Overview of Modified Gravity

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Abstract. This paper provides an introduction to modified gravity. Firstly, we review the current best fit model of cosmology (ΛCDM model) and present the cosmological constant problem. As opposed to having a cosmological constant to account for dark energy, we motivate an alternative solution which is modifying Einstein's theory of general relativity (GR). We review GR to provide a basis to study modified theories of gravity. We will then introduce $F(R)$ gravity by first deriving the field equations. From this, we will look at $F(R)$ gravity in metric formalism. Then we will show how this theory can be mapped to a scalar tensor theory, the consequences this can pose, and the potential mechanisms that can be used to save it. Finally, we introduce massive gravity as another way to modify GR. We outline the history of massive gravity from Fierz-Pauli massive gravity to the construction of the unique non-linear theory of massive gravity (dRGT), named after the authors de Rham, Gabadadze and Tolley.

Contents

1 Introduction

Theories of modified gravity aim to address in part or full some of the current issues facing the ΛCDM model. These theories can take any form of modification to general relativity. The main goal of these theories is to provide a theory that performs in line with GR over small distances for astronomical masses (where GR cosmology shows success) [1], and that neatly describes large scale observations such as the universal acceleration. This can be tricky to achieve but interesting methods like screening mechanisms can be employed to hide these modifications on solar system scales to satisfy experimental data [2].

Whether they can achieve this or not, these theories provide interesting ways of thinking about general relativity and can give a deeper understanding of the theory. By pulling apart and fiddling with the model it is easier to understand how those pieces work.

There are quite a few ways to modify gravity, and so this paper will only focus on two main examples. One method will look at finding invariant higher orders of the Ricci scalar in the action in $F(R)$ gravity [3]. The other will look to modify the massless tensor field of GR by giving it mass in massive gravity [4].

Section 1 introduces general relativity and the field equations that govern it. We then provides a brief history of the ΛCDM model; the current most widely accepted model of cosmology. We discuss the main issues with the ΛCDM model, in particular the issues with the cosmological constants use in dark energy. We then provide motivation for studying modified theories of gravity to solve these the problem of dark energy. Furthermore, modified gravity can also be used to explain dark matter. Section 2 studies general relativity in more detail, preparing for its modification in later chapters. We give a derivation of the field equations from the action. After this, we will look into linearising GR, how gauge transformations can be used to simplify the theory, and a solution for a generic source coupled with Einstein's theory. From the linearisation, the degrees of freedom of the theory are counted and used to describe GR as a massless spin-2 field theory. The non-linear extensions are also discussed. Section 3 introduces $F(R)$ gravity by modifying the Einstein-Hilbert action in GR and deriving the field equations. By manipulating the action we present the theory as a scalar tensor theory akin to Brans-Dicke. Brans-Dicke has a problem at being heavily constrained in the solar-system, meaning the theory basically looks like GR. This results in the theory not being interesting for dark energy purposes. Section 4 presents massive gravity by introducing the Fierz-Pauli action motivated from linearised GR. It then shows how the degrees of freedom are found and confirm the expectation of 5 degrees of freedom expected from a massive spin-2 field theory. Solutions for the massless limit of Fierz-Pauli theory and GR are then looked at for a static source. The name of the issue that causes a discrepancy in physical observation between these two theories is the vDVZ discontinuity. We derive the reason for the discontinuity utilising the Stückelberg trick and briefly mention how to solve the problem by extending Fierz-Pauli massive gravity to a non-linear theory.

In this report we will work with a $(-+++)$ metric signature.

1.1 General Relativity

General relativity is widely considered to be the correct theory of gravity for large masses at solar system scales. It accurately predicts the orbits of planets in the solar system and still continues to stand many rigorous astrophysical tests today. One of the main assumptions made by the current model of cosmology (the ΛCDM model) is that general relativity describes gravitational interactions in our universe

The theory of GR considers how the curvature of spacetime affects the motion of matter and how the presence of matter affect the curvature of spacetime [5]. This motivates the Einstein-Hilbert action which contains curvature and matter terms that look like

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(2R - \Lambda) + \int d^4x\mathcal{L}_M[g_{\mu\nu}, \Psi^i], \tag{1.1}$$

where the constant $1/16\pi G$ can be obtained by comparing the classical limit of the theory with Newtonian gravity. Λ is the cosmological constant allowed by the symmetries of the theory. The $\mathcal{L}_M$ is the Lagrangian density that describes matter and energy (radiation, baryons, dark matter) inside the universe whose motion is described by the fields Ψ^i.

Applying the principal of least action to the action in eq (1.1) results in the Einstein field equations that define how the metric (curvature) relates to the enegy-momentum tensor (matter). The Einstein field equations are

$$8\pi G T^{\mu\nu} = G^{\mu\nu} + \Lambda g^{\mu\nu}. \tag{1.2}$$

Here, the Einstein tensor $G^{\mu\nu}$ contains the information of the curvature from the metric $g^{\mu\nu}$ and the energy-momentum tensor $T^{\mu\nu}$ contains information on the matter in the model. This will be explored more deeply in Section 2

1.2 A History of the Cosmological Model

For this review it is important to look at the cosmological constant as it is a key cause of the issues with the current cosmological model. In this section we will look at the history of our current cosmological model and the part that the cosmological constant has played in it's creation.

Einstein found great success with his theory of general relativity and by 1916 he had consolidated his four papers on the subject into "The Foundation of the General Theory of Relativity" [6]. In 1917 Einstein looked to create a cosmological model based on GR [7]. Before the observational evidence provided by Hubble in 1929 [8] he assumed a closed static universe to satisfy Mach's principle; that local physical laws are determined by the large-scale structure of the universe . The metric of this universe took the form

$$ds^2 = -dt^2 + R^2 \sin^2(r/R) d{\Omega_2}^2 \tag{1.3}$$

to describe a homogeneous, isotropic, and static universe. The R represents the curvature radius of the 3-dimensional space, and r is the spacial distance from the origin. He discovered this metric was not in fact a solution to his Field equations, and so the introduction of a cosmological constant was required for GR to be compatible with a static, homogeneous and isotropic universe. This positive constant was used as an ad-hoc fix to counter the attractive pull that gravity would have on matter, causing the universe to collapse in on itself.

The next step beyond Einstein's model would be to generalise it to allow the curvature radius R a time dependence. This still maintains homogeneity and isotropy on a constant t slice but abandons the assumption of a static universe. Friedmann in 1922, 1924 [9, 10] and Lemaître in 1927 [11] independently discovered solutions to Einstein's field equations for a positive, zero and negative curvature universe. Depending on the form of the expansion rate, the universe could expand or contract. This resolved the issue of a collapsing universe without the need of the cosmological constant. The observational evidence provided by Hubble in the late 1920s showed that the universe was in fact expanding. With the theory of an expanding universe, along with the observational evidence gave rise to the Big Bang cosmological model. This brought into question the cosmological constant and Einstein's model.

In the 1930s, Robertson [12–15] and Walker [16, 17] showed independently of each other that there are only three types of metric that satisfy each of the three scenarios. In a spherically symmetric coordinate system they take the form

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\omega^2\right] \tag{1.4}$$

where $k = \pm 1, 0$. For $k = 1$, the $t = const$ slices are positively curved hyper-surfaces, for $k = 0$ they are flat, and $k = -1$ has negative curvature. The function $a(t)$, known as the scale factor, describes the relative change in the size of the universe with time. It is taken to be unitary at the time of measurement $a(t = 0) = a_0 = 1$. This metric is known as the Friedmann–Lemaître–Robertson–Walker (FLRW) metric.

Over the years more observations caused the cosmological constants necessity to fall in and out of favour [18]. In the 1980s, one prevalent issue with the cosmological model was the horizon problem; regions of space that should have no causal contact share similarities. If the universe began with the slightest difference in temperature in different regions, by now there should not be such isotropy in the cosmic microwave background radiation (CMBR); the relic radiation left over from the early stages of the universe. This could be explained by a universe with an acceleration phase (inflation), that would cause regions previously in causal contact to move outside of this range [19]. The arguments for an inflationary epoch and observational results from CMBR provided reason to believe the universe was flat [20].

Now by applying the metric in eq (1.4) to the Einstein field equations in eq (1.2) one can derive the Friedman equation

$$H^2 = 8\pi G\frac{\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}. \tag{1.5}$$

where the Hubble parameter is defined by the rate of change in scale factor $H \coloneqq \dot{a}/a$. The Λ was initially not present in the equation as previously mentioned. But the matter density $\rho \simeq 9 \times 10^{-27} kg/m^3$ of the universe is not enough alone to achieve a flat universe $k = 0$ for the observed Hubble parameter at the time $H_0 \coloneqq H(0) \simeq 70(km/s)/Mpc$. So a vacuum energy provided by the cosmological constant Λ was needed to make up the shortfall for this.

By late 1990s, observational results strengthened the case for dark energy in the cosmological model. Some of these included measurements of the Luminosity distance of type Ia supernovae [21]. More recent estimates put the cosmological equation of state (ratio between the pressure and energy density of the dark energy fluid) to be $w \simeq -1.061^{+0.069}_{-0.068}$ [22]. Another observational piece of evidence for dark energy comes from measuring CMBR anisotropies [23] with Sloan Sky Survey measurements [24] which found $w = -0.99 \pm 0.11$.

1.3 The ΛCDM Model

The ΛCDM model is the most widely accepted cosmological model. It follows a universe that begins with inflation, enters a hot radiation epoch, then next follows a matter (Baryonic and Dark Matter) epoch and finally a dark energy dominated epoch at late times. It is the balance of these components that contribute to the energy density in the universe that describes the ΛCDM model of today. To see this balance, the Friedmann equation can be rewritten in terms of the separate components, scaled so they sum to 1.

We start by assuming matter is a perfect fluid that has cosmological symmetry (isotropic and homogeneous) and by using conservation of energy ($\nabla^\mu T_{\mu\nu} = 0$), the FLRW metric can be used to show that

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1 + w_i) = 0, \tag{1.6}$$

where w_i is defined through a perfect fluid $P = w_i\rho$. A $w_M = 0$ defines cold matter ($P = 0$), $w_R = 1/3$ defines radiation pressure ($P = 1/3\rho$) and a $w_\Lambda = -1$ the vacuum pressure ($P = -\rho$). The solution to this equation can be written in terms of present time ($t = t_0$) as

$$\rho_i = \rho_{i,0}\left(\frac{a_0}{a}\right)^{3(1+w_i)}, \tag{1.7}$$

where the a_0,ρ_0 integration constants are their respective values at present time. The energy density in the Friedmann equation eq (1.5) can then be separated into cold matter and radiation parts using eq (1.7). The cosmological constant term is given by the vacuum energy part as it is a constant in eq (1.7). By defining the vacuum energy $\rho_\Lambda \coloneqq \frac{\Lambda}{8\pi G}$ and the contribution from the curvature $\rho_{k,0} \coloneqq \frac{3}{8\pi G}\frac{-k}{a_0^2}$ Friedmann equation can be written as

$$H^2 = \frac{8\pi G}{3}\left(\rho_\Lambda + \rho_{k,0}\left(\frac{a_0}{a}\right)^2 + \rho_{M,0}\left(\frac{a_0}{a}\right)^3 + \rho_{R,0}\left(\frac{a_0}{a}\right)^4\right). \tag{1.8}$$

In order to contain k it is useful to start by considering a flat universe with zero Cosmological Constant Λ. The Friedman equation eq (1.5) and more specifically the Hubble Constant at present time ($H_0 \coloneqq H(t_0)$) can then defined through the critical density

$$\rho_c \coloneqq \frac{3}{8\pi G}H_0^2. \tag{1.9}$$

By making use of the density parameter, defined by

$$\Omega_i = \frac{\rho_{i,0}}{\rho_c}, \tag{1.10}$$

the Friedmann equation of eq (1.8) can be rewritten as

$$\left(\frac{H}{H_0}\right)^2 = \left(\Omega_\Lambda + \Omega_k\left(\frac{a_0}{a}\right)^2 + \Omega_M\left(\frac{a_0}{a}\right)^3 + \Omega_R\left(\frac{a_0}{a}\right)^4\right). \tag{1.11}$$

It is easy to see from this equation how there will be an era in which each term is dominant in the universal expansion. For present time $t = t_0$ this equation gives the percentage contributions to our universe

$$1 = \Omega_\Lambda + \Omega_k + \Omega_M + \Omega_R. \tag{1.12}$$

The exact shape of the universe is still a matter of debate in physical cosmology, but experimental data from various independent sources studying the CMBR from WMAP [23], BOOMERanG [20], and the Planck telescope [25] confirm that the universe is flat. So the contribution from the curvature can effectively be ignored: $|\Omega_k| << 1$.

Data from the CMB that shows radiation density is past its epoch i.e $|\Omega_r| << 1$. This means we have we have $\Omega_M + \Omega_\Lambda = 1$. By studying the CMBR we can obtain best-fit values where $\Omega_M \sim 0.3$ and $\Omega_\Lambda \sim 0.7$. One of the more recent studies obtains for a flat ΛCDM cosmology, found $\Omega_m = 0.295 \pm 0.034$ and a constant dark-energy equation of state parameter $w = -1.018 \pm 0.057$ [26].

