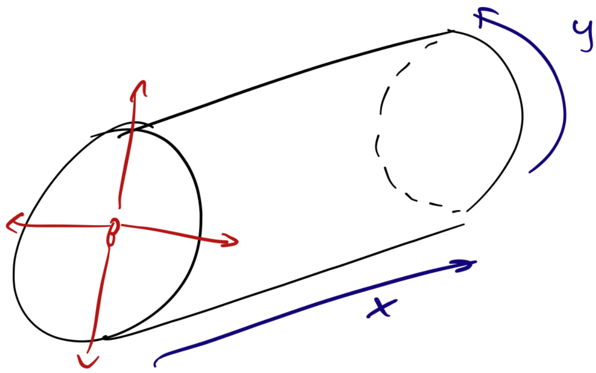

Figure 3.2: The setup of the periodic boundary, and confining x potential. The system has geometry $L_x \times L_y$.

to a strip in the x direction by a potential $V(x)$, shown in Fig. 3.2. In the system without a boundary in x (taking $L_x \to \infty$) in linear gauge, the Hamiltonian becomes degenerate in the quantum number m which represents the position of the wavefunction in the x direction. Adding a confining potential will give the spectrum a quadratic m^2 dependence around the edges; as the Landau level is occupied, first the bulk states will fill and then the states around the boundaries will be filled later. Although this is the typical behaviour of a Fermi surface this phenomenon is in real space — called the quantum Hall droplet.

Consider now filling the inside of the droplet by setting the Fermi level at an appropriate value (Fig. 3.3). We can always approximate the confining potential at the boundary with $V(x) = E_{\text{edge}}x + \cdots$, and hence the electron eigenstates are shifted $\omega(k) = E_{\text{edge}}k/B$ at the droplet surface (with momentum of edge excitations in the periodic y direction). Semi-classically these describe skipping orbits along the edge with Fermi velocity $v = E_{\text{edge}}/B$ (Halperin, 1982). The sign of the velocity depends entirely on the sign of the electric field at that boundary E_{edge}, and necessarily gives opposing edges oppositely pointed velocities.

The excitations about the Fermi level are gapless and have a linear dispersion; they are also chiral, and each chirality is localised at opposite ends of the sample. Now to bosonise this theory we should focus on the model close to the Fermi level, and note its similarity to the Dirac theory: there are two chiral branches which are well approximated by a filled Dirac sea for $E < E_F$. Of course as one looks away from the immediate vicinity of the edge of the droplet, the dispersion will deviate from linearity and the vacuum is a Fermi sea with a finite charge. However we seek a description for only low energies, where the Dirac approximation will suffice.

To quantify the currents carried by the edge excitations, define the charge operator $\mathcal{J}_\Lambda$, which is the charge density $\Psi^\dagger\Psi$ integrated over a window $\Lambda \ll L_x$ of the edge,

$$\mathcal{J}(y) = \int_{-\Lambda}^{\Lambda} \mathrm{d}x\, \Psi^\dagger(x,y)\Psi(x,y). \tag{3.50}$$

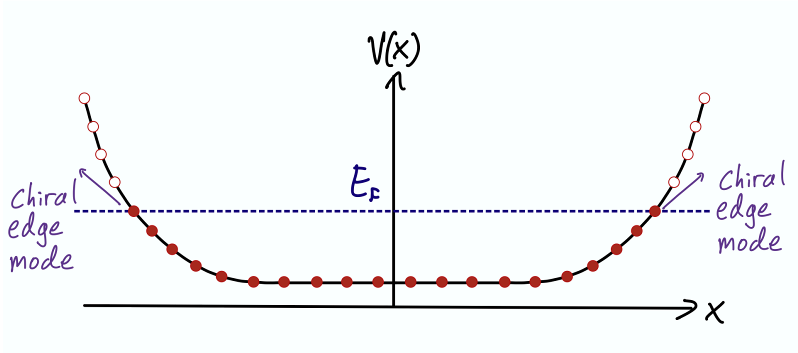

FIGURE 3.3: The quantum number m labels the x position of states, which are filled up to the Fermi level. Excitations are about the edge and have an approximately linear dispersion.

In the linear gauge $A_y = Bx$ the field operator is given

$$\Psi(x, y) = \sqrt{\frac{B}{\pi L_y}} \sum_n a_n e^{ik_n y} e^{-B(x-k_n/B)^2/2} \tag{3.51}$$

where $k_n = 2\pi n/L_y$. Hence in the limit where the cutoff length Λ is much greater than the Gaussian width one can evaluate the current carried by the edge at $x = 0$

$$\mathcal{J}(y) = \sum_n e^{-ik_n y - k_n^2/4B} \sum_m a^\dagger_{m+n} a_m. \tag{3.52}$$

This is wholly reminiscent of the Luttinger charge density $\mathcal{J}_n$ defined in Eq. 3.4, with the addition of a Gaussian factor which localises the charge around the droplet edge. Moreover, there is only one chirality of fermion included in this sum, which suggests that the edge theory can be identified with a *chiral* Luttinger theory and $\mathcal{J}(y)$ with its current $\mathcal{J}_+(y)$. The chiral Luttinger model of the droplet edge has only one chirality of fermion in the Dirac operator $\Psi = \Psi_+ = (\psi_+, 0)$, and the theory therefore has the Hamiltonian

$$H = v \int \mathrm{d}y\, \psi_+(y)(-i\partial_y)\psi_+(y). \tag{3.53}$$

In terms of this chiral boundary fermion, the current is $\mathcal{J}(y) = :\psi^\dagger_+(y)\psi_+(y):$ and so H can be written in the Sugawara form with one chirality

$$H = v\pi \int \mathrm{d}y\, \mathcal{J}(y)^2. \tag{3.54}$$

This theory is well defined with the same Dirac sea vacuum as the Luttinger model, and hence inherits a Schwinger term in its current commutator, as for Eq. (3.27),

$$[\mathcal{J}(y), \mathcal{J}(y')] = -\frac{i}{2\pi}\partial_y \delta(y - y'). \tag{3.55}$$

Now the currents $\mathcal{J}(y)$ can be used to explicitly generate an excitation along the boundary of the quantum Hall fluid. Define the unitary operator

$$U[\theta(y)] = \exp\Big(+i \int \mathrm{d}y\, \theta(y)\, \mathcal{J}(y)\Big), \tag{3.56}$$

which generates an excitation charge profile $\theta(y)$ along the periodic boundary. In the Heisenberg picture, under the isomorphism defined by U, the current transforms as

$$U[\theta(y)]\, \mathcal{J}(y)\, U^{\dagger}[\theta(y)] = \mathcal{J}(y) + \frac{1}{2\pi}\partial_y \theta(y) \tag{3.57}$$

using the anomalous commutator (3.55). The vacuum state of the quantum Hall fluid $|\text{Dirac}\rangle$ has a zero Fermi level and uniform charge profile at the boundary. Add excitations with the profile $\theta(y)$ using $U[\theta(y)]$ to give the excited state

$$|\theta(y)\rangle = U[\theta(y)]\, |\text{Dirac}\rangle \tag{3.58}$$

which has current profile

$$\mathcal{J}(y)\, |\theta(y)\rangle = \frac{1}{2\pi}\partial_y \theta(y)\, |\text{Dirac}\rangle\,. \tag{3.59}$$

Note that we used the fact that the current density is normal-ordered with respect to $|\text{Dirac}\rangle$.

Physically, these waves correspond to ripples on the surface of the quantum Hall fluid. The change in charge density at the left/right ($\pm$) edge is $\delta n = \partial_y \theta(y)/2\pi$ which is due to a change in occupation of the real-space 'band' above the Fermi level. The Fermi level correspondingly shifts up by the amount $\delta E_F = \pm v \partial_y \theta(y)/L_y$. Because of the linear dispersion, an increase in the Fermi level causes a corresponding shift in the surface position like $\sim \partial_y \theta(y)$.

Due to the linear dispersion around the Fermi point, we expect all low-energy excitations to propagate with the same group velocity. This can be formalised quantum-mechanically by time-evolving the Heisenberg-picture current operator $U[\theta(y)] = \exp[G(\theta)]$

$$e^{-iHt} e^{G(\theta)} e^{-iHt} = \exp\big(e^{-iHt}\, G(\theta)\, e^{+iHt}\big) = \exp(G(\theta) + it[G(\theta), H] + \cdots) \tag{3.60}$$

using the fact that e^{-iHt} is unitary. Now evaluate the commutator $[G, H]$ with H given (3.54)

$$[G(\theta), H] = \pi i\, v \int \mathrm{d}y\, \mathrm{d}y' \left[\theta(y)\mathcal{J}(y), \mathcal{J}(y')^2\right] = \pi i\, v \int \mathrm{d}y\, \mathrm{d}y'\, 2\mathcal{J}(y')\, \theta(y)\, [\mathcal{J}(y), \mathcal{J}(y')] \tag{3.61}$$

and using the commutator (3.55) and then integrating by parts gives

$$[G(\theta), H] = v \int \mathrm{d}y\, \mathrm{d}y'\, t(y)\mathcal{J}(y)\partial_y \delta(y - y') \; = -v \int \mathrm{d}y\, \partial_y \theta(y). \tag{3.62}$$

Resumming the Taylor expansion shows the wave-like time evolution

$$e^{-iHt} U[\theta(y)] e^{-iHt} = \exp\Big[i \int \mathrm{d}y\, \big(\theta(y) - tv\partial_y \theta(y) + \cdots\big)\, \mathcal{J}(y)\Big] = U[\theta(y - vt)] \tag{3.63}$$

to leading order. Note tha this may be performed to all orders using the BCH identity. As expected, this describes the rigid evolution of the density fluctuation profile, as in Fig. 3.4.

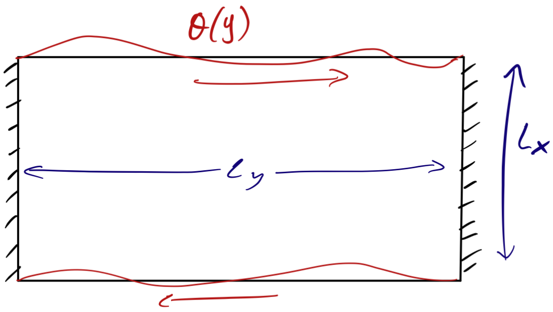

Figure 3.4: Propagation of edge mode profile $\theta(y)$ along the y coordinate at the edges.

3.3 FQHE Boundary Excitations

3.3.1 FQHE Boundary Anomaly

We will now consider the boundary theory of the FQHE; in a similar story to its integer counterpart, the fractional CS action has a gauge anomaly when placed on a compact manifold with boundaries which leads to a gapless edge excitation (Laughlin, 1981; Halperin, 1982). However a difference in the calculation arises due to the presence of a dynamical gauge field which will have an interesting interaction with the edge modes.

Previously we use the fact that the gauge anomaly generally seeks a cancellation — which may be provided by the axial anomaly of a chiral fermion — to derive the physics of the quantum Hall edge. This is indeed applicable to the fractional model, but differs from the historical route that the literature took. In this section we will reproduce this result by instead fixing the gauge degrees of freedom on the edge and showing that there emerge physical degrees of freedom which describe the same chiral fermion. Different ways of dealing with the bulk's gauge theory reliably yield the same physical picture. This new method will provide us with a motivating bosonised description which is appealing for the later application of conformal field theory techniques.

Following the methods of Hansson and Viefers (2000), we will construct the edge theory by requiring that the anomaly of the edge theory must cancel the gauge-noninvariant terms at the boundary. The description of the edge mode will lead to a profound understanding of this edge mode, and indeed many general arguments can be made which specify its gapless nature (Wen, 1990b, 1991, 1995). Once again, there will emerge a chiral fermion at the edge of the quantum Hall droplet which we can bosonise to get a similar Luttinger liquid description.

Recall that the FQHE action (2.86) is controlled by the level-m CS term

$$\frac{m}{4\pi}\int_{\mathcal{M}} a \wedge \mathrm{d}a \,. \tag{3.64}$$

Using the general results of Section 2.1.2, the gauge anomaly of the dynamical statistical gauge field is given by (Elitzur et al., 1989)

$$\delta S_{\text{anom}} = \frac{m}{4\pi}\int_{\partial\mathcal{M}} a \wedge \delta a \tag{3.65}$$

Take δa to be a gauge transform $\delta a = \mathrm{d}\lambda$. Then place the system on the manifold $\mathcal{M} = \text{Disk} \times R$, and using integration by parts and Stokes' theorem we get an integral over its boundary $S^1 \times R$,

$$\delta S_{\text{anom}} = -\frac{m}{4\pi}\int_{S^1\times R} \lambda \, \mathrm{d}a = -\frac{m}{4\pi}\int_{S^1\times R} \lambda(t,\theta)\, \varepsilon^{r\nu\rho}\partial_\nu a_\rho \, \mathrm{d}\theta \wedge \mathrm{d}t\,. \tag{3.66}$$

To get $\delta S_{\text{anom}} = 0$ we must take the gauge transformations to be zero $\lambda(t,\theta) = 0$ on the boundary, which causes dynamical gauge degrees of freedom on the boundary to become physical.

The dynamics of these degrees of freedom are found by taking the temporal gauge $a_t = 0$ and $a = a_i\,\mathrm{d}x^i$, and applying it to the gauge field in the bulk. The bulk action then becomes

$$S = \frac{m}{4\pi}\int a \wedge \mathrm{d}a = \frac{m}{4\pi}\int a_j\, \partial_t a_i \,\mathrm{d}t \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j\,. \tag{3.67}$$

The equation of motion $\mathrm{d}a = 0$ (a is closed) in this gauge implies we can write $a = \mathrm{d}\varphi$ (a is exact) globally, and so the (non-zero) spatial components are given $a_i = \partial_i\varphi$. There is a boundary at $r = 1$ and so there is a finite boundary action which comes from its integration by parts

$$S = \frac{m}{4\pi}\int_{r=1} \partial_t\varphi\, \partial_\theta\varphi \,\mathrm{d}t \wedge \mathrm{d}\theta\,. \tag{3.68}$$

This is the action of a free scalar field which evolves on the circular boundary of the disk. To recap: the condition that the gauge transforms are zero on the boundary means the statistical gauge field becomes physical here. Next, the gauge fixing allowed us to represent the physical degrees of freedom of this field with a scalar φ. We have now shown that the only contribution of a to the action in this gauge is through a boundary action which describes a 1+1 dimensional scalar φ. We must next use a clever trick which involves changing coordinates to find the behaviour of the field φ and therefore the gauge field a at the boundary.

A generalisation of this temporal gauge choice, which will be of particular use for discussing boundary dynamics, is the axial gauge condition

$$a_t - v a_\theta = 0. \tag{3.69}$$

Now perform a coordinate transform

$$(t', \theta', y') = (t, \theta - vt, r) \tag{3.70}$$

such that the original temporal gauge $a_t = 0$ implies the axial gauge (3.69) in the new primed coordinates $a_{t'} - v a_{\theta'} = 0$. This next step exploits the fact that the CS action is gauge invariant — thus the temporal gauge in one set of coordinates implies the axial gauge in another set of coordinates. Use this to transform the action (3.68) written in terms of a φ degree of freedom. This holds in the new coordinates, so dropping the primes we may write

$$S_{\mathrm{CS}}[a] = \frac{m}{4\pi} \int_{r=1} (\partial_t + v\partial_\theta)\varphi\, \partial_\theta \varphi\, \mathrm{d}t \wedge \mathrm{d}\theta\,. \tag{3.71}$$

This is the Floreanini–Jackiw action of a free scalar field in 1+1 dimensions, and its quantisation is a subtle process due to the first-order nature of the action in time derivatives (Floreanini and Jackiw, 1987). Defining a new boundary field $\rho = \partial_\theta \varphi / 2\pi$ we get the equation of motion

$$\partial_t \rho + v\partial_\theta \rho = 0 \tag{3.72}$$

which has chiral wavelike solutions $\rho = \rho(\theta - vt)$. This therefore presents our bosonised excitation of the fractional QH edge: the field ρ is a completely chiral scalar which represents the excitations of the underlying fermionic electrons which cause the emergent statistical gauge field.

This result holds even in the presence of interactions, which we have ignored thus far; impurities or electron-electron interactions which could cause backscattering are forbidden, as only one chirality of edge mode is supported by the theory in the magnetic field and therefore there are no opposite-chirality states for the backscattered scalars to move to. Forward-scattering is permitted, but as was seen for the Luttinger liquid, their only effect is to renormalise the Fermi velocity of the chiral modes (Fisher and Glazman, 1996).

The physical gauge field enters the fractional CS action (2.86) as $A \wedge \star J$ where $\star J = \mathrm{d}a/2\pi$. Choose the physical gauge where $A = A(t, \theta)$ and $A_r = 0$ (recall $\mathrm{d}A = 0$ on shell)

$$\int_{\mathcal{M}} A \wedge \star J = \frac{1}{2\pi} \int_{\mathcal{M}} A \wedge \mathrm{d}a = \frac{1}{2\pi} \int_{\mathcal{M}} \mathrm{d}\,(A \wedge a) = \frac{1}{2\pi} \int_{S^1 \times R} A \wedge a \tag{3.73}$$

then using $a_t = 0$ and $a = \mathrm{d}\varphi$

$$\int_{\mathcal{M}} A \wedge \star J = \frac{1}{2\pi} \int_{r=1} \varepsilon^{r\nu\rho} A_\nu a_\rho\, \mathrm{d}\theta \wedge \mathrm{d}t = \frac{1}{2\pi} \int_{r=1} A_t\, a_\theta\, \mathrm{d}\theta \wedge \mathrm{d}t\,. \tag{3.74}$$

Again generalise this by transforming coordinates and combine with (3.71) to get

$$S_{\mathrm{edge}}[\varphi] = \int_{r=1} \left[\frac{m}{4\pi} (\partial_t + v\partial_\theta)\varphi + \frac{1}{2\pi} (A_t - vA_\theta) \right] \partial_\theta \varphi\, \mathrm{d}\theta \wedge \mathrm{d}t\,. \tag{3.75}$$

Its edge current is defined through

$$\rho(\theta, t) = \frac{\delta S_{\mathrm{edge}}}{\delta A_t} = \frac{1}{2\pi} \partial_\theta \varphi, \quad j(\theta, t) = \frac{\delta S_{\mathrm{edge}}}{\delta A_\theta} = -\frac{v}{2\pi} \partial_\theta \varphi, \tag{3.76}$$

which confirms that $\rho(\theta, t)$ is the boundary EM charge density, and $j(\theta, t) = -v\rho(\theta, t)$ is its edge current which is clearly chiral. The equations of motion for the action (3.75) are

$$(\partial_t + v\partial_\sigma)\rho(\sigma) = -\frac{m}{2\pi} E. \tag{3.77}$$

When $m = 1$ this reproduces the chiral anomaly derived from the domain-wall fermion model of the IQHE, but this result extends the previous calculation and is valid for all Laughlin states with odd-integer m. We have now shown, following Wen (1991), that the gauge anomaly of the fractional QHE at the boundary describes a theory of a chiral fermion. As in the integer case, this anomaly must be cancelled by a physical chiral fermion residing on the boundary of the sample.

We reiterate that even in the fractional QHE when the gauge field is dynamical, the appearance of a chiral fermion at the quantum Hall edge is totally general and is a result of the chiral anomaly. Cappelli et al. (1992) even derive a conformal chiral fermion action directly from a simple model of electrons in Landau levels — without using a Chern–Simons effective theory. This process makes a direct connection with the underlying physics of the QHE. Bosonising this model gives the standard Floreanini–Jackiw action.

Because this theory is defined in terms of a periodic coordinate, using the same argument as before, we can prove the compact-nature of the boson, and the resulting quantisation of charge (3.46).

3.3.2 Hydrodynamic Theory of Edge States

Let us now attempt to quantise the boundary theory (3.75) with zero sources $A_t = A_\theta = 0$. For the Laughlin states recall that the conductivity is $\nu = \frac{1}{m}$, so first substitute m in the action and next rescale the disk to have circumference L and use coordinate $\sigma = \theta L$:

$$S_{\text{edge}}[\varphi] = \frac{1}{4\pi\nu} \int \mathrm{d}\sigma\, \mathrm{d}t\, (\partial_t + v\partial_\sigma)\varphi\, \partial_\sigma\varphi \tag{3.78}$$

$$= \alpha \int \mathrm{d}\sigma\, \mathrm{d}t\, \varphi' \dot{\varphi} + v\varphi'^2 \tag{3.79}$$

with $\alpha = (4\pi\nu)^{-1}$. This theory is difficult to quantise because it is first order in time derivatives — here we follow the method developed by Faddeev and Jackiw (1988). In such systems the 'conjugate momentum' $\varpi = \partial\mathcal{L}/\partial\varphi$ is not a physical momentum, and indeed for the Floreanini–Jackiw this is dependant only on *spatial* derivatives of the field $\varpi = \alpha\varphi'$. The Hamiltonian of this theory takes the form of a potential (which is independent of time derivatives)

$$\mathcal{H} = \varpi\dot{\varphi} - \mathcal{L} = v\alpha\varphi'^2. \tag{3.80}$$

Hence this Lagrangian can be written in terms of a combined variable $\xi = (\varphi, \varpi) = (\varphi, \alpha\varphi')$

$$\mathcal{L} = \alpha\varphi'\dot{\varphi} + v\alpha\varphi'^2 = \frac{1}{2}\xi_i\, \varepsilon_{ij}\, \dot{\xi}_j + V(\varphi) \tag{3.81}$$

where the potential V is written explicitly. The canonical commutator is therefore given by

$$[\xi_i(\sigma), \xi_j(\sigma')] = -\frac{i}{2}\varepsilon_{ij}\delta(\sigma - \sigma') \tag{3.82}$$

and hence

$$[\varphi(\sigma), \partial_{\sigma'}\varphi(\sigma')] = -(2\pi\nu)i\delta(\sigma - \sigma'). \tag{3.83}$$

In terms of the field $\rho(\sigma) = \partial_\sigma\varphi(\sigma)/2\pi$ we get the following commutators through differentiating and integrating

$$[\varphi(\sigma), \varphi(\sigma')] = \pi i\nu\,\mathrm{sign}(\sigma - \sigma') \tag{3.84}$$

$$[\varphi(\sigma), \rho(\sigma')] = -i\nu\delta(\sigma - \sigma') \tag{3.85}$$

$$[\rho(\sigma), \rho(\sigma')] = -\frac{i\nu}{2\pi}\partial_\sigma\delta(\sigma - \sigma'). \tag{3.86}$$

In fact, in Floreanini and Jackiw (1987), Eq. (3.86) is derived directly from an action written in terms of ρ. This is the important Kac–Moody algebra which naturally emerges when we quantise the bosonised form of the chiral Luttinger theory (Wen, 1990b).

Now moving to momentum space

$$\rho(x) = \frac{1}{\sqrt{L}}\sum_n e^{2i\pi n/L}\rho_n \tag{3.87}$$

we can express the Hamiltonian (3.80) as

$$H = \frac{\pi v}{\nu}\int \mathrm{d}\sigma = \frac{\pi v}{\nu}\sum_n \rho_n\rho_{-n} = \frac{2\pi v}{\nu}\sum_{k>0}\rho_k\rho_{-k}. \tag{3.88}$$

In this language the ρ-ρ commutator (3.86) makes explicit the Kac–Moody structure (Wen, 1995)

$$[\rho_p, \rho_q] = \frac{\nu}{2\pi}p\,\delta_{p+q} \tag{3.89}$$

and

$$[H, \rho_p] = v\rho_p. \tag{3.90}$$

Even though it is not explicit in this derivation, this result is indeed dependent on the Schwinger term from the quantised Dirac sea; this is because the bosonised description only exists as a consequence of the stable ground state containing infinite particles. The duality we derived in Section 3.1.3 is an important result which undergirds this line of thinking (the bosonic theory has a different action because the Luttinger liquid has two chiralities). This result can be obtained by counting states inside the Fermi surface in the UV picture, as was done in Section 3.1.1. The appendix of Hansson et al. (2017) contains a comparison of the UV method with our method wich uses an effective IR description of bosonised currents about the vacuum. Edge operators living in the Kac–Moody algebra (which only has one chiral species) are a general feature of FQH states. Because the bulk system is described by a Chern–Simons theory with a gauge anomaly, the edge system has gapless excitations which are representations of the Kac–Moody algebra (Wen, 1992).

The bosonised field has a logarithmic correlator $\langle \varphi(\sigma, t)\varphi(0,0)\rangle \sim \log(\sigma - vt)$ so that it satisfies the equation of motion (integral of Eq. 3.72)

$$(\partial_t + v\partial_\sigma)\langle \varphi(\sigma, t)\varphi(0,0)\rangle = 0. \tag{3.91}$$

Using Wick's theorem in the quantum theory the correlator is

$$\langle \varphi(\sigma, t)\varphi(0,0)\rangle = -\nu \log(\sigma - vt), \tag{3.92}$$

which is a correlator of the chiral compact boson. The normalisation of ν arises the propagator is the inverse of the kinetic operator in the action (3.79), and this can be traced through the normalisation of the Kac–Moody algebra too.

3.3.3 Charge Excitations

The excitations ρ correspond to ripples — or *phonons* — on the surface of the quantum Hall droplet. We can introduce a different form of charged excitation which corresponds to adding an electron to the edge.

Represent the edge-electron by an operator $\Psi^\dagger_{\text{elec}}$; this must have a localised charge equal to 1, which is calculated from the commutator with the charge density $J^0(\sigma) = \rho(\sigma)$

$$[\rho(\sigma), \Psi^\dagger_{\text{elec}}(\sigma')] = \Psi^\dagger_{\text{elec}}(\sigma)\delta(\sigma - \sigma'). \tag{3.93}$$

We can work in a representation of this operator in the chiral boson theory; introduce a family of operators labelled γ,

$$\Psi_\gamma(\sigma) = :\exp(i\gamma\varphi(\sigma)/\sqrt{\nu}): = :\exp(i\gamma\phi(\sigma)):, \tag{3.94}$$

where we defined the renormalised field

$$\phi(\sigma) = \varphi(\sigma)/\sqrt{\nu}. \tag{3.95}$$

In this representation, the ν factor in the commutators (3.84–3.86) are absorbed into the field definitions, and the correlator (3.92) is

$$\langle \phi(\sigma, t)\phi(0,0)\rangle = -\log(\sigma - vt). \tag{3.96}$$

Furthermore, the current operator changes normalisation $J(\sigma) = (\sqrt{\nu}/2\pi)\partial_\sigma\phi(\sigma)$.

These electrons Ψ_{elec} must anticommute

$$\{\Psi_{\text{elec}}(\sigma), \Psi_{\text{elec}}(\sigma')\} = 0; \tag{3.97}$$

using the BCH formula we see

$$:e^A::e^B: = e^{[A,B]}:e^B::e^A: \tag{3.98}$$

(which holds when the commutator $[A, B]$ commutes with A, B). Consider now two particles with general label γ; exchanging two of them gives the following phase

$$\Psi_\gamma(\sigma)\Psi_\gamma(\sigma') = e^{-\gamma^2[\phi(\sigma),\phi(\sigma')]}\Psi_\gamma(\sigma')\Psi_\gamma(\sigma) = e^{\pm i\pi\gamma^2}\Psi_\gamma(\sigma')\Psi_\gamma(\sigma), \tag{3.99}$$

where we used the commutator (3.84) in terms of the new $\phi(\sigma)$ field. This expression is fermionic if $\gamma^2 = m$ is an odd integer. Now explicitly we can evaluate the physical charge of the fermion field Ψ_γ using the chain-rule property of the brackets and (3.85)

$$[\sqrt{\nu}\partial_\sigma\phi(\sigma), \Psi^\dagger_\gamma(\sigma')] = \sqrt{\nu}[\partial_\sigma\phi(\sigma), i\gamma\phi(\sigma)] :e^{i\gamma\phi(\sigma)}: \tag{3.100}$$

$$= \sqrt{\nu}\gamma\,\Psi^\dagger_\gamma(\sigma)\delta(\sigma - \sigma'). \tag{3.101}$$

Requiring this is compatible with the definition (3.94) implies that $\nu\gamma^2 = 1$ and together with the constraint from the exchange phase calculation we get $\nu = m^{-1}$. Remarkably, this is the Laughlin condition for the filling fraction, and so we find that Laughlin edge theories necessarily have electron-like particles in their edge spectrum! For other filling fractions (m not necessarily an odd integer) this particle can still be present, but the edge description is complicated by many branches of gapless bosonic excitations at the edge (Wen, 1990b). Explicitly in this representation

$$\Psi_{\text{elec}}(\sigma) = \Psi_{\sqrt{m}}(\sigma) = :e^{i\sqrt{m}\phi(\sigma)}: . \tag{3.102}$$

From the logarithmic form of the bosonic charge correlator (3.96) and the definition of the electrons (3.102) we can evaluate the electron-electron correlator on the boundary

$$G_{\text{elec}}(\sigma, t) = \left\langle T\Psi^\dagger_{\text{elec}}(\sigma, t)\Psi_{\text{elec}}(0,0)\right\rangle = \exp[m\left\langle\phi(\sigma,t)\phi(0,0)\right\rangle] = \frac{1}{(\sigma - vt)^m}. \tag{3.103}$$

This physical correlation function of an observable operator is a power-law function; such functions are scale invariant, and associated with a gapless *quantum critical point*. This invariance under scale should motivate the discussion presented in the next chapter, where this scale invariance is seen to be only a part of a larger 'conformal group'.

The fundamental charged excitation of the system is not an electron, but a 'vortex' quasiparticle with fractional $\frac{1}{m}$ charge. We can define such an operator as

$$\Psi_{\text{vor}}(\sigma) = \Psi_1(\sigma) = :e^{i\phi(\sigma)/\sqrt{m}}:, \tag{3.104}$$

and its fractional charge is identified through its commutator

$$[\rho(\sigma), \Psi^\dagger_{\text{vor}}(\sigma')] = \frac{1}{m}\Psi^\dagger_{\text{vor}}(\sigma)\delta(\sigma - \sigma'). \tag{3.105}$$

Their correlation function is

$$G_{\text{vor}}(\sigma, t) = \left\langle T\Psi^\dagger_{\text{vor}}(\sigma, t)\Psi_{\text{vor}}(0,0)\right\rangle = \exp\left[m^{-1}\left\langle\phi(\sigma,t)\phi(0,0)\right\rangle\right] = \frac{1}{(\sigma - vt)^{1/m}}. \tag{3.106}$$

The quasiparticle has a branch-cut singularity of order $\frac{1}{m}$ and therefore is not single-valued on the σ-t surface. Under a monodromy, or a loop of one particle around the other in this space, the state changes by a non-trivial phase. Repeating the exchange-phase calculation using the BCH identity (3.98) shows the vortices have statistical phase $\delta = \pi/m$.

As we saw in Section 2.4, the fundamental excitations of the fractional quantum Hall system are vortices with fractional $\frac{1}{m}$ charge and non-trivial braiding statistics. These could be produced or destroyed only when a single electron splits or fuses into groups of m bulk anyons. The fractionally-charged edge excitations are a non-trivial superposition of the bulk anyons, which due to the gapless nature of the edge may even be massless.

Bulk-Boundary Correspondence 4

4.1 Conformal Field Theory

In this thesis thus far we have investigated the bulk and boundary theories of the integer and fractional quantum Hall effects. The bulk was described by a topological gauge theory — the Chern–Simons action — which has a gauge anomaly at the boundary. We showed that this anomaly necessarily gives rise to a chiral fermion at the edge, which was then bosonised in order to quantise.

Now we will aim to bridge the gap between the degrees of freedom of these two theories: we will relate the fermionic and anyonic excitations in the bulk and the boundary in this chapter by forming a *correspondence* between the two quantum theories. In a fantastic result, the two theories will be shown to be different representations of the same conformal field theory (simply using different Lorentzian/Euclidian signatures). This will connect with the work of Witten (1989) which draws a more complex duality between the bulk and boundary in *non-Abelian* Chern–Simons theories, which represent a type of quantum Hall state that we have not met yet with non-commutative anyon braiding properties.

4.1.1 Conformal Group in d Dimensions

CFTs are quantum field theories with additional 'conformal' symmetries. We will now introduce the fundamentals of conformal symmetry, and then apply it to 1+1 dimensional systems, in order to apply it to the quantum Hall edge.

The general d-dimensional local Weyl group is defined as the group of transformations which only vary the scale of the metric,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = e^{2\sigma(x)} g_{\mu\nu}(x). \tag{4.1}$$

The conformal group is then the group of *coordinate transformations* $x \to x'$ which rescale the metric as in (4.1), a restriction which reduces the group of allowed scaling functions $\sigma(x)$.

Clearly the Poincarè group $SO(1, d-1)$ is a subgroup which leaves $g_{\mu\nu}(x)$ invariant (Di Francesco et al., 1997; Wiseman, 2020). The additional generators in the conformal group are the dilatation D and the special conformal transform K_μ. Firstly, the dilatation acts to rescale the coordinates

$$x^\mu \to \lambda x^\mu, \tag{4.2}$$

and functions rescale as

$$Df(x) = ix^\mu \partial_\mu f(x), \tag{4.3}$$

and obeys

$$[M_{\alpha\beta}, D] = 0, \quad [P_\mu, D] = iP_\mu, \tag{4.4}$$

where $M_{\alpha\beta}$ and $P_\mu = i\partial_\mu$ generate Lorentz transforms and translations, respectively. However fields which are called 'quasi-primary' scale with the dilatation as

$$\Phi(x) \to \Phi'(x) = \lambda^\Delta \Phi(\lambda x), \tag{4.5}$$

where Δ is the field's scaling dimension. The generator on the field therefore satisfies

$$[D, \Phi(x)] = i(\Delta + x^\mu \partial_\mu)\Phi(x). \tag{4.6}$$

The special conformal transform is defined through

$$K_\mu f(x) = i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) f(x), \tag{4.7}$$

and together the commutators of all generators show that the conformal group is isomorphic to $SO(2, N)$.

