Imperial College London

Department of Physics

Theoretical Physics Group

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Consistent truncations: from Kaluza and Klein to generalised geometry

Author:

Gerlando G. Alfarano

Supervisor:

Prof. Daniel J. Waldram

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Abstract

In this paper, we review the issue of consistency in the context of dimensional Kaluza-Klein reductions with a focus towards supergravity truncations. Thus we present the progress made from a group theoretical point of view by looking at the formalism of manifold group reduction as conceived by Scherk and Schwarz as well as discussing the issues arising in coset spaces reductions. We then provide an example of consistent sphere reduction by reviewing the $\mathcal{N} = 8$ gauged $SO(6)$ truncation of type IIB supergravity on the five sphere S^5. We conclude with a discussion of G-structure and their role in truncations and finally look at the formalism of generic supersymmetric consistent truncation from generalised G-structures with singlet intrinsic torsion which was recently found.

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Contents

Chapter 1

Introduction

The topic of unification has played a very central role in theoretical physics research over the last century and it certainly does now more than ever before. Of all the models attempting to reconcile the geometrical picture of spacetime with the quantum mechanical nature of the microscopic world, in our opinion string theory seems at present to be the most promising option, containing the key ingredients of the Standard model such as gauge interactions and parity violation but also beyond it, such as unification of the gauge interactions, supersymmetry and extra dimensions, although more work is required in order to formulate a string model from which the physics of the Universe we live can be extrapolated as a limiting case. At present we know of five distinct superstring theories [1] which are defined in a 10 dimensional spacetime, conjectured to further unify in a non perturbative formulation called M-theory [2] whose low energy limit gives 11-dimensional supergravity.

Ultimately one is interested in extracting 4-dimensional information out of those models and one way to do so is to introduce a mechanism of compactification of some of the spatial dimensions so that the low dimensional physics emerges as an effective theory in the low-energy limits of string theory which are supergravities, hence one wishes to find out exactly how the Standard Model and gravity embed into those models. For the heterotic string, compactification was mainly achieved on

Calabi Yau manifolds or backgrounds with exact $(2,0)$ supersymmetry, while type II theories have made use of D-branes and orientifolds.

As a result of compactification, the 10-dimensional spacetime ground state is locally described by a product space of a lower dimensional spacetime and a compact internal manifold. One of the main issues with compactification is the emergence of unobserved massless scalar fields with no potential term called moduli, and flux compactification attempts to address the issue via introducing v.e.v. for the field strength fields of supergravity which render the moduli very massive, i.e. it stabilises them [3]. A side effect of the introduction of flux is that they produce a back-reaction on the geometry itself, thus restricting the class of manifolds onto which the compactification can be achieved.

This is then followed by a procedure known as Kaluza-Klein reduction [4, 5], in which the field content of the theory is expanded in terms of eigenfunctions of the compact space in a way that its dimensions appear in the lower dimensional theory as an infinite tower of modes. Usually, a truncation of the field content to a finite subset of the modes is subsequently performed producing a lower-dimensional effective action describing gauge interactions among the fields. It then becomes a question of finding an appropriate truncation such that the solutions of the truncated theory's equations of motion are also solutions of the original equation of motion, that is finding a $consistent\ \ truncation$ for the reduction.

Kaluza-Klein reductions made their appearance long before string theory was conceived, although interest in them was revived in the 70's with the advent of supergravity. The idea was first conceived by Kaluza [4] in 1919 where he attempted to unify Einstein's gravity with electromagnetism by reducing a 5-dimensional pure gravity theory on a circle $U(1)$, at the expense of introducing a massless scalar field, which was subsequently set to be constant in the work of Klein [5] and others, until Jordan and Thiry showed that such condition on the scalar would be inconsistent with the 5-dimensional equations of motions as it would require the Maxwell field to vanish as well [6].

The first attempt at obtaining non-Abelian Yang Mills equation from a reduction scheme was by DeWitt [7] and was further elaborated by Kerner [8] using the language of fibre bundle theory. To achieve it, the reduction would be carried on a group manifold, thus generalising the case of Kaluza to a more complex scenario. The theory of consistent group manifold reductions was finally developed by Cho and Freund [9] in the 70's and the consistency condition emphasised by Scherk and Schwarz [10], that is the group manifold must have structure constants that satisfy the unimodularity condition, namely $C^{\alpha}_{\beta\gamma} = 0$, condition always satisfied by compact Lie groups.

On the other hand, dimensional reduction can also be performed in the case where the compact manifold is a coset space G/H of which spheres S^n are an example of interest. However, the coset space reductions are more involved and prior to generalised geometry, no algorithmic prescription for coset reductions was known although a few cases were found to be consistent, notably the S^5 reduction of type IIB supergravity [11] which was at the centre of the first example of AdS/CFT correspondence proposed by Maldacena in [12].

The situation has changed over the last decade with the introduction of generalised geometry by Hitchin [13] and Gualtieri [14], which is framework that incorporates both complex and symplectic geometry by introducing a new bundle structure on a manifold. More importantly, the reformulation of supergravity using generalised geometry has allowed for consistent truncations to maximally supersymmetric theories on both group-manifold and spheres [15], based on the realisation that spheres are generalised parallelisable spaces and thus the known consistent truncations simply reduce to generalised Scherk-Schwarz reductions.

Furthermore, a generic procedure was developed in [16] where the above was generalised to truncations with any amount of supersymmetry, including non-supersymmetric truncations. The truncation is then achieved via the choice of a generalised G-

structures on the manifold $\mathcal{M}$ characterised by a set of $G-$invariant generalised tensor fields in terms of which the bosonic degrees of freedom of the theory are expanded and by keeping all possible singlets in truncating the fields, with the requirement that the intrinsic torsion of the G-structure contains only singlets.

The structure of this review is as follows. In Chapter 2 we briefly introduce the mathematical tools required for the discussion of local group manifold reductions, that is the concepts of manifolds, Lie groups and their action on manifold. We also provide a brief discussion on fibre bundle theory and conclude the chapter with a brief account of the tetrad formalism. In Chapter 3 we start by treating the $U(1)$ reduction of 5-dimensional Einstein gravity and discuss the definition of a consistent truncation. We then move onto describing the local group manifold reduction formalism and the unimodularity condition. We also give a discussion of the geometrical aspect of coset spaces and look at the maximally supersymmetric reduction of type IIB supergravity on S^5.

Finally, in Chapter 4 we begin by introducing some key concepts of complex geometry such as G-structures and torsion classes before introducing generalised geometry. We first give an overview of the key elements of the framework in the original formulation and then look at how the geometries can be extended for application to supergravity. We then look at how reductions work in the case of ordinary G-structures before concluding with the formalism for generic supersymmetric reductions of supergravity from generalised G-structures with singlet intrinsic torsion.

Chapter 2

Mathematical Preliminaries

In this chapter we shall introduce the basic mathematical tools of which extensive use will be made throughout the paper, although we shall leave the treatment of generalised geometry to the final chapter of this paper where the topic is treated.

2.1 Differentiable manifolds and Lie Groups

Given a m-dimensional differentiable manifold $\mathcal{M}$, the *tangent space* $T_q\mathcal{M}$ is defined as the set of all tangent vectors at the point $q \in \mathcal{M}$ and structurally it is a real vector space. The union of the tangent space of every point q in the manifold forms the *tangent bundle* $T\mathcal{M}$, namely $T\mathcal{M} := \bigcup_{q\in\mathcal{M}} T_q M$, which itself has the structure of a $2m$-dimensional manifold. Similarly, the *tangent vector* at a point q on the manifold $\mathcal{M}$ is a linear map $b : T_q\mathcal{M} \mapsto \mathbb{R}$ and the sent of all such co-vectors at any point on the manifold forms the dual of $T_q\mathcal{M}$, the *cotangent space* $T_q^*\mathcal{M}$. The *cotangent bundle* is naturally defined as $T^*\mathcal{M} := \bigcup_{q\in\mathcal{M}} T_q^*\mathcal{M}$ and has the structure of a $2m$-dimensional manifold [17].

Starting from two manifolds $\mathcal{M}$ and $\mathcal{N}$ and a map between them $h : \mathcal{N} \mapsto \mathcal{M}$, a natural linear map between the tangent spaces arises called the *push-forward* of h, denoted by

$$h_* : T_p\mathcal{M} \mapsto T_{h(p)}\mathcal{N}. \tag{2.1}$$

Furthermore, given a product manifold $\mathcal{M} \times \mathcal{N}$, a natural isomorphism is the following

$$\begin{aligned} T_{(p,q)}(\mathcal{M} \times \mathcal{N}) &\simeq T_p\mathcal{M} \oplus T_q\mathcal{N} \\ v &\mapsto (pr_{1*}(v), pr_{2*}(v)) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} pr_1 : \mathcal{M} \times \mathcal{N} \mapsto \mathcal{M} \qquad & pr_2 : \mathcal{M} \times \mathcal{N} \mapsto \mathcal{N} \\ (p, q) \mapsto p \qquad & (p, q) \mapsto q \end{aligned} \tag{2.3}$$

are projection maps from the product manifold into the respective component. Assigning a tangent vector $Y_p \in T_p\mathcal{M}$ to every point $p \in \mathcal{M}$ results in the *vector field* *Y*. The set that all such vector fields on a manifold form is a real vector space and is denoted by $VFld(\mathcal{M})$. Given two vector fields X and Y, a third vector field can be defined via their $commutator$ as

$$[X, Y] = X \circ Y - Y \circ X \tag{2.4}$$

where $[X, Y]$ is antisymmetric and satisfies the Jacobi identity [17].

Analogously to vector fields, assigning to every point on $\mathcal{M}$ a cotangent vector ω_q results in the *one-form* ω. Taking the dual of Eq. (2.1) gives the *pull-back* map

$$h^* : T^*_{h(p)}\mathcal{N} \mapsto T^*_p\mathcal{M} \tag{2.5}$$

defined by

$$\langle h^*g, r\rangle_p := \langle g, h_*r\rangle_{h(p)} \tag{2.6}$$

where $g \in T^*_{h(p)}$ and $r \in T_p\mathcal{M}$ and the definition is extendable to one-forms. More generally, using the tensor product operation, one can construct $tensors\ of\ type\ (s,t)$ which are elements of the tensor product space $T_p^{s,t}\mathcal{M}$ given by $[\otimes^s T_p\mathcal{M}]\otimes[\otimes^t T_p^*\mathcal{M}]$, where we have taken the tensor product of the tangent and cotangent space s and t times respectively. The idea of a *tensor field of type (s,t)* then follows. A special case of tensors are *n-forms*, defined as totally anti-symmetric $(0,n)$ tensor fields, where the symmetry property is with respect to the order of n the vector fields on which it naturally acts [17].

Another key concept is that of a *Lie group*, namely a set that is both a topological group and a differentiable manifold, with additional structure so that taking the inverse and the group product between two elements are both smooth operations. For an element g in the Lie group G, one can construct two diffeomorphisms called the *right* or *left translations of G* with the respective actions [18]

$$r_g : G \mapsto G, \qquad g' \mapsto g'g \tag{2.7}$$

$$l_g : G \mapsto G, \qquad g' \mapsto gg' \tag{2.8}$$

where $g, g' \in G$, satisfying

$$l_{g_1} \circ l_{g_2} = l_{g_1 g_2} \tag{2.9}$$

and

$$r_{g_1} \circ r_{g_2} = r_{g_2 g_1}. \tag{2.10}$$

The existence of those translations is crucial as they allow for local structures such as the tangent space to be mapped around the whole group manifold. To appreciate this, we begin by labelling a vector field X on G as *left-invariant* if $l_{g^*}X = X$ for all $g \in G$ with set of all such vector fields on G forming the vector space $L(G)$, which is closed under the commutator operation of any two elements and forms the *Lie algebra* of G. Furthermore there exists a theorem that states that the Lie algebra is isomorphic to the tangent space at the identity e of G, namely $T_e\mathcal{M}$ and as such they have the same dimensionality as vector spaces. Provided the set $\{T_1, T_2, ..., T_n\}$ forms a basis for $L(G)$ with n being the dimension of $L(G)$, then we have

$$[T_a, T_b] = \sum_{d=1}^{n} C_{ab}{}^{d} T_d \tag{2.11}$$

where $C_{ab}{}^{d}$ are the *structure constants* of $L(G)$ [18].