The fact that these results come from widely different observational experiments and yet have a great concordance with one another gave ΛCDM the name concordance model.

1.4 Problems with the ΛCDM model

The fact that the ΛCDM model is able to constantly provide success with cosmological observations should make it a satisfying model. The big issue with it is in the energy density contributions. It predicts around 70% of the universe is dominated by dark energy; an unknown component that exerts a negative pressure to provide universal acceleration. This is still something that has yet to be observed by any experiment. The same goes for dark matter which provides the next biggest contribution of around 25%, dominating Baryonic matter by about 85%. This model suggests we have only ever directly observed the matter that makes up less then 5% of the universe. Even though these components have yet to be detected, it does not imply the model is inaccurate.

This may change our whole view on the composition of our universe, but this can just be seen as observational evidence that suggests this models validity. On the other hand, an absence of evidence does not always prove evidence of absence. This may be somewhat a case of history repeating itself. When Newtonian gravity was discovered no one knew that it was the weak field limit of some more complex theory (GR). In the 19th century, French mathematician Urbain Le Verrier hypothesized that peculiarities in Mercury's orbit were the result of a dark planet, which he named "Vulcan". A number of searches were made for Vulcan, but despite occasional claimed observations, no such planet was ever confirmed. Peculiarities in Mercury's orbit are now explained by Einstein's theory of general relativity [27]. It could be that the current most accepted model of cosmology covers up a more fundamental theory that describes our universe. While speculative, it feels compelling to assume that GR is the limit of some more complex gravity theory that we do not know.

One of the real issues with the ΛCDM model is that we do not have a fundamental view on how these components interact. For example, we know how 5% of the matter we see in the standard model interact, and how photons from the CMB interact. It is hard to accept a model that fundamentally we know so little about just because it works.

Some of the more direct issues focus around the cosmological constant itself. One of the most well known is the cosmological coincidence problem. It notes that extreme fine tuning in initial conditions would be necessary for both the special events of matter-vacuum equality: $\Omega_M \sim \Omega_\Lambda$ and the formation of large scale structures happen roughly at the same time today. But this may be as much of an issue as any other fine tuned constant in physical models e.g. the strength of the strong nuclear force. An argument can be made that the statistical probability of these events occurring together are in fact reasonable [28]. If planetary formation occurred in most other eras they would be unable to sustain life in the first place. So our existence could be thought of as a solution to this.

One of the biggest problems in the ΛCDM model is the eponymous cosmological constant problem or vacuum catastrophe [29]. The model uses Λ for the purpose of another source term ($g_{\mu\nu}\Lambda$) in the field equations of general relativity. In QFT, the vacuum-state $|0\rangle$ provides the ground state. This typically would be empty but virtual particles produced by quantum fluctuations gives rise to a non-zero vacuum energy. These theories are built with Lorentz invariance and so the vacuum energy should provide a contribution to the stress-energy tensor given by

$$T_{\mu\nu}^{vac} = \langle 0| T_{\mu\nu} |0\rangle =< \rho_{vac} > g_{\mu\nu}. \tag{1.13}$$

This is a quantum expectation value and although GR is a classical field theory it should be present in the stress energy tensor as it can emerge through classical observations like the

Casimir effect. It is reasonable to set the cut-off limit of QFT to the Planck energy

$$E_p = \left(\frac{\hbar c^5}{G}\right)^{1/2} \sim 10^{19} GeV, \tag{1.14}$$

where we have specifically reintroduced c, and $\hbar$ for clarity. By treating these virtual particles as a set of harmonic oscillators the vacuum energy can be witten as

$$\rho_{vac} = \frac{E}{V} = \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{2}\hbar\omega_{\mathbf{k}} \approx \frac{\hbar}{2\pi^2 c^3}\int_0^{\omega_{max}} \omega^3 d\omega = \frac{\hbar}{8\pi^2 c^3}\omega_{max}^4. \tag{1.15}$$

This means that using the plank energy as the cut-off ($E_p = \hbar\omega_{max}$), the vacuum energy of QFT is roughly

$$\rho_{vac}^{Plank} \sim 10^{114} erg/cm^3. \tag{1.16}$$

These theoretical predictions are widely different of the observational evidence for the value of a cosmological constant which puts $\Lambda < 10^{-56} cm^-2$ and so provides a vacuum energy density of

$$\rho_{vac}^{\Lambda} \sim 10^{-9} erg/cm^3. \tag{1.17}$$

With over 120 orders of magnitude difference this truly is a vacuum catastrophe. Even the electroweak scale $\sim 100\ GeV$ cutoff will have a vacuum energy; $\rho_{vac}^{EW} \sim 10^{46} erg/cm^3$, almost 60 orders of magnitude difference. This means that the gravitational strength of the vacuum is much weaker then predicted by QFT.

1.5 Modified Gravity

Instead of using the cosmological constant for dark energy to provide the solution to the universal acceleration, we could look to the left hand side of the Einstein equations and modify gravity. Since we have only tested GR in our universe we do not necessarily know how gravity might work at universal scales. Dark matter is required to explain rotation curves of galaxies; why stars rotation speed does not tend to vary with distance from the center. Though this is not discussed in this review, modified gravity theories have been used to explain dark matter; massive graviton geons, bimetric gravity, and even Modified Newtonian dynamics (MOND) [30–32]. We will focus on the applications of modified gravity theories to tackle the problem of dark energy.

This report will try and show how these modified gravity theories may be used to take on the cosmological constant problem and provide answers for late time acceleration of the universe. The initial criteria for a successful theory of modified gravity would be to replicate the results of GR around a micrometer to solar system scales, with gravitational interactions weakening at universal scales to account for acceleration. In this paper we look at two theories that try and achieve this; massive gravity which modifies the graviton to be massive and $F(R)$ which modifies the action to higher orders of curvature invariants.

2 General Relativity

General relativity provides a geometrical interpretation of gravity. It replaces the Newtonian concepts of force fields that require action at a distance for a field defined by the curvature of a spacetime manifold M (with Lorentzian signature), whose source is the energy of matter and radiation.

This chapter will look at the main features of general relativity. More specifically; how it can be motivated and the path to its solutions. It will also cover how perturbation theory (linearised gravity) can be applied to provide solutions and a deeper understanding of GR.

2.1 The Einstein-Hilbert Action

Like most field theories, general relativity starts with an action. The theory is defined by the curvature of the spacetime which restricts the action to inherently depend on the metric field tensor $g_{\mu\nu}$; which describes the geometrical structure of a spacetime manifold. A first step would be to consider that the metric tensor will be second order in the action as most physical theories. By applying the insights of Einstein from the equivalence principle and general covariance, the action can be derived. The action is formed from the Ricci scalar which is the simplest curvature invariant that can be written. The Ricci scalar is written in terms of derivatives of the metric by

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^{\rho}{}_{\mu\rho\nu} = g^{\mu\nu} \left(\partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu} \right), \tag{2.1}$$

where Γ is the Christoffel connection and is given by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right) \tag{2.2}$$

The Ricci scalar forms the main body of the action which is defined by equation (1.1).

The first step in recovering the field equations of eq (1.2) uses the variation of the action with respect to the metric tensor ($g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$), this takes the form

$$\delta S = \frac{1}{16\pi G} \int d^4x \left(\delta(\sqrt{-g})(R - 2\Lambda) + \sqrt{-g}\delta R \right) + \delta S_M, \tag{2.3}$$

where S_M is the action of the matter fields from the lagrangian density $\mathcal{L}_M$. By factoring out the variation in $\delta g^{\mu\nu}$ from each term the field equations can be retrieved. In order to achieve this, the variation of the determinant $\delta(\sqrt{-g})$ can be written as

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \tag{2.4}$$

The Christoffel connection is simply

$$\delta(\Gamma^\rho{}_{\mu\nu}) = \frac{1}{2} g^{\rho\lambda} (\nabla_\mu \delta g_{\lambda\nu} + \nabla_\nu \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\nu}). \tag{2.5}$$

The Ricci scalar can be written as the tensor $R = g_{\mu\nu} R^{\mu\nu}$, so that the scalar variation can be given by

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}. \tag{2.6}$$

The Ricci tensor $\delta R_{\mu\nu}$ variation can be written as

$$\begin{aligned} g^{\mu\nu} \delta R_{\mu\nu} &= g^{\mu\nu} (\nabla_\nu \delta \Gamma^\rho_{\rho\nu} - \nabla_\rho \delta \Gamma^\rho_{\mu\nu}) \\ &= \frac{1}{2} g^{\mu\nu} g^{\rho\lambda} \left(\nabla_\nu (\nabla_\rho \delta g_{\lambda\mu} + \nabla_\mu \delta g_{\rho\lambda} - \nabla_\lambda \delta g_{\rho\mu}) - \nabla_\rho (\nabla_\mu \delta g_{\lambda\nu} + \nabla_\nu \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\nu}) \right) \\ &= \frac{1}{2} \left((\nabla_\nu \nabla_\rho \delta g^{\rho\nu} + g_{\rho\lambda} \Box \delta g_{\rho\lambda} - \nabla_\nu \nabla_\lambda \delta g_{\lambda\mu}) - (\nabla_\rho \nabla_\mu \delta g^{\rho\mu} + \nabla_\rho \nabla_\nu \delta g_{\nu\rho} - g_{\mu\nu} \Box \delta g_{\mu\nu}) \right) \\ &= g_{\mu\nu} \Box \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu}. \end{aligned} \tag{2.7}$$

which in the action integral can be transformed into a surface term via integration by parts and Stokes theorem. By requiring that the metric and its first derivative vanish on the boundary, the last term will vanish in the integral. The stress-energy tensor can be defined as

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{2.8}$$

along with equations (2.5) and (2.6), the metric variation can be factored out of the action (2.3) so that

$$\delta S = \int d^4x\sqrt{-g}\left(\frac{1}{34\pi G}(-g^{\mu\nu}(R-2\Lambda)+2R^{\mu\nu}) - \frac{1}{2}T^{\mu\nu}\right)\delta g_{\mu\nu}. \tag{2.9}$$

By minimising the action ($\delta S = 0$), the field equation are recovered

$$8\pi G T^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \Lambda g^{\mu\nu}. \tag{2.10}$$

The Einstein tensor, defined by

$$G^{\mu\nu} \coloneqq R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} \tag{2.11}$$

can be used to shorten the expression to eq (1.2).

2.2 Linearised Gravity

To perform tests of GR experimentally, the Einstein field equations must be simplified. Birkhoff's theorem gives justification to treat gravity in the weak-field limit as a perturbation around Minkowski space. The metric tensor $g_{\mu\nu}$ can be written as

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \\ &= \eta_{\mu\nu} + h^{\Lambda}_{\mu\nu} + h^{GW}_{\mu\nu}\,, \end{aligned} \tag{2.12}$$

where $\eta_{\mu\nu}$ is the background Minkowski metric and $h^{\Lambda}_{\mu\nu}, h^{GW}_{\mu\nu}, h_{\mu\nu} << 1$ are small perturbations. $h_{\mu\nu}$ is split up into a perturbation from gravitational waves ($h^{GW}_{\mu\nu}$) and a background from the cosmological constant ($h^{\Lambda}_{\mu\nu}$). When using linearised analysis, all expressions raise and lower indices using the background metric $\eta_{\mu\nu}$. This paper will only look at linear order in the variation h for analysis.