Focusing on a subset of this algebra, we can identify how the K_μ, P_μ fields act as raising and lowering operators of with respect to D:

$$[D, K_\mu] = +iK_\mu, \quad [D, P_\mu] = -iP_\mu. \tag{4.8}$$

This action increases or decreases the scaling dimension Δ by 1; the lowest-weight state of D is known as the primary operator, and its descendents are generated with P_μ. These descendents are made up of derivatives of the operator, and so if $\Psi(0)$ is a primary operator then due to its Taylor expansion, the whole field is a primary field.

The scaling of a correlation function of N primary fields is entirely fixed by the conformal symmetry. Under dilatations this correlator transforms as

$$\langle \Phi_1(x_1) \cdots \Phi_N(x_N) \rangle \to \lambda^{\Delta_1 + \cdots + \Delta_N} \langle \Phi_1(\lambda x_1) \cdots \Phi_N(\lambda x_N) \rangle. \tag{4.9}$$

Poincaré invariance means the two-point correlator is only a function of $x - y$, and so when the fields have the same scaling dimension the scaling under conformal dilatation allows us to fix the functional form to be

$$\langle \Phi(x)\Phi(y) \rangle = \frac{c}{(x-y)^{2\Delta}}. \tag{4.10}$$

Furthermore, it is important to note that the transformation under the special conformal symmetry implies that the correlator is zero if the scaling dimensions differ (Polyakov, 1974). Consequently, correlation functions of primary fields with different scaling dimensions are zero.

4.1.2 Conformal Group in 2 Dimensions

Field theories in two dimensions have the special property that the coordinate $x = (x_0, x_1)$ can be written as a complex coordinate z, with the real and imaginary degrees of freedom representing the two dimensions.

These theories are special because their operators can be written as functions over the complex plane, but because there exists an infinite number of transformations which map the complex plane onto itself, we must therefore be careful when defining which transformations define the conformal group (Di Francesco et al., 1997). We will now show that the group of conformal transformations in $d = 2$ is the group of functions which are holomorphic in the complex representation of the coordinates.

Under coordinate transformations $x^\mu \to x^\mu + e^\mu$, the (Minkowski) metric changes as

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + \partial_\mu e_\nu + \partial_\nu e_\mu. \tag{4.11}$$

The conformal group means the metric must only differ by a scale $\delta\eta_{\mu\nu} = 2\sigma\eta_{\mu\nu}$ and so (Belavin et al., 1984; Ginsparg, 1988)

$$\partial_\mu e_\nu + \partial_\nu e_\mu = \frac{d}{2}\partial_\rho e^\rho \eta_{\mu\nu} \tag{4.12}$$

which, where $d = 2$, is equivalent to the Cauchy–Riemann equations

$$\partial_0 e^1 = -\partial_1 e^0, \quad \partial_0 e^0 = +\partial_1 e^1. \tag{4.13}$$

These are solved with complex holomorphic and antiholomorphic transformations

$$e(z) = e^0 + ie^1 \tag{4.14}$$
$$\bar{e}(\bar{z}) = e^0 - ie^1, \tag{4.15}$$

in terms of the corresponding complex coordinates $z = \sigma + it$ and $\bar{z} = \sigma - it$. Note that in performing this definition we have Wick rotated into Euclidian signature. The complex coordinates therefore transform as $z \to z + e$ and $\bar{z} \to \bar{z} + \bar{e}$ under infinitesimal diffeomorphisms. Conformal transformations are therefore just holomorphic transformations on the complex plane $z \to z'(z) = z + e(z)$ (and its complex conjugate equation). This is the group that preserves angles on the complex plane.

In our chiral field theory these will naturally be represented by right- and left-moving variables (like lightcone coordinates under a Wick rotation). General gapless $1+1$-dimensional models have a conformal symmetry, and the two chiralities decouple. Let us focus on the chiral (or holomorphic) states, as these are relevant for the bosonised FQHE theory we have developed.

The generators of these transformations on the complex plane are classically

$$\ell_n = -z^{n+1}\partial_z, \tag{4.16}$$
$$\bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{4.17}$$

These generators live in the Witt algebra

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m} \tag{4.18}$$
$$[\bar{\ell}_n, \bar{\ell}_m] = (n-m)\bar{\ell}_{n+m} \tag{4.19}$$
$$[\ell_n, \bar{\ell}_m] = 0, \tag{4.20}$$

and form irreducible representations of the conformal algebra labelled by integer n. This infinite dimensional group is larger than the conformal group we aimed to describe; the global conformal group is generated by $\ell_0, \ell_1, \ell_{-1}$. Explicitly, $\ell_{-1} = -\partial_z$ generates translations and $\ell_1 = -z^2\partial_z$ generates special conformal transformations. The combinations $\ell_0 + \bar{\ell}_0$ and $i(\ell_0 - \bar{\ell}_0)$ generate dilatations and rotations, respectively.

These two (anti)holomorphic algebras are independent (in the sense that they have a trivial cross-commutator), and so we can truly consider the conformal action as being dependent on z and $\bar{z}$ seperately. The physical space of solutions is on the 'real sheet' where $\bar{z} = z^*$.

Given a field with scaling dimension Δ and spin s we can define the (anti)holomorphic conformal weight (or conformal dimension) h ($\bar{h}$) as

$$h = \frac{1}{2}(\Delta + s), \quad \bar{h} = \frac{1}{2}(\Delta - s). \tag{4.21}$$

These weights are the eigenvalues of ℓ_0 and $\bar{\ell}_0$, and so because dilatations and rotations are generated by $\ell_0 + \bar{\ell}_0$ and $i(\ell_0 - \bar{\ell}_0)$ the eigenvalues are conformal dimension $\Delta = h + \bar{h}$ and spin $s = h - \bar{h}$. Under a conformal transform $z' = w(z)$ a quasi-primary field transforms as

$$\Phi'(z', \bar{z}') = \left[\frac{\mathrm{d}w}{\mathrm{d}z}\right]^{-h} \left[\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right]^{-\bar{h}} \Phi(z, \bar{z}). \tag{4.22}$$

For now, assume the holomorphic and antiholomorphic components separate so that we can write the holomorphic primary field is $\Phi(z)$. It therefore transforms only under the variable $z \to w(z)$ as

$$\Phi'(w) = \left[\frac{\mathrm{d}w}{\mathrm{d}z}\right]^{-h} \Phi(z), \tag{4.23}$$

and the equivalent for the anti-holomorphic primary $\overline{\Phi}(\bar{z})$.

4.1.3 Operator Product Expansion

In two dimensions the notion of an operator product expansion (OPE) is a powerful tool for calculating general correlation functions (Năstase, 2015; Cardy, 2008). The correlation function of two fields is singular as their points are moved together, and the scaling of this function can be expressed as a Laurent series. The singular behaviour when field operators are brought together must hold on the operator level too — this is what the OPE describes. It defines the leading divergence as two holomorphic operators approach, defined in terms of the 'conformal data' tensor $C_{ij}^k(z - w)$

$$\lim_{z \to w} \Phi_i(z)\Phi_j(w) = C_{ijk}(z - w)\Phi_k(w). \tag{4.24}$$

When this operator relation is applied to general fields, the right hand side can contain primary and descendent fields, the latter of which contain higher scaling dimensions.

When applied to primary field, the leading contribution to the singularity therefore does not contain descendent fields.

For massless holomorphic theories the form of the OPE simplifies to

$$\lim_{z\to w} \Phi_i(z)\Phi_j(w) = \frac{c_{ijk}}{(z-w)^{h_i+h_j-h_k}}\Phi_k(w). \tag{4.25}$$

Now this recovers the constraint on the two-point function that we found in higher-dimension

$$\left\langle \Phi_i(z_i)\Phi_j(z_j)\right\rangle = \frac{1}{|z_i - z_j|^{2h_i}} \tag{4.26}$$

and gives the more general reduction relation

$$\left\langle \Phi_i(z_i)\Phi_j(z_j)\cdots\right\rangle = \frac{c_{ijk}}{|z_i - z_j|^{2h_i}}\left\langle \Phi_j\left(\frac{z_i+z_j}{2}\right)\cdots\right\rangle \tag{4.27}$$

A general correlation function can be expressed in terms of 'conformal blocks' (Belavin et al., 1984)

$$\langle \Phi_i(z_i,\bar{z}_i)\cdots\rangle = \sum_p \left|\mathcal{F}_p(z_i,\cdots)\right|^2. \tag{4.28}$$

Each block is a product of a holomorphic function $\mathcal{F}_p$ and its antiholomorphic conjugate. This expresses that the general correlation function is a sum of blocks, which are linearly independent functions (labelled p) which span a vector space.

The stress tensor in any field theory is a symmetric divergence-free rank-2 tensor $T_{\mu\nu}$ which generates local coordinate transforms. The dilatation current can be expressed in terms of this tensor $j_\mu = T_{\mu\nu}x^\nu$, and its conservation in a general CFT implies that the stress tensor is traceless $T^\mu{}_\mu = 0$. Indeed this is an important result, for a classically scale-invariant theory has no defining energy scale Λ. For this to be true, the stress tensor cannot define a natural scale through its trace. In this sense, the traceless condition of the stress tensor is an important prerequisite for classical conformal invariance.

This is possible to represent the stress tensor in holomorphic coordinates $(z,\bar{z})$ (Ginsparg, 1988),

$$T_{zz} = \frac{1}{4}\left(T_{00} - 2iT_{01} - T_{11}\right), \tag{4.29}$$

plus $\overline{T}_{zz} = T_{\bar{z}\bar{z}}$ and $T_{z\bar{z}} = 0$. The conservation law simplifies this, and imposes pure (anti)holomorphic dependence:

$$\bar{\partial}T_{zz}(z,\bar{z}) = 0, \quad \partial\overline{T}(z,\bar{z}) = 0. \tag{4.30}$$

Writing this concisely, we will use the notation $T(z) = T_{zz}(z)$ and $\overline{T}(\bar{z}) = T_{\bar{z}\bar{z}}(\bar{z})$.

This allows us to formalise the notion of a primary field: these are operators which have an OPE with the stress-energy tensor which is a double-pole

$$\lim_{z\to w} T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{(z-w)}\partial\Phi(w,\bar{w}) + \cdots. \tag{4.31}$$

Because $T(z)$ generates conformal transformations, this OPE defines the scaling behaviour of the field; in particular we will now show that the h which appears in (4.31) is the conformal dimension of $\Phi(w, \overline{w})$ (Tong, 2009).

The field transforms under infinitesimal rescaling $w = z + \epsilon(z)$ as (4.23). This diffeomorphism generates an infinitesimal change in the field

$$\Phi(w(z)) = [1 - h\epsilon'(z) + \cdots]\, \Phi(z). \tag{4.32}$$

Expressing as a variation at the point z using $\Phi(w(z)) = \Phi(z) + \epsilon(z)\partial\Phi(z) + \cdots$ gives

$$\delta\Phi(z) = -h\epsilon'(z)\Phi(z) - \epsilon(z)\partial\Phi(z). \tag{4.33}$$

Take the limit $z \to w$ where the derivative can be written $\epsilon'(z) = [\epsilon(z) - \epsilon(w)]/(z - w)$ and so

$$\delta\Phi(z) = -h\epsilon(z)\Phi(z) - \epsilon(z)\partial\Phi(z). \tag{4.34}$$

The conformal Ward identity expresses the variation of an operator $\mathcal{O}(z)$ under a symmetry transformation $\mathcal{J}(z)$ and is given as a residue

$$\delta\mathcal{O}(z, \bar{z}) = -\operatorname{Res}\left[\mathcal{J}(w)\mathcal{O}(z, \bar{z})\right]. \tag{4.35}$$

In our case, we are considering variations $\delta z = \epsilon(z)$ and so $\mathcal{J}(z) = \epsilon(z)T(z)$. Using the primary field's OPE with the stress tensor (4.31) we get

$$\delta\Phi(z, \bar{z}) = -\operatorname{Res}\left[\epsilon(w)\, T(z)\mathcal{O}(z, \bar{z})\right] = -\operatorname{Res}\left[\epsilon(w)\left(h\frac{\Phi(z, \bar{z})}{(w - z)^2} + \frac{\partial\Phi(z, \bar{z})}{(w - z)} + \cdots\right)\right]. \tag{4.36}$$

Taylor expanding $\epsilon(w) = \epsilon(z) + (w - z)\epsilon'(z) + \cdots$ gives a residue to the first term in this expression and we recover exactly (4.34), which is the infinitesimal form of the primary scaling transform

$$\Phi(z) \to \left[\frac{\mathrm{d}w}{\mathrm{d}z}\right]^{-h} \Phi(z). \tag{4.37}$$

Consequently we may identify h with the scaling dimension of the primary field.

4.1.4 Conformal Anomaly

The conformal anomaly breaks the conformal invariance of the system by introducing an energy scale Λ. We will now show that this arises because the OPE of stress-energy tensor receives an additional higher-order pole which renders it non-primary. Consider the OPE of the stress tensor with itself (Itzykson and Drouffe, 1989)

$$\lim_{z\to w} T(z)T(w) = \frac{c/2}{(z - w)^4} + \frac{2}{(z - w)^2}T(w) + \frac{1}{(z - w)}\partial T(w) + \cdots, \tag{4.38}$$

which has a term proportional to the *central charge* c which renders the field non-primary. This constant arises whenever there are particle-contents in the theory, and it roughly counts the number of degrees of freedom in the theory. Therefore all non-trivial CFTs are generally expected to have a conformal anomaly quantified by a central extension c, as can be shown manually by calculating the TT OPE for free fermions or bosons. Given this OPE we may calculate its non-primary scaling

$$T(z) \to T'(w) = [w'(z)]^2 T(z) + \frac{c}{12} S(z,w), \tag{4.39}$$

where $S(z,w)$ is a non-trivial function of the two coordinates called the Schwartzian derivative.

As stated in (Fradkin, 2020b), this directly implies the emergence of an anomalous Schwinger term of the stress tensor commutator, and consequently the non-vanishing correlator

$$\langle T(z)T(w)\rangle = \frac{c/2}{(z-w)^4}. \tag{4.40}$$

Just as the chiral anomaly led to the central extension of the current algebra (which was the Kac–Moody term, proportional to k), the conformal anomaly imbues the Witt algebra with a central extension to give the *Virasoro algebra* (Goddard and Olive, 1986). In the quantisation of a two-dimensional CFT, the de Witt generators instead live in two copies of the Virasoro algebra, given

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m} + \frac{c}{12} m(m^2-1)\delta_{m,-m}. \tag{4.41}$$

We stated before that the traceless nature of the stress energy tensor is responsible for the classical conformal invariance, so we should expect there to arise a quantum energy scale proportional to the central charge c in the presence of an anomaly.

In fact, in free space $T_{\mu\nu}$ is always traceless, but by putting the theory on a weakly curved background the quantum theory can violate conformal invariance by depending on the new energy scale governed by this weak curvature. The scale which emerges is

$$\Lambda = \left\langle T^{\mu}{}_{\mu} \right\rangle = c\frac{R}{12}, \tag{4.42}$$

where R is the Ricci curvature of the background.

4.2 Quantum Hall Correspondence

4.2.1 Motivation

Let us now explain why the edge action derived previously is a conformal field theory, as well as motivating why general quantum Hall edges are CFT's. This will lead us to discussing a general and powerful correspondence between QH bulk states and conformal field theories. The end result of this duality is that one may predict wavefunctions of physical quantum Hall states simply by understanding their edge theory (Hansson et al., 2017).

The chiral anomaly generally implies that there must exist gapless chiral fermions at the edge. The explicit emergence of such states was demonstrated at the Chern–Simons edge (Wen, 1991, 1990b, 1992), but this result can be explicitly derived independently of this effective-theory apparatus by considering the dynamics of confined electrons in Landau levels (Cappelli et al., 1992). It is shown that there universally exists chiral fermions at the quantum Hall edge which are described by a conformal field theory.

The actions describing these systems can all be described as *quantum critical systems* because they exist at a fine-tuned point in parameter space which is scale invariant (Fradkin, 2013). Such a system naturally arises at a second-order phase transition, where the system is described by only one characteristic length scale which diverges. Indeed in the general parameter space of a field theory work by Zamolodchikov (1986); Ludwig and Cardy (1987) even show how, at the different fixed points (described by CFTs) of the phase diagram, the central charges c are related. The renormalisation group flow (into the IR) induces a flow in parameter space which must reduce the value of c which — in a sense — links the different CFTs.

In terms of renormalisation group flow, the phase transition is a fixed point of dilatation and is therefore scale invariant. For 2-dimensional classical (Gross and Wess, 1970) and quantum (Polchinski, 1988) field theories this implies conformal invariance.

The Mermin–Wagner theorem states that in 1+1 dimensions there can be no spontaneous symmetry breaking of a continuous group (Mermin and Wagner, 1966; Coleman, 1973). Expressed in terms of quantum observables, this theorem requires that all correlation functions must not decay more quickly than a power law in distance.

Because we are interested in the quantum Hall edge theory at zero temperature, the power-law fluctuations at the critical point of the phase transition are necessarily quantum-mechanical. These distinctions from classical Landau theory are why one can call the Luttinger theory a quantum critical phase. Recall the correlator of electron operators is a power-law (3.103), which saturates the bound defined by the Mermin–Wagner theorem. Indeed, one can even explicitly formulate a scale transform for the edge theory which is a classical symmetry of the action: transforming the fields $\Psi \to e^{\ell/2}\Psi$ and coordinates $x^\mu \to e^\ell x^\mu$ (so that $\partial_\mu \to e^{-\ell}\partial_\mu$) leaves the Luttinger action invariant. Sachdev (2011) shows that the Luttinger theory can be defined as a single point on an extended phase diagram for all temperature $T > 0$ and filling away from the band-crossing $\mu \neq 0$. At zero temperature, order parameters and charge fluctuations diverge as the Fermi level is restored to zero $\mu \to 0$. Hence the Luttinger phase at $T = 0$ and $\mu = 0$ is a quantum critical point of this phase diagram.

Further evidence that such theories should be conformal is that the bulk topological theory must be gapped, with dynamics having much lower energy than the energy of massive excitations. In this regime there are no physical energy scales of the system, which is a hint that $\operatorname{tr} T_{\mu\nu} = 0$ and the system is classically conformal.

In physical quantum Hall systems, the disorder inherent to physical samples and experiments could break this argument by introducing an energy scale. The chiral anomaly is responsible for the emergence of the chiral fermion at the edge, and the resulting action is anomalous in the sense that it cannot be written in terms of local operators at the edge. Therefore our arguments for a CFT are robust against disorder because edge perturbations due to dust and disorder must be local, and cannot remove the non-local and *topological* chiral fermion edge theory. The topological nature of the anomaly is said to protect the edge states against disorder, and the state is said to have *topological order* (Wen, 1989).

4.2.2 Chiral Boson

Conformal field theories are a ubiquitous tool used for describing scale invariant systems which exist at a critical point — it is no surprise that we have just been able to argue that the chiral edge state fermions are a CFT. We also know that the edge's degrees of freedom can be described by a chiral boson, which we also expect to be generally conformally invariant. We can now use our CFT formalism to analyse the simple case of a free boson.

Consider the free boson in $1 + 1$-dimensions, which is a conformal field theory. We will find that a chiral boson appears in the holomorphic sector, which justifies us considering this sector as a CFT on its own. In holomorphic coordinates the achiral CFT action is

$$S_{\mathrm{FB}} = \frac{1}{2\pi} \int \mathrm{d}^2 x \, \partial\Phi\bar{\partial}\Phi. \tag{4.43}$$

The equation of motion is $\partial\bar{\partial}\Phi(z, \bar{z}) = 0$ with a solution that decouples left- and right-moving fields

$$\Phi(z, \bar{z}) = \frac{1}{2} \left(\phi(z) + \bar{\phi}(\bar{z}) \right). \tag{4.44}$$

In terms of the chiral fields, the action is simply $(1/8\pi)\partial\phi\bar{\partial}\bar{\phi}$. Evaluated by functionally differentiating the path integral and solving a differential equation, the two-point correlation function of the free theory is

$$\langle \Phi(z, \bar{z})\Phi(w, \bar{w}) \rangle = -\frac{1}{4} \log |w - z|^2. \tag{4.45}$$

Splitting this into its two components $\log |w - z|^2 = \log(w - z) + \log(\bar{w} - \bar{z})$ shows explicitly the chiral contribution to the propagator by both the fields ϕ and $\bar{\phi}$. Focusing on the holomorphic part, the correlator is the same as the chiral correlation function (3.92) we calculated for the edge boson of the quantum Hall effect

$$\langle \phi(z)\phi(w) \rangle = -\log(w - z). \tag{4.46}$$

Taking derivatives of this chiral field, we notice its correlator is

$$\langle \partial\phi(z)\partial\phi(w) \rangle = -\frac{1}{(w - z)^2}, \tag{4.47}$$

which appears to be a conformal field theory with an operator product expansion given by (4.27). This implies the conformal field is $\partial\phi(z)$ and not $\phi(z)$. Indeed, calculating the stress tensor that $\partial\phi$ transforms with conformal dimension $h = 1$ (Ginsparg, 1988).

The holomorphic stress tensor of this theory is

$$T(z) = -2\partial\Phi\partial\Phi = -\frac{1}{2}\partial\phi\partial\phi. \tag{4.48}$$

In the quantum theory, this operator should be normal-ordered $:\partial\phi\partial\phi:$ in order to subtract infinities from its expectation values. We can explicitly demonstrate that $\partial\phi$ is a primary field and find its conformal dimension by evaluating its OPE with $T(z)$. This is

$$T(z)\partial\phi(w) = -\frac{1}{2} :\partial\phi\partial\phi: \partial\phi. \tag{4.49}$$

The LHS of this expression is time-ordered, and the right contains normal-ordered elements. Using Wick's theorem (detailed in Di Francesco et al. (1997)) the RHS will evaluate to a sum of all contractions, each given by (4.47). Retaining only the divergent pole this is

$$T(z)\partial\phi(w) = \frac{\partial\phi}{(w-z)^2} + \frac{\partial^2\phi}{(w-z)} + \cdots \tag{4.50}$$

which shows $\partial\phi(z)$ is a primary operator of the CFT with $h = 1$.

Taking the system to have periodic coordinates (by placing the complex coordinates on a torus) we can explicitly evaluate the partition function of the free boson. Here we see that the partition function factorises into two sectors: its holomorphic and antihomomorphic components $Z = Z_l \times Z_r$ (Cardy, 1986). The chiral theory's Hilbert space forms a closed subspace of the full theory, and this is the basis for why we may hope to use this model to describe the chiral edge modes of the QHE. Theories of the quantum Hall edge which include chiral fermions not only contain a conformal anomaly, but also a Lorentz anomaly. This means that the conformal holomorphic and antiholomorphic charges c and $\bar{c}$ may be different. In our case specifically $\bar{c} = 0$ when the theory has only a chiral (and not antichiral) sector.

This primary operator $\partial\phi$ generates the chiral subalgebra, which is related to the edge action derived for the FQHE droplet. More concretely, Read (2009) shows that the *general* edge theory is a local, unitary chiral CFT by defining valid Virasoro operators in terms of well-defined charges on the edge. Even more explicitly, see (Fuentealba et al., 2019) for the explicit construction of this algebra for the Floreanini–Jackiw action.

The surprising result that is the focus of this Chapter is that the edge CFT also totally describes universal properties of the bulk theory. This is possible because of the topological nature of the quantum Hall phases. These systems have long-range topological order which leads to robust features like the ground state degeneracy (on a closed system with non-trivial topology). Such results are independent of microscopic details. There exist robust global observables defined through Wilson loops which are 'topological invariants' in these systems, and indeed these operators will be key in formalising the bulk-boundary correspondence.

From another perspective, the form of the Laughlin wavefunction suggests that the quantum Hall system is related to a CFT: the wavefunction can be expressed as a purely holomorphic function of the coordinates in a way reminiscent of correlators of primary CFT operators. Furthermore, more complex wavefunctions of QHE phases — including of non-Abelian phases — all have a wavefunction description in terms of conformal blocks.

Indeed, in a remarkable result Moore and Read (1991) showed that a conformal field theory approach to the quantum Hall edge is able to recover the full wavefunction of the quantum Hall bulk through its high-order correlation functions. Now it is thought that all QHE phases have an equivalent CFT description, which has successfully been used to predict the form of wavefunctions.

4.2.3 Vertex Operators & Fusion Rules

Let us again consider a chiral field with correlator $\langle \phi(x)\phi(y)\rangle = -\log(x-y)$. Because the conformal dimension of the derivative $\partial\phi$ is 1, the conformal dimension of the chiral field ϕ is zero. Therefore we can generate the series of 'vertex operators' from ϕ (Hansson et al., 2017), given

$$\mathcal{V}_\alpha(x) =: \exp(i\alpha\phi(x)) :, \tag{4.51}$$

without introducing a scale. These are called Fubini–Venazeano operators in their applications in string theory (Fubini and Veneziano, 1969; Fubini et al., 1969), and in the landscape of the quantum Hall effect they represent operators which generate physical (quasi)particle excitations (Fubini, 1991; Fubini and Lutken, 1991). We can calculate the conformal dimension of the vertex operators (Ginsparg, 1988) by calculating their OPE with the stress tensor. Evaluating with Wick's theorem gives

$$T(z)\mathcal{V}_\alpha(w) = -\frac{1}{2} :\partial\phi(z)\partial\phi(z)::e^{i\alpha\phi(w)}: \tag{4.52}$$

$$= -\frac{1}{2}(i\alpha \langle \partial\phi(w)\, \phi(z)\rangle)^2 :e^{i\alpha\phi(w)}: -i\alpha\partial\phi(z) \langle \partial\phi(z)\, \phi(w)\rangle :e^{i\alpha\phi(w)}: \tag{4.53}$$

$$= \frac{\alpha^2/2}{(z-w)^2}\mathcal{V}_\alpha(w) + \frac{1}{(z-w)}\partial\mathcal{V}_\alpha(w) \tag{4.54}$$

and hence it is a primary field with conformal dimension $h = \alpha^2/2$.

The operator product expansion of two vertex operators as they are brought together gives their 'fusion rules', written generically as

$$\mathcal{V}_\alpha \times \mathcal{V}_\beta = N_{\alpha\beta}{}^{\rho}\mathcal{V}_\rho. \tag{4.55}$$

This can be evaluated explicitly for the Abelian case of a free chiral theory that we are focused on. Using the identity relating normal-ordered products

$$:e^A::e^B:=:e^{A+B}: e^{\langle AB\rangle}, \tag{4.56}$$

allows us to evaluate the product, using the logarithmic $\langle\phi\phi\rangle$ correlator (Di Francesco et al., 1997)

$$\mathcal{V}_\alpha(z)\mathcal{V}_\beta(w) = |z-w|^{2\alpha\beta}\mathcal{V}_{\alpha+\beta}(w) + \cdots. \tag{4.57}$$

This demonstrates the simple fusion rule in this case: fusing two quasiparticles together gives a new particle with combined charge $\mathcal{V}_{\alpha+\beta}$.

Note that we will need to consider a compactified boson ϕ, which acts like an angular coordinate $\phi = \phi + 2\pi\sqrt{m}$. Note that this periodicity relation arises when we compactify the boson with a radius $R = \sqrt{m}$. In this case the vertex operators remain single-valued if $\alpha = n/R = n/\sqrt{m}$ with integer n. We saw before that the boson was naturally compactified when we placed it on a periodic system, and this result produced a quantised total charge of the system. The effect of compactifying the chiral field will be important in restricting the extended algebra to contain a finite number of (equivalence classes of) vertex operators (which are distinguishable up to the number of electrons) (Hansson et al., 2017).

Recall that in the edge CFT generated by the electrons, the chiral Kac–Moody algebra is proportional to a level m. Importantly, this level is guaranteed to be a rational number, as was found when deriving FQHE states. The vertex operator $\mathcal{V}_\alpha(x)$ has the same form as the charged excitations of the edge states in (3.94). In this formalism, the field $\mathcal{V}_m$ was the electron with charge 1, and $\mathcal{V}_1$ was the quasihole with charge $1/m$ (Fradkin, 2013). Because $m = p/q$ is rational, the fusion of an integer number q of quasiparticle vertex operators will generate a state containing only electrons.

Next, we require that all primary fields $\mathcal{V}_n$ are local with respect to the electron operator $\mathcal{V}_m$, meaning looping $\mathcal{V}_n$ around the electron does not pick up any phase. This loop is called a monodromy, as calculated in Section 1.3, and involves taking the closed path in 1+1 dimensions (which involves a closed path in time). The monodromy of two vertex operators $\mathcal{V}_n$,$\mathcal{V}_{n'}$ has statistical phase $\exp(2\pi i\, nn'/m)$ and vanishes for all n *only* when $n' = m$. Therefore all operators are local with respect to the electron (but not necessarily each other). We can extend the Kac–Moody algebra by looking for its representations which are also physical states and are local with respect to the electron. Next, all fields which only differ by one or more electrons can be considered physically equivalent in the sense that these electron trivially interact with the electron which defines the extended algebra. Furthermore, because there exists an integer number q of quasiholes which combine to give an electron, we can restrict our algebra to be $n = 1 \cdots q - 1$. This implies the physical subalgebra of vertex operators has a finite number of (distinguishable) representations, and it generates a *rational* conformal field theory (RCFT) (Moore and Read, 1991).

The full RCFT derived from the free boson has both holomorphic and antiholomorphic vertex operators, but it is a general property that one can define a purely holomorphic subalgebra (generated by $\mathcal{V}_1$) (Moore and Seiberg, 1988; Moore and Read, 1991). This chiral algebra is fully consistent when considered on its own (as it contains its own holomorphic stress tensor and identity), and this is the algebra which will be applied to the quantum Hall effect. This is since the QHE edge is purely chiral, and it demands a description in terms of purely holomorphic vertex operators.

For the simple FQHE state $m \in \mathbb{Z}$, the RCFT fusion rule is

$$\mathcal{V}_n \times \mathcal{V}_{n'} = \mathcal{V}_{n+n' \bmod m}. \tag{4.58}$$

The quasiparticles are in representations of the level-m Kac–Moody algebra, which is what restricts the allowed vertex representations. Therefore we call this state the $U(1)_m$ compactified chiral boson RCFT with m different primary vertex operators in representations of KM. Each operator $\mathcal{V}_n$ has conformal dimension

$$h_n = \frac{n^2}{2m}. \tag{4.59}$$

Because the theory is chiral, the conjugate scaling dimension is $\bar{h} = 0$, and the spin of the particle is simply given by $s = h$. The fundamental quasiparticle $\mathcal{V}_1$ therefore has spin $s = 1/2m$, and the electron $\mathcal{V}_m$ has spin $m/2$ which is bosonic/fermionic for even/odd m (Hansson et al., 2017). As expected, it is therefore fermionic in the Laughlin state with odd m.

4.2.4 Correspondence

This formalism now allows us to detail the correspondence between the edge CFT and the bulk Chern–Simons theory; first let us follow the operator-map laid out by Witten (1989). The gauge-invariant observables of the bulk effective theory are Wilson loop operators $W_n[\Gamma]$ defined over a closed loop Γ. To recap: these represent heavy anyon-particle insertions which are moved around a closed cycle, and their expectation value measures the phase accumulated. This is written in terms of the emergent gauge field a and an integer n (which labels a representation of the $U(1)$ group) as

$$W_n[\Gamma] = P\exp\Big(in \oint_\Gamma a\Big). \tag{4.60}$$

In Section 3.3.1 where we derived the boundary theory, the physical (temporal) gauge constraint was equivalent to considering 'pure gauge' configurations $a = \mathrm{d}\phi$ (an exact connection). This would imply $W_n[\Gamma]$ is zero over bulk cycles, however in a theory with a boundary we can terminate the path $\Gamma(x, y)$ on the edge at points x, y, as shown in Fig. 4.1. Evaluating the Wilson line shows that it is the correlation functions of level-n vertex operators

$$\langle W_n[\Gamma(x,y)]\rangle = \Big\langle Te^{-in\phi(x)}e^{in\phi(y)}\Big\rangle = \Big\langle T\mathcal{V}_n^\dagger(x)\mathcal{V}_n(y)\Big\rangle. \tag{4.61}$$

These operators have a charge n/m and conformal scaling dimension $n^2/2m$.

As was found for the vertex operators, there are only m distinguishable Wilson loop operators, where m here is the level of the Chern–Simons theory. Witten (1989) lays out a deep correspondence between 2+1 dimensional Chern–Simons theories and operators in rational CFTs. The topological nature of CS theories, meaning that the observables are only dependent upon global properties of the manifold, are generally responsible for the conformal properties of the dual edge excitations.

But because of the general covariance of the Chern–Simons theories, 'slicing' the theory in different ways will give the same result. Instead of slicing the theory to get a 1+1 dimensional dynamical theory of the edge, we can take a spacelike slice (at fixed time) to get a 2+0 dimensional theory which describes the wavefunction of the bulk. The previous correspondence (4.61) allows us to think of the operator $\mathcal{V}_n(x)$ creating a particle on the boundary, and then drawing it through the bulk of the 2+1 dimensional CS theory along the path Γ. Taking a slice at t after the creation event, the path of the

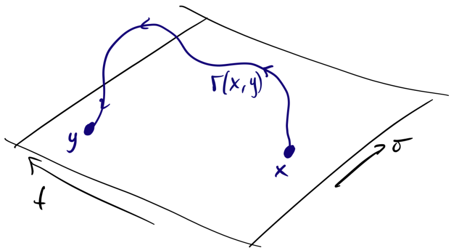

Figure 4.1: Taking an edge excitation on a path Γ through the bulk and terminating it on the boundary.