In theoretical physics, transformations of the space representing a system are best analysed in terms of groups, hence it is natural to study how a group acts on a given space (G-action). Hence, the *left action* of a Lie group G on $\mathcal{M}$ is a homomorphism $g \mapsto \gamma_g$ defined by the smooth map

$$\begin{aligned} \Gamma : G \times \mathcal{M} &\mapsto \mathcal{M} \\ (g, p) &\mapsto \gamma_g(p) \equiv gp. \end{aligned} \tag{2.12}$$

A similar argument would lead to the definition of a *right action*. The set of points in $\mathcal{M}$ that can be reached from p via a G-action is called the *orbit* O_p and given by

$$O_p := \{q \in \mathcal{M} \mid \exists\, g \in G \; with \; q = gp\}. \tag{2.13}$$

The *orbit space*, denoted by $\mathcal{M}/G$, is then the set of equivalence classes defined by the equivalence relation for which two points on $\mathcal{M}$ are equivalent if they lie on the same orbit. In the case of the right action of $H \subset G$ on G itself, the orbit space is the space of (left) cosets G/H [18].

Within this context, for any point p one can also define *little group* G_p as the closed subgroup of G

$$G_p := \{g \in G \mid gp = p\} \tag{2.14}$$

and also the *kernel* of a G-action as the normal subgroup of G

$$K := \{g \in G \mid gp = p \,\forall\, p \in \mathcal{M}\}. \tag{2.15}$$

G is then said to act *freely* on the manifold in the instance where every element of the manifold is moved away from itself by the group action except for the identity element e action [17, 18].

2.2 Fibre bundles

A very important role in the context of dimensional reduction is played by $bundles$ which provide a framework used to describe fields carrying more than one index, whether the index structure refers to the tensorial properties of the field with respect to the underlying spacetime or to the its symmetry transformation properties with respect to the symmetry group. While a vector-valued field maps each point $x \in \mathcal{M}$ to an element v living in a vector space V, more generally one can consider a manifold-valued field where the target spaces varies at every point in $\mathcal{M}$, resulting in a bundle of spaces $\mathcal{N}_x$ each one labelled by the manifold point x [17].

Given two topological spaces E and $\mathcal{M}$ and a continuous map $\pi : E \mapsto \mathcal{M}$, a $bundle$ is defined as the triple $(E, \pi, \mathcal{M})$, with E and $\mathcal{M}$ becoming the bundle space and base space respectively and the inverse image of the projection π being the fibre $\pi^{-1}(x)$ over $x \in \mathcal{M}$. In the case where the fibres $\pi^{-1}(x)$ are diffeomorphic to a common space F, then the bundle is referred to as a $fibre\ bundle$ and F as the $fibre$. If the fibre is a vector space, the bundle is called a $vector\ bundle$. Furthermore, given a bundle $(E, \pi, \mathcal{M})$ a $cross\text{-}section$ (or simple a $section$) is a map $s : \mathcal{M} \mapsto E$ whose image $s(x) \in E$ is contained in $\pi^{-1}(\{x\})$, the fibre. The space of al sections on E is denoted by $\Gamma(E)$. In the case of the tangent and cotangent bundle, the sections are the *vector fields* and *one-forms* respectively.

Finally, a fibre bundle $(E, \pi, \mathcal{M})$ is said to be $trivial$ is there exist a $\mathcal{M}$-isomorphism to the product bundle $(\mathcal{M} \times F, pr_1, \mathcal{M})$ where F is a general space, with the bundle being $locally\ trivial$ when the bundle map is a local isomorphism [17, 18]. Of direct relevance to our discussion are *principal bundles*, a special class of fibre bundles where the fibre is a Lie group. Firstly the notion of a G-bundle is needed, which is a bundle $(E, \pi, \mathcal{M})$ isomorphic to $(E, \rho, E/G)$with E being a right G-space and ρ the usual projection map in to the base manifold, E/G in this case. Then, a principal G-bundle is a G-bundle $(E, \pi, \mathcal{M})$ on which G acts freely, with G becoming the bundle's $structure\ group$. As an example, given a Lie group G and a closed subgroup H, we get a principal H-bundle $(G, \pi, G/H)$ via the right action of H on G, where H is the fibre [18].

2.3 Tetrad formalism

Gravitational theories are characterised by the dynamics of metric tensor $g_{\mu\nu}$. It gives a notion of distance on the manifold of interest and it is used to define the Christoffel symbols and the Riemann curvature tensor. Although pure gravitational theories of interest to us will be generally covariant, i.e. invariant under a general local differentiable transformation, the gauge theory aspect of the theories is hardly made manifest in the usual metric formalism and hence a different approach is needed in order to make this feature manifest. The tedrad formalism, or vielbein formalism, provides such a framework.

A manifold is locally at very small scales, thus we have a local Lorentz invariance. To make it manifest, we can choose a non coordinate basis in which the mnetric takes the form

$$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab} \tag{2.16}$$

with η_{ab} being the Minkowski metric. The tetrad $e^a_\mu(x)$ is a object which transforms the fundamental representation of the Lorentz gauge group as well as a form under spacetime transformation

$$\delta_\xi e^a_\mu(x) = (\xi^\rho \partial_\rho)\, e^a_\mu + (\partial_\mu \xi^\rho)\, e^a_\rho. \tag{2.17}$$

The tetrad almost look like a gauge field connection, except that the latter would transform under the adjoint representation of the gauge group [19].

We can define the *spin connection* ω_μ^{ab}, which defines the action of the covariant derivative on spinors ψ

$$D_\mu \psi = \partial_\mu \psi + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\psi \tag{2.18}$$

with $1/4\Gamma_{ab} = 1/2[\Gamma_a, \Gamma_b]$ the generator of the action of the group on spinors and Γ_a being the gamma matrices. The $torsion$ is then defined in terms of the tetrad as

$$T^a_{[\mu\nu]} \equiv 2D_{[\mu}e^a_{\nu]} = 2\partial_{[\mu}e^a_{\nu]} + 2\omega^{ab}_{[\mu}e^b_{\nu]} \tag{2.19}$$

In the absence of fermions, this can be made to vanish as it is the case in general relativity. The field strength $R^{ab}_{\mu\nu}(\omega)$ is then given by

$$R^{ab}_{\mu\nu}(\omega) = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^{ab}_\mu \omega^{bc}_\nu - \omega^{ac}_\nu \omega^{\omega b}_\mu \tag{2.20}$$

which is connected to the usual Riemann $R^\mu_{\nu\rho\sigma}(\Gamma(e))$ tensor by [19]

$$R^{ab}_{\rho\sigma}(\omega(e)) = e^a_\mu (e^{-1})^{\nu b} R^\mu_{\nu\rho\sigma}(\Gamma(e)). \tag{2.21}$$

Chapter 3

Consistent truncations

3.1 The Kaluza-Klein S^1 reduction and the consistency issue

We begin the chapter by studying the original Kaluza-Klein (KK) reduction on a circle S^1 [4, 5], hence we consider the dimensional reduction of a pure gravity Einstein theory in $(D+1)$ dimensions, whose dynamics is contained in the Lagrangian $\mathcal{L} = \sqrt{-\hat{g}}\hat{R}$ which enters the Einstein-Hilbert action as

$$S = -\frac{1}{2\pi\kappa^2}\int \mathrm{d}x^{D+1}\sqrt{-\hat{g}}\hat{R}, \tag{3.1}$$

where hatted quantities and indices refer to the higher dimensional space and we are using a mostly . The reduction is a consequence of the compactification of one of the spatial coordinates, denoted by y, into a circle S^1 of radius L. The higher-dimensional theory describes the evolution of a symmetric spin 2 tensor, i.e. the metric tensor $g_{\hat{\mu}\hat{\nu}}(x, y)$ and we assume the classical solution of the equation of motion resulting from varying Eq. (3.1) in the case of $D = 4$ to be the product space $M^4 \times S^1$ parameterised by coordinates $x^{\hat{\mu}} = (x^\mu, y)$ with $0 \leq \hat{\mu} \leq 4$ and $0 \leq \mu \leq 3$, where M^4 is the 4 dimensional Minkowski spacetime and $0 \leq y \leq 2\pi L$ [4].

Due to the periodicity of the compact dimension, the metric can be expanded following Fourier's theorem as

$$\hat{g}_{\hat{\mu}\hat{\nu}}(x,y) = \sum_n g^{(n)}_{\hat{\mu}\hat{\nu}}(x) e^{iny/L} \tag{3.2}$$

obtaining the so called Kaluza-Klein tower of modes each labelled by n. It turns out [20] that all modes with $n \neq 0$ correspond to massive fields in the reduced theory with the mass being inversely proportional to the size of the compact dimension L. Since this is usually assumed to be of the order of Plank length in this context, it results in fields with masses of the order of the Plank mass, hence a truncation to the massless sector is usually taken, corresponding to the higher dimensional metric being independent of the compact dimension y, namely we take $\hat{g}_{\hat{\mu}\hat{\nu}}(x,y) = g^{(0)}_{\hat{\mu}\hat{\nu}}(x)$, with components [21]

$$\hat{g}_{\hat{\mu}\hat{\nu}}(x) = \begin{pmatrix} \hat{g}_{\mu\nu}(x) & \hat{g}_{\mu z}(x) \\ \hat{g}_{\mu z}(x) & \hat{g}_{zz}(x) \end{pmatrix} . \tag{3.3}$$

Although one could take these components to be the D-dimensional fields of the reduced theory and carry on with the computation of the action Eq. (3.1), there is a parameterization [22] that makes the resulting equation of motions much neater and the content of the lower dimensional theory more explicit, namely the non-linear reduction ansatz

$$\hat{g}_{\hat{\mu}\hat{\nu}}(x) = \phi^{-1/3}(x) \begin{pmatrix} g_{\mu\nu}(x) + \kappa^2 \phi(x) A_\mu(x) A_\nu(x) & \kappa\phi(x) A_\mu(x) \\ \kappa\phi(x) A_\nu(x) & \phi(x) \end{pmatrix} , \tag{3.4}$$

where the fields $g_{\mu\nu}(x)$, $A_\mu(x)$ and $\phi(x)$ are the zero modes of their Fourier expansion in the sense of Eq. (3.2). It is a well known-result in the literature [22] that when computing the Ricci curvature $\hat{R}$ resulting from the 5-dimensional metric given in Eq. (3.4) and its determinant $\hat{g}$, one can integrate out the y coordinate as no quantity depends on it in the action, thus obtaining a 4-dimensional effective action of the form

$$S = \int d^4x\sqrt{-g}\Big[-\frac{1}{\kappa^2}R(g) - \frac{1}{4}\phi F_{\mu\nu}F^{\mu\nu} + \frac{1}{6\kappa^2\phi^2}\partial^\mu\phi\partial_\mu\phi\Big], \tag{3.5}$$

where $g = det(g_{\mu\nu})$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with the indices raised and lowered by $g_{\mu\nu}$.

The 5-dimensional theory was originally invariant under a general coordinate transformation $x^{\hat{\mu}} \mapsto x^{\hat{\mu}} - \sigma^{\hat{\mu}}(x^{\hat{\nu}})$. If we also expand the transformation parameter around the compact dimension y and retain only the $n = 0$ mode, i.e. $\sigma^{\hat{\mu}}(x^{\hat{\nu}}) = \sigma^{\hat{\mu}}(x^\mu)$, the remnants of the higher dimensional symmetry in the effective action of Eq. (3.5) appear as invariance under the field transformations with parameter $\sigma^\mu(x)$

$$\delta g_{\mu\nu} = \partial_\mu\sigma^\rho g_{\rho\nu} + \partial_\nu\sigma^\rho g_{\rho\mu} + \sigma^\rho\partial_\rho g_{\mu\nu} \tag{3.6}$$

$$\delta A_\mu = \partial_\mu\sigma^\rho A_\rho + \sigma^\rho\partial_\rho A_\mu \tag{3.7}$$

$$\delta\phi = \sigma^\rho\partial_\rho\phi \tag{3.8}$$

and under a local gauge transformation

$$\delta A_\mu = \kappa^{-1}\partial_\mu\sigma^4 \tag{3.9}$$

with parameter $\kappa^{-1}\sigma^4(x)$ [23]. Hence we note that the reduced action describes the dynamics of a spin 2 field, $g_{\mu\nu}$, a spin 1 gauge field, A_μ, and a scalar field ϕ, thus providing a mechanism in which gravity and electromagnetism can be embedded in a purely geometrical theory in higher dimensions, at the expense of introducing a scalar field. This was Klein's revolutionary idea [5].

Although one could be tempted to set the scalar field to a constant value ($\phi = 1$ with the parameterization used above) and recover the Einstein-Maxwell action from pure gravity in 5 dimensions, there would be an issue of consistency arising from doing so. In other words, if we compute the equation of motion (e.o.m.) obtained from varying Eq. (3.1), one obtains the 5-dimensional Einstein's field equations

$$\hat{R}_{\hat{\mu}\hat{\nu}} - \frac{1}{2}\hat{g}_{\hat{\mu}\hat{\nu}}\hat{R} = 0. \tag{3.10}$$

Focusing on the yy or 44 component, this would reduce to an equation of motion for the scalar ϕ [24]

$$\Box(ln\phi) = \frac{3}{4}\kappa^2\phi F_{\mu\nu}F^{\mu\nu}, \tag{3.11}$$

hence setting $\phi = constant$ would ultimately result in the vanishing of the Yang-Mills term $F_{\mu\nu}F^{\mu\nu}$, thus a truncation of the scalar is prevented by the details of the interactions between the lower dimensional fields [21]. In this case consistency is restored by retaining the scalar field as a dynamical degree of freedom and recognising that an S^1 reduction of 5-dimensional pure gravity and the subsequent truncation to the massless sector yields an Einstein-Maxwell-scalar system [23].