2.2.1 Curvature and Einstein

To recover the linearised field equations one can start with the action in eq (1.1) and work through the derivation or more easily start with the field equation in eq (1.2) and find each component in terms of eq (2.24). For the latter, the only components required are $R_{\mu\nu}$ and R. A good starting point is to linearise the metric connection

$$\begin{aligned} \Gamma^{\rho}{}_{\mu\nu} &= \frac{1}{2}\eta^{\rho\lambda}\left(h_{\nu\lambda,\mu} + h_{\lambda\mu,\nu} - h_{\mu\nu,\lambda}+\right) + \mathcal{O}(h^2)\,, \\ &= \frac{1}{2}\left(h^{\rho}{}_{\nu,\mu} + h^{\rho}{}_{\mu,\nu} - h_{\mu\nu}{}^{,\rho}\right) + \mathcal{O}(h^2)\,, \end{aligned} \tag{2.13}$$

where the notation for partial derivatives are shortened by $\partial_\rho h_{\mu\nu} = h_{\mu\nu,\rho}$. The Ricci tensor in terms of the connection

$$R_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu,\rho} - \Gamma^{\rho}{}_{\mu\rho,\nu} + \Gamma^{\lambda}{}_{\mu\nu}\Gamma^{\rho}{}_{\nu\lambda} - \Gamma^{\lambda}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\lambda}, \tag{2.14}$$

when linearised to first order, the last two terms which are $\mathcal{O}(h^2)$ will be dropped. The Ricci tensor linearised will be given by

$$\begin{aligned} R_{\mu\nu} &= \Gamma^{\rho}{}_{\mu\nu,\rho} - \Gamma^{\rho}{}_{\mu\rho,\nu} + \mathcal{O}(h^2) \\ &= \frac{1}{2}\left(h^{\rho}{}_{\nu,\mu\rho} + h^{\rho}{}_{\mu,\nu\rho} - h_{\mu\nu}{}^{,\rho}{}_{\rho} - h^{\rho}{}_{\rho,\mu\nu} - h^{\rho}{}_{\mu,\rho\nu} + h_{\mu\rho}{}^{,\rho}{}_{\nu}\right) + \mathcal{O}(h^2) \\ &= \frac{1}{2}\left(h^{\rho}{}_{\nu,\mu\rho} - h_{\mu\nu}{}^{,\rho}{}_{\rho} - h^{\rho}{}_{\rho,\mu\nu} + h_{\mu\rho}{}^{,\rho}{}_{\nu}\right) + \mathcal{O}(h^2) \\ &= \frac{1}{2}\left(h^{\rho}{}_{\nu,\mu\rho} - \Box h_{\mu\nu} - h_{,\mu\nu} + h_{\mu\rho}{}^{,\rho}{}_{\nu}\right) + \mathcal{O}(h^2)\,, \end{aligned} \tag{2.15}$$

where the last row uses $h \equiv h^{\rho}{}_{\rho}$ and $\Box = \partial^\rho\partial_\rho$ for shorthand. Finally, by taking the trace, the scalar will be

$$\begin{aligned} R &= \eta^{\mu\nu}R_{\mu\nu} \\ &= \frac{1}{2}\left(h^{\rho\mu}{}_{,\mu\rho} - \Box h - \Box h + h^{\rho\mu}{}_{,\mu\rho}\right) + \mathcal{O}(h^2) \\ &= h^{\rho\mu}{}_{,\mu\rho} - \Box h + \mathcal{O}(h^2)\,. \end{aligned} \tag{2.16}$$

With eq (2.15) and (2.16) the Einstein tensor of eq (2.11) can be linearised to first order

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \\ &= \frac{1}{2}\left(-\Box h_{\mu\nu} + h^{\rho}{}_{\nu,\rho\mu} + h^{\rho}{}_{\mu,\rho\nu} - h_{,\mu\nu} + \eta_{\mu\nu}\Box h - \eta_{\mu\nu}h^{\rho\mu}{}_{,\mu\rho}\right), \end{aligned} \tag{2.17}$$

So that the field equations will be of the form

$$8\pi G T_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} + h^{\rho}{}_{\nu,\rho\mu} + h^{\rho}{}_{\mu,\rho\nu} - h_{,\mu\nu} + \eta_{\mu\nu}\Box h - \eta_{\mu\nu}h^{\rho\mu}{}_{,\mu\rho}\right), \tag{2.18}$$

with the cc absorbed into the energy tensor. It is easy to see in this form that the mass-energy tensor is a conserved quantity $\partial^\mu T_{\mu\nu} = 0$ under spacetime translation symmetry. This form can be simplified by substituting for the trace-reversed metric perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}. \tag{2.19}$$

In this form, the linearised Einstein tensor will look like

$$G_{\mu\nu} = \frac{1}{2}\left(\bar{h}^{\rho}_{\nu,\rho\mu} + \bar{h}^{\rho}_{\mu,\rho\nu} - \Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\bar{h}^{\rho\mu}{}_{,\mu\rho}\right). \tag{2.20}$$

2.2.2 Gauge Transformations

This derivation has looked at perturbations from Minkowski due to infinitesimal changes of spacetime. These perturbations can also be from changes in the coordinate system. This is because changes in the coordinate system will not effect the measurable quantities (gauge invariance). Before solving eq (2.20) for the metric perturbation, it is important to show how infintesimal coordinate changes (gauge transformations) will look.

The coordinate systems x^μ and $x^{\mu'}$ are separated by a element ξ^μ, defined by

$$x'^\mu = x^\mu + \xi^\mu(x^\nu), \tag{2.21}$$

the elements are small so that $|\xi^\mu{}_{,\nu}| << 1$. The derivatives of each coordinate system will be

$$x'^\rho{}_{,\mu} = \delta^\rho_\mu + \xi^\rho{}_{,\mu} \tag{2.22a}$$

$$\frac{\partial x^\rho}{\partial x'^\mu} = \delta^\rho_\mu - \xi^\rho{}_{,\mu} + \mathcal{O}((\partial\xi)^2), \tag{2.22b}$$

where eq 2.22b is found by expanding $\xi^\mu(x^\nu)$ about x'^ν. The metric relation between the two coordinate systems is given by

$$\begin{aligned} g'_{\mu\nu} &= \frac{\partial x^\rho}{\partial x'_\mu}\frac{\partial x^\lambda}{\partial x'_\nu} g_{\rho\lambda} \\ &= (\delta^\rho_\mu - \xi^\rho{}_{,\mu})(\delta^\lambda_\nu - \xi^\lambda{}_{,\nu}) g_{\rho\lambda} + \mathcal{O}((\partial\xi)^2) \\ &= g_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu} + \mathcal{O}((\partial\xi)^2)\ . \end{aligned} \tag{2.23}$$

By reintroducing the metric perturbation the relation becomes

$$h'_{\mu\nu} = h_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu} + \mathcal{O}((\partial\xi)^2) \tag{2.24}$$

or in terms of the trace reversed metric perturbation

$$\begin{aligned} \bar{h}'_{\mu\nu} &= h'_{\mu\nu} - \frac{1}{2}h'\eta_{\mu\nu} \\ &= \bar{h}_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu} + \eta_{\mu\nu}\xi^\rho{}_{,\rho} + \mathcal{O}((\partial\xi)^2)\ . \end{aligned} \tag{2.25}$$

This equation shows how the metric perturbation varies under a gauge transformation. Gauge conditions can be imposed on the metric tensor to simplify the work needed to solve Einstein's equations. Four conditions can be imposed on the metric by the degrees of freedom the coordinate system has. These conditions must be such that there exists some $\xi^\mu(x^\nu)$ that satisfies eq (2.25).

2.2.3 Harmonic Gauge

The most useful gauge condition to solve the linearised Einstein equations in the pure gravity sector (without cc) is the Harmonic Gauge

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0. \tag{2.26}$$

It is easy to prove there exists some ξ^μ for this condition by looking at the divergence of eq (2.25);

$$\begin{aligned} \bar{h}'_{\mu\nu}{}^{,\nu} &= \bar{h}_{\mu\nu}{}^{,\nu} - \xi_{\nu,\mu}{}^\nu - \xi_{\mu,\nu}{}^\nu + \eta_{\mu\nu}\xi^\rho{}_{,\rho}{}^\nu + \mathcal{O}((\partial\xi)^2) \\ &= \bar{h}_{\mu\nu}{}^{,\nu} - \Box\xi^\mu + \mathcal{O}((\partial\xi)^2)\ . \end{aligned} \tag{2.27}$$

With the harmonic gauge condition (eq 2.26) the transformed (primed) system must be by a perturbed by some ξ^μ that can satisfy

$$\bar{h}_{\mu\nu}{}^{,\nu} = \Box \xi^\nu, \tag{2.28}$$

to first order. So for any function $\bar{h}_{\mu\nu}$, there will always exist a perturbation ξ^μ that takes the system into the harmonic gauge. With the system in the harmonic gauge, the linearised Einstein tensor (eq 2.20) is simplified to

$$G_{\mu\nu} = -\frac{1}{2}\Box \bar{h}_{\mu\nu} + \mathcal{O}(h^2). \tag{2.29}$$

It is important to note that these coordinates are not unique. There are many functions ξ^μ that bring the system into the harmonic gauge. A further condition can be applied from eq (2.21) that functions ξ^μ must satisfy

$$\Box \xi^\mu = 0. \tag{2.30}$$

2.2.4 Generalised Solution

Given eq (2.29), the equations of motion in the harmonic gauge will be

$$8\pi G T_{\mu\nu} = -\frac{1}{2}\left(\delta^\rho_\mu \delta^\lambda_\nu \Box - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\lambda}\Box\right) h_{\rho\lambda}. \tag{2.31}$$

It is interesting to study the propagator as it provides information on the probability amplitude of the particle and whether its renormalisablility. The propagator will be defined by the solution to

$$-\frac{1}{2}\left(\delta^\rho_\mu \delta^\lambda_\nu \Box - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\lambda}\Box\right) D_{\rho\lambda}{}^{\sigma\gamma}\left(x; x'\right) = \frac{1}{2}\left(\delta^\sigma_\mu \delta^\gamma_\nu + \delta^\gamma_\nu \delta^\sigma_\mu\right) \delta^{(4)}\left(x - x'\right). \tag{2.32}$$

It is possible to split the propagator up into a scalar part and a numerical part that contains the tensor properties $D_{\rho\lambda}{}^{\sigma\gamma} = S_{\rho\lambda}{}^{\sigma\gamma} D_S(x - x')$, where the parts are defined by

$$\begin{aligned} S_{\rho\lambda}{}^{\sigma\gamma} &= \frac{1}{2}\left(\delta^\sigma_\rho \delta^\gamma_\lambda + \delta^\sigma_\lambda \delta^\gamma_\rho - \eta_{\rho\lambda}\eta^{\sigma\gamma}\right) \\ \Box D_S(x - x') &= -2\delta^{(4)}\left(x - x'\right). \end{aligned} \tag{2.33}$$

To isolate the field in eq (2.31), take the trace and substitute it back in to get

$$\Box h_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T\right) \tag{2.34}$$

The field and energy-momentum tensor can be substituted for their Fourier transform in momentum space

$$-\int \frac{d^4p}{(2\pi)^4} e^{ip.x} p^2 \tilde{h}_{\mu\nu}(p) = -8\pi G \int \frac{d^4p}{(2\pi)^4} e^{ip.x}\left(\tilde{T}_{\mu\nu}(p) - \frac{1}{2}\eta_{\mu\nu}\tilde{T}(p)\right) \tag{2.35}$$

This will gives the solution in momentum space

$$\tilde{h}_{\mu\nu}(p) = \frac{8\pi G}{p^2}\left(\tilde{T}_{\mu\nu}(p) - \frac{1}{2}\eta_{\mu\nu}\tilde{T}(p)\right) \tag{2.36}$$

By taking the Fourier transform again, the solution is given by

$$h_{\mu\nu}(x) = 8\pi G \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip.x}}{p^2} \left(\tilde{T}_{\mu\nu}(p) - \frac{1}{2}\eta_{\mu\nu}\tilde{T}(p) \right) \tag{2.37}$$

With the general solution for $h_{\mu\nu}$, a defined source $T_{\mu\nu}$ is now required for integration to find an exact solution.

It will be useful for later sections on Massive Gravity to look at a static, spherically symmetric source of mass M so that the energy-momentum is

$$T_{\mu\nu}(\mathbf{x}') = M\delta^0_\mu\delta^0_\nu\delta^{(3)}(\mathbf{x}'), \qquad \tilde{T}^{\mu\nu}(p) = 2\pi M\delta^\mu_0\delta^\nu_0\delta(p_0), \tag{2.38}$$

with trace

$$\tilde{T}(p) = -2\pi M\delta(p_0), \tag{2.39}$$

with this source and using complex analysis(Cauchy's theorem), the exact solution for the field is given

$$h_{\mu\nu}(x) = \begin{cases} h_{00}(x) = 8\pi GM \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}.\mathbf{x}}}{\mathbf{p}^2} = \frac{2GM}{r} \\ h_{0i}(x) = 0 \\ h_{ij}(x) = 8\pi GM \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}.\mathbf{x}}}{\mathbf{p}^2}\delta_{ij} = \frac{2GM}{r}\delta_{ij} \end{cases} \tag{2.40}$$

by substituting the conventional Newtonian potential, $\Phi(r) = -GM/r$ the equations are simply given by

$$\begin{aligned} h_{00}(r) &= -2\Phi(r) \\ h_{ij}(r) &= -2\Phi(r)\delta_{ij}\ . \end{aligned} \tag{2.41}$$

2.2.5 Λ Gauge

In the case of a non vanishing cosmological constant, a better choice of gauge would be one such that

$$\bar{h}^{\mu\nu}{}_{,\mu} = \Lambda x^\nu. \tag{2.42}$$

In this case using eq (2.27), there must exist a ξ^μ such that

$$\bar{h}_{\mu\nu}{}^{,\nu} - \Lambda x^\mu = \Box\xi^\mu, \tag{2.43}$$

to first order. The solution roughly looks like

$$\xi^\mu = \int dx^\nu \left(\bar{h}_{\mu\nu} - \frac{1}{2}\Lambda x^\mu x_\nu \right), \tag{2.44}$$

which is easy to see that there will always exist some ξ^μ that can move into the Λ gauge. In this gauge, the linearised Einstein tensor (eq 2.20) becomes

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} - \Lambda\eta_{\mu\nu} + \mathcal{O}(h^2). \tag{2.45}$$

In this case, the further condition that can be applied without leaving this gauge will be

$$\Box\xi^\mu = \Lambda\xi^\mu. \tag{2.46}$$

2.2.6 Degrees of Freedom

In order to build up to massive gravity, it is useful to look at linearised GR. The previous sections have linearised the Einstein equations by expanding them in terms of the perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

While this is the easiest way to arrive at the linearised equations of motion it is more beneficial to start from the action. This is because the Lagrangian can give information on the degrees of freedom and so the type of particle that carries GR. Starting with Einstein-Hilbert action eq (1.1), $\mathcal{O}(h)$ terms in the R part of the action can be removed as surface terms. The eq (2.16) provides R and the metric determinant is given by

$$\sqrt{-g} = \sqrt{-\eta}\left(1 + \frac{h}{2} + \frac{h^2}{8} - \frac{h_{\mu\nu}h^{\mu\nu}}{4}\right) + \mathcal{O}(h^3) \tag{2.47}$$

which is simplified by $\sqrt{-\eta} = 1$ for Minkowski space. By preemptively removing the below second order terms, this will give an action of the form

$$\begin{aligned} S = \frac{1}{16\pi G}\int d^4x \Big[& -\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} - \partial_\mu h^{\mu\nu}\partial_\nu h \\ & + \frac{1}{2}\partial_\rho h \partial^\rho h - \Lambda\left(\frac{h^2}{2} - h_{\mu\nu}h^{\mu\nu}\right) + \mathcal{O}(h^3)\Big] + S_M[g_{\mu\nu}, \Psi] \ . \end{aligned} \tag{2.48}$$

This can then be varied to obtain the same linearised equations of motion as derived from above.