Wilson line Γ will puncture the spacelike slice; Witten quantises the CS theory on this Euclidian 2+0 dimensional slice in the presence of these Wilson lines which pierce the spacelike surface. The space of solutions of the CS theory is the space of conformal blocks in a 1+1 dimensional CFT which motivates the identification of the quantum theory of Wilson lines W_n (in a given representation n) with correlation functions of primary vertex operators on the edge CFT (Witten, 1989).

Using this method, one may even recover the bulk wavefunction at a given constant-time slice at t_0. Choose the representation of the boundary vertex operators so that they represent electron creation operators, then the Wilson lines $\Gamma(t)$ represent the path of these electrons after they are pulled into the bulk. The spatial slice has the geometry of a disk, which will be punctured by a number of 'electron' Wilson lines which were produced before t_0. In order to identify the coordinates of the boundary with those on the slice, we must choose a 'conformal mapping' which is an analytic mapping of the Lorentzian edge coordinates $x = (t, \sigma)$ to the Euclidian slice coordinates $\Gamma = (\sigma, \eta)$ (where η is the radial and η the azimuthal coordinate). Using the complex representation of the edge coordinate $z = \sigma + it$, the conformal mapping is

$$w = e^{iz}e^{-t_0} = e^{t-t_0}e^{i\sigma}, \tag{4.62}$$

where the complex coordinate system on the disk is $w = \eta e^{i\sigma}$. In a sense, this defines a choice of the path Γ_i of each electron created at the boundary in order to simply relate the complex coordinates z and w, as shown in Fig. 4.2.

Let us now explicitly recover the Laughlin wavefunction from correlation functions of the $\nu = \frac{1}{m}$ edge RCFT (Moore and Read, 1991). Choose the primary field $\mathcal{V}_m(z)$ as the electron of the theory, which will be used to generate this wavefunction. The previous discussion motivates the identification of the expectation value of $\mathcal{V}_m$ insertions at the boundary points z_i with the Wilson lines puncturing the spatial disk at w_i (defined through the conformal mapping). Evaluating the correlation function of the edge CFT we find

$$\langle W_m(w_1)\cdots W_m(w_N)\rangle = \langle \mathcal{V}_m(z_1)\cdots\mathcal{V}_m(z_N)\rangle = \exp\left(-m\sum_{i<j}^{N}\Big\langle \phi(z_i)\phi(z_j)\Big\rangle\right). \tag{4.63}$$

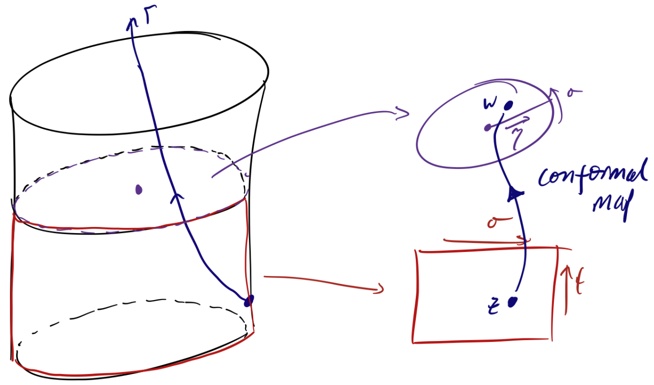

FIGURE 4.2: The path of a boundary excitation defines a conformal map between the boundary and a spacelike slice.

This expression is evaluated using Wick's theorem applied to free chiral bosons, and then taking the correlator $\langle \phi(z)\phi(w) \rangle = -\log(z-w)$ gives the wavefunction on the spatial slice

$$\Psi(w_1, \dots, w_N) = \prod_{i<j} (w_i - w_j)^m. \tag{4.64}$$

However evaluating (4.63) violated the neutrality condition imposed on conformal field theories: the total scaling dimension of primary fields in a correlation function must be zero. Inserting a background neutralising charge density to the system allows us to regain a meaningful expression for the wavefunction by cancelling this scaling dimension. Define the background charge operator

$$\mathcal{O}_{\text{bg}} = \exp\Big(-i\rho_m \sqrt{m} \int \mathrm{d}^2 z\, \phi(z)\Big), \tag{4.65}$$

which defined a constant areal charge density of a fractionally filled Landau level $\rho_m = B/2\pi m = 1/2\pi m l_B^2$ (Hansson et al., 2009). The wavefunction is finally recovered

$$\Psi(w_1, \dots, w_N) = \Big\langle \mathcal{V}_m(w_1) \cdots \mathcal{V}_m(w_N)\, \mathcal{O}_{\text{bg}} \Big\rangle = \prod_{i<j} (w_i - w_j)^m e^{-\sum_i |w_i|^2/4l_B^2}. \tag{4.66}$$

This incredible result has used the conformal mapping to evaluate the wavefunction on the spatial slice by using correlation functions of the vertex operators (which create the electrons in the system) at the boundary. The full Laughlin wavefunction is therefore simply a many-body correlation function of an edge CFT.

Excited states containing quasiparticles may even be included in the wavefunction through this procedure: simply inserting a CFT operator $\mathcal{V}_1(\tilde{z}_i)$ creates a quasihole in the Laughlin system (Hansson et al., 2017), and dragging it through the bulk to puncture the spatial slice inserts it into the Laughlin wavefunction

$$\Psi(\tilde{w}; w_1, \dots, w_N) = \Big\langle \mathcal{V}_1(\tilde{z})\, \mathcal{V}_m(z_1) \cdots \mathcal{V}_m(z_N)\, \mathcal{O}_{\text{bg}} \Big\rangle \tag{4.67}$$

$$= \prod_i (w_i - \tilde{w}) \prod_{i<j} (w_i - w_j)^m e^{-\sum_i |w_i|^2/4l_B^2 - |\tilde{w}|^2/4l_B^2}. \tag{4.68}$$

This fractionally charged edge operator therefore created an anyon in the bulk! Adding a second one allows us to more clearly see its braiding statistics appearing:

$$\Psi(\tilde{w}_1, \tilde{w}_2; w_1, \dots, w_N) = \left\langle \mathcal{V}_1(\tilde{w}_1)\mathcal{V}_1(\tilde{w}_2)\, \mathcal{V}_m(w_1) \cdots \mathcal{V}_m(w_N)\, \mathcal{O}_{\text{bg}} \right\rangle \tag{4.69}$$

$$= (\tilde{w}_1 - \tilde{w}_2)^{1/m} \prod_i (w_i - \tilde{w}_1)(w_i - \tilde{w}_2) \prod_{i<j} (w_i - w_j)^m e^{-\sum_i |w_i|^2/4l_B^2 - (|\tilde{w}_1| + |\tilde{w}_2|)^2/4l_B^2}. \tag{4.70}$$

There is now a further non-analytic term which is responsible for the relative fractional statistics (recall Section 1.3). The concept of operator fusion at the edge implied that as we take $\mathcal{V}_1(z_1)$ and $\mathcal{V}_2(z_2)$ to the same point, they will behave as a fused operator $\mathcal{V}_2$. This phenomenon is reproduced in the Laughlin wavefunction when the electron-quasihole interaction term approaches

$$\prod_i (w_i - \tilde{w}_1)(w_i - \tilde{w}_2) \to \prod_i (w_i - \tilde{w})^2 \tag{4.71}$$

as $\tilde{w}_{1,2} \to \tilde{w}$.

This conformal mapping between coordinates on the boundary and on the spatial slice allowed us to evaluate the constant-time CS wavefunction through correlation functions of the edge CFT. This result is not purely dependent upon the coordinate mapping we chose, however; by using a 'holomorphic quantisation' of the bulk CS action, it is possible to identify the bulk wavefunctions with the edge's partition function (Moore and Read, 1991; Eliashvili, 1996).

4.3 Non-Abelian Quantum Hall Systems

So far we have considered only *Abelian* fractional quantum Hall states, represented by the level-m Chern–Simons action. However there is a whole class of interesting theories that our discussion has thus far omitted: promoting the gauge group of the emergent gauge field a to be $SU(N)_k$ (at level k) we can engineer a quantum Hall phase with non-Abelian particles.

In fact, there are many observations which suggest the physics of quantum Hall plateaus above the lowest Landau level are governed by non-Abelian statistics (Eisenstein et al., 2002; Xia et al., 2004). In particular, it is suggested that the $\nu = \frac{5}{2}$ level is an example of such a phase which hosts non-Abelian anyons (Moore and Read, 1991; Greiter et al., 1992). These non-Abelian CS theories are applicable more broadly in condensed matter: there are proposed non-Abelian vortex excitations in strontium ruthenate superconducting atomic layers (Das Sarma et al., 2006) and superfluid helium-3, for example.

There has been a strong experimental interest in realising this phase and attempting to measure the non-Abelian quasiparticles in such phases (Bonderson et al., 2006; Stern and Halperin, 2006) because control over them would allow for the advent of a new field: topological quantum computation (Das Sarma et al., 2005). The feature limiting progress of quantum computation over the last few decades has been the intolerance of entangled quibits to decoherence and environmental noise. The use of topological phases in quantum computation provides an exciting notion — since generally these phases have ground states which are only dependent upon global properties of the system, the computer can be made resillient against local perturbations. The use of non-Abelian anyonic excitations of the FQHE has been proposed as such a method of fault-tolerant computation and data-encoding (Kitaev, 2003) (see (Nayak et al., 2008) for a review).

It has been suggested that the $\nu = \frac{5}{2}$ FQHE state lies in the non-Abelian class of $SU(2)_2$ (although it may also be described by a Wen–Zee K-matrix state, and experimental work is currently seeking to determine its nature). All amplitudes for processes which loop and knot a set of anyon worldlines in this $SU(2)_2$ theory (with their antiparticles) is given by the Kauffmann knot invariant (Simon, 2016). The calculation of the Kauffman knot invariant is exponentially hard in the number of crossings, but a quantum computer made from this FQHE state would be able to calculate it in constant time simply by braiding the anyons into a given knot structure, and then measuring the amplitude of annihilation. The ability to perform this calculation of the Kauffmann invariant allows the system to generally perform any quantum computation (Freedman, Kitaev, and Wang, 2002). Control over quantum Hall anyons with this level of precision is years away, but harnessing the power of these systems would prove an astonishing new development in the field of quantum computation.

The non-Abelian CS action is similar to the familiar Abelian case except with an additional term to ensure gauge invariance

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \text{tr}\left[A \wedge \text{d}A + \frac{2}{3} A \wedge A \wedge A\right]. \tag{4.72}$$

The theory is invariant under $SU(N)_k$ gauge transformations, given

$$A \to gAg^{-1} + g(\text{d}g^{-1}), \tag{4.73}$$

with $g \in SU(N)$. This is only gauge invariant on compact spaces when $k \in \mathbb{Z}$. Similarly to before, quantising this action on a manifold with boundaries leads to an anomaly at the edge, which is now a level-k WZW chiral CFT theory, due to Wess and Zumino (1971); Witten (1983)

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int_{\partial\mathcal{M}} \text{tr}\left[\text{d}g \wedge \star\text{d}g^{-1}\right] + \frac{k}{12\pi} \int_{\mathcal{M}} \text{tr}\left[(g^{-1}\,\text{d}g) \wedge (g^{-1}\,\text{d}g) \wedge (g^{-1}\,\text{d}g)\right], \tag{4.74}$$

which only depends on the value of the field g at the boundary edge (Nayak et al., 2008).

This action also arises through the bosonisation of fermions with a non-Abelian symmetry (Witten, 1984), once again demonstrating that there are chiral fermion excitations which will appear at the quantum Hall edge. Moreover, it was shown that at this value of the WZW coupling constant, the theory is conformally invariant (Knizhnik and Zamolodchikov, 1984) as expected generally of the quantum Hall edge. The chiral anomaly of the boundary fermions leads to a non-Abelian $SU(N)_k$ Kac–Moody current algebra, and the matter content of the theory also leads to a conformal anomaly.

A non-Abelian CS action can be derived as an effective theory of a set of N 'partons', which are fermions which transform under the $SU(N)$ symmetry group (Wen, 1999). Integrating out the fermions and taking the IR limit of the gauge field derives an effective theory which has non-Abelian excitations, since other terms are irrelevant and vanish. In particular, integrating out the fermions with two filled Landau levels generates the $SU(2)_2$ state which may describe the $\nu = 5/2$ FQHE state. Alternatively, one may derive the bulk theory by considering the inverse of the anomaly inflow argument (Hansson et al., 2017). If we wish to describe the $\nu = 1$ state with bosonic edge excitations, it can be argued that these must live in a chiral $SU(2)_k$ Kac–Moody algebra. The fermionic dual theory will be generated by the corresponding WZW theory. Now crucially, in order to generate this chiral anomaly, the bulk must be described by a $SU(2)_2$ CS theory (Fradkin et al., 1998).

The fermionic vertex operators in the non-Abelian boundary theory (such as the WZW theory) obey the general fusion rules given (4.55). Indeed the simple case where two operators fused to form only one product is unique to Abelian theories and is generally not the result of non-Abelian CFTs. For example, in an $SU(2)_k$ theory the vertex operators live in the spin algebra with a highest weight state labelled by a 'spin' $s = k/2$. The fusion rules are therefore simply the Clebsch–Gordan decomposition of the spin group with $\mathcal{V}_n$ living in the module $n = 2s + 1$, *i.e.*

$$\mathcal{V}_2 \times \mathcal{V}_2 = \mathcal{V}_1 + \mathcal{V}_3. \tag{4.75}$$

The general $SU(N)_k$ fusion rules are given by the Young tableaux tensor product rules.

The space of constants $N_{\alpha\beta}{}^{\rho}$ which define the relative fraction of each of these possible states in the fusion define a Hilbert space of possible fusion results. In this $SU(2)_2$ case for example, the 4-point correlator of quasiparticles will have two independent components

$$\Psi(\tilde{w}_1, \tilde{w}_2) = \langle \mathcal{V}_1(0)\mathcal{V}_1(\tilde{w}_1)\mathcal{V}_1(\tilde{w}_2)\mathcal{V}_1(1)\rangle \tag{4.76}$$

$$= a_+\Psi_+(\tilde{w}_1, \tilde{w}_2) + a_-\Psi_-(\tilde{w}_1, \tilde{w}_2). \tag{4.77}$$

In fact the general high-point correlation function can be decomposed into conformal blocks Ψ_α which span a degenerate vector space (Moore and Seiberg, 1988, 1989),

$$\Psi(\tilde{w}_1, \dots, w_1, \dots) = \langle \mathcal{V}_1(\tilde{w}_1) \cdots \mathcal{V}_m(w_1) \cdots \rangle \tag{4.78}$$

$$= \sum_\alpha c_\alpha \Psi_\alpha(\tilde{w}_1, \dots, w_1, \dots). \tag{4.79}$$

The non-Abelian nature of the state arises when the quasiholes are taken to loop around each other in a braid. In this circumstance, after the quasiholes have been returned to the same positions, the wavefunction has varied after the coefficients change

$$c_\alpha \to U[\text{braid}]_\alpha{}^\beta c_\beta. \tag{4.80}$$

The matrix $U[\text{braid}]$ is called the monodromy matrix and causes the non-Abelian statistics of the quasiholes (Moore and Read, 1991).

Recalling the quantisation of the Chern–Simons action in Section 2.4, we will now follow the work of Witten (1989) who quantised and exactly solved the non-Abelian model. In this work, he draws a fascinating link between the non-Abelian CS theory and the WZW edge theory. The added feature of this duality is the non-trivial nature of fusion; canonically quantising the CS theory on a spatial slice which is penetrated by Wilson lines gives the same Hilbert space as the set of conformal blocks of the WZW theory (Williams, 2019).

Thus the non-Abelian generalisation of the bulk-boundary correspondence has been achieved: the correlation functions of all conformal edge theories can be exactly mapped to the bulk theory of Wilson lines and therefore reproduce the wavefunction. This program has been successful in describing more complex quantum Hall states which include multiple gauge or auxilliary fields in terms of the edge chiral RCFTs (Fradkin et al., 1998, 2001). The use of CFT tools allows for the computation of wavefunctions of a great number of quantum Hall states, and indeed other states with intrinsic topological order.

Moreover, in the context of the non-Abelian quantum Hall effect the braiding of edge excitations are dual to physically braiding the non-Abelian anyons. Their braiding properties are defined by the monodromy matrix of the edge CFT.

Particle-Vortex Duality in the FQHE 5

5.1 Dualities in Quantum Field Theory

Our study of the quantum Hall effect has so far led us to use several dualities — let us briefly reflect on how they have been used to highlight their relevance. In 1+1 dimensions we used bosonisation dualities to reformulate fermionic edge theories in terms of bosonic degrees of freedom. This duality allowed the edge to be canonically quantised in a fashion which allowed us to describe it as a conformal theory. Finally a duality was drawn up between the boundary correlation functions and the bulk wavefunction using an analysis of Wilson lines in the bulk.

There are still many unanswered questions of the FQHE theory which we will now develop tools to better understand: Can we formulate a theory wherein the bulk electron excitations are first-class particles? How does bosonisation apply in higher dimensions? And are there any properties of even-filling m FQHE states which we can describe using field theory?

In order to do this, we will need to develop tools of *particle-vortex duality* in 2+1 dimensions. In 2-spatial dimensional field theories there are two categories of 'particles' which can be described. The first category is what one usually refers to as a particle: local field excitations which are momentum-eigenstates. The second category is that of 'vortices' which are solitonic solutions to the field equations which are topological field configurations that connect multiple vacuums. They are topological objects and are categorised by integer winding numbers, so they cannot be described in perturbation theory. We will find that quasiholes are naturally particle-like particles, and dualising the theory of the fractional QHE will provide a theory of quasiholes as vortices but with the electrons as true particles.

The dualising process will involve changing the statistics of particles, a process which connects this story to bosonisation in higher than 1+1 dimensions. As is usual in the process of added-dimensionality, the correspondences can not be made as rigorously as before, but we can still make convincing arguments by analysing such theories in tractable limits. We will later propose a bosonisation duality which equates the path integrals of a fermionic and bosonic theory. This duality has only recently been discovered, and has led to a surge of recent work which aims to propose a consistent theory of the half-filled Landau level — for a review see (Senthil et al., 2018; Turner, 2019).

The $\nu = 1/2$ filling fractional quantum Hall state has been the centre of recent work in the field of quantum Hall physics, and indeed will be the focus of the end of this Chapter. This is not a gapped state, and so does not exhibit the FQHE, but it has interesting fermionic quasiparticles which are expected to be CFs. We will introduce the Halperin–Lee–Reed theory of the composite fermion for this state, and then we will discuss how recent work has shown that the Dirac fermion may be a better model.

5.1.1 Photon – Compact Scalar Duality

The first duality we will consider will be a reformulation of a simple photon in 2+1 dimensions, where we will show that the 'dual photon' is simply a compact scalar. The path integral of the free photon theory a is

$$Z = \int \mathcal{D}a\, e^{iS[a]}, \quad S[a] = -\frac{1}{4g^2}\int f \wedge \star f, \tag{5.1}$$

where $f = \mathrm{d}a$. Its equation of motion is $\mathrm{d} \star f = 0$ and the Bianchi identity is trivially $\mathrm{d}f = 0$. This path integral only depends upon f and so we can replace $\mathcal{D}a$ with $\mathcal{D}f\, \delta_F[\mathrm{d}f]$ as the path-integral measure, where the functional-Dirac delta functional imposes the Bianchi identity. Now using the standard representation of the Dirac functional in terms of Nakanishi–Lautrup field σ (which acts as a Lagrange-multiplier in the derivation of the equations of motion), the path integral becomes

$$Z = \int \mathcal{D}\sigma\mathcal{D}f \exp\left[i\int\left(-\frac{1}{g^2}f \wedge \star f + \frac{1}{2\pi}\sigma\, \mathrm{d}f\right)\right]. \tag{5.2}$$

The equations of motion are

$$f = -\frac{g^2}{2\pi} \star \mathrm{d}\sigma\,, \tag{5.3}$$

$$\mathrm{d}\sigma = 0. \tag{5.4}$$

We therefore recover $\mathrm{d}f = 0$ (which now comes from the $\mathrm{d}\sigma = 0$ equation of motion) and $\mathrm{d} \star f = 0$ (which follows trivially from Bianchi). The roles of the equation of motion and Bianchi identity have been reversed in this duality.

Because f appears quadratically in the action it can be integrated out (at tree-level using the classical equation of motion) to give an action in terms of σ, the 'dual photon':

$$Z = \int \mathcal{D}\sigma \exp\left[i\frac{g^2}{4\pi}\int \mathrm{d}\sigma \wedge \star\, \mathrm{d}\sigma\right]. \tag{5.5}$$

This action has a global $U(1)_{\mathrm{topo}}$ symmetry with the current

$$J_{\mathrm{topo}} = -\frac{g^2}{\pi}\,\mathrm{d}\sigma \tag{5.6}$$

which acts by shifting $\sigma \to \sigma + \mathrm{const}$.

Thus we have provided a duality of this 2+1 dimensional photon theory by writing the same theory in terms of a different point of view: the one-form theory of 2+1 dimensional electromagnetism is dual to a zero-form theory of a scalar. This new formulation even highlighted that there is a global topological $U(1)$ symmetry, which was obfuscated in the original writing-down of the theory. Similar dualities exist in higher dimensions too; in the familiar case of 3+1 dimensional Maxwell a general electromagnetic duality transform (Montonen and Olive, 1977) relates 1-form theory to another 1-form theory with the role of the electric and magnetic fields reversed (hence the general name of "electric-magnetic duality") (Turner, 2019). Moreover, the equivalent of this compact boson duality in 3+1 dimensions is a map from a scalar (with a shift symmetry) to a 3-form field strength.

5.1.2 XY Model

The next duality we will consider is our first example of a particle-vortex duality (Peskin, 1978). One side of this duality is the familiar XY model, which will later be related to the Abelian Higgs model. Firstly, the XY model is a theory of a complex scalar ϕ with the action

$$S_{\mathrm{XY}} = \int \mathrm{d}^3x \left[|\partial\phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4 \right], \tag{5.7}$$

which has a global $U(1)_{\mathrm{XY}}$ symmetry generated by the current

$$J^\mu_{\mathrm{XY}} = i\left(\phi^* \partial^\mu \phi - \phi\partial^\mu\phi^*\right). \tag{5.8}$$

This has two distinct phases

- $m^2 > 0$:

  The vacuum is $\phi = 0$ and the excitations are ϕ with mass m. The global $U(1)_{\mathrm{XY}}$ symmetry remains unbroken. The excitations are massive particles ϕ.

- $m^2 < 0$:

  The system undergoes spontaneous symmetry breaking of the $U(1)_{\mathrm{XY}}$ group and ϕ receives a vacuum expectation value $|\phi|^2 = v^2 = m^2/\lambda$. Expanding around the vacuum, the field is given

  $$\phi = (v + \rho)e^{i\sigma}. \tag{5.9}$$

  Expanding the action in this form gives a kinetic term $(\partial\rho)^2 + (\partial\sigma)^2$, but the expansion of the potential $V(|\phi|) = V(v + \rho)$ is independent of the phase σ. Therefore ρ is massive and σ is the massless Goldstone field.

  Because the field must be single-valued, the field σ must be unchanged up to a $2\pi n$ difference under a closed loop. We can define a topological winding number for this field

  $$n = \frac{1}{2\pi}\oint_C \mathrm{d}\sigma = \frac{1}{2\pi}\oint_C \mathrm{d}x^i\, \partial_i\sigma. \tag{5.10}$$

  Vortices of the theory are topological defects which have non-zero winding number, when the loop C encloses its core (Tong, 2016).

  Let us construct an example vortex solution to understand some features of their dynamics. The XY Hamiltonian with constant ρ is

  $$H_{\mathrm{vor}} = \frac{1}{2}\int \mathrm{d}^2\mathbf{x}\, |\partial_i\sigma|^2. \tag{5.11}$$

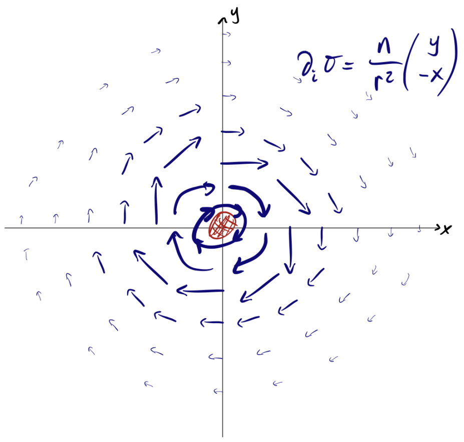

Figure 5.1: Vortex field gradient $\partial_i\sigma$ shown in blue; the 'core' of the vortex is shaded out.

Now take the example vortex solution with winding number n (in polar coordinates)

$$d\sigma = \frac{n}{r^2} d\theta \tag{5.12}$$

as shown in Fig. 5.1. This solution is valid outside of the 'core' of the vortex — at some short distance around the core there is spontaneous symmetry breaking. This short distance acts as a regulator, which is needed since the field configuration would be divergent if extended to the origin. The distance scale of this core size is determined by the equation of motion of the XY theory, and we will only consider the behaviour of the field configuration at scales r_0 greater than the core size.

Evaluating the energy of an isolated vortex involves a radial integral from r_0 to R, where these are short- and long-distance cutoffs, respectively. The energy is given

$$E_{\text{vor}} = \pi n^2 \log\left(\frac{R}{r_0}\right), \tag{5.13}$$

which is logarithmically divergent in R/r_0, and quadratic in winding number n (Tong, 2017).

Now to develop a model of interacting vortices, we must use a method which relates the field σ to an effective 'electric field' $E_i = \varepsilon_{ij}\partial_j\sigma = -\partial_i\chi$. In this language, the system energy is

$$H_{\text{vor}} = \frac{1}{2}\int d^2\mathbf{x}\, E_i E_i = -\frac{1}{2}\int d^2\mathbf{x}\, \chi\partial_i E_i. \tag{5.14}$$

Evaluating the divergence of the electric field shows that vortices are point sources $\partial_i E_i = 2\pi n\delta^{(2)}(\mathbf{x}-\mathbf{x}_i)$, and so solving this gives the 'potential' in terms of the

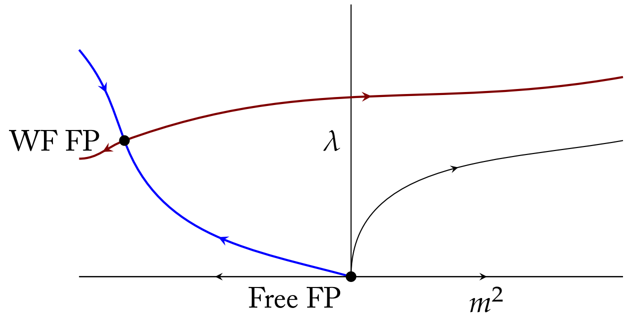

FIGURE 5.2: Wilson–Fisher FP of the RG flows in the XY model. Adapted from (Turner, 2019).

pairwise radius $r_i = |\mathbf{x} - \mathbf{x}_i|$,

$$\chi = -n \log(r_i/r_0) + \text{const}, \tag{5.15}$$

which is again defined in terms of the short-distance cutoff r_0 which represents the size of the vortex core. The energy is therefore

$$H_{\text{vor}} = \pi \sum_{i<j} n_i n_j \log\left(r_{ij}/r_0\right) + \text{const}, \tag{5.16}$$

which shows the logarithmic interactions are attractive for vortex-antivortex pairs (when $n_{i,j}$ have opposite signs).

In-between the phases there is a second-order phase transition which lives in the universality class of the Wilson–Fisher (WF) critical point. A schematic of the RG flow of the system is shown in Fig. 5.2, which demonstrates that the WF fixed point does not lie exactly at $m^2 = 0$ but in fact is slightly away from it; this will not matter for our analysis but it is worth mentioning that the crossover between the broken and unbroken phases lies slightly away from zero.

The correspondence will be drawn by identifying a corresponding critical point in the dual theory, and then tuning the same relevant operators to both sides of the theory one may follow the same RG flow away from the critical point. In the XY model, such a relevant operator is ϕ^2 which flows the theory away from the WF fixed point along the red line. We also have dual currents at our disposal, and we will use this to add new CS couplings to the theories.

5.1.3 Abelian Higgs Model

Consider the theory of a complex scalar field with a *local* $U(1)_{\text{gauge}}$ gauge symmetry

$$S_{\text{AH}} = \int \mathrm{d}^3x \left[-\frac{1}{4g^2} f_{\mu\nu}^2 + |D\phi|^2 - \tilde{m}^2|\phi|^2 - \tilde{\lambda}|\phi|^4 \right], \tag{5.17}$$

where ϕ has charge g under the gauge group: $D_\mu\phi = \partial_\mu\phi + iga_\mu\phi$, and f is the field strength of the connection a.

Calling on our prior analysis of the photon, we can dualise the Maxwell sector and write it in terms of a dual scalar σ. This local theory in 2+1 dimensions has a global 'topological' $U(1)_{\text{topo}}$ symmetry, which motivates us to identify this phase with the $m^2 < 0$ phase of the XY model. The charge of this global group, written in terms of the original gauge field is $\star\, \mathrm{d}a$ (Tong, 2018). Explicitly in components

$$J^{\mu}_{\text{topo}} = -\frac{g^2}{\pi}\partial^{\mu}\sigma = \frac{1}{4\pi}\varepsilon^{\mu\nu\rho}f_{\nu\rho}, \tag{5.18}$$

and the conserved charge is

$$Q_{\text{topo}} = \frac{1}{2\pi}\int \mathrm{d}^2\mathbf{x} f_{12} = \frac{1}{2\pi}\int \mathrm{d}^2\mathbf{x}\, B. \tag{5.19}$$

Operators charged under this current are monopole operators $\mathcal{M} = e^{iQ_{\text{topo}}\sigma}$, which transform $\mathcal{M} \to \exp\left(iQ_{\text{topo}}\theta\right)\mathcal{M}$. Now consider in detail the two phases of the theory:

- $\tilde{m}^2 > 0$ Coulomb phase:

  The ϕ is a massive field with vacuum $\phi = 0$ and so the $U(1)_{\text{gauge}}$ symmetry is unbroken. The vacuum therefore contains a massless photon field σ which spontaneously breaks the $\sigma \to \sigma + \text{const}$ symmetry; the Coulomb phase breaks the global $U(1)_{\text{topo}}$ symmetry and has the photon as its Goldstone (Kovner et al., 1991). For this reason we will aim to identify the Coulomb phase of the Abelian Higgs model with the global-symmetry-broken $m^2 < 0$ phase of the XY model.

  The identification of the photon as the Goldstone as a broken symmetry may seem like an unfamiliar identification. The discussion of higher-form symmetry allows this result to be generalised to higher dimensions (Kalb and Ramond, 1974; Savit, 1977). For example, in 3+1 dimensions there is a corresponding 1-form symmetry instead of the global 0-form symmetry of σ translations. The Goldstone bosons in this higher dimensional setting are Wilson line operators, and in general higher dimensions they can be higher-dimensional topological defects like branes (Gaiotto et al., 2015).

  If we are to recover all properties of the global-symmetry-broken XY phase, then we expect to find massive excitations which are logarithmically confined. We find that particle excitations in this model have such behaviour. Let us examine the Abelian Higgs action (5.17) by taking the limit $\tilde{m}^2 \gg 1 \gg g^2$ and $\tilde{\lambda} \sim 1$; here the action is approximately

  $$S = \int \mathrm{d}^3x \left[-\frac{1}{4g^2}f_{\mu\nu}^2 + |\partial\phi|^2 - a_{\mu}\left(\phi^*\partial_{\mu}\phi - \phi\partial_{\mu}\phi^*\right) - \tilde{m}^2|\phi|^2\right]. \tag{5.20}$$

  The scalar field is quadratic in this action which allows it to be integrated out, giving a sourced Maxwell action, where the scalar field acts as a source of EM excitations

  $$S = \int \mathrm{d}^3x \left[-\frac{1}{4g^2}f_{\mu\nu}^2 + a_{\mu}j^{\mu}\right], \qquad j_{\mu} = 2ig\,\phi^*\partial_{\mu}\phi. \tag{5.21}$$

The current-current interaction in 2+1 dimensions is a logarithmic Coulomb potential; to see this we can find the Green's function of the Maxwell equation $d \star F = 0$

$$\nabla^2 A_0(\mathbf{x}) = Q\delta(\mathbf{x}) \tag{5.22}$$

which in 2+1 dimensions has logarithmic polar solutions $A_0(r) = Q\log(r/r_0)$. This form holds in the quantum model too (Zee, 2010)

$$W[j] = -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{|j(k)|^2}{k^2 + i\epsilon}, \tag{5.23}$$

and given two delta-function point sources of unit charge the interaction energy is given by the 2-dimensional integral over space

$$E = -\int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2 + i\epsilon} \tag{5.24}$$

which scales logarithmically in distance $E = Q\log r + \text{const.}$ Hence we have shown that particle-like excitations in the theory have a logarithmic interaction Coulomb potential in the Abelian Higgs model.