Nevertheless one may ask whether the truncation to the massless sector is consistent or not: as it turns out, such truncation is consistent so long as one truncates out all the non-zero modes. The reason is that the Fourier coefficients in Eq. (3.2) are $U(1)$ representations, with the $n = 0$ mode being the $singlet$ representation. Hence by keeping only the singlets, the truncation is guaranteed to be consistent [21].

Although the following sections will explore generalisations of Kaluza's key idea to theories with more than 5-dimension and beyond Einstein's gravity, the key common features among them will be to consider a Lagrangian $\mathcal{L}$ in a given number of spacetime dimensions which are then reduced via a dimensional reduction while keeping the number of degrees of freedom unvaried by truncating the field content to a subset, usually the massless sector. This differs from a pure compactification where the extra dimensions are traded for an infinite number of degrees of freedom. There are other instances in which the truncation is achieved via the introduction of constraints [see 25], however in this paper we will only deal with truncations of the KK type.

As was the case for the reduction discussed above, there is an issue of $consistency$ of the truncation that arises from such procedure. In a stronger sense than that explored in the S^1 case, in order for a truncation to be consistent, one has to ensure that the implementation of the truncation at the level of the variational principle is in agreement with that at the level of the e.o.m.. In other words, finding the e.o.m. from $\mathcal{L}$ and truncating them must yield the same result as when we first truncate $\mathcal{L}$ and find the e.o.m. from $\mathcal{L}_R$ [26]

$$\left(\frac{\delta\mathcal{L}}{\delta\Phi}\right)_R = \frac{\delta\mathcal{L}_R}{\delta\Phi} \tag{3.12}$$

where Φ indicates any type of field content of the theory. It is this latter criterion that is used in the majority of the literature on the topic [23, 24] and as such it will be adopted in the rest of the paper.

3.2 Local Group Manifold Reductions

Dimensional KK reduction of pure gravity on S^1 is the simplest among its kind. In fact it belongs to a broader category of consistent truncations, named *local group manifold reductions*, in which one starts with a theory in a curved $(D + E)$ dimensional space that is invariant under the action of an arbitrary E-dimensional Lie group G, subject to some conditions. Then, assuming a phenomenon of compactification of the E dimensions is achieved in some way, a richer D-dimensional theory emerges in which the extra E dimensions appear as massive multiplets of the Lie group G coupled to non-Abelian Yang-Mills fields [27].

From a geometric point view, the starting point is a D-dimensional Riemannian spacetime manifold over which a principal fibre bundle is constructed. Then a finite-dimensional E-dimensional Lie group is the structure group of the bundle and the existence of a Lie algebra valued connection 1-form $\hat{\Omega}$ on the bundle is assumed, the covariant derivative of which gives the curvature 2-form and is viewed as a generalised Y-M field tensor [8].

3.2.1 The formalism

In the tetrad formalism, where the Greek indices refer to the curved space and Latin ones to the flat tangent space, the Einstein-Hilbert action is given by

$$S = -\frac{1}{4\kappa^2}\int \mathrm{d}^D x \int \frac{d^E y}{\rho(E)} \hat{V}\hat{R}(\hat{\Omega}) \tag{3.13}$$

where κ is such that $\kappa^2/4\pi$ is the D-dimensional Newton's constant, $\hat{V}$ is he determinant of the vielbein which in this formalism represent the degrees of freedom in the theory and is defined by $\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{V}_{\hat{\mu}}^{\hat{m}}\hat{V}_{\hat{\nu}}^{\hat{n}}\hat{\eta}_{\hat{m}\hat{n}}$, with $\rho(E)$ being the internal space's invariant volume that can be written in terms of a measure $T(y)$ as $\rho(E) = \int d^E y T(y)$ and $\hat{R}(\hat{\Omega})$ is the Ricci scalar built out of the connection 1-form $\hat{\Omega}_{\hat{\mu}\hat{s}}^{\hat{r}}$ [10]. This action is invariant under general coordinate transformation $x^{\hat{\mu}} \mapsto x^{\hat{\mu}} - \sigma^{\hat{\mu}}(x^{\hat{\rho}})$ with the Lie algebra given by $[\delta_{\sigma_1}, \delta_{\sigma_2}] = \delta_{\sigma_3}$ and

$$\sigma_3^{\hat{\mu}}(x,y) = \sigma_2^{\hat{\rho}}(x,y)\partial_{\hat{\rho}}\sigma_1^{\hat{\mu}}(x,y) - \sigma_1^{\hat{\rho}}(x,y)\partial_{\hat{\rho}}\sigma_2^{\hat{\mu}}(x,y). \tag{3.14}$$

This transformation acts on the vielbein as

$$\delta\hat{V}_{\hat{\mu}}^{\hat{r}} = \sigma^{\hat{\rho}}\partial_{\hat{\rho}}\hat{V}_{\hat{\mu}}^{\hat{r}} + \partial_{\hat{\mu}}\sigma^{\hat{\rho}}\hat{V}_{\hat{\rho}}^{\hat{r}} \tag{3.15}$$

and due to local Lorentz invariance in the $(D+E)$-dimensional space [9] one can choose a triangular parameterization for the vielbein as

$$\hat{V}_{\hat{\mu}}^{\hat{r}} = \begin{pmatrix} \theta^{\gamma}V_{\mu}^{r} & 2\kappa A_{\mu}^{\alpha}\Phi_{\alpha}^{a} \\ 0 & \Phi_{\alpha}^{a} \end{pmatrix}, \tag{3.16}$$

where $\theta = det(\Phi_{\alpha}^{a})$, γ is a free-parameter to be chosen representing Weyl invariance of $\hat{V}_{\hat{\mu}}^{\hat{r}}$ and all the field components depend on the full set of coordinates (x^{μ}, y^{α}).

At this point, one could produce a normal mode expansion [1] of the vielbein components [20] in terms of the compact space coordinates and then truncate the field content and the parameter to the massless sector by retaining only the x^{α}-

[1] The mode expansion will be explained in more details in Section 3.3.1.

dependence as it was done for the S^1 reduction, thus we have $\hat{V}_{\hat{\mu}}^{\hat{r}}(x^\mu, y^\alpha) = \hat{V}_{\hat{\mu}}^{\hat{r}}(x^\mu)$ and $\sigma^{\hat{\mu}}(x^\mu, y^\alpha) = \sigma^{\hat{\mu}}(x^\mu)$. In this case, from Eq. (3.15) one would find that the transformation properties of the components are respectively that of a D-dimensional vielbein V_μ^r, $E(E+1)/2$ scalars Φ_α^a and finally E vector fields A_α^μ corresponding to gauge fields of $U(1)^E$ invariance of theory [10]. In fact, proceeding through with the computation and by choosing $\gamma = -(D-2)^{-1}$, the action in Eq. (3.13) reduces to

$$S = \int \mathrm{d}^D x V \Big[-\frac{1}{4\kappa^2} R - \frac{1}{4}\theta^{2/(D-2)} F^{\mu\nu\alpha} F_{\mu\nu}^\beta h_{\alpha\beta} - \frac{1}{16\kappa^2} g^{\rho\lambda}\partial_\rho h_{\alpha\beta}\partial_\lambda h^{\alpha\beta} + \frac{1}{4\kappa^2(D-2)\theta^2} g^{\rho\lambda}\partial_\rho\theta\partial_\lambda\theta \Big] \tag{3.17}$$

where $h_{\alpha\beta} = \Phi_\alpha^a \delta_{ab} \Phi_\beta^b$ is the curved metric in the compact space[2] and $F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha$ [10].

A more interesting scenario is that in which the fields are given a y^α-dependence of a simple enough form such that to allow the y dependence to ultimately cancel in the action Eq. (3.13). It turns out that for a pure gravity theory it is sufficient to specify the y-dependence of the transformation parameter $\sigma^{\hat{\mu}}(x, y)$ in order to fully specify the type of reduction, which we take as

$$\sigma^\mu(x, y) = \sigma^\mu(x) \tag{3.18}$$

$$\sigma^\alpha(x, y) = \left[T^{-1}(y)\right]^\alpha_\beta \sigma^\beta(x). \tag{3.19}$$

The Lie Algebra $[\delta_{\sigma_1}, \delta_{\sigma_2}] = \delta_{\sigma_3}$ now has

$$\sigma_3^\mu(x) = \sigma_2^\rho(x)\partial_\rho\sigma_1^\mu(x) - \sigma_1^\rho(x)\partial_\rho\sigma_2^\mu(x) \tag{3.20}$$

for two D-spacetime general coordinate transformations,

$$\sigma_3^\alpha(x) = -\sigma_1^\rho(x)\partial_\rho\sigma_2^\alpha(x) \tag{3.21}$$

[2]Here δ_{ab} is the Euclidean flat metric, hence Φ_α^a is the vielbein for the compact E-dimensional space.

for σ_1 a spacetime transformation and $\sigma_2(x)$ an internal transformation and finally

$$\sigma_3^\gamma(x) = C_{\alpha\beta}{}^\gamma \sigma_1^\alpha(x)\sigma_2^\beta(x) \tag{3.22}$$

for two internal space transformations, with U matrices chosen such that

$$C_{\alpha\beta}{}^\gamma \equiv \left(T^{-1}\right)_\alpha^\delta \left(T^{-1}\right)_\beta^\epsilon \left(\partial_\epsilon T_\delta^\gamma - \partial_\delta T_\epsilon^\gamma\right) \tag{3.23}$$

are constant coefficients [10, 28].

By choosing the y coordinates to parameterise the manifold of an E-dimensional Lie group G, the above is satisfied and a representation of the group generators is given by

$$e_\alpha(y) = (T^{-1}(y))_\alpha^\beta \partial_\beta \tag{3.24}$$

so that $[e_\alpha, e_\beta] = C_{\alpha\beta}{}^\gamma e_\gamma$. This choice would correspond to selecting a subalgebra of the set of isometries of the higher dimensional theory comes equipped with. We are left to specify how the fields themselves depend on the y coordinates which according to [10] should be chosen as

$$V_\mu^r(x,y) = V_\mu^r(x) \tag{3.25}$$

$$A_\mu^\alpha(x,y) = \left[T^{-1}(y)\right]_\beta^\alpha A_\mu^\beta(x) \tag{3.26}$$

$$\Phi_\alpha^a(x,y) = T_\alpha^\beta(y)\Phi_\beta^a(x). \tag{3.27}$$

Then under the symmetries Eqs. (3.20) to (3.22), factoring out the y-dependence they transform as

$$\delta\Phi_\alpha^a(x) = C_{\alpha\beta}{}^\gamma \sigma^\beta(x)\Phi_\gamma^a(x) \tag{3.28}$$

$$\delta\Phi_a^\alpha(x) = C_{\beta\gamma}{}^\alpha \sigma^\beta(x)\Phi_a^\gamma(x) \tag{3.29}$$

$$\delta A_\mu^\alpha(x) = \frac{1}{2k}\partial_\mu\sigma^\alpha(x) + C_{\beta\gamma}{}^\alpha \sigma^\beta(x) A_\mu^\gamma(x) \tag{3.30}$$

$$\delta V_\mu^r(x) = 0, \tag{3.31}$$

hence from Eq. (3.30) we recognise the the transformation of E gauge fields $A^{\alpha}_{\mu}(x)$ of the non-Abelian Lie group G with structure constants $C_{\beta\gamma}{}^{\alpha}$ [10].