In this case the perturbations have been from a flat (Minkowski space) background $\eta_{\mu\nu}$. The field equations of the background are vacuum which sets $T_{\mu\nu} = 0$ this means that the S_M term can be removed. The vacuum field equations

$$\bar{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{R} + \Lambda\eta_{\mu\nu} = 0, \tag{2.49}$$

can be traced to obtain equations

$$\Lambda = \frac{\bar{R}}{4}, \qquad \bar{R}_{\mu\nu} = \frac{\bar{R}}{4}g^{\mu\nu}. \tag{2.50}$$

Now that the field is moved to Minkowski space it will be invariant under Lorentz transformations. This means that it can be looked at in terms of QFT and classified as a particle with properties like mass, and spin. It must be massless as there is no mass term in the action and a boson as only long range forces can be described by them. This means that the particle will have a spin of $s = 0, 1, 2, 3$, etc. It is possible to obtain the spin from a Lagrangian. This can be done by counting the degrees of freedom.

It is easier to take the Hamiltonian approach when counting degrees of freedom (dof) as it allows us to identify constraints. The following discussion is taken without extensive details on the analysis of gauge theories and their constraints. The reference [33] has a more detailed look on gauge theory analysis.

The switch is made via a Legendre transform

$$\mathcal{H} = \dot{h}_{\mu\nu}\frac{\partial\mathcal{L}}{\partial\dot{h}_{\mu\nu}} - \mathcal{L}. \tag{2.51}$$

The next step would be to work out the canonical momenta $\pi^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{\mu\nu}}$. But it is important to note that two Lagrangians that may differ by a surface term, giving different canonical momenta can have the same equations of motion. Thus by performing an integration by parts on the time derivatives will change the canonical momenta. The best course of action would be to constrain this freedom by removing the time derivatives of h_{00} and h_{0i} [34]. This just leaves a Legendre transform in spatial components only

$$\mathcal{H} = \dot{h}_{ij}\frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} - \mathcal{L}. \tag{2.52}$$

For the purpose of counting the degrees of freedom the cosmological constant will not have any effect in counting, so it will be faster to work in the pure gravity sector, setting $\Lambda = 0$. This will leave a conjugate momenta of the form

$$\pi_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - \partial_{(i}h_{j)0s} + 2\partial_k h_{0k}\delta_{ij} \tag{2.53}$$

by taking the π_{kk} components it can be rewritten as

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{2}\pi_{kk}\delta_{ij} + \partial_{(i}h_{j)0} \tag{2.54}$$

with this, and using partial integration on the spatial derivatives, the Lagrangian can then be rewritten in terms of canonical momenta [34] as

$$\mathcal{L} = \dot{h}_{ij}\pi_{ij} - \mathcal{H} + 2h_{0i}\partial_j\pi_{ij} + h_{00}(\partial_i\partial_i h_{jj} - \partial_i\partial_j h_{ij}), \tag{2.55}$$

with the equation for the Hamiltonian shown to be

$$\mathcal{H} = \frac{1}{2}\pi_{ij}^2 - \frac{1}{4}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j h_{kk} - \frac{1}{2}\partial_i h_{jj}\partial i h_{kk}. \tag{2.56}$$

By looking at the reverse of eq (2.51) and the Lagrangian of eq (2.55), the $h_{0\mu}$ fields can be thought of as Lagrange multipliers, giving the four constraints

$$\chi_i = \partial_j\pi_{ij} = 0, \qquad \chi_4 = \partial_i\partial_i h_{jj} - \partial_i\partial_j h_{ij} = 0 \tag{2.57}$$

The Poisson bracket for any combination of $a, b = 1, 2, 3, 4$ vanishes

$$\{\chi_a(t,\mathbf{x}), \chi_b(t,\mathbf{y})\} = \int d^3\mathbf{x}' \left\{ \frac{\partial \chi_a(t,\mathbf{x})}{\partial h_{ij}(\mathbf{x}')}\frac{\partial \chi_b(t,\mathbf{y})}{\partial \pi_{ij}(\mathbf{x}')} - \frac{\partial \chi_a(t,\mathbf{x})}{\partial \pi_{ij}(\mathbf{x}')}\frac{\partial \chi_b(t,\mathbf{y})}{\partial h_{ij}(\mathbf{x}')} \right\} = 0. \tag{2.58}$$

This is particularly true on a surface where $\chi_a = 0$ for all $a = 1, 2, 3, 4$. This is a constraint that holds true off-shell; a first class constraint. On top of this, the Poisson brackets with the Hamiltonian are

$$\{H(t), \chi_4(t,\mathbf{x})\} = \int d^3\mathbf{x}' \left\{ \frac{\partial H(t)}{\partial h_{ij}(\mathbf{x}')}\frac{\partial \chi_4(t,\mathbf{x})}{\partial \pi_{ij}(\mathbf{x}')} - \frac{\partial H(t)}{\partial \pi_{ij}(\mathbf{x}')}\frac{\partial \chi_4(t,\mathbf{y})}{\partial h_{ij}(\mathbf{x}')} \right\} = \partial_i\partial_j\pi_{ij}(t,\mathbf{x}), \tag{2.59}$$

and

$$\{H(t), \chi_i(t,\mathbf{x})\} = 0. \tag{2.60}$$

This means that on the surface where $\chi_a = 0$ for all $a = 1, 2, 3, 4$, these will vanish too. This means that the Hamiltonian is first class and along with the constraints, makes this a first class gauge system. The four conditions above provide the gauge invariances of the system.

The degrees of freedom can now be counted with these gauge conditions. The h_{ij} and π_{ij} are symmetric 3×3 matrices in spatial dimensions, providing 12 degrees of freedom. The four constraints of eq (2.57) reduce the space which the fields span to an eight-dimensional surface. The four gauge invariances generated by the constraints reduce the system further to a four dimensional hypersurface. This leaves four degrees of freedom in the phase space and so two physical degrees of freedom i.e two polarisations along with their conjugate momenta.

2.2.7 Non-Linear Interactions

In section 2.2.2, gauge transformations were introduced as a way to simplify the linearised Einstein equations through the Harmonic gauge in section 2.2.3 so that it was solvable, as shown in section 2.2.4. This can be done as it does not change the underlying physics of the system; that is to say it is gauge invariant. It can be shown that the linearised action of eq (2.48) is invariant under these gauge transformations. In General Relativity this is known as the general covariance of the theory; that the coordinates are a measuring device and should not affect the fundamental physics of the system.

Alternatively, one can start by looking for an action of a symmetric field $h_{\mu\nu}$ on a flat background field theory the most general quadratic theory. By requiring locality, Lorentz invariance, and gauge invariance, the most general quadratic action that contains up to two derivatives can only be eq (2.48). These properties not only provide linearised GR but can fix non-linear. This can be seen by the higher order $\mathcal{O}(h^2)$ terms in the field equations. The importance of the requirement for gauge invariance to produce a non-linear theory becomes apparent when considering massive gravity. In that case there will be no invariance and so a non-linear theory cannot be constructed from it.

2.3 Lovelock and Birkhoff theorems

There are two important theorems that restrict the structure of General Relativity. These theories help describe how gravity interacts in different systems. These theorems underpin a lot of the acquired intuition on how gravity should function in different environments, and what the resulting phenomenology should be. In alternative theories of gravity, however, the theorems of General Relativity often fail, allowing new behaviours that would otherwise be impossible. The next sections consider two important theorems of General Relativity.

2.3.1 Lovelock's Theorem

The Einstein-Hilbert action is the most simple form of action constructed using $g_{\mu\nu}$. The most general four dimensional Lagrangian that contains only the metric tensor and its derivatives is

$$\mathcal{L} = \alpha\sqrt{-g}R - 2\lambda\sqrt{-g} + \beta\epsilon^{\mu\nu\rho\lambda}R^{\alpha\beta}_{\mu\nu}R_{\alpha\beta\rho\lambda} + \gamma\sqrt{-g}\left(R^2 - 4R^{\mu}_{\nu}R^{\nu}_{\mu} + R^{\mu\nu}_{\rho\lambda}R^{\rho\lambda}\right), \tag{2.61}$$

where $\alpha, \beta, \gamma, \lambda$ are constants [35]. The equation of motion of this can be found using the second order Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} - \frac{d}{dx^\rho}\left(\frac{\partial\mathcal{L}}{\partial g_{\mu\nu,\rho}}\right) + \frac{d^2}{dx^\rho dx^\lambda}\left(\frac{\partial\mathcal{L}}{\partial g_{\mu\nu,\rho\lambda}}\right) = 0. \tag{2.62}$$

The third and fourth terms of eq (2.62) vanish in the Euler-Lagrange equation. The resulting equation of motion is

$$\alpha \left[R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right] + \lambda g^{\mu\nu} = 0, \tag{2.63}$$

which is of the exact same form as the solution to the Einstein-Hilbert action of equation (2.10) without the added energy term.

Lovelock's theorem states that the Einstein Field equations are the only possible solution to an action which contains up to second derivatives of the four-dimensional spacetime metric $g^{\mu\nu}$. This theorem sets a good framework for alternate theories of gravity.

2.3.2 Birkhoff's Theorem

Any spherically symmetric solution of the vacuum field equations will be static and asymptotically flat. This theorem is difficult to apply to observations as complete spherical symmetry and vacuums are theoretical. It can provide a good approximation for the behaviour of gravity around large masses (Schwarzschild black hole). The asymptotic flatness means that in the weak-field approximation of GR, the field can be treated as a perturbation from Minkowski space. Without this theorem there is less justification to assume the weak-field approximation as a perturbation around Minkowski space. Beyond GR, this theorem is not always true; even far from the source the background curvature might not be asymptotically flat.

2.4 Tests of Gravity

In this section we will look at important observational tests that can be performed on theories of gravity. For a theory of gravity to be considered, it must be able to work within the constraints they set.

The first test is predicting the effect from gravitational lensing. The light from distant stars will have a measurable deflection due to the mass of the Sun. A competitive gravitational theory should be able to accurately predict this deflection. In general relativity, this deflection angle α due to the suns mass M will be given by

$$\alpha = \frac{2M}{b}(1 + \cos\phi), \tag{2.64}$$

where b is the impact parameter and ϕ is the angle measured between the path of the photon and the sun to the observer. In GR the angle predicted for a photon trajectory that just touches the edge of the sun is $\alpha \simeq 1.75''$. The most rigorous observational tests on the angle to date [36] give values

$$\alpha = (0.99992 \pm 0.00023) \times 1.75''. \tag{2.65}$$

Another prediction test is on the Shapiro time-delay effect. It instead considers the deflection time or time dilation a photon would experience due to the suns gravitational field. In GR this time will be given by

$$\Delta t = 4M \ln \left(\frac{4 r_o r_p}{d^2} \right), \tag{2.66}$$

where r_O, r_p are the distances between observer and photon to the gravitating mass M. So far, the best experiment measuring this [37] measured the effect to agree with GR by

$$\Delta t = (1.00001 \pm 0.00001) \Delta t_{GR}. \tag{2.67}$$

where δt_{GR} is the time delay predicted by GR using the equation above. The other test which was touched on earlier looks at the perihelion of Mercury. In Newtonian gravity of an orbiting satellite will stay fixed unless other large bodies are introduced into the system to perturb the orbit. This is what lead to the prediction of Vulcan from observations of the peculiar orbit of Mercury before GR. In relativistic theories, the potential contributions from other planets allows for this effect as they do not follow the $1/r^2$ relationship due to perihelion precession. In GR this precession will be given by

$$\Delta\omega = \frac{6\pi M}{p}, \tag{2.68}$$

where M is the total mass of the gravitating mass with the orbiting mass and p is the semi-latis rectum of orbit. For the perihelion of Mercury this value is predicted to be $\Delta\omega \simeq 42.98''$. Observational experiments [38] put this value around $\Delta\omega = 42.969 \pm 0.052$.