- $\tilde{m}^2 < 0$ Higgs phase:

  In the Higgs phase the scalar field ϕ gains a vacuum expectation value and therefore gauge symmetry is broken. The Higgs mechanism gives the photon mass; explicitly, the expansion of $|D_\mu\phi|^2$ around the ground state $\phi = (v + \rho)e^{i\sigma}$ is

$$|D_\mu\phi|^2 = (\partial_\mu\rho)^2 + v^2\left(\partial_\mu\sigma + ea_\mu\right)^2 + \cdots \tag{5.25}$$

  Under local gauge transformations $ga_\mu \to ga_\mu - \partial_\mu\theta$ the field transforms $\phi \to e^{i\theta}\phi$, and thus the combination $b_\mu = a_\mu + g^{-1}\partial_\mu\sigma$ is gauge invariant. Rewriting the action in terms of b leaves $F[a] = F[b]$ unchanged since a and b differ by a gauge transformation. Therefore the expansion of the covariant derivative (5.25) simply becomes

$$|D_\mu\phi|^2 = (\partial_\mu\rho)^2 + v^2g^2b^2 + \cdots \tag{5.26}$$

  which gives the gauge field a mass $m_b = gv$ and the Goldstone field of the ungauged model is removed from the spectrum altogether. In gauging the XY model, the vector boson a_μ gains mass as it 'eats' the Goldstone σ.

  There exist massive excitations in the theory, which we described by the modified gauge field b_μ. We will now aim to show that local excitations in this field are necessarily vortices in σ, which is the phase of ϕ. Take a field configuration in the Higgs phase at fixed $\rho = v$, so $\phi = ve^{i\sigma}$. The Hamiltonian of the configuration is

$$H_{\text{vor}} = \frac{1}{2}\int d^2\mathbf{x}\,|D_i\phi|^2 = \frac{g^2}{2}\int d^2\mathbf{x}\,|b_i\phi|^2 = \frac{g^2v^2}{2}\int d^2\mathbf{x}\,(b_i)^2. \tag{5.27}$$

In the XY model with only a global symmetry, the energy density scales with radius $\partial_i \phi \to 0$ as r^{-2} and the spatial integral was logarithmically divergent. This result meant pairs of vortices are confined. In contrast, in the gauged model, since b_i is the physical field, taking $ga_i \to -\partial_i \sigma$ as $r \to \infty$ allows us to take $b_i \to 0$ faster than r^{-2} and hence the energy of the vortex solution is convergent and finite (Zee, 2010).

Generally, the localised and charged massive excitations in b must involve winding of σ. These are called Nielson–Olesen vortices of σ and are dually monopoles of the $U(1)_{\text{topo}}$ field (Nielsen and Olesen, 1973). First, note that these vortex solutions are charged under $U(1)_{\text{topo}}$ with

$$Q_{\text{topo}} = \frac{1}{2\pi} \int d^2x f_{12} = \frac{1}{2\pi} \oint dx^i\, a_i. \tag{5.28}$$

This can be interpreted as the magnetic charge of b excitations, and hence these are monopoles. Now using $ga_i \to -\partial_i \sigma$ as $r \to \infty$, enclose the vortex in a loop at infinity such that

$$n = \frac{1}{2\pi} \oint dx^i\, \partial_i \sigma \to \frac{-g}{2\pi} \oint dx^i\, a_i = -gQ_{\text{topo}}. \tag{5.29}$$

This implies that monopole operators of the b field must induce a vortex by winding the (unphysical) σ field. However in the vacuum where there are no monopoles, the system preserves the $U(1)_{\text{topo}}$ symmetry. We may therefore identify this phase and its unbroken global symmetry with the $m^2 > 0$ phase of the XY model; for this phase too has massive excitations which are non-confining. In the duality between these phases, the XY model's particles become the Abelian Higgs model's gauged vortices.

At $\tilde{m}^2 = 0$ there is also a phase transition in this model, which we postulate to be in the same Wilson–Fisher class as the XY model. This has been proven on the lattice (Banks et al., 1977) and numerical evidence which shows equal critical exponents supports this (Dasgupta and Halperin, 1981).

5.1.4 XY – Abelian Higgs Duality

Let us summarise the duality between the phases of these two models by requiring dual phases have the same global symmetry breaking scheme (Turner, 2019). In the Abelian Higgs model the local symmetry is broken when $\tilde{m}^2 < 0$, however this is the phase where the local symmetry is preserved in the vacuum. It is important that these features are not confused when constructing the duality. Firstly the unbroken phase of the XY model is dual to the Higgs phase of the Abelian Higgs model, as shown in the table below

	Phase	Global Symmetry	Massive field	Gapped excitation
XY	$m^2 > 0$	Unbroken $U(1)_{\text{XY}}$	Massive ϕ	ϕ particle
Abelian Higgs	$\tilde{m}^2 < 0$	Unbroken $U(1)_{\text{topo}}$	Massive b	σ vortex / b monopole

Both phases preserve the global $U(1)$ group and in the Abelian Higgs model the Higgs mechanism gives the photon b mass. The interesting feature of the duality is that the ϕ particle-like excitations are shown to be dual to σ Nielson–Olesen vortices of the AH model.

The broken phase of the XY model is dual to the Coulomb phase of the Abelian Higgs model, as shown in the following table

	Phase	Global Symmetry	Goldstone	Confining excitation
XY	$m^2 < 0$	Broken $U(1)_{\text{XY}}$	Scalar σ	σ vortices
Abelian Higgs	$\tilde{m}^2 > 0$	Broken $U(1)_{\text{topo}}$	Dual photon σ	ϕ particle

Both phases are characterised by a global symmetry breaking, and an emergent Goldstone mode. In the XY model, vortices of this Goldstone mode are dual to particle-like excitations of the ϕ-field. Both of these excitations are confining: the vortices because of their divergent self-energies, and the ϕ-excitations of the AH model because they are charged under the unbroken $U(1)_{\text{gauge}}$ field and the Coulomb potential in 2+1 dimensions is confining.

The duality between the two theories equates the currents of the global symmetries $J_{\text{XY}} \leftrightarrow J_{\text{topo}}$, and it equates the phases with opposite masses $m^2 \leftrightarrow -m\tilde{m}^2$. Alternatively, this implies that the dual relevant operators which drive the system away from the Wilson–Fisher fixed point under RG flow are $|\phi|^2 \leftrightarrow -|\phi|^2$ in the XY/AH models respectively (Turner, 2019). We may therefore identify the RG flow of the two diagrams if we swap the sign of their mass coefficients. Addition of these dual relevant operators to their respective theories will push the theories along the same RG flow. Assuming such a mean-field description is correct along the whole RG flow, one may identify the fixed points at the end of these flows.

An equivalent representation of this duality which will be useful for the application to quantum Hall systems can be derived by coupling these theories to a background gauge field A. A suitable gauge field-current coupling is found in the term $A \wedge \star J_{\text{global}}$. Coupling the gauge field to the current of the global $U(1)$ charge preserves the duality because these currents are themselves dual. For the XY action this gives

$$S_{\text{XY}}[\phi; A] = \int \mathrm{d}^3x\, |\partial\phi|^2 - V(\phi) + A \wedge \star J_{\text{XY}}. \tag{5.30}$$

Using $J_{\text{XY}} = ie(\phi\, \mathrm{d}\phi^* - \phi^*\, \mathrm{d}\phi)$, up to $\mathcal{O}(e)$ this can be written

$$S_{\text{XY}}[\phi; A] = \int \mathrm{d}^3x\, |(\partial_\mu - ieA_\mu)\phi|^2 - V(\phi) + \cdots. \tag{5.31}$$

Its partition function is

$$Z_{\text{XY}}[A] = \int \mathcal{D}\phi\mathcal{D}A\, e^{iS_{\text{XY}}[\phi;A]}. \tag{5.32}$$

Now the dual theory to (5.30) must be formed by coupling to the dual topological current $J_{\text{topo}} = (1/2\pi) \star \mathrm{d}a$ to give

$$S_{\text{AH}}[\phi, a; A] = \int \mathrm{d}^3x \left[-\frac{1}{4g^2} f_{\mu\nu}^2 + |D\phi|^2 + \frac{1}{2\pi} A \wedge \mathrm{d}a - V(\phi) \right], \tag{5.33}$$

Its partition function is

$$Z_{\mathrm{AH}}[A] = \int \mathcal{D}\phi \mathcal{D}a \mathcal{D}b \, e^{iS_{\mathrm{AH}}[\phi,a;A]}, \tag{5.34}$$

and the duality can be written

$$Z_{\mathrm{XY}}[A] = Z_{\mathrm{AH}}[A]. \tag{5.35}$$

5.2 Vortices in the Zhang–Hansson–Kivelson Model

Now this formalism can be directly used to develop a duality for the fractional quantum Hall system. We will compare the nature of the quasihole excitations in the two theories, and show that their dual representations are indeed particles and vortices. The construction of a new model will link with the 'ZHK' model of the FQHE, which provided a Ginzburg–Landau description of the phase.

This result will be based on the duality (5.35), where we will now promote A to be a dynamical gauge field b, and deform each action with a marginal level-m Chern–Simons kinetic term. This level will make connection with the FQHE level when applied to the XY model, and the dual theory will give us an alternative theory to work with. Beginning with the XY model (5.33) this transformation gives,

$$S_{\mathrm{XY}}[\phi, b] = \int \mathrm{d}^3x \left[|(\partial - ieb)\phi|^2 - \frac{m}{4\pi} b \wedge \mathrm{d}b - V(\phi) + \cdots \right]. \tag{5.36}$$

The Chern–Simons term looks like the action for a statistical gauge field b at CS level m. In fact, the particles ϕ are charged under this dynamical field with a coupling given by e/m (by inspection of the $e\,b\phi$ gauged derivative and the canonically normalised CS term). One can also find that the (Nielson–Olesen) vortices of this field ϕ have a charge e and are electrons. Explicitly now, its partition function involves the path integral over all b

$$Z_{\mathrm{XY}} = \int \mathcal{D}\phi \mathcal{D}b \, e^{iS_{\mathrm{XY}}[\phi;b]}. \tag{5.37}$$

The dual theory to (5.36) must be formed by introducing the same (marginal) level-m CS couplings to the Abelian Higgs model:

$$S_{\mathrm{AH}}[\phi, a, b] = \int \mathrm{d}^3x \left[-\frac{1}{4g^2} f_{\mu\nu}^2 + |D\phi|^2 - \frac{m}{4\pi} b \wedge \mathrm{d}b + \frac{1}{2\pi} b \wedge \mathrm{d}a - V(\phi) \right]. \tag{5.38}$$

Its partition function is

$$Z_{\mathrm{AH}} = \int \mathcal{D}\phi \mathcal{D}a \mathcal{D}b \, e^{iS_{\mathrm{AH}}[\phi,a,b]}, \tag{5.39}$$

and integrating out b gives the following action

$$S_{\mathrm{AH}}[\phi, a] = \int \mathrm{d}^3x \left[|(\partial - iea)\phi|^2 - \frac{1}{4\pi m} a \wedge \mathrm{d}a - V(\phi) + \cdots \right], \tag{5.40}$$

where now

$$Z_{\rm AH} = \int \mathcal{D}\phi\mathcal{D}a\, e^{iS_{\rm AH}[\phi,a]}. \tag{5.41}$$

Its vortices (which are dual to the particles of the original theory) are quasiholes. This can be see by evaluating the charge of the vortices (Zhang, 1992): the charge density in the theory is given

$$\rho(x) = \frac{\delta S}{\delta a_0} = -\frac{e}{m}\varepsilon^{ij}\partial_i a_j. \tag{5.42}$$

Because the flux of a_i around the loop must be quantised (take the lowest-order vortex excitation with one unit), the excess charge is

$$Q_{\rm vor} = \int {\rm d}^2\mathbf{x}\, \delta\rho(\mathbf{x}) = -\frac{e}{m}, \tag{5.43}$$

which has the charge of a quasihole.

This relativistic field Φ in (5.40) has particle and antiparticle excitations with charge $\pm e$ which are electrons and holes. Taking the non-relativistic limit of this action, as outlined in (Nastase and Rojas, 2016), gives the Jackiw–Pi action (Jackiw and Pi, 1990, 1992)

$$S_{\rm AH}[\phi,a] = \int {\rm d}^3x \left[i\Phi^*(\partial_0 - iea_0)\Phi - \frac{1}{2M}|\partial_i\Phi - iea_i\Phi|^2 - \frac{1}{4\pi m} a\wedge {\rm d}a - V(\Phi) + \cdots \right]. \tag{5.44}$$

The resulting action has only particle-like electron excitations. This non-relativistic limit is performed by turning on a mass for the scalar — which gaps the spectrum — and then breaking Lorentz invariance by adding a chemical potential term a_0. It is instructive to note that the duality of the non-relativistic models can be separately formulated in the context of superfluids (Fisher and Lee, 1989).

An interesting feature of this model is that the physical excitations are electron-like, despite the field Φ being bosonic. How can this be possible? The critical feature is the inclusion of the dynamical gauge field: the scalar coupling to a CS gauge field acts to couple flux of the magnetic field to each particle, which transmutes its statistics (Wilczek and Zee, 1983).

The equation of motion of the gauge field a_0 shows that

$$\frac{1}{2\pi} b = -m|\Phi|^2, \tag{5.45}$$

where $b = f_{12} = \varepsilon^{0ij}\partial_i a_j$ is the magnetic field strength of the dynamic gauge field a. Eq. 5.45 implies that the local magnetic field depends upon the particle distribution, and is localised at excitations of the boson — this process is called *flux attachment*. We therefore call these Φ particles 'composite bosons', since they are constituted of a boson and m units of flux. For the Laughlin states m is odd; since the particle-like excitations of Φ have m fractional units of a-charge attached to them, they will have fermionic braiding statistics. This is despite Φ obeying commutation relations — due to the flux attachment, the composite bosons have changed their statistics!

The non-relativistic Lagrangian (5.44), derived as the dual theory of the Chern–Simons effective description of quasiholes, is the effective field theory derived by Zhang, Hansson, and Kivelson (1989) from a microscopic Hamiltonian (called the ZHK model). In this model, the composite bosons are the electronic excitations of the FQHE. This formulation is particularly useful because it allows one to define a order parameter for the FQHE state (Girvin and MacDonald, 1987). The FQHE state is identified by algebraic 'off-diagonal long-range order' in the electrons. The FQHE phase is characterised by the non-vanishing of the correlator $\left\langle \Phi^\dagger(x)\Phi(y)\right\rangle \to \rho_0$, when the operators $\Phi(x)$ are separated to infinity. This order parameter allows for the construction of a Ginzburg–Landau theory of the FQHE in analogy with superfluid states (Read, 1989). This theory was discovered before the Chern–Simons description and is able to describe the phenomenology of the FQHE.

5.3 Composite Fermions

5.3.1 Halperin–Lee–Reed Theory

Now we will investigate a field theory of the fractional quantum Hall system which is written in terms of a fermionic field. This will allow us to develop a field theory of composite fermions (CFs) called the Halperin–Lee–Reed (HLR) theory. This model is particularly suited to the description of the half-filled Landau level, which is a theory that has gained strong interest in recent years.

We begin with a non-relativistic theory of interacting electrons coupled to an external gauge field A,

$$S_{\mathrm{e}}[\psi_{\mathrm{e}}] = \int \mathrm{d}^3x \left[i\psi_{\mathrm{e}}^\dagger \left(\partial_0 - iA_0\right) \psi_{\mathrm{e}} - \frac{1}{2M}|(\partial_i - iA_i)\psi_{\mathrm{e}}|^2 + (\text{interactions}) \right]. \tag{5.46}$$

To transform this into a theory of composite fermions, we must change the statistics of the electron field. This can be done by coupling the action to a dynamical Chern–Simons gauge field a. Therefore the resultant CF action, called the HLR theory (Halperin, Lee, and Read, 1993), is

$$S_{\mathrm{CF}}[\psi] = \int \mathrm{d}^3x \left[i\psi^\dagger \left(\partial_0 - iA_0 + ia_0\right) \psi - \frac{1}{2M}|(\partial_i - iA_i + ia_i)\psi|^2 \right.$$
$$\left. + \frac{1}{2}\frac{1}{4\pi} a \wedge \mathrm{d}a + (\text{interactions}) \right]. \tag{5.47}$$

The changing of statistics is achieved through 'flux attachment', wherein the CS coupling pairs each electron with lines of magnetic flux to produce the composite fermions (Lopez and Fradkin, 1991). A technical problem with this HLR theory is due to the $\frac{1}{2}$ level of the CS term. This leads to a gauge anomaly on closed manifolds and renders the theory non-gauge invariant here.

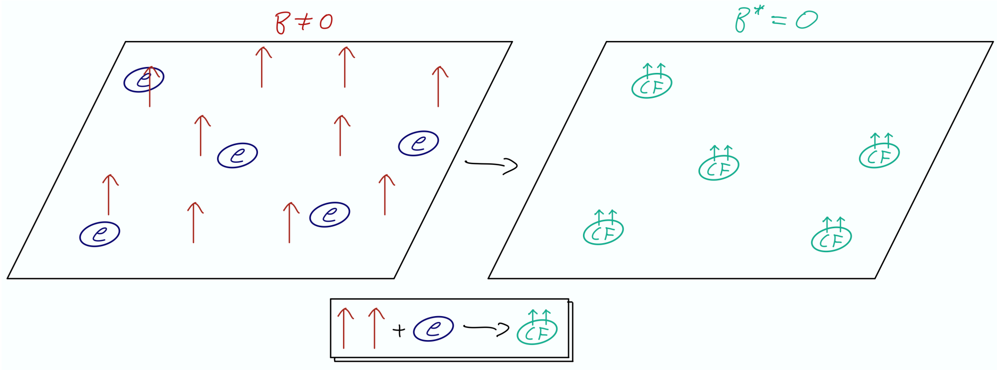

FIGURE 5.3: Flux attachment to electrons in HLR theory creates composite fermions which move in a zero effective magnetic field.

The local magnetic field can be evaluated using the a_0 equation of motion (as was done in the ZHK model)

$$2\psi^\dagger \psi = \frac{\mathrm{d}a}{2\pi}. \tag{5.48}$$

Now the magnetic field is $b = (\mathrm{d}a)_0 = f_{12}$ and write the fermion density $\rho_{\mathrm{CF}} = \psi^\dagger \psi$, so

$$\frac{b}{2\pi} = 2\rho_{\mathrm{CF}}. \tag{5.49}$$

The flux of the dynamic field depends on the local density of the CF field ρ_{CF}, which is localised around excitations. Each CF carries two units of flux, which does not change the statistics of the particle, as shown in Fig. 5.3. Because of the difference of sign in the action, the effective magnetic field in the sample is $B^* = B - b$, and at the mean field level, we see

$$B^* = B - \langle b \rangle = B - 4\pi\rho_{\mathrm{CF}}. \tag{5.50}$$

This reproduces the result (1.40) which relates the electron and CF filling fractions.

A particularly intriguing phase of the FQHE is the $\nu = \frac{1}{2}$ state, which has some very unique features. Firstly, because the lowest Landau level is half full, the composite fermion density is $B/4\pi$ and therefore they move in an effective zero magnetic field. The spectrum is hence gapless at this filling fraction, and indeed there is no quantum Hall plateau seen here.

This unique FQHE phase has a Fermi surface which has allowed easy analysis in experiments; the HLR theory is both extremely accurate and predictive for work done in the lab. For example, Willett et al. (1990) find the expected Fermi liquid behaviour at $\nu = 1/2$ and were able to measure the Fermi surface with Shubnikov–de Haas oscillations in conductivity. Away from half-filling there is a small effective magnetic field, and other works have found particle resonances which have the correct semiclassical behaviour to be CFs moving in much larger orbits than the electrons (Goldman et al., 1994; Kang et al., 1993). The CF effective mass has also been measured using this process (Willett et al., 1995). See Willett (1997) for a review of experimental tests of this remarkably robust theory.

However there is a shortcoming of the HLR theory, which was for a long time dismissed as a technicality: it is not particle-hole (PH) symmetric. This symmetry is well-defined in the limit of the theory with only the lowest Landau level projected out, where it exchanges all electrons for holes and therefore swaps the filling fraction $\nu \to 1 - \nu$ (Girvin, 1984).

More specifically, taking the electron mass to zero means the cyclotron energy goes to infinity; taking this limit with the strength of electron-electron interactions held fixed acts to project out all higher Landau levels. In this context, one expects there to be an exact PH symmetry which reflects about $\nu = \frac{1}{2}$. To formalise this symmetry, the natural representation for the particle-hole symmetry is an (anti-Hermitian) operator $\mathcal{PH}$ with the following action:

$$\mathcal{PH} : |\text{full}\rangle \to |\text{empty}\rangle \tag{5.51}$$

$$\mathcal{PH} : c^\dagger_{\mathbf{k}} \to c_{\mathbf{k}}. \tag{5.52}$$

This swaps the occupation of states to generate the $\nu \to 1 - \nu$ transform, and acts on fermions to exchange particles for holes. This is a good attempt, but it is in fact not a consistent PH symmetry away from the lowest Landau level limit in the HLR theory.

In fact, it is generally not possible to find a discrete antiunitary symmetry of the HLR theory which can be identified with PH symmetry. The Chern–Simons coupling in (5.47) does not allow for attaching flux to holes — it only allows coupling to CFs — which must couple to both excitations in order to respect the symmetry. This manifestly violates particle-hole symmetry. Kivelson et al. (1997) show that the particle-hole symmetry is incompatible with the Fermi liquid interpretation of the HLR composite fermions. It could be possible that there is spontaneous breaking of the PH symmetry, but numerical evidence suggests against this for the half-filled LLL (Rezayi and Haldane, 2000), although it has been suggested as a mechanism for the $\nu = \frac{5}{2}$ state. There has been speculation that the HLR theory has an emergent PH symmetry in the infrared, but recent evidence shows that it is not possible to recover fully PH-symmetric correlation functions from the theory (Wang et al., 2017; Nguyen et al., 2018). Modern work has even highlighted that the density of electrons and composite fermions can differ, producing an acute contradiction with the HLR theory (Kamburov et al., 2014).

Despite the astounding successes of the HLR theory, evidence suggests we must seek an alternative which is capable of containing a particle-hole symmetry. Having been contemplated for over two decades, PH has now been realised to be a critical feature of a theory needed to replace the HLR model — signalling a way forward in the field.

5.3.2 Dirac Composite Fermion

A groundbreaking recent work provided a potential answer which will turn out to have remarkable roots in particle-vortex duality. Son (2015) has provided an alternative theory of the half-filled lowest Landau level, wherein the composite fermion is a Dirac spinor. Let us first write down the theory of a Dirac particle localised on a plane, interacting with a background gauge field A defined through the bulk. This setup is reminiscent of the fermion zero-mode described in Section 2.2.3, and we can indeed think of Ψ_e as a massless zero-mode localised on a domain wall. The action of these electrons is

$$S_{\text{Dirac}}[\Psi; A] = \int \mathrm{d}^3x \, i\overline{\Psi}_e(\partial\!\!\!/ - iA\!\!\!/)\Psi_e - \frac{1}{2}\frac{1}{4\pi} A \wedge \mathrm{d}A - \frac{1}{4e^2}\int \mathrm{d}^4x \, F_{\mu\nu}^2. \tag{5.53}$$

Hypothesise a relativistic dual theory which we call QED_3, constituted of CF fermions ψ coupled to a dynamical gauge field a

$$S_{\text{QED}_3}[\psi, a; A] = \int \mathrm{d}^3x \, i\overline{\psi}(\partial\!\!\!/ - ia\!\!\!/)\psi + \frac{1}{2}\frac{1}{2\pi} a \wedge \mathrm{d}A - \frac{1}{2}\frac{1}{4\pi} A \wedge \mathrm{d}A + \cdots - \frac{1}{4e^2}\int \mathrm{d}^4x \, F_{\mu\nu}^2. \tag{5.54}$$

The suppressed terms include interactions and a possible Maxwell (kinetic) term for the a emergent gauge field. The absense of a Chern–Simons term for a is a distinguishing feature from the HLR theory of classical CFs, which is necessary for the particle-hole symmetry to be present in this new model. The dynamical gauge field is charged under the background Maxwell field using a BF coupling.

This 2+1 dimensional relativistic duality is assumed to hold at zero external magnetic field, but we will call on the analysis of Sec. 2.1.5 in order to justify how the application of an external magnetic field affects the dual theory. The duality provides a simple prediction: if we identify the Dirac theory with a fundamental description of electrons localised to a plane in the FQHE system, then the duality immediately gives the theory of composite fermions. The behaviour away from zero external field is critical to making this duality relevant to the quantum Hall effect.

This sort of domain wall fermion appears in the different context of 3 (spatial) dimensional topological insulators (TIs): the surface of which contain similar gapless Dirac fermions coupled to a background gauge field. These states are not *intrinsically* topological in the same way as the QHE (for example, they don't have the same long-range entanglement), but the gapless edge modes are protected by a group of discrete symmetries.

The prototypical example of such models is due to Fu, Kane, and Mele (2007), which has states localised on the 2-dimensional surface which are gapless and 'topologically protected', meaning a mass cannot be introduced without breaking its $\mathcal{T}$ symmetry. The proposal by Son (2015) has spurred other authors to propose the same duality can be applied to understand the surface Dirac fermion coupled to an emergent gauge field acts as a dual description for the topological suface states (Wang and Senthil, 2015; Metlitski and Vishwanath, 2016).

Recall that the principal downfall of the HLR theory of the half-filled LLL was that it contained no (discrete antiunitary) symmetry operators which could be identified with particle-hole symmetry. Indeed the construction manifestly violated PH symmetry by only coupling flux to particles and not holes. Our fundamental Dirac theory (5.53) is naturally endowed with the time reversal symmetry $\mathcal{T}$ which acts on the Dirac spinors as $\mathcal{T} : \Psi(t, \mathbf{x}) \to -i\sigma^2\Psi(-t, \mathbf{x})$ and leaves the action invariant. The time reversal and charge conjugation operators act differently on the dual QED_3 spinors as

$$\mathcal{T} : \psi(t, \mathbf{x}) \to \sigma^3\psi^*(-t, \mathbf{x}), \tag{5.55}$$

$$\mathcal{C} : \psi(t, \mathbf{x}) \to \sigma^1\psi^*(t, \mathbf{x}). \tag{5.56}$$

Our dual theory QED_3 therefore contains a dual of this symmetry which we may identify as the PH symmetry $\mathcal{PH} = \mathcal{C} \times \mathcal{T}$ which acts on the CFs as

$$\mathcal{PH} : \psi(t, \mathbf{x}) \to -i\sigma^2\psi(-t, \mathbf{x}). \tag{5.57}$$

Importantly, this symmetry does not exchange particles for holes, but simply inverts the momentum of the Dirac particle $\mathbf{p} \to -\mathbf{p}$ (through the time-inversion of the Dirac spinor), and mixes the components. A fermion mass term $m\bar{\psi}\psi$ is not invariant under this symmetry, and should therefore be excluded.

Likewise, more detailed analysis of the transform of the gauge fields a and A shows that they transform under this new PH symmetry as

$$\mathcal{PH} : a_0(t, \mathbf{x}) \to a_0(-t, \mathbf{x}) \qquad \mathcal{PH} : A_0(t, \mathbf{x}) \to -A_0(-t, \mathbf{x}), \tag{5.58}$$

$$\mathcal{PH} : a_i(t, \mathbf{x}) \to -a_i(-t, \mathbf{x}) \qquad \mathcal{PH} : A_i(t, \mathbf{x}) \to A_i(-t, \mathbf{x}). \tag{5.59}$$

It is now clear that the dynamic CS term $a \wedge \mathrm{d}a$ also violates the PH symmetry and may not be included in this theory.

The coupling of (5.54) to the external gauge field A appears as $A \wedge \star J_{\mathrm{EM}}$, where the EM current is explicitly

$$J_{\mathrm{EM}} = \frac{1}{2\pi} \star (\mathrm{d}a - \mathrm{d}A)\,. \tag{5.60}$$

The EM charge density is therefore determined by the emergent field strength $b = (\mathrm{d}a)_0$ and $B = (\mathrm{d}A)_0$ by

$$\rho_{\mathrm{EM}} = \frac{1}{2\pi}(b - B) = \frac{B^*}{2\pi}, \tag{5.61}$$

where B^* is the effective magnetic field strength. The EM charge filling of the Dirac cone is therefore set by the effective magnetic field. Alternatively, one can say that deviations from charge neutrality of the Dirac theory corresponds to applying an effective magnetic field B^* to the composite Fermion theory, splitting its spectrum into Landau levels; see Fig. 5.4(a).

The equation of motion for a_0 show that the CF density is fixed in terms of the external magnetic field

$$\rho_{\mathrm{CF}} = \psi^\dagger\psi = \frac{B}{2\pi}. \tag{5.62}$$

Note that in this QED_3 theory, the role of the applied magnetic field and the effective magnetic field are reversed compared to the HLR theory of CFs (5.48): in this case the

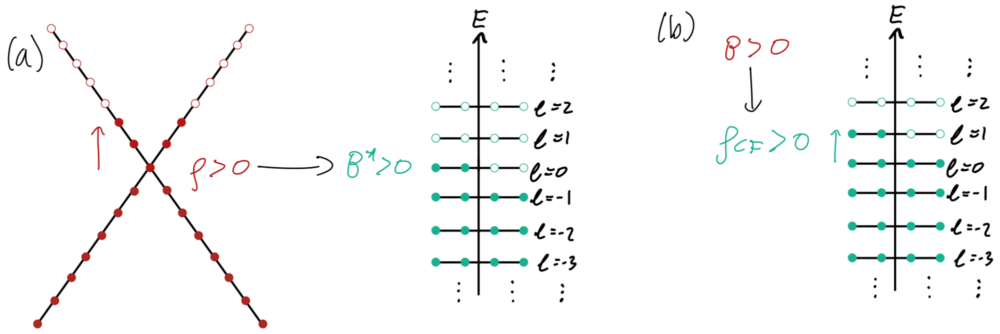

FIGURE 5.4: (a) Deviation of charge density causes an effective magnetic field B^* which splits the CF spectrum into Landau levels. (b) An external magnetic field B causes an increase in ρ_{CF} and therefore CFs to occupy higher Landau levels.

CF density is set by the external field B and not B^* as in the HLR theory. Similarly, the physical charge density of the theory is set by the effective field B^* and not the applied field B.

This result is critical to understanding the duality away from zero external field. Application of a magnetic field B to the fundamental Dirac theory causes an increase in the CF density of the dual theory, as shown in Fig. 5.4(b) Son (2018). This reversal of roles of the fields, compared to the Dirac theory, is another sign that the QED_3 theory is a dual theory, since bosonic particle-vortex duality has this effect.

How can we relate these charge density expressions ρ_{EM} and ρ_{CF} to the respective particles' filling factors? In the HLR theory this calculation (1.40) implied the fractional CF states were simply integer QH states of the composite fermion. However to achieve this result for the Dirac theory, we must acknowledge a difference between charge densities defined using relativistic and non-relativistic models. Namely, the Fermi sea of occupied states in the relativistic theory contribute to the conductivity through their uniform charge density $-B/4\pi$ (Son, 2015). This result was also found in Section 2.1.5, where the Dirac ladder vacuum of *virtual* fermions contributed an extra $\frac{1}{2}$ to the conductivity, compared to the non-relativistic model. Accounting for this shift, we find the relation between the filling fractions is

$$\nu_{\text{EM}} = \nu_{\text{Dirac}} + \frac{1}{2}. \tag{5.63}$$

Explicitly in our model

$$\nu_{\text{EM}} = \frac{\rho_{\text{EM}}}{B/2\pi} + \frac{1}{2} = -\frac{B^*}{B} + \frac{1}{2}, \tag{5.64}$$

$$\nu_{\text{CF}} = \frac{\rho_{\text{EM}}}{B^*/2\pi} + \frac{1}{2} = \frac{B}{B^*} + \frac{1}{2}. \tag{5.65}$$

This implies the Jain sequence states of electrons $\nu_{\text{Jain}} = \frac{n}{2n+1}$ are dual to the half-integer tower of CF states $\nu_{\text{CF}} = n + \frac{1}{2}$. In a remarkable result, the particle-hole symmetry of the theory guarantees that the PH conjugate-Jain state sequence $\nu_{\text{Jain PH}} = 1 - \nu_{\text{Jain}} = \frac{n+1}{2n+1}$ is simply given by the PH conjugate sequence of the dual theory $\nu_{\text{CF PH}} = -(n + \frac{1}{2})$.

Beyond reproducing the Jain sequence there is a slew of recent works which provide evidence for the composite Dirac fermion model. Firstly, we naturally find that correlation functions of the model are particle-hole symmetric (Nguyen et al., 2018), which agrees with the experimental work presented in Section 5.3.1 and has now been established with renewed precision (Pan et al., 2020). Moreover, numerical work using density matrix renormalisation group (DMRG) simulations strongly affirms the notion that a Fermi liquid of a Dirac CFs appears at the half-filled LLL (Geraedts et al., 2015).

5.4 Composite Fermion Duality

5.4.1 2+1D Bosonisation

The Dirac CF model was based on the Dirac–QED_3 duality, which we extended to finite magnetic fields by showing that the external field B simply changes the occupation of Dirac CFs. We will now work to prove the duality between these two theories through equating their partition functions

$$Z_{\text{QED}_3}[A] = Z_{\text{Dirac}}[A] \tag{5.66}$$

when the theories approach corresponding critical points. The actions (5.53, 5.54) can be written in minimal forms to make clearer the duality which we hope to prove:

$$\int \mathcal{D}\Psi_e \, \exp\Big(i \int \mathrm{d}^3x \, i\overline{\Psi}_e(\partial\!\!\!/ - iA\!\!\!/)\Psi_e\Big)$$
$$= \int \mathcal{D}a\mathcal{D}\psi \, \exp\bigg(i \int \mathrm{d}^3x \, \overline{\psi}(i\partial\!\!\!/ + a\!\!\!/)\psi + \frac{1}{2}\frac{1}{2\pi} a \wedge \mathrm{d}A\bigg). \tag{5.67}$$

It has recently been shown that this CF duality, the XY-Abelian Higgs duality, and indeed many others, can be derived from a single 'seed duality' (Karch and Tong, 2016). This duality explicitly defines an equality (at the level of the partition function) of a theory of fermions and a theory of a $U(1)$ gauged scalar (Polyakov, 1988), which has been understood for some time but only recently used to produce boson-boson and fermion-fermion dualities which can be applied to the composite fermion.