One would then proceed with the computation of the quantities that enter the action. It turns out [10] that $\hat{R}$ is y-independent and that $\hat{V} = T(y)\theta^{\gamma D+1}V$, with $T(y) = det(T^{\alpha}_{\beta}(y))$ and $V = detV^{r}_{\mu}$, where the $T(y)$ gives the integration measure for the compact space and cancels in the action, which then becomes [9]

$$S = \int \mathrm{d}^D x V \left\{ -\frac{1}{4\kappa^2}R - \frac{1}{4}\theta^{2/(D-2)}F^{\mu\nu\alpha}F^{\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{16\kappa^2}g^{\rho\lambda}\mathcal{D}_{\rho}h_{\alpha\beta}\mathcal{D}_{\lambda}h^{\alpha\beta} \right. \\ \left. + \frac{1}{4\kappa^2(D-2)\theta^2}g^{\rho\lambda}\partial_{\rho}\theta\partial_{\lambda}\theta - \frac{1}{16\kappa^2}\theta^{-2/(D-2)}C_{\beta\gamma}{}^{\alpha}\left[2C_{\alpha\iota}{}^{\beta}h^{\gamma\iota} + C_{\epsilon\iota}{}^{\delta}h_{\alpha\delta}h^{\beta\epsilon}h^{\gamma\iota}\right] \right\}, \tag{3.32}$$

where $F^{\alpha}_{\mu\nu} = \partial_{\mu}A^{\alpha}_{\nu} - \partial_{\nu}A^{\alpha}_{\mu} - 2\kappa C_{\beta\gamma}{}^{\alpha}A^{\beta}_{\mu}A^{\gamma}_{\nu}$ is the non Abelian strength field for G and $\mathcal{D}_{\mu}\Phi^{a}_{\alpha} = \partial_{\mu}\Phi^{a}_{\alpha} - 2\kappa C_{\alpha\beta}{}^{\gamma}A^{\beta}_{\mu}\Phi^{\gamma}_{a}$ is the covariant derivative with respect of the Lie group [9].

In the effective action resulting from the truncation in Eq. (3.32) we can distinguish four different terms: the first describes a $(D \times D)$ symmetric massless tensor field $g_{\mu\nu}(x)$, the second is a Yang-Mills term for the gauge field A^{α}_{μ} coupled to gravity and the last three terms describe $E(E+1)/2$ scalar fields $h_{\alpha\beta}$ together with their self interactions and their interactions with gravity and the Yang-Mills fields, which are governed by the choice of the Lie group G [9].

One would need to show that the higher dimensional action is invariant under transformations in Eq. (3.19) and this restricts the type of gauge group one can choose for the reduction to Lie groups which satisfy the condition $C_{\alpha\beta}{}^{\alpha} = 0$. A proof will be presented in Section 3.2.2 by using the metric formalism. Furthermore the potential term in Eq. (3.32) needs to be unbounded from below which imposes further conditions in the choice of the gauge group, rendering ”flat groups” the only viable choice in order to obtain a consistent truncation [10].

Finally one could examine how the procedure just described changes if the pure gravity action in Eq. (3.13) is enriched by introducing matter multiplets of the Lie group G.[3] This becomes of interest in $11-$ or $10-$dimensional supersymmetric theories of gravity where the presence of matter multiplets is required by the symmetries of the theory themselves [27]. Furthermore such theories seems to provide a mechanism under which a spontaneous compactification is admitted by the field equations [24].

3.2.2 Consistency of group reductions: unimodularity condition

Here we present the proof for the unimodularity condition for the group G as given by Pons and Talavera in [25], expressed in a more geometrical language which will also be used in Chapter 4. We begin with the same setup as the previous section, namely a $(D+E)$-dimensional Lorentzian spacetime invariant under a E-dimensional group of isometries generated by the set[4] of Killing vector fields $\mathbf{K}_{a'}$ satisfying the Lie algebra $[\mathbf{K}_{a'}, \mathbf{K}_{b'}] = C_{a'b'}{}^{c'}\mathbf{K}'_c$, which in components are $\mathbf{K}_{'a} = K^{\alpha}_{a'}(x,y)\partial_\alpha$. We choose a set of E independent vector fields $\mathbf{L}_{a'} = L^{\alpha}_{a'}(x,y)\partial_\alpha$ which are left-invariant under the Lie algebra and satisfy $[\mathbf{L}_{a'}, \mathbf{L}_{b'}] = C_{a'b'}{}^{c'}\mathbf{L}_{c'}$ and the corresponding dual 1-form $\omega^{a'} = \omega^{a'}_{\alpha}(x,y)\mathrm{d}x^\alpha$, so that $\omega^{b'}\cdot\mathbf{L}_{a'} = \delta^{b'}_{a'}$ [29].

In this section the metric components $\hat{g}_{\hat{\mu}\hat{\nu}}$ are parameterised using the mixed base $\{\mathrm{d}x^\mu, \omega^{a'}\}$ as

$$\hat{\mathbf{g}} = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\upsilon + g_{a'b'}\left(A^{a'}_{\mu}\mathrm{d}x^\mu + \omega^{a'}\right)\left(A^{b'}_{\nu}\mathrm{d}x^\nu + \omega^{b'}\right). \tag{3.33}$$

where in principle the fields all have both and x and y dependence, with $det(\mathbf{g}) = det(g_{\mu\nu})det(g_{a'b'})det(\omega^{a'}_{\alpha})^2$. However, requiring the metric be invariant under the set of isometries, condition expressed via the vanishing of the Lie derivative with respect to $\mathbf{K}_{a'}$ of $\mathbf{g}$

$$\mathfrak{L}_{\mathbf{K}_{a'}}(\mathbf{g}) = 0 \tag{3.34}$$

[3] The details have being worked out by Scherk and Schwarz in [10].
[4] Primed Latin indices refer to the basis chosen for the Lie algebra and not to the vielbein.

results in $g_{a'b'}$ and $g_{\mu\nu}$ being y-independent, while the of $A^{a'}_{\mu}$ is dictated by

$$\left(\partial_\mu \mathbf{K}^{\alpha}_{a'}\right)\partial_\alpha = -\mathbf{K}_{a'}\left(A^{b'}_{\mu}\right)\mathbf{Y}_{b'} \tag{3.35}$$

coming from $\mathfrak{L}_{\mathbf{K}_a}\left(A^b_v \mathrm{d}x^v + \omega^b\right) = 0$. This will also be taken to be y-independent in the truncation which follows, hence it is appropriate to take $\mathbf{K}_{a'}$, $\mathbf{Y}_{a'}$ and $\omega^{a'}$ also y-independent. Than any p-forms expressed in terms of the mixed basis with y-independent components will be automatically invariant under the group of isometries [25].

If we denote the field content of Eq. (3.33) with Φ and the first a second derivative by Φ_μ and $\Phi_{\mu\nu}$, the action is written as

$$S = \int d^D x d^E y \mathcal{L}(\Phi, \Phi_\mu, \Phi_{\mu\nu}) \equiv \int d^D x d^E y |\omega| \tilde{\mathcal{L}}(\Phi, \Phi_\mu, \Phi_{\mu\nu}) \tag{3.36}$$

where $\omega^{a'}$ and $\mathbf{Y}_{b'}$ are not considered fields variables and $|\omega| = det(\omega^{a'}_{\alpha})$. Ultimately we define the truncated theory by the reduced Lagrangian

$$\mathcal{L}_R(\Phi, \Phi_\mu, \Phi_{\mu\nu}) = \tilde{\mathcal{L}}(\Phi, \Phi_\mu, \Phi_{\mu\nu}; \mathbf{Y}_{a'}\Phi = 0, \mathbf{Y}_{a'}\mathbf{Y}_{b'}\Phi = 0). \tag{3.37}$$

The reduced Euler-Lagrange equations obtained from the variation of Eq. (3.36) by setting the y-derivatives of the field to zero in the higher dimensional equations are finally given by [26]

$$\left(\frac{\delta\tilde{\mathcal{L}}}{\delta\Phi}\right)_R = \left\{\frac{\delta\mathcal{L}_R}{\delta\Phi} - C_{a'c'}{}^{c'}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\mathbf{Y}_{a'}\Phi}\right)_R + \frac{1}{2}C_{a'c'}{}^{c'}C_{b'd'}{}^{d'}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\mathbf{Y}_{a'}\mathbf{Y}_{b'}\Phi}\right)_R\right\}. \tag{3.38}$$

Recalling from Eq. (3.12) the statement of consistency for the truncation is the following

$$\left(\frac{\delta\tilde{\mathcal{L}}}{\delta\Phi}\right)_R = \frac{\delta\mathcal{L}_R}{\delta\Phi} \tag{3.39}$$

which is satisfied where the structure constants obey the condition $C_{a'c'}{}^{c'} = 0$. Lie algebras which belong to this category are the Abelian Lie algebras, semi-simple Lie algebras and compact Lie algebras, and as such they provide the starting point in obtaining consistent truncations via local group reductions [25].

The analysis extends to the case where fermion are considered in the starting Lagrangian: one would define the vielbein basis $\hat{e}^{\hat{m}}_{\hat{\mu}}$ and choose the components so that the Killing condition is satisfied, namely $\mathfrak{L}_{\mathbf{K}_{a'}}\hat{e}^{\hat{m}}_{\hat{\mu}} = 0$, and y-independent spinors. Then the procedure discussed above would be implemented with the Killing conditions on spinor fields and the consistency requirement results in the same condition on the structure constants [26].

For the type of truncations seen in this section, it was essential that the generators of the Lie group under which the theory was reduced were independent Killing vector fields. Group theoretical arguments cannot be used in cases where the Killing vectors are linearly dependent, as it is the case for spheres reduction [29] and there are only a limited set of such truncations which are known to be consistent. However, as we will see in Chapter 4, generalised geometry can be used to define consistent generalised Scherk-Schwartz reductions on spaces such as spheres.

3.3 Coset space dimensional reductions

As anticipated, group manifold reductions are not the only possibility when it comes to dimensional reductions. There are many other internal manifold that could be used as the compact space and would deliver different fields content in the truncated theory. One such case are coset spaces G/H, of which unit spheres S^n are the most common examples: they are coset spaces $SO(n+1)/SO(n)$. One reason one would perform a dimensional reduction of a theory on the coset space G/H rather then on the group manifold G is that to obtain the same lower dimensional gauge field content, a coset space reduction would require less extra dimensions. More precisely if a G reduction requires $dimG$ extra dimensions, a G/H reduction would require $dimG - dimH$ extra dimensions [21].

As explained below, the reduction procedure follows a similar path of that taken in group manifold reductions, involving a Fourier mode expansion of the metric (and any possible higher dimensional matter fields) around the compact space coordinates using a complete set of eigenfunctions followed by a truncation to a subset of such fields, usually the massless sector. This is where the consistency issue comes in: it is generally not possible to achieve consistency for coset space reductions due to the nature of the interactions between modes resulting from the reduction. More explicitly, products of zero modes eigenfunctions will generally generate non-zero modes so that setting the latter to zero produces inconsistency [21].

On the other hand there have been cases in the literature where consistent reduction ansatz for supergravity theories have been found. Some examples include S^5 reduction of type IIB supergravity [11], S^3 reduction for the NS-NS sector of type IIA and type IIB supergravity [30], the S^4 reduction of 11D supergravity [31] and of type IIA supergravity [32], to list a few.

3.3.1 G/H coset space geometry and dimensional reduction

A metric space that admits the transitive action of a group G as its isometry is called $homogeneous$. Furthermore, an homogeneous space is a coset space G/H. For G a Lie group, coset manifolds with a Riemannian structure are obtained. One can split $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} \bigoplus \mathfrak{K}$ where $\mathfrak{K}$ is the coset generator subalgebra. As Lie coordinates we can choose $y^{a'}, x^{i'}$, with y parameterising the coset manifold with respect to H. We label the coset representative by $L_y = exp[y^a K_a]$ and the whole geometry on G/H can be built out of coset representative via left multiplication by $g \in G$ [33].

For K compact or semi-simple, the coset space is reductive [34] and the structure constants $C_{a'b'}{}^{c'}$ can always be made antisymmetric. The there are two left invariant metric that can be chosen on the coset space

$$g_{\alpha\beta} = \delta_{ab} V_\alpha^a V_\beta^b \tag{3.40}$$

$$g_{\alpha\beta} = \gamma_{ab} V_\alpha^a V_\beta^b \tag{3.41}$$

where $V^a = V_\alpha{}^a(y)dy^\alpha$ is the Vielbein 1-form, γ_{ab} is the Killing metric obtained from the structure constants and restricted to G/H.

Consider a $(D+E)$-dimensional spacetime which undergoes compactification of E dimensions to a ground state spacetime which can locally be written as a product space $M \times G/H$ which is invariant under the action of a Lie group G which is an internal symmetry of the theory. The space is parameterised locally by a set of coordinates (x^μ, y^α). Then the lower D-dimensional theory can be obtained by expanding higher dimensional fields, the metric components and any matter fields, in terms of a complete set of harmonics on G/H, with the coefficients transforming under the action of G in different representations [27].