3 Introduction to $F(R)$

There are several ways to modify general relativity. This can by done choosing at least one of the following modifications: 1) By adding more fields including or instead of the metric tensor (as well as giving it a mass), 2) Higher than second order derivatives of the metric, 3) Higher than four dimensions, 4) Abandon Lorentz invariance, 5) Abandon locality.

In extended theories of gravity, additional fields can be added to the spin-2 field of GR. This can include scalar, vector and tensor fields. For the theory to be valid, it must return to general relativity at scales where experiments confirm GR at high confidence. This generally involves choosing additional fields that weakly couple to the spin-2 field of GR. There are other methods like the chameleon mechanism that can achieve this [39]. In section 2.3.1 Lovelock's theorem states that the Einstein Field equations are the only solution for an action with up to second order derivatives of a four-dimensional metric. An obvious modification would be to consider higher order derivatives. Another modification would be to consider higher dimensions as well [40]. The last two are more extreme measures that will not be considered in this review, see [35, 41] for more information on 4,5.

3.1 The $F(R)$ Equations

When working on areas of unknown physics, it is always important to start with a simple generalised model. One can explore the theory and is limitations by working on constraints. This will help give motivation for using modified gravity to remedy the issues with the current cosmological model.

$F(R)$ theories that provide a good entrance to modified theories of gravity, with a generalised structure to test on. It relaxes the condition that the Einstein-Hilbert action in eq (1.1) must be linear in Ricci scalar R. This means that the curvature term $(R - 2\Lambda)$ in the action will be some function $F(R)$ that will have the general form

$$F(R) = ... + c_{-2}R^{-2} + c_{-1}R^{-1} - 2\Lambda + R + c_2R^2 + ..., \tag{3.1}$$

where the c_i coefficients determine the particular model of $F(R)$ gravity. The same variation method applied in section 2.1 can be applied to determine the field equations. Starting with the action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}F(R) + S_M[g_{\mu\nu}, \Psi], \tag{3.2}$$

the variation with respect to the metric $g_{\mu\nu}$ will be

$$\delta S = \frac{1}{16\pi G}\int d^4x \left(\delta(\sqrt{-g})F(R) + \sqrt{-g}\delta F(R)\right) + \delta S_M. \tag{3.3}$$

The $\delta F(R)$ term can be expanded using eq (2.6)

$$\begin{aligned}\delta F(R) &= \delta(R)F'(R) \\ &= (R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu})F'(R)\,.\end{aligned} \tag{3.4}$$

With eq (2.4) and eq (2.7) the variation in eq (3.4) can be written as

$$\delta S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(-\frac{1}{2}F(R)g_{\mu\nu} + R_{\mu\nu}F'(R) + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu F'(R)\right)\delta g^{\mu\nu} + \delta S_M. \tag{3.5}$$

Minimising the action and using the identity for $T_{\mu\nu}$ from eq (2.8) yields the field equation

$$8\pi G T_{\mu\nu} = -\frac{1}{2}F(R)g_{\mu\nu} + R_{\mu\nu}F'(R) + g_{\mu\nu}\Box F'(R) - \nabla_\mu\nabla_\nu F'(R). \tag{3.6}$$

This can be remodelled in terms of the Einstein field tensor by substituting terms

$$8\pi G T_{\mu\nu} = -\frac{1}{2}F(R)g_{\mu\nu} + R_{\mu\nu}F'(R) - \frac{1}{2}F'(R)Rg_{\mu\nu} + \frac{1}{2}F'(R)Rg_{\mu\nu} + g_{\mu\nu}\Box F'(R) - \nabla_\mu\nabla_\nu F'(R), \tag{3.7}$$

rearranging in terms of the field tensor

$$G_{\mu\nu} = \frac{1}{F'(R)}\left[g_{\mu\nu}\frac{F(R) - F'(R)R}{2} - g_{\mu\nu}\Box F'(R) + \nabla_\mu\nabla_\nu F'(R) + 8\pi G T_{\mu\nu}\right] \tag{3.8}$$

This can be considered a generalisation of the Einstein field equations for curvature R. In the limit that $F(R) \to R$ the theory returns to General Relativity. The central terms are fourth order in the metric.

3.2 Mapping $F(R)$ to Scalar Tensor Theories

Scalar fields are interesting for cosmology because they could be used to explain the interactions described by dark energy. Instead of adding a Cosmological Constant to the action, a scalar field can be added to mediate gravitational interaction along with the metric.

$F(R)$ gravity is a simple modification to GR that generalises the curvature of the metric tensor beyond first order. On initial inspection this extension does not look like any extra fields are added. However, this theory can be related to a scalar–tensor theory known as Brans-Dicke gravity, where the metric tensor is accompanied by a scalar field.

The $F(R)$ action of eq (3.2) can be shown to be dynamically equivalent to the equation

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[f(\chi) + f'(\chi)(R-\chi)\right] + S_M[g_{\mu\nu}, \Psi], \tag{3.9}$$

where χ is a newly introduced field yet to be determined. It is easy to see that the by variation with respect to this newly introduced field and minimising the action leads to the equation

$$f''(\chi)(R-\chi) = 0. \tag{3.10}$$

This means that for regions where $f''(\chi) \neq 0$, the fields will be the same $\chi = R$. This reduces the equation to the original $F(R)$ action. By a tactical redefinition of the field χ to $\psi = f'(\chi)$ and introducing a field potential defined by

$$V(\psi) = \chi(\psi)\psi - f(\chi(\psi)) \tag{3.11}$$

the action can be rewritten as

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, [\psi R - V(\psi)] + S_M[g_{\mu\nu}, \Psi]. \tag{3.12}$$

This action is the same form as that of a general scalar tensor theory

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[\psi R - \frac{\omega}{\psi} \partial_\mu \psi \partial^\mu \psi\right] + S_M[g_{\mu\nu}, \Psi]. \tag{3.13}$$

for a vanishing coupling constant $\omega = 0$ and a redefinition of ψ to absorb G. This type of action is also known as Brans-Dicke (BD) theory which is discussed in section 3.3.

Solar system measurements on the coupling constant in eq for BD theory constrain it to $\omega > 40,000$. Since $F(R)$ theory is related to Brans-Dicke for $\omega = 0$ it implies that the theory is incompatible with measurements. However, by tactical choices of $F(R)$ in the potential V it is possible to screen the field via the chameleon mechanism, avoiding the constraint on ω. This is because the effective mass is roughly the second derivative of the potential which depends on $F(R)$. Since it depends on the curvature scalar R, its mass can be large at small scales (solar system) which makes the mediated force short range; removing the issues with ω. On large scales (cosmological) the mass can be small, making the force longer ranged to explain cosmological interactions.

The general scalar tensor theory action of eq (3.13) is in the Jordan frame where a scalar field multiplies the Ricci scalar and the scalar field does not interact with matter fields. By using a conformal transformation of the metric and a field redefinition

$$g_{\mu\nu} = A^2(\phi)\bar{g}_{\mu\nu}, \ \log\psi = 2\sqrt{\frac{8\pi G}{3 + 2\omega}}\phi, \tag{3.14}$$

the equation can be moved to the Einstein frame

$$S = \int d^4x \sqrt{-\bar{g}} \left[\frac{\bar{R}}{16\pi G} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \bar{V}(\phi)\right] + S_M[A^2(\phi)\bar{g}_{\mu\nu}, \Psi], \tag{3.15}$$

where $A^2(\phi) = \exp\left(2\sqrt{\frac{8\pi G}{3+2\omega}}\phi\right)$ [42]. In this frame it is easy to see that in the absence of matter fields the theory is related conformally to GR with the presence of a scalar field in a potential.

3.3 Brans-Dicke Gravity

Brans-Dicke gravity is an example of a scalar tensor theory of gravity [43]. A scalar tensor theory of gravity is a theory in which, in the gravity sector, the force of gravity is mediated not only by a tensor field, like in GR, but an extra scalar field also contributes to the gravitational dynamics. We can consider a non-relativistic fluid $T^0_0 = -\rho$ in the action of eq (3.13). In the non-relativistic regime we can consider the weak field approximation with perturbations $\psi = \psi_0 + \varphi$ in which the metric takes the form

$$ds^2 = -(1 + 2\Psi)\, dt^2 + (1 - 2\Phi)\, \delta_{ij} dx_i dx_j, \tag{3.16}$$

where $\nabla^2 = \delta_{ij}\partial_i\partial_j$ and we have assumed the potentials time independence; $\Psi = \Psi(x_i)$ and $\Phi = \Phi(x_i)$. The perturbation will satisfy a Poisson equation of the form

$$(3 + 2\omega)\nabla^2\varphi = -8\pi G\rho. \tag{3.17}$$

The potentials are also related by

$$\begin{aligned} \nabla^2 (2\Psi + \varphi) &= 8\pi G\rho \\ \Psi - \Phi &= \varphi \ . \end{aligned} \tag{3.18}$$

In this form it is easy to see how the introduced scalar field ψ modifies gravity in the weak field limit. The Newtonian potential in the time and spatial components now differ by φ. The potentials can be written individually as Poisson equations of the form

$$\nabla^2\Phi = 4\pi G\mu\rho, \ \ \Psi = \eta\Phi, \tag{3.19}$$

where constants are given by

$$\mu = \frac{4 + 2\omega}{3 + 2\omega}, \ \ \eta = \frac{1 + \omega}{2 + \omega}. \tag{3.20}$$

We can see from this equation that for large ω that $\Psi = \Phi$ and we recover GR. So it is already obvious we will require a large ω to agree with our solar system measurements. From our tests on GR [44], we have that $|\eta - 1| = (2.1 \pm 2.3)^{-5}$ which constrains $\omega > 40,000$. Once this constraint is placed on our model it successfully agrees with GR on solar system measurements. The only issue is that it barely allows us to modify away from GR on other scales.

4 History of Massive Gravity

This section will detail dRGT massive gravity; the unique, non-linear theory of massive gravity models. The Cascading DGP model is another interesting method that uses brane cosmology. This model is not discussed in this paper but has plenty of resources that detail the method [45–47].

As detailed in the introduction, the current model of cosmology has difficulties explaining the late time acceleration of the universe. The approximate amount of matter observed to be in our universe roughly coincides with a flat curvature. The consequence of this would be a universe that expands forever at a slowing rate as gravity tries to pull it back. However, from observations the universe has reached a crossing point where the deceleration of universal expansion has been reversed and begun to instead accelerate. The ΛCDM model proposes an undetected energy with unusual properties that dominates the universe. However, there are more satisfying but challenging alternatives possible.

The idea behind massive gravity is to modify gravity by weakening the field strength at universal scales to explain the acceleration. This would require the theory to have a field potential like $1/r$ at solar system scales in order to maintain GR and it's observational successes at that level. As it reaches some crossover point, the strength would experience an additional decay function. This might sound radical but is already a familiar presence in particle physics.

In 1935 the only understood interactions of the atomic model were the electromagnetic forces. At the atomic scale EM forces between protons should pull repel the atom apart.

To balance the EM forces, an attractive force was required. By using Heisenberg's ideas on a short range force with Fermi's of particle exchange, Hideki Yukawa was able to provide a force that would preserve atomic structure.

By introducing a decay of the form e^{-mr} to a $1/r$ term, the potential strength decreases even more rapidly with distance. Just like the photon in the EM force interaction, this force also has an exchange particle. In this case the particle has a mass which relates to the strength of the interaction; the lighter the mass, the weaker the interaction. This is also a good illustration of how massive particles govern short ranged forces (Strong Nuclear and Weak Nuclear) and massless particles long (EM and GR Gravity. The carrier of this attractive nuclear force interaction was given the name meson.

The principle behind the Yukawa potential and its uses for gravity is that the strength of interactions are linked to the mass of the particle. The mass essentially fixes the range the interaction mediates. For a small mass the potential will look like $1/r$ for a large distance $r \lesssim 1/m$. As it passes the threshold $r_c = 1/m$ (Compton radius), the decay term kicks in, rapidly weakening the $1/r$ term. This mechanic could be used to solve the acceleration problem by setting this crossover point to the order of the Hubble radius today $r_c \sim H_0^{-1}$.