In the non-relativistic context, we developed the notion of flux attachment to understand the statistical transmutation induced by a gauge field coupling to a scalar (Wilczek and Zee, 1983). However a different picture is needed for relativistic theories, and this seed duality provides the correct framework to describe this process.

Writing a general action of scalar 2+1 dimensional bosons ϕ in a background of gauge field A as

$$S_{\text{scalar}}[\phi; A] = \int \mathrm{d}^3x \, |(\partial_\mu - iA_\mu)\phi|^2 + \cdots, \tag{5.68}$$

where the additional terms include $|\phi|^2, |\phi|^4$ deformations. As before, the duality will hold at gapless critical theories, which means the scalar sector will be either at the

Wilson–Fisher fixed point, or at the free theory point. Choice of fixed point will determine the nature of scalar excitations, and the nature of excitations in the dual theory. The fermion action is

$$S_{\text{fermion}}[\psi; A] = \int \mathrm{d}^3x\, i\bar{\psi}(\partial\!\!\!/ - iA\!\!\!/)\psi + \cdots, \tag{5.69}$$

where again the deformations determine the phase of the theory. The details of this will be discussed below when the duality between the phases of the seed duality are more formally equated.

In this language, the seed duality can be written (Fradkin and Schaposnik, 1994; Aharony, 2015):

$$\begin{aligned}\int \mathcal{D}\psi\, \exp\!\left(i \int \mathrm{d}^3x\, \mathcal{L}_{\text{fermion}}[\psi; A] - \frac{1}{2}\frac{1}{4\pi} A \wedge \mathrm{d}A\right) \\ = \int \mathcal{D}\phi \mathcal{D}a\, \exp\!\left(i \int \mathrm{d}^3x\, \mathcal{L}_{\text{scalar}}[\phi; a] + \frac{1}{4\pi} a \wedge \mathrm{d}a + \frac{1}{2\pi} a \wedge \mathrm{d}A\right).\end{aligned} \tag{5.70}$$

This describes the duality between a fermion interacting with a background gauge field A, and a scalar which is charged under a dynamical gauge field a. The scalar action contains a 'BF' term $a \wedge \mathrm{d}A$ which couples the dynamical gauge field to the background field. Focusing on the left hand side of the duality, the second term is a half-quantised Chern–Simons term for the background field $\frac{1}{2}S_{\text{CS}}[A]$, as seen in the HLR theory which is not gauge invariant and produces a gauge anomaly on a closed manifold. However the fermion theory has the same anomaly when it is integrated out, so the anomalies cancel and this theory is consistent.

More concisely, this seed duality can be written

$$Z_{\text{fermion}}[A] e^{-\frac{1}{2}S_{\text{CS}}[A]} = Z_{\text{scalar+flux}}[A]. \tag{5.71}$$

We can recover an equation which is reminiscent of the non-relativistic flux attachment identities by taking the equation of motion of a_0 at zero background flux $A = 0$,

$$\rho_\phi = |\phi|^2 = -\frac{f_{12}}{2\pi}. \tag{5.72}$$

This attaches one unit of flux to the boson, and therefore we can predict the excitations will be fermionic.

We can consider the duality at $A = 0$ to get a sense of what form the excitations of the gauged boson theory take (Dunne, 1999). In this case the theory resembles the Abelian Higgs model, except the kinetic term for the gauge field is of Chern–Simons form and not Maxwell (K. Paul and Khare, 1987; Jatkar and Khare, 1990). This distinction, and the effect of (5.72), is to give the Nielson–Olesen vortices an intrinsic angular momentum

$$J = \frac{kQ_{\text{topo}}}{2} \tag{5.73}$$

proportional to the Chern–Simons level k. In the fermion theory defined here, we are considering the level-1 CS theory, and so excitations have half-integer spin and therefore fermionic statistics (Wu and Yang, 1976). This result demonstrates explicitly that there is one unit of flux attachment to vortex excitations and motivates why the duality (5.70) could be expected to hold: both theories have fermionic excitations. A more motivated proof of the duality of these two field theories is offered in Appendix E, and it is separately supported by numerical works on the lattice (Karthik and Narayanan, 2016).

Moreover, the $k = 0$ fermion theory (simply removing the CS term) is exactly the Abelian Higgs model $\mathcal{L}_{\text{fermion}}[\phi]$. Eq. 5.73 confirms that in this theory the Nielson–Olesen vortices have zero spin and therefore bosonic statistics, which are needed for them to be dual to the XY model's particle-like excitations.

Having demonstrated flux attachment to scalars producing a fermion, let us now investigate how (5.70) implies a seperate gauged-fermion to boson duality. This gives us a way of prescribing a 'bosonisation' procedure in 2+1 dimensions which allows fermionic path integrals to be equated to path integrals over gauge fields and bosons.

The first step is related to the procedure of formulating the ZHK particle-vortex duality from the XY-Abelian Higgs duality in Section 5.2. Called the 'ST procedure' by Witten (2003), in the case of Eq. 5.70 this involves the following steps: (1) promote the background field A to be a dynamical field and (2) couple it to a new background gauge field B with a BF term. Renaming variables $A \to b$ and $B \to A$ this combined operation takes

$$Z[A] \quad \to \quad Z'[A] = \int \mathcal{D}b\, Z[b] \exp\left(-\frac{1}{2\pi} i \int \mathrm{d}^3x\, b \wedge \mathrm{d}A \right). \tag{5.74}$$

Therefore the action changes as

$$S[\phi, \ldots; A] \quad \to \quad S'[b, \phi, \ldots; A] = S[\phi, \ldots; b] - \frac{1}{2\pi} i \int \mathrm{d}^3x\, b \wedge \mathrm{d}A\,. \tag{5.75}$$

Note that we took the coefficient of the BF term $\frac{1}{2\pi} i \int \mathrm{d}^3x\, b \wedge \mathrm{d}A$ to be -1; because of flux quantisation conditions $\int \mathrm{d}a\,, \int \mathrm{d}A \in 2\pi\mathbb{Z}$, the term is only gauge invariant for integer BF coupling.

Let us perform this procedure to both sides of (5.70) to generate a new duality; first perform the ST transform on the left side of this identity to yield

$$Z_{\text{fermion}}[A] e^{-\frac{1}{2} S_{\text{CS}}[A]} \to \int \mathcal{D}b \mathcal{D}\psi\, \exp\left(i \int \mathrm{d}^3x\, \mathcal{L}_{\text{fermion}}[\psi; b] - \frac{1}{2}\frac{1}{4\pi} b \wedge \mathrm{d}b - \frac{1}{2\pi} b \wedge \mathrm{d}A \right). \tag{5.76}$$

The δb_0 equation of motion gives the fermion's relativistic flux attachment formula (on a $\mathrm{d}A = 0$ background)

$$\rho_\psi = \psi^\dagger \psi = \frac{1}{2}\frac{f_{12}}{2\pi}, \tag{5.77}$$

where $f = \mathrm{d}b$. This attaches two units of flux to the fermions and therefore leaves their statistics unchanged. Performing the same transformation on the right hand side of

(5.70) gives

$$Z'_{\text{scalar+flux}}[A] = \int \mathcal{D}a\mathcal{D}b\mathcal{D}\phi \; \exp\left(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{scalar}}[\phi; a] + \frac{1}{4\pi} a \wedge \mathrm{d}a + \frac{1}{2\pi} a \wedge \mathrm{d}b - \frac{1}{2\pi} b \wedge \mathrm{d}A \right). \tag{5.78}$$

The field b appears linearly in the action; its variation constrains $\mathrm{d}a = \mathrm{d}A$ so integrating it out sets $b = 0$ and fixes $a = A$

$$Z'_{\text{scalar+flux}}[A] = \int \mathcal{D}\phi \; \exp\left(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{scalar}}[\phi; A] + \frac{1}{4\pi} A \wedge \mathrm{d}A \right). \tag{5.79}$$

Together Eqns. 5.76 and 5.79 are dual

$$\begin{aligned} \int \mathcal{D}b\mathcal{D}\psi \; \exp\left(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{fermion}}[\psi; b] - \frac{1}{2}\frac{1}{4\pi} b \wedge \mathrm{d}b - \frac{1}{2\pi} b \wedge \mathrm{d}A \right) \\ = \int \mathcal{D}\phi \; \exp\left(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{scalar}}[\phi; A] + \frac{1}{4\pi} A \wedge \mathrm{d}A \right), \end{aligned} \tag{5.80}$$

which is schematically

$$Z_{\text{fermion+flux}}[A] = Z_{\text{scalar}}[A] e^{iS_{\text{CS}}[A]}. \tag{5.81}$$

Finally, another set of dualities can be generated directly from Eqns. 5.70 and 5.80 by taking their time reversal. The action is invariant under this symmetry, except the Chern–Simons and BF terms which are odd. For 2+1 dimensional theories this time reversal is the same action as parity inversion. The ability to time-reverse a duality will be a key ingredient in generating the fermion-fermion duality and expanding the web of dualities beyond. Explicitly, the time reversal of the seed duality (5.70), which relates a fermion to a scalar+flux, is

$$\begin{aligned} \int \mathcal{D}\psi \; \exp\left(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{fermion}}[\psi; A] + \frac{1}{2}\frac{1}{4\pi} A \wedge \mathrm{d}A \right) \\ = \int \mathcal{D}\phi\mathcal{D}a \; \exp\left(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{scalar}}[\phi; a] - \frac{1}{4\pi} a \wedge \mathrm{d}a - \frac{1}{2\pi} a \wedge \mathrm{d}A \right). \end{aligned} \tag{5.82}$$

Similarly, the time reversal of the fermion+flux to scalar duality (5.80) is

$$\begin{aligned} \int \mathcal{D}b\mathcal{D}\psi \; \exp\left(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{fermion}}[\psi; b] + \frac{1}{2}\frac{1}{4\pi} b \wedge \mathrm{d}b + \frac{1}{2\pi} b \wedge \mathrm{d}A \right) \\ = \int \mathcal{D}\phi \; \exp\left(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{scalar}}[\phi; A] - \frac{1}{4\pi} A \wedge \mathrm{d}A \right). \end{aligned} \tag{5.83}$$

5.4.2 Fermionic Particle-Vortex Duality

Using the tools we have so far — ST operation, time reversal, integrating out dynamical fields, and dividing by the $A \wedge \mathrm{d}A$ CS term — we can finally prove the Dirac-QED$_3$ duality. Before that, we may first reproduce the original bosonic particle-vortex duality using the results we have derived thus far. Using the fermion+flux to scalar duality derived from the seed duality (5.81), divide by the external field's Chern–Simons action $e^{iS_{\mathrm{CS}}[A]}$ to give

$$Z_{\text{fermion+flux}}[A]e^{-iS_{\mathrm{CS}}[A]} = Z_{\text{scalar}}[A]. \tag{5.84}$$

Again perform the ST transform, with a BF coupling of +1 to transform this duality. The right hand side becomes a theory we will call scalar QED_3

$$Z_{\text{scalar}\,\mathrm{QED}_3}[A] = Z'_{\text{scalar}}[A] = \int \mathcal{D}b\mathcal{D}\phi\, \exp\!\left(i \int \mathrm{d}^3x\, \mathcal{L}_{\text{scalar}}[\phi; b] + \frac{1}{2\pi} b \wedge \mathrm{d}A \right). \tag{5.85}$$

The left hand side of this duality becomes

$$\begin{aligned} &Z_{\text{fermion+flux}}[A]e^{-iS_{\mathrm{CS}}[A]} \to \\ &\int \mathcal{D}a\mathcal{D}b\mathcal{D}\psi\, \exp\!\left(i \int \mathrm{d}^3x\, \mathcal{L}_{\text{fermion}}[\psi; b] - \frac{1}{2}\frac{1}{4\pi} b \wedge \mathrm{d}b - \frac{1}{2\pi} b \wedge \mathrm{d}a - \frac{1}{4\pi} a \wedge \mathrm{d}a + \frac{1}{2\pi} a \wedge \mathrm{d}A \right). \end{aligned} \tag{5.86}$$

Integrating out a using the equation of motion

$$\mathrm{d}a = \mathrm{d}A - \mathrm{d}b \tag{5.87}$$

and substituting $a = A - b$ gives

$$\int \mathcal{D}b\mathcal{D}\psi\, \exp\!\left(i \int \mathrm{d}^3x\, \mathcal{L}_{\text{fermion}}[\psi; b] - \frac{1}{2}\frac{1}{4\pi} b \wedge \mathrm{d}b - \frac{1}{2\pi} b \wedge \mathrm{d}A + \frac{1}{4\pi} A \wedge \mathrm{d}A \right). \tag{5.88}$$

Now the first three terms in this action are the time reversed fermion+flux to scalar duality (5.83); substituting this duality cancels the final $A \wedge \mathrm{d}A$ term giving the purely boson-to-boson duality, and equating to Eq. 5.85

$$\int \mathcal{D}\phi\, \exp\!\left(i \int \mathrm{d}^3x\, \mathcal{L}_{\text{scalar}}[\phi; A] \right) = \int \mathcal{D}b\mathcal{D}\phi\, \exp\!\left(i \int \mathrm{d}^3x\, \mathcal{L}_{\text{scalar}}[\phi; b] + \frac{1}{2\pi} b \wedge \mathrm{d}A \right). \tag{5.89}$$

Schematically this states

$$Z_{\text{scalar}}[A] = Z_{\text{scalar}\,\mathrm{QED}}[A], \tag{5.90}$$

which is the original bosonic particle-vortex duality (5.35).

Starting by dividing through the seed duality (5.71)

$$Z_{\text{fermion}}[A] = Z_{\text{scalar+flux}}[A]e^{\frac{1}{2}S_{\mathrm{CS}}[A]}. \tag{5.91}$$

Similarly to (5.76), we perform the ST operation on the left hand side (which is now without a CS term) and use the BF coupling +1/2. We call the resulting theory QED_3,

$$Z_{\mathrm{QED}_3}[A] = Z'_{\text{fermion}}[A] = \int \mathcal{D}b\mathcal{D}\psi\, \exp\!\left(i \int \mathrm{d}^3x\, \mathcal{L}_{\text{fermion}}[\psi; b] + \frac{1}{2}\frac{1}{2\pi} b \wedge \mathrm{d}A \right). \tag{5.92}$$

Because of the non-integral BF coupling, we must impose the more restrictive background flux quantisation condition

$$\int_{S^2} \frac{\mathrm{d}A}{2\pi} = 2\mathbb{Z}. \tag{5.93}$$

This technicality is trivial to impose on the background gauge field A because we have total control over its value, however promoting it to be dynamical may pose an issue. The solution is that the new connection b is not a $U(1)$ gauge connection but a Spin_c connection which is consistent with this quantisation condition (Seiberg et al., 2016).

The right hand side of (5.91) becomes

$$Z_{\text{scalar+flux}}[A]e^{\frac{1}{2}S_{\text{CS}}[A]} \to$$
$$\int \mathcal{D}a\mathcal{D}b\mathcal{D}\phi \, \exp\Big(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{scalar}}[\phi; a] + \frac{1}{4\pi} a \wedge \mathrm{d}a + \frac{1}{2}\frac{1}{4\pi} b \wedge \mathrm{d}b + \frac{1}{2}\frac{1}{2\pi} b \wedge \mathrm{d}A \Big). \tag{5.94}$$

Integrating out b using the equation of motion

$$\mathrm{d}b = -(2\,\mathrm{d}a + \mathrm{d}A) \tag{5.95}$$

and substituting $b = -(2a + A)$ gives

$$\int \mathcal{D}a\mathcal{D}\phi \, \exp\Big(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{scalar}}[\phi; a] - \frac{1}{4\pi} a \wedge \mathrm{d}a - \frac{1}{2\pi} a \wedge \mathrm{d}A - \frac{1}{2}\frac{1}{4\pi} A \wedge \mathrm{d}A \Big). \tag{5.96}$$

Now note that the time reversed seed duality (5.83) multiplied through by $e^{\frac{1}{2}S_{\text{CS}}[A]}$ is equal to (5.96). Therefore, recalling that this expression (5.96) is dual to (5.92), we find

$$\int \mathcal{D}\psi \, \exp\Big(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{fermion}}[\psi; A] \Big) = \int \mathcal{D}b\mathcal{D}\psi \, \exp\Big(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{fermion}}[\psi; b] + \frac{1}{2}\frac{1}{2\pi} b \wedge \mathrm{d}A \Big) \tag{5.97}$$

or

$$Z_{\text{fermion}}[A] = Z_{\text{QED}_3}[A]. \tag{5.98}$$

This completes the derivation of the fermion-to-fermion duality, derived from the seed duality, which forms the foundation of the interpretation of the Dirac composite fermion model.

Discussion & Conclusion 6

This dissertation has provided a review of the use of the Chern–Simons action in describing both the integer and fractional quantum Hall effects. This is the effective theory of a system of electrons in 2+1 dimensions, split into Landau levels by a strong magnetic field.

The tools of topology, quantum field theory (including knowlege of anomalies, effective theories, vortices, *etc.*), conformal field theory, and dualities — which have been developed in the works cited by this thesis — have provided a much deeper understanding of the quantum Hall system.

The Chern–Simons action was first motivated by renormalisation group arguments, providing grounds for the universality of this phase in quantum 2+1 dimensional systems. The action was specifically derived in the zero magnetic field case before more recent work was studied which were used to derive the same general form of the action in the case where the background $B \neq 0$. The universality of the Chern–Simons action as an effective theory in such a scenario was upheld: we indeed found that the same action arose in this different context. In the calculation of integrating out massive fermions, the effect of the background field was to change the Hilbert space of states accessible to intermediate virtual photons. The Dirac spectrum was split into a tower of Landau levels which extended to negative infinity, and the Dirac sea was replaced by a filled sea of negative Landau levels, and a half-filled zero level. Compared to a non-relativistic model without such a 'Dirac ladder' of negative energy states, the relativistic model contributes $\frac{1}{2}$ to the CS level.

The fractional quantum Hall effect is characterised by an emergent and dynamic $U(1)$ gauge field, which endows it with a significantly richer structure than its integer counterpart. Excitations of this statistical gauge field were shown to be very massive particles, and its quantisation in the presence of Wilson lines clearly presented how fractional statistics arise in this model. In contrast with the earlier analysis of the Laughlin wavefunction, this perspective gave a much deeper understanding of the topological nature of the phase and how anyonic braiding could be generalised to non-Abelian theories.

This dissertation included a discussion of a novel proposal to deal with the incompatibility inherent in the flux quantisation of the two gauge fields of the FQHE. We reviewed a novel approach to integrating out 1-form gauge fields in 3 dimensions, and outlined the steps needed to use this calculation to evaluate the partition function for classically forbidden charge-flux configurations. Evaluation of this path integral holds promise to clarify the incompatibility of the EM and statistical gauge field's flux quantisation conditions. The common understanding in the literature is that the only physical solutions are ones which contain m units of statistical flux for every unit of EM flux, which implies a picture where anyons can only be produced by EM excitations in groups of m. Any non-trivial behaviour of the path integral which allow for more exotic behaviours would be a fascinating route of further study.

Next, the role of the chiral anomaly played a central role in our discussion of the conformal edge. The Nielson–Ninomiya theorem forbids chiral fermions from existing in a purely 1+1 dimensional theory, but we observed how they naturally emerged from the quantum Hall boundary. This was solved when we described it as being a fermion zero mode at the boundary of a Chern–Simons theory, and showed how the chiral anomaly was cancelled by the anomaly inflow of the bulk's gauge anomaly. In the setting of the fractional QHE, the edge anomaly could be understood by imposing a gauge constraint on the statistical gauge field. This promoted some degrees of freedom to become dynamical, and we explicitly got to see how its quantisation led to a chiral fermion emerging in this context as well.

The 1+1 dimensional bosonisation duality was used to quantise this edge fermion as a chiral Luttinger liquid, and in this process the formalism of Schwinger terms had to be used. We drew the connection between these terms and the chiral anomaly, and showed that this bosonised theory's excitations necessarily obey the Kac–Moody algebra.

The study of two dimensional phases of matter arising in condensed matter systems remains the focus of an active and diverse research community. Topological phases have a unique connection to dimensionality, and the field of quantum materials offers abundant scope to customise these systems.

We have seen many times throughout this dissertation how 2+1 dimensional systems are unique and lead to distinctive physics. Firstly, anyons are exclusive to two spatial dimensions, a fact which can be motivated by noticing how braiding one particle around another is only a well-defined topological operation in this context (in higher dimensions one may lift the loop out of the plane to close the loop). For this reason, the action of braiding may not return the state to itself, permitting so-called non-Abelian statistics. In the picture we have formulated in Chapter 2, this looping ties a knot in the worldlines of the two particles, and for the special topological field theories we focus on, this is categorised by a knot invariant which fully determines the amplitude of the braiding process.

Moreover, 2+1 dimensions were special because, given a gauge field A, we could define an identically conserved topological current J. This was used at several points, most notably to construct the CS action as the unique three-form marginal operator in 2+1 dimensions. Later in the story of dualities, the global symmetry associated with this current (in the Abelian Higgs model) was able to be identified with a global symmetry of the XY model, forming the basis of our first particle-vortex duality. Such dualities are not possible to formulate in higher dimensions, and the programme of applying these dualities to the QHE which followed relied upon these unique non-perturbative tools.

The anomaly cancellation of the bulk by a chiral anomaly at the edge is also only possible (for physical systems) in 2+1 dimensions. The chiral anomaly is responsible for the presence of robust gapless edge modes, but it only exists in even dimensional boundary spacetimes (such as 1+1, 3+1, *etc.*). Although the same physics emerges on the boundary of a 4+1 dimensional spacetime, this situation is not relevant for any physical experiments that one could easily imagine.

Therefore such topological phases — characterised by such an anomaly at their boundary — can only exist in odd dimensions. In order to reproduce such protected gapless edge modes in different dimensions requires more engineering, but is still possible. Topological insulators in 3+1 dimensions have recently been proposed, and are capable of hosting surface Dirac cones and even surface Weyl fermions. However these phases do not possess the long-range entanglement which classifies intrinsic topological phases, but rather the edge fermions are protected more loosely by a set of discrete symmetries of the bulk. These 3+1 systems have interesting properties driven by the chiral anomaly too, but it is still not possible to derive holography in such a system.

There is recent progress in the application of relativistic field theories to derive condensed matter phenomena. Notably Kaplan and Sen (2020) present a relativistic $U(1) \times U(1)$ gauge theory containing three species of fermions, which reproduces the fractional QHE in the IR (*i.e.* it recovers the CS effective action with fractional level after integrating out the fermions and one of the gauge fields). This model also describes the quantum spin Hall effect, wherein the Hall current does not transport charge, but instead spins (Kane and Mele, 2005).

This work draws an interesting connection with the study of of chiral gauge theories using domain-wall fermions; the tool of constructing anomalous theories by localising fermions in higher dimensions along a lower-dimensional surface was first used to describe this spin Hall effect. This effect was noticed before it gained traction in condensed matter: using this localisation method one may construct a gauge theory where the bulk current associated with the gauge anomaly does not carry a charge under the global group, but instead lives in a non-trivial representation of some other 'flavour group' (Kaplan, 1992). In the language of the QHE, the Hall current is therefore not electrically charged, but through choosing the flavours of the chiral fermion to be the spins, this clearly describes the quantum spin Hall effect. Furthermore, the emergence of Majorana zero modes were first introduced in the context of Lattice supersymmetry (Kaplan and Schmaltz, 2000) through such a localisation argument. Now these modes are subject to widespread interest in condensed matter, particularly their emergence in superconducting topological nanowires due to their ability to act as the non-Abelian anyons needed for topological quantum computation (Sarma et al., 2015).

In the bulk, topological order arose as a ground state degeneracy which depends on the topology of the manifold. We briefly commented on how more recent work has highlighted the long-range entanglement of topological phases as being another defining feature of topological order. We should now highlight that the study of the entanglement entropy has been pioneered in other topological models like the toric code or Kagome spin liquids (Jiang et al., 2012), and the generalisation to the entanglement spectrum by Li and Haldane (2008) in the case of the non-Abelian FQHE. These techniques provide numerically accessible measures of topological order and are now considered an indispensable tool in the study of general topological phases.

The deep structure of the 2+1 dimensional TQFT formed the foundation of the bulk-boundary correspondence. After arguing that the boundary boson theory had a conformal symmetry group, we identified the vertex operators which were shown to form a RCFT algebra through their fusion properties. In the description of the Laughlin state, the level-1 operator was the quasihole operator, and at level-m the excitation was an electron. The rational nature of this algebra and Abelian fusion rules were derived using the OPE of these vertex operators using a simple chiral boson theory.

We then considered a conformal mapping which inserts operators on the boundary and takes them on a path along a Wilson line in the bulk. The evaluation of correlation functions of vertex functions at the 1+1 dimensional boundary could recover the Laughlin wavefunction on a 2+0 dimensional bulk slice. This suggests a duality which is simple to understand for the Laughlin case, but its generalisation is possible: the Hilbert space of the Wilson lines at some constant-time surface is exactly the Hilbert space of conformal blocks of the boundary theory. These blocks are the spaces of possible fusion pathways between non-Abelian conformal boundary operators.

This conjectured general correspondence is lauded as a method of predicting novel QHE phases, and has been successfully applied to the Moore–Read, Read–Rezayi, and many more states (Moore and Read, 1991; Read and Rezayi, 1999).

The demonstration of such a duality involved inserting a macroscopic number of vertex operators on the edge and drawing them along a path through the bulk which defines a conformal map. This nicely recovered the Laughlin wavefunction, although the duality is expected to hold more generally too. Instead of inserting the electrons at the edge at some finite time, consider the $t = -\infty$ slice as the initial boundary condition. Threading a flux B through this state will naturally set the number of electrons in the system at this initial time, and evolving the system under its unitary time evolution will recover the same Laughlin wavefunction at a later time slice.

In this sense it does not matter on which boundary we choose to define the boundary conditions — the TQFT/CFT duality will hold up whatever we choose. Indeed, the system should also be independent of the mapping we choose to relate the coordinates of the edge and the slice. Different coordinates will not recover exactly the Laughlin wavefunction, but it will represent the same physical system. The new correlation function will be related to the Laughlin wavefunction through the symmetry of the state.

Bulk–boundary correspondences are a hot topic in string theory and cosmology, particularly centred around the study of the proposed anti-de Sitter space to conformal field theory (AdS/CFT) correspondence. There is also a field of theoreticians who are interested in forming dualities between emergent conformal field theories of condensed matter to gravitational theories (AdS/CMT correspondence), with work particularly focused around tensor models (such as the SYK model) (Rosenhaus, 2019) and around superconductors (Hartnoll et al., 2008). The excitement around these proposals have spurred an intense and rapid development of this work over the space of only a decade — perhaps motivated by the tantalising possibility of experimental realisation (Danshita et al., 2017).

There has therefore naturally been much study of the use of the AdS/CFT correspondence to form dual gravitational descriptions of the quantum Hall effect. A specific work by Keski-Vakkuri and Kraus (2008) draws a parallel between the ZHK model of the quantum Hall anyons to a model of a superconductor, and leverages the framework of the AdS/CFT correspondence in that context to propose a AdS/QHE correspondence.

The final chapter of this dissertation focused upon the description of particle-vortex dualities in the quantum Hall system. These dualities have been known in their most basic form since the late '70s, but only recently has it been realised that these are implied by a different 'bosonisation duality', which equates theories of a Wilson–Fisher scalar plus a dynamic flux to a theory of a free fermion. In our discussion of the ZHK model it became clear how the Chern–Simons term attaches flux to scalars and may change their statistics. This result, generalised to the relativistic case, forms the foundation of the bosonisation duality. This proposed duality represents a novel extension of bosonisation, which has previously been derived only in 1+1 dimensions.

This work explicitly follows the recent literature and derives a particle-vortex duality between two fermion models from the seed bosonisation duality, which was the crucial ingredient needed by Son to write down the Dirac composite fermion theory of the half-filled Landau level. The merits of this discovery were discussed extensively, and we particularly highlighted how particle-hole symmetry can be used to make the case for this novel theory over the HLR model. We also showed that applying a background magnetic field upholds the duality, and simply corresponds to changing the filling of the composite fermions. The calculation of conductivities was however adjusted in the usual way by the presence of the Dirac ladder vacuum.

The Dirac CF model and the duality which underpins it has a deep connection to the physics of topological insulators in 3 spatial dimensions (Alicea, 2015). There has long been evidence of some form of connection between the half-filled Landau level and TIs (Ludwig et al., 1994), but more recent work which constructed gapped surface states of a TI which are similar to the Moore–Read FQHE phase has refocused interest (Bonderson et al., 2013). Now, as stated in the body of Chapter 5, Son's duality of the Dirac composite fermion provides a framework to understand this connection (Wang and Senthil, 2015; Metlitski and Vishwanath, 2016). The work which followed this has made fascinating connections to other aspects of condensed matter theory — even deriving connections between topological insulators to topological superconductors (Murugan and Nastase, 2017) and quantum spin liquids (Wang and Senthil, 2016).

Furthermore, the bosonisation duality itself has a history in the high-energy theory community. There has been interest in dualising Chern–Simons theories since the '90s: starting with dualities of supersymmetric theories (Intriligator and Seiberg, 1996; Aharony et al., 1997) which satisfy numerous consistency checks, including exact calculation of partition functions and operator dimensions, and the convergence of the theories in the large-$\mathcal{N}$ limit. One may then break supersymmetry to arrive at TQFT dualities by introducing masses to scalars of the theory, a procedure which results in the 'level-rank' dualities of CS theories (including motivation for our seed duality) (Aharony, 2015).

The recent profound addition from the literature is the use of this seed duality to produce a fermionic particle-vortex duality, as previously detailed in this work. Similarly, through introducing other deformations to both sides of the duality, one may generate a 'duality web' containing an unlimited number of new dualities which can be constructed from combining different species of fermions and different charges of the gauge fields (Seiberg et al., 2016; Karch et al., 2017).

The extension of bosonisation to 2+1 dimensions provides a rare and valued non-perturbative tool which may be applied to typically intractable regimes of gauge theories. One such case is the strong coupling regime of the 2+1 dimensional Thirring model, which has now been successfully bosonised for the first time (Santos et al., 2020). This duality extends the seed duality in a familiar way (by introducing deformations), and one must therefore be careful to check the result still holds. It stands up in certain tractable limits, and the new UV fixed point of the 2+1 dimensional Thirring model which is predicted by this duality is consistent with other works (Gies and Janssen, 2010).

This work has inspired proofs of the bosonisation duality which take diverse approaches: including recalling the historical supersymmetry-breaking scheme (Kachru et al., 2016) to a more modern 'quantum wires' formalism (Mross et al., 2016). In this process, one of the theory's 2+1 dimensions are discretised on a lattice such that the resultant problem becomes a set of 2 dimensional ones. Regular 2 dimensional bosonisation is then leveraged to demonstrate the duality. Indeed this wires formalism has been applied to the 2+1 dimensional Thirring model and verified the correspondence in that case (Hernaski and Gomes, 2018). Most recently however, the duality was shown by equating the partition functions of both theories on a fully 3 dimensional lattice (Chen et al., 2018).

The quantum Hall effect is a cornerstone of condensed matter physics which remains at the forefront of research. It provides a very real bridge between the exotic theory of modern high-energy physics to the details of experiment. This work has brought into focus the connections between effective theories, topology, bosonisation, conformal field theories, and the dualities, and applied them to this system of electrons in a magnetic field.

Fundamentals — Appendix A

Berry Phase

Consider a Hamiltonian which depends on a set of external parameters $H(\vec{\lambda})$; variation of the parameters $\lambda_i(t)$ will lead to an evolution of the eigenstates by the adiabatic theorem $|phi(t)\rangle = U(t)\left|n, \vec{\lambda}(t)\right\rangle$. Under a closed loop, we must return to the same state up to a phase,

$$\left|n, \vec{\lambda}(0)\right\rangle \to e^{i\gamma_{\text{dyn}}+i\gamma}\left|n, \vec{\lambda}(0)\right\rangle \tag{A.1}$$

where γ is the topological Berry phase, and the dynamical phase depends on the path taken $\gamma_{\text{dyn}} = -\int_0^t E(t')\,\mathrm{d}t'$.

The Berry phase (for a given n) can be written in terms of a Berry connection using the time-dependent Schrödinger equation (Berry, 1984; Witten, 2015),

$$\mathrm{d}\gamma = \dot{\gamma}\,\mathrm{d}t = i\dot{\vec{\lambda}} \cdot \vec{A}(\lambda)\,\mathrm{d}t\,, \tag{A.2}$$

$$\vec{A}(\lambda) = i\left\langle n(\vec{\lambda})\right|\vec{\nabla}_\lambda\left|n(\vec{\lambda})\right\rangle. \tag{A.3}$$

Hence integrating over a closed curve gives (in terms of the one-form over parameter space $A = \mathrm{d}\gamma$)

$$\gamma = \oint \mathrm{d}\vec{\lambda} \cdot \vec{A}(\lambda) = \oint A. \tag{A.4}$$

To draw an analogy with a more familiar physical system, this Berry phase γ would be the same as the Arhanov–Bohm (AB) phase, if the vector $\vec{A}$ were a real vector potential. This new phase has many of the same properties as the AB phase, but the connection is a one-form defined over parameter space and not real space. The flux of this fictional gauge field is the Berry curvature, given $F = \mathrm{d}A$, which is equivalent to the field strength in the EM analogy.

