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The Wave Function of The Universe

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Abstract

The wave function of the Universe is a solution to the Wheeler-DeWitt equation. As this equation permits infinitely many solutions, selecting the one that corresponds to our Universe requires that we impose appropriate boundary conditions. It is hoped that such boundary conditions will give rise to a classical spacetime in the late Universe and provide an initial condition for the inflationary period required to solve the flatness and horizon problems of classical cosmology. This dissertation is mainly concerned with the no-boundary proposal of Hartle and Hawking as a candidate for those boundary conditions. After discussing the basic principles of quantum cosmology, the recent criticisms of this proposal are addressed.

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To my mother and father for supporting me in pursuing something I have been so passionate about for so many years.

Contents

Chapter 1

Introduction

The Penrose-Hawking singularity theorems of general relativity lead to the conclusion of an initial singularity before the Big Bang [1]. As the laws of thermodynamics tell us that isolated systems such as the Universe shall naturally evolve towards states of higher disorder, the initial configuration of the Universe must have had a very high degree of order to it. This gives rise to the thermodynamic arrow of time. An explanation as to why the Universe began in a highly ordered state was put forth by Penrose in [2] based on the predictions of general relativity that singularities have occurred in the past and shall once again occur in the future. Future singularities could be a result of the Big Crunch if the entire Universe is to recollapse, or at the centre of black holes if only local regions are to collapse. Penrose proposes that the Weyl tensor vanishes at the initial singularity so that the geometry is highly ordered and equivalently, the gravitational entropy is small. The Weyl tensor however need not vanish at singularities in the future which are free to be arbitrarily disordered.

It is both a cause for concern and a valid criticism of Penrose's proposal that at this singularity at the beginning of the Universe, the classical equations of general relativity likely no longer apply [1]. This has led many to suggest that the origin of the

[1]Though significantly less popular, the alternative view is that there are particles whose histories did not exist before a certain time.

Universe must be treated using an alternative theory. As we lack a satisfactory theory of quantum gravity, we shall deal with what is sometimes argued to simply be an effective theory known as quantum cosmology.

If quantum mechanics is truly a fundamental theory, one would expect that we can apply it to the Universe as a whole. It therefore seems wise to seek a description of the quantum creation of the Universe. The quantum state that the Universe occupies is then an object of great interest. The relevant functional, $\Psi[\tilde{h}_{ij}(\mathbf{x}), \tilde{\Phi}(\mathbf{x}), \Sigma]$, which shall henceforth be referred to as the wave function of the Universe, was first introduced by DeWitt in 1967 [3] and describes the probability amplitude that the Universe contains a three-surface Σ on which the three-metric is $\tilde{h}_{ij}(\mathbf{x})$ and the matter field configuration is $\tilde{\Phi}(\mathbf{x})$. Computing this object which should describe the past, present, and future of a closed Universe is the main concern of quantum cosmology.

Unlike the familiar particle wave functions of quantum mechanics, the wave function of the Universe is not defined on spacetime but rather, actually being a functional, is defined on an infinite dimensional manifold known as superspace. The governing equation of such a function, and the central equation to quantum cosmology is known as the Wheeler-DeWitt equation. This equation, which the wave function must satisfy, takes steps towards a theory of quantum gravity as it blends ideas from both quantum mechanics and general relativity by Dirac quantizing the Hamiltonian constraint of a gravity plus matter system. The equation takes the form of a second order hyperbolic functional differential equation and permits infinitely many solutions. As we shall soon see, the explicit form of this equation is rather complicated due to the fact it is defined on an infinite dimensional manifold. In an attempt to understand properties of its solutions, we usually restrict ourselves to a finite dimensional manifold known as minisuperspace. Upon doing so, the Wheeler-DeWitt equation is reduced to a wave equation which can be solved by standard techniques

[4].

Similarly to how models from classical cosmology usually require various types of initial conditions in order to have any kind of predictive power, the wave functions of quantum cosmology require a choice of boundary conditions. Even then, more is sometimes required to fully determine a wave function. We shall soon see this in the case of the no-boundary proposal where choosing a contour of integration remains a contentious issue [5, 6, 7, 8]. In ordinary single-particle quantum mechanics, the Schrödinger equation is solved and boundary conditions dependent on the external set up of the system are imposed to fully determine the particle's wave function. As there is nothing external to the Universe the boundary conditions cannot be derived from quantum cosmology itself. They are to be introduced as new laws of physics in their own right. Therefore, when proposing boundary conditions there are certain sobriety tests that they should be able to withstand. The two most basic requirements are that the Universe should behave classically whilst large and that the wave function should be consistent with quantum field theory on a fixed background for small matter fluctuations about a homogenous Universe. The later of which implies that a suitable wave function should take the form of a decaying gaussian in the semiclassical approximation.

Many physicists seem to believe that the boundary conditions of the Universe are arbitrary to an extraordinary high degree and for this reason cannot be known exactly. Regardless of whether or not it is possible to know them with precision, it seems that they certainly have large restrictions placed on them. Observations place the density parameter $\Omega = \rho/\rho_{crit}$ very close to one and suggest that the large scale structure of the Universe is homogenous and isotropic. Therefore being described by the Friedmann-Robertson-Walker (FRW) metric [9, 10]

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega_2\right]. \tag{1.1}$$

In quantum cosmology we usually study closed Universes and thus set the parameter k equal to one. One possible route to achieving a Universe with the properties described above is through a phenomena known as inflation where the Universe undergoes an initial period of exponential expansion driven by the energy of a false vacuum [11]. Models of this sort claim to provide answers to the following two problems of classical cosmology [2], but as inflation must be a consequence of the boundary conditions the models are incomplete without making statements about what those boundary conditions happen to be.

1. The flatness problem: For the density parameter of the Universe to be as close to 1 as it is in the present day, at the Planck time it must have differed from 1 by an amount smaller than 10^{-60}.

2. The horizon problem: Regions of the Universe which appear to have never been in causal contact with one another are in thermal equilibrium.

Numerous proposals have been put forth, the most studied of which being the proposals of Hartle and Hawking who claim that the boundary condition of the Universe is that it has no boundary [13], and the tunnelling boundary condition of Vilenkin who attempts to draw analogy between quantum tunnelling and the quantum creation of the Universe from nothing [14]. Both sets of authors claim that their boundary condition predicts sufficient inflation to explain the flatness and horizon problems, but in the case of the no-boundary condition we shall see that this is disputable as it requires a volume-weighting to first be introduced.

[2] Alternative solutions involving a time varying speed of light have been proposed by Albrecht and Magueijo [12].

In recent years the no-boundary proposal has become the subject of much controversy after Neil Turok and collaborators wrote a series of papers claiming that it cannot possibly describe the emergence of a realistic cosmology [5, 6, 7, 8]. More specifically, the authors believe that the proposal necessarily leads to unsuppressed matter fluctuations. In writing these papers, they were unable to convince everyone and subsequent papers by Halliwell *et al.* were published in defence of the proposal [15, 16, 17, 18]. This led to a dispute lasting multiple years which at the time of writing remains unresolved. The two sets of authors were unable to come to an agreement and simply published final papers reiterating their most recent positions on the matter [8, 18]. This dispute shall be discussed in detail in chapter four after developing the basic tools required for quantum cosmology.

Chapter 2

The Wheeler-DeWitt Equation

2.1 The Action Principle

It has been suggested by Sir Roger Penrose that a physically appropriate spacetime must be globally hyperbolic [2]. Any manifold of this form admits a smooth time function t, such that the set of points satisfying t = constant form a spacelike Cauchy hypersurface Σ. We may split a manifold $\mathcal{M}$, of this type in terms of the orthogonal product [19]

$$\mathcal{M} = \mathbb{R} \times \Sigma. \tag{2.1}$$

As we are mainly concerned with closed Universes in quantum cosmology we can take the three-surface to be compact. It then becomes possible to choose coordinates such that our line element can be written as

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(N^2 - N_i N^i\right)\mathrm{d}t^2 + 2N_i \mathrm{d}x^i \mathrm{d}t + h_{ij}\mathrm{d}x^i \mathrm{d}x^j. \tag{2.2}$$

The quantity N is known as the lapse function and is fully determined by the difference between the elapsed coordinate time t, and proper time τ on curves normal to the hypersurfaces. Explicitly, the defining equation of this function takes the form

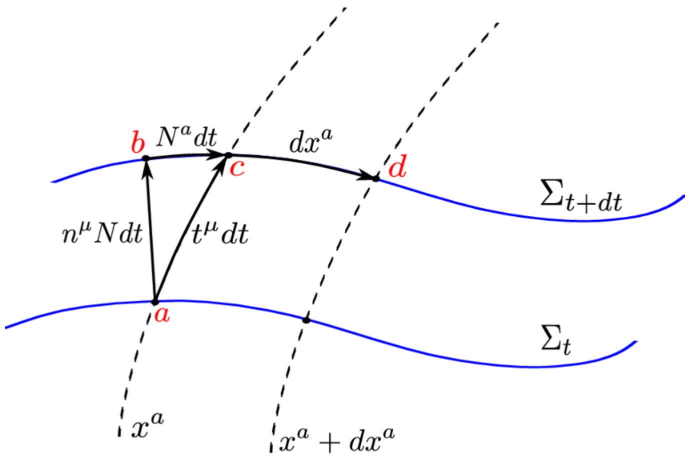

Figure 2.1: This is a graphical illustration of the lapse function N and the shift vector N^a. By starting at point $\mathbf{a}$ on Σ_t, moving by $n^\mu N dt$ leads to the point $\mathbf{b}$ on Σ_{t+dt}. We then shift by $N^a dt$ to arrive at the point $\mathbf{c}$, this is the time evolution of $\mathbf{a}$. The graphic has been taken from [20].

$d\tau = Ndt$. The shift vector N^i describes how the hypersurface Σ_t differs from the neighbouring hypersurface Σ_{t+dt}. Consider a point a on Σ_t. The shift vector gives the difference between the point one would end up at if instead of following a from one hypersurface to the next, you instead followed a curve tangent to the normal vector n^μ, of the hypersurface Σ_t. For the special case in which $N^i = 0$, the spatial coordinates are said to be "comoving".

We shall now derive the central equation of quantum cosmology. We start by introducing the Einstein-Hilbert action which in the presence of matter is coupled to the corresponding matter field Φ

$$S = \frac{m_p^2}{16\pi}\left[\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}(R-2\Lambda) + 2\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}K\right] + S_m \tag{2.3}$$

with the term due to the matter coupling being given by

$$S_m = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + 2V(\Phi)\right]. \tag{2.4}$$

The letter g is used to denote the determinant of the metric $g_{\mu\nu}$ while K denotes the trace of the purely spatial part of the extrinsic curvature tensor K_{ij}. This can be written in terms of the three-metric as $K = K_{ij}h^{ij}$. Explicitly, these components satisfy the relation

$$K_{ij} = \frac{1}{2N}\left[-\frac{\partial h_{ij}}{\partial t} + 2D_{(i}N_{j)}\right]. \tag{2.5}$$

Using our new variables, the entire action S may be rewritten as

$$S = \frac{m_p^2}{16\pi}\int \mathrm{d}^3x\mathrm{d}tN\sqrt{h}\left[K_{ij}K^{ij} - K^2 + {}^3R - 2\Lambda\right] + S_{\mathrm{m}}. \tag{2.6}$$

We now calculate the conjugate momenta

$$\pi^{ij} = \frac{\delta\mathcal{L}}{\delta\dot{h}_{ij}} = -\frac{m_p^2}{16\pi}\sqrt{h}\left(K^{ij} - h^{ij}K\right) \tag{2.7}$$

$$\pi_\Phi = \frac{\delta\mathcal{L}}{\delta\dot{\Phi}} = \frac{1}{N}\sqrt{h}\left(\dot{\Phi} - N^i\partial_i\Phi\right). \tag{2.8}$$

Using standard techniques, the Hamiltonian form of the action is found to be [21]

$$S = \int \mathrm{d}^3x\mathrm{d}t\left[\dot{h}_{ij}\pi^{ij} + \dot{\Phi}\pi_\Phi - N\mathcal{H} - N^i\mathcal{H}_i\right] \tag{2.9}$$

where

$$\mathcal{H} = \frac{16\pi}{m_p^2}G_{ijkl}\pi^{ij}\pi^{kl} - \frac{m_p^2}{16\pi}\sqrt{h}\left({}^3R - 2\Lambda\right) + \mathcal{H}_m \tag{2.10}$$

and

$$\mathcal{H}^i = -2D_j\pi^{ji} + \mathcal{H}^i_m. \tag{2.11}$$

In this, we have introduced the Wheeler-DeWitt metric G_{ijkl} which has signature (-+++++) at every point on the hypersurface. It has the explicit form

$$G_{ijkl} = \frac{1}{2} h^{-1/2} \left(h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl} \right). \tag{2.12}$$

In (2.9), the lapse function N and the shift vector N^i are acting as Lagrange multipliers. Recognising this leads us to the equations of motion. We have the Hamiltonian constraint

$$\mathcal{H} = \frac{16\pi}{m_p^2} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{m_p^2}{16\pi} \sqrt{h} \left({}^3R - 2\Lambda \right) + \mathcal{H}_m = 0 \tag{2.13}$$

which happens to be the (00) component of Einstein's field equations, and also the momentum constraints

$$\mathcal{H}^i = -2 D_j \pi^{ji} + \mathcal{H}^i_m = 0. \tag{2.14}$$

which are the $(0i)$ components.

2.2 Canonical Quantisation

At this point, we shall make note of an interesting and important problem that arises from our analysis. From equation (2.9), we may use the standard relation between the Lagrangian and Hamiltonian density

$$\mathcal{L} = \sum_i \dot{\phi}_i \pi_i - \mathcal{H} \tag{2.15}$$

to see that the Hamiltonian of the system is just a linear combination of the constraints, multiplied by some Lagrange multipliers

$$H = \int \mathrm{d}^3x \left(\mathcal{N} \mathcal{H} + \mathcal{N}_i \mathcal{H}^i \right). \tag{2.16}$$

When our constraints are taken on-shell, the integrand and thus the Hamiltonian vanish. This is rather suspicious as in quantum mechanics, the Hamiltonian is responsible for generating translations in time which we interpret as being the flow of time. A vanishing Hamiltonian therefore implies that there is no such flow. The issue that has arisen is one of the main difficulties encountered when attempting to construct theories of quantum gravity and is one aspect of the more general "*Problem of Time*" [22]. At its core, this is a conflict due to the fact that quantum mechanics and general relativity each have independent and incompatible notions of time. In quantum mechanics time is treated as a background parameter with an absolute and rigid flow that is external to the system itself. On the other hand, in general relativity time is a coordinate that can flow in a malleable way depending on the motion and position of the system under consideration. Finding a resolution to this conflict is vital if we are to ever replace quantum theory and general relativity with a unified framework to treat situations where the effects of both are important i.e. during the early Universe or inside of a black hole.

From the Hamiltonian constraint (2.13), we may construct the Wheeler-DeWitt equation by using the process of Dirac quantisation [23]. We start by quantising our operators in the standard way

$$\pi^{ij} \to -i\frac{\delta}{\delta h_{ij}} \tag{2.17}$$

$$\pi_{\phi} \to -i\frac{\delta}{\delta \phi}. \tag{2.18}$$

By making the appropriate replacements and operating on the wave function with the quantised Hamiltonian constraint, we finally obtain the Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\Psi = \left[-\frac{16\pi}{m_p^2}G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} - \frac{m_p^2}{16\pi}\sqrt{h}\left({}^3R - 2\Lambda\right) + \hat{\mathcal{H}}_m\right]\Psi = 0. \tag{2.19}$$

While time certainly plays a role in equations describing subsystems of the Universe such as Schrödinger's equation, it appears that equations describing the Universe as a whole are stationary. One is then faced with having to explain how notions of time come to be, this is another aspect of the problem of time known as "The Frozen Formalism Problem". While the Wheeler-DeWitt equation is static and seemingly suggests that the Universe does not evolve in time (in clear contradiction with observation), if this wave function is to contain an all encompassing description of the Universe, then time itself being part of the Universe, should be described by its wave function. Thus we should be able to construct a monotonic functional $t(h_{ij}, \Phi)$ from solutions of the equation [3]. Although a sensible prescription is not obvious and has not been found, the supposed paradox seems apparent rather than actual.

In this derivation, the issues associated with the ordering of operators have been ignored. Make no mistake though, the Wheeler-DeWitt equation suffers from issues of operator ordering as is often the case in quantum theory. Although solutions to this equation will clearly depend on how we choose to resolve these issues, it will not be too big of a concern to us as predictions in quantum cosmology can only be trusted to leading semiclassical order. That is to say, the operator ordering will only affect the prefactor and not the exponential contribution to the wave function in which we are interested.

The significance of the momentum constraint was first realised and proven by Peter Higgs in 1958 [24]. From the quantised version of (2.14), it is found that the theory is invariant under three-dimensional diffeomorphisms. To demonstrate this, we consider a change in coordinates

$$x_i \to x_i' = x_i - \eta_i \tag{2.20}$$

under which our wave function transforms as

$$\Psi[h_{ij} + D_{(i}\eta_{j)}] = \Psi[h_{ij}] + \int \mathrm{d}^3x D_{(i}\xi_{j)} \frac{\delta\Psi}{\delta h_{ij}}. \tag{2.21}$$

As we are interested in compact geometries, we may now integrate by parts and drop the boundary term. The above is now rewritten as

$$\Psi[h_{ij} + D_{(i}\eta_{j)}] = \Psi[h_{ij}] - \int \mathrm{d}^3x \eta_j D_i \left(\frac{\delta\Psi}{\delta h_{ij}}\right) = \Psi[h_{ij}] + \frac{1}{2i}\int \mathrm{d}^3x \eta_i \mathcal{H}^i \Psi. \tag{2.22}$$

If the momentum constraint is satisfied, the second term on the far RHS of the above equation vanishes, and we are simply left with

$$\Psi[h_{ij} + D_{(i}\eta_{j)}] = \Psi[h_{ij}] \tag{2.23}$$

proving the original claim.

Similarly to the wave functions from non-relativistic particle mechanics, there is a probability interpretation associated with our wave function. As the Wheeler-DeWitt equation is a Klein-Gordon type equation, there is a corresponding conserved current

$$J = \frac{i}{2}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi). \tag{2.24}$$

Although tempting to naively interpret J as a probability flux, just as in the case of the Klein-Gordon equation for the complex scalar it is not positive definite. For this reason, many physicists reject probabilities constructed from the conserved current J. Instead, many adopt Hawking's proposal that $|\Psi[\tilde{h}_{ij}, \tilde{\phi}, \Sigma]|^2$ is to be interpreted as being proportional to the probability of the Universe containing a three-surface Σ on which the metric is $\tilde{h}_{ij}$ and matter field is $\tilde{\Phi}$. Explicitly, the probability of finding that the Universe is in a configuration contained within a region $\mathcal{V}$ of our superspace is then

$$P(\mathcal{V}) \propto \int_{\mathcal{V}} |\Psi|^2 \star 1 \tag{2.25}$$

with $\star 1$ being the volume element [10].

In quantum field theory, solutions to the Klein-Gordon equation are quantized and turned into field operators, thinking in a similar manner about the Wheeler-DeWitt equation has led to proposals that Ψ should be "third quantised" and turned into an operator $\hat{\Psi}$. This operator creates and annihilates Universes in the same way that the familiar ladder operators associated with the Klein-Gordon field create and annihilate particles. Difficulties with this method arise due to the fact that we clearly cannot make measurements on a statistical ensemble of Universes the same way we can for particles. Therefore, it is not known how using this method could lead to measurable probabilities [25, 26, 27].

2.3 Superspace

The space on which our wave function is defined is known as superspace. To construct this space, we start by considering the configuration space of all Riemannian three-metrics $h_{ij}(\mathbf{x})$ and matter field configurations $\Phi(\mathbf{x})$ on a spatial hypersurface Σ

$$\mathrm{Riem}(\Sigma) = \{h_{ij}(\mathbf{x}), \Phi(\mathbf{x}) \mid \mathbf{x} \in \Sigma\}. \tag{2.26}$$

If we can find a diffeomorphism relating a set of configurations, those configurations must have the same intrinsic geometry and we consider them to be equivalent. We now proceed by partitioning this space into equivalence classes such that if two configurations are related by a diffeomorphism, they belong to the same equivalence class. We identify superspace as

$$\mathrm{Sup}(\Sigma) = \mathrm{Riem}(\Sigma)/\mathrm{Diff}_0(\Sigma) \tag{2.27}$$

where the subscript zero indicates that we only consider diffeomorphisms that are connected to the identity. The metric on our infinite dimensional superspace is the Wheeler-DeWitt metric (2.12).

2.4 Minisuperspace

As was previously mentioned, superspace is notoriously difficult to work with due to its infinite dimensionality. The Wheeler-DeWitt equation is in fact not a single equation, but an infinite number of equations, one for each point $\mathbf{x}$ on our spatial hypersurface Σ. It is not known how to solve the Wheeler-DeWitt equation on superspace with modern techniques [26]. Necessity therefore dictates that we seek some simplification. This can be done by freezing all but a finite number of degrees of freedom of the metric and matter fields to obtain what is then known as a minisuperspace model. Although these models are unable to tell us what the exact solution to the Wheeler-DeWitt equation is, they will allow us to learn about the properties it will possess.

2.5 The Path Integral Approach

At this point, it is worth explicitly noting that the Wheeler-DeWitt equation is a second order hyperbolic functional differential equation and therefore requires boundary conditions to specify a particular solution. What those boundary conditions could be will be discussed momentarily. Solving this equation is however not the only path one could take to obtain the wave function of the Universe. To understand the alternative to the canonical quantization approach, we start by thinking back to single-particle quantum mechanics. Working in units where $\hbar = 1$, the propagator taking a particle with position x' at a time t' to a position x at a time t is given by the path integral

$$\psi = \langle x, t | x', t' \rangle = N \int \mathcal{D}x(t) e^{iS[x(t)]}. \tag{2.28}$$

In the expression above, N is a normalisation constant and S is the classical Lorentzian action associated with a path the particle can take. Using this path integral, we can obtain the wave function by summing over all possible paths from a point x' at time t' that pass through x at time t and weighting each path by its associated classical action.

Ground states in quantum mechanics are eigenstates associated with the smallest eigenvalue of the system's Hamiltonian. A system's ground state is calculated by Wick rotating as $t \to -i\tau$ so that we now are left with the Euclidean version of the integral

$$\psi_0 = N \int \mathcal{D}x(\tau) e^{-I[x(\tau)]} \tag{2.29}$$

in which I is the Euclidean action. Here, we sum over the class of paths that have vanishing action in the far past. In analogy with particle mechanics we may write the wave function of the Universe in terms of a path integral. Formally,

$$\Psi[\tilde{h}_{ij}, \tilde{\Phi}, \Sigma] = \sum_{\mathcal{M}} \int_{\mathcal{C}} \mathcal{D}g_{\mu\nu} \mathcal{D}\Phi e^{iS[g_{\mu\nu}, \Phi]}. \tag{2.30}$$

Here, we are summing over a set of manifolds $\mathcal{M}$ whose boundary contains Σ, a three-surface on which the three-metric is $\tilde{h}_{ij}(\mathbf{x})$ and the matter field configuration is $\tilde{\Phi}(\mathbf{x})$. It was naturally suggested by Hartle and Hawking that the ground state of the Universe should be the cosmological analog of a quantum mechanical ground state. However, in the case of quantum cosmology, we do not have the notion of a ground state being the state of lowest energy. The energy of a closed Universe isn't even well defined. Instead, the ground state is associated with the minimum excitation corresponding to the classical notion of a geometry of high symmetry [13]. By once again performing a Wick rotation, we are left with

$$\Psi[\tilde{h}_{ij}, \tilde{\Phi}, \Sigma] = \sum_{\mathcal{M}} \int_{\mathcal{C}} \mathcal{D}g_{\mu\nu} \mathcal{D}\Phi e^{-I[g_{\mu\nu}, \Phi]} \tag{2.31}$$

with I being the Euclidean action of the gravity plus matter system. One reason for doing this is that the Euclidean version of the integral has nicer convergence properties, this is analogous to Wick rotated quantum field theory. Another reason is that a Euclidean approach allows for the inclusion of topologically non-trivial manifolds [13]. Although we have two reasons for formulating the path integral this way, it does come with its own set of issues. Those being that the Euclidean action is unbounded from below leading to the conformal factor problem [28], and the action for a topologically non-trivial manifold can be unbounded from both above and below [29].

Chapter 3

Boundary Conditions

DeWitt, in his 1967 paper speculates that the Wheeler-DeWitt equation could permit just a single unique solution [3]. This conjecture has never come to fruition and it seems that as with most differential equations, when paired with our usually theories of dynamics the Wheeler-DeWitt equation permits not just one solution, but an infinite number. Singling out just one solution to be the wave function of the Universe requires that we apply boundary conditions. There could be many different sets of boundary conditions leading to a Universe in a similar state to our own, but we would like to find the boundary conditions of our Universe. As to what those boundary conditions are is not universally agreed upon. One possibility known as the Tunnelling Proposal was introduced in 1982 by Alexander Vilenkin [14].

3.1 The Tunnelling Proposal

Vilenkin's approach to quantum cosmology involves the Universe spontaneously nucleating from nothing into de Sitter spacetime and then entering an inflationary period. "*Nothing*" in this context refers to the absence of matter, space and also time. Vilenkin originally formulated his proposal in terms of a path integral approach [30, 31] by regarding the transition amplitude between two three-geometries

h_{ij}^1 and h_{ij}^2 with corresponding matter fields Φ^1 and Φ^2 as being given by

$$\sum_{\mathcal{M}} \int_{(h_{ij}^1,\Phi^1)}^{(h_{ij}^2,\Phi^2)} \mathcal{D}g_{\mu\nu}\mathcal{D}\Phi e^{iS[g_{\mu\nu},\Phi]}. \tag{3.1}$$

The wave function of the Universe can then be computed by shrinking the geometry corresponding to h_{ij}^1 to a single point and calculating the transition amplitude between this vanishing three-geometry and the observed configuration of the Universe $(\tilde{h}_{ij}, \tilde{\Phi})$, that is

$$\Psi[\tilde{h}_{ij}, \tilde{\Phi}, \Sigma] = \sum_{\mathcal{M}} \int_{\emptyset}^{(\tilde{h}_{ij},\tilde{\Phi})} \mathcal{D}g_{\mu\nu}\mathcal{D}\Phi e^{iS[g_{\mu\nu},\Phi]}. \tag{3.2}$$

Here the integration is performed over histories lying in the past of $(\tilde{h}_{ij}, \tilde{\Phi})$ so that the Universe evolves from a vanishing three-geometry and not to one. There is also no restriction to compact Euclidean geometries as there is in the Hartle-Hawking proposal. We are to integrate over Lorentzian metrics. An issue that Vilenkin acknowledges with defining our wave function this way is that any Lorentzian geometry that connects our vanishing three-geometry to $\tilde{h}_{ij}$ is inherently singular. He suggests that this singularity could be avoided with an adequate theory of quantum gravity or by discretising spacetime manifolds on scales smaller than the Planck length [30]. At present, this is purely speculative.

Several years after its inception, the proposal was reformulated in terms of a boundary condition on superspace [32]. While not obviously equivalent to the original proposal, the condition is that at singular boundaries of superspace, the wave function includes only outgoing modes (carrying flux out of superspace). Here, ingoing and outgoing modes are playing a similar role to that of positive and negative frequency modes for solutions to the Klein-Gordon equation, although they are not as well defined outside of the semiclassical case. As an additional constraint, the wave function should be bounded everywhere.

A boundary of superspace consists of singular configurations. This be can be due to points or regions where Φ, $\partial_i\Phi$, or the three-curvature ${}^3R$ is infinite, as well as configurations with infinite three-volume. We now consider dividing the boundary of our superspace up into two regions

1. A non-singular boundary consisting of three-geometries in which the singularities are attributed to slicing of regular four-geometries.
2. A singular boundary consisting of everything not in the non-singular boundary.

A semi-classical complex wave function can be expressed in terms of a superposition

$$\Psi = \sum_n C_n e^{iS_n} \tag{3.3}$$

where each S_n is a solution to the Hamilton-Jacobi equation on superspace

$$\frac{1}{2}\left(\nabla S_n\right)^2 + U = 0. \tag{3.4}$$

To each oscillatory WKB mode $C_n e^{iS_n}$, there is an associated current

$$J_n = -|C_n|^2 \nabla S_n. \tag{3.5}$$

What the tunnelling boundary condition is telling us is that the congruence of classical paths defined by each S_n are permitted to end at the singular boundary of superspace but are forbidden from beginning there. Equivalently, $-\nabla S_n$ is required to point out of superspace at the boundaries.

We now consider a model of a closed Universe described by the FRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left(\mathrm{d}\chi^2 + sin^2\chi(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\varphi^2)\right) \tag{3.6}$$

where $0 \leq \chi \leq \pi$, $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$ and an action of the form

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{3a^{-2}}{8\pi}(1 + \dot{a}^2 + a\ddot{a}) + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi)\right]. \tag{3.7}$$

By restricting our metric and scalar fields to be homogenous and isotropic, our only two remaining degrees of freedom are $a(t) \geq 0$ and $\Phi(t) \in \mathbb{R}$ where $a(t)$ is the usual scale factor from (1.1). These two variables shall define our minisuperspace. We can easily show that

$$\int \mathrm{d}^3x\sqrt{-g} = 2\pi^2a^3 \tag{3.8}$$

which allows us to recast our action in the form

$$S = \int \mathrm{d}t\left(\frac{3\pi a}{4}(1 + \dot{a}^2 + a\ddot{a}) + \pi^2a^3\dot{\Phi}^2 - 2\pi^2a^3V(\Phi)\right). \tag{3.9}$$

To simplify, we integrate by parts and drop the boundary term to obtain

$$S = \int \mathrm{d}t\left(\frac{3\pi}{4}(1 - \dot{a}^2)a + \pi^2a^3\dot{\Phi}^2 - 2\pi^2a^3V(\Phi)\right) \tag{3.10}$$

from which we can simply read off the Lagrangian

$$L = \frac{3\pi}{4}(1 - \dot{a}^2)a + \pi^2a^3\dot{\Phi}^2 - 2\pi^2a^3V(\Phi). \tag{3.11}$$

We make use of this expression to find the canonical momenta

$$\pi_a = \frac{\partial L}{\partial \dot{a}} = -\frac{3\pi}{2}a\dot{a} \tag{3.12}$$

$$\pi_\Phi = \frac{\partial L}{\partial \dot{\Phi}} = 2\pi^2a^3\dot{\Phi} \tag{3.13}$$

and apply the standard relationship between the Lagrangian and Hamiltonian to get

$$H = -\frac{1}{3\pi a}\pi_a^2 + \frac{1}{4\pi^2 a^3}\pi_\Phi - \frac{3\pi}{4}a(1 - \frac{8\pi}{3}a^2 V(\Phi)). \tag{3.14}$$

The Wheeler-DeWitt equation for this model can be obtained using standard procedure of canonical quantization where we make the replacements $\pi_a \to -i\frac{\partial}{\partial a}$ and $\pi_\Phi \to -i\frac{\partial}{\partial \Phi}$. We note that there will clearly be some issue of operator ordering as the first term will involves a factor of $1/a$ as well as derivatives with respect to a. Hawking and Page have argued that this issue should be resolved by requiring that the differential operator that appears in the Wheeler-DeWitt equation is the Laplacian operator in the natural metric on superspace [10]. This leads to a Wheeler-DeWitt equation of the form

$$\left(a\frac{\partial}{\partial a}\left(a\frac{\partial}{\partial a}\right) - \frac{3}{4\pi}\frac{\partial^2}{\partial \Phi^2} - \frac{9\pi^2}{4}a^4\left(1 - \frac{8\pi}{3}a^2 V(\Phi)\right)\right)\Psi = 0. \tag{3.15}$$

Instead of this approach, we shall follow Vilenkin by introducing a parameter p, to write down a family of orderings

$$\left(\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - \frac{3}{4\pi a^2}\frac{\partial^2}{\partial \Phi^2} - \frac{9\pi^2}{4}a^2\left(1 - \frac{8\pi}{3}a^2 V(\Phi)\right)\right)\Psi = 0. \tag{3.16}$$

For reasons discussed in section 2.2, this alternative choice of ordering is largely irrelevant to the analysis as we are only interested in the lowest order semiclassical approximation. Note that when $p = 1$, our equation reduces to the form that Hawking and Page have argued for. To simplify, we shall introduce a suitable rescaling of Φ and $V(\Phi)$ as so that our equation is now in the form [32]

$$\left(\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial \Phi^2} - a^2\left(1 - a^2 V(\Phi)\right)\right)\Psi = 0. \tag{3.17}$$

The term $U = a^2\left(1 - a^2 V(\Phi)\right)$ is known as the superpotential and divides our minisuperspace up into a Euclidean region where $U > 0$ and a Lorentzian region where $U < 0$ which roughly coincide with the classically forbidden and classically allowed regions [33]. In the classically allowed region the wave function is oscillatory and in the classically forbidden region the wave function is exponential. The boundary between these two regions occurs at $U = 0$, or equivalently $V(\Phi) = 1/a^2$.

We now make the assumption that the potential is far from this boundary and varies slowly with Φ. We will find that the following condition is required for our approximation to be valid [34]

$$\left|\frac{\mathrm{d}V(\Phi)}{\mathrm{d}\Phi}\right| \ll max\{|V(\Phi)|, \frac{1}{a^2}\}. \tag{3.18}$$

Due our assumptions, it is reasonable to neglect derivatives with respect to Φ allowing us to rewrite the Wheeler-DeWitt equation as

$$\left(\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - a^2\left(1 - a^2 V(\Phi)\right)\right)\Psi = 0. \tag{3.19}$$

It now becomes convenient to set the parameter $p = -1$, as the resulting equation can be solved exactly. We introduce a new variable

$$\kappa = (-2V(\Phi))^{-\frac{2}{3}}\left(1 - a^2 V(\Phi)\right) \tag{3.20}$$

so that the equation to be solved now takes the form of Airy's equation

$$\left(\frac{\partial^2}{\partial \kappa^2} + \kappa\right)\Psi = 0. \tag{3.21}$$

The general solution to this is a linear combination of Airy functions where the coefficients are arbitrary functions of Φ. These functions have a characteristic turning point where the solution changes from exponential to oscillatory. Just as we expect our wave function to have when passing between the classically forbidden and classically allowed region. We shall only be interested in the asymptotic forms of these functions where $\kappa \to \infty$, in this case these are

$$\begin{aligned}
Ai(\kappa) &\approx \frac{1}{2\sqrt{\pi}}\kappa^{-\frac{1}{4}}e^{-\frac{2}{3}\kappa^{\frac{3}{2}}} \\
Bi(\kappa) &\approx \frac{1}{\sqrt{\pi}}\kappa^{-\frac{1}{4}}e^{\kappa^{\frac{3}{2}}} \\
Ai(-\kappa) &\approx \frac{1}{\sqrt{\pi}}\kappa^{-\frac{1}{4}}\sin\left(\frac{2}{3}\kappa^{\frac{3}{2}}+\frac{\pi}{4}\right) \\
Bi(-\kappa) &\approx \frac{1}{\sqrt{\pi}}\kappa^{-\frac{1}{4}}\cos\left(\kappa^{\frac{3}{2}}+\frac{\pi}{4}\right).
\end{aligned} \tag{3.22}$$

As the Universe is expanding, we are only interested in wave functions describing an expanding Universe. In the classically allowed region with $V(\Phi) > 1/a^2$ we require the constraint

$$\frac{i}{\Psi}\frac{\partial\Psi}{\partial a} > 0. \tag{3.23}$$

To see how this arises consider

$$\frac{i}{\Psi}\frac{\partial\Psi}{\partial a} = -\frac{1}{\Psi}\hat{\pi}_a\Psi = \frac{3\pi}{2}a\dot{a}. \tag{3.24}$$

For an expanding Universe, the right hand side of the above expression must be greater than zero. The constraint (3.23) therefore follows.

This is now the point where we impose that condition that the wave function includes only outgoing modes in the classically allowed region where $i\Psi^{-1}\frac{\partial\Psi}{\partial a} > 0$ for $V(\Phi) > 1/a^2$. We find that for negative values of $V(\Phi)$

$$\Psi_T = \frac{Ai(|\kappa|)}{Ai(|\kappa|_{a=0}|)}. \tag{3.25}$$

If we consider the classically allowed region far from the boundary, we see that $\kappa \to \infty$ and $\kappa_{a=0} \to -\infty$. Therefore, as an approximation for the case $a^2V(\Phi) > 1$

$$\Psi_T \propto \exp\left(-\frac{1+i(a^2V(\Phi)-1)^{\frac{3}{2}}}{3V(\Phi)}+\frac{i\pi}{4}\right) \tag{3.26}$$

and for the classically forbidden region where $a^2V(\Phi) < 1$ we have

$$\Psi_T \propto \exp\left(\frac{(1-a^2V(\Phi))^{\frac{3}{2}}-1}{3V(\Phi)}\right). \tag{3.27}$$

The conserved current associated with this wave function has two components

$$\begin{aligned} J^a &= \frac{i}{2}a^p\left(\Psi\partial_a\Psi^* - \Psi^*\partial_a\Psi\right) \\ J^\Phi &= -\frac{i}{2}a^{p-2}\left(\Psi\partial_\Phi\Psi^* - \Psi^*\partial_\Phi\Psi\right). \end{aligned} \tag{3.28}$$

When this current was introduced in a previous section, it was mentioned that J is not positive definite for arbitrary wave functions and issues with negative probabilities can arise. Vilenkin however does not believe this to be an issue [32], his argument being that the wave function Ψ_T is not arbitrary and corresponds to an expanding de Sitter space with scale factor

$$a \approx V^{-\frac{1}{2}}\cosh\left(V^{\frac{1}{2}}t\right). \tag{3.29}$$

If we take a to be our time variable, it can be shown that the current is positive for expanding Universes and negative for contracting ones. In our context, the Universe is expanding so interpretations based on the current are just.

For $p = -1$ and sufficiently large values of a we can construct a conserved probability measure

$$\mathrm{d}P_T = J\cdot\mathrm{d}\Sigma =\propto \exp\left(-\frac{2}{3V(\Phi)}\right)\mathrm{d}\Phi \tag{3.30}$$

so that

$$P_T\left(\Phi_2, \Phi_1\right) = \int_{\Phi_1}^{\Phi_2} J \cdot \mathrm{d}\Sigma \tag{3.31}$$

is the probability for the scalar field to lie in the interval (Φ_1, Φ_2) at a given value of the scale factor.

3.2 The No-Boundary Proposal

The singularity theorems of Hawking and Penrose show that the Universe cannot have a classical beginning corresponding to a Lorentzian geometry. This provided partial motivation for the idea which evolved into the no-boundary proposal. During a conference in the Vatican in 1981, Hawking made the conjecture that the Universe began with a regular Euclidean geometry having four spatial dimensions. Later, this made a quantum transition to a Lorentzian geometry with three spatial dimensions and one time dimension [35].

It was previously discussed that as an alternative to the canonical quantization procedure, one may use the path integral formalism to determine the wave function. It is through this method that the no-boundary proposal is most naturally expressed. In essence, the proposal is a topological statement concerning the class of histories to be summed over. To calculate the wave function of the Universe, Hartle and Hawking would have you restrict the sum in (2.31) strictly to compact Euclidean geometries for which the only boundary of the geometry is Σ, a compact three-surface on which the three-metric is $\tilde{h}_{ij}(\mathbf{x})$ and the matter field configuration is $\tilde{\Phi}(\mathbf{x})$. Halliwell and Hawking have shown that this provides a boundary condition on superspace [36]. For the following, it is convenient to include the lapse function in our FRW metric so that it now reads

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\mathrm{d}\Omega_3^2. \tag{3.32}$$

In the gauge $\dot{N} = 0$, the no-boundary wave function (NBWF) for a closed Universe is given by

$$\Psi_{NB} = \int \mathrm{d}N \int \mathcal{D}\Phi\mathcal{D}a e^{-I[a,\Phi,N]}. \tag{3.33}$$

It is convenient to impose a parameterisation where the three-surface Σ corresponds to $\tau = 1$, that means

$$\begin{aligned} &h_{ij}(\mathbf{x}, 1) = \tilde{h}_{ij}(\mathbf{x}) \\ &\Phi(\mathbf{x}, 1) = \tilde{\Phi}(\mathbf{x}). \\ &a(1) = \tilde{a} \end{aligned} \tag{3.34}$$

If we also choose the initial point to correspond to $\tau = 0$, it can be shown that the Euclidean action for this model is

$$I = \frac{1}{2}\int_0^1 \mathrm{d}\tau N\left[-\frac{a}{N^2}\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)^2 + \frac{a^3}{N^2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}\right)^2 - a + a^3V(\Phi)\right]. \tag{3.35}$$

To obtain the two field equations for this action, we vary with respect to a and Φ, a third equation is obtained by imposing the saddle-point condition $\frac{\partial I}{\partial N} = 0$.

$$\frac{1}{N^2}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\tau^2} + \frac{3}{Na}\frac{\mathrm{d}a}{\mathrm{d}\tau}\frac{\mathrm{d}\Phi}{\mathrm{d}\tau} - \frac{1}{2}\frac{\mathrm{d}V(\Phi)}{\mathrm{d}\Phi} = 0 \tag{3.36}$$

$$\frac{1}{N^2 a}\frac{\mathrm{d}^2 a}{\mathrm{d}\tau^2} + \frac{2}{N^2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}\right)^2 + V(\Phi) = 0 \tag{3.37}$$

$$\frac{1}{N^2}\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)^2 - \frac{a^2}{N^2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}\right)^2 - 1 + a^2V(\Phi) = 0 \tag{3.38}$$

To obtain a wave function, we must impose a condition at the initial point of the class of histories to sum over in (3.33). For the no-boundary proposal, we require the four-geometry to close off in a regular way. This means that $a(\tau) \approx N\tau$ as $\tau \to 0$ so that the Euclidean form of the metric approaches the metric of flat space in spherical coordinates. We therefore must have the condition

$$a(0) = 0. \tag{3.39}$$

This can immediately be seen to be problematic if no further conditions are imposed since it will cause the second term of equation (3.36) to blow up. To prevent this we also require the scalar field to satisfy

$$\left.\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}\right|_{\tau=0} = 0. \tag{3.40}$$

The solution to (3.37) that is compatible with the above conditions is

$$a(\tau) \approx \frac{\sin(V^{\frac{1}{2}}N\tau)}{\sin(V^{\frac{1}{2}}N)}. \tag{3.41}$$

We now substitute this back into the saddle-point condition (3.38)

$$\frac{V\tilde{a}^2\cos^2(V^{\frac{1}{2}}N\tau)}{\sin^2(V^{\frac{1}{2}}N)} - 1 + a^2V = 0 \tag{3.42}$$

and evaluate at $\tau = 1$ to obtain

$$\tilde{a}^2V\left(1 + \cot^2(V^{\frac{1}{2}}N)\right) = 1 \tag{3.43}$$

which finally simplifies down to give us

$$\sin^2(V^{\frac{1}{2}}N) = \tilde{a}^2V. \tag{3.44}$$

As we are only interested in real values of the potential and scale factor, we shall consider the case $\tilde{a}^2V < 1$ for which there are a countably infinite number of solutions that we shall parameterise by $n \in \mathbb{Z}$

$$N_n^{\pm} = \frac{1}{V^{\frac{1}{2}}}\left[\left(n+\frac{1}{2}\right)\pi \pm \cos^{-1}(\tilde{a}V^{\frac{1}{2}})\right]. \tag{3.45}$$

If we set $n = 0$ and take $\cos^{-1}(\tilde{a}V^{\frac{1}{2}})$ to lie in the principal range $(0, \frac{\pi}{2})$, we find [37]

$$a(\tau) \approx \frac{1}{V^{\frac{1}{2}}}\sin\left[\left(\frac{\pi}{2} \pm \cos^{-1}(\tilde{a}V^{\frac{1}{2}})\right)\tau\right]. \tag{3.46}$$

We can then use this to evaluate our action, we have now obtained two possible solutions

$$I_{\pm} = -\frac{1}{3V(\tilde{\Phi})}\left[1 \pm \left(1-\tilde{a}^2V(\tilde{\Phi})\right)^{\frac{3}{2}}\right]. \tag{3.47}$$

Here, the (-) solution corresponds to the three-sphere being closed off by less than half of a four-sphere while the (+) solution corresponds to it being closed off by more than half of a four-sphere. Despite all of this work, we still find that this is not enough to fully specify a wave function. There is still the issue of choosing a contour that one is to perform the integration over and it has been shown that different convergent contours are dominated by different saddle-points leading to different wave function [38, 39, 40]. The no-boundary proposal offers no guidance in making a choice. Although no contour stands out as being preferred above all else, some are clearly better than others. Halliwell and Hartle argued in [41] that any sensible contour should satisfy the following five physical constraints:

1. The integral should converge.
2. The wave function should be compatible with the diffeomorphims invariance implemented by the momentum constraint.
3. When the Universe is large, classical spacetime should be predicted.
4. The correct theory in a curved spacetime should be reproduced in this spacetime.

5. To the extent that wormholes make the cosmological constant dependent on initial conditions the wave function should predict its vanishing.

It was shown in the same paper that contours dominated by saddle-points corresponding to negative lapse functions lead to difficulties in recovering quantum field theory in a curved space time. While this provides us with a good reason for excluding contours for which the dominating contribution is from a saddle-point with $n < 0$, there does not appear to be any good reason for preferring a contour for which the dominating contribution comes from a saddle-point associated with any particular $n \geq 0$.

When Hartle and Hawking were originally faced with this problem in [13], they argued that the steepest-descent contour does not pass through the extremum corresponding to I_+ and so it is I_- that gives the dominant contribution. Although not all have found this argument totally convincing [34], Hartle and Hawking obtained a wave function

$$\Psi_{NB} \propto \exp\left[\frac{1-(1-a^2V(\Phi))^{\frac{3}{2}}}{3V(\Phi)}\right] \tag{3.48}$$

in the classically forbidden region where $a^2V(\Phi) < 1$, and

$$\Psi_{NB} \propto \exp\left[\frac{1}{3V(\Phi)}\right]\cos\left[\frac{(a^2V(\Phi)-1)^{\frac{3}{2}}}{3V(\Phi)}-\frac{\pi}{4}\right] \tag{3.49}$$

in the classically allowed region where $a^2V(\Phi) > 1$. Finally, in the semiclassical approximation it is possible to construct a probability measure on a set of paths $J \cdot d\Sigma$, where Σ is a surface of constant a

$$dP_{NB} = J \cdot d\Sigma \propto \exp\left(\frac{2}{3V(\Phi)}\right) d\Phi. \tag{3.50}$$

Although we have taken the standard approach found in the literature to obtain the

NBWF of the Universe [26, 34], it is possible to specify a NBWF without relying on a functional integral at all. The semiclassical NBWF can be defined as a collection of appropriately regular saddle-points of the action of the dynamical theory coupled to gravity [17]. In the following we shall keep the factors of $\hbar$ explicit and write a wave function in the form

$$\Psi[\tilde{h}_{ij}, \tilde{\Psi}, \Sigma] = \sum_n C_n \exp\left(-I_n[\tilde{h}_{ij}, \tilde{\Phi}, \Sigma]/\hbar\right). \tag{3.51}$$

The wave function must satisfy the Wheeler-DeWitt equation which in minisuperspace is often written in the covariant form

$$\hat{H}\Psi = \left(-\frac{1}{2}\hbar^2\nabla^2 + U\right)\Psi = 0. \tag{3.52}$$

If we now Taylor expand this in powers of $\hbar$,

$$\sum_n e^{-I_n/\hbar}\left\{C_n\left[-\frac{1}{2}(\nabla I_n)^2 + U\right] + \hbar\left[\nabla I_n \cdot \nabla C_n + \frac{1}{2}C_n\nabla^2 I_n\right] + \mathcal{O}(\hbar^2)\right\} = 0 \tag{3.53}$$

so that this holds true for any choice of coefficients, we can see that from the terms of lowest order that each I_n is a solution to the Euclidean Hamilton-Jacobi equation

$$-\frac{1}{2}(\nabla I_n)^2 + U = 0. \tag{3.54}$$

If we demand that each I_n is the action $I_{sp}[\tilde{h}_{ij}, \tilde{\Phi}, \Sigma]$ of a saddle-point of the action defining the dynamical theory, the equation is satisfied. This means our wave function is a linear combination of saddle-points of the action associated with our dynamical theory where the geometries have a spacelike boundary Σ on which the three-metric and matter field match the data $(\tilde{h}_{ij}, \tilde{\Phi})$

$$\Psi[\tilde{h}_{ij}, \tilde{\Phi}, \Sigma] = \sum_{sp} C_{sp} \exp\left(-I_{sp}\left[\tilde{h}_{ij}, \tilde{\Phi}, \Sigma\right]/\hbar\right). \tag{3.55}$$

The resulting wave function depends on the collection of saddle-points we sum over and different sets shall lead to different wave functions and different predictions. Admittedly this scenario is incomplete in an analogous way to the functional integral formulation of the no-boundary proposal in that choosing that collection remains an open problem. In the same way that Halliwell and Hartle suggested restrictions to be placed on the contour of integration for the NBWF in [13], three similar constraints on the collection of saddle-points are provided in [17]:

1. The wave function must satisfy the constraints of general relativity and the matter theory.

2. The wave function must be consistent with the principles of the quantum framework used to make predictions. For example, it must be normalisable in an appropriate Hilbert space inner product.

3. At least at the semiclassical level, the wave function must provide predictions that are consistent with observation.

It is important that the set only includes saddle-points with damped fluctuations so that we do not risk the normalisation principle. Clearly, if I_{sp} is a solution to (3.54), $-I_{sp}$ and I_{sp}^* are also solutions. One must be careful not to include both I_{sp} and $-I_{sp}$ in our sum of saddle-points as if one is associated with damped fluctuations, the other will be associated with anti-damped fluctuations. On the other hand, if for every I_{sp} included in the sum we also include I_{sp}^* and weight them with equal coefficients, the result shall be a real wave function. Wave functions of this form such as the NBWF originally obtained by Hartle and Hawking can be shown to be CPT-invariant.

3.3 CPT-Invariance of the NBWF

In 1985, Hawking published a paper on the arrow of time in cosmology [42] in which he made what he considered to be his "*greatest mistake*" [43]. Previously, at the 11^{th} Solvay Conference on Physics, Thomas Gold had proposed that the thermodynamic arrow of time and the cosmological arrow of time must always point in the same direction [44]. With this proposal in mind, Hawking concluded that if the Universe were to eventually recollapse, the thermodynamic arrow, at the moment of maximum expansion and entropy would reverse as to agree with the cosmological arrow. As the psychological arrow of time is presumably a consequence of the thermodynamic arrow, this would bizarrely imply that a conscious observer would then remember the future but not the past. An influencing factor that led him to this conclusion was the fact that his no-boundary quantum state was CPT-invariant. The CPT-theorem is recognised as a property of all fundamental physical laws and states that those laws are invariant under the combination of charge conjugation, space inversion, and time reversal. Therefore, the fact that the no-boundary wave function exhibits this property is an encouraging sign. To prove this feature, we will follow Hawking in considering the Laplace transformation

$$\xi(\tilde{h}_{ij}, K_E, \tilde{\Phi}) = \int \mathrm{d}h^{\frac{1}{2}} \exp\left[-\frac{m_p^2 K_E}{18\pi}\int K_E h^{\frac{1}{2}} \mathrm{d}^3x\right]\Psi(h_{ij}, \tilde{\Phi}). \tag{3.56}$$

Here, $h^{\frac{1}{2}}$ is the square root of the determinant of the three-metric and K_E is the trace of the Euclidean extrinsic curvature tensor. The Laplace transformation ξ is holomorphic for $Re(K_E) > 0$ so we may analytically continue ξ to Lorentzian values in K_E by defining $K_L = iK_E$ which is interpreted as the rate of expansion of the boundary Σ. Upon doing so it is easy to show that for real values of k_L and a real wave function Ψ

$$\xi(\tilde{h}_{ij}, K_L, \tilde{\Phi}) = \xi^*(\tilde{h}_{ij}, -K_L, \tilde{\Phi}) \tag{3.57}$$

From this statement, we have recovered T-invariance for the wave function of the Universe from the no-boundary proposal. As Hawking interprets $|\xi(\tilde{h}_{ij}, K_L, \Phi)|^2$ as being the probability that the Universe contains a three surface Σ with conformal metric $\tilde{h}_{ij}$ and rate of expansion K_L he concludes that "*if the wave function represents an expanding phase of the Universe, then it will also represent a contracting one*".

Next, we consider a situation in which our field Φ is complex so that it can permit charge. As is often convenient, in this situation we shall treat Φ and Φ^* as independent variables instead of using an equivalent method where we treat the real and imaginary part of our scalar field independently of one another. We now must integrate over both variables in our path integral to construct our wave function, that is

$$\Psi[\tilde{h}_{ij}, \tilde{\Phi}, \tilde{\Phi}^*] = \sum_M \int \mathcal{D}g_{\mu\nu}\mathcal{D}\Phi\mathcal{D}\Phi^* e^{-I[g_{\mu\nu},\Phi,\Phi^*]} \tag{3.58}$$

Due to the fact that our scalar field is no longer real, the Euclidean action is no longer required to be real, however as our two variables must appear in a rather symmetric configuration within the action, it is true that

$$I[g_{\mu\nu}, \Phi, \Phi^*] = I^*[g_{\mu\nu}, \Phi^*, \Phi]. \tag{3.59}$$

from which it can easily be shown

$$\Psi[\tilde{h}_{ij}, \tilde{\Phi}, \tilde{\Phi}^*] = \Psi^*[\tilde{h}_{ij}, \tilde{\Phi}^*, \tilde{\Phi}]. \tag{3.60}$$

Using the above, one can arrive at the statement of CT-invariance

$$\xi(\tilde{h}_{ij}, K_L, \tilde{\Phi}) = \xi^*(\tilde{h}_{ij}, -K_L, \tilde{\Phi}^*) \tag{3.61}$$

To finally arrive at CPT-invariance, we introduce fermion fields ψ and a set of three co-vectors e^a_μ with a directed unit normal e^0_μ to Σ. The wave function $\Psi[e^a_\mu, \tilde{\psi}]$ is a function of the co-vectors and fermion fields evaluated on the three-surface. Using the fact that

$$I[e^a_\mu, \psi, \psi^\dagger] = I^*[-e^a_\mu, \hat{C}\psi, (\hat{C}\psi)^\dagger] \tag{3.62}$$

where $\hat{C}$ is operator that implements the charge-conjugation transformation, Hawking then arrived at

$$\Psi[e^a_\mu, \tilde{\psi}] = \Psi[-e^a_\mu, \hat{C}\tilde{\psi}] \tag{3.63}$$

which is the expression for CPT-invariance for the no-boundary quantum state of the Universe. This means that if Ψ_{NB} is a solution to the Wheeler-DeWitt equation, then so is $\hat{\theta}\Psi_{NB}$ where $\hat{\theta}$ is the anti-unitary operator that implements the CPT transformation. Not long after Hawking's paper was published, Page published a paper correcting him [45] where he pointed out CPT-invariance of the wave function does not imply that an individual history is also CPT-invariant. It only implies that if a particular history is contained by the wave function, then the wave function must also contain the CPT image of that history with the same probability. This in no way suggests that the thermodynamic arrow of time will reverse in accordance with the cosmological arrow during the Universe's contracting phase. While Hawking accepted the mistake and abandoned the idea, he still believed that the no-boundary proposal is very consequential for the thermodynamic arrow of time and argues in [46] that it naturally arises from these boundary conditions.

3.4 Inflation

Clearly the Universe has achieved a sufficient amount of inflation so that large scale structure and even observers can emerge. At least in the case of a minisuperspace model, both sets of boundary conditions previously mentioned in this chapter give rise to a wave function predicting a period of inflation. The amount of inflation a Universe undergoes is determined by the initial value Φ_0 of our scalar field. It is the potential energy of this field which allows the Universe to expand in an exponential manner for a short while. If this value is too small, the model will predict that the Universe will expand and recollapse over a period of time that is too short for the emergence of large scale structure. Therefore, we shall make the same restriction as Hawking and Page have done in previous work [10] and not concern ourselves with values of Φ_0 less than a given small value which we shall call Φ_{min}. We would like our model to predict a sufficient amount of inflation, roughly 60 e-folds [47, 48] as to provide a satisfactory explanation to the flatness and horizon problems of classical cosmology. This motivates us to define a value Φ_{suff} such that the Universe experiences sufficient inflation for $\Phi_0 > \Phi_{suff}$. To find an approximate value for this parameter we shall use that the number of e-folds N_e can be written as

$$N_e = 6 \int_{\Phi_e}^{\Phi_0} \mathrm{d}\Phi \frac{V(\Phi)}{V'(\Phi)} \tag{3.64}$$

where Φ_e is the value of the scalar field at the end of the inflationary period. If we now take our potential to be a chaotic type of the form $V(\Phi) = m^2\Phi^2$, the integral (3.64) evaluates to

$$N_e = \frac{3}{2}(\Phi_0^2 - \Phi_e^2). \tag{3.65}$$

Achieving $N_e > 60$ requires that $\Phi_0^2 - \Phi_e^2 > 40$. We now use the slow-rolling condition that $|V(\Phi)/V''(\Phi)| < 9$ to give us $\Phi_e^2 > \frac{2}{9}$. This implies that $\Phi_0^2 > \frac{362}{9}$ allowing us to conclude that a sufficient amount of inflation occurs at $\Phi_{suff} \approx 6.3$. Vilenkin has pointed out [32] that for large values of Φ, the potential $V(\Phi)$, is likely to far exceed the Planck energy density unless it is of a very special shape. As the derivation of our probability density has been based on a semiclassical approximation, it would not be wise to trust the predictions of our minisuperspace model in this region. We are therefore once again motivated to define another value Φ_{max} which is the largest value of Φ_0 for which we shall trust our model. Given that the initial value for the scalar field lies somewhere within the range $\Phi_{min} < \Phi_0 < \Phi_{max}$, we would like to calculate the probability that it is greater than Φ_{suff}. That is, we would like to evaluate the conditional probability

$$P\left(\Phi_0 > \Phi_{suff} | \Phi_{min} < \Phi_0 < \Phi_{max}\right) = \frac{\int_{\Phi_{suff}}^{\Phi_{max}} \mathrm{d}\Phi \exp\left(\pm\frac{2}{3V(\Phi)}\right)}{\int_{\Phi_{min}}^{\Phi_{max}} \mathrm{d}\Phi \exp\left(\pm\frac{2}{3V(\Phi)}\right)} \tag{3.66}$$

which we rewrite in a form that shall prove to be more useful momentarily

$$P\left(\Phi_0 > \Phi_{suff} | \Phi_{min} < \Phi_0 < \Phi_{max}\right) = 1 - \frac{\int_{\Phi_{min}}^{\Phi_{suff}} \mathrm{d}\Phi \exp\left(\pm\frac{2}{3V(\Phi)}\right)}{\int_{\Phi_{min}}^{\Phi_{max}} \mathrm{d}\Phi \exp\left(\pm\frac{2}{3V(\Phi)}\right)}. \tag{3.67}$$

Instead of evaluating this integral directly, it is useful to think about it graphically. Figure 3.1 shows a plot of the probability distribution for both wave functions. For the tunnelling wave function, the probability distribution is small in the region about Φ_{min} meaning

$$\int_{\phi_{min}}^{\Phi_{max}} \mathrm{d}\Phi \exp\left(\frac{2}{3V(\Phi)}\right) >> \int_{\Phi_{min}}^{\Phi_{suff}} \mathrm{d}\Phi \exp\left(\frac{2}{3V(\Phi)}\right) \tag{3.68}$$

and $P\left(\Phi_0 > \Phi_{suff} | \Phi_{min} < \Phi_0 < \Phi_{max}\right) \approx 1$ so that sufficient inflation is overwhelmingly likely. For the no-boundary wave function this is not the case. The region about Φ_{min} represents a peak so that

$$\int_{\Phi_{min}}^{\Phi_{suff}} \mathrm{d}\Phi \exp\left(-\frac{2}{3V(\Phi)}\right) \approx \int_{\phi_{min}}^{\Phi^{max}} \mathrm{d}\Phi \exp\left(-\frac{2}{3V(\Phi)}\right) \tag{3.69}$$

and $P\left(\Phi_0 > \Phi_{suff} | \Phi_{min} < \Phi_0 < \Phi_{max}\right) << 1$. While any non-vanishing probability means that sufficient inflation is still possible, the no-boundary wave function seems strongly biased towards little inflation. In [10] Hawking and Page attempt to escape

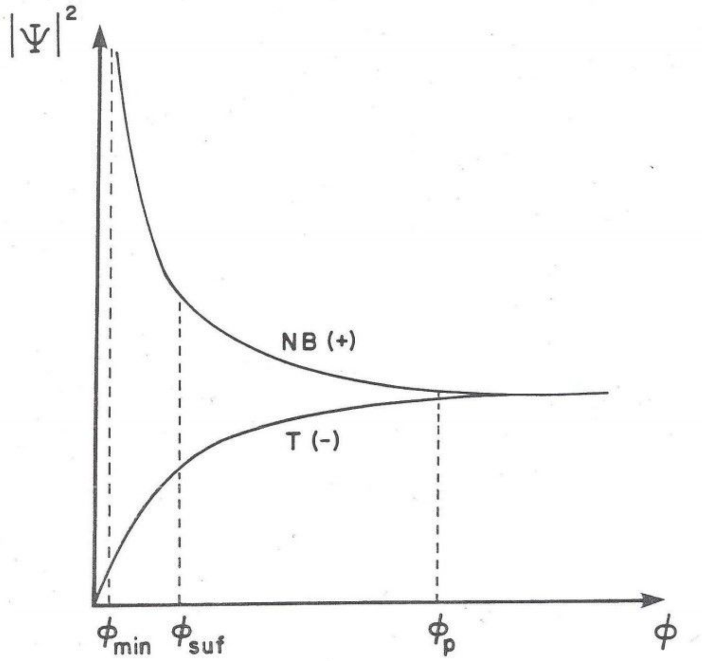

Figure 3.1: This is a graphical illustration comparing the probability distributions associated with the tunnelling and no-boundary wave functions. It has been taken from [34].

from this by taking $\Phi_{max} \to \infty$ but as Vilenkin has pointed out, this is flawed.

A further and prehaps more realistic attempt to rescue inflation as a prediction of the no-boundary proposal was made by Hawking and collaborators [49, 50] where it is argued that the measure that has previously been used is in need of an amendment. The measure $\mathrm{d}P$ corresponds to entire cosmological histories. As many of these Universes recollapse or fail to leave the quantum regime, they do not evolve into classical Universes such as our own. The proposal Hawking argues for is that we should condition on the fact that the Universe contains a Hubble volume that corresponds to our data so that the predictions made are relevant to the Universe we exist in. The probability measure should therefore be proportional to the number of such Hubble volumes, which in turn is proportional to e^{3N_e}. This implies that the probability measure should have been "*volume weighted*" by this factor

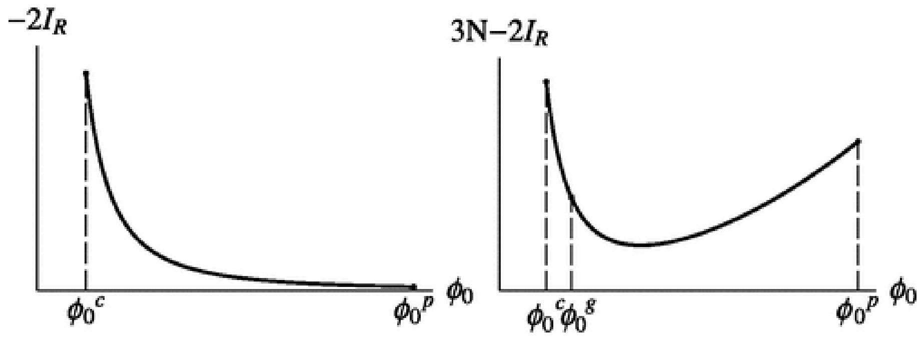

Figure 3.2: This is a heuristic plot of how the argument of the probability distribution changes upon application of volume-weighting. The graphic has been taken from [51].

$$\mathrm{d}P_{volume-weighted} = e^{3N_e}\mathrm{d}P. \tag{3.70}$$

In some crude sense, the volume-weighting is how we have accounted for the fact that our existence and our observations are more likely to occur in a larger Universe rather than a smaller one. While this transforms our probability distribution in a way that the theory now favours larger amounts of inflation necessary for explaining the Universe's homogeneity and spatial flatness, a new issue arises. The theory now predicts that the Universe has emerged from a region of eternal inflation [50, 52, 53].

The prospect of eternal inflation is problematic for cosmological models as it predicts inflation may end locally, but never globally. This results in a picture where the inflating regions undergo eternal exponential growth while pieces known as bubble Universes break off ad infinitum. Eternal inflation therefore produces an infinite fractal-like multiverse. If this is the case, anything permitted to occur by the laws of physics does so an infinite number of times and conditioning on any observational data in such a scenario becomes trivial. Theories of such a Universe are often severely defective as when attempting to extract a meaningful prediction one is often met with ratios of infinite quantities. Alan Guth has famously said "*In a single Universe, cows born with two heads are rarer than cows born with one head. But in an infinitely branching multiverse, there are an infinite number of one-headed cows and* *an infinite number of two-headed cows. What happens to the ratio?*" This is a typical way in which the "*Measure Problem*" of cosmology arises.

Chapter 4

Lorentzian Quantum Cosmology

Despite the success of Euclidean geometry in understanding effects of quantum gravity [54], in recent years some have proposed that a Lorentzian path integral is a better starting point for quantum cosmology than the Euclidean version [5]. Feldbrugge, Lehners and Turok believe that if one is to formulate the wave function of the Universe in terms of a Lorentzian path integral, causal and unitary behaviour shall follow. They also claim that the steepest-descent contour running through the saddle-point of the Euclidean action corresponding to Hartle and Hawking's NBWF bears no relation to Lorentzian path integral. Therefore, similar behaviour should not be expected to follow from the Euclidean analog which they believe to be a "*meaningless divergent integral*".

4.1 Picard-Lefschetz Theory

Although tempting to immediately dismiss the Lorentzian path integral (2.30) on the grounds that it is highly oscillatory and not absolutely convergent, the authors were highly aware of this issue and employ techniques of Picard-Lefschetz theory as a method of dealing with integrals such as

$$\int_D \mathrm{d}x e^{iS[x]/\hbar}. \tag{4.1}$$

Much of this discussion shall follow [5] as we are only interested in building the minimum tool kit necessary in order to understand the arguments of Turok *et al.*. However, a more complete overview of Picard-Lefschetz theory can be found in [55]. In (4.1), the integration is over a real domain D, $\hbar$ is a small parameter, and $S[x]$ is a real valued function which we shall now interpret as being holomorphic in the complex plane of the variable x. Keeping the end points of our contour fixed, Cauchy's theorem licenses us to deform D into a contour of steepest-descent passing through the critical points of $S[x]$, i.e. points that satisfy $\partial_x S[x] = 0$. Taking the derivative of $S[x]$ with respect to x along the $Re[x]$-axis we have

$$\frac{\partial S}{\partial x} = \frac{1}{2}\left(\frac{\partial Re(S)}{\partial Re(x)} + i\frac{\partial Im(S))}{\partial Re(x)}\right) \tag{4.2}$$

for this to equal zero, both terms on the RHS must separately be equal to zero. By virtue of the Cauchy-Reimann equations, it also must be the case that $\frac{\partial Re(S)}{\partial Im(x)} = 0$ from which we can conclude that a critical point of $S[x]$ corresponds to a saddle-point of $Re[iS[x]]$ in the complex plane of x. This is significant as $Re[iS[x]]$ governs the magnitude of the integrand via the relationship $|e^{iS[x]/\hbar}| = e^{Re(iS[x])/\hbar}$. Thus, the contour of steepest-descent corresponds to the path over which $Re[iS[x]]$ decreases most rapidly. These contours shall lead to a convergent integral and in this case are called Lefschetz thimbles $\mathcal{J}_\sigma$.

We now write $\mathcal{I} = iS/\hbar$ and split both this and x in to its real and imginary parts, $\mathcal{I} = h + iH$ and $x = u^1 + iu^2$. Introducing λ to parameterise our paths, we define a downward flow to be the one which satisfies

$$\frac{\mathrm{d}u^i}{\mathrm{d}\lambda} = -g^{ij}\frac{\partial h}{\partial u^j} \tag{4.3}$$

where g_{ij} is a Riemannian metric on the complex plane. The real part of $\mathcal{I}$ is known as the Morse function h. With the exception of the trivial solution that remains at the critical point for all λ, the Morse function is strictly decreasing along a flow which can easily be seen under application of the chain rule

$$\frac{\mathrm{d}h}{\mathrm{d}\lambda} = \sum_i \frac{\partial h}{\partial u^i}\frac{\mathrm{d}u^i}{\mathrm{d}\lambda} = -\sum_i \left(\frac{\partial h}{\partial u^i}\right)^2. \tag{4.4}$$

We now see that the term on the RHS is negative due to the fact that we are dealing with a Riemannian metric, our claim then follows. The path of steepest-descent is now identified with the flow because the gradient takes the largest possible magnitude. Similarly, it is possible to define upward flows $\mathcal{K}_\sigma$, from the equation

$$\frac{\mathrm{d}u^i}{\mathrm{d}\lambda} = g^{ij}\frac{\partial h}{\partial u^j} \tag{4.5}$$

which we shall identify with paths of steepest-ascent. As we are now required to choose a metric, we shall pick the obvious Kähler metric $\mathrm{d}s^2 = |\mathrm{d}x|^2$ so that from the downward flow equation we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}\lambda} = -\frac{\partial I^*}{dx^*}. \tag{4.6}$$

Using this, we can clearly see that the imaginary part of $\mathcal{I}$ is conserved along the flow

$$\frac{\mathrm{d}H}{\mathrm{d}\lambda} = \frac{1}{2i}\frac{\mathrm{d}(\mathcal{I}-\mathcal{I}^*)}{\mathrm{d}\lambda} = \frac{1}{2i}\left(\frac{\partial I}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\lambda} - \frac{\partial I^*}{\partial x^*}\frac{\mathrm{d}x^*}{\mathrm{d}\lambda}\right) = 0. \tag{4.7}$$

As we have seen that the imaginary part of the integrand $e^{iS[x]/\hbar}$ remains fixed along the downward flow while the real part decreases monotonically, the integral converges absolutely. Although unlikely, there is a subtlety here that shall be addressed. That being the special case in which the steepest-descent contour for a saddle-point P_σ matches the steepest-ascent contour for the a separate saddle-point P'_σ. While uncommon, this may occur as a result of a symmetry. It can be treated by adding infinitesimal perturbations to $S[x]$ that violate the symmetry so that the degeneracy between the imaginary part of the exponent H, at the two saddle-points is broken. Equation (4.7) tells us that H is conserved along the flow so that the imaginary parts can no longer coincide. We then define the contour $\mathcal{C}$ in the limit that these perturbations are taken to zero so that they have negligible contribution to the integral. Upon doing this, we shall find that each saddle-point P_σ corresponds to a unique steepest-descent contour $\mathcal{J}_\sigma$, and steepest-ascent contour $\mathcal{K}_\sigma$. As was our original goal, we now deform our contour of integration from the real axis so that it becomes a linear combination of Lefschetz thimbles

$$\mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma. \tag{4.8}$$

The coefficients n_σ are known as the intersection numbers and take values ± 1 or 0. Although you may expect all critical points to contribute to the integral with $n_\sigma = 1$, in general the Lefschetz thimbles do not have natural orientations so we must also allow for the possibility $n_\sigma = -1$ so to avoid pathological exponential growth. For critical points that would make very large contributions to the integral, we set $n_\sigma = 0$ [55]. From this discussion, we can rewrite the integral of interest as

$$\int_D \mathrm{d}x e^{iS[x]/\hbar} = \int_{\mathcal{C}} \mathrm{d}x e^{iS[x]/\hbar} = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathrm{d}x e^{iS[x]/\hbar} \tag{4.9}$$

which is absolutely convergent if

$$\int_{\mathcal{J}_\sigma} |\mathrm{d}x| e^{h(x)} < \infty \tag{4.10}$$

for all Lefschetz thimbles $\mathcal{J}_\sigma$. Expanding the right hand side of (4.9) in powers of $\hbar$, we obtain our final result

$$\int_D \mathrm{d}x e^{iS[x]/\hbar} = \sum_\sigma n_\sigma e^{iH(p_\sigma)} \int_{\mathcal{J}_\sigma} e^h \mathrm{d}x \approx \sum_\sigma n_\sigma e^{iS(p_\sigma)/\hbar}[A_\sigma + \mathcal{O}(\hbar)]. \tag{4.11}$$

In the above $S(p_\sigma)$ is the action associated with p_σ and A_σ is the value of the leading-order Gaussian integral about the same critical point.

4.2 The No-Boundary Dispute

We begin by considering a model of the Universe with a positive cosmological constant Λ, a metric of the form of (3.32), and an Einstein-Hilbert action (2.3) with $S_m = 0$. We shall work in units where $m_p = 1$ and $8\pi G = 1$ so that our action takes the form

$$S = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}(R - 2\Lambda) + \int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}K. \tag{4.12}$$

Instead of Wick rotating as Hartle and Hawking did in [13], Turok *et al.* attempt to evaluate the path integral (2.30) directly. It has long been known integrating the lapse function over the entire real number line leads to a solution of the Wheeler-DeWitt equation, this is what was done in section 3.2. On the other hand, if we restrict the integration to a half-infinite contour extending from zero to infinity, we obtain a Green's function for the Wheeler-DeWitt equation instead [56]. This is the approach the Turok and his collaborators have taken. Their starting point, is the Feynman path integral

$$G[a_1; a_0] = \int \mathcal{D}N\mathcal{D}\pi\mathcal{D}a\mathcal{D}p\mathcal{D}C\mathcal{D}\bar{P}e^{\frac{i}{\hbar}\int_0^1[\dot{N}\pi + \dot{a}p + \dot{C}\bar{P} - NH]dt} \tag{4.13}$$

where a fermionic ghost C, along with conjugate momenta p, π, and $\bar{P}$ have been introduced. Working in a minisuperspace, it then becomes possible to evaluate this path integral analytically. The expression above reduces to

$$G[a_1;a_0] = \int_0^\infty \mathrm{d}N \int_{a_0}^{a_1} \mathcal{D}a e^{iS(N,a)/\hbar} \tag{4.14}$$

and the action is now

$$S = 2\pi^2 \int_0^1 \mathrm{d}tN \left(-3a\frac{\dot{a}^2}{N^2} + 3a - a^3\Lambda\right). \tag{4.15}$$

If we now redefine the lapse function $N \to N/a$ and introduce $q \equiv a^2$ the action becomes quadratic in q

$$S = 2\pi^2 \int_0^1 \mathrm{d}t \left(-\frac{3}{4N}\dot{q}^2 + N(3 - \Lambda q)\right). \tag{4.16}$$

From the Lagrange-Euler equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$, we obtain an equation of motion

$$\ddot{q} = \frac{2\Lambda}{3}N^2 \tag{4.17}$$

for $q(0) = q_0$ and $q(1) = q_1$ the solution can easily be found to be

$$\bar{q} = \frac{\Lambda}{3}N^2t^2 + \left(-\frac{\Lambda}{3}N^2 + q_1 - q_0\right)t + q_0. \tag{4.18}$$

We also have a Hamiltonian constraint $\frac{\partial L}{\partial N} = 0$ which gives us

$$\frac{3}{4N^2}\dot{q}^2 + 3 = \Lambda q. \tag{4.19}$$

Imposing the constraint gives us the full solution which we write in the form

$$q(t) = \bar{q}(t) + Q(t) \tag{4.20}$$

so that our path integral now becomes

$$G[q_1;q_0] = \int_0^\infty \mathrm{d}Ne^{2\pi^2 iS_0/\hbar} \int_{Q[0]=0}^{Q[1]=0} \mathcal{D}Qe^{2\pi^2 iS_2/\hbar} \tag{4.21}$$

such that

$$S_0 = \int_0^1 \mathrm{d}t \left(-\frac{3}{4N}\dot{\bar{q}}^2 + 3N - N\Lambda\bar{q} \right). \tag{4.22}$$

This is simply evaluated to be

$$S_0 = \frac{\Lambda^2}{36}N^3 + \left(3 - \frac{\Lambda}{2}(q_0 + q_1)\right) N - \frac{3}{4N}(q_1 - q_0)^2 \tag{4.23}$$

and

$$S_2 = -\frac{3}{4N}\int_0^1 \mathrm{d}t\dot{Q}^2. \tag{4.24}$$

The saddle-points of S_0 can be found by solving a quartic equation resulting from $\frac{\partial S_0}{\partial N} = 0$, noticing that it happens to be quadratic in N^2 allows us to solve it routinely. Explicitly,

$$\Lambda^2 N^4 + (36 - 6\Lambda(q_0 + q_1))\, N^2 + 9(q_1 - q_0)^2 = 0 \tag{4.25}$$

solving this yields the four solutions

$$N_{sp} = \pm\frac{3}{\Lambda}\left[\left(\frac{\Lambda}{3}q_0 - 1\right)^{\frac{1}{2}} \pm \left(\frac{\Lambda}{3}q_1 - 1\right)^{\frac{1}{2}} \right]. \tag{4.26}$$

From this we can see that the saddle-points need not be real for the case in which q_0 or $q_1 > \frac{3}{\Lambda}$. This is an encouraging sign as complex saddle-points can lead to a region in which the wave function behaves in an oscillatory manner, thus predicting a classical-spacetime. The authors of [5] stray from this common understanding in making the claim "*complex saddle-points imply non-classical behaviour since the propagator becomes dominated by non-Lorentzian geometries*". This seems to be confusion between the outputs and inputs. While it is true that the propagator will be dominated by non-Lorentzian geometries, this is not an issue as the geometries it predicts shall be Lorentzian. Just as we would like them to be.

We separately substitute each solution back into (4.23) to find the action associated with each of these saddle-points, the result being

$$S_0^{sp} = \mp\frac{6}{\Lambda}\left[\left(\frac{\Lambda}{3}q_0 - 1\right)^{\frac{3}{2}} \pm \left(\frac{\Lambda}{3}q_1 - 1\right)^{\frac{3}{2}}\right]. \tag{4.27}$$

The path integral in (4.21) is of a Gaussian form, therefore we can use the saddle-point approximation to find

$$\int_{Q[0]=0}^{Q[1]=0} \mathcal{D}Q e^{2\pi^2 i S_2/\hbar} = \sqrt{\frac{3\pi i}{2N\hbar}} \tag{4.28}$$

so that our remaining integral is just over the lapse

$$G[q_1; q_0] = \sqrt{\frac{3\pi i}{2\hbar}} \int_{0^+}^{\infty} \frac{\mathrm{d}N}{N^{\frac{1}{2}}} e^{2\pi^2 i S_0/\hbar}. \tag{4.29}$$

In the semiclassical limit we may evaluate an oscillatory integral of this form by making use of the Picard-Lefshetz theory that was previously discussed. To each saddle-point, there is an associated Lefschetz thimble $\mathcal{J}_\sigma$. We deform the half-infinite contour along the positive real axis so that it is a linear combination of the Lefschetz thimbles as in (4.8). The integral we are interested in solving now takes the form

$$G[q_1; q_0] = \sum_\sigma n_\sigma \sqrt{\frac{3\pi i}{2\hbar}} \int_{\mathcal{J}_\sigma} \frac{\mathrm{d}N}{N^{\frac{1}{2}}} e^{2\pi^2 i S_0/\hbar}. \tag{4.30}$$

We now Taylor expand S_0 about the point N_{sp} . Note that the first order term shall vanish as S_0 is a saddle-point

$$S_0 \approx S_0^{sp} + \frac{1}{2} S_{0,NN}(N - N_{sp})^2 + \dots . \tag{4.31}$$

Substituting this back into our integral,

$$
\begin{aligned}
G[q_1;q_0] &\approx \sum_\sigma n_\sigma \sqrt{\frac{3\pi i}{2\hbar}} \frac{e^{2\pi^2 i S_0^{\rm sp}/\hbar}}{N_{sp}^{1/2}} \int_{\mathcal{J}_\sigma} \mathrm{d}N e^{\frac{i\pi^2}{\hbar} S_{0,NN}(N-N_{sp})^2} \left[1+\mathcal{O}\left(\hbar^{1/2}\right)\right] \\
&\approx \sum_\sigma n_\sigma \sqrt{\frac{3\pi i}{2h}} \frac{e^{2\pi^2 i S_0^{sp}/\hbar}}{N_{sp}^{1/2}} e^{i\theta_\sigma} \int_{\mathcal{J}_\sigma} \mathrm{d}n e^{-\frac{\pi^2}{\hbar}|S_{0,NN}|n^2} \left[1+\mathcal{O}\left(\hbar^{1/2}\right)\right] \\
&\approx \sum_\sigma n_\sigma \sqrt{\frac{3i}{2N_{sp}\,|S_{0,NN}|}} e^{i\theta_\sigma} e^{2\pi^2 i S_0^{\rm sp}/\hbar} \left[1+\mathcal{O}\left(\hbar^{1/2}\right)\right].
\end{aligned}
\tag{4.32}
$$

Here, $N - N_{sp} = ne^{i\theta_\sigma}$ where n is a real number and θ_σ is the angle between the Lefschetz thimble and the positive real N axis. As was mentioned previously, equation (4.27) gives rise to the possibility of complex saddle-points. In fact, the no-boundary proposal requires complex saddle-points and is encompassed by the case