To see how this potential is acquired think of the Klein-Gordon equation; a relativistic wave equation for a scalar field along with a source ρ.

$$(\Box - m^2)\phi = \rho, \tag{4.1}$$

By choosing a source of the form $\rho = G\delta(\mathbf{x})$ (since objects in the solar system generally don't change mass with time) where G is some constant and search for a static solution $\phi = \phi(\mathbf{x})$, the equation becomes

$$(\nabla^2 - m^2)\phi = G\delta(\mathbf{x}). \tag{4.2}$$

The field can be substituted for its Fourier transform and inverse

$$\begin{aligned} \phi(\mathbf{x}) &= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}.\mathbf{x}} \tilde{\phi}(\mathbf{k}) \\ \tilde{\phi}(\mathbf{k}) &= \int \frac{d^3x}{(2\pi)^3} e^{-i\mathbf{k}.\mathbf{x}} \phi(\mathbf{x}) \,. \end{aligned} \tag{4.3}$$

This admits the solution

$$\tilde{\phi}(\mathbf{k}) = -\frac{G}{(2\pi)^3} \frac{1}{k^2 + m^2}. \tag{4.4}$$

By taking the Fourier transform and using contour analysis, the Yukawa potential is found

$$\phi(\mathbf{x}) = -\frac{G}{4\pi} \frac{e^{-mr}}{r}. \tag{4.5}$$

The aim of this section will be to look at the history of massive gravity and build up to one of the most successful massive gravity models known as dRGT from its main proponents; de Rham, Gabadadze and Tolley.

4.1 The Fierz-Pauli Action

Section 2 looked at the action of linearised General Relativity to see how it can be described as an interaction theory of a massless helicity-2 field on a flat (Minkowski) background. The modification that Massive Gravity makes to the action will be such to describe a massive spin-2 field on a flat background. This action was discovered in 1939 by Wolfgang Pauli and Markus Fierz [4].

So on a flat background, this particle described by field $h_{\mu\nu}$ have the action

$$S = \int d^4x \left[-\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\rho h \partial^\rho h - \frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right)\right]. \tag{4.6}$$

On first inspection it is easy to see it is a small extension to the linearised GR action as shown in eq (2.48), omitting the cc and mass terms to study pure gravity. This would make sense for a theory that should return to GR when $m = 0$. The most obvious feature is that this equation contains all possible combinations of $\mathcal{O}(h^2)$ with up to two derivatives. The Fierz-Pauli fine tuning fixes the mass terms to the same magnitude with -1 relative difference in their coefficients. The most important feature of this action is that it describes a spin-2 massive field only. Any changes to this would introduce new degrees of freedom (new particles) or remove the properties of the spin-2 massive particle from the theory.

As before in section 2.2.6, the time derivative time components ($\dot{h}_{00}$ and $\dot{h}_{0i}$) are removed by integration by parts. While the h_{00} component still appears linearly after this process, the h_{0i} now appears quadratically with the introduction of the mass terms.

To see the importance of this, consider a more generalised mass term of the form

$$\alpha h_{\mu\nu}h^{\mu\nu} + \beta h^2 = (\alpha+\beta)h_{00}^2 - 2\alpha h_{0i}^2 + \alpha h_{ij}^2 - 2\beta h_{00}h_{ii} + \beta h_{ii}^2. \tag{4.7}$$

To make sure that h_{00} continues to remain linear in the action it will require $a = -b$. If this was not true, there would be no constraint to remove a degree of freedom as shown in the next section. If $\beta = 0$, then h_{0i} would appear linearly like the GR case, and would provide at least three conditions. This would remove extra degrees of freedom that are required for a massive spin-2 particle (5 degrees).

4.1.1 Degrees of Freedom

The linearised Einstein-Hilbert action was shown to contain two physical degrees of freedom of a massless helicity-2 field in section 2.2.6. This section will show how this action contains 5 degrees of freedom of a massive spin-2 field. Since the mass term contains no time derivatives, the conjugate momenta will also be the same as eq (4.8). More explicitly

$$\pi_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - \partial_{(i}h_{j)0s} + 2\partial_k h_{0k}\delta_{ij} \tag{4.8}$$

which will give the same result when inverted

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{2}\pi_{kk}\delta_{ij} + \partial_{(i}h_{j)0}. \tag{4.9}$$

With the mass term the Lagrangian in terms of Hamiltonian and conjugate momenta will be

$$\mathcal{L} = \dot{h}_{ij}\pi_{ij} - \mathcal{H} + m^2 h_{0i}^2 + 2h_{0i}\partial_j\pi_{ij} + h_{00}(\partial_i\partial_i h_{jj} - \partial_i\partial_j h_{ij} - m^2 h_{ii}^2), \tag{4.10}$$

with the Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2}\pi_{ij}^2 - \frac{1}{4}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j h_{kk} - \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2(h_{ij}h_{ij} - h^2). \tag{4.11}$$

The Lagrangian still has h_{00} appear linearly and so it remains a Lagrange multiplier with the constraint

$$\chi = \partial_i\partial_i h_{jj} - \partial_i\partial_j h_{ij} - m^2 h_{ii} = 0 \tag{4.12}$$

In this case h_{0i} now appears quadratically and so can no longer be represented by Lagrange multipliers. It is still non-dynamical and so the quadratic term will lead to the algebraic equation of motion

$$h_{0i} = -\frac{1}{m^2}\partial_j\pi_{ij}. \tag{4.13}$$

It prompts an introduction of another Hamiltonian of the form

$$\mathcal{H}' = \mathcal{H} + \frac{1}{m^2}\partial_j\pi_{ij}\partial_k\pi_{ik}. \tag{4.14}$$

So that the h_{0i} term can be contained in the Hamiltonian, leaving an action of the form

$$S = \int d^4x \left[\dot{h}_{ij}\pi_{ij} - \mathcal{H}' + m^2 h_{0i}^2 + 2h_{0i}\partial_j\pi_{ij} + h_{00}(\partial_i\partial_i h_{jj} - \partial_i\partial_j h_{ij} - m^2 h_{ii}^2)\right]. \tag{4.15}$$

The changes lead to a Hamiltonian that is not first class as the constraint is no longer conserved. This can be seen by the Poisson bracket

$$\left\{H'(t), \chi(t,\mathbf{x})\right\} = \frac{1}{2}m^2\pi_{ii} + \partial_i\partial_j\pi_{ij}, \tag{4.16}$$

which does not vanish on the surface $\chi = 0$ as the condition in eq (4.12) cannot remove the term. This means a secondary constraint is required for the bracket to vanish

$$\chi' = \left\{H'(t), \chi(t,\mathbf{x})\right\} = 0. \tag{4.17}$$

The Poisson bracket between the two constraints no longer vanishes

$$\left\{\chi'(t,\mathbf{x}), \chi(t,\mathbf{y})\right\} = \frac{3}{2}m^4\delta(\mathbf{x}-\mathbf{y}), \tag{4.18}$$

and so no longer generate a gauge symmetry. This means that the four degrees of freedom that where removed by the four gauge symmetries in the GR case are now returned. So in total from the twelve degrees provided by h_{ij} and π_{ij}, two constraints removing two degrees, there are ten degrees of freedom of the phase space, and so five degrees of freedom of the field; as is required for a massive spin-2 particle.

4.1.2 Non-Linear Interactions

Unlike the linearised action of GR, the Fierz-Pauli action is not gauge invariant and so the field cannot transform under eq (2.24). This means that non-linear interactions cannot be constructed from requiring gauge invariance in the same way as that discussed in section 2.2.7.

The previous section details how modifications of the linear action would produce varying degrees of freedom of the theory. Modifying the action in any other way at the linear level to retain it's massive spin-2 property will produce another degree of freedom, that is known as a ghost [48]; additional field(s) employed to retain gauge invariance. This means that Fierz-Pauli is the only action free of any ghost for the theory. So to extend the theory to higher orders only Fierz-Pauli action can be considered.

4.2 Generalised Solution

This section will look at solutions to a fixed external symmetric source in the Fierz-Pauli action. The action will take the form

$$S = \int d^4x \Big[-\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\rho h\partial^\rho h - \frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) + 8\pi G h_{\mu\nu}T^{\mu\nu}\Big] . \tag{4.19}$$

The coupling strength ($8\pi G$) of the source is chosen in such a way to provide an equation of motion of the same form as GR. By varying the action to obtain the equation

$$8\pi G T_{\mu\nu} = \frac{1}{2}\left[-\Box h_{\mu\nu} + h^\rho{}_{\nu,\rho\mu} + h^\rho{}_{\mu,\rho\nu} - h_{,\mu\nu} + \eta_{\mu\nu}\Box h - \eta_{\mu\nu}h^{\rho\mu}{}_{,\mu\rho} + m^2\left(h_{\mu\nu} - \eta_{\mu\nu}h\right)\right] , \tag{4.20}$$

which is indeed of similar form to eq (2.18) with the addition of the mass term. The mass term means that the conservation equation $\partial^\mu T_{\mu\nu} = 0$ is no longer a condition. The conservation is associated with spacetime translational symmetry no longer applies. This means that gauge transformations cannot be automatically applied to simplify the equations of motion. The condition now takes the form

$$\frac{16\pi G}{m^2}\partial^\mu T_{\mu\nu} = (\partial^\mu h_{\mu\nu} - \partial_\nu h). \tag{4.21}$$

This can be used on eq (4.20) to get the equation

$$\Box h_{\mu\nu} - \partial_\mu\partial_\nu h - m^2\left(h_{\mu\nu} - \eta_{\mu\nu}h\right) = 16\pi G\left[-T_{\mu\nu} + \frac{1}{m^2}\left(\partial^\rho\partial_\mu T_{\nu\rho} + \partial^\rho\partial_\nu T_{\mu\rho} - \eta_{\mu\nu}\partial_\rho\partial_\lambda T^{\rho\lambda}\right)\right] , \tag{4.22}$$

and by taking the trace

$$h = -\frac{16\pi G}{3m^2}\left(T + \frac{2}{m^2}\partial_\rho\partial_\lambda T^{\rho\lambda}\right) . \tag{4.23}$$

This can be put back into eq (4.23) to remove the h term to give

$$\partial^\mu h_{\mu\nu} = -\frac{16\pi G}{m^2}\left[\partial^\mu T_{\mu\nu} - \frac{1}{3}\left(\partial_\nu T + \frac{2}{m^2}\partial_\nu\partial_\rho\partial_\lambda T^{\rho\lambda}\right)\right] . \tag{4.24}$$

Now both eq (4.23) and eq (4.24), can be used to rewrite equations of motion in terms of the energy-momentum tensor

$$(\Box - m^2)h_{\mu\nu} = -16\pi G\left[T_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu\right)T\right] + \frac{16\pi G}{m^2}\left[\partial^\rho\partial_\mu T_{\nu\rho} + \partial^\rho\partial_\nu T_{\mu\rho} - \frac{1}{3}\left(\eta_{\mu\nu} + \frac{2}{m^2}\partial_\mu\partial_\nu\right)\partial_\rho\partial_\lambda T^{\rho\lambda}\right] . \tag{4.25}$$

The equation of motion thus provides these three equations (4.23, 4.24, 4.25). However, they are not independent as by tracing eq (4.25) and some manipulation will imply the other two. This means that only this equation will be needed to find a solution for the field $h_{\mu\nu}$.