One can create a monopole of such a Berry curvature which — just as for the AB effect — leads to a phase being acquired when a particle is taken on a closed loop, proportional to the subtended flux. The expression for the Berry (or AB) phase is equal to the expectation value of the Wilson operator moving in an Abelian 'geometric' gauge field with connection $A(\vec{\lambda})$. In accordance with gauge invariance of the Wilson loop observable, the phase is indeed invariant under gauge transform in $\left|n, \vec{\lambda}\right\rangle$ which is equivalent to $A_i \to A_i + \mathrm{d}\chi_i$.

This motivates defining a Berry curvature $F = \mathrm{d}A$ which is itself gauge invariant. Using Stoke's theorem for any surface $\mathcal{S}$ such that $\mathcal{C} = \partial\mathcal{S}$ and expressing the surface element as $\mathrm{d}S^i$ we derive the expression for the Berry phase

$$\gamma = \oint_{\mathcal{C}} A = \oint_{\mathcal{C}} A_i\,\mathrm{d}\lambda_i = \int_{\mathcal{S}} F_i\,\mathrm{d}S^i\,. \tag{A.5}$$

Written in terms of states, using (A.2), the Berry curvature is

$$\vec{F} = i\vec{\nabla}_\lambda \times \left\langle n(\vec{\lambda})\middle|\vec{\nabla}_\lambda\middle|n(\vec{\lambda})\right\rangle . \tag{A.6}$$

This flux is sourced by a 'monopole' in parameter space, which we can show must be quantised using the AB setup. Consider passing a particle around a total flux

$$Q = \frac{1}{2\pi}\int_S F_i\,\mathrm{d}S^i\,, \tag{A.7}$$

then the Berry phase picked up along a closed loop is $e^{2\pi i Q}$. For the monopole to be physical it cannot affect any observables, so Q must be quantised. This is called the Dirac quantisation condition. A useful expression for the Berry curvature is found by expanding on a basis of states, this gives in components

$$F_i = -\operatorname{Im}\varepsilon_{ijk}\sum_{m\neq n}\frac{\langle n|\,\partial_j H\,|m\rangle\,\langle n|\,\partial_k H\,|n\rangle}{[E_n - E_m]^2}. \tag{A.8}$$

Differential Geometry

We will now introduce the necessary language of differential forms which will be used extensively in this dissertation (Nakahara, 2003). A p-form simply lives in the subspace of $(0,p)$-tensors that are fully antisymmetric on indices. We call the space of p-forms Ω_p. Almost trivial examples of such forms are zero-forms, which are scalar functions, and one-forms, which are simply covector functions. Expressing the general one-form in a *coordinate basis*, we may write $A = A_\mu\,\mathrm{d}x^\mu$. Using this basis makes the construction of forms particularly simple to understand.

Introduce the Cartan wedge product, which is the antisymmetrised tensor product of basis elements

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu\,. \tag{A.9}$$

The general Cartan product of p basis elements forms a basis for Ω_p, the general p-forms:

$$\omega = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\,. \tag{A.10}$$

The Cartan product therefore forms a product on the space of forms $\wedge : \Omega_p \times \Omega_q \to \Omega_{p+q}$, for example

$$\omega\wedge\chi = \frac{1}{p!q!}\omega_{\mu_1\cdots\mu_p}\chi_{\mu_1\cdots\mu_q}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{p+q}}\,, \tag{A.11}$$

$$= \frac{1}{(p+q)!}(\omega\wedge\chi)_{\mu_1\cdots\mu_{p+q}}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{p+q}} \tag{A.12}$$

and so we may identify the components

$$(\omega\wedge\chi)_{\mu_1\cdots\mu_{p+q}} = \frac{(p+q)!}{p!q!}\omega_{\mu_1\cdots\mu_p}\chi_{\mu_1\cdots\mu_q}. \tag{A.13}$$

This product has a graded commutivity $\omega \wedge \chi = (-1)^{pq} \chi \wedge \omega$, and satisfies $\omega \wedge \omega = 0$. The product is only defined when $p+q \leq m$, where m is the dimension of the manifold.

We may now define the exterior derivative $\mathrm{d} : \omega_p \to \Omega_{p+1}$ which is defined through its action as a differential operator. For example, on a one-form it acts like

$$\mathrm{d}A = \mathrm{d}A_\mu \wedge \mathrm{d}x^\mu = \partial_\nu A_\mu \, \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \, . \tag{A.14}$$

On a general form it acts like

$$\mathrm{d}\omega = \frac{1}{p!} (\partial_\mu \omega_{\mu_1 \cdots \mu_p}) \, \mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \, . \tag{A.15}$$

It satisfies a graded Leibniz rule and it is nilpotent, so $\mathrm{d}^2 = 0$.

A top-form is a m-form on an m dimensional manifold $\mathcal{M}$. A volume form v is a nowhere-vanishing top form on $\mathcal{M}$, which can be integrated over the manifold to provide a measure of its volume

$$\mathrm{vol} = \int_\mathcal{M} v. \tag{A.16}$$

In the familiar coordinate basis, there is a natural volume form on a Lorentizian manifold defined using the determinant of the metric and the totally antisymmetric Levi–Civita symbol

$$\begin{aligned} \mathrm{vol} &= \int_\mathcal{M} v_{\mu_1 \cdots \mu_m} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_m} & \text{(A.17)} \\ &= \int_\mathcal{M} \sqrt{-\det g_{\mu\nu}(x)} \, \varepsilon_{\mu_1 \cdots \mu_m} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_m} \, . & \text{(A.18)} \end{aligned}$$

We may generally integrate any top-form over the manifold, which can be evaluated explicitly in the coordinate basis. We will also use Stokes' theorem, which relates the integral over an *exact* form (which is the exterior derivative of lower-rank form) to an integral of the lower-rank form over the boundary $\partial\mathcal{M}$,

$$\int_\mathcal{M} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega. \tag{A.19}$$

The Hodge dual $\star\omega$ of a form defines a map $\star : \omega_p \to \omega_{m-p}$, which can be taken in components by dualising with the volume form

$$(\star\omega)_{\mu_{p+1} \cdots \mu_m} = \frac{1}{p!} v_{\mu_1 \cdots \mu_m} \omega^{\mu_1 \cdots \mu_p}. \tag{A.20}$$

Importantly, the Hodge dual of any scalar function is proportional to the volume form, and the square of the Hodge dual operator is proportional to the identity

$$\star^2 \, \omega = (-1)^{m(p+1)+1} \omega. \tag{A.21}$$

One may also define the adjoint derivative $\mathrm{d}^\dagger : \Omega_p \to \Omega_{p-1}$, through $\mathrm{d}^\dagger \omega = (-1)^{m(p+1)+1} \star \mathrm{d} \star \omega$. Importantly, on one forms w, this acts as a divergence operator

$$\mathrm{d}^\dagger w = -\partial_\mu w^\mu. \tag{A.22}$$

The Laplacian may also be defined in this language:

$$\Delta = \mathrm{d} \star \mathrm{d}\star - \star \mathrm{d} \star \mathrm{d}\,, \tag{A.23}$$

which acts on closed forms ($\mathrm{d}w = 0$) as

$$\Delta w = -\partial^2 w. \tag{A.24}$$

Quantum Field Theory

We will now briefly introduce some of the tools of QFT which will be used in this work. Theories are defined by an action $S[\phi, \psi, \dots]$ as a functional of sole set of bosonic and fermionic fields. This action is minimised on the classical solution

$$\delta S[\phi(x)] = \int \mathrm{d}^d y \, \frac{\delta S}{\delta\phi(x)} \delta\phi(y) = 0, \quad \Longrightarrow \quad \frac{\delta\phi(x)}{\delta\phi(y)} = 0, \tag{A.25}$$

which implies the classical equations of motion.

Under a general variation $\delta\phi$, the action will also vary by total-derivative terms

$$\delta S = \int \mathrm{d}^d y \, \frac{\delta S}{\delta\phi(x)} \phi(y)\delta\phi(y) + \int \mathrm{d}^d y \, \partial_\mu J^\mu. \tag{A.26}$$

If $\delta\phi$ generates an internal symmetry of the action, then the bulk term is automatically zero, but for the boundary term to also be zero then it must be conserved $\partial_\mu J^\mu = 0$. This is the conserved current associated with the symmetry operation.

The quantum theory is defined by the path integral of the sourced action (Srednicki, 2007)

$$Z[J] = \int \mathcal{D}\phi \exp\Big(iS[\phi] + i \int \mathrm{d}^d x \, J(x)\phi(x)\Big). \tag{A.27}$$

The connected correlation functions are generated through

$$\exp(iW[J]) = \frac{Z[J]}{Z[0]} \tag{A.28}$$

by taking functional derivatives

$$\langle \phi(x_1) \cdots \phi(x_N) \rangle = \frac{\delta^N \exp(iW[J])}{\delta J(x_1) \cdots \delta J(x_N)}. \tag{A.29}$$

Writing the classical solution as $\overline{\psi}$ one may perform a loop expansion in the action by expanding around this path $\phi = \overline{\psi} + \chi$. Performing a Legendre transform of the action to turn it into a function of the new variable, we find the quantum effective action which has the following loop expansion

$$\Gamma[\overline{\phi}] = S[\overline{\phi}] + \Gamma^{(1)}[\overline{\phi}] + \cdots, \tag{A.30}$$

where $S[\overline{\phi}]$ is the classical (or *tree-level* action). Calculating derivatives of $\exp\big(i\Lambda[\overline{\phi}]\big)$ will calculate the 1-point irreducible correlation functions, which includes all interacting diagrams which contribute to the amplitude. The term $\Gamma^{(1)}$ includes contributions from one-loop diagrams, and will act to renormalise the potential of the action $V(\phi)$.

At one-loop order we need the expansion of the action around a minimum

$$S[\psi] = S[\overline{\psi}] + \frac{1}{2}\int \mathrm{d}^d x \int \mathrm{d}^d y\, \chi(x) \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\chi(y) + \cdots, \tag{A.31}$$

which gives the following expression for the 1-loop quantum effective action

$$e^{i\Gamma^{(1)}[\overline{\phi}]} = \frac{\int \mathcal{D}\chi \exp\left(\frac{i}{2}\,\mathrm{d}^d x\, \chi[\partial^2 - m^2 + i\epsilon - V''(\overline{\phi})]\chi\right)}{\int \mathcal{D}\chi \exp\left(\frac{i}{2}\,\mathrm{d}^d x\, \chi[\partial^2 - m^2 + i\epsilon]\chi\right)}. \tag{A.32}$$

Evaluating this bosonic Gaussian integral gives the effective action at 1-loop

$$e^{i\Gamma^{(1)}[\overline{\phi}]} = \sqrt{\frac{\det\left(\partial^2 - m^2 + i\epsilon - V''(\overline{\phi})\right)}{\det\left(\partial^2 - m^2 + i\epsilon\right)}}, \tag{A.33}$$

which may be written

$$\Gamma^{(1)}[\overline{\phi}] = \frac{i}{2}\operatorname{tr}\log\left[1 - (\partial^2 - m^2 + i\epsilon)^{-1}V''(\overline{\phi})\right], \tag{A.34}$$

which is amenable to perturbative expansion in powers of the coupling of the potential.

Generally this one-loop action will be divergent, and we must employ a regularisation scheme to separate and then manage the divergent component from a finite term. In systems with odd dimension, like the 2+1 dimensional cases we focus on, the tools of 'dimensional regularisation' are not so helpful. This is because this scheme tracks only logarithmic divergences and their contribution to the renormalisation of parameters. However in odd dimensions one often comes across linear divergences, which have no sub-leading logarithmic components. We will instead use the Paili–Villars regulator which involves adding massive unphysical particles and integrating them out, through a process described later.

One may also write down path integrals of fermions ψ using Grassmann numbers, which are classical numbers which anticommute (as expected for the fermion). The symmetry properties of the theory may be derived in the same way, and a similar expression for the one-loop effective action can be obtained — this is done explicitly in the main body.

Fermions in Lower Dimensions

There are more differences when doing QFT in 2+1 dimensions: most notably the Dirac matrices obey different relations. Choose the chiral basis of 2×2 Dirac matrices, which can be written in terms of the Pauli matrices

$$\gamma^0 = -i\sigma_y, \quad \gamma^1 = \sigma_x, \quad \gamma^2 = \sigma_z. \tag{A.35}$$

These satisfy $(\gamma^0)^\dagger = -\gamma^0$ and $(\gamma^i)^\dagger = \gamma^i$ and live in the Clifford algebra. Furthermore they satisfy the following identities — crucially the trace of odd numbers of matrices does not vanish

$$\operatorname{tr}(\gamma^\mu\gamma^\nu) = 2g^{\mu\nu}, \quad \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = -2\varepsilon^{\mu\nu\rho}. \tag{A.36}$$

In this basis one may also define the chirality operator $\gamma^5 = \sigma_z$ which is used for the construction of

We will also briefly work with 1+1 dimensional QFT (notably in the next Appendix); the only relation which will be important is the generalisation of the standard chiral trace identity (from $d = 4$) which is in $d = 2$

$$\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^5) = -2i\varepsilon^{\mu\nu}. \tag{A.37}$$

Effective Theories and Renormalisation Group

The act of including ϕ interactions in some renormalisation scheme will adjust the potential of the scalar and make it dependent upon some energy scale μ. Since the physical observables like mass must not depend upon this scale, the couplings which appear in the action will be forced to take up μ-dependence.

The behaviour of this dependence can be predicted using crude dimensional arguments: the coupling constants of dimension-N operators $\mathcal{O}_N$ have dimension $[\lambda_N] = d - N$ will scale as μ^{d-N}. Taking the limit $\mu \to 0$ will manifest the behaviour of the coupling at low energies, and clearly it will only be *relevant* (or increasing) in this IR regime when $N < d$. Operators with $N > d$ are termed *irrelevant* in the IR, but grow in the UV. In between live *marginal operators* which scale logarithmically and their behaviour depend on details of loop calculations.

The renormalisation of a theory can be specified by integrating out high-energy degrees of freedom; the removal of these modes acts to 'renormalise' the coupling constants of a theory, generating a flow in the space of coupling constants. Fixed points under the renormalisation group are points where integrating out high-momentum modes does not modify the constants of the theory, and they are scale invariant with power-law correlation functions. The action of irrelevant operators under the renormalisation group flow is to drive the system towards a fixed point, and relevant operators move the theory away from one.

Suppose we have a theory with multiple degrees of freedom with a hierarchy of masses: there are heavy and light particles in the theory. Dividing the functional integral's measure into an integral over all heavy and light degrees of freedom (relative to a scale Λ) and then 'integrating out' the heavy degrees of freedom will produce an *effective theory* which is valid at energy scales $E \ll \Lambda$. Any heavy particles with masses $m > \Lambda$ will be removed from the spectrum entirely, and heavy-light interactions will generate a tower of couplings which renormalise the light degrees of freedom. This effect is encoded in the Wilsonian effective action, defined in terms of the path integral (in analogy with the connected generating functional $W[J]$) by

$$e^{iS_{\mathrm{eff}}[\phi_{\mathrm{light}}]} = \frac{\int \mathcal{D}\phi_{\mathrm{heavy}}\, e^{iS[\phi_{\mathrm{light}},\phi_{\mathrm{heavy}}]}}{\int \mathcal{D}\phi_{\mathrm{heavy}}\, e^{iS[0,\phi_{\mathrm{heavy}}]}}. \tag{A.38}$$

Such a Wilsonian action $iS_{\mathrm{eff}}[\phi_{\mathrm{light}}]$ includes a tower of irrelevant operators, but by repeatedly integrating out the high-energy degrees of freedom of the light-mass fields one may approach the IR limit of the theory where only the relevant (or marginal) Wilsonian operators dominate the physics (Burgess, 2007).

1+1D Chiral Anomaly — Appendix B

This Appendix will derive the chiral anomaly of the chiral fermion theory, following the methods of Fujikawa (1979) but specialised to the case of 1+1 dimensions. The action for the fermion is

$$S = \int \mathrm{d}^2 x\, \overline{\psi}(i\not{D})\psi, \tag{B.1}$$

where the covariant derivative in components is $D_\mu = \partial_\mu - ieA_\mu$. This is classically invariant under the vector and axial tranformations

$$\text{Vector}: \quad \psi(x) \to e^{ie\lambda(x)}\psi \tag{B.2}$$

$$\text{Axial}: \quad \psi(x) \to e^{ie\gamma^5\lambda(x)}\psi. \tag{B.3}$$

Because of the corresponding transform of the vector potential $A_\mu \to A_\mu - i\partial_\mu\lambda(x)/e$, the vector transform is seen to be the gauge transform of the theory. Under a general field variation which is a symmetry, the only variation of the action is a boundary term

$$\delta S = \int \mathrm{d}^2 x\, \partial_\mu(i\overline{\psi}\gamma^\mu\delta\psi), \tag{B.4}$$

where the term in brackets defines the current of the symmetry. Specifically for the axial symmetry, the current is

$$J_A^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi, \tag{B.5}$$

which is seen to be conserved classically $\partial_\mu J_A^\mu = 0$.

The aim of this Appendix is to formalise how the quantum theory may violate this current conservation using the path integral formalism. Given that the action is explicitly invariant, the non-conservation must arise from the functional path integral measure. Under the axial field transformation $\psi \to \psi' = \psi + \delta\psi$, the fermionic measure changes by a Jacobian

$$\mathcal{D}\overline{\psi}\mathcal{D}\psi \to \mathcal{D}\overline{\psi}'\mathcal{D}\psi' = \mathcal{J}^{-2}\mathcal{D}\overline{\psi}\mathcal{D}\psi. \tag{B.6}$$

In order to evaluate this Jacobian of the functional measure, we must expand the fermionic fields in terms of a bosonic eigenbasis ϕ_n with Grassmann coefficients

$$\psi = \sum_n \alpha_n\phi_n \tag{B.7}$$

$$\overline{\psi} = \sum_n \overline{\beta}_n\phi_n^\dagger. \tag{B.8}$$

The eigenvalue equation satisfied is $(i\not{D})\phi_n = \lambda_n\phi_n$. The measure in this basis is

$$\mathcal{D}\psi = \sum_n \mathrm{d}\alpha_n\,, \tag{B.9}$$

and since the variation of one component of the measure is under one power of the Jacobian,

$$\mathcal{D}\psi \to \mathcal{D}\psi' = \sum_n \mathrm{d}\alpha'_n = \sum_{nm} \mathcal{J}_{nm}\,\mathrm{d}\alpha_m\,, \tag{B.10}$$

we may hope to evaluate the Jacobian.

Evaluating the following integral over the transformed field ψ' and a dual basis vector $\phi_m^\dagger$ gives

$$\int \mathrm{d}^2x\,\phi_m^\dagger\psi' = \sum_n \int \mathrm{d}^2x\,\phi_m^\dagger\,[\alpha'_n\phi_n] \tag{B.11}$$

$$= \sum_n \int \mathrm{d}^2x\,\phi_m^\dagger\,[\alpha_n\phi_n + i\lambda\gamma^5\alpha_n\phi_n]\,. \tag{B.12}$$

Using the natural completeness relation $\int \mathrm{d}^2x\,\phi_m^\dagger\phi_m = \delta_{nm}$ we find

$$\int \mathrm{d}^2x\,\phi_m^\dagger\psi' = \alpha'_m = \sum_n(\delta_{nm} + C_{nm})\alpha_n, \tag{B.13}$$

where the matrix in the parentheses is the infinitessimal form of the Jacobian. This allows us to evaluate its determinant — explicitly using the definition of C_{nm} from (B.12) we get

$$\log\mathcal{J} = \log\det(1+C) = \operatorname{tr} C = i\int \mathrm{d}^2x\,\lambda(x)\left[\sum_n \phi_n^\dagger\gamma^5\phi_n\right]. \tag{B.14}$$

Now focusing on the sum in the brackets, regulating this expression by using the fact that the high momentum mode square-eigenvalues are very negative ($\lambda_n^2 \to -\infty$ as $n \to \infty$). We will modulate this regulator with a mass M that gets taken to infinity, and then express this using the operator $i\not{D}$ as follows

$$\text{sum} = \lim_{M\to\infty}\sum_n \phi_n^\dagger\gamma^5\phi_n \exp\!\left[\lambda_n^2/M^2\right] = \lim_{M\to\infty}\sum_n \phi_n^\dagger\gamma^5 \exp\!\left[(i\not{D})^2/M^2\right]\phi_n \tag{B.15}$$

$$= \lim_{M\to\infty}\langle x|\,\gamma^5\exp\!\left[(i\not{D})^2/M^2\right]|x\rangle\,. \tag{B.16}$$

This square Dirac operator can be simplified to be written as

$$(i\not{D})^2 = -D^2 + \frac{e}{2}S^{\mu\nu}F_{\mu\nu}, \tag{B.17}$$

where $D^2 = D_\mu D^\mu$, $S^{\mu\nu} = -\frac{1}{2}[\gamma^\mu,\gamma^\nu]$ lives in the spinor representation of the Poincarè algebra, and $F_{\mu\nu}$ is the field strength tensor. Using the action of the operator

$$-D^2\,|x\rangle = \int \frac{\mathrm{d}^2k}{(2\pi)^2}(-k^2)\,|x\rangle \tag{B.18}$$

on the position basis, we may express the sum as

$$\text{sum} = \lim_{M\to\infty}\operatorname{tr}\left[\gamma^5\exp\!\left(\frac{e}{2}S^{\mu\nu}F_{\mu\nu}/M^2\right)\right]\int\frac{\mathrm{d}^2k}{(2\pi)^2}e^{-k^2/M^2}. \tag{B.19}$$

Now evaluating the trace using $\operatorname{tr}\left[\gamma^5 S^{\mu\nu}\right] = (-2i)\varepsilon^{\mu\nu}$, and the integral is $iM^2/4\pi$; the sum is

$$\text{sum} = -\frac{e}{4\pi}\varepsilon^{\mu\nu}F_{\mu\nu} = -\frac{e}{2\pi}F_{01}. \tag{B.20}$$

The Jacobian is now written

$$\log\mathcal{J} = -i\int \mathrm{d}^2x\,\lambda(x)\frac{e}{2\pi}F_{01}, \tag{B.21}$$

and therefore the variation of the full measure is

$$\log\mathcal{J}^{-2} = i\int \mathrm{d}^2x\,\lambda(x)\frac{e}{\pi}F_{01} = iS_{\text{anom}} \tag{B.22}$$

and the measure changes

$$\mathcal{D}\overline{\psi}\mathcal{D}\psi \to \mathcal{D}\overline{\psi}\mathcal{D}\psi e^{iS_{\text{anom}}}, \tag{B.23}$$

which looks like an anomalous addition to the action. Explicitly, written in terms of the chiral current,

$$\mathcal{D}\psi\mathcal{D}\overline{\psi} \to \mathcal{D}\Psi\mathcal{D}\overline{\Psi}\exp\Big(i\int \mathrm{d}^2x\,\lambda(x)\,\partial_\mu J^\mu_A\Big). \tag{B.24}$$

which is not conserved in the quantum theory, but obeys

$$\partial_\mu J^\mu_A = \frac{eF_{01}}{\pi}. \tag{B.25}$$

The Wen–Zee Model & Hierarchy States
Appendix C

So far we have described the series of states $1/m$, of which the odd-m states are experimentally realised. A further hierarchy of fractional quantum Hall states can be constructed by including κ additional gauge fields in the model due to Wen and Zee (1992a,b)

$$S = \int \left[-\frac{K_{ij}}{4\pi} a_i \wedge \mathrm{d}a_j + \frac{t_i}{2\pi} A \wedge \mathrm{d}a_i \right]. \tag{C.1}$$

The additional gauge fields a_i are coupled by the $\kappa \times \kappa$ matrix K, and the charge vector t specifies the physical EM current $J = \star(t_i \, \mathrm{d}a_i)$. In general the filling fraction is

$$k = t_i (K^{-1})_{ij} t_j. \tag{C.2}$$

We can demonstrate that generating the Jain series $\nu = n/(2n+1)$ from a combination of two forms a, b given by

$$K = \begin{pmatrix} m & -1 \\ -1 & n \end{pmatrix}, \quad t = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{C.3}$$

gives the action

$$S[a,b;A] = \frac{1}{4\pi} \int \left[-ma \wedge \mathrm{d}a + 2A \wedge \mathrm{d}a + (2a - nb + 2A) \wedge \mathrm{d}b \right] \tag{C.4}$$

Manually integrate out b by finding its equation of motion by varying δb

$$-\, n\delta\,(b \wedge \mathrm{d}b) + 2(A + a) \wedge \mathrm{d}\delta b = 0 \qquad \Longrightarrow \qquad n\,\mathrm{d}b = \mathrm{d}A + \mathrm{d}a \tag{C.5}$$

which is solved locally to give $b = n^{-1}(A + a)$. Place b on-shell by substituting into the effective action

$$S[a;A] = \frac{1}{4\pi} \int \left[-ma \wedge \mathrm{d}a + 2A \wedge \mathrm{d}a + n^{-1}\,(A + a) \wedge (\mathrm{d}A + \mathrm{d}a) \right] \tag{C.6}$$

$$= \frac{1}{4\pi} \int \left[n^{-1} A \wedge \mathrm{d}A - (m - n^{-1}) a \wedge \mathrm{d}a + 2(1 + n^{-1}) A \wedge \mathrm{d}a \right]. \tag{C.7}$$

The latter two terms of (C.7) are the same as (2.86) with different coefficients. Integrating out a this time gives $(1 + n^{-1})A = (m - n^{-1})a$ and so

$$S[A] = \frac{1}{4\pi} \int A \wedge \mathrm{d}A \left[n^{-1} + \frac{(1 + n^{-1})^2}{m - n^{-1}} \right] = \frac{1}{4\pi} \int A \wedge \mathrm{d}A \left[\frac{m + n + 2}{mn - 1} \right] \tag{C.8}$$

which is a CS action with filling fraction $\nu = (m + n + 2)/(mn - 1)$. Choosing appropriate coefficients $n = n(m)$ gives the Jain series $\nu_m = m/(2m + 1)$ that we wanted to describe.

This explicit example elucidates how the general Wen–Zee actions can describe general hierarchy states: integrating out each field generates a new set of (fractional) couplings for the rest. In this way we are able to generate arbitrary filling fraction, including the so-called hierarchy states (Blok and Wen, 1990). Another important series that the Wen–Zee theory is capable of reproducing is the particle-hole inverted partner of the Laughlin states. Particle-hole symmetry, when we consider only the lowest Landau level, acts to exchange filled states for unfilled ones and thus the filling fraction changes $\nu \to 1 - \nu$. Consider the charge vector $t = (1, 1)^T$ and $K = \text{diag}(1, -m)$ with odd m, which give a Hall conductivity

$$\sigma = 1 - \frac{1}{m}. \tag{C.9}$$

These are the conductivities of the PH-inverted Laughlin states, which form the series $2/3, 5/7, \ldots$.

This construction is similar to the construction of branes in string theory (Susskind, 2001; Belhaj et al., 2015), and this has even motivated work which produces a fractional quantum Hall effect in string theory models (Bergman, 2004).

Diagonalising the Luttinger Hamiltonian
Appendix D

Diagonalising the UV Theory

To diagonalise the interacting theory, perform a Bogoliubov transformation generated by the Hermitian operator

$$S = \frac{2\pi i}{L} \sum_{k>0} \frac{\varphi_p}{p} \mathcal{J}_+(p)\mathcal{J}_-(-p), \tag{D.1}$$

where we defined a real even function φ_p. This is fully determined by requiring that the unitary operator $U = \exp(iS)$ diagonalises H_I in the form given in (3.13) (Mattis and Lieb, 1965). First note $U^\dagger H_0 U = H_0$, and that

$$U^\dagger \mathcal{J}_\pm(p) U = \mathcal{J}_\pm(p)\cosh\varphi_p + \mathcal{J}_\mp(p)\sinh\varphi_p \tag{D.2}$$

which conserves the commutation relations (3.7), (3.8). Choose $\tanh 2\varphi_k = -\lambda V_k/\pi v$ to diagonalise H_I and which produces

$$H = \frac{2\pi}{L} \sum_{k>0} \operatorname{sech} 2\varphi_k \left[\mathcal{J}_+(k)\mathcal{J}_+(-k) + \mathcal{J}_-(-k)\mathcal{J}_-(k)\right]. \tag{D.3}$$

One can recover canonical commutation relations from (3.7), (3.8) by defining the canonical raising operators

$$c^\dagger_{\pm k} = \sqrt{\frac{2\pi}{L\,|k|}}\mathcal{J}_\pm(k). \tag{D.4}$$

Transforming (D.3) into the canonical form

$$H = \sum_{k>0} \omega_k \tilde{c}^\dagger_k \tilde{c}_k, \qquad \omega_k = |k| \operatorname{sech} 2\varphi_k = |k|\sqrt{v^2 - \lambda^2 V_k^2}. \tag{D.5}$$

which elucidates the spectrum of the bosonic excitations.

There exists a representation of the fermion excitations in terms of these $\mathcal{J}_\pm(\sigma)$ bosonic ones; Luther and Peschel (1974) finds the form

$$\psi_\pm(\sigma) = \frac{1}{\sqrt{2\pi a}} e^{ik_F\sigma + i\varphi_\pm(\sigma)}, \qquad \varphi_\pm(\sigma) = \frac{-2\pi i}{L}\sum_{k>0}\left[\mathcal{J}_\pm(-k)e^{ik\sigma} - \mathcal{J}_\pm(k)e^{-ik\sigma}\right] \tag{D.6}$$

which is defined in terms of the $\varphi(\sigma)$ field used in the Bogoliubov transform, and a short-distance cutoff a. Inverting this to get $\mathcal{J}_\pm$ in real space, and take the continuum ($a \to 0$) limit to recover

$$\mathcal{J}_\pm(\sigma) = \frac{1}{L}\sum_{k>0}\left[\mathcal{J}_\pm(-k)e^{ik\sigma} + \mathcal{J}_\pm(k)e^{-ik\sigma}\right] \to \frac{1}{2\pi}\partial_\sigma\varphi_\pm(\sigma). \tag{D.7}$$

Diagonalising the IR Theory

In the interacting Luttinger theory we must add an interparticle interaciton potential, called V_k in (3.12); alternatively move the Hamiltonian (3.39) into momentum space

$$\mathcal{J}_\pm(k) = \int_0^L d\sigma\, \mathcal{J}_\pm(\sigma)e^{\mp ik\sigma} \tag{D.8}$$

with $k_n = 2\pi n/L$. Hence

$$H_0 = \frac{\pi v}{L}\sum_k \left(\mathcal{J}_+(k)\mathcal{J}_+(-k) + \mathcal{J}_-(k)\mathcal{J}_-(-k)\right). \tag{D.9}$$

Adding the interaction Hamiltonian with two terms, which represent forward (g_4) and backward (g_2) scattering (Degiovanni et al., 1998)

$$H_{\text{int}} = \frac{\pi}{L}\sum_k 2g_2\mathcal{J}_+(k)\mathcal{K}_-(k) + g_4\left[\mathcal{J}_+(k)\mathcal{J}_+(-k) + \mathcal{J}_-(k)\mathcal{J}_-(-k)\right]. \tag{D.10}$$

As before, a Bogoliubov transformation diagonalises the full Hamiltonian given

$$\tanh 2\varphi = -\frac{g_2}{v+g_4}, \quad \text{and} \quad \begin{pmatrix}\mathcal{G}_+\\ \mathcal{G}_-\end{pmatrix} = \begin{pmatrix}\cosh\varphi & -\sinh\varphi\\ -\sinh\varphi & \cosh\varphi\end{pmatrix}\begin{pmatrix}\mathcal{J}_+\\ \mathcal{J}_-\end{pmatrix}. \tag{D.11}$$

This diagonalises the Hamiltonian back to the same form as the free theory

$$H = \frac{\pi}{L}\sum_k \left(\mathcal{G}_+(k)\mathcal{G}_+(-k) + \mathcal{G}_-(k)\mathcal{G}_-(-k)\right). \tag{D.12}$$

The velocity is renormalised $v \to v_S$ and currents are

$$\mathcal{J}_\pm(k) = \alpha^{-1/2}\mathcal{G}_\pm(k) \implies \begin{pmatrix}\rho(\sigma)\\ j(\sigma)\end{pmatrix} = \begin{pmatrix}\alpha^{-1/2}\left[\mathcal{G}_+(\sigma) + \mathcal{G}_+(\sigma)\right]\\ \alpha^{-1/2}v_S\left[\mathcal{G}_+(\sigma) - \mathcal{G}_+(\sigma)\right]\end{pmatrix} \tag{D.13}$$

where

$$\alpha = e^{-2\varphi} = \sqrt{\frac{v+g_2+g_4}{v-g_2+g_4}}, \tag{D.14}$$

$$v_S = \sqrt{(v+g_4)^2 - g_2}. \tag{D.15}$$

The Hamiltonian of the new theory is now equivalent to the original under these transformations

$$H = \pi v_S \int d^2\sigma \left[\mathcal{G}_+(\sigma)^2 + \mathcal{G}_-(\sigma)^2\right]. \tag{D.16}$$

Deriving the Seed Duality Appendix E

The 'seed' bosonisation duality which formed the core of deriving particle-vortex duality has a derivation proposed by Seiberg, Senthil, Wang, and Witten (2016). Although this result does not constitute a proof, it is an interesting result which strongly motivates the result, and is based upon only a few assumptions. Let us begin by reciting the bosonsation duality:

$$\int \mathcal{D}\psi \, \exp\left(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{fermion}}[\psi; A] - \frac{1}{2}\frac{1}{4\pi} A \wedge \mathrm{d}A \right)$$
$$= \int \mathcal{D}\phi \mathcal{D}a \, \exp\left(i \int \mathrm{d}^3x \, \mathcal{L}_{\text{scalar}}[\phi; a] + \frac{1}{4\pi} a \wedge \mathrm{d}a + \frac{1}{2\pi} a \wedge \mathrm{d}A \right), \quad \text{(E.1)}$$

where $\mathcal{L}_{\text{fermion}}[\psi; A]$ and $\mathcal{L}_{\text{salar}}[\phi; a]$ are defined at corresponding fixed points. The derivation will proceed as follows: a parent field theory will be written down, and then its various phases discussed in limits which make it amenable. Next, a symmetry-breaking transition will be induced by introducing monopole operators which break its global symmetry. Finally, we will hypothesise that there is a unique fixed point of this transition and equate the field theories at limiting points on the phase-transition line.

