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MSc Thesis

Quantum Fields and Fundamental Forces

Generalised Geometry and Type II Supergravity

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Abstract

By considering Hitchin's generalised geometry, we discuss the construction of the necessary structure on the weighted generalised tangent bundle to induce a generalised metric parameterised by the ordinary metric, a two-form field and a scalar. It is shown that, under a combined diffeomorphism of the manifold and a gauge transformation of the two-form field, the variation of the generalised metric fields are given by the Dorfman derivative and reduce to the symmetry variations of the bosonic fields of type II supergravity: the graviton, B-field and dilaton respectively. Using this structure, the generalised Levi-Civita connection is derived and found to be non-unique, but a uniquely defined generalised Ricci tensor allows for the construction of a generalised Einstein-Hilbert action. It is shown that this action exactly coincides with that of the bosonic sector of type II supergravity.

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Chapter 1

Introduction

Differential geometry is the study of calculus on manifolds and provides the most accurate known mathematical description for gravitational physics through Einstein's general theory of relativity. Einstein's famous field equations are the statement that spacetime, as a manifold, is dynamical and its curvature is determined by its energy content and distribution. The fact that such a simple statement so successfully describes large-scale physics that, one hundred years after its discovery, general relativity continues to make correct predictions makes it particularly elegant.

So, the potential to build upon these ideas using a generalisation of differential geometry is appealing from the outset. While ordinary differential geometry is the study of differential structures relating to the tangent bundle [1], Hitchin's generalised geometry [2] extends this bundle and studies analogous objects. Aside from interesting mathematical discoveries that may occur, the possibility of unifying other physical phenomena into generalised geometry motivates its study.

Of course, unification is ubiquitous in fundamental physics and the key unsolved problem theoretical physicists face is the unification of the classical theory of Einstein gravity with quantum field theory – the framework describing all other observed forces and interactions in the Standard Model of Particle Physics. The latter assumes a non-dynamical background in front of which quantised scalar, vector and spinor fields interact without disturbing the underlying spacetime. This is an extraordinarily good approximation for the energy scales attainable in collider experiments at which various interactions are measured, but clearly fundamentally at odds with Einstein gravity. Unfortunately, any attempt to introduce a dynamical spacetime with which the standard model fields can interact inevitably fails. The divergent loops in Feynman diagrams require an infinite number of counter-terms to tame, giving rise to a non-renormalisable theory with no predictive power as a so-called theory of everything.

String theory is the study of one-dimensional quantum objects with no internal structure and aims to evade the non-renormalisability of gravity. Arising from the study of strong interactions but subsequently superseded by quantum chromodynamics, the quantised string spectrum consists of an infinite tower of spin states of increasing mass classified by highest-weight representations of the Virasoro algebra. At spin-two, the state decomposes into an antisymmetric field $B_{\mu\nu}$, a scalar field ϕ and a traceless symmetric field, g, identified with the graviton – the quantum of spacetime disturbances. For these massless states to transform in representations of the Poincaré group, they must form massless representations of the little group $SO(d-2)$ of the Lorentz group, which requires the dimensionality of spacetime to be twenty-six – the critical dimension of the bosonic string [3]. Since the mass increases with spin in the bosonic string spectrum, there is a tachyonic state at spin-one which causes yet more doubt that the theory is realised in nature. This latter issue is evaded in the theory of quantum superstrings whose world-sheet theory is supersymmetric and whose critical dimension is ten.

Supersymmetry is the unique additional symmetry in nature that evades the Coleman-Mandula theorem [4]. In a certain sense, any bosonic charges other than scalar quantum numbers, energy-momentum and the Lorentz generator would restrict the scattering angle in an interacting quantum field theory to be non-analytic. Fermionic charges satisfying anticommutation relations are allowed, but since the anticommutator of a conserved spin-$n/2$ fermionic charge is a conserved spin-n bosonic charge subject to the restrictions of Coleman-Mandula, there is only one possibility: $n = 1$. These spin-$1/2$ supercharges form a supersymmetry algebra whose representations are the supermultiplets of fermions and bosons, giving the well-known symmetry between two superpartners.

Initially, there is no restriction on the number of copies of this supersymmetry. If there are any more than thirty-two supersymmetry generators, however, massless supermultiplets necessarily contain fields with spin greater than two, which pose difficulties in interacting theories [5].

Supergravity is a theory of local supersymmetry. As with any local symmetry, supergravity gives rise to a "gauge" field called the gravitino whose superpartner is the vielbein: the field which makes manifest the local diffeomorphism invariance of general relativity. Of the various supergravity theories, eleven-dimensional maximal supergravity is notable. All higher dimensional supergravity theories contain massless fields with spin greater than two and in all lower dimensional theories, one cannot obtain all the standard model gauge fields via Kaluza-Klein dimensional reduction [6].

The inclusion of fermionic worldsheet modes is not unique and leads to distinct superstring theories [3]. The theories of most relevance to this thesis are the two type II superstring theories which describe quantised closed superstrings with left- and right-moving fermions [7]. The worldsheet fermionic theory has a $\mathbb{Z}_2$ symmetry allowing a choice of boundary conditions for the fermions on the cylinder. These are the Neveu-Schwarz (NS) and Ramond (R) conditions and must be consistent among left- and right-moving fermions giving four possible sectors of type II theories. The NS-NS sector, in which both left- and right-moving fermionic modes have NS boundary conditions, contains the bosons of the theory, transforming in tensor representations of the Lorentz group. In particular, the lowest energy state decomposes into the graviton, the antisymmetric B-field, and the dilaton. The absence of tachyons from the type II spectrum and the inclusion of spinors in the other sectors makes type II superstring theory attractive to study.

The various superstring theories were found to correspond to Kaluza-Klein compactifications of a theory in eleven dimensions whose low-energy effective field theory is maximal supergravity. This is M-theory and describes supersymmetric, extended, two- and five-dimensional objects called branes [8]. As such, the low-energy limit of ten-dimensional type IIA superstring theory is a ten-dimensional supergravity, also labelled type IIA [9]. Type IIB supergravity isn't obtained this way, however for our purposes there will be no need to distinguish between the two type II theories which differ only in their fermionic content.

The NS-NS sector of type II supergravity describes the dynamics of the low-energy, type II, NS-NS superstring excitations. The action is given by

$$S_{NS} \propto \int d^{10}x\sqrt{-g}e^{-2\phi}\Big[\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{12}H^2\Big] \tag{1.1}$$

where $\mathcal{R}$ is the Ricci curvature scalar corresponding to the Levi-Civita connection for g and $H = dB$ is the field strength tensor for the B-field [9].

In what follows, in the setting of generalised geometry, we review the construction of a generalised Einstein-Hilbert action – the central object in Einstein gravity whose stationary point satisfies the field equations of general relativity – which is equal to the action for the NS-NS sector of type II supergravity. The fact that simple, geometric objects can neatly encode the physics of supergravity is attractive for the same reasons that general relativity is attractive as a theory of gravity. The equations of motion for the NS-NS fields, found by extremising Eq. 1.1, elegantly reduce to the statement of vanishing generalised Ricci curvature on the generalised tangent bundle, which is also found to encode the diffeomorphism and gauge symmetries of supergravity.

We first revisit some concepts in differential geometry whose generalisation will play an important role throughout. In Chapter 3, generalised geometry is introduced in it original $O(d,d)$ form and some of its subtleties and central objects are discussed. Consideration of generalised G-structures and connections then lead to an explicit construction of a generalised Riemannian metric, which contains the ordinary metric g and the B-field. In order to include the dilaton, in Chapter 4 we review the extension [10] to generalised geometry, in which the tangent bundle becomes weighted. The additional degree of freedom can be included as a conformal factor in classes of generalised frames and the subsequent generalised structures are shown to induce a generalised "metric" encoding all NS-NS fields. In order to construct curvature quantities in analogy to Einstein gravity, the torsion-free, generalised connection compatible with these structures is derived and shown to be non-unique. The generalised Ricci tensor is then shown to evade this ambiguity and, in Chapter 5, we show this leads to a uniquely defined, generalised Einstein-Hilbert action equal to the NS-NS action in Eq. 1.1. In the final Chapter, we conclude and discuss various other areas of application for this type of generalised geometric construction.

Chapter 2

Differential Geometry and General Relativity

In this Chapter, we introduce some key definitions in the geometric description of gravity that will feature throughout this thesis.

2.1 Fibre bundles and frames

Following [1], we define a smooth *fibre bundle* (E, M, F, π, G), often simply denoted E. E, M and F are all smooth manifolds – the *total space*, *base space* and *fibre* respectively. For an open neighbourhood (and element of an open cover U_i of M) $U_i \subset M$ of x, its preimage under the smooth, surjective *projection* $\pi : E \to M$ must be diffeomorphic to the local product space $U_i \times F$. The corresponding diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i) \subset E$ is called the *trivialisation*. Restricted to each point $x \in U_i$, the map $\phi_i(x, \;) : F \to \pi^{-1}(x)$ is a diffeomorphism called the *fibre at x*. In the non-empty intersection of two open neighbourhoods of x, $U_i \cap U_j$, the *transition function*

$$g_{ij} := \phi_j^{-1}(x, \;) \circ \phi_i(x, \;) \tag{2.1}$$

must be an element of the group G, called the *structure group*. For consistency with differentiability, the group elements g_{ij} must satisfy

$$g_{ii} = e \tag{2.2a}$$

$$g_{ij} = g_{ji}^{-1} \;\; \forall x \in U_i \cap U_j \tag{2.2b}$$

$$g_{ij} g_{jk} = g_{ik} \;\; \forall x \in U_i \cap U_j \cap U_j \tag{2.2c}$$

If $G = e$, i.e. all transition functions are identity elements, then the bundle is globally a product $E = M \times F$ and known as the *trivial bundle*. A *vector bundle* is a bundle with fibres that form vector spaces [1]. Two important examples are the *tangent bundle* and *cotangent bundle* of a smooth manifold M, respectively defined as the disjoint unions of the tangent and cotangent spaces of M

$$TM = \bigsqcup_{x \in M} T_x M \quad \text{and} \quad T^*M = \bigsqcup_{x \in M} T_x^* M \tag{2.3}$$

Elements of TM can be thought of as pairs of vectors and points (X, x), the projection map π takes $(X, x) \to x$ and the structure group $G \simeq GL(d, \mathbb{R})$ where d is the dimension of M. If $F \simeq G$ for some group G, then E is called a *principal G-bundle* [1].

A *frame* $\{e\}$ at a point $x \in M$ for real vector bundle (E, M, F, π, G) is an ordered basis for F at x and an element of the *set of frames* $Fr(T_x M)$ for the tangent space at x. A choice of frame for each point x induces a bundle called the *structure bundle* $\mathcal{F}$, with points $(x, \{e\})$ where $\{e\} \in Fr(T_x M)$ is a particular frame, and projection $\pi : (x, \{e\}) \mapsto x$. It can be shown that $\mathcal{F}$ is a principal $GL(d, \mathbb{R})$-bundle due to the action of $GL(d, \mathbb{R})$ giving the transformation between two frames [1]. The structure bundle is an example of a *frame bundle*, $\mathcal{F}_G$ which has fibre $G \in GL(d, \mathbb{R})$. For $(x, \{e_a\})$ and $(\tilde{x}, \{\tilde{e}_a\}) \in \mathcal{F}_G$, the bases are related by a transformation in G

$$\tilde{e}_a = M_a{}^b e_b \quad : \quad M \in G \tag{2.4}$$

A *coordinate basis* for TM at $x \in M$ is given by $\{\partial / \partial x^\mu\}$ and is motivated by the equivalence of vectors and directional derivatives [11]. The dual basis for T^*M at x is denoted $\{dx^\mu\}$.

A *section* of a bundle is a map $s : M \to E$ such that

$$s(x) \in \pi^{-1}(x) \;\; \forall x \in M \tag{2.5}$$

The set $\Gamma(U_i, E)$ of sections over some neighbourhood U_i represent elements of the fibres at each point $x \in M$ and, if global sections are admitted by E, they are denoted $\Gamma(E)$. The global sections of the tangent and cotangent bundles, $\Gamma(TM)$ and $\Gamma(T^*M)$, are identified with the sets of vector fields and covector fields respectively.

Tensor fields can be defined as smoothly assigned linear maps on sections of the tangent and cotangent bundles at a point $x \in M$ [11], or as representations of the structure group: $GL(d, \mathbb{R})$ in the case of TM [10].

2.2 Lie derivatives and the Lie bracket

We require a derivative operator that allows us to compare geometric objects along integral curves in manifolds so that we can construct curvature tensors. Any differentiable manifold admits such an operator and we follow the conventions of [1] and [11].

The *Lie derivative* of a vector field $Y \in \Gamma(TM)$ with respect to another, X at $x \in M$ is defined as

$$\mathcal{L}_X Y|_x := \lim_{\epsilon \to 0} \left[\frac{\sigma_X(-\epsilon)_* Y|_{x'} - Y|_x}{\epsilon} \right] \tag{2.6}$$

where $x' = \sigma_X(\epsilon, x)$ is given by the infinitesimal *flow* diffeomorphism along the integral curve defined by X, and $\sigma_X(-\epsilon)_*$ is its push-forward if thought of as a function on the manifold. In a coordinate basis, the Lie derivative has components

$$(\mathcal{L}_X Y)^\nu = X^\mu \frac{\partial}{\partial x^\mu} Y^\nu - Y^\mu \frac{\partial}{\partial x^\mu} X^\nu \tag{2.7}$$

Imposing the following action on a smooth function f

$$\mathcal{L}_X f = X[f] \tag{2.8}$$

gives the components of the Lie derivative of any tensor field via the inner product

$$\begin{aligned} \langle \cdot , \cdot \rangle : \quad \Gamma(TM) &\longrightarrow \Gamma(T^*M) \\ (X, \xi) &\longmapsto \xi(X) = X^\mu \xi_\mu \end{aligned} \tag{2.9}$$

The variation of a tensor field T under an infinitesimal flow diffeomorphism is given by the Lie derivative

$$x^\mu \mapsto x^\mu + \epsilon X^\mu + \mathcal{O}(\epsilon^2) \;\Rightarrow\; T \mapsto T + \epsilon \mathcal{L}_X T + \mathcal{O}(\epsilon^2) \tag{2.10}$$

The *Lie bracket* of two vector fields is given by the Lie derivative

$$[X, Y] = \mathcal{L}_X Y \tag{2.11}$$

It gives the closure of two infinitesimal flows along the vector fields X and Y.

2.3 Differential forms

A *differential r-form field* is a tensor field $\omega \in \wedge^r \Gamma(T^*M) = \Omega^r(M)$ and has components given by

$$\omega = \frac{1}{r!}\omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r} \tag{2.12}$$

in a coordinate basis. The *exterior derivative* is a nilpotent, differential operator

$$\begin{aligned} d: \quad \Omega^r(M) &\longrightarrow \Omega^{r+1}(M) \\ \omega &\longmapsto d\omega \end{aligned} \tag{2.13}$$

which, in a coordinate basis, is given by

$$d\omega = \partial_\mu \omega_{\nu_1 \ldots \nu_r} dx^\mu \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_r} \tag{2.14}$$

Given a vector field $X \in \Gamma(TM)$, the *interior product* is a nilpotent, linear map

$$\begin{aligned} i_X: \quad \Omega^r(M) &\longrightarrow \Omega^{r-1}(M) \\ \omega &\longmapsto i_X\omega \end{aligned} \tag{2.15}$$

such that for $V_i \in \Gamma(TM)$

$$i_X\omega(V_1, \ldots, V_{r-1}) = \omega(X, V_1, \ldots, V_{r-1}) \tag{2.16}$$

Cartan's magic formula [1] relates the Lie derivative of an r-form ω to d and i_X

$$\mathcal{L}_X\omega = \{d, i_X\}\omega \tag{2.17}$$

An r-form ω is *closed* if $d\omega = 0$ and *exact* if $\omega = d\xi$ for any $(r-1)$-form ξ. We denote the space of closed r-forms as $\Omega^r(M)$ and exact r-forms as $Z^r(M)$. Two closed r-forms are *cohomologous* if they differ by an exact r-form.

2.4 Connections, covariant derivatives and torsion

In order to correctly take directional derivatives of tensor fields, one needs to introduce a notion of parallel transport in the manifold M so that vectors in different vector spaces can be meaningfully compared. This is achieved by defining a *connection*: a linear differential operator [1]

$$\begin{aligned} \nabla : \quad \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM) \\ (X,Y) &\longmapsto \nabla_X Y \end{aligned} \tag{2.18}$$

which, for all vector fields X and Y and all smooth functions f, satisfies

$$\nabla_{fX} Y = f \nabla_X Y \tag{2.19a}$$

$$\nabla_X (fY) = X[f]Y + f\nabla_X Y \tag{2.19b}$$

In a coordinate basis $\{e_\mu\}$ for TM at $x \in M$, the connection acts as

$$\nabla_X Y = X^\mu (\partial_\mu Y^\nu + \omega_\mu{}^\nu{}_\rho Y^\rho) e_\nu \tag{2.20}$$

where the connection coefficients are defined by

$$\omega_\mu{}^\nu{}_\rho e_\nu = \nabla_{e_\mu} e_\rho =: \nabla_\mu e_\rho \tag{2.21}$$

The generalisation to a directional derivative of a tensor field is given by the *covariant derivative*. It is defined [11] by it action on a smooth function f

$$\nabla_X f = X[f] \tag{2.22}$$

and the Leibniz property

$$\nabla_X (T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2) \tag{2.23}$$

Given a connection ∇, the *torsion map* is an antisymmetric, linear map

$$\begin{aligned} \mathcal{T} : \quad \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM) \\ (X,Y) &\longmapsto \mathcal{T}(X,Y) \end{aligned} \tag{2.24}$$

with

$$\mathcal{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{2.25}$$

The *torsion* is a (1,2)-tensor given, in a general basis, by

$$\mathcal{T}^a_{\ bc} := \mathcal{T}(e_a, e_b)[e^a] = \omega_b{}^a{}_c - \omega_a{}^a{}_b + [e_b, e_c]^a \tag{2.26}$$

Alternatively, and more conveniently, the torsion can be neatly defined using the interior product [10]. Contracting with the first index defines a (1,1)-tensor with the following components in a coordinate basis

$$(i_X \mathcal{T})^\nu{}_\rho = X^\mu \mathcal{T}^\nu_{\ \mu\rho} \tag{2.27}$$

Its action on a tensor α is given by

$$(i_X \mathcal{T})\alpha = \mathcal{L}^\nabla_X \alpha - \mathcal{L}_X \alpha \tag{2.28}$$

where we have defined $\mathcal{L}^\nabla$ as the Lie derivative with ∂_μ replaced with ∇_μ.

2.5 Riemannian metrics and G-structures

A *pseudo-Riemannian metric* on a manifold M is a (0,2)-tensor field g that is symmetric and non-vanishing [1]. Equivalently, it is the smooth assignment of an inner product to each tangent space in the manifold, with

$$\langle X, Y \rangle = g(X, Y) =: g_{\mu\nu} X^\mu Y^\nu \tag{2.29}$$

The *signature* of the metric is a pair (p, q) with $p + q = d$ such that the matrix $g_{\mu\nu}$ in any basis has p negative and q positive eigenvalues.

Einstein gravity is the study of a manifold M equipped with a metric g and the corresponding Levi-Civita connection ∇ – the unique torsion-free connection compatible with the metric. This means that, for all vector fields X and Y

$$\nabla g = 0 \ \text{ and } \ \mathcal{T}(X, Y) = 0 \tag{2.30}$$

The requirement that g is smooth means it is locally flat. That is, it is possible to define a frame $\{\hat{e}\}$ for M related to the coordinate frame e_μ by

$$\hat{e}_a = e_a{}^\mu e_\mu \tag{2.31}$$

and such that

$$g(\hat{e}_a, \hat{e}_b) = \eta_{ab} \tag{2.32}$$

where η_{ab} is the flat $O(p,q)$ metric [11]. In particular, two such frames $\{\hat{e}\}$ and $\{\hat{\hat{e}}\}$ are related to one another by an $O(p,q)$ transformation

$$\hat{\hat{e}}_a = M_a{}^b \hat{e}_b \quad : \quad M \in O(p,q) \tag{2.33}$$

In some sense, it is more general and more useful in the generalised geometry setting to work in reverse. That is to say, a pseudo-Riemannian metric of signature (p,q) is *defined* by a so-called $O(p,q)$-structure on M.

A *G-structure* on a manifold M is a sub-bundle of the structure bundle with fibre G [10]. Topological obstructions may prevent a G-structure being defined. For an orientable manifold M, a $GL^+(d,\mathbb{R})$-structure defines an orientation and an $SL(d,\mathbb{R})$-structure defines a volume form. If G is the trivial group, the G-structure defines a global frame on M called a *parallelisation* [12].

As alluded to above, a particular $O(p,q)$-structure with points $(x,\{\hat{e}\})$ defines a symmetric, non-vanishing (0,2)-tensor field g by

$$g(\hat{e}_a,\hat{e}_b) = \eta_{ab} \tag{2.34}$$

since, for $M \in O(p,q)$

$$g(M_a{}^c\hat{e}_c, M_b{}^d\hat{e_d}) = M_a{}^c M_b{}^d \eta_{cd} = \eta_{ab} \tag{2.35}$$

g is called a *pseudo-Riemannian metric* on M.

As mentioned above, a connection ∇ is *compatible* with a metric g if

$$\nabla_\mu g = 0 \tag{2.36}$$

Equivalently, using the frame $\{\hat{e}\}$ which defines the $O(p,q)$-structure, the connection coefficients are antisymmetric in their final two indices

$$\omega_{\mu(ab)} = 0 \tag{2.37}$$

where $\omega_{\mu ab} = \eta_{ac}\omega_\mu{}^a{}_b$. In fact, given a connection ∇ compatible with a G-structure, the *connection form* $\omega_b{}^a$ defined by

$$\omega_b{}^a = \omega_\mu{}^a{}_b dx^\mu \tag{2.38}$$

transforms in the adjoint representation of G [1]

$$\omega \mapsto M\omega M^{-1} + MdM^{-1} \tag{2.39}$$

and so $\omega_b{}^a$ takes values in $\mathfrak{g}$. For $\mathfrak{o}(p,q)$-valued connection forms, we arrive at the antisymmetry condition 2.37.

2.6 Riemannian geometry and general relativity

Equipped with a connection, we can construct the following curvature quantities for which we follow [11]. The *curvature map*

$$\begin{aligned}\mathcal{R}: \quad & \Gamma(TM)^3 \longrightarrow \Gamma(TM) \\ & (X, Y, Z) \longmapsto \mathcal{R}(X, Y, Z)\end{aligned} \tag{2.40}$$

is defined by commuting connections along closed integral curves defined by vector fields X and $Y \in \Gamma(TM)$:

$$\mathcal{R}(X, Y, Z) = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \tag{2.41}$$

In some basis $\{e_a\}$, the *Riemann tensor* has components

$$\mathcal{R}^a{}_{bcd} = \left\langle e^a, \mathcal{R}(e_b, e_c, e_d)\right\rangle \tag{2.42}$$

or, in terms of the torsion

$$\mathcal{R}^a{}_{bcd}Z^d = [\nabla_b, \nabla_c]Z^a - \mathcal{T}^d_{\ bc}\, Z^a \tag{2.43}$$

The *Ricci tensor* is a (0,2)-tensor field given by the unique, non-trivial contraction

$$\mathcal{R}_{bd} := \mathcal{R}^a{}_{bad} \tag{2.44}$$

and can also be defined by the following action on a vector:

$$\mathcal{R}_{ab}Z^a = [\nabla_a, \nabla_b]Z^a \tag{2.45}$$

The vacuum field equations of general gelativity, or the condition of Ricci flatness

$$\mathcal{R}_{ab} = 0 \tag{2.46}$$

is the equation of motion obtained by extremising the *Einstein-Hilbert action*

$$\mathcal{S} \propto \int \Omega_g \mathcal{R} = \int d^d x \sqrt{-g}\mathcal{R} \tag{2.47}$$

with respect to variations of the metric g_{ab}. In Eq. 2.47, the *Ricci Scalar* $\mathcal{R}$ is the following unique scalar curvature quantity

$$\mathcal{R} := g^{ab}\mathcal{R}_{ab} \tag{2.48}$$

Ω_g is the canonical volume form for the metric and $g = \det g_{ab}$.

Chapter 3

Generalised Geometry

In this Chapter, we introduce the formalism of generalised geometry, which incorporates the diffeomorphism invariance of the manifold and gauge symmetries of the B-field in the NS-NS action 3.2. We largely follow the original ideas of Hitchin [2] and Gualtieri [13] in what follows.

Since the B-field is a gauge field, it only defined up to cohomology, so on the intersection $U_i \cap U_j$ on M, it is patched as

$$B_{(i)} = B_{(j)} - d\Lambda_{(ij)} \tag{3.1}$$

for some patching one-form $\Lambda_{(ij)} \in \Omega^1(M)$. To correctly capture the following diffeomorphism (see 2.10) and gauge variation of B on U_i parameterised by vector X and one-form ξ

$$\delta_{X+\xi} B_{(i)} = \mathcal{L}_X B_{(i)} + d\xi_{(i)} \tag{3.2}$$

we require

$$\delta_{X+\xi} B_{(i)} = \delta_{X+\xi} B_{(j)} \;\Rightarrow\; d\xi_{(i)} = d\xi_{(j)} + \mathcal{L}_X d\Lambda_{(ij)} \tag{3.3}$$

Using 2.17, we see that for vectors and one-forms to correctly parameterise the variation 3.2, they must be patched as

$$X_{(i)} + \xi_{(i)} = X_{(j)} + \xi_{(j)} - i_{X_{(j)}} d\Lambda_{(ij)} \tag{3.4}$$

It is these objects we attempt to repackage into one mathematical formalism in this Chapter.

3.1 The generalised tangent bundle, frames and G-structures

A natural first step in unifying these concepts is to extend the tangent bundle TM of a manifold M to a *generalised tangent bundle* E given by [2]

$$E := TM \oplus T^*M \tag{3.5}$$

where T^*M is the usual cotangent bundle. Sections of E are generalised vector fields V, denoted either by a sum or a pair of ordinary vectors and one-forms, depending on which is notationally neater

$$V = X + \xi = \begin{pmatrix} X \\ \xi \end{pmatrix} \tag{3.6}$$

In either case, we will use Latin characters for the vector part and Greek characters for the one-form part of generalised vectors. Given a *generalised frame* $\{E_A\} \in Fr(E)$ for M that splits into a frame $\{e_a\}$ for TM and $\{e^a\}$ for T^*M, a generalised vector has components

$$V^A = \begin{cases} X^a & : \ A = a \\ \xi_a & : \ A = a + d \end{cases} \tag{3.7}$$

For example, we could use the *generalised coordinate basis* defined as

$$\{E_M\} = \{\partial_M\} \cup \{dx^\mu\} \tag{3.8}$$

A *generalised frame bundle* F is defined exactly analogously to the ordinary geometry case

$$F = \big\{(x, \{E_A\}) \ \ : \ \ x \in M \text{ and } \{E_A\} \text{ is a basis for } E \text{ at } x\big\} \tag{3.9}$$

Also analogously, a generalised frame bundle is a principal $GL(2d, \mathbb{R})$-bundle. We then define a *generalised G-structure* as a principal G-bundle given by a sub-bundle of F.

There is a natural inner product for generalised vectors [14] given by

$$\big\langle X + \xi, Y + \eta \big\rangle = \eta(V, U) := \frac{1}{2}(i_X \eta + i_Y \xi) \tag{3.10}$$

which, in the basis 3.7, evidently corresponds to a metric with components

$$\eta_{AB} = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{3.11}$$

This is a signature (d, d) metric and thus equivalently an $O(d, d)$-structure defined by the choice of frame $\{\hat{E}_A\}$ which gives a basis of the form 3.7 at each point

$$\eta(\hat{E}_A, \hat{E}_B) = \eta_{AB} \tag{3.12}$$

The corresponding frame bundle is called the *generalised structure bundle*, since the inner product is defined very naturally [10].

It is worth noticing the fact that η defines an isomorphism between E and E^* [10], meaning generalised cotangent vectors can be equivalently thought of as generalised tangent vectors. Generalised indices $A, B, \ldots$ are raised and lowered with η.

The Lie algebra $\mathfrak{so}(d, d)$ of the generalised structure group $O(d, d)$ consists of elements m which generate matrices that leave η invariant. This property is equivalent to

$$m^T \eta = -\eta m \tag{3.13}$$

Thus, a general element takes the form

$$m = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix} \tag{3.14}$$

A is an arbitrary $d \times d$ matrix generating the $GL(d, \mathbb{R})$ subgroup containing matrices of the form

$$\begin{pmatrix} M & 0 \\ 0 & M^{-T} \end{pmatrix} \quad : \quad M \in GL(d, \mathbb{R}) \tag{3.15}$$

Note that, as expected, this correctly embeds the action of the structure group $GL(d, \mathbb{R})$ on the vector and one-form components of a generalised vector into the $O(d, d)$ generalised structure group. $B_{\mu\nu}$ are the components of a two-form and generate the group of matrices of the form [15]

$$\exp(B) = \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix} \tag{3.16}$$

which act on generalised vectors to give the so-called B-field transformation

$$X + \xi \mapsto X + \xi + i_X B \tag{3.17}$$

The bivector $\beta \in \wedge^2(TM)$ generates an action that does not play a significant role in our discussion.

3.2 The Dorfman derivative and Courant bracket

In order to construct differential operators on the generalised tangent bundle, we define an object which generalises the Lie derivative, or so-called *Dorfman derivative.* Defined as

$$\begin{aligned} L: \quad \Gamma(E)\times\Gamma(E) &\longrightarrow \Gamma(E) \\ (V,U) &\longmapsto L_V U \end{aligned} \tag{3.18}$$

with

$$L_V U = L_{X+\xi}(Y+\eta) = \mathcal{L}_X Y + \mathcal{L}_Y \eta - i_Y d\xi \tag{3.19}$$

it appears to encapsulate the combined action of diffeomorphisms and gauge transformations 3.2 [16].

By defining the generalised partial derivative in a coordinate basis $\{E_M\}$ as

$$\partial_M = \begin{cases} \partial_\mu & : \ M = \mu \\ 0 & : \ M = \mu + d \end{cases} \tag{3.20}$$

we can rewrite the Dorfman derivative of a generalised vector $U = Y + \eta$ with respect to $V = X + \xi$ in an explicitly $O(d,d)$-covariant form [10]

$$(L_V U)^M = V^N \partial_N U^M + (\partial^M V^N - \partial^N V^M) U_N \tag{3.21}$$

On a function, the Dorfman derivative is defined to act as [16]

$$L_V f = \mathcal{L}_X f \tag{3.22}$$

and a Leibniz property defines its action on any generalised, rank-n tensor field T

$$\begin{aligned} (L_V T)^{M_1\ldots M_n} =& V^N \partial_N T^{M_1\ldots M_n} + (\partial^{M_1} V^N - \partial^N V^{M_1}) T_N{}^{M_2\ldots M_n} + \ldots \\ &+ (\partial^{M_n} V^N - \partial^N V^{M_n}) T^{M_1\ldots M_{n-1}}{}_N \end{aligned} \tag{3.23}$$

The generalisation of the Lie bracket to generalised geometry, which is most natural in the sense that it commutes with diffeomorphisms and B-field transformations, is called the *Courant bracket.* It is an antisymmetric, bilinear on generalised vector fields [2]

$$\begin{aligned} [[\,.\,,\,.\,]]: \quad \Gamma(E)\times\Gamma(E) &\longrightarrow \Gamma(E) \\ (V,U) &\longmapsto [[V,U]] \end{aligned} \tag{3.24}$$

and can be defined as the antisymmetrisation of the Dorfman derivative [10]

$$[[V,U]] = \frac{1}{2}(L_V U - L_U V)$$
$$= [X,Y] + \mathcal{L}_Y \eta - \mathcal{L}_X \xi - \frac{1}{2}(d i_X \eta - d i_Y \xi) \tag{3.25}$$

We can see that, given the constituent parts of the Courant bracket, it commutes with diffeomorphisms [1]. Specifically, given a diffeomorphism $f : M \to M$, the action of the "generalised pushforward", $f_* \oplus f^* : E_x \to E_{f(x)}$ commutes with the Lie derivative, exterior derivative and interior product. The Courant bracket also commutes with a closed B-field transformation. To see this, let $B \in Z^2(M)$ be a closed two-form with action on a generalised vector given by 3.17. The claim is that

$$[[X + \xi + i_X B, Y + \eta + i_Y B]] = [[X + \xi, Y + \eta]] + i_{[X,Y]} B \tag{3.26}$$

First note that the left-hand side of 3.26 gives

$$[[X + \xi, Y + \eta]] + \mathcal{L}_X i_Y B - \mathcal{L}_Y i_X B - \frac{1}{2} d(i_X i_Y B - i_Y i_X B) \tag{3.27}$$

Now, using the antisymmetry of the inner product and Eq. 2.17, one finds

$$-\frac{1}{2} d(i_X i_Y B - i_Y i_X B) = d(i_Y i_X B) = \mathcal{L}_Y i_X B - i_Y d(i_X B) \tag{3.28}$$

With this, Eqn. 3.26 becomes

$$[[X + \xi, Y + \eta]] + [\mathcal{L}_X, i_Y] B + i_Y i_X dB \tag{3.29}$$

Finally, using the fact that B is closed and the identity [1]

$$[\mathcal{L}_X, i_Y] B = i_{[X,Y]} B \tag{3.30}$$

we arrive at the result [14].

We have seen that both the inner product η and the Courant bracket are preserved by the action of the following semi-direct product group

$$\text{Diff}(M) \ltimes Z^2(M) \tag{3.31}$$

It can be shown that this is the *full* group that preserves the Courant bracket [13]. So, these are appropriate objects with which to build generalised curvature quantities that correctly encode the combined diffeomorphism and gauge invariance 3.2.

3.3 Generalised connections and generalised torsion

In this Section, we define a generalisation of the connection: a directional derivative of one generalised vector field with respect to another or, equivalently, a notion of parallel transport of generalised vectors in the manifold. It is this additional structure that is needed to define differential operators and discuss the notion of generalised curvature.

A *generalised connection* is a linear, differential map [10]

$$\begin{aligned} D: \quad \Gamma(E) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (V, U) &\longmapsto D_V U \end{aligned} \tag{3.32}$$

which, for generalised vectors V and $U \in \Gamma(E)$ and smooth function f, satisfies

$$D_{fV} U = f D_V U \tag{3.33a}$$

$$D_V(fU) = U[f] + f D_V U \tag{3.33b}$$

Since the action of a generalised vector on a function is given by the generalised directional derivative 3.20, in a frame $\{E_A\}$, with generalised connection coefficients

$$D_M E_A = \Omega_M{}^B{}_A E_B \tag{3.34}$$

we have following action of D_M on a generalised vector

$$(D_M U)^A = \partial_M U^A + \Omega_M{}^A{}_B U^B \tag{3.35}$$

The *generalised covariant derivative* is defined by the Liebniz requirement

$$D_V(U \otimes W) = (D_V U) \otimes W + U \otimes (D_V W) \tag{3.36}$$

Its action on a rank-n tensor field has components [10]

$$(D_M T)^{A_1 \dots A_n} = \partial_M T^{A_1 \dots A_n} + \Omega_M{}^B{}_{A_1} T^{B A_2 \dots A_n} + \dots + \Omega_M{}^B{}_{A_n} T^{A_1 \dots A_{n-1} B} \tag{3.37}$$

The generalised tangent space naturally admits an $O(d, d)$-structure meaning it is natural to impose compatibility of the generalised connection with this structure. Since the metric η_{AB} is used to raise and lower indices, we see immediately that

$$D_M \eta = 0 \;\Leftrightarrow\; \Omega_{M(AB)} = 0 \tag{3.38}$$

Given a dual basis $\{E^A\}$, the *generalised torsion map* is defined to act on generalised basis vectors as

$$T(E_B, E_C)[E^A] = T^A{}_{BC} \tag{3.39}$$

Here, the *generalised torsion tensor* is given by contracting with a generalised vector in analogy to 2.28

$$V^N T^M{}_{NP} U^P = (L^D_V U)^M - (L_V U)^M \tag{3.40}$$

where L^D is the Dorfman derivative 3.18 with ∂_M replaced with D_M [10]. This gives the following form for the generalised torsion tensor

$$T^M{}_{NP} = \Omega_N{}^M{}_P + \Omega^M{}_{PN} - \Omega_P{}^M{}_N + \Omega_Q{}^Q{}_N \delta^M_P \tag{3.41}$$

Lowering the indices with η, we find the decomposition

$$T^M{}_{NP} = (T_1)^M{}_{NP} - (T_2)_N \delta^M_P \in \Gamma(\wedge^3 E \oplus E) \tag{3.42}$$

with [10]

$$(T_1)_{MNP} = -3\Omega_{[MNP]} \tag{3.43a}$$

$$(T_2)_M = -\Omega_Q{}^Q{}_M \tag{3.43b}$$

Equipped with the generalised connection and torsion, we could define a torsion-free generalised connection that is compatible with the $O(d,d)$-structure and use it to construct curvature quantities. However, we need to introduce further structure that defines a (pseudo-)Riemannian metric g on the manifold and, ideally, the other NS-NS fields too so that they are contained in generalised curvature quantities.

3.4 The generalised Riemannian metric

As mentioned in Section 2.5, a Riemannian metric on a manifold is defined by an $O(d)$-structure [12]. $O(d)$ is also the maximal subgroup of $GL(d,R)$ that is compact with respect to the subspace topology. According to the Cartan-Iwasawa-Malcev theorem, such a maximal compact subgroup is admitted by every locally compact Lie group [17]. For the structure group $O(d,d)$ of the generalised tangent bundle, the maximal compact subgroup is $O(d) \times O(d)$ [18]. So naturally, defining a generalised Riemannian metric G is equivalent to defining an $O(d) \times O(d)$ structure on E.

Following [16] we construct a generalised Riemannian metric G explicitly. Consider a manifold M equipped with a Riemannian metric g. Notice that g can be defined by the sub-bundle containing vectors $X + g\xi$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(T^*M)$ since g maps vector fields to one-form fields. Also note that E can be split into two sub-bundles that contain generalised vectors of positive and negative norms with respect to the $O(d,d)$ metric η respectively, each of dimension d

$$E = C_+ \oplus C_- \tag{3.44}$$

Define the generalised Riemannian metric as a linear map

$$\begin{aligned} G: \quad \Gamma(E) \times \Gamma(E) &\longrightarrow \mathbb{R} \\ (V, U) &\longmapsto G(V, U) \end{aligned} \tag{3.45}$$

with

$$G(V, U) = \eta(V, U)|_{C_+} - \eta(V, U)|_{C_-} \tag{3.46}$$

which is clearly symmetric and positive definite by definition. Also note that the structure that separately preserves the $O(d,d)$ metric on each sub-bundle is an $O(d) \times O(d)$ structure, as required. Since $\eta(V, V) = 0$, if V is either pure vector or pure one-form

$$C_\pm \cap TM = C_\pm \cap T^*M = 0 \tag{3.47}$$

A generic section of C_+ is then given by $X + MX$ for some $d \times d$ matrix M. Decomposing M into a symmetric matrix g and an antisymmetric matrix B, notice the patching condition 3.4 implies that

$$g_{(i)} = g_{(j)} \quad \text{and} \quad B_{(i)} = B_{(j)} - d\Lambda_{(ij)} \tag{3.48}$$

Thus, g and B can be identified with the Riemannian metric and a B-field since they have the correct patching.

Now, following [19] we construct the explicit matrix form of G. First set $B = 0$ and note that a generic generalised vector in the $C_\pm$ eigenspaces is written

$$X_{g\pm} = X \pm gX \tag{3.49}$$

Then, notice that

$$2X = X_{g+} + X_{g-} \quad \text{and} \quad 2gX = X_{g+} - X_{g-} \tag{3.50}$$

These imply that, in a basis 3.7, the generalised Riemannian metric with $B = 0$, which we denote G_g, satisfies

$$G_g \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ gX \end{pmatrix} \quad \text{and} \quad G_g \begin{pmatrix} 0 \\ gX \end{pmatrix} = \begin{pmatrix} g^{-1}gX \\ 0 \end{pmatrix} \tag{3.51}$$

This gives the general form of G_g in this basis

$$G_g = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} = G_g^{-1} \tag{3.52}$$

To reintroduce the B-field, we recall the form of the B-field transformation 3.17. Its action on $X_{g\pm}$ is

$$X_\pm := \exp(B) X_{g\pm} = X \pm gX \pm BX \tag{3.53}$$

Using the fact that $X_\pm$ are elements of the $\pm$ eigenspace of the Riemannian metric G (now including the B-field), or

$$G(X_\pm) = \pm X_\pm \tag{3.54}$$

we see that

$$X_\pm = \pm \exp(B) G_g \exp(-B)(X_\pm) \tag{3.55}$$

Along with the two-component form of the B-field transformation, this gives the following form of G [19]

$$G = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \tag{3.56}$$

We have successfully generalised the notion of a Riemannian metric and it contains the ordinary Riemannian metric g and the B-field. One could now impose compatibility with G on the generalised connection D defined in Section 3.3, use the form of the generalised torsion to define a generalised Levi-Civita connection and consider the resulting generalised curvature. However, to introduce the dilaton ϕ, we must extend the generalised tangent bundle further to account for the extra degree of freedom.

Chapter 4

Extended Generalised Geometry

In this Chapter, we show that by extending the generalised tangent bundle E introduced in Section 3.1, the resulting $O(d) \times O(d)$ structure induces an additional object with one degree of freedom which can be identified with the dilaton.

4.1 Extended generalised bundles

The correct way to extend the tangent bundle was discovered by the authors of [10] and we follow their construction below. Define the *extended generalised tangent bundle* as

$$\tilde{E} = \tilde{L} \oplus E \tag{4.1}$$

where E is the generalised tangent bundle defined in Section 3.1 and $\tilde{L}$ is an $\mathbb{R}^+$-bundle. Under this construction, sections of $\tilde{E}$ are weighted generalised tensors of rank-1 (or weighted generalised vectors). That is, we define $\tilde{L}$ such that, for generalised vector $V = X + \xi \in \Gamma(E)$, the patching conditions 3.4 are modified to

$$X_{(i)} + \xi_{(i)} = (\det M)^{-1}(X_{(j)} - i_{X_{(j)}} d\Lambda_{(ij)}) \tag{4.2}$$

where M is the $GL(d, \mathbb{R})$ transformation matrix. This implies that, as required

$$\tilde{L} \simeq \det T^*M = \wedge^d(T^*M) \tag{4.3}$$

There is a natural structure on $\tilde{E}$ given the natural $O(d, d)$-structure induced by the metric η called the *(extended) generalised structure bundle* [10], defined by

$$\tilde{F} = \big\{(x, \{E_A\}) \;:\; x \in M \text{ and } \eta(E_A, E_B) = \Phi^2 \eta_{AB}\big\} \tag{4.4}$$

where $\Phi \in \Gamma(\tilde{L})$ and the $O(d,d)$-invariant metric is given by 3.11. The basis $\{E_A\}$ is called a *conformal basis*. Analogously to ordinary structure bundles introduced in 3.1, the generalised structure bundle forms a principal $O(d,d) \times \mathbb{R}^+$-bundle [10]. Extended generalised tensors now form representations of the structure group $O(d,d)\times\mathbb{R}^+$, that is, they are representations of $O(d,d)$ with some weight p and are sections of the bundle

$$E^{\otimes n}_{(p)} = \tilde{L}^p \otimes E \otimes ... \otimes E \tag{4.5}$$

We want to construct a generalised Riemannian metric in this extended generalised geometry setting which contains the dilaton field ϕ. To do this, we first investigate an additional structure on $\tilde{E}$ induced by so-called *split frames*. Given a generic basis $\{\hat{e}_a\}$ for TM and its dual $\{e^a\}$ for T^*M, a split frame is defined by [10]

$$\hat{E}_A = \begin{cases} \hat{E}_a = (\det e)(\hat{e}_a + i_{\hat{e}_a} B) & : \ A = a \\ E^a = (\det e) e^a & : \ A = a + d \end{cases} \tag{4.6}$$

where e is the vielbein 2.31. This is a subset of the set of conformal frames since, using the antisymmetry of the interior product, we find

$$\eta(\hat{E}_A, \hat{E}_B) = (\det e)^2 \eta_{AB} \tag{4.7}$$

and so, the corresponding split frame bundle is a sub-bundle of the generalised structure bundle. The structure group of the split frame bundle, which we denote G_{split}, is found by noting the transformation properties of split frames

$$\begin{aligned} \hat{E}'_A &= \begin{cases} \hat{E}'_a = (\det e')(\hat{e}'_a + i_{\hat{e}'_a} B') & : \ A = a \\ E'^a = (\det e') e'^a & : \ A = a + d \end{cases} \\ &= \begin{cases} (\det e') M^b{}_a (\hat{e}_b + i_{\hat{e}_b}(B + \omega)) & : \ A = a \\ (\det e')(M^{-1})_b{}^a) e^b & : \ A = a + d \end{cases} \end{aligned} \tag{4.8}$$

We see that a generic element of G_{split} is

$$(\det M)^{-1} \begin{pmatrix} \mathbb{1} & 0 \\ \omega & \mathbb{1} \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & (M^{-1})^T \end{pmatrix} \tag{4.9}$$

where ω is a closed two-form and $M \in GL(d, \mathbb{R})$ are elements of the ordinary structure group.

From this, we see that [15]

$$G_{\text{split}} \simeq GL(d,\mathbb{R}) \ltimes \mathbb{R}^{d(d-1)/2} \subset O(d,d) \times \mathbb{R}^+ \tag{4.10}$$

that is, a choice of split frame induces a G_{split}-structure on $\tilde{E}$. In addition, we can define a $G_{\text{split}} \times \mathbb{R}^+$-structure by introducing a *conformal split frame*, which is defined as

$$\hat{E}_A = \begin{cases} \hat{E}_a = e^{-2\phi}(\det e)(\hat{e}_a + i_{\hat{e}_a}B) & : \ A = a \\ E^a = e^{-\phi}(\det e)e^a & : \ A = a+d \end{cases} \tag{4.11}$$

The $\mathbb{R}^+$ factor acts on the conformal factor between fibres of the frame bundle defined by the conformal split frame [10]. We anticipate that this form can now be used to specify a frame bundle such that ϕ and B give the dilaton and B-fields of supergravity.

4.2 $O(p,q) \times O(q,p)$-structures and the NS-NS fields

Since supergravity motivates this whole construction, the metric g we aim for is pseudo-Riemannian. As such, we now consider an additional $O(p,q) \times O(q,p)$-structure on E (with $p+q=d$) [12] induced by a smaller sub-bundle of the extended generalised structure bundle 4.4. Analogously to the construction of an $O(d) \times O(d)$ structure in Section 3.4, this is equivalent to defining a splitting of E as in 3.44 and additionally fixing a particular weight $\Phi \in \Gamma(\tilde{L})$. We therefore anticipate that, while the splitting will define the metric G as in Section 3.4, this additional freedom will define a dilaton ϕ.

Rather than explicitly constructing these objects as we did in Section 3.4, we will construct the particular conformal split frame $\{\hat{E}_A\}$ which induces the $O(p,q) \times O(q,p)$-structure. The metric G is defined from this frame in the usual way

$$G = \Phi^{-2}\eta^{AB}\hat{E}_A \otimes \hat{E}_B \tag{4.12}$$

where Φ is the conformal factor associated to $\{\hat{E}_A\}$ which also defines the dilaton [10].

The particular conformal split frame $\{\hat{E}_A\}$ must split into

$$\{\hat{E}_A\} = \{\hat{E}^+_a\} \cup \{\hat{E}^-_{\bar{a}}\} \tag{4.13}$$

with $a, \bar{a} = 1, ..., d$ such that $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$ form a conformal basis for C_+ and C_- respectively, that is

$$\eta(\hat{E}^+_a, \hat{E}^+_b) = \Phi^2\eta_{AB} \tag{4.14a}$$

$$\eta(\hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}}) = -\Phi^2 \eta_{AB} \tag{4.14b}$$

$$\eta(\hat{E}^+_a, \hat{E}^-_{\bar{a}}) = 0 \tag{4.14c}$$

for fixed density Φ. Note that we change basis, so now the $O(d,d)$ metric η has components which differ from 3.11, given by

$$\eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix} \tag{4.15}$$

where η_{ab} and $\eta_{\bar{a}\bar{b}}$ are flat metrics of signature (p,q) [10].

Explicitly, we choose the conformal split frame to take the form

$$\hat{E}_A = \begin{cases} \hat{E}^+_a = e^{-2\phi}\sqrt{-g}(\hat{e}^+_a + e^+_a + i_{\hat{e}^+_a} B) & : \ A = a \\ \hat{E}^-_{\bar{a}} = e^{-2\phi}\sqrt{-g}(\hat{e}^-_{\bar{a}} - e^-_{\bar{a}} + i_{\hat{e}^-_{\bar{a}}} B) & : \ A = \bar{a} + d \end{cases} \tag{4.16}$$

where $\Phi = e^{-2\phi}\sqrt{-g}$ is the fixed weight, $\hat{e}^+_a$ and $\hat{e}^-_{\bar{a}}$ are independent orthonormal ordinary frames with duals e^{+a} and $e^{-\bar{a}}$, that is

$$g(\hat{e}^+_a, \hat{e}^+_b) = \eta_{ab} \ \text{ and } \ g(\hat{e}^-_{\bar{a}}, \hat{e}^-_{\bar{b}}) = \eta_{\bar{a}\bar{b}} \tag{4.17}$$

g is the ordinary Riemannian metric, given by

$$g = \eta_{ab} e^{+a} \otimes e^{+b} = \eta_{\bar{a}\bar{b}} e^{-\bar{a}} \otimes e^{-\bar{b}} \tag{4.18}$$

Note that now we have fixed the conformal factor Φ and built the frame from ordinary orthonormal frames, the structure group of the frame bundle defined by 4.16 is clearly $O(p,q) \times O(q,p)$, giving the required structure.

Finally, from 4.16 we see that the two objects G and Φ induced by the $O(p,q)\times O(q,p)$ structure contain all the NS-NS fields: g, B and ϕ. In the next Section, we will construct a connection compatible with this $O(p,q) \times O(q,p)$-structure in addition to the original $O(d,d) \times \mathbb{R}^+$-structure. The torsion-free such compatible connection will be the generalised Levi-Civita connection from which we can construct curvature quantities.

4.3 Generalised Levi-Civita connection

In this Section, we use the $O(p,q) \times O(q,p)$-structures to construct a family of compatible, generalised connections. The simplest way to do this is to lift the ordinary Levi-Civita connection ∇ on TM to $\tilde{E}$ [10]. Despite ∇ being torsion-free, we will find that the corresponding lifted connection is not by finding the components explicitly. Finally, we will define the Levi-Civita connection by removing the torsion, but we will find it is not unique.

First, let the (extended) generalised connection act on a weighted generalised vector $U \in \Gamma(\tilde{E})$ as

$$D_M U^A = \partial_M U^A + \tilde{\Omega}_M{}^A{}_B U^B \tag{4.19}$$

where M is a coordinate basis index 3.8 and $\{A, B\}$ are conformal split frame indices

$$U = U^A \hat{E}_A = u^a \hat{E}_a + \zeta_a E^a \tag{4.20}$$

Compatibility with the $O(d,d)\times\mathbb{R}^+$-structure implies that the generalised connection forms $\tilde{\Omega}^A{}_B \in \Gamma(E^*)$ are $\mathfrak{o}(d,d) \oplus \mathbb{R}$-valued [1]

$$\tilde{\Omega}_M{}^A{}_B = \Omega_M{}^A{}_B - \Lambda_M \delta^A{}_B \tag{4.21}$$

where $\Omega^A{}_B$ are the $\mathfrak{o}(d,d)$-valued connection forms as in Eq. 3.38 and $\Lambda_M \in \Gamma(E^*)$ are real-valued forms.

The action on a generalised weighted rank-n tensor $\alpha \in \Gamma(\tilde{E}^{\otimes n}_{(p)})$ is

$$\begin{aligned} D_M \alpha^{A_1 \ldots A_n} =& \partial_M \alpha^{A_1 \ldots A_n} + \Omega_M{}^{A_1}{}_B \alpha^{B A_2 \ldots A_n} + \ldots + \Omega_M{}^{A_n}{}_B \alpha^{A_1 \ldots A_{n-1} B} \\ & - p \Lambda_M \alpha^{A_1 \ldots A_n} \end{aligned} \tag{4.22}$$

where p is the weight of α or equivalently the charge labelling its representation of $\mathbb{R}^+$ [10].

To explicitly construct the connection, we use the conformal split frame 4.11 and lift the ordinary connection ∇, defined on weighted vectors and one-forms, to a generalised connection D^∇ with components

$$(D^\nabla_M U^A)\hat{E}_A = \begin{cases} (\nabla_\mu u^a)\hat{E}_a + (\nabla_\mu \zeta_a) E^a & : \ M = \mu \\ 0 & : \ M = \mu + d \end{cases} \tag{4.23}$$

Exactly analogously to the construction of the generalised torsion in Section 3.3, we use the slightly modified form of the Dorfman derivative for a weighted tensor $\alpha \in \Gamma(\tilde{E}^{\otimes n}_{(p)})$ given by

$$\begin{aligned} (L_V \alpha)^{M_1 \ldots M_n} =& V^N \partial_N \alpha^{M_1 \ldots M_n} + (\partial^{M_1} V^N - \partial^N V^{M_1}) \alpha_N{}^{M_2 \ldots M_n} + \ldots \\ & + (\partial^{M_n} V^N - \partial^N V^{M_n}) \alpha^{M_1 \ldots M_{n-1}}{}_N + p(\partial_N V^N) \alpha^{M_1 \ldots M_n} \end{aligned} \tag{4.24}$$

together with an analogous quantity L^D_V to construct the (extended) generalised torsion tensor as in 3.40. Using the antisymmetry properties of Ω, we find that [10]

$$T_{ABC} = -3\tilde{\Omega}_{[ABC]} + \tilde{\Omega}_D{}^D{}_B \eta_{AC} - \Phi^{-2} \eta(\hat{E}_A, L_{\Phi^{-1}\hat{E}_B} \hat{E}_C) \tag{4.25}$$

Once again, the torsion tensor decomposes as $T \in \Gamma(\wedge^3 E \oplus E)$. Explicitly, in a coordinate basis, Eq. 4.25 implies that

$$T^M{}_{NP} = (T_1)^M{}_{NP} - (T_2)_P \delta^M{}_N \tag{4.26}$$

with

$$(T_1)_{MNP} = -3\tilde{\Omega}_{[MNP]} = -3\Omega_{[MNP]} \tag{4.27a}$$

$$(T_2)_M = -\tilde{\Omega}_Q{}^Q{}_M = \Lambda_M - \Omega_Q{}^Q{}_M \tag{4.27b}$$

Now, using the conformal split frame 4.11, we calculate T_1 and T_2 for the generalised connection D^∇ in Eq. 4.23. First, we utilise the Leibniz rule for the Dorfman derivative

$$L_{\Phi^{-1}\hat{E}_A} \hat{E}_B = L_{\Phi^{-1}\hat{E}_A} (\Phi \Phi^{-1} \hat{E}_B) = (L_{\Phi^{-1}\hat{E}_A} \Phi) \Phi^{-1} \hat{E}_B + \Phi L_{\Phi^{-1}\hat{E}_A} (\Phi^{-1} \hat{E}_B) \tag{4.28}$$

For $\Phi = e^{-2\phi} (\det e)$ we have

$$L_{\Phi^{-1}\hat{E}_A} \Phi = \begin{cases} L_{\hat{e}_a + i_{\hat{e}_a} B} (e^{-2\phi} \det e) & : \; A = a \\ L_{e^a} (e^{-2\phi} \det e) & : \; A = a + d \end{cases} = \begin{cases} \mathcal{L}_{\hat{e}_a} (e^{-2\phi} \det e) & : \; A = a \\ 0 & : \; A = a + d \end{cases} \tag{4.29}$$

where we also used property 3.22. Using 2.17 we see that the $A = a$ components are

$$-e^{-2\phi} (\det e)(2 i_{\hat{e}_a} d\phi + i_{\hat{e}_a} i_{\hat{e}_b} de^b) \tag{4.30}$$

Next, we calculate the second Dorfman derivative on the right-hand side of Eq. 4.28 in two-component blocks. For $A, B = a, b$ we have

$$L_{\Phi^{-1}\hat{E}_A} \Phi^{-1} \hat{E}_B = \mathcal{L}_{\hat{e}_a} \hat{e}_b + \mathcal{L}_{\hat{e}_a} (i_{\hat{e}_b} B) - i_{\hat{e}_b} (d i_{\hat{e}_b} B) \tag{4.31}$$

using the definition 3.18. Now, using 3.30 and defining the B-field strength

$$H := dB \tag{4.32}$$

Eq. 4.31 becomes

$$[\hat{e}_a, \hat{e}_b] + i_{[\hat{e}_a, \hat{e}_b]} B - i_{\hat{e}_a} i_{\hat{e}_b} H \tag{4.33}$$

For $A, B = a, b+d$ and $a+d, b$ we simply find that the Dorfman derivative gives the Lie derivative of the dual basis vectors e^b and e^a respectively, and for $A, B = a+d, b+d$, the Dorfman derivative vanishes as it is zero on two pure one-forms. Overall, we have [10]

$$L_{\Phi^{-1}\hat{E}_A}\Phi^{-1}\hat{E}_B = \begin{pmatrix} [\hat{e}_a, \hat{e}_b] + i_{[\hat{e}_a,\hat{e}_b]}B - i_{\hat{e}_a}i_{\hat{e}_b}H & \mathcal{L}_{\hat{e}_a}e^b \\ -\mathcal{L}_{\hat{e}_b}e^a & 0 \end{pmatrix} \tag{4.34}$$

To simplify the calculation, we will assume that the ordinary connection ∇ is torsion-free. Equivalently, we see from Eq. 2.26 that the coefficients of ∇ satisfy

$$\omega_{[\mu}{}^{a}{}_{b]} = 0 \tag{4.35}$$

This implies that

$$\tilde{\Omega}_{[ABC]} = \Omega_{[ABC]} = \frac{1}{6}(\eta_{DB}\tilde{\Omega}_{[A}{}^{D}{}_{C]} + \eta_{DA}\tilde{\Omega}_{[C}{}^{D}{}_{B]} + \eta_{DC}\tilde{\Omega}_{[A}{}^{D}{}_{B]}) = 0 \tag{4.36}$$

and, using symmetry properties, we also find

$$\tilde{\Omega}_{D}{}^{D}{}_{A} = \tilde{\Omega}_{[D}{}^{D}{}_{A]} = 0 \tag{4.37}$$

This simplifies the generalised torsion tensor 4.25 to

$$T_{ABC} = -\Phi^{-2}\eta\big(\hat{E}_A, (L_{\Phi^{-1}\hat{E}_B}\Phi)\Phi^{-1}\hat{E}_C\big) - \Phi^{-2}\eta\big(\hat{E}_A, \Phi L_{\Phi^{-1}\hat{E}_B}(\Phi^{-1}\hat{E}_C)\big) \tag{4.38}$$

The first term on the right-hand side can be calculated using the result 4.30

$$-\Phi^{-2}\eta\big(\hat{E}_A, (L_{\Phi^{-1}\hat{E}_B}\Phi)\Phi^{-1}\hat{E}_C\big) = \begin{cases} (i_{\hat{e}_b}i_{\hat{e}_d}de^d + 2i_{\hat{e}_b}d\phi)\eta_{AC} & : \ B = b \\ 0 & : \ B = b+d \end{cases} \tag{4.39}$$

This clearly forms the $\Gamma(E^*)$ part of the decomposition, yielding

$$(T_2)_B = \begin{cases} -(i_{\hat{e}_b}i_{\hat{e}_d}de^d + 2i_{\hat{e}_b}d\phi) & : \ B = b \\ 0 & : \ B = b+d \end{cases} \tag{4.40}$$

The generalised covector T_2 can be evaluated in a generalised coordinate basis 3.8, simplifying the above expression considerably:

$$T_2 = (T_2)_M E^M = -2(i_{E_M}d\phi)E^M = -4d\phi \tag{4.41}$$

where the factor of 1/2 in the metric η_{AB}, which is used to raise and lower indices, gives the extra factor of 2 [10].

Next, the second term in the right-hand side of Eq. 4.38 is calculated using 4.34 which, in the coordinate basis, simplifies significantly. The only non-zero component is in the $A, B = a, b$ block, and the Lie brackets of coordinate basis vectors vanish. Taking the inner product with $\hat{E}_M$, we find the tensor T_1 takes the simple form [10]

$$T_1 = -4H \tag{4.42}$$

Importantly, we have found that although the ordinary connection ∇ is torsion free, the corresponding connection lifted to $\tilde{E}$ 4.23 is not free of generalised torsion.

The connection we wish to use in building generalised curvature quantities is the generalisation of the Levi-Civita connection. As such, we must further restrict D^{∇} to be compatible with the $O(p,q) \times O(q,p)$-structure. We defined this structure via the splitting of the conformal split frame $\hat{E}_A$ into two conformal frames on $C_{\pm}$ as in Eq. 4.16. Under this split, a weighted generalised vector $U \in \Gamma(\tilde{E})$ takes the form

$$U = u_+^a \hat{E}_a^+ + u_-^{\bar{a}} \hat{E}_{\bar{a}}^- \tag{4.43}$$

and the (extended) generalised connection becomes

$$D_M U^A = \begin{cases} \partial_M u_+^a + \tilde{\Omega}_M{}^a{}_b & : \ A = a \\ \partial_M u_-^{\bar{a}} + \tilde{\Omega}_M{}^{\bar{a}}{}_{\bar{b}} & : \ A = \bar{a} \end{cases} \tag{4.44}$$

Compatibility with the the $O(p,q) \times O(q,p)$-structure implies that $\tilde{\Omega}^A{}_B$ splits into an $\mathfrak{o}(p,q)$-valued connection form on the a, b indices and a $\mathfrak{o}(q,p)$-valued connection form on the $\bar{a}, \bar{b}$ indices:

$$\tilde{\Omega}_{M(bc)} = \tilde{\Omega}_{M(\bar{b}\bar{c})} = 0 \tag{4.45}$$

Now we explicitly construct such a connection. Consider an ordinary Levi-Civita connection ∇ for g. In terms of the two independent orthonormal frames $\{\hat{e}_a^+\}$ and $\{\hat{e}_{\bar{a}}^-\}$, this has the action

$$\nabla_\mu u_+^a = \partial_\mu u_+^a + (\omega^+)_\mu{}^a{}_b u_+^b \tag{4.46a}$$

$$\nabla_\mu u_-^{\bar{a}} = \partial_\mu u_-^{\bar{a}} + (\omega^-)_\mu{}^{\bar{a}}{}_{\bar{b}} u_-^{\bar{b}} \tag{4.46b}$$

which we can straightforwardly lift to connections on $\tilde{E}$ [10]

$$D^{\nabla}_M U^a = \begin{cases} \nabla_\mu u^a_+ & : M = \mu \\ 0 & : M = \mu + d \end{cases} \tag{4.47a}$$

$$D^{\nabla}_M U^{\bar{a}} = \begin{cases} \nabla_\mu u^{\bar{a}}_- & : M = \mu \\ 0 & : M = \mu + d \end{cases} \tag{4.47b}$$

The fact that ∇ is a Levi-Civita connection implies that $(\omega^{\pm})_{\mu(bc)} = 0$. Therefore Eq. 4.45 is satisfied meaning D^{∇} is compatible with the $O(p,q) \times O(q,p)$-structure.

As anticipated, despite the Levi-Civita connection ∇ being torsion-free, its lift D^{∇} to E is not free of generalised torsion. This is most easily seen in the case where the two independent orthogonal bases align

$$\hat{e}^+_a = \hat{e}^-_a =: \hat{e}_a \tag{4.48}$$

Now, comparing the conformal split frame 4.11, with which we calculated the generalised torsion 4.41 and 4.42, with the particular splitting frame 4.16 that defined the $O(p,q) \times O(q,p)$ structure, we see that

$$U = u^a_+ \hat{E}^+_a + u^{\bar{a}}_- \hat{E}^-_{\bar{a}} = (u^a_+ + u^a_-)\hat{E}^+_a + (u_{+a} - u_{-a})\hat{E}^a \tag{4.49}$$

In this case, the lift of the Levi-Civita connection 4.47 and the explicit $O(d,d) \times \mathbb{R}^+$-compatible generalised connection 4.23 are the same [10]. The decomposed generalised torsion quantities 4.41 and 4.42 are frame-independent, so the generalised torsion associated to the lift of the Levi-Civita connection D^{∇} is the same. Therefore, D^{∇} is not the generalised Levi-Civita connection as it is not generalised torsion-free.

We may add to this particular generalised connection 4.47, the components a generic $O(p,q) \times O(q,p)$-compatible generalised connection

$$D_M U^A = \partial_M U^A + \Sigma_M{}^A{}_B U^B \tag{4.50}$$

to obtain another

$$\mathcal{D}_M U^A = D^{\nabla}_M U^A + \Sigma_M{}^A{}_B U^B = \partial_M U^A + (\tilde{\Omega}_M{}^A{}_B + \Sigma_M{}^A{}_B) U^B \tag{4.51}$$

The compatibility of this generic connection D places the following restriction on its components

$$\Sigma_{M(ab)} = \Sigma_{M(\bar{a}\bar{b})} = 0 \tag{4.52}$$

Evidently, Eq. 4.27 implies that the addition of this generic generalised connection modifies the torsion as follows [10]

$$(T_1)_{ABC} = -4H_{ABC} - 3\Sigma_{[ABC]} \tag{4.53a}$$

$$(T_2)_A = -4d\phi_A - \Sigma_C{}^C{}_A \tag{4.53b}$$

The generalised Levi-Civita connection should be free of generalised torsion, requiring the following assignments to the coefficients

$$\begin{aligned}
\Sigma_{[abc]} &= -\frac{1}{6}H_{[abc]} & \Sigma_a{}^a{}_b &= -2\partial_b\phi \\
\Sigma_{[\bar{a}bc]} &= -\frac{1}{2}H_{[\bar{a}bc]} & \Sigma_{\bar{a}}{}^{\bar{a}}{}_{\bar{b}} &= -2\partial_{\bar{b}}\phi \\
\Sigma_{[a\bar{b}\bar{c}]} &= \frac{1}{2}H_{[a\bar{b}\bar{c}]} \\
\Sigma_{[\bar{a}\bar{b}\bar{c}]} &= \frac{1}{6}H_{[\bar{a}\bar{b}\bar{c}]}
\end{aligned} \tag{4.54}$$

Putting everything together, the *generalised Levi-Civita* connection has the following action in the frame 4.16

$$\begin{aligned}
\mathcal{D}_a u_+^b &= \nabla_a u_+^b - \frac{1}{6}H_a{}^b{}_c u_+^c - \frac{2}{9}(\delta_a{}^b\partial_c\phi - \eta_{ac}\partial^b\phi)u_+^c + (A^+)_a{}^b{}_c u_+^c \\
\mathcal{D}_{\bar{a}} u_+^b &= \nabla_{\bar{a}} u_+^b - \frac{1}{2}H_{\bar{a}}{}^b{}_c u_+^c \\
\mathcal{D}_a u_-^{\bar{b}} &= \nabla_a u_-^{\bar{b}} + \frac{1}{2}H_a{}^{\bar{b}}{}_{\bar{c}}{}^{\bar{c}}{}_- \\
\mathcal{D}_{\bar{a}} u_-^{\bar{b}} &= \nabla_{\bar{a}} u_-^{\bar{b}} + \frac{1}{6}H_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} u_-^{\bar{c}} - \frac{2}{9}(\delta_{\bar{a}}{}^{\bar{b}}\partial_{\bar{c}}\phi - \eta_{\bar{a}\bar{c}}\partial^{\bar{b}}\phi)u_-^{\bar{c}} + (A^-)_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} u_-^{\bar{c}}
\end{aligned} \tag{4.55}$$

where, unlike the ordinary Levi-Civita connection, $\mathcal{D}$ is not unique as the tensors $A^{\pm}$ can be added without affecting compatibility. [10]. They must satisfy the following properties in order not to contribute to the generalised torsion

$$(A^+)_{a(bc)} = (A^+)_{[abc]} = (A^+)_a{}^a{}_b = 0 \tag{4.56a}$$

$$(A^-)_{\bar{a}(\bar{b}\bar{c})} = (A^-)_{[\bar{a}\bar{b}\bar{c}]} = (A^-)_{\bar{a}}{}^{\bar{a}}{}_{\bar{b}} = 0 \tag{4.56b}$$

4.4 Generalised curvature quantities

Equipped with the generalised Levi-Civita connection, we now attempt to form generalised curvature expressions needed to study generalised Einstein gravity. These quantities should be unique which requires evading the ambiguity introduced to the generalised Levi-Civita connection 4.55 by the $A^{\pm}$ terms.

The construction of generalised curvature appears to fall at the first hurdle: there is no generalised Riemann tensor. This is due to the non-linearity of the naturally defined generalised curvature map for a generalised connection D

$$R(V, U, W) = [D_V, D_U]W - D_{[[V,U]]}W \tag{4.57}$$

To see this, consider the smooth functions f, g and h and let $V = X + \xi$ and $U = Y + \eta$. Then using

$$[[fV, gU]] = fg\Big([X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)\Big) - \frac{1}{2}d(fg)(i_X\eta - i_Y\xi) \tag{4.58}$$

we find non-linearity [10]

$$R(fV, gU, hW) = fghR(V, U, W) - \frac{1}{2}h\eta(V, U)D_{fdg-gdf}W \tag{4.59}$$

However, restricted to act on V and U in *different* orthogonal subspaces $C_\pm$, the generalised curvature map *is* linear since $\eta(V, U) = 0$. So, given the $O(p, q) \times O(q, p)$-structure, there is a natural generalised Riemann tensor, non-zero only if its index structure is one of the following

$$\{R_{a\bar{b}}{}^{c}{}_{d},\ R_{a\bar{b}}{}^{\bar{c}}{}_{\bar{d}},\ R_{\bar{a}b}{}^{c}{}_{c},\ R_{\bar{a}b}{}^{\bar{c}}{}_{\bar{d}}\} \tag{4.60}$$

Unfortunately, it is precisely these components of the generalised Riemann tensor that cannot be uniquely defined for the generalised Levi-Civita connection $\mathcal{D}$. The first two indices always correspond to mixed commutators, for example $[\mathcal{D}_a, \mathcal{D}_{\bar{b}}]$, and so there can be no cancellation of $A_\pm$ tensors in the curvature map.

It is not the Riemann tensor that appears in the ordinary field equations though, so we attempt to construct a generalised Ricci tensor using the index structures for which the generalised Levi-Civita connection is uniquely defined, namely

$$\begin{aligned}
\mathcal{D}_{\bar{a}}u_+^b &= \nabla_{\bar{a}}u_+^b - \frac{1}{2}H_{\bar{a}}{}^{b}{}_{c}u_+^c \\
\mathcal{D}_a u_-^{\bar{b}} &= \nabla_a u_-^{\bar{b}} + \frac{1}{2}H_a{}^{\bar{b}}{}_{\bar{c}}u_-^{\bar{c}} \\
\mathcal{D}_a u_+^a &= \nabla_a u_+^a - 2(\partial_a\phi)u_+^a \\
\mathcal{D}_{\bar{a}}u_-^{\bar{a}} &= \nabla_{\bar{a}}u_-^{\bar{a}} - 2(\partial_{\bar{a}}\phi)u_-^{\bar{a}}
\end{aligned} \tag{4.61}$$

In analogy to Eq. 2.45, we define the generalised Ricci tensor via

$$R_{a\bar{b}}u^a_+ = [\mathcal{D}_a, \mathcal{D}_{\bar{b}}]u^a_+ \;\text{ or }\; R_{\bar{a}b}u^{\bar{a}}_- = [\mathcal{D}_{\bar{a}}, \mathcal{D}_b]u^{\bar{a}}_- \tag{4.62}$$

To calculate the tensor explicitly, note that

$$\begin{aligned}\mathcal{D}_a\mathcal{D}_{\bar{b}}u^a_+ = \nabla_a\nabla_{\bar{b}}u^a_+ + \Big[&-\frac{1}{2}\nabla_a H_{\bar{b}}{}^a{}_c - \frac{1}{2}H_{\bar{b}}{}^a{}_c\nabla_a + \frac{1}{4}H_{a\bar{b}}{}^{\bar{c}}H_{\bar{c}}{}^a{}_c + (\partial_a\phi)H_{\bar{b}}{}^a{}_c \\ &-\frac{1}{2}H_{c\bar{b}}{}^{\bar{c}}\nabla_{\bar{c}} - 2(\partial_c\phi)\nabla_{\bar{b}}\Big]u^c_+\end{aligned} \tag{4.63a}$$

$$\begin{aligned}\mathcal{D}_{\bar{b}}\mathcal{D}_a u^a = \nabla_{\bar{b}}\nabla_a u^a_+ + \Big[&- 2\nabla_{\bar{b}}(\partial_c\phi) - 2(\partial_c\phi)\nabla)\bar{b} + \frac{1}{2}H_{\bar{b}c}{}^a\nabla_a - (\partial_a\phi)H_{\bar{b}c}{}^a \\ &-\frac{1}{2}H_{\bar{b}}{}^a{}_c\nabla_a + (\partial_a\phi)H_{\bar{b}}{}^a{}_c\Big]u^c_+\end{aligned} \tag{4.63b}$$

Using the total antisymmetry of the field strength H, we find

$$R_{a\bar{b}}u^a_+ = [\nabla_a, \nabla_{\bar{b}}]u^a_+ + \Big[-\frac{1}{2}\nabla_a H_{\bar{b}}{}^a{}_c + (\partial_a\phi)H_{\bar{b}c}{}^a + \frac{1}{4}H_{a\bar{b}}{}^{\bar{c}}H_{\bar{c}}{}^a{}_c + 2\nabla_{\bar{b}}(\partial_c\phi)\Big]u^c_+ \tag{4.64}$$

Now, using the fact that $\nabla_a\phi = \partial_a\phi$, it is straightforward to verify that the second and fifth terms in Eq. 4.64 can be combined as

$$\frac{1}{2}e^{2\phi}\nabla^c(H_{\bar{b}ca}e^{-2\phi}) \tag{4.65}$$

Finally, we see that the generalised Ricci tensor is given by [10]

$$R_{a\bar{b}} = \mathcal{R}_{a\bar{b}} - \frac{1}{4}H_{ac\bar{c}}H_{\bar{b}}{}^{c\bar{c}} + 2\nabla_a\nabla_{\bar{b}}\phi + \frac{1}{2}e^{2\phi}\nabla^c(H_{ca\bar{b}}e^{-2\phi}) \tag{4.66}$$

where $\mathcal{R}_{a\bar{b}}$ is the ordinary Ricci tensor for the Levi-Civita connection ∇. If the two distinct orthonormal frames $\{\hat{e}^+_a\}$ and $\{\hat{e}^-_{\bar{a}}\}$ are chosen to coincide, the generalised Ricci tensor has the index structure R_{ab}.

Importantly, since $R_{a\bar{b}}$ is the matrix representation of a linear map on $C_+ \otimes C_-$ only, at first glance there is no way to infer the off-diagonal components of the full generalised Ricci tensor, R_{AB} in the full frame 4.13. However, if we demand that the generalised Ricci scalar R is defined in analogy to 2.48

$$R = G^{AB}R_{AB} \tag{4.67}$$

and recall the form of the generalised metric 4.12 in this frame

$$G^A B = \begin{pmatrix} \eta^{ab} & 0 \\ 0 & \eta^{\bar{a}\bar{b}} \end{pmatrix} \tag{4.68}$$

we can naturally define the full generalised Ricci tensor as [20]

$$R_{AB} = \begin{pmatrix} R_{ab} & R_{a\bar{b}} \\ R_{\bar{a}b} & R_{\bar{a}\bar{b}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2d} R\delta_{ab} & R_{a\bar{b}} \\ R_{\bar{a}b} & \frac{1}{2d} R\delta_{\bar{a}\bar{b}} \end{pmatrix} \tag{4.69}$$

The generalised Ricci scalar is thus given by [10]

$$R = \mathcal{R} + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{12}H^2 \tag{4.70}$$

where $\mathcal{R}$ is the ordinary Ricci scalar in Eq. 2.48. This is a unique, tensorial expression for the generalised Ricci scalar curvature for a generalised Levi-Civita connection, which should appear in the generalisation of the Einstein-Hilbert action.

Chapter 5

The Generalised Einstein-Hilbert Action and the NS-NS Sector

We now have all the ingredients to form a generalised Einstein-Hilbert action whose stationary condition is that of vanishing generalised Ricci curvature. In this short Chapter, we construct this action and find it is exactly equivalent to the NS-NS action of type II supergravity in Eq. 1.1.

5.1 The generalised Einstein-Hilbert action

Let M be a ten-dimensional manifold and $\tilde{E}$ be the extended generalised tangent bundle, admitting a natural $O(10,10)\times\mathbb{R}^{+}$-structure. The generalised Riemannian metric is defined by an $O(9,1)\times O(1,9)$-structure 4.16 and parameterised by an ordinary metric g of signature (9,1), a two-form field B patched as in 3.48 and a scalar ϕ. Respectively, these correspond exactly to the graviton, B-field and dilaton in the NS-NS sector of type II supergravity.

Under combined diffeomorphisms of M and gauge transformations of B (parameterised by a generalised vector $U = X + \xi \in \Gamma(\tilde{E})$), the invariant tensors of the $O(9,1)\times O(1,9)$-structure: G and Φ, transform via the Dorfman derivative [10]

$$\delta_U G = L_U G \quad \text{and} \quad \delta_U \Phi = L_U \Phi \tag{5.1}$$

Reformulating this in terms of g, B and ϕ, we see that these fields possess the correct variational properties for two ordinary tensors 2.10 and one gauge tensor field

$$\delta_{X+\xi} g = \mathcal{L}_X g \tag{5.2a}$$

$$\delta_{X+\xi} \phi = \mathcal{L}_X \phi \tag{5.2b}$$

$$\delta_{X+\xi} B_{(i)} = \mathcal{L}_X B_{(i)} - d\xi_{(i)} \tag{5.2c}$$

In a coordinate basis, the canonical volume form for the pair (G, Φ) is given by [10]

$$\Omega_{G,\Phi} = \sqrt{-g} e^{-2\phi} dx^0 \wedge ... \wedge dx^9 \tag{5.3}$$

So, using the generalised Ricci scalar in Eq. 4.70, the generalised Einstein-Hilbert action is

$$S_{EH} \propto \int_M \Omega_{G,\Phi} R = \int d^{10}x \sqrt{-g} e^{-2\phi} \left[\mathcal{R} + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{12} H^2 \right] \tag{5.4}$$

Integrating by parts on the second term above yields

$$S_{EH} \propto \int d^{10}x \sqrt{-g} e^{-2\phi} \left[\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right] \tag{5.5}$$

We see that the generalised Einstein-Hilbert action in Eq. 5.5 agrees with the action for the NS-NS sector fields of ten-dimensional, type II supergravity 1.1. We have successfully reformulated the dynamics of the bosonic fields of type II supergravity as the vanishing of Ricci curvature in generalised geometry.

Chapter 6

Conclusion, Discussion and Further Research

In this work, the generalised geometry formalism, which extends the tangent bundle by the cotangent bundle, was introduced. A natural $O(d,d)$-structure, induced by the inner product between the vector and one-form parts of generalised vectors, was shown to exist on the generalised tangent bundle. The Dorfman derivative was defined as a natural generalisation of the Lie derivative and shown to correctly encapsulate the parameters of a diffeomorphism of the manifold and a gauge transformation of a two-form field, motivating the reformulation of type II supergravity as generalised geometry. The generalised connection was introduced and an additional $O(d) \times O(d)$-structure was shown to induce a generalised metric incorporating the ordinary metric and a two-form field identified with the B-field of type II supergravity.

To further incorporate the final bosonic supergravity field, the dilaton, we introduced extended generalised geometry which studies the weighted generalised tangent bundle. The natural structure bundle was shown to be a principal $O(d,d) \times \mathbb{R}^{+}$-bundle and, motivated by the generalised metric in $O(d,d)$ generalised geometry, an $O(p,q) \times O(q,p)$-structure was defined by a splitting of the generalised tangent bundle into positive and negative eigenspaces of the $O(d,d)$ metric. The generalised connection compatible with both the $O(d,d) \times \mathbb{R}^{+}$- and $O(p,q) \times O(q,p)$-structures was defined by a lifting of the ordinary Levi-Civita connection and its generalised torsion was found. The generalised Levi-Civita connection was derived by counteracting this torsion by adding coefficients of a different compatible generalised connection, but it was shown to be non-unique. The generalised curvature map, which is non-linear unless restricted to act on different eigenspaces of the $O(d,d)$ metric, is subsequently not uniquely defined. However, a natural generalised Ricci tensor was given, allowing for the construction of a generalised Einstein-Hilbert action. Finally, it was shown that this action exactly coincides with the action for the NS-NS sector of type II supergravity, validating the use of extended generalised geometry to describe this physics.

An obvious further issue to address is the inclusion of the fermionic sector fields of type II supergravity. The authors of [10] included the R-R field strengths F as representations of $\mathrm{Spin}(d,d)$ and used the generalised Levi-Civita connection to write the spinor equations of motion and supersymmetry variations in a $\mathrm{Spin}(p,q) \times \mathrm{Spin}(q,p)$-covariant form. The question of whether other supergravity theories could be reformulated using generalised geometry was subsequently addressed in [21]. In particular, the authors consider the further extension to the generalised tangent bundle [15] which admits a natural $E_{d(d)} \times \mathbb{R}^+$-structure. The analogue of the Levi-Civita connection compatible with the H_d maximally compact subgroup of $E_{d(d)}$ and the construction of corresponding curvature quantities gives a geometric description of bosonic, eleven-dimensional supergravity restricted to a d-dimensional manifold for $d < 7$. In a follow-up paper [22], the full eleven-dimensional supergravity action, restricted to a warped product $\mathbb{R}^{(10-d),d} \times M$ for d-dimensional spin manifold M and $d < 7$, was derived.

More generally, generalised geometry and, in particular, the $O(d,d)$ action and T-duality has been used to describe non-geometric string backgrounds [16,23] which is an indication of the wider applicability the formalism has in theoretical physics.

Finally, as standard differential geometry is a rich field of mathematics in its own right, with many branches and areas of active research, the exploration of analogous concepts in a generalised geometry setting could lead to many interesting mathematical discoveries.

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