It is a little tricky proceeding from this point on with the general case that includes non-conserved sources. In the case of conserved sources $\partial^\mu T_{\mu\nu} = 0$ the equation will reduce to

$$(\Box - m^2)h_{\mu\nu} = -16\pi G\left[T_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu\right)T\right] . \tag{4.26}$$

The field and energy-momentum tensor can be substituted for their Fourier counterparts so that the equation becomes

$$-\int \frac{d^4p}{(2\pi)^4}(p^2+m^2)e^{ip.x}\tilde{h}_{\mu\nu}(p) = -16\pi G\int \frac{d^4p}{(2\pi)^4}\left[\tilde{T}_{\mu\nu}(p) - \frac{1}{3}\left(\eta_{\mu\nu} + \frac{1}{m^2}p_\mu p_\nu\right)\tilde{T}(p)\right], \tag{4.27}$$

which simplifies to the solution of the field in momentum space

$$\tilde{h}_{\mu\nu}(p) = \frac{16\pi G}{p^2+m^2}\left[\tilde{T}_{\mu\nu}(p) - \frac{1}{3}\left(\eta_{\mu\nu} + \frac{1}{m^2}p_\mu p_\nu\right)\tilde{T}(p)\right]. \tag{4.28}$$

Taking the Fourier transform again yields the solution in position space

$$h_{\mu\nu}(x) = 16\pi G\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+m^2}e^{ip.x}\left[\tilde{T}_{\mu\nu}(p) - \frac{1}{3}\left(\eta_{\mu\nu} + \frac{1}{m^2}p_\mu p_\nu\right)\tilde{T}(p)\right]. \tag{4.29}$$

Now with the general solution for conserved sources $T_{\mu\nu}$ it is useful to look at explicit solutions for a particular source. A simple and obvious choice would be a point source of the form

$$T_{\mu\nu}(\mathbf{x}') = M\delta^0_\mu\delta^0_\nu\delta^{(3)}(\mathbf{x}'), \qquad \tilde{T}^{\mu\nu}(p) = 2\pi M\delta^\mu_0\delta^\nu_0\delta(p_0), \qquad \tilde{T}(p) = -2\pi M\delta(p_0), \tag{4.30}$$

so that the general solution is reduced to

$$h_{\mu\nu}(x) = 16\pi GM\int \frac{d^3p}{(2\pi)^3}\int dp_0\frac{e^{-i(p_0.x_0-\mathbf{p}.\mathbf{x})}}{-p_0^2+\mathbf{p}^2+m^2}\left[\delta^0_\mu\delta^0_\nu + \frac{1}{3}\left(\eta_{\mu\nu} + \frac{1}{m^2}p_\mu p_\nu\right)\right]\delta(p_0). \tag{4.31}$$

The next step involves contour integration using Cauchy's integral theorem. Working this out, the exact solution for the field will be

$$h_{\mu\nu}(x) = \begin{cases} h_{00}(x) = \dfrac{32\pi GM}{3}\displaystyle\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{i\mathbf{p}.\mathbf{x}}}{\mathbf{p}^2+m^2} = \frac{8GM}{3r}e^{-mr} \\ h_{0i}(x) = 0 \\ h_{ij}(x) = \dfrac{16\pi GM}{3}\displaystyle\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{i\mathbf{p}.\mathbf{x}}}{\mathbf{p}^2+m^2}\left(\delta_{ij} + \frac{p_ip_j}{m^2}\right) \\ \qquad = \dfrac{4GM}{3r}e^{-mr}\left[\dfrac{1+mr+m^2r^2}{m^2r^2}\delta_{ij} - \dfrac{1}{m^2r^4}(3+3mr+m^2r^2)x_ix_j\right]. \end{cases} \tag{4.32}$$

4.3 The vDVZ Discontinuity

As already discussed, it is important that this new theory will comply with GR in the $m \to 0$ limit. This can now be tested using the solutions of the field in a point source for GR in eq (2.40) and for the Fierz-Pauli action eq (4.33).

It is useful to compare the two using the gravitational lensing that a test particle will experience in the presence of the field $h_{\mu\nu}$. If the field components can be represented as

$$h_{\mu\nu}(r) = \begin{cases} h_{00}(r) = -2\phi(r) \\ h_{0i}(r) = 0 \\ h_{ij}(r) = -2\psi(r)\delta_{ij} \end{cases} \tag{4.33}$$

for some potentials $\phi(r)$ and $\psi(r)$ then the potential time component potential is given by the Newtonian potential $\phi(r) = \Phi(r)$. In the PPN formalism $\gamma(r)$ represents how much space curvature is produced by a unit rest mass [49].

If the field components can be related through this parameter like

$$\gamma(r) = \frac{h_{ij}(r)|_{i=j}}{h_{00}(r)} = \frac{\phi(r)}{\psi(r)}, \tag{4.34}$$

and the time component is of the form $\phi(r) = -k/r$ then the lensing angle α is given by the equation

$$\alpha = \frac{2k}{b}(1+\gamma), \tag{4.35}$$

for impact parameter b.

First, consider the solutions of GR and their relation to the Newtonian potential in eq (2.41). Since the time and spatial components are the same, $\gamma = 1$ for GR. This means lensing will be given by

$$\alpha = \frac{4GM}{b}. \tag{4.36}$$

Now consider the Fierz-Pauli solution of eq (4.33). This can be reduced for the task of working out the lensing. While the Fierz-Pauli action is not gauge invariant, the source energy-momentum tensor has been chosen in such a way to be conserved. This means that a gauge transformation on the solution; eq (4.33) will be invariant. The $\frac{1}{m^2}\rho_\mu\rho_\nu$ term in the general solution of eq (4.29) will not produce any observable changes to a test particle with a conserved energy-momentum. It can therefore be removed in this case. This leaves a solution of the form

$$h_{\mu\nu}(x) = \begin{cases} h_{00}(x) = \dfrac{8GM}{3r}e^{-mr} \\ h_{0i}(x) = 0 \\ h_{ij}(x) = \dfrac{4GM}{3r}e^{-mr}\delta_{ij}, \end{cases} \tag{4.37}$$

and in terms of the potentials

$$\Phi(r) = \phi(r) = -\frac{4GM}{3r}e^{-mr}, \qquad \psi = -\frac{2GM}{3r}e^{-mr}. \tag{4.38}$$

This means that in this case, $\gamma = 1/2$ and so the angle will also be given by

$$\alpha = \frac{4GM}{b}. \tag{4.39}$$

So the lensing angle will be the same in massive gravity as with GR. The problem with this is that the Newtonian potential $\Phi(r)$ is twice the size in the limit $m \to 0$. By rescaling $G \to 3/4G$ this can be remedied but there will now be a 25% error in the lensing angle.

This is unusual as there is no discontinuity present in the action as taking $m \to 0$ returns to GR. But the issue manifests itself in the the physical predictions from the solutions. The mass is just a parameter and should not cause any conflict in the underlying physics between the action and the solution. This issue was identified by both van Dam and Veltman [50], and Zakharov [51] and so named the vDVZ discontinuity.

This reason this issue appears in the solutions is due to the degrees of freedom available at $m = 0$. Previous sections have shown that the action of GR has two degrees of freedom while Fierz-Pauli has five. While GR is a gauge invariant, it looses this property outside of $m = 0$ and so gains these degrees of freedom. This is the heart of the issue with this discontinuity. The way forward to remediate will be to construct a new theory that does display gauge-invariance while also maintaining the same physical results that the Fierz-Pauli solutions provided.

4.4 The Stückelburg Trick

The previous section has shown how the $m \to 0$ limit of the Fierz-Pauli theory does not match GR. This section will show how a new scalar field can be introduced to provide gauge symmetry for the action without directly changing the theory behind it.

4.4.1 Massive Photon Test Case

Instead of jumping into the Fierz-Pauli action, this section will take some time to display the trick on a simple case. Consider the case of a massive photon with field A_μ coupled to a source j^μ. In this case the source does not have to be conserved. The action will take the form

$$S = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu \right], \tag{4.40}$$

with the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.41}$$

The theory returns to classical electromagnetism for $m = 0$ which contains 2 degrees of freedom from the photons transverse and longitudinal polarisation states; representing a massless spin-1 particle. In this case the Lagrangian is invariant under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \lambda \tag{4.42}$$

where λ is an arbitrary scalar function of spacetime. As in the case for the graviton, the introduction of mass breaks the symmetry. This leaves a theory describing three degrees of freedom of a massive spin-1 particle. So once again the taking the limit $m \to 0$ is not continuous as a degree of freedom is lost in the process.

To fix this problem a trick is used [52]; a scalar field ϕ is introduced in the action so that it maintains it's dynamics while acquiring gauge invariance. The new field is added through the original via the transformation

$$A_\mu \to A_\mu + \partial_\mu \phi, \tag{4.43}$$

which aims to replicate the form of the gauge transform. This is so that the EM term $F_{\mu\nu} F^{\mu\nu}$ will be invariant under this transform (as this part is gauge invariant) but it will alter the other terms in the action. This new action takes the form

$$S = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 (A_\mu + \partial_\mu \phi)^2 + (A_\mu + \partial_\mu \phi) J^\mu \right], \tag{4.44}$$

By integrating by parts on the last term and removing the boundary term, the action is

$$S = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 (A_\mu + \partial_\mu \phi)^2 + (A_\mu + \phi \partial_\mu) J^\mu \right]. \tag{4.45}$$

Now if a gauge transformation of eq (4.42) is performed on this new action, if the field transforms as $\phi \to \phi - \lambda$ then the action will be invariant, and so the desired gauge symmetry has been achieved. Since this new field ϕ has been added simply as a trick, it can always be fixed as $\phi = 0$ to recover the original action.

This trick is a good display of the main concept of gauge theory. A gauge invariant system is one with redundancies of the physical description. Any gauge theory can be reduced by removing it's degrees of freedom from the gauge symmetry, thus losing the symmetry. And any non-gauge theory can have redundant fields added to induce a gauge symmetry, increasing it's degrees of freedom.

In the previous section the problem encountered was due to the theory chosen; the action contained no gauge symmetry but when reduced to $m \to 0$ it returned to GR with gauge symmetry. This trick can now be applied to the action so that the action remains smooth in the limit.

Before the limit can be taken in this example, ϕ must be normalised in the action by taking $\phi \to \frac{1}{m}\phi$. The normalised action will therefore be given by

$$S = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 \left(A_\mu + \frac{1}{m} \partial_\mu \phi \right)^2 + \left(A_\mu + \frac{1}{m} \phi \partial_\mu \right) J^\mu \right], \tag{4.46}$$

with symmetry under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \lambda, \qquad \phi \to \phi - m\lambda. \tag{4.47}$$

This equation now has an issue when attempting to take the limit $m \to 0$. The last term will cause a divergence and so this trick cannot work for a non-conserved source.

Now assuming the source is conserved by $\partial_\mu J^\mu = 0$, the last term can be removed. In the limit action will tend to

$$S = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + A_\mu J^\mu \right], \tag{4.48}$$

with gauge symmetry of the form

$$A_\mu \to A_\mu + \partial_\mu \lambda. \tag{4.49}$$

This new theory has comprised a massive vector field A_μ, with transverse and longitudinal degrees of freedom. The field couples to a scalar field ϕ with one degree of freedom. In the massless limit the degrees of freedom remain the same and the fields decouple from each other. The requirement that the degrees of freedom and gauge invariance remain unchanged in the limit is satisfied by this new theory.

4.4.2 Massive Gravity Case

The Stüekelberg trick can be used to improve the theory of massive gravity developed in eq (4.19). It is easier to contain the linearised GR terms in a separate Lagrangian so that the action will look like

$$S = \int d^4x \left[\mathcal{L}_{GR} - \frac{1}{2} m^2 \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right) + 8\pi G h_{\mu\nu} T^{\mu\nu} \right]. \tag{4.50}$$

Recall how the field $h_{\mu\nu}$ transforms under gauge transformations from eq (2.24). The linearised action of GR in eq (2.48) was invariant and so had gauge symmetry. But just like the massive photon case, the mass terms break the symmetry of the Fierz-Pauli action in eq (4.19). Just like before, the Stüekelberg trick can be used by now introducing a vector field A_μ via the transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu. \tag{4.51}$$

Since the trick introduces the field via a gauge transform, $\delta\mathcal{L}_{GR} = 0$ due to it's gauge invariance. By using integration by parts, the mass and source terms will transform in the action as

$$\begin{aligned} S = \int d^4x \bigg[&\mathcal{L}_{GR} - \frac{1}{2}m^2 \left(h_{\mu\nu}h^{\mu\nu} - h^2\right) - \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} \\ &- 2m^2 \left(h_{\mu\nu}\partial^\nu - h\partial_\mu\right) A^\mu - +8\pi G \left(h_{\mu\nu} - 2A_\mu \partial_\nu\right) T^{\mu\nu} \bigg], \end{aligned} \tag{4.52}$$

where $F_{\mu\nu}$ is the EM field tensor, defined in eq (4.41). This new action will be invariant under gauge transformations of the form

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad A_\mu \to A_\mu - \xi_\mu. \tag{4.53}$$

where ξ_μ is some element of the spacetime. The trick allows the user to fix the Stüeckelberg field to $A_\mu = 0$ at any time, recovering the original Fierz-Pauli action of eq (4.50). This means that the two theories and their physical results are equivalent. By following the massive photon case, the next logical step would be to rescale the field to normalise the action before taking the limit. But in this case taking $m \to 0$ for this action is still not smooth as it lead to a loss of a degree of freedom. This is because it leaves a massless graviton $h_{\mu\nu}$ and photon A_μ providing two degrees of freedom each; falling one degree short of the Fierz-Pauli action. The previous section showed how an extra degree of freedom was added by introducing the ϕ field to bring the massless case of 2 dof up to the 3 dof of the massive case. Introducing the field A_μ has only brought the massless case up from 2 dof to 4. So the trick must be used again; now introducing a scalar field ϕ via the typical gauge transformation of EM as in eq (4.47). Once again the EM tensor term will be invariant, leaving the mass and source terms to transform (with some integration by parts) in the action as

$$\begin{aligned} S = \int d^4x \bigg[&\mathcal{L}_{GR} - \frac{1}{2}m^2 \left(h_{\mu\nu}h^{\mu\nu} - h^2\right) - \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} \\ &- 2m^2 \left(h_{\mu\nu}\partial^\nu - h\partial_\mu\right) A^\mu - 2m^2 \left(h_{\mu\nu}\partial^\mu\partial^\nu - h\partial_\mu\partial^\mu\right)\phi \\ &+ 8\pi G \left(h_{\mu\nu} - 2A_\mu\partial_\nu + 2\phi\partial_\mu\partial_\nu\right) T^{\mu\nu} \bigg]. \end{aligned} \tag{4.54}$$

This action is now invariant under the two gauge transformations

$$\begin{aligned} &h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad A_\mu \to A_\mu - \xi_\mu \\ &A_\mu \to A_\mu + \partial_\mu \lambda, \qquad \phi \to \phi - m\lambda. . \end{aligned} \tag{4.55}$$

As in previous cases, fixing both $\phi = 0$ and $A_\mu = 0$ returns the Fierz-Pauli action; proving the two theories equivalence. This theory can be still thought of as Fierz-Pauli with extra redundancies in the form of gauge symmetries in the theory.

The kinetic term in the action are normalised by having no dependence on mass. This can be achieved by redefining $A_\mu \to \frac{1}{m}A_\mu$, as well as $\phi \to \frac{1}{m^2}\phi$ preemptively. With this, the action will take the form

$$S = \int d^4x \Big[\mathcal{L}_{GR} - \frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} \\ - 2m\left(h_{\mu\nu}\partial^\nu - h\partial_\mu\right)A^\mu - 2\left(h_{\mu\nu}\partial^\mu\partial^\nu - h\partial_\mu\partial^\mu\right)\phi \\ + 8\pi G\left(h_{\mu\nu} - \frac{2}{m}A_\mu\partial_\nu + \frac{2}{m^2}\phi\partial_\mu\partial_\nu\right)T^{\mu\nu}\Big] , \tag{4.56}$$

and contain gauge symmetries under

$$\begin{aligned} &h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \qquad A_\mu \to A_\mu - m\xi_\mu \\ &A_\mu \to A_\mu + \partial_\mu m\lambda, \qquad \phi \to \phi - m^2\lambda. \; . \end{aligned} \tag{4.57}$$

As in the previous case, taking $m \to 0$ for a non-conserved source will cause the introduce fields to strongly couple to the divergence of the source. Assuming a conserved source, the limit will become

$$S = \int d^4x \left[\mathcal{L}_{GR} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2\left(h_{\mu\nu}\partial^\mu\partial^\nu - h\partial_\mu\partial^\mu\right)\phi + 8\pi G h_{\mu\nu}T^{\mu\nu}\right] . \tag{4.58}$$

The next step is to check that this massless limit matches the five degrees of freedom of the massive theory. The first step will be to separate the scalar-field tensor mixing term. This can be done by redefining the field as

$$h_{\mu\nu} = h'_{\mu\nu} + \omega\eta_{\mu\nu} \tag{4.59}$$

for a scalar function $\omega(x)$ that re-scales the metric (a conformal transformation). The linearised Lagrangian of GR will transform as

$$\mathcal{L}_{GR}(h) = \mathcal{L}_{GR}(h') + 2\left(\partial_\mu\omega\partial^\mu h' - \partial_\mu\omega\partial_\nu h'^{\mu\nu} + \frac{3}{2}\partial_\mu\omega\partial^\mu\omega\right) \tag{4.60}$$

so the action in eq (4.58) becomes

$$S = \int d^4x \Big[\mathcal{L}_{GR}(h') + 2\left[\partial_\mu\omega\partial^\mu h' - \partial_\mu\omega\partial_\nu h'^{\mu\nu} + \frac{3}{2}\partial_\mu\omega\partial^\mu\omega\right] \\ - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2\left(h'_{\mu\nu}\partial^\mu\partial^\nu - h'\partial_\mu\partial^\mu - 3\omega\partial_\mu\partial^\mu\right)\phi \\ + 8\pi G\left(h'_{\mu\nu} + \omega\eta_{\mu\nu}\right)T^{\mu\nu}\Big] . \tag{4.61}$$

By setting the conformal function to the scalar field $\omega = \phi$ and by utilising integration by parts, the cross terms in the action can be canceled providing a kinetic term for *phi* and another coupled to the source. It is also nice to have the typical coefficients in front of the kinetic terms for both the scalar and vector fields; so they can be rescaled via $\phi \to \frac{1}{\sqrt{6}}\phi$, and $A_\mu \to \frac{1}{\sqrt{2}}A_\mu$. In this picture the action will look like

$$S = \int d^4x \left[\mathcal{L}_{GR}(h') - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial^\mu\phi\partial^\nu\phi + 8\pi G\left(h'_{\mu\nu} + \frac{1}{\sqrt{6}}\phi\eta_{\mu\nu}\right)T^{\mu\nu}\right], \tag{4.62}$$

and the gauge transformation symmetries now take the form

$$\begin{aligned} h'_{\mu\nu} &\rightarrow h'_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \\ A_\mu &\rightarrow A_\mu + \partial_\mu \lambda \ . \end{aligned} \tag{4.63}$$

From this point of view there is no mixing of fields and it is clear to see the degrees of freedom in the system. The massles graviton $h_{\mu\nu}$ provides two degrees of freedom, a massless vector A_μ with another two, and finally a scalar with one degree of freedom. So indeed the massless limit of this theory does provide the same degrees of freedom as the massive.

This form also has the added benefit of neatly displaying the cause of the vDVZ discontinuity in a single term. The very last term is a mixing between the introduced scalar field and the trace of the stresss-energy tensor. Its introduction will observe no change when measuring the effect of gravitational lensing since $T = 0$ but it will alter the Newtonian potential through this new potential term. In a sense the ϕT term is the vDVZ discontinuity.

4.5 Solution of vDVZ and dRGT Massive Gravity

The vDVZ discontinuity was solved by Arkady Vainshtein in 1972 [53]. His idea was to extend linear Fierz-Pauli massive gravity to a fully non-linear theory of massive gravity, the result of which allowed a smooth GR limit. For this theory we now generalise to curved spaces where the background is now used as the background metric which is not necessarily Minkowski. The perturbation is given by

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \tag{4.64}$$

where $g^{(0)}_{\mu\nu}$ the background metric. A typical non-linear theory takes the form,

$$S = \frac{1}{16\pi G} \int \left(\sqrt{-g} R - \sqrt{-g^{(0)}} \frac{1}{4} m^2 U(g^{(0)}, h) \right), \tag{4.65}$$

where $U(g^{(0)}, h)$ is the most general potential given by

$$\begin{aligned} &U(g^{(0)}, h) = U_2(g^{(0)}, h) + U_3(g^{(0)}, h) + U_4(g^{(0)}, h) + \dots, \\ &U_2(g^{(0)}, h) = tr(h^2) - tr(h)^2 , \\ &U_3(g^{(0)}, h) = C_1 tr(h^3) + C_2 tr(h^2) tr(h) + C_3 tr(h)^3 , \\ &U_4(g^{(0)}, h) = D_1 tr(h^4) + D_2 tr(h^3) tr(h) + D_3 tr(h^2)^2 + D_4 tr(h^2) tr(h)^2 + D_5 tr(h)^4 \ . \end{aligned} \tag{4.66}$$

where the trace is taken using the background metric $g^{(0)\mu\nu}$ and we omit orders higher than 4. Since this is a generalisation of the linear theory, the equation reduces to Fierz-Pauli at linear order.

To see the behaviour of this non-linear theory, we will look for static spherical solutions. We introduce the following metric ansatz:

$$g^{(0)}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dr^2 + r^2, d\Omega^2 \tag{4.67}$$

where we have considered the case of the background metric being Minkowski. To find a static spherically symmetric solution, consider the dynamical metric ansatz

$$g_{\mu\nu} dx^\mu dx^\nu = -B(r) dt^2 + C(r) dr^2 + A(r) r^2, d\Omega^2 \tag{4.68}$$

plugging into the equations of motion, which we omit here but arise from varying the action in eq (4.65) with respect to $g_{\mu\nu}$, we obtain for the tt component

$$\begin{aligned}
&4BC^2m^2r^2A^3 + \left(2B(C-3)C^2m^2r^2 - 4\sqrt{A^2BC}(C-rC')\right)A^2 \\
&+ 2\sqrt{A^2BC}\left(2C^2 - 2r(3A'+rA'')C + r^2A'C'\right)A + C\sqrt{A^2BC}r^2(A')^2 = 0\ ,
\end{aligned} \tag{4.69}$$

the rr component

$$\frac{4(B+rB')A^2 + (2r^2A'B' - 4B(C-rA'))A + Br^2A'^2}{A^2BC^2r^2} - \frac{2(2A+B-3)m^2}{\sqrt{A^2BC}} = 0, \tag{4.70}$$

and due to spherical symmetry both the $\theta\theta$ and $\phi\phi$ components

$$\begin{aligned}
&-2B^2C^2m^2rA^4 - 2B^2C^2(B+C-3)m^2rA^3 \\
&-\sqrt{A^2BC}\left(2C'B^2 + (rB'C' - 2C(B'+rB''))B + rCB'^2\right)A^2 \\
&-B\sqrt{A^2BC}\left(rCA'B' + B(4CA' - rC'A' + 2rCA'')\right)A - B^2C\sqrt{A^2BC}rA'^2 = 0\ .
\end{aligned} \tag{4.71}$$

To see the behaviour away from GR, we introduce expansions in the functions A, B, C of the form

$$\begin{aligned}
A(r) &= A_0(r) + \epsilon A_1(r) + \epsilon^2 A_3(r) + \mathcal{O}(\epsilon^3) \\
B(r) &= B_0(r) + \epsilon B_1(r) + \epsilon^2 B_3(r) + \mathcal{O}(\epsilon^3) \\
C(r) &= C_0(r) + \epsilon C_1(r) + \epsilon^2 C_3(r) + \mathcal{O}(\epsilon^3)\ .
\end{aligned} \tag{4.72}$$

Upon doing this, and solving each equation order by order we obtain,

$$\begin{aligned}
A(r) - 1 &= \frac{4}{3}\frac{GM}{4\pi m^2r^3}\left(1 - 4\frac{GM}{m^4r^5} + ...\right) \\
B(r) - 1 &= -\frac{8}{3}\frac{GM}{r}\left(1 - \frac{1}{6}\frac{GM}{m^4r^5} + ...\right) \\
C(r) - 1 &= -\frac{8}{3}\frac{GM}{m^2r^3}\left(1 - 14\frac{GM}{m^4r^5} + ...\right)\ .
\end{aligned} \tag{4.73}$$

where the preceding terms in each are from higher orders of the parameter ϵ. The expansions above are in terms of r_V/r where

$$r_V = \left(\frac{GM}{m^4}\right)^{1/5}, \tag{4.74}$$

is the Vainshtein radius. Beyond this radius is the region in which the solution is accurate. As $m \to 0$ this radius increases to infinity which causes the theory to break down. The potential will feel different below the Vanshtein radius. This will be given by

$$\begin{aligned}
\phi &\sim \frac{M}{8\pi G}\frac{1}{r}, \qquad r >> r_V\ , \\
\phi &\sim \sqrt{\frac{M}{8\pi G}}\Lambda_5^{5/2}r^{3/2}, \qquad r << r_V\ .
\end{aligned} \tag{4.75}$$

where Λ_5 is the cuttoff of the theory, given by $\Lambda_5 = (8\pi G m^4)^{1/5}$This would mean that the vDVZ discontinuity is caused by linear perturbations and a non-linear perturbation would give a smooth massless limit.

Unfortunately, the generic non-linear theory given by the action in eq (4.65) suffers from a ghost like pathology. It turns out, that generic non-linear theories of massive gravity have 1 extra propagating mode called the Boulware-Deser ghost. It was not until 2011 that the Boulware-Deser ghost was exorcised by de Rham, Gabadadze and Tolley [54, 55] via a specific construction of the non-linear potential for details see the aforementioned references.

We have outlined the construction of a consistent, non-linear theory of massive gravity which propagates 5 degrees of freedom. This theory can be used to explain dark energy, where the interaction potential contributes to the dark energy budget of the universe, and self-accelerating solutions have been found [56].

5 Conclusions

In this paper we have discussed some of the ideas behind modifying gravity. We first took a fundametnal look at general relativity as a helicity-2 massless field theory by linearising the action. We then went on to look at $F(R)$ gravity which generalises the action to higher orders of curvature invariants. We were then able to link this to a scalar-tensor theory known as Brans-Dicke which is heavily constrained by solar system measurements. Finally, we looked at the path followed for the theory of massive gravity; from a linear Fierz-Pauli action to a non-linear interacting theory that could describe a massive spin-2 particle and smoothly return to the massless case in the limit.

The cosmological constant problem, the identity of both dark energy and dark matter are some of the biggest unsolved problems in modern day physics. Some physicists view our fundamental understanding of gravity to be cause of these problems, and to reconcile them, have taken up the task of modifying gravity. While this is a challenging task with many pitfalls like ghost instabilities, it is also a rewarding endeavour, offering up interesting solutions and new insights into the theory of gravity.

The success of modified gravity theories will be decided by their ability to stand up to our current tests and predict future cosmological observations. With the advent of gravitational wave astronomy, we are at the beginning of a whole new field of opportunities to test these theories. The aim then is to be able to find and test solutions to these theories [57, 58]. Future experiments such as LISA [59, 60] will be able shed light onto what theory is responsible for the gravitational interactions in our universe.

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