The proposed parent field theory contains a global a gauged interacting scalar ϕ, a gauged free fermion χ, a dynamic gauge field a, and a background gauge field A:

$$S[\chi, \phi, a; A] = \int \mathrm{d}^3x \left[\overline{\chi} \, (i\partial\!\!\!/ + m + A\!\!\!/ + a\!\!\!/) \, \chi + |(\partial_\mu - ia_\mu)\phi|^2 - V(|\phi|^2) - \frac{1}{4} f_{\mu\nu}^2 \right]. \quad \text{(E.2)}$$

The scalar has charge 1 under the dynamic gauge field a, and the fermion χ also has a charge 1 under the background field. The equations of motion show that there are two global charges: $U(1)_A$ given by $J_A^\mu = i\overline{\chi}\gamma^\mu\chi$ associated with the conservation of fermion number and $U(1)_a$, the topological charge $J_a = \star \, \mathrm{d}a$.

From our previous discussion of the abelian Higgs model in Section 5.1.3, we know the scalar sector should have two distinct phases when $\langle \phi \rangle = 0$ and $\langle \phi \rangle = v \neq 0$ in the vacuum. Again let us call these phases the Coulomb and Higgs phases of the theory. The coupling to a fermion will provide additional structure to the model.

First consider the Higgs phase. Expanding around the vacuum with $\phi = (v+\rho)e^{i\sigma}$, the Higgs mechanism gives the gauge boson a a mass and removes a (massless) degree of scalar freedom σ from the theory. The resultant theory, taking the result of the covariant derivative expansion in the Abelian Higgs model (5.26), is

$$\int \mathcal{D}a\mathcal{D}\chi\mathcal{D}\rho \exp\left[i \int \mathrm{d}^3x \, \overline{\chi} \, (i\partial\!\!\!/ + m + A\!\!\!/ + a\!\!\!/) \, \chi - \frac{1}{4} f_{\mu\nu}^2 + v^2 a^2 + (\partial_\mu\rho)^2 + \tilde{m}\rho^2 + \cdots \right]. \quad \text{(E.3)}$$

Here, the scalar mass is $\tilde{m} = V''(\rho)|_{\rho=0}$ and the '$\cdots$' contains higher-order operators from the expansion of the potential. These induce interactions, many of which will be irrelevant in the UV of the 2+1-dimensional theory. Explicitly, this theory contains a Higgsed massive boson a and a massive real scalar — choose a regime where we can integrate out a. This can be explicitly performed at tree-level using the equation of motion $(-\partial^2 + 2v^2)a^\mu = iJ_A^\mu$ (in Coulomb gauge $\partial_\mu a^\mu = 0$) to give

$$\int \mathcal{D}\chi\mathcal{D}\rho \exp\left[\int \mathrm{d}^3x\, \chi\,(\not{\partial} + m + \not{A})\,\chi - \frac{1}{4v^4}(\overline{\chi}\gamma^\mu\chi)^2 + \rho(-\partial^2 + \tilde{m})\rho + \cdots\right], \tag{E.4}$$

which is the massive Thirring model (Thirring, 1958) and a scalar. See discussion of this duality in (Kondo, 1995). Note that A is non-dynamical and therefore the excitations of this phase are completely gapped due to the Higgs mechanism of a.

Taking the Higgs mass $v \to \infty$ when a is integrated out removes the interaction term and recovers a free-fermion effective theory. This result is a general feature of effective field theories called decoupling, where in this case the fermion loses its a-mediated interactions if the a particle being integrated out is infinitely massive. Next, the massive scalar will not be important here so assume $\tilde{m} \gg m$ and integrate it out, to give the tree-level effective action

$$S_{\text{Higgs}}[\chi; A] = \int \mathrm{d}^3x\, [\overline{\chi}\,(i\not{\partial} + m + \not{A})\,\chi]\,. \tag{E.5}$$

Expanding the scalar around the zero-vacuum of the Coulomb phase means that there is no symmetry breaking and no Higgs mechanism. The photon remains gapless and the scalar is described by the action $\mathcal{L}_{\text{scalar}}[\phi; a]$,

$$S_{\text{Coulomb}}[\chi, \phi, a; A] = \int \mathrm{d}^3x \left[\overline{\chi}\,(i\not{\partial} + m + \not{A} + \not{a})\,\chi - \frac{1}{4}f_{\mu\nu}^2 + \mathcal{L}_{\text{scalar}}[\phi; a]\right]. \tag{E.6}$$

This phase has the possibility of gapless a-excitations, but this could be broken by quantum corrections due to interactions with χ.

The final step is to integrate out the fermion of both of these theories to give the one-loop effective action. This can be done explicitly for the Coulomb phase (E.6), and the Higgs phase (E.5) in the limiting case $v \to \infty$ where it becomes free. This calculation is identical to the result of Section 2.1.4 where the fermionic path integral gives a topological half-quantised Chern–Simons term which depends on the sign of the fermion mass m (2.36).

We have the freedom to add a CS term to the original action (E.5) so that the resulting induced CS terms are integer-quantised. The new action is

$$S'_{\text{Higgs}}[\chi; A] = \int \mathrm{d}^3x \left[\overline{\chi}\,(i\not{\partial} + m + \not{A})\,\chi - \frac{1}{2}\frac{1}{4\pi}A \wedge \mathrm{d}A\right]. \tag{E.7}$$

The fermionic path integral therefore gives a trivial theory when $m > 0$, and the action

$$S_{\text{Higgs}}^{m<0}[A] = -\frac{1}{4\pi}\int \mathrm{d}^3x\, A \wedge \mathrm{d}A\,. \tag{E.8}$$

when $m < 0$. This result only holds when the fermion mass is nonzero $m \neq 0$; then both of these phases are gapped because of the Higgs mechanism.

Now focusing on the Coulomb phase, and taking heed of the previous result, we must similarly introduce the CS term to this theory

$$S'_{\text{Coulomb}}[\chi, \phi, a; A] = \int \mathrm{d}^3x \Bigg[\overline{\chi}\,(i\partial\!\!\!/ + m + A\!\!\!/ + a\!\!\!/)\,\chi - \frac{1}{4}f_{\mu\nu}^2 + \mathcal{L}_{\text{scalar}}[\phi; a] \\ - \frac{1}{2}\frac{1}{4\pi}(A + a) \wedge \mathrm{d}(A + a) + \frac{1}{4\pi}A \wedge \mathrm{d}A \Bigg]. \quad \text{(E.9)}$$

The resulting integral gives for $m > 0$

$$S^{m>0}_{\text{Coulomb}}[\chi, \phi, a; A] = \int \mathrm{d}^3x \left[\mathcal{L}_{\text{scalar}}[\phi; a] - \frac{1}{4}f_{\mu\nu}^2 + \frac{1}{4\pi}A \wedge \mathrm{d}A \right], \quad \text{(E.10)}$$

which is gapless (and the dual photon is the Goldstone mode, as before). For $m < 0$

$$S^{m<0}_{\text{Coulomb}}[\chi, \phi, a; A] = \int \mathrm{d}^3x \left[\mathcal{L}_{\text{scalar}}[\phi; a] - \frac{1}{4}f_{\mu\nu}^2 - \frac{1}{4\pi}a \wedge \mathrm{d}a - \frac{1}{2\pi}A \wedge \mathrm{d}a \right]. \quad \text{(E.11)}$$

The phase diagram is depicted in the following table, and only the $m < 0$ Coulomb phase has gapless excitations.

	$m < 0$	$m > 0$
$v = \infty$ Higgs	No excitations Induced $A \wedge \mathrm{d}A$	No excitations
$v = 0$ Coulomb	Gapped ϕ Gapped a from induced $(A + a) \wedge \mathrm{d}(A + a)$	Gapped ϕ Gapless a

Now let us introduce monopole operators to the theory, which will explicitly break the $U(1)_a$ symmetry. The immediate consequence of breaking this is that the massless goldstone mode of the gapless theory is gapped-out and the theory because the SSB procedure can no-longer occur. Analysing the theory in the limits $v = 0, \infty$ and $m < 0$, $m > 0$ in the presence of monopole operators shows that all phases except $m < 0, v \to \infty$ preserve the $U(1)_A$ group. All points of the phase diagram are insulating, but this limit forms a quantum hall insulator state. Now assume that the three 'trivial' states are identical (in the presence of monopoles), then one generally expects there to be a finite-v transition between the two states for $m < 0$, as shown in Fig. E.1.

By universality different points on the phase transition are equivalent; therefore taking two limits (as prescribed on Fig. E.1) which approach different ends of the transition line, we may derive two dual field theories. For the first limit we remain in the Coulomb sector, and simply take $m \to -\infty$; using Eq. E.11 this gives

$$S^{m<0}_{\text{Coulomb}}[\chi, \phi, a; A] = \int \mathrm{d}^3x \left[\mathcal{L}_{\text{scalar}}[\phi; a] - \frac{1}{4\pi}a \wedge \mathrm{d}a - \frac{1}{2\pi}A \wedge \mathrm{d}a \right]. \quad \text{(E.12)}$$

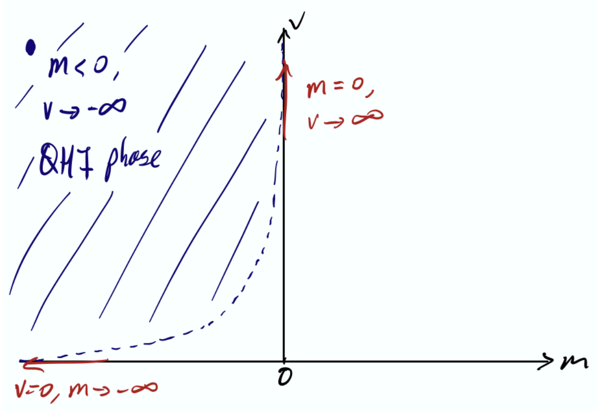

Figure E.1: Phase transition.

Next, the dual theory is obtained by the $v \to \infty$ Higgs action (E.7) when $m = 0$, given by

$$S'_{\text{Higgs}}[\chi; A] = \int d^3x \left[\overline{\chi}\,(i\partial\!\!\!/ + A\!\!\!/)\,\chi - \frac{1}{2}\frac{1}{4\pi} A \wedge dA \right] \tag{E.13}$$

$$= \int d^3x \left[\mathcal{L}_{\text{fermion}}[\chi; A] - \frac{1}{2}\frac{1}{4\pi} A \wedge dA \right]. \tag{E.14}$$

This result motivates the duality in Eq. E.1.

The quantum numbers of the monopole operators in different theories are important to understand to formalise the operator dictionary between the two phases (Seiberg et al., 2016). Details about the operator matching and evaluation of spin are presented by Turner (2019), and further details about the vortices in these types of models are given in (Horvathy, 2007).

Bibliography

A. G. Abanov. On the effective hydrodynamics of the fractional quantum Hall effect. *Journal of Physics A: Mathematical and Theoretical*, 46(29):292001, jun 2013. doi: 10.1088/1751-8113/46/29/292001. URL https://doi.org/10.1088%2F1751-8113%2F46%2F29%2F292001.

A. G. Abanov and A. Gromov. Electromagnetic and gravitational responses of two-dimensional non-interacting electrons in background magnetic field. 2014. doi: 10.1103/PhysRevB.90.014435.

C. Adam, R. Bertlmann, and P. Hofer. Overview on the anomaly and Schwinger term in two-dimensional QED. *La Rivista Del Nuovo Cimento Series 3*, 16:1–52, 08 1993. doi: 10.1007/BF02724505.

S. L. Adler. Axial-vector vertex in spinor electrodynamics. *Phys. Rev.*, 177:2426–2438, Jan 1969. doi: 10.1103/PhysRev.177.2426. URL https://link.aps.org/doi/10.1103/PhysRev.177.2426.

I. Affleck. Exact critical exponents for quantum spin chains, non-linear σ-models at $\theta = \pi$ and the quantum Hall effect. *Nuclear Physics B*, 265(3):409 – 447, 1986. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(86)90167-7. URL http://www.sciencedirect.com/science/article/pii/0550321386901677.

O. Aharony. Baryons, monopoles and dualities in Chern-Simons-matter theories. 2015. doi: 10.1007/JHEP02(2016)093.

O. Aharony, A. Hanany, K. Intriligator, N. Seiberg, and M. J. Strassler. Aspects of $n = 2$ supersymmetric gauge theories in three dimensions. *Nuclear Physics B*, 499(1):67–99, 1997.

J. Alicea. Composite fermions meet dirac. Journal Club for Condensed Matter Physics: Commentary, 2015. URL https://www.condmatjclub.org/uploads/2015/09/JCCM_SEP_2015_01.pdf.

J. Ambjørn, J. Greensite, and C. Peterson. The axial anomaly and the lattice Dirac sea. *Nuclear Physics B*, 221(2):381 – 408, 1983. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(83)90585-0. URL http://www.sciencedirect.com/science/article/pii/0550321383905850.

D. Arovas, J. R. Schrieffer, and F. Wilczek. Fractional statistics and the Quantum Hall Effect. *Phys. Rev. Lett.*, 53:722–723, Aug 1984. doi: 10.1103/PhysRevLett.53.722. URL https://link.aps.org/doi/10.1103/PhysRevLett.53.722.

M. Atiyah. Topological quantum field theories. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 68(1):175–186, 1988.

T. Banks, R. Myerson, and J. Kogut. Phase transitions in Abelian lattice gauge theories. *Nuclear Physics B*, 129(3):493 – 510, 1977. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(77)90129-8. URL `http://www.sciencedirect.com/science/article/pii/0550321377901298`.

H. Bartolomei, M. Kumar, R. Bisognin, A. Marguerite, J.-M. Berroir, E. Bocquillon, B. Plaçais, A. Cavanna, Q. Dong, U. Gennser, Y. Jin, and G. Fève. Fractional statistics in anyon collisions. *Science*, 368(6487):173–177, 2020. ISSN 0036-8075. doi: 10.1126/science.aaz5601. URL `https://science.sciencemag.org/content/368/6487/173`.

A. Belavin, A. Polyakov, and A. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Physics B*, 241(2):333 – 380, 1984. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(84)90052-X. URL `http://www.sciencedirect.com/science/article/pii/055032138490052X`.

A. Belhaj, Z. Benslimane, M. E. Idrissi, B. Manaut, A. Sebbar, and M. B. Sedra. Fractional quantum Hall filling factors from string theory using toric geometry, 2015.

J. Bell and R. Jackiw. A PCAC puzzle: $\pi^0 \to \gamma\gamma$ in the σ model. *Nuovo Cim. A*, 60:47–61, 1969. doi: 10.1007/BF02823296.

O. Bergman. Quantum Hall physics in string theory. 2004. doi: 10.1142/9789812702340_0024.

M. V. Berry. Quantal phase factors accompanying adiabatic changes. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 392(1802):45–57, 1984. ISSN 00804630. URL `http://www.jstor.org/stable/2397741`.

W. Bishara, P. Bonderson, C. Nayak, K. Shtengel, and J. K. Slingerland. Interferometric signature of non-Abelian anyons. *Phys. Rev. B*, 80:155303, Oct 2009. doi: 10.1103/PhysRevB.80.155303. URL `https://link.aps.org/doi/10.1103/PhysRevB.80.155303`.

B. Blok and X. G. Wen. Effective theories of the fractional quantum Hall effect: Hierarchy construction. *Phys. Rev. B*, 42:8145–8156, Nov 1990. doi: 10.1103/PhysRevB.42.8145. URL `https://link.aps.org/doi/10.1103/PhysRevB.42.8145`.

P. Bonderson, A. Kitaev, and K. Shtengel. Detecting non-Abelian statistics in the $\nu = 5/2$ fractional quantum Hall state. *Phys. Rev. Lett.*, 96:016803, Jan 2006. doi: 10.1103/PhysRevLett.96.016803. URL `https://link.aps.org/doi/10.1103/PhysRevLett.96.016803`.

P. Bonderson, C. Nayak, and X.-L. Qi. A time-reversal invariant topological phase at the surface of a 3D topological insulator. *Journal of Statistical Mechanics: Theory and Experiment*, 2013(09):P09016, sep 2013. doi: 10.1088/1742-5468/2013/09/p09016. URL `https://doi.org/10.1088%2F1742-5468%2F2013%2F09%2Fp09016`.

C. P. Burgess. Introduction to effective field theory. 2007. doi: 10.1146/annurev.nucl.56.080805.140508.

D. Cabra, E. Fradkin, G. L. Rossini, and F. A. Schaposnik. Gauge invariance and finite temperature effective actions of Chern-Simons gauge theories with fermions. *Physics Letters B*, 383(4):434 – 438, 1996. ISSN 0370-2693. doi: https://doi.org/10.1016/0370-2693(96)00613-2. URL `http://www.sciencedirect.com/science/article/pii/0370269396006132`.

C. Callan and J. Harvey. Anomalies and fermion zero modes on strings and domain walls. *Nuclear Physics B*, 250(1):427 – 436, 1985. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(85)90489-4. URL `http://www.sciencedirect.com/science/article/pii/0550321385904894`.

A. Cappelli, G. V. Dunne, C. A. Trugenberger, and G. R. Zemba. Conformal symmetry and universal properties of quantum Hall states. 1992. doi: 10.1016/0550-3213(93)90603-M.

J. Cardy. Conformal field theory and statistical mechanics, 2008.

J. L. Cardy. Operator Content of Two-Dimensional Conformally Invariant Theories. *Nucl. Phys. B*, 270:186–204, 1986. doi: 10.1016/0550-3213(86)90552-3.

A. M. Chang. Chiral Luttinger liquids at the fractional quantum Hall edge. *Rev. Mod. Phys.*, 75:1449–1505, Nov 2003. doi: 10.1103/RevModPhys.75.1449. URL `https://link.aps.org/doi/10.1103/RevModPhys.75.1449`.

J.-Y. Chen, J. H. Son, C. Wang, and S. Raghu. Exact boson-fermion duality on a 3D Euclidean lattice. *Phys. Rev. Lett.*, 120:016602, Jan 2018. doi: 10.1103/PhysRevLett.120.016602. URL `https://link.aps.org/doi/10.1103/PhysRevLett.120.016602`.

Y.-H. Chen, F. Wilczek, E. Witten, and B. I. Halperin. On anyon superconductivity. *Int. J. Mod. Phys. B*, 3:1001, 1989. doi: 10.1142/S0217979289000725.

S. Coleman. There are no Goldstone bosons in two dimensions. *Communications in Mathematical Physics*, 31(4):259–264, 1973. doi: 10.1007/BF01646487. URL `https://doi.org/10.1007/BF01646487`.

S. Coleman. Quantum sine-Gordon equation as the massive Thirring model. *Phys. Rev. D*, 11:2088–2097, Apr 1975. doi: 10.1103/PhysRevD.11.2088. URL `https://link.aps.org/doi/10.1103/PhysRevD.11.2088`.

S. Coleman and B. Hill. No more corrections to the topological mass term in QED $_3$. *Physics Letters B*, 159(2-3):184–188, Sept. 1985. doi: 10.1016/0370-2693(85)90883-4.

S. Coleman, D. Gross, and R. Jackiw. Fermion avatars of the Sugawara model. *Phys. Rev.*, 180:1359–1366, Apr 1969. doi: 10.1103/PhysRev.180.1359. URL `https://link.aps.org/doi/10.1103/PhysRev.180.1359`.

I. Danshita, M. Hanada, and M. Tezuka. Creating and probing the Sachdev–Ye–Kitaev model with ultracold gases: Towards experimental studies of quantum gravity. *Progress of Theoretical and Experimental Physics*, 2017(8), 08 2017. ISSN 2050-3911. doi: 10.1093/ptep/ptx108. URL `https://doi.org/10.1093/ptep/ptx108`. 083I01.

S. Das Sarma, M. Freedman, and C. Nayak. Topologically protected qubits from a possible non-Abelian fractional quantum Hall state. *Phys. Rev. Lett.*, 94:166802, Apr 2005. doi: 10.1103/PhysRevLett.94.166802. URL `https://link.aps.org/doi/10.1103/PhysRevLett.94.166802`.

S. Das Sarma, C. Nayak, and S. Tewari. Proposal to stabilize and detect half-quantum vortices in strontium ruthenate thin films: Non-Abelian braiding statistics of vortices in a $p_x + ip_y$ superconductor. *Phys. Rev. B*, 73:220502, Jun 2006. doi: 10.1103/PhysRevB.73.220502. URL `https://link.aps.org/doi/10.1103/PhysRevB.73.220502`.

C. Dasgupta and B. I. Halperin. Phase transition in a lattice model of superconductivity. *Phys. Rev. Lett.*, 47:1556–1560, Nov 1981. doi: 10.1103/PhysRevLett.47.1556. URL https://link.aps.org/doi/10.1103/PhysRevLett.47.1556.

P. Degiovanni, C. Chaubet, and R. Melin. Conformal field theory approach to gapless 1-D fermion systems and application to the edge excitations of neutrino = 1 / (2p + 1) quantum Hall sequences. *Theor. Math. Phys.*, 117:1113–1181, 1998. doi: 10.1007/BF02557158.

S. Deser, R. Jackiw, and S. Templeton. Three-dimensional massive gauge theories. *Phys. Rev. Lett.*, 48:975–978, Apr 1982. doi: 10.1103/PhysRevLett.48.975. URL https://link.aps.org/doi/10.1103/PhysRevLett.48.975.

S. Deser, R. Jackiw, and S. Templeton. Topologically massive gauge theories. *Annals of Physics*, 281(1):409 – 449, 2000. ISSN 0003-4916. doi: https://doi.org/10.1006/aphy.2000.6013. URL http://www.sciencedirect.com/science/article/pii/S0003491600960135.

P. Di Francesco, P. Mathieu, D. Sénéchal, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer, 1997. ISBN 9780387947853. URL https://books.google.co.uk/books?id=keUrdME5rhIC.

G. V. Dunne. Aspects of Chern-Simons theory, 1999.

G. V. Dunne, R. Jackiw, and C. Trugenberger. Chern-Simons theory in the Schrödinger representation. *Annals of Physics*, 194(1):197 – 223, 1989. ISSN 0003-4916. doi: https://doi.org/10.1016/0003-4916(89)90036-5. URL http://www.sciencedirect.com/science/article/pii/0003491689900365.

J. P. Eisenstein, K. B. Cooper, L. N. Pfeiffer, and K. W. West. Insulating and fractional quantum Hall states in the first excited Landau level. *Phys. Rev. Lett.*, 88:076801, Jan 2002. doi: 10.1103/PhysRevLett.88.076801. URL https://link.aps.org/doi/10.1103/PhysRevLett.88.076801.

M. Eliashvili. On the holomorphic gauge quantization of the Chern-Simons theory and Laughlin wave functions. 5 1996.

S. Elitzur, G. W. Moore, A. Schwimmer, and N. Seiberg. Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory. *Nucl. Phys. B*, 326:108–134, 1989. doi: 10.1016/0550-3213(89)90436-7.

L. Faddeev and R. Jackiw. Hamiltonian reduction of unconstrained and constrained systems. *Phys. Rev. Lett.*, 60:1692–1694, Apr 1988. doi: 10.1103/PhysRevLett.60.1692. URL https://link.aps.org/doi/10.1103/PhysRevLett.60.1692.

M. P. A. Fisher and L. I. Glazman. Transport in a one-dimensional Luttinger liquid, 1996.

M. P. A. Fisher and D. H. Lee. Correspondence between two-dimensional bosons and a bulk superconductor in a magnetic field. *Phys. Rev. B*, 39:2756–2759, Feb 1989. doi: 10.1103/PhysRevB.39.2756. URL https://link.aps.org/doi/10.1103/PhysRevB.39.2756.

R. Floreanini and R. Jackiw. Self-dual fields as charge-density solitons. *Phys. Rev. Lett.*, 59:1873–1876, Oct 1987. doi: 10.1103/PhysRevLett.59.1873. URL https://link.aps.org/doi/10.1103/PhysRevLett.59.1873.

E. Fradkin. *Field Theories of Condensed Matter Physics*. Cambridge University Press, 2 edition, 2013. doi: 10.1017/CBO9781139015509.

E. Fradkin. *Anomalies in Quantum Field Theory*. 2020a. URL `http://eduardo.physics.illinois.edu/phys583/ch20.pdf`.

E. Fradkin. *Conformal Field Theory*. 2020b. URL `http://eduardo.physics.illinois.edu/phys583/ch21.pdf`.

E. Fradkin. *Topological Field Theory*. 2020c. URL `http://eduardo.physics.illinois.edu/phys583/ch22.pdf`.

E. Fradkin and F. A. Schaposnik. The fermion-boson mapping in three-dimensional quantum field theory. *Physics Letters B*, 338(2):253 – 258, 1994. ISSN 0370-2693. doi: https://doi.org/10.1016/0370-2693(94)91374-9. URL `http://www.sciencedirect.com/science/article/pii/0370269394913749`.

E. Fradkin, C. Nayak, A. Tsvelik, and F. Wilczek. A Chern-Simons effective field theory for the Pfaffian quantum Hall state. *Nuclear Physics B*, 516(3):704 – 718, 1998. ISSN 0550-3213. doi: https://doi.org/10.1016/S0550-3213(98)00111-4. URL `http://www.sciencedirect.com/science/article/pii/S0550321398001114`.

E. Fradkin, M. Huerta, and G. R. Zemba. Effective Chern–Simons theories of pfaffian and parafermionic quantum Hall states, and orbifold conformal field theories. *Nuclear Physics B*, 601(3):591 – 606, 2001. ISSN 0550-3213. doi: https://doi.org/10.1016/S0550-3213(01)00077-3. URL `http://www.sciencedirect.com/science/article/pii/S0550321301000773`.

M. H. Freedman, A. Kitaev, and Z. Wang. Simulation of topological field theories by quantum computers. *Communications in Mathematical Physics*, 227(3):587–603, 2002.

J. Frohlich and A. Zee. Large scale physics of the quantum Hall fluid. *Nucl. Phys. B*, 364: 517–540, 1991. doi: 10.1016/0550-3213(91)90275-3.

L. Fu, C. L. Kane, and E. J. Mele. Topological insulators in three dimensions. *Phys. Rev. Lett.*, 98:106803, Mar 2007. doi: 10.1103/PhysRevLett.98.106803. URL `https://link.aps.org/doi/10.1103/PhysRevLett.98.106803`.

S. Fubini. Vertex operators and quantum Hall effect. *Mod. Phys. Lett. A*, 6:347–358, 1991. doi: 10.1142/S0217732391000336.

S. Fubini and C. Lutken. Vertex operators in the fractional quantum Hall effect. *Mod. Phys. Lett. A*, 6:487–500, 1991. doi: 10.1142/S0217732391000506.

S. Fubini and G. Veneziano. Level structure of dual-resonance models. *Il Nuovo Cimento A (1965-1970)*, 64(4):811–840, 1969. doi: 10.1007/BF02758835. URL `https://doi.org/10.1007/BF02758835`.

S. Fubini, D. Gordon, and G. Veneziano. A general treatment of factorization in dual resonance models. *Phys. Lett. B*, 29:679–682, 1969. doi: 10.1016/0370-2693(69)90109-9.

O. Fuentealba, H. A. González, M. Pino, and R. Troncoso. The anisotropic chiral boson. 2019. doi: 10.1007/JHEP11(2019)123.

K. Fujikawa. Path-integral measure for gauge-invariant fermion theories. *Phys. Rev. Lett.*, 42: 1195–1198, Apr 1979. doi: 10.1103/PhysRevLett.42.1195. URL https://link.aps.org/doi/10.1103/PhysRevLett.42.1195.

D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett. Generalized global symmetries. *Journal of High Energy Physics*, 2015(2):172, 2015.

S. D. Geraedts, M. P. Zaletel, R. S. K. Mong, M. A. Metlitski, A. Vishwanath, and O. I. Motrunich. The half-filled Landau level: the case for dirac composite fermions. 2015. doi: 10.1126/science.aad4302.

H. Gies and L. Janssen. Uv fixed-point structure of the three-dimensional Thirring model. *Phys. Rev. D*, 82:085018, Oct 2010. doi: 10.1103/PhysRevD.82.085018. URL https://link.aps.org/doi/10.1103/PhysRevD.82.085018.

P. Ginsparg. Applied conformal field theory. 1988.

S. M. Girvin. Particle-hole symmetry in the anomalous quantum Hall effect. *Phys. Rev. B*, 29:6012–6014, May 1984. doi: 10.1103/PhysRevB.29.6012. URL https://link.aps.org/doi/10.1103/PhysRevB.29.6012.

S. M. Girvin and A. H. MacDonald. Off-diagonal long-range order, oblique confinement, and the fractional quantum Hall effect. *Phys. Rev. Lett.*, 58:1252–1255, Mar 1987. doi: 10.1103/PhysRevLett.58.1252. URL https://link.aps.org/doi/10.1103/PhysRevLett.58.1252.

S. M. Girvin, A. H. MacDonald, and P. M. Platzman. Magneto-roton theory of collective excitations in the fractional quantum Hall effect. *Phys. Rev. B*, 33:2481–2494, Feb 1986. doi: 10.1103/PhysRevB.33.2481. URL https://link.aps.org/doi/10.1103/PhysRevB.33.2481.

P. Goddard and D. I. Olive. Kac-Moody and Virasoro Algebras in Relation to Quantum Physics. *Int. J. Mod. Phys. A*, 1:303, 1986. doi: 10.1142/S0217751X86000149.

V. J. Goldman, B. Su, and J. K. Jain. Detection of composite fermions by magnetic focusing. *Phys. Rev. Lett.*, 72:2065–2068, Mar 1994. doi: 10.1103/PhysRevLett.72.2065. URL https://link.aps.org/doi/10.1103/PhysRevLett.72.2065.

M. Greiter, X. Wen, and F. Wilczek. On paired Hall states. *Nucl. Phys. B*, 374:567–614, 1992. doi: 10.1016/0550-3213(92)90401-V.

D. Gross and R. Jackiw. Construction of covariant and gauge invariant t^* products. *Nuclear Physics B*, 14(2):269 – 284, 1969. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(69)90207-7. URL http://www.sciencedirect.com/science/article/pii/0550321369902077.

D. J. Gross and J. Wess. Scale invariance, conformal invariance, and the high-energy behavior of scattering amplitudes. *Phys. Rev. D*, 2:753–764, Aug 1970. doi: 10.1103/PhysRevD.2.753. URL https://link.aps.org/doi/10.1103/PhysRevD.2.753.

F. Haldane. Luttinger liquid theory of one-dimensional quantum fluids. I. Properties of the Luttinger model and their extension to the general 1D interacting spinless Fermi gas. *J. Phys. C*, 14:2585–2609, 1981. doi: 10.1088/0022-3719/14/19/010.

F. D. M. Haldane. Coupling between charge and spin degrees of freedom in the one-dimensional Fermi gas with backscattering. *Journal of Physics C: Solid State Physics*, 12(22): 4791–4799, nov 1979. doi: 10.1088/0022-3719/12/22/020. URL `https://doi.org/10.1088%2F0022-3719%2F12%2F22%2F020`.

B. I. Halperin. Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential. *Phys. Rev. B*, 25:2185–2190, Feb 1982. doi: 10.1103/PhysRevB.25.2185. URL `https://link.aps.org/doi/10.1103/PhysRevB.25.2185`.

B. I. Halperin, P. A. Lee, and N. Read. Theory of the half-filled Landau level. *Phys. Rev. B*, 47: 7312–7343, Mar 1993. doi: 10.1103/PhysRevB.47.7312. URL `https://link.aps.org/doi/10.1103/PhysRevB.47.7312`.

T. H. Hansson and S. Viefers. Edge theories for polarized quantum Hall states. *Phys. Rev. B*, 61:7553–7563, Mar 2000. doi: 10.1103/PhysRevB.61.7553. URL `https://link.aps.org/doi/10.1103/PhysRevB.61.7553`.

T. H. Hansson, M. Hermanns, and S. Viefers. Quantum Hall quasielectron operators in conformal field theory. *Phys. Rev. B*, 80:165330, Oct 2009. doi: 10.1103/PhysRevB.80.165330. URL `https://link.aps.org/doi/10.1103/PhysRevB.80.165330`.

T. H. Hansson, M. Hermanns, S. H. Simon, and S. F. Viefers. Quantum Hall physics: Hierarchies and conformal field theory techniques. *Rev. Mod. Phys.*, 89:025005, May 2017. doi: 10.1103/RevModPhys.89.025005. URL `https://link.aps.org/doi/10.1103/RevModPhys.89.025005`.

S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz. Building a holographic superconductor. *Phys. Rev. Lett.*, 101:031601, Jul 2008. doi: 10.1103/PhysRevLett.101.031601. URL `https://link.aps.org/doi/10.1103/PhysRevLett.101.031601`.