The expansion is slightly more involved than the case where the internal space is the entire group manifold G. When expanding over the whole group G in terms of coefficients, a sum over all the irreducible n-dimensional representations of G, $D^n{}_{pq}$, and over all the d_n-dimensional matrix components indices is carried out, hence a function is expanded as

$$\phi(g) = \sum_n \sum_{p,q} \sqrt{d_n} D^n_{pq}(g) \phi^n_{qp} \tag{3.42}$$

where the coefficients are obtained by projecting

$$\phi^n_{pq} = \frac{\sqrt{d_n}}{V_G} \int_G d\mu D^n_{pq}\left(g^{-1}\right) \phi(g). \tag{3.43}$$

Looking now at the coset space expansion, for a set of fields $\psi_i(x,y)$ which transform under a combined left translations $y \mapsto y'$ a tangent space rotation h as

$$\psi_i(x,y) = \mathbb{D}_{ij}(x,y')\psi_j(x,y) \tag{3.44}$$

with $\mathbb{D}_{ij}$ being a $d_{\mathbb{D}}$-dimensional representation of H, the set of unitary representation is restricted to those that satisfy [27]

$$\mathbb{D}^n(hg) = \mathbb{D}(h) D^n(g). \tag{3.45}$$

Then taking $\mathbb{D}$ irreducible and taking into account that the $\mathbb{D}(h)$ may be contained more than once in $D^n(g)$ by introducing an extra labelζ [27], the expansion is given by

$$\psi_i(x,y) = \sum_n \sum_{\zeta,q} \sqrt{\frac{d_n}{d_{\mathbb{D}}}} D^n_{i\zeta,q}\left(L_y^{-1}\right)\psi^n_{q\zeta}(x), \tag{3.46}$$

where n labels the excitations. Finally expansion coefficients are obtained via integration over G/H

$$\psi^n_{q\zeta} = \frac{1}{V_K}\sqrt{\frac{d_n}{d_{\mathbb{D}}}} \int_{G/H} d\mu D^n_{q,i\zeta}\left(L_y\right)\psi_i\left(L_y^{-1}\right) \tag{3.47}$$

and transform as

$$\psi'^n_{p\zeta}(x) = D^n_{pq}(g)\psi^n_{q\zeta}(x). \tag{3.48}$$

One would then proceed by expanding all the fields in the higher dimensional Lagrangian in a similar fashion thus obtain a lower dimensional formulation which is so far consistent with he higher dimensional theory. However, a truncation on the expansions is usually carried out and only a subset of the field excitations is kept. There is not a systematic approach [6] that can be taken that is guaranteed to produce a consistent truncation in this case and one is left to guess what possible truncation ansatz could produce a truncation. We next look at one such example which has been of major interest due to its application in the field of the AdS-CFT correspondence [12].

3.3.2 SO(6) reduction of type IIB supergravity on S^5

Superstring theories of type IIB require a spacetime with 10 dimensions and have 2 supersymmetrisies in the 10-dimensional sense of opposite chirality. In the low energy limit, these are known to reduce to the chiral $\mathcal{N} = (2,0)$ supergravity in 10D [1]. When looking at compactifications, one usually sets the fermionic fields to zero

[35]. Thus looking at the bosonic sector alone, the Lagrangian [36] becomes

$$\begin{aligned}\mathcal{L}_{10}^{\mathrm{IIB}} =& \hat{R}\hat{*}\mathbb{1} - \frac{1}{2}\hat{*}d\hat{\phi}\wedge d\hat{\phi} - \frac{1}{2}\mathrm{e}^{2\hat{\phi}}\hat{*}d\hat{\chi}\wedge d\hat{\chi} - \frac{1}{4}\hat{*}\hat{H}_{(5)}\wedge\hat{H}_{(5)} \\ &- \frac{1}{2}\mathrm{e}^{-\hat{\phi}}\hat{*}\hat{F}^2_{(3)}\wedge\hat{F}^2_{(3)} - \frac{1}{2}\mathrm{e}^{\hat{\phi}}\hat{*}\hat{F}^1_{(3)}\wedge\hat{F}^1_{(3)} - \frac{1}{2}\hat{B}_{(4)}\wedge d\hat{A}^1_{(2)}\wedge d\hat{A}^2_{(2)}\end{aligned} \tag{3.49}$$

where $\hat{*}$ denotes the 10-dimensional Hodge dual, $\wedge$ denotes the wedge product, $\hat{H}_{(5)} = d\hat{B}_{(4)} - \frac{1}{2}\hat{A}^1_{(2)}\wedge d\hat{A}^2_{(2)} + \frac{1}{2}\hat{A}^2_{(2)}\wedge d\hat{A}^1_{(2)}$ is the self dual 5-form[5] field strength while $\hat{F}^2_{(3)} = d\hat{A}^2_{(2)}$ and $\hat{F}^1_{(3)} = d\hat{A}^1_{(2)} - \hat{\chi}d\hat{A}^2_{(2)}$ the Ramond-Ramond 3-form and the Neveu-Schwarz-Neveu-Schwarz 3-form field strenghts. The Lagrangian also describes a dilaton field $\hat{\phi}$ and a scalar $\hat{\chi}$ [36]. The equations of motion arising form the variational principle were found in [35]

$$\begin{aligned}\hat{R}_{\hat{\mu}\hat{\nu}} =& \frac{1}{2}\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi} + \frac{1}{2}\mathrm{e}^{2\hat{\phi}}\partial_{\hat{\mu}}\hat{\chi}\partial_{\hat{\nu}}\hat{\chi} + \frac{1}{96}\hat{H}^2_{\hat{\mu}\hat{\nu}} + \frac{1}{4}\mathrm{e}^{\hat{\phi}}\left(\left(\hat{F}^1_{(3)}\right)^2_{\hat{\mu}\hat{\nu}} - \frac{1}{12}\left(\hat{F}^1_{(3)}\right)^2\hat{g}_{\hat{\mu}\hat{\nu}}\right) \\ &+ \frac{1}{4}\mathrm{e}^{-\hat{\phi}}\left(\left(\hat{F}^2_{(3)}\right)^2_{\hat{\mu}\hat{\nu}} - \frac{1}{12}\left(\hat{F}^2_{(3)}\right)^2\hat{g}_{\hat{\mu}\hat{\nu}}\right)\end{aligned} \tag{3.50}$$

$$d\hat{*}d\hat{\phi} = -\mathrm{e}^{2\hat{\phi}}\hat{*}d\hat{\chi}\wedge d\hat{\chi} - \frac{1}{2}\mathrm{e}^{\hat{\phi}}\hat{*}\hat{F}^1_{(3)}\wedge\hat{F}^1_{(3)} + \frac{1}{2}\mathrm{e}^{-\hat{\phi}}\hat{*}\hat{F}^2_{(3)}\wedge\hat{F}^2_{(3)} \tag{3.51}$$

$$d\left(\mathrm{e}^{2\hat{\phi}}\hat{*}d\hat{\chi}\right) = \mathrm{e}^{\hat{\phi}}\hat{*}\hat{F}^1_{(3)}\wedge\hat{F}^2_{(3)} \tag{3.52}$$

$$d\left(\mathrm{e}^{\hat{\phi}}\hat{*}\hat{F}^1_{(3)}\right) = \hat{H}_{(5)}\wedge\hat{F}^2_{(3)} \tag{3.53}$$

$$d\left(\mathrm{e}^{-\hat{\phi}}\hat{*}\hat{F}^2_{(3)} - \hat{\chi}\mathrm{e}^{\hat{\phi}}\hat{*}\hat{F}^1_{(3)}\right) = -\hat{H}_{(5)}\wedge\left(\hat{F}^1_{(3)} + \hat{\chi}\hat{F}^2_{(3)}\right) \tag{3.54}$$

$$d\left(\hat{*}\hat{H}_{(5)}\right) = -\hat{F}^1_{(3)}\wedge\hat{F}^2_{(3)} \tag{3.55}$$

$$\hat{H}_{(5)} = \hat{*}\hat{H}_{(5)} \tag{3.56}$$

where the last equation is inserted by hand and expresses the self duality of the 5-form, required here by the form of the Lagrangian.

A first consistent truncation in 10D can be made to the IIB theory itself by only keeping gravity and the 5-form and setting to zero the rest of the fields with the

[5]The numeric index in parenthesis indicates label the order of the form with respect of the 10-dimensional spacetime.

equations of motions reducing to

$$\widehat{R}_{\hat{\mu}\hat{\nu}} = \frac{1}{96}\widehat{H}_{\hat{\mu}\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\upsilon}}\widehat{H_{\hat{\nu}}}^{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\upsilon}} \tag{3.57}$$

$$d\widehat{H}_{(5)} = 0. \tag{3.58}$$

Then the dimensional reduction ansatz is made on this truncated theory and the one we present is the first full non-linear ansatz that was first found by Cvetič et al in [11] and no further truncation is performed on the field content but only a parameterisation of the higher dimensional field in terms of lower dimensional ones which are representation of the $SO(6)$ gauge group. Hence, the 10-dimensional metric is parameterised in terms of 5 dimensional fields as

$$d\hat{s}_{10}^2 = \Delta^{1/2}ds_5^2 + g^{-2}\Delta^{-1/2}T^{-1}_{\alpha\beta}D\mu^\alpha D\mu^\beta \tag{3.59}$$

where $\Delta \equiv T_{\alpha\beta}\mu^\alpha\mu^\beta$, $U \equiv 2T_{\alpha\beta}T_{\beta\gamma}\mu^\alpha\mu^\gamma - \Delta T_{\beta\beta}$ and ds_1^20 represent the 5-dimensional metric of the reduced theory. $T_{\alpha\beta}$ is taken to be a unimodular symmetric tensor and $D\mu^\alpha \equiv d\mu^\alpha + gA^{\alpha\beta}_{(1)}\mu^\beta$ is the covariant derivative acting on the local coordinates μ^α spanning the internal 5-sphere[6] and thus are subject to the condition $\mu^\alpha\mu^\alpha = 1$. Similarly, the self-dual 5 form is parameterised as

$$\widehat{H}_{(5)} = \widehat{G}_{(5)} + \hat{*}\widehat{G}_{(5)} \tag{3.60}$$

$$with\ \ \widehat{G}_{(5)} = -gU\epsilon_{(5)} + g^{-1}\left(T^{-1}_{\alpha\beta} * DT_{\beta\gamma}\right)\wedge(\mu^\gamma D\mu^\alpha) \tag{3.61}$$

$$-\frac{1}{2}g^{-2}T^{-1}_{\alpha\gamma}T^{-1}_{\beta\delta} * F^{\alpha\beta}_{(2)}\wedge D\mu^\gamma\wedge D\mu^\delta \tag{3.62}$$

$$and\ \ \hat{*}\widehat{G}_{(5)} = \frac{1}{5!}\epsilon_{\alpha_1\cdots\alpha_6}\left[g^{-4}U\Delta^{-2}D\mu^{\alpha_1}\wedge\cdots\wedge D\mu^{\alpha_5}\mu^{\alpha_6}\right. \tag{3.63}$$

$$-5g^{-4}\Delta^{-2}D\mu^{\alpha_1}\wedge\cdots\wedge D\mu^{\alpha_4}\wedge DT_{\alpha_5\beta}T_{\alpha_6\gamma}\mu^\beta\mu^\gamma \tag{3.64}$$

$$\left.-10g^{-3}\Delta^{-1}F^{\alpha_1\alpha_2}_{(2)}\wedge D\mu^{\alpha_3}\wedge D\mu^{\alpha_4}\wedge D\mu^{\alpha_5}T_{\alpha_6\beta}\mu^\beta\right] \tag{3.65}$$

where $F^{\alpha\beta}_{(2)} = dA^{\alpha\beta}_{(1)} + gA^{\alpha\gamma}_{(1)}\wedge A^{\gamma\beta}_{(1)}$ and $DT_{\alpha\beta} \equiv dT_{\alpha\beta} + gA^{\alpha\gamma}_{(1)}T_{\gamma\beta} + gA^{\beta\gamma}_{(1)}T_{\alpha\gamma}$ [36].

[6] As per convention, early Greek alphabet indices refer to the compact space, in this case they are in the **6** representation of $SO(6)$, the symmetry group of S^5.

Plugging the ansatz above into Eq. (3.57) results into equations of motions the 5-dimensional fields need to satisfy in order to have consistency [11]

$$\begin{aligned} D\left(T^{-1}_{\alpha\gamma}T^{-1}_{\beta\lambda} * F^{\gamma\lambda}_{(2)}\right) =& -2gT^{-1}_{\gamma\alpha} * DT_{\beta]\gamma} - \frac{1}{8}\epsilon_{\alpha\beta\gamma_1\cdots\gamma_4}F^{\gamma_1\gamma_2}_{(2)} \wedge F^{\gamma_3\gamma_4}_{(2)} \\ D\left(T^{-1}_{\alpha\gamma} * DT_{\gamma\beta}\right) =& -2g^2\left(2T_{\alpha\gamma}T_{\beta\gamma} - T_{\alpha\beta}T_{\gamma\gamma}\right)\epsilon_{(5)} + T^{-1}_{\alpha\gamma}T^{-1}_{\lambda m} * F^{\lambda\gamma}_{(2)} \wedge F^{m\beta}_{(2)} \\ & - \frac{1}{6}\delta_{\alpha\beta}\left[-2g^2\left(2T_{\gamma\lambda}T_{\gamma\lambda} - (T_{\gamma\gamma})^2\right)\epsilon_{(5)} + T^{-1}_{\zeta\gamma}T^{-1}_{\lambda\delta} * F^{\lambda\gamma}_{(2)} \wedge F^{\delta\zeta}_{(2)}\right] \end{aligned} \tag{3.66}$$

as well the corresponding Einstein field equations. It turns out [11] that these equation of motions can be derived from the 5-dimensional Lagrangian

$$\begin{aligned} \mathcal{L}_5 =& R * \mathbb{1} - \frac{1}{2}g^2\left(2T_{\alpha\beta}T_{\alpha\beta} - (T_{\alpha\alpha})^2\right) * \mathbb{1} \\ & - \frac{1}{4}T^{-1}_{\alpha\beta} * DT_{\beta\gamma} \wedge T^{-1}_{\gamma\gamma}DT_{\gamma\alpha} - \frac{1}{4}T^{-1}_{\alpha\gamma}T^{-1}_{\beta\gamma} * F^{\alpha\beta}_{(2)} \wedge F^{\gamma\gamma}_{(2)} \\ & - \frac{1}{48}\epsilon_{\alpha_1\cdots\alpha_6}\Big(F^{\alpha_1\alpha_2}_{(2)} \wedge F^{\alpha_3\alpha_4}_{(2)} \wedge A^{\alpha_5\alpha_6}_{(1)} - gF^{\alpha_1\alpha_2}_{(2)} \wedge A^{\alpha_3\alpha_4}_{(1)}A^{\alpha_5\beta}_{(1)} \wedge A^{\beta\alpha_6}_{(1)} \\ & \qquad + \frac{2}{5}g^2 A^{\alpha_1\alpha_2}_{(1)} \wedge A^{\alpha_3\beta}_{(1)} \wedge A^{\beta\alpha_4}_{(1)} \wedge A^{\alpha_5\gamma}_{(1)} \wedge A^{\gamma\alpha_6}_{(1)}\Big) \end{aligned} \tag{3.67}$$

which is a truncated version of the full $\mathcal{N} = 8$ gauged $SO(6)$ supergravity in 5 dimensions [37]. It was noted in [11] in order for the ansatz to be consistent, the 5-form $\widehat{H}_{(5)}$ has to be self-dual and thus the term $\hat{*}\widehat{G}_{(5)}$ had to be included in Eq. (3.60). If that were not the case, Eq. (3.58) would give, among other acceptable equations, the constraint $\epsilon_{\alpha\beta\gamma_1\cdots\gamma_4}F^{\gamma_1\gamma_2}_{(2)} \wedge F^{\gamma_3\gamma_4}_{(2)} = 0$ which cannot be satisfied.

Looking a the transformation properties, one can deduce [11] that the truncated theory obtained describes a graviton field given by the 5-dimensional metric, 20 scalar fields in the form of the unimodular symmetric tensor $T_{\alpha\beta}$ and the 15 $SO(6)$ Yang-Mills gauge fields 1-forms $A^{\alpha\beta}$, obtained from the truncation of IIB supegravity in the 10D starting with gravity and a self-dual 5-form only.

Chapter 4

Consistent truncations from generalised geometry

4.1 Ordinary complex geometry

4.1.1 G-structures

In Section 2.2 we defined the notion of a fibre bundle, that is roughly speaking a bundle $(E, \pi, \mathcal{M})$ where the fibres $\pi^{-1}(x)$ are isomorphic to a common space F which is then referred to as the fibre, for $x \in \mathcal{M}$. Now, given two patches U_α and U_β on the base manifold $\mathcal{M}$, the transformations of the fibres among patches is dictated by $transition\ functions$ so that globally the total space is not generally a product space. Vector bundles are fibre bundles whose fibre is a vector space, example of interest being the $tangent$ and $cotangent\ bundles$ $T\mathcal{M}$ and $T^*\mathcal{M}$. Furthermore, to every vector bundle there is an associated principal fibre bundle $F(E)$, called the $frame\ bundle$, with the fibre over x being the set of all $frames$ or $ordered\ basis$ at x [38].

Associated to tangent bundle TM is the $tangent\ frame\ bundle$ $F\mathcal{M}$, a principal bundle where the fibre is the set of frames of the tangent space $T_p\mathcal{M}$. At a local level, element on $F\mathcal{M}$ are labelled by the $local\ trivialisation$ (p, e_a) where $p \in U_\alpha$ and

$e_a = e^i{}_a\partial_i$ is a set of independent vectors providing a frame for $T_p\mathcal{M}$. Also there is a natural action of $GL(d, \mathbb{R}$ on the $a-$index for $a = 1, ..., d$. Considering two patches on the base manifold $\mathcal{M}$,U_α and U_β, and local trivialisations (p, e_a) and (p', e'_a), then on the overlap we have $e'^i a = \mathrm{e}^i{}_b(t_{\beta\alpha})^b{}_a$ where $(t_{\beta\alpha})^b{}_a$ are transition functions which takes one basis into the other and form a group, the *structure group* of the bundle, $GL(d, \mathbb{R})$ in the case of tangent frame bundle [38].

Given the structure group $GL(d, \mathbb{R})$, one can choose a local frame in each of the patches such that a *reduced* tangent frame bundle is defined that has as the structure group the proper subgroup $G \subset GL(d, \mathbb{R})$. Provided this can be done, then a manifold $\mathcal{M}$ is said to have a G-*structure* with $G \subset GL(d, \mathbb{R})$ as the reduced structure group. A manifold is said *parallelizable* if one can find a global section of the whole tangent frame bundle such that by an appropriate local frame redefinition the global section takes the same form $s^a{}_b \in GL(d, \mathbb{R})$ everywhere, which results in the structure group being *trivial,* i.e. contains only the identity[18].

One way to characterise G-structures is by using globally defined, non-degenerate G-invariant tensors (or spinors). Since they are globally defined, one can choose the e_a so that the objects take the same form over the manifold. The transitions functions that leave their form invariant then form the reduced structure group G or a subgroup of G. Decomposing the representation of $GL(d, \mathbb{R})$ into irreducible representations of G one can then select those invariant under G, i.e. the G-singlets, which will give the G-invariant tensors needed to classify the G-structures [18].

An *almost complex structure* is a tensor $J \in \Gamma(T\mathcal{M} \otimes T^*\mathcal{M})$ satisfying $J^2 = \mathbb{1}$ which reduces the structure group to $GL(d/2, \mathbb{C})$. Considering the complexified tangent bundle $T\mathcal{M} \otimes \mathbb{C}$, J acts on $T_p\mathcal{M}$ with eigenvalues $-i$ and $+i$, giving a decomposition of the tangent bundle in the sub-bundles L and $\bar{L}$, spanned by two separate bases of vector fields, that have as fibres the $(+i)$ and $(-i)$ eigenspaces respectively. Since the J is preserved under G, the decomposition in sub-bundles is conserved and L and $\bar{L}$ are so called *distributions* and have equal rank. If the distribution L is integrable, J is said to be a *complex structure*. Similarly the cotangent bundle $T^{*}\mathcal{M}$ splits into sub-bundles as

$$\Lambda^{1}T^{*}M \otimes \mathbb{C} = \Lambda^{1,0}T^{*}M \oplus \Lambda^{0,1}T^{*}M, \tag{4.1}$$

with the sections $\Gamma(\Lambda^{p,q}T * \mathcal{M})$ labelled by $\Omega^{p,q}(\mathcal{M})$ for convenience. A higher form would then decompose as [39]

$$\Lambda^{l}T^{*}M = \bigoplus_{0\leq p\leq l} \left(\Lambda^{p}T^{*(1,0)}M \otimes \Lambda^{l-p}T^{*(0,1)}M\right) = \bigoplus_{0\leq p\leq l} \Lambda^{p,l-p}T^{*}M. \tag{4.2}$$

Similarly, a globally defined symmetric two-tensor provides a *metric* g with reduced structure group $O(d,\mathbb{R})$, which combined with the existence of a volume form gives an orientable Riemannian manifold with structure group $SO(d,\mathbb{R}$. A $pre-symplectic\ structure$ is given by non-degenerate two-form $\omega \in \Gamma(\Lambda^{2}T^{*}\mathcal{M})$ with structure group $Sp(d,\mathbb{R})$. This becomes a *symplectic structure* if ω is integrable, namely if $d\omega = 0$. Finally an *Hermitian metric* is given by a metric g and an *almost complex structure* J satisfying $J^{i}{}_{k}g_{ij}J^{j}{}_{l} = g_{kl}$ and has $U(d/2)$ as the structure group. These two in turn imply the existence of a symplectic structure [39].

Given an almost complex structure, one can define a local frame span by $d/2$ independent (1,0) forms $\theta^{a} \in \Lambda^{1,0}T^{*}\mathcal{M}$ and the corresponding local section Ω of the *canonical bundle* $\Lambda^{d/2,0}T^{*}\mathcal{M}$ as $\Omega = \theta^{1} \wedge ... \wedge \theta^{d}/2$ associated to the almost complex structure. If Ω is globally defined decomposible or simple form then structure group further reduces down to $SU(d/2)$ [39].

4.1.2 Forms, torsion classes and spinors

To begin with, we give a precise definition of the *exterior derivative* which is an object that act on a (l)-form χ to produce an (l+1)-form $d\chi$

$$\begin{aligned} \mathrm{d}\chi\left(V_0,\ldots,V_l\right) &= \sum_{0\leq a\leq l}(-1)^{a}V_a\left(\chi\left(V_0,\ldots,\hat{V}_a,\ldots,V_l\right)\right) \\ &+ \sum_{0\leq a<b\leq l}(-1)^{a+b}\chi\left(\left[V_a,V_b\right],V_0,\ldots,\hat{V}_a,\ldots,\hat{V}_b,\ldots,V_l\right) \end{aligned} \tag{4.3}$$

Torsion classes	Name
$\mathcal{W}_1 = \mathcal{W}_2 = 0$	Complex
$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0$	Symplectic
$\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$	Nearly Kähler
$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$	Kähler
$\operatorname{Im}\mathcal{W}_1 = \operatorname{Im}\mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$	Half-flat
$\mathcal{W}_1 = \operatorname{Im}\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$	Nearly Calabi-Yau
$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$	Calabi-Yau
$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = 0, (1/2)\mathcal{W}_4 = (1/3)\mathcal{W}_5 = -\mathrm{d}A$	Conformal Calabi-Yau

Table 4.1: Geometries classification by vanishing torsion classes of SU(3). Table taken from [39].

where $V_i \in \Gamma(T\mathcal{M})$ and $\hat{V}_i$ indicates the absence of the vector field with the given index and $[\cdot,\cdot]$ denotes the Lie bracket. In components this becomes

$$(\mathrm{d}\chi)_{i_0 i_1 \dots i_l} = (l+1)\partial_{[i_0}\chi_{i_1\dots i_l]}. \tag{4.4}$$

Returning to the decomposition in Eq. (4.2), on a complex manifold the exterior derivative of a (p,q)-form χ decomposes as

$$\mathrm{d}\left(\chi^{p,q}\right) \in \Omega^{p+2,q-1}(M) \cup \Omega^{p+1,q}(M) \cup \Omega^{p,q+1}(M) \cup \Omega^{p-1,q+2}(M) \tag{4.5}$$

and in the case of a complex manifold it becomes simply

$$\mathrm{d}\left(\chi^{p,q}\right) \in \Omega^{p+1,q}(M) \cup \Omega^{p,q+1}(M). \tag{4.6}$$

Let's consider an $SU(3)$-structure provided by a globally defined complex $(3,0)$ form Ω and a pre-symplectic real $(1,1)$-form ω, then the exterior derivative decomposes as

$$\begin{aligned} \mathrm{d}\omega &= -\tfrac{3}{2}\operatorname{Im}\left(\overline{\mathcal{W}}_1\Omega\right) + \mathcal{W}_4\wedge\omega + \mathcal{W}_3 \\ \mathrm{d}\Omega &= \mathcal{W}_1\omega^2 + \mathcal{W}_2\wedge\omega + \overline{\mathcal{W}}_5\wedge\Omega \end{aligned} \tag{4.7}$$

where $\mathcal{W}_i$ are the so called *torsion classes* [40], different type of forms[41] which determine the type of manifold, as shown in Table 4.1. It turns out [42] that the torsion classes are $SU(3)$ representations[1] in which the *intrinsic torsion* decomposes.

[1] In this case $SU(4)$, but generally they are G representations.

Given a set of $O(d)$-invariant non degenerate tensors Ξ_i which define a metric on $\mathcal{M}$, the covariant derivative with respect of the metric g acting on each invariant tensor is given by [16]

$$\begin{aligned}\nabla_m \Xi_i{}^{n_1\ldots n_r}{}_{p_1\cdots p_s} =& K_m{}^{m_1}{}_q \Xi_i{}^{q\ldots n_r}{}_{p_1\ldots p_s} + \cdots + K_m{}^{n_r}{}_q \Xi_i{}^{n_1\ldots q}{}_{p_1\ldots p_s} \\ & - K_m{}^q{}_{p_1} \Xi_i{}^{n_1\ldots n_r}{}_{q\ldots p_s} + \cdots - K_m{}^q{}_{p_s} \Xi_i{}^{n_1\ldots n_r}{}_{p_1\ldots q}\end{aligned} \tag{4.8}$$

and the intrinsic torsion is then defined as $(\mathcal{T}_{int})_{mn}{}^p = K_m{}^p{}_n - K_n{}^p{}_m$. Geometrically, given a choice of metric compatible with the G-structure whose connection acting on the defining invariant tensors vanishes, the intrinsic torsion of the G-structure is the component of the torsion which does not depend on the choice of compatible metric [43].

Finally, an $SU(d/2)$-structure can alternatively be described by an invariant spinor (and the associated conjugate spinor) by introducing a metric and orientation. This reduces structure group to $SO(d,\mathbb{R})$, subsequently lifted[2] to its double cover $Spin(d,\mathbb{R})$ which has a spinor representation.

4.2 Generalised Complex Geometry

4.2.1 The original formulation

A first generalisation of the concepts explored thus far is to replace the tangent bundle $T\mathcal{M}$ with the *generalised tangent bundle* $T\mathcal{M} \oplus T^*\mathcal{M}$ [13, 14] and then extend the same concepts to this framework, the difference being that this bundle comes with a *canonical metric* η and an associated volume-form $vol_\eta \in \Gamma\left(\Lambda^{2d}\left(TM \oplus T^*M\right)\right)$, thus reducing the structure group to $SO(d,d)$. Given two generalised vector fields $\mathbb{X} = X + \xi$ and $\mathbb{Y} = Y + \chi \in \Gamma\left(TM \oplus T^*M\right)$, with X, Y vector fields and ξ, χ forms,

[2] This lift requires that the manifold be a *spin manifold*.

the $Courant\ \ bracket$ is a generalisation of the Lie bracket and is given by

$$[X+\xi, Y+\chi]_C = [X,Y] + \mathcal{L}_X\chi - \mathcal{L}_Y\xi - \frac{1}{2}d\left(\iota_X\chi - \iota_Y\xi\right). \tag{4.9}$$

The structure group of the generalised tangent bundle is generated by elements of the type

$$\begin{pmatrix} A & 0 \\ 0 & \left(A^T\right)^{-1} \end{pmatrix}, \quad e^B = \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix}, \quad e^\beta = \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix} \tag{4.10}$$

where $A \in GL(d,\mathbb{R})$, B is a two-form and β an anti-symmetric 2-vector, which act on the generalised vector as

$$\mathrm{e}^B : X+\xi \to X + (\xi - \iota_X B) \tag{4.11}$$

$$\mathrm{e}^\beta : X+\xi \to (X - \iota_\xi\beta) + \xi \tag{4.12}$$

with i_X denoting the interior product [39]. This bracket is invariant under general diffeomorphism (generated by A) and $B-$transforms where $dB = 0$. For $dB \leq 0$, one introduces a closed three-form H and the H-twisted Courant bracket is then defined by

$$[\mathbb{X},\mathbb{Y}]_H = [\mathbb{X},\mathbb{Y}]_C + \iota_X\iota_Y H, \tag{4.13}$$

which under a general B transformation satisfies $\left[\mathrm{e}^B\mathbb{X}, \mathrm{e}^B\mathbb{Y}\right]_{H-dB} = \mathrm{e}^B[\mathbb{X},\mathbb{Y}]_H$. In the case where $H = dB$, B is the curving of a connection on a gerbe [44]. Performing a B-transform in every patch to eliminate the H field allows for the gerbe structure to be carried over to the generalised bundle resulting in the twisted bundle E whose structure group is the generalised diffeomorphism group

$$G_{\text{gendiff}} = GL(d,\mathbb{R}) \rtimes G_{B,\,\text{closed}} \tag{4.14}$$

which is actually the set of transformations that leave the Courant bracket invariant. Such twisted bundle has been called the generalised tangent bundle E [45].

We can now define a generalised almost complex structure $\mathcal{J}$ as the map transforming the generalised tangent bundle into itself that reduces the structure group from $SO(d,d)$ to $U(d/2,d/2)$, whose action on the fibres decomposes the generalised tangent bundle into two sub-bundles $L_{\mathcal{J}}, \bar{L}_{\mathcal{J}} \subset (TM \oplus T^*M) \otimes \mathbb{C}$ with eigenvalues $(\pm i)$ under the action of $\mathcal{J}$. The map has to satisfy $\mathcal{J}^2 = 1$ and $\eta(\mathcal{J}\mathbb{X}, \mathcal{J}\mathbb{Y}) = \eta(\mathbb{X}, \mathbb{Y})$. Then a generalised complex structure is obtained by making $\mathcal{J}$ H-integrable [39]. In a similar fashion, all ordinary G-structures can be constructed in the generalised framework.

Consider the bundle of formal sums of differential forms (or bundle of forms) denoted by $\Lambda^\bullet T^*\mathcal{M}$ on $\mathcal{M}$ on which the $Spin(d,d)$ naturally acts. Given an element $\phi \in \Lambda^\bullet T^*\mathcal{M}$ and a generalised vector $V = v + \xi$, for any ϕ we have a map which takes

$$\Gamma_V : \phi \mapsto \iota_v \phi + \xi \wedge \phi \tag{4.15}$$

which satisfy the Clifford algebra, whose action on $\Lambda^\bullet T^*\mathcal{M}$ gives a representation of $Spin(d,d)$ on it. Under the action of $GL(d,\mathbb{R}) \subset Spin(d,d)$ however, we have

$$\phi \mapsto |\det M|^{1/2} M^* \phi \tag{4.16}$$

where M denoted the standard $GL(d,\mathbb{R})$ action on $\Lambda^\bullet T^*\mathcal{M}$, hence the spin bundle S is $S = \Lambda^\bullet T^* \otimes \left(\Lambda^d T\right)^{1/2}$. Finally, we have a split in the bundle of forms odd and even forms corresponding to the spin bundle decomposition $S^\pm = \Lambda^\pm T^* \otimes \left(\Lambda^d T\right)^{1/2}$ [46].

As first proposed by Gaultieri [14], we can introduce a *generalised metric* $\mathcal{H}$ on $T\mathcal{M} \oplus T^*\mathcal{M}$ which is compatible with the canonical metric η seen earlier, namely $\eta^{-1}\mathcal{H}\eta^{-1} = \mathcal{H}^{-1}$. This condition means that $\mathcal{H}$ has d^2 independent components and can be parameterised as

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \tag{4.17}$$

where G and B a symmetric and anti-symmetric matrices respectively with the norm of a generalise d vector given by

$$\mathcal{H}(V,V) = G(v,v) + G^* \left(\xi + \iota_v B, \xi + \iota_v B\right) \tag{4.18}$$

where G^* is the metric on $T^*\mathcal{M}$.

4.2.2 Extended geometries

We are now ready to consider more general bundles E over $\mathcal{M}$ with structure group $O(d,d)$ or $SO(d,d)$, called type I extended geometries, which are generalised geometries in the sense of Hitchin only when the structure group G is in the Courant bracket preserving geometric subgroup and reduce to $T\mathcal{M} \oplus T^*\mathcal{M}$ when $G = GL(d, \mathbb{R}$. The bundle E can be spilt into sub-bundles $E = E^+ \oplus E^-$ where $\pm$ labels the sub-bundle based on whether the canonical metric η is positive or negative definite on it. This corresponds to a reduction of G to $O(d) \times O(d)$ or $SO(d) \times SO(d)$, with the reduction defining the generalised metric by the restrictions of the canonical metric on the sub-bundles as follows [46]

$$\mathcal{H} = \eta|_{E^+} - \eta|_{E^-} \,. \tag{4.19}$$

The choice of a generalised metric corresponds to the choice a a reduction, with the space of such reductions at $x \in \mathcal{M}$ forming the space $\frac{O(d,d)}{O(d)\times O(d)}$ or its special orthogonal counterpart. Given projection maps $\mathcal{V}^\pm : E \to E^\pm$, then we can construct an object which maps $E \to E^+ \oplus E^-$

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}^+ \\ \mathcal{V}- \end{pmatrix} \tag{4.20}$$

represented by two sets of $d \times 2d$ matrices as

$$\mathcal{V}_I^A = \begin{pmatrix} \mathcal{V}^a{}_I \\ \mathcal{V}^{a'}{}_I \end{pmatrix} \tag{4.21}$$

which is a vielbein acting on a general basis I to transform it into the basis for $E^+ \oplus E^-$ labelled by $A = (a', a)$, with the generalised metric taking the form $\mathcal{H} = \mathcal{V}^t\mathcal{V}$ or in components $\mathcal{H}_{IJ} = \delta_{AB}\mathcal{V}^A I^B$. $\mathcal{H}$ will vary around $\mathcal{M}$, hence we have map

$$\mathcal{H} : M \to O(d,d)/O(d) \times O(d). \tag{4.22}$$

defined by $\mathcal{H}(x)$ for $x \in \mathcal{M}$ [46].

Furthermore there is a local symmetry under $O(d) \times O(d)$ which can be used to parameterise the vielbein as

$$\mathcal{V} = \begin{pmatrix} e^t & 0 \\ -e^{-1}B & e^{-1} \end{pmatrix} \tag{4.23}$$

so that the generalise metric becomes

$$\mathcal{H} = \mathcal{V}^t\mathcal{V} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \tag{4.24}$$

with $G = e^t e$ or in components $G_{ij} = e_i^a e_j^b \delta_{ab}$, where e_i^a are d-bein and B_{ij} are anti-symmetric matrices [46]. Finally, the discussion above generalises to vector bundles $\mathcal{E}$ with structure groups G non-compact, which however has a maximal compact subgroup H_d. Then $\mathcal{E}$ reduces to the bundle $\bar{\mathcal{E}}$ and H_d becomes its structure group.

4.3 Consistent truncations formalism

4.3.1 Ordinary G-structures

Before looking at the generalised case, we shall briefly extend Scherk-Schwarz reductions to the language of $G-$structures. Conventionally, as seen in Chapter 3, the main procedure for group manifold G reductions involves an expansion of the higher dimensional fields into representation of G and a subsequent truncation of the field content to the singlet representations only. More generally, the argument extends to the reduction of the structure group G_S so that higher dimensional fields can be decomposed into G_S irreducible representations. A truncation to the singlet representations would then provide a consistent truncation [16].

Focusing on gravity and recalling that choosing a metric on $\mathcal{M}$ results in a parameterization of the coset space $GL(d, \mathcal{R})/O(d)$, the number of scalars resulting from the reduction of the structure group to G_S is found via the centraliser $C_K(A)$, which counts the number of elements in A which commutes with every element of A. Hence the scalar metric space would be given by $H \in \frac{\mathrm{C}_{\mathrm{GL}(d,\mathrm{R})}(G_S)}{\mathrm{C}_{\mathrm{O}(d)}(G_S)}$. Given that the set of G_S-invariant tensors $\{\Xi_i\}$ defines the G_S-structure and vice versa, by counting the number of invariant one forms $\eta^a \in \{\Xi_i\}$ we can obtain the number of gauge fields A^a which result from the reduction. For singlet torsion, the gauge symmetry algebra of the metric gauge fields is fully determined by the intrinsic torsion of the G_S-structure [16].

Using this language, Scherk-Schwarz reduction are simply cases in which the structure group is trivial $G_s = \mathbb{1}$ due to the choice of a globally defined set of left-invariant one-forms e^a which provides a parallelisation of the manifold. Hence the scalars belong to the coset space $GL(d, \mathbb{R})/SO(d)$ while the one-forms provide d gauge fields with a Lie algebra given by $[\bar{e}_a, \bar{e}_b] = C_{ab}{}^c \bar{e}_c$, with $\bar{e}_a$ being the dual of e^a and $C_{ab}{}^c$ being constants fixed by the intrinsic torsion from the Lie derivatives of the invariant tensors

$$\mathcal{L}_{\hat{\eta}_a} \Xi_i = C_{ai}{}^j \Xi_j. \tag{4.25}$$

4.3.2 Generalised G_S-structures

In order to generalise the construction above, conventional G_S-structures are replaced with generalised G_S-structures on the generalised tangent space E on the manifold $\mathcal{M}$. If we restrict our focus on supergravity, whether it be 10- or 11-dimensional, the focus shifts on generalised tangent spaces E on which we have a natural action of the exceptional group $E_{d(d)}$ with H_d as the compact subgroup encoding the R-symmetries of the theory, that is the set of transformations under which supercharges transform into each other.

In a recent paper [16] it was shown that given a d- or $(d-1)$-dimensional manifold $\mathcal{M}$ on which a generalised structure $G_S \subset H_d$ with only constant singlet intrinsic torsion defined by a set of invariant tensors Q_i exists, then using the invariant tensors to expand the bosonic fields in the 10D or type II supergravity respectively and keeping all possible singlets produces a consistent truncation. Similarly given H_d has double cover $\bar{H}_d$ which acts on fermionic fields, then the structure group becomes $\bar{G}_S \subset \bar{H}_d$ and the truncation can be applied to the fermionic sector as well by expanding the fermionic field in terms of $\bar{G}_S$-singlets [16].

To see how the truncation works, let's consider an 11D or type II supergravity theory reformulated on a product space $X \times \mathcal{M}$ where $\mathcal{M}$ is an internal d- or $(d-1)$ dimensional manifold, whose generalised tangent bundle E has a structure group extended to $E_{d(d)}$ [46] so that the bosonic fields in the theory are organised in $GL(D, \mathbb{R}) \times E_{d(d)}$ representations [47], with D denoting the dimensions of the spacetime X.

The scalar degrees of freedom under the structure group of X, namely $GL(D, \mathbb{R}$, are then captured by a generalised metric defined over E represented by a symmetric tensor $G_{MN}(x, y) \in \Gamma(S^2E^*)$ which is invariant under the R-symmetry and thus defines an H_d-structure over $\mathcal{M}$, as first discovered in [48, 49]. Furthermore the vector/one-form d.o.f. are captured by $\mathcal{A}_\mu^M(x, y) \in \Gamma(T^*X \otimes E)$ which are sections of E. Finally the two-forms under $GL(D, \mathbb{R})$ are given by $\mathcal{B}_{\mu\nu}^{MN}(x, y) \in \Gamma(\Lambda^2T^*X \otimes N)$, where $N \subset S^2E^*$.

For the truncation, we considered the reduced structure group $G_S \subset H_d$ and the corresponding G_S-invariant tensors Q_i. The generalised metric G_{MN} defined an H_d structure on E, hence it must be encoded in the reduced $G_S \subset H_d$ structure. The G_S-singlets we want to keep in the truncation are only those which deform the structure but not the metric. Thus one finds [16] that the scalars parameterise the coset space

$$h^I(x) \in M_{\text{scalar}} = \frac{\mathrm{C}_{\mathrm{E}_{d(d)}}(G_S)}{\mathrm{C}_{H_d}(G_S)} := \frac{\mathcal{G}}{\mathcal{H}} \tag{4.26}$$

while vectors and two-forms are given by

$$\mathcal{A}_\mu^{\mathcal{A}}(x) K_{\mathcal{A}} \in \Gamma\left(T^*M\right) \otimes \mathcal{V} \tag{4.27}$$

$$\text{and} \qquad \mathcal{B}_{\mu\nu}^{\Sigma}(x) J_\Sigma \in \Gamma\left(\Lambda^2 T^* X\right) \otimes \mathcal{B} \tag{4.28}$$

respectively, with $\{K_{\mathcal{A}}\}$ and $\{J_\Sigma\}$ being basis spanning the vector spaces $\mathcal{V} \subset \Gamma(E)$ and $\mathcal{B} \subset \Gamma(N)$, which are representation spaces of the centraliser $\mathcal{G}$.

In order to define the structure of the gauge symmetries in the reduced theory, we make use of the singlet intrinsic torsion argument. In analogy with ordinary structures, one can define [43] the intrinsic torsion of the generalised G_S-structure as follows. Consider a generalised connection $\tilde{D}$ which satisfies $\tilde{D}Q_i = 0$ for all Q_i and define the generalised torsion T of the connection via the action on a generalised tensor α as

$$(L_V^{\tilde{D}} - L_V)\alpha = T(V) \cdot \alpha. \tag{4.29}$$

with L denoting the generalised Lie derivative.

Here T is a map $T : \Gamma(E) \to \Gamma(\mathrm{ad}\,\tilde{F})$ which acts via the adjoint action and whose $\tilde{D}$-independent component defines the generalised intrinsic torsion and can be decomposed into irreducible representations of the reduced structure group G_S. For singlet intrinsic torsion, one can define the generalised Levi-Civita connection so that the action on the invariant tensors, which is determined by the intrinsic torsion, gives us $D_M Q_i = \Sigma_M \cdot Q_i$, with $\Sigma_M \in \Gamma(E^* \otimes \mathrm{ad}\, P_{H_d})$ and $P_{H_d})$ is the bundle of tensors transforming in the adjoint representation of H_d [16].

Now recall that $K_{\mathcal{A}}$ is the basis spanning $\mathcal{V}$, from Eq. (4.29) due to the compatibility condition $\tilde{D}K_{\mathcal{A}} = 0$, one has

$$L_{K_{\mathcal{A}}} Q_i = -T_{\text{int}}\left(K_{\mathcal{A}}\right) \cdot Q_i \tag{4.30}$$

and $T_{int}(K_{\mathcal{A}})$ is a singlet of $ad\tilde{F}$ which are just the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. Hence $-T_{int}$ defines a map $\Theta : \mathcal{V} \mapsto \mathfrak{g}$ called the embedding tensor [50], which is a constant tensor normally describing the explicit embedding of a gauge subgroup into its parent group. Looking at the generalised Lie derivative acting on $K_{\mathcal{A}}$ we then find

$$L_{K_{\mathcal{A}}} K_{\mathcal{B}} = \Theta_{\mathcal{A}} \cdot K_{\mathcal{B}} = \Theta_{\mathcal{A}}{}^{\tilde{\alpha}} \left(t_{\tilde{\alpha}}\right)_{\mathcal{B}}{}^{\mathcal{C}} K_{\mathcal{C}} \tag{4.31}$$

and define $X_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}} = \Theta_{\mathcal{A}}{}^{\tilde{\alpha}} \left(t_{\tilde{\alpha}}\right)_{\mathcal{B}}{}^{\mathcal{C}}$ where $(t_{\tilde{\alpha}})_{\mathcal{B}}{}^{\mathcal{C}}$ represent the action of $\mathfrak{g}$ on the vector space $\mathcal{V}$ which is span by $\{K_{\mathcal{A}}\}$. Finally one has the quadratic condition on the embedding tensor resulting from the properties of L [16]

$$[X_{\mathcal{A}}, X_{\mathcal{B}}] = -X^{\mathcal{C}}_{\mathcal{A}\mathcal{B}} X_{\mathcal{C}} \tag{4.32}$$

which means that we can view $K_{\mathcal{A}}$ as the generators of $\mathfrak{g}$ and thus $\mathfrak{g}_{\mathfrak{gauge}}$ is given by the set of elements in g which can be reached from $\mathcal{V}$ via the embedding tensor map, hence $\mathcal{G}_{gauge} \subseteq \mathcal{G}$.

Finally, if $k_{\hat{\alpha}}$ are Killing vectors generating the action of $\mathfrak{g}$ on $\mathcal{M}_{scalar}$, then covariant derivative acting on the scalars is given by [16]

$$\hat{D}_\mu h^I = \partial_\mu h^I - \mathcal{A}_\mu{}^{\mathcal{A}} \Theta_{\mathcal{A}}{}^{\hat{\alpha}} k^I_{\hat{\alpha}} \tag{4.33}$$

and the gauge transformations are [16]

$$\begin{aligned}
\delta \mathcal{A}^{\mathcal{A}}_\mu &= \partial_\mu \Lambda^{\mathcal{A}} + X_{\mathcal{B}\mathcal{C}}{}^{\mathcal{A}} \left(\mathcal{A}^{\mathcal{B}}_\mu \Lambda^{\mathcal{C}} - \Xi^{\mathcal{B}\mathcal{C}}_\mu\right) \\
\delta \mathcal{B}^{\Sigma}_{\mu\nu} &= 2 d_{\mathcal{A}\mathcal{B}}{}^{\Sigma} \left(\partial_{[\mu} \Xi^{\mathcal{A}\mathcal{B}}_{\nu]} + 2 X_{\mathcal{C}\mathcal{D}}{}^{\mathcal{A}} \mathcal{A}^{\mathcal{C}}_{[\mu} \Xi^{\mathcal{D}\mathcal{B}}_{\nu]} - \Lambda^{\mathcal{A}} \mathcal{H}^{\mathcal{B}}_{\mu\nu} - \mathcal{A}^{\mathcal{A}}_{[\mu} \delta \mathcal{A}^{\mathcal{B}}_{\nu]}\right)
\end{aligned} \tag{4.34}$$

where $\mathcal{H}^{\mathcal{A}} = \mathrm{d}\mathcal{A}^{\mathcal{A}} + X_{\mathcal{B}\mathcal{C}}{}^{\mathcal{A}} \left(\mathcal{A}^{\mathcal{B}} \wedge \mathcal{A}^{\mathcal{C}} + \mathcal{B}^{\Sigma} \tilde{d}^{\mathcal{B}\mathcal{C}}_{\Sigma}\right)$ and $\Xi^{\mathcal{A}\mathcal{B}}_\mu = \Xi^{\Sigma}_\mu \tilde{d}^{\mathcal{A}\mathcal{B}}_{\Sigma}$, with the constants $d_{\mathcal{A}\mathcal{B}}{}^{\Sigma}$ and $\tilde{d}_{\Sigma}{}^{\mathcal{A}\mathcal{B}}$ being defined by $K_{\mathcal{A}} \times_N K_{\mathcal{B}} = d_{\mathcal{A}\mathcal{B}}{}^{\Sigma} J_{\Sigma}$ and $J_{\Sigma} = \tilde{d}_{\Sigma}{}^{\mathcal{A}\mathcal{B}} K_{\mathcal{A}} \otimes K_{\mathcal{B}}$ respectively. Then given H_d has a double cover $\tilde{H}_d$, so that the structure group is lifted to $\tilde{G}_S \subseteq \tilde{H}_d$, then the number of $\tilde{G}_S$-singlets in the generalised spinor bundle $\mathcal{S}$ fully determines the number of supersymmetries which survive the truncation [16].

Hence by specifying the reduced generalised structure both the field content of the truncated theory and their transformations under the gauge group are fully determined. The argument for consistency is then following: since the all the fields and their derivative transform as singlets under G_S, so long as the whole set of singlets is kept one has a consistent truncation since the truncated non-singlet representations cannot be sourced by products of singlets ones [16].

Chapter 5

Conclusions

In this paper we looked at the concept of dimensional reduction and consistent truncation. We first showed how Kaluza embedded Einstein's gravity and electromagnetism into a 5-dimensional pure gravity theory which reduces to the latter via a compactification of one of the dimensions on a circle.

We looked at the more general group manifold dimensional reductions of pure gravity and at the requirement of unimodularity for the structure constants in order to deliver a consistent truncations as developed by Scherk and Schwarz. The key idea there was to expand the degrees of freedom in the higher dimensional theory in terms of eigenfunctions of the compact manifold and the truncate the modes to the singlets under the group action or allow only for a very specific dependence on the compact space coordinates that would factor out of the lower dimensional equation of motion.

We gave an account of the geometry of coset spaces and at the failure to find a purely group theoretical formalism for consistent truncations on them. We then showed one coset space reduction which is known to be consistent, that is the $SO(6)$ gauged reduction of type IIB supergravity on S^5 preserving maximal supersymmetry.

Finally we reviewed the framework of generalised geometry and how it was recently used to construct a formalism for obtaining generic supersymmetric consistent reduction for supergravity. It was found that starting from a generalised tangent space E that has a natural action of the group $E_{d(d)}$ with maximal compact subgroup H_d, a generalised $G_s \subseteq H_d$ structure can be used to produce consistent truncations via expanding the fields in the supergravity theory in term of the G_S-invariant tensors that define the structure.

So long us one restricts the attention to generalised structures whose intrinsic torsion can be at best decomposed into singlet representations of the generalised structure group, then the a truncation of the field content to all of the possible singlets would then be consistent. Then given the double cover $\tilde{H}_d$, an uplift of the generalised structure group to $\tilde{G}_S \subseteq \tilde{H}_d$ would allow for the same argument to be applied to the fermionic degrees of freedom, with the number of $\tilde{G}_S$-singlets given the number of preserved supersymmetries under the truncation. Thus the geometry of the generalised structure would completely determine the content and structure of the truncated theory.

It would be interesting to see how the generalised geometries here reviewed will be used in the future to construct new truncations of M-theory and perhaps provide new results that can be used in the search of more dualities in the holographic front. More interesting still would be to see this formalism being used to shed some new light into the nature of M-theory itself.

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