M. Hashisaka, T. Ota, K. Muraki, and T. Fujisawa. Shot-noise evidence of fractional quasi-particle creation in a local fractional quantum Hall state. *Phys. Rev. Lett.*, 114:056802, Feb 2015. doi: 10.1103/PhysRevLett.114.056802. URL `https://link.aps.org/doi/10.1103/PhysRevLett.114.056802`.

C. A. Hernaski and P. R. S. Gomes. Duality between 3D massive Thirring and Maxwell-Chern-Simons models from 2D bosonization. *Phys. Rev. Lett.*, 121:041601, Jul 2018. doi: 10.1103/PhysRevLett.121.041601. URL `https://link.aps.org/doi/10.1103/PhysRevLett.121.041601`.

G. T. Horowitz. Exactly Soluble Diffeomorphism Invariant Theories. *Commun. Math. Phys.*, 125:417, 1989. doi: 10.1007/BF01218410.

P. A. Horvathy. Lectures on (Abelian) Chern-Simons vortices, 2007.

K. Intriligator and N. Seiberg. Mirror symmetry in three dimensional gauge theories. *Physics Letters B*, 387(3):513–519, 1996.

C. Itzykson and J.-M. Drouffe. *Conformal invariance*, volume 2 of *Cambridge Monographs on Mathematical Physics*, page 501–645. Cambridge University Press, 1989. doi: 10.1017/CBO9780511622786.005.

R. Jackiw. *FIELD THEORETIC INVESTIGATIONS IN CURRENT ALGEBRA*, pages 81–210. doi: 10.1142/9789814503044_0002. URL https://www.worldscientific.com/doi/abs/10.1142/9789814503044_0002.

R. Jackiw and S.-Y. Pi. Classical and quantal nonrelativistic Chern-Simons theory. *Phys. Rev. D*, 42:3500, 1990. doi: 10.1103/PhysRevD.42.3500. [Erratum: Phys.Rev.D 48, 3929 (1993)].

R. Jackiw and S.-Y. Pi. Selfdual Chern-Simons solitons. *Prog. Theor. Phys. Suppl.*, 107:1–40, 1992. doi: 10.1143/PTPS.107.1.

R. Jackiw and C. Rebbi. Solitons with fermion number ½. *Phys. Rev. D*, 13:3398–3409, Jun 1976. doi: 10.1103/PhysRevD.13.3398. URL https://link.aps.org/doi/10.1103/PhysRevD.13.3398.

J. K. Jain. Composite-fermion approach for the fractional quantum Hall effect. *Phys. Rev. Lett.*, 63:199–202, Jul 1989. doi: 10.1103/PhysRevLett.63.199. URL https://link.aps.org/doi/10.1103/PhysRevLett.63.199.

J. K. Jain. *Composite Fermions*. Cambridge University Press, 2007. doi: 10.1017/CBO9780511607561.

D. P. Jatkar and A. Khare. Peculiar charged vortices in higgs models with pure Chern-Simons term. *Physics Letters B*, 236(3):283 – 286, 1990. ISSN 0370-2693. doi: https://doi.org/10.1016/0370-2693(90)90983-D. URL http://www.sciencedirect.com/science/article/pii/037026939090983D.

H.-C. Jiang, Z. Wang, and L. Balents. Identifying topological order by entanglement entropy. *Nature Physics*, 8(12):902–905, 2012.

S. K. Paul and A. Khare. Chern-Simons term by spontaneous symmetry breaking in an Abelian Higgs model. *Physics Letters B*, 191(4):389 – 392, 1987. ISSN 0370-2693. doi: https://doi.org/10.1016/0370-2693(87)90627-7. URL http://www.sciencedirect.com/science/article/pii/0370269387906277.

S. Kachru, M. Mulligan, G. Torroba, and H. Wang. Bosonization and Mirror Symmetry. *Phys. Rev. D*, 94(8):085009, 2016. doi: 10.1103/PhysRevD.94.085009.

M. Kalb and P. Ramond. Classical direct interstring action. *Phys. Rev. D*, 9:2273–2284, Apr 1974. doi: 10.1103/PhysRevD.9.2273. URL https://link.aps.org/doi/10.1103/PhysRevD.9.2273.

D. Kamburov, Y. Liu, M. A. Mueed, M. Shayegan, L. N. Pfeiffer, K. W. West, and K. W. Baldwin. What determines the Fermi wave vector of composite fermions? *Phys. Rev. Lett.*, 113:196801, Nov 2014. doi: 10.1103/PhysRevLett.113.196801. URL https://link.aps.org/doi/10.1103/PhysRevLett.113.196801.

C. L. Kane and E. J. Mele. Quantum spin Hall effect in graphene. *Phys. Rev. Lett.*, 95:226801, Nov 2005. doi: 10.1103/PhysRevLett.95.226801. URL https://link.aps.org/doi/10.1103/PhysRevLett.95.226801.

W. Kang, H. L. Stormer, L. N. Pfeiffer, K. W. Baldwin, and K. W. West. How real are composite fermions? *Phys. Rev. Lett.*, 71:3850–3853, Dec 1993. doi: 10.1103/PhysRevLett.71.3850. URL https://link.aps.org/doi/10.1103/PhysRevLett.71.3850.

D. B. Kaplan. A method for simulating chiral fermions on the lattice. *Physics Letters B*, 288(3): 342–347, 1992.

D. B. Kaplan and M. Schmaltz. Supersymmetric Yang-Mills theories from domain wall fermions. 2000.

D. B. Kaplan and S. Sen. Fractional quantum Hall effect in a relativistic field theory. *Phys. Rev. Lett.*, 124:131601, Mar 2020. doi: 10.1103/PhysRevLett.124.131601. URL `https://link.aps.org/doi/10.1103/PhysRevLett.124.131601`.

A. Karch and D. Tong. Particle-vortex duality from 3d bosonization. *Phys. Rev. X*, 6:031043, Sep 2016. doi: 10.1103/PhysRevX.6.031043. URL `https://link.aps.org/doi/10.1103/PhysRevX.6.031043`.

A. Karch, B. Robinson, and D. Tong. More Abelian dualities in 2 + 1 dimensions. *Journal of High Energy Physics*, 2017(1):17, 2017.

L. H. Karsten. Lattice fermions in euclidean space-time. *Physics Letters B*, 104(4):315 – 319, 1981. ISSN 0370-2693. doi: https://doi.org/10.1016/0370-2693(81)90133-7. URL `http://www.sciencedirect.com/science/article/pii/0370269381901337`.

N. Karthik and R. Narayanan. Absence of bilinear condensate in three-dimensional QED. *PoS*, LATTICE2016:245, 2016. doi: 10.22323/1.256.0245.

E. Keski-Vakkuri and P. Kraus. Quantum Hall Effect in AdS/CFT. *JHEP*, 09:130, 2008. doi: 10.1088/1126-6708/2008/09/130.

A. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2 – 30, 2003. ISSN 0003-4916. doi: https://doi.org/10.1016/S0003-4916(02)00018-0. URL `http://www.sciencedirect.com/science/article/pii/S0003491602000180`.

A. Kitaev and J. Preskill. Topological entanglement entropy. *Phys. Rev. Lett.*, 96:110404, Mar 2006. doi: 10.1103/PhysRevLett.96.110404. URL `https://link.aps.org/doi/10.1103/PhysRevLett.96.110404`.

S. A. Kivelson, D.-H. Lee, Y. Krotov, and J. Gan. Composite-fermion Hall conductance at ν=. *Phys. Rev. B*, 55:15552–15561, Jun 1997. doi: 10.1103/PhysRevB.55.15552. URL `https://link.aps.org/doi/10.1103/PhysRevB.55.15552`.

H. Kjønsberg and J. Leinaas. Charge and statistics of quantum Hall quasi-particles — a numerical study of mean values and fluctuations. *Nucl. Phys. B*, 559:705–742, 1999. doi: 10.1016/S0550-3213(99)00353-3.

I. R. Klebanov, S. S. Pufu, S. Sachdev, and B. R. Safdi. Renyi entropies for free field theories. 2011. doi: 10.1007/JHEP04(2012)074.

V. Knizhnik and A. Zamolodchikov. Current algebra and Wess-Zumino model in two dimensions. *Nuclear Physics B*, 247(1):83 – 103, 1984. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(84)90374-2. URL `http://www.sciencedirect.com/science/article/pii/0550321384903742`.

K.-I. Kondo. Thirring model as a gauge theory. *Nuclear Physics B*, 450(1):251 – 266, 1995. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(95)00316-K. URL `http://www.sciencedirect.com/science/article/pii/055032139500316K`.

A. Kovner, B. Rosenstein, and D. Eliezer. Photon as a Goldstone boson in (2 + 1)-dimensional abelian gauge theories. *Nuclear Physics B*, 350(1):325 – 354, 1991. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(91)90263-W. URL `http://www.sciencedirect.com/science/article/pii/055032139190263W`.

R. Laughlin. Anomalous quantum Hall effect: An Incompressible quantum fluid with fractionally charged excitations. *Phys. Rev. Lett.*, 50:1395, 1983. doi: 10.1103/PhysRevLett.50.1395.

R. B. Laughlin. Quantized Hall conductivity in two dimensions. *Phys. Rev. B*, 23:5632–5633, May 1981. doi: 10.1103/PhysRevB.23.5632. URL `https://link.aps.org/doi/10.1103/PhysRevB.23.5632`.

M. Levin and X.-G. Wen. Detecting topological order in a ground state wave function. *Phys. Rev. Lett.*, 96:110405, Mar 2006. doi: 10.1103/PhysRevLett.96.110405. URL `https://link.aps.org/doi/10.1103/PhysRevLett.96.110405`.

H. Li and F. D. M. Haldane. Entanglement spectrum as a generalization of entanglement entropy: Identification of topological order in non-Abelian fractional quantum Hall effect states. *Phys. Rev. Lett.*, 101:010504, Jul 2008. doi: 10.1103/PhysRevLett.101.010504. URL `https://link.aps.org/doi/10.1103/PhysRevLett.101.010504`.

A. Lopez and E. Fradkin. Fractional quantum Hall effect and Chern-Simons gauge theories. *Phys. Rev. B*, 44:5246–5262, Sep 1991. doi: 10.1103/PhysRevB.44.5246. URL `https://link.aps.org/doi/10.1103/PhysRevB.44.5246`.

A. W. Ludwig and J. L. Cardy. Perturbative evaluation of the conformal anomaly at new critical points with applications to random systems. *Nuclear Physics B*, 285:687 – 718, 1987. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(87)90362-2. URL `http://www.sciencedirect.com/science/article/pii/0550321387903622`.

A. W. W. Ludwig, M. P. A. Fisher, R. Shankar, and G. Grinstein. Integer quantum hall transition: An alternative approach and exact results. *Phys. Rev. B*, 50:7526–7552, Sep 1994. doi: 10.1103/PhysRevB.50.7526. URL `https://link.aps.org/doi/10.1103/PhysRevB.50.7526`.

A. Luther and I. Peschel. Single-particle states, Kohn anomaly, and pairing fluctuations in one dimension. *Phys. Rev. B*, 9:2911–2919, Apr 1974. doi: 10.1103/PhysRevB.9.2911. URL `https://link.aps.org/doi/10.1103/PhysRevB.9.2911`.

J. M. Luttinger. An Exactly Soluble Model of a Many-Fermion System. *Journal of Mathematical Physics*, 4(9):1154–1162, Sept. 1963. doi: 10.1063/1.1704046.

C.-T. Ma. Parity Anomaly and Duality Web. *Fortsch. Phys.*, 66(8-9):1800045, 2018. doi: 10.1002/prop.201800045.

N. Maeda. Chiral anomaly and effective field theory for the quantum Hall liquid with edges. *Phys. Lett. B*, 376:142–147, 1996. doi: 10.1016/0370-2693(96)00274-2.

S. Mandelstam. Soliton operators for the quantized sine-Gordon equation. *Phys. Rev. D*, 11: 3026–3030, May 1975. doi: 10.1103/PhysRevD.11.3026. URL `https://link.aps.org/doi/10.1103/PhysRevD.11.3026`.

D. C. Mattis and E. H. Lieb. Exact solution of a many-fermion system and its associated boson field. *Journal of Mathematical Physics*, 6(2):304–312, 1965. doi: 10.1063/1.1704281. URL https://doi.org/10.1063/1.1704281.

N. D. Mermin and H. Wagner. Absence of ferromagnetism or antiferromagnetism in one- or two-dimensional isotropic heisenberg models. *Phys. Rev. Lett.*, 17:1133–1136, Nov 1966. doi: 10.1103/PhysRevLett.17.1133. URL https://link.aps.org/doi/10.1103/PhysRevLett.17.1133.

M. A. Metlitski and A. Vishwanath. Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators. *Phys. Rev. B*, 93 (24):245151, 2016. doi: 10.1103/PhysRevB.93.245151.

C. Montonen and D. I. Olive. Magnetic Monopoles as Gauge Particles? *Phys. Lett. B*, 72: 117–120, 1977. doi: 10.1016/0370-2693(77)90076-4.

G. W. Moore and N. Read. Nonabelions in the fractional quantum Hall effect. *Nucl. Phys. B*, 360:362–396, 1991. doi: 10.1016/0550-3213(91)90407-O.

G. W. Moore and N. Seiberg. Polynomial equations for rational conformal field theories. *Physics Letters B*, 212(4):451 – 460, 1988. ISSN 0370-2693. doi: https://doi.org/10.1016/0370-2693(88)91796-0. URL http://www.sciencedirect.com/science/article/pii/0370269388917960.

G. W. Moore and N. Seiberg. Naturality in Conformal Field Theory. *Nucl. Phys. B*, 313:16–40, 1989. doi: 10.1016/0550-3213(89)90511-7.

D. F. Mross, J. Alicea, and O. I. Motrunich. Explicit derivation of duality between a free dirac cone and quantum electrodynamics in $(2+1)$ dimensions. *Phys. Rev. Lett.*, 117:016802, Jun 2016. doi: 10.1103/PhysRevLett.117.016802. URL https://link.aps.org/doi/10.1103/PhysRevLett.117.016802.

J. Murugan and H. Nastase. Particle-vortex duality in topological insulators and superconductors. *Journal of High Energy Physics*, 2017(5):159, 2017. doi: 10.1007/JHEP05(2017)159. URL https://doi.org/10.1007/JHEP05(2017)159.

M. Nakahara. *Geometry, topology and physics*. 2003.

J. Nakamura, S. Liang, G. C. Gardner, and M. J. Manfra. Direct observation of anyonic braiding statistics at the ν=1/3 fractional quantum Hall state, 2020.

H. Nastase and F. Rojas. Vortices with source, fqhe and nontrivial statistics in 2+1 dimensions, 2016.

C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma. Non-abelian anyons and topological quantum computation. *Rev. Mod. Phys.*, 80:1083–1159, Sep 2008. doi: 10.1103/RevModPhys.80.1083. URL https://link.aps.org/doi/10.1103/RevModPhys.80.1083.

D. X. Nguyen and A. Gromov. Exact electromagnetic response of Landau level electrons. 2016. doi: 10.1103/PhysRevB.95.085151.

D. X. Nguyen, S. Golkar, M. M. Roberts, and D. T. Son. Particle-hole symmetry and composite fermions in fractional quantum Hall states. *Phys. Rev. B*, 97:195314, May 2018. doi: 10.1103/PhysRevB.97.195314. URL https://link.aps.org/doi/10.1103/PhysRevB.97.195314.

H. Nielsen and M. Ninomiya. Absence of neutrinos on a lattice: (ii). intuitive topological proof. *Nuclear Physics B*, 193(1):173 – 194, 1981a. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(81)90524-1. URL http://www.sciencedirect.com/science/article/pii/0550321381905241.

H. Nielsen and M. Ninomiya. Absence of neutrinos on a lattice: (i). proof by homotopy theory. *Nuclear Physics B*, 185(1):20 – 40, 1981b. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(81)90361-8. URL http://www.sciencedirect.com/science/article/pii/0550321381903618.

H. Nielsen and P. Olesen. Vortex-line models for dual strings. *Nuclear Physics B*, 61:45 – 61, 1973. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(73)90350-7. URL http://www.sciencedirect.com/science/article/pii/0550321373903507.

H. Năstase. *Elements of conformal field theory*, page 142–153. Cambridge University Press, 2015. doi: 10.1017/CBO9781316090954.010.

A. W. Overhauser. Note on the band theory of magnetism. *Physics Physique Fizika*, 1:307–309, May 1965. doi: 10.1103/PhysicsPhysiqueFizika.1.307. URL https://link.aps.org/doi/10.1103/PhysicsPhysiqueFizika.1.307.

W. Pan, W. Kang, M. P. Lilly, J. L. Reno, K. W. Baldwin, K. W. West, L. N. Pfeiffer, and D. C. Tsui. Particle-hole symmetry and the fractional quantum Hall effect in the lowest Landau level. *Phys. Rev. Lett.*, 124:156801, Apr 2020. doi: 10.1103/PhysRevLett.124.156801. URL https://link.aps.org/doi/10.1103/PhysRevLett.124.156801.

M. E. Peskin. Mandelstam 't Hooft Duality in Abelian Lattice Models. *Annals Phys.*, 113:122, 1978. doi: 10.1016/0003-4916(78)90252-X.

J. Polchinski. Scale and conformal invariance in quantum field theory. *Nuclear Physics B*, 303(2):226 – 236, 1988. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(88)90179-4. URL http://www.sciencedirect.com/science/article/pii/0550321388901794.

A. Polyakov. Nonhamiltonian approach to conformal quantum field theory. *Zh. Eksp. Teor. Fiz.*, 66:23–42, 1974.

A. M. Polyakov. Fermi-bose transmutations induced by gauge fields. *Modern Physics Letters A*, 03(03):325–328, 2020/08/09 1988. doi: 10.1142/S0217732388000398. URL https://doi.org/10.1142/S0217732388000398.

B. J. Powell. Emergent particles and gauge fields in quantum matter, 2020.

N. Read. Order parameter and Ginzburg-Landau theory for the fractional quantum Hall effect. *Phys. Rev. Lett.*, 62:86–89, Jan 1989. doi: 10.1103/PhysRevLett.62.86. URL https://link.aps.org/doi/10.1103/PhysRevLett.62.86.

N. Read. Conformal invariance of chiral edge theories. *Phys. Rev. B*, 79:245304, Jun 2009. doi: 10.1103/PhysRevB.79.245304. URL https://link.aps.org/doi/10.1103/PhysRevB.79.245304.

N. Read and E. Rezayi. Beyond paired quantum Hall states: Parafermions and incompressible states in the first excited Landau level. *Phys. Rev. B*, 59:8084–8092, Mar 1999. doi: 10.1103/PhysRevB.59.8084. URL `https://link.aps.org/doi/10.1103/PhysRevB.59.8084`.

A. N. Redlich. Parity violation and gauge noninvariance of the effective gauge field action in three dimensions. *Phys. Rev. D*, 29:2366–2374, May 1984a. doi: 10.1103/PhysRevD.29.2366. URL `https://link.aps.org/doi/10.1103/PhysRevD.29.2366`.

A. N. Redlich. Gauge noninvariance and parity nonconservation of three-dimensional fermions. *Phys. Rev. Lett.*, 52:18–21, Jan 1984b. doi: 10.1103/PhysRevLett.52.18. URL `https://link.aps.org/doi/10.1103/PhysRevLett.52.18`.

E. H. Rezayi and F. D. M. Haldane. Incompressible paired Hall state, stripe order, and the composite fermion liquid phase in half-filled Landau levels. *Phys. Rev. Lett.*, 84:4685–4688, May 2000. doi: 10.1103/PhysRevLett.84.4685. URL `https://link.aps.org/doi/10.1103/PhysRevLett.84.4685`.

V. Rosenhaus. An introduction to the SYK model. *Journal of Physics A: Mathematical and Theoretical*, 52(32):323001, jul 2019. doi: 10.1088/1751-8121/ab2ce1. URL `https://doi.org/10.1088%2F1751-8121%2Fab2ce1`.

S. Sachdev. *Dilute Fermi and Bose gases*, page 293–331. Cambridge University Press, 2 edition, 2011. doi: 10.1017/CBO9780511973765.018.

R. C. B. Santos, P. R. S. Gomes, and C. A. Hernaski. Bosonization of the Thirring model in $2+1$ dimensions. *Phys. Rev. D*, 101:076010, Apr 2020. doi: 10.1103/PhysRevD.101.076010. URL `https://link.aps.org/doi/10.1103/PhysRevD.101.076010`.

S. D. Sarma, M. Freedman, and C. Nayak. Majorana zero modes and topological quantum computation. *npj Quantum Information*, 1(1):15001, 2015.

R. Savit. Topological excitations in u(1)-invariant theories. *Phys. Rev. Lett.*, 39:55–58, Jul 1977. doi: 10.1103/PhysRevLett.39.55. URL `https://link.aps.org/doi/10.1103/PhysRevLett.39.55`.

H. J. Schulz. Fermi liquids and non–Fermi liquids. 1995.

J. Schwinger. On gauge invariance and vacuum polarization. *Phys. Rev.*, 82:664–679, Jun 1951. doi: 10.1103/PhysRev.82.664. URL `https://link.aps.org/doi/10.1103/PhysRev.82.664`.

J. Schwinger. Field theory commutators. *Phys. Rev. Lett.*, 3:296–297, Sep 1959. doi: 10.1103/PhysRevLett.3.296. URL `https://link.aps.org/doi/10.1103/PhysRevLett.3.296`.

N. Seiberg, T. Senthil, C. Wang, and E. Witten. A duality web in 2+1 dimensions and condensed matter physics. *Annals of Physics*, 374:395 – 433, 2016. ISSN 0003-4916. doi: https://doi.org/10.1016/j.aop.2016.08.007. URL `http://www.sciencedirect.com/science/article/pii/S0003491616301531`.

T. Senthil, D. T. Son, C. Wang, and C. Xu. Duality between $(2+1)d$ quantum critical points. 2018. doi: 10.1016/j.physrep.2019.09.001.

S. H. Simon. Topological quantum: Lecture notes, 2016.

D. T. Son. Is the composite fermion a dirac particle? *Phys. Rev. X*, 5:031027, Sep 2015. doi: 10.1103/PhysRevX.5.031027. URL https://link.aps.org/doi/10.1103/PhysRevX.5.031027.

D. T. Son. The dirac composite fermion of the fractional quantum Hall effect. 2018. doi: 10.1146/annurev-conmatphys-033117-054227.

M. Srednicki. *Quantum Field Theory*. Cambridge Univ. Press, Cambridge, 2007. URL https://cds.cern.ch/record/1019751.

A. Stern and B. I. Halperin. Proposed experiments to probe the non-Abelian $\nu = 5/2$ quantum Hall state. *Phys. Rev. Lett.*, 96:016802, Jan 2006. doi: 10.1103/PhysRevLett.96.016802. URL https://link.aps.org/doi/10.1103/PhysRevLett.96.016802.

M. Stone. Edge waves in the quantum Hall effect. *Annals of Physics*, 207(1):38 – 52, 1991. ISSN 0003-4916. doi: https://doi.org/10.1016/0003-4916(91)90177-A. URL http://www.sciencedirect.com/science/article/pii/000349169190177A.

L. Susskind. The quantum hall fluid and non-commutative chern simons theory, 2001.

W. E. Thirring. A soluble relativistic field theory. *Annals of Physics*, 3(1):91 – 112, 1958. ISSN 0003-4916. doi: https://doi.org/10.1016/0003-4916(58)90015-0. URL http://www.sciencedirect.com/science/article/pii/0003491658900150.

D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs. Quantized Hall conductance in a two-dimensional periodic potential. *Phys. Rev. Lett.*, 49:405–408, Aug 1982. doi: 10.1103/PhysRevLett.49.405. URL https://link.aps.org/doi/10.1103/PhysRevLett.49.405.

S.-i. Tomonaga. Remarks on Bloch's Method of Sound Waves applied to Many-Fermion Problems. *Progress of Theoretical Physics*, 5(4):544–569, 07 1950. ISSN 0033-068X. doi: 10.1143/ptp/5.4.544. URL https://doi.org/10.1143/ptp/5.4.544.

D. Tong. String theory, 2009.

D. Tong. The quantum Hall effect, 2016.

D. Tong. Statistical field theory, 2017.

D. Tong. Gauge theory, 2018.

S. A. Trugman and S. Kivelson. Exact results for the fractional quantum hall effect with general interactions. *Phys. Rev. B*, 31:5280–5284, Apr 1985. doi: 10.1103/PhysRevB.31.5280. URL https://link.aps.org/doi/10.1103/PhysRevB.31.5280.

D. C. Tsui. Interplay of disorder and interaction in two-dimensional electron gas in intense magnetic fields, 1998. URL https://www.nobelprize.org/prizes/physics/1998/tsui/lecture/.

D. C. Tsui, H. L. Stormer, and A. C. Gossard. Two-dimensional magnetotransport in the extreme quantum limit. *Phys. Rev. Lett.*, 48:1559–1562, May 1982. doi: 10.1103/PhysRevLett.48.1559. URL https://link.aps.org/doi/10.1103/PhysRevLett.48.1559.

C. Turner. Lectures on dualities in 2+1 dimensions. 2019. doi: 10.22323/1.349.0001.

J. Voit. One-dimensional Fermi liquids. *Reports on Progress in Physics*, 58(9):977–1116, sep 1995. doi: 10.1088/0034-4885/58/9/002. URL https://doi.org/10.1088%2F0034-4885%2F58%2F9%2F002.

K. von Klitzing. Quantum Hall effect: Discovery and application. *Annual Review of Condensed Matter Physics*, 8(1):13–30, 2017. doi: 10.1146/annurev-conmatphys-031016-025148. URL https://doi.org/10.1146/annurev-conmatphys-031016-025148.

K. von Klitzing, G. Dorda, and M. Pepper. New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance. *Phys. Rev. Lett.*, 45:494–497, Aug 1980. doi: 10.1103/PhysRevLett.45.494. URL https://link.aps.org/doi/10.1103/PhysRevLett.45.494.

C. Wang and T. Senthil. Dual Dirac Liquid on the Surface of the Electron Topological Insulator. *Phys. Rev. X*, 5(4):041031, 2015. doi: 10.1103/PhysRevX.5.041031.

C. Wang and T. Senthil. Half-filled Landau level, topological insulator surfaces, and three-dimensional quantum spin liquids. *Phys. Rev. B*, 93:085110, Feb 2016. doi: 10.1103/PhysRevB.93.085110. URL https://link.aps.org/doi/10.1103/PhysRevB.93.085110.

C. Wang, N. R. Cooper, B. I. Halperin, and A. Stern. Particle-hole symmetry in the fermion-Chern-Simons and Dirac descriptions of a half-filled Landau level. *Phys. Rev. X*, 7:031029, Aug 2017. doi: 10.1103/PhysRevX.7.031029. URL https://link.aps.org/doi/10.1103/PhysRevX.7.031029.

X. Wen. Topological Order in Rigid States. *Int. J. Mod. Phys. B*, 4:239, 1990a. doi: 10.1142/S0217979290000139.

X. G. Wen. Vacuum degeneracy of chiral spin states in compactified space. *Phys. Rev. B*, 40:7387–7390, Oct 1989. doi: 10.1103/PhysRevB.40.7387. URL https://link.aps.org/doi/10.1103/PhysRevB.40.7387.

X. G. Wen. Chiral Luttinger liquid and the edge excitations in the fractional quantum Hall states. *Phys. Rev. B*, 41:12838–12844, Jun 1990b. doi: 10.1103/PhysRevB.41.12838. URL https://link.aps.org/doi/10.1103/PhysRevB.41.12838.

X. G. Wen. Gapless boundary excitations in the quantum Hall states and in the chiral spin states. *Phys. Rev. B*, 43:11025–11036, May 1991. doi: 10.1103/PhysRevB.43.11025. URL https://link.aps.org/doi/10.1103/PhysRevB.43.11025.

X.-G. Wen. Theory of the edge states in fractional quantum Hall effects. *Int. J. Mod. Phys. B*, 6:1711–1762, 1992. doi: 10.1142/S0217979292000840.

X.-G. Wen. Topological orders and edge excitations in fractional quantum Hall states. *Advances in Physics*, 44(5):405–473, 1995. doi: 10.1080/00018739500101566. URL https://doi.org/10.1080/00018739500101566.

X.-G. Wen. Projective construction of non-Abelian quantum Hall liquids. *Phys. Rev. B*, 60:8827–8838, Sep 1999. doi: 10.1103/PhysRevB.60.8827. URL https://link.aps.org/doi/10.1103/PhysRevB.60.8827.

X. G. Wen and A. Zee. Classification of Abelian quantum Hall states and matrix formulation of topological fluids. *Phys. Rev. B*, 46:2290–2301, Jul 1992a. doi: 10.1103/PhysRevB.46.2290. URL https://link.aps.org/doi/10.1103/PhysRevB.46.2290.

X.-G. Wen and A. Zee. Neutral superfluid modes and “magnetic” monopoles in multilayered quantum Hall systems. *Phys. Rev. Lett.*, 69:1811–1814, Sep 1992b. doi: 10.1103/PhysRevLett.69.1811. URL https://link.aps.org/doi/10.1103/PhysRevLett.69.1811.

J. Wess and B. Zumino. Consequences of anomalous Ward identities. *Phys. Lett. B*, 37:95–97, 1971. doi: 10.1016/0370-2693(71)90582-X.

F. Wilczek. Quantum mechanics of fractional-spin particles. *Phys. Rev. Lett.*, 49:957–959, Oct 1982. doi: 10.1103/PhysRevLett.49.957. URL https://link.aps.org/doi/10.1103/PhysRevLett.49.957.

F. Wilczek and A. Zee. Linking numbers, spin, and statistics of solitons. *Phys. Rev. Lett.*, 51:2250–2252, Dec 1983. doi: 10.1103/PhysRevLett.51.2250. URL https://link.aps.org/doi/10.1103/PhysRevLett.51.2250.

R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. H. English. Observation of an even-denominator quantum number in the fractional quantum Hall effect. *Phys. Rev. Lett.*, 59:1776–1779, Oct 1987. doi: 10.1103/PhysRevLett.59.1776. URL https://link.aps.org/doi/10.1103/PhysRevLett.59.1776.

R. L. Willett. Experimental evidence for composite fermions. *Semiconductor Science and Technology*, 12(5):495–524, may 1997. doi: 10.1088/0268-1242/12/5/001. URL https://doi.org/10.1088%2F0268-1242%2F12%2F5%2F001.

R. L. Willett, M. A. Paalanen, R. R. Ruel, K. W. West, L. N. Pfeiffer, and D. J. Bishop. Anomalous sound propagation at $\nu = 1/2$ in a 2d electron gas: Observation of a spontaneously broken translational symmetry? *Phys. Rev. Lett.*, 65:112–115, Jul 1990. doi: 10.1103/PhysRevLett.65.112. URL https://link.aps.org/doi/10.1103/PhysRevLett.65.112.

R. L. Willett, K. W. West, and L. N. Pfeiffer. Apparent inconsistency of observed composite fermion geometric resonances and measured effective mass. *Phys. Rev. Lett.*, 75:2988–2991, Oct 1995. doi: 10.1103/PhysRevLett.75.2988. URL https://link.aps.org/doi/10.1103/PhysRevLett.75.2988.

I. Williams. Aspects of topological quantum fields, strings dualities. Master's thesis, 2019.

T. Wiseman. An introduction to ads-cft, 2020.

E. Witten. Global Aspects of Current Algebra. *Nucl. Phys. B*, 223:422–432, 1983. doi: 10.1016/0550-3213(83)90063-9.

E. Witten. Nonabelian bosonization in two dimensions. *Comm. Math. Phys.*, 92(4):455–472, 1984. URL https://projecteuclid.org:443/euclid.cmp/1103940923.

E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3): 351–399, 1989. URL https://projecteuclid.org:443/euclid.cmp/1104178138.

E. Witten. $sl(2,z)$ action on three-dimensional conformal field theories with Abelian symmetry, 2003.

E. Witten. Three lectures on topological phases of matter. 2015. doi: 10.1393/ncr/i2016-10125-3.

T. T. Wu and C. N. Yang. Dirac Monopole Without Strings: Monopole Harmonics. *Nucl. Phys. B*, 107:365, 1976. doi: 10.1016/0550-3213(76)90143-7.

J. S. Xia, W. Pan, C. L. Vicente, E. D. Adams, N. S. Sullivan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West. Electron correlation in the second Landau level: A competition between many nearly degenerate quantum phases. *Phys. Rev. Lett.*, 93:176809, Oct 2004. doi: 10.1103/PhysRevLett.93.176809. URL `https://link.aps.org/doi/10.1103/PhysRevLett.93.176809`.

A. Zamolodchikov. Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory. *JETP Lett.*, 43:730–732, 1986.

A. Zee. *Quantum Field Theory in a Nutshell*. Princeton University Press, 2010.

S.-C. Zhang. The Chern-Simons-Landau-Ginzburg theory of the fractional quantum Hall effect. *Int. J. Mod. Phys. B*, 6:25–58, 1992. doi: 10.1142/S0217979292000037.

S. C. Zhang, T. H. Hansson, and S. Kivelson. Effective-field-theory model for the fractional quantum Hall effect. *Phys. Rev. Lett.*, 62:82–85, Jan 1989. doi: 10.1103/PhysRevLett.62.82. URL `https://link.aps.org/doi/10.1103/PhysRevLett.62.82`.

B. Zumino, Y.-S. Wu, and A. Zee. Chiral Anomalies, Higher Dimensions, and Differential Geometry. *Nucl. Phys. B*, 239:477–507, 1984. doi: 10.1016/0550-3213(84)90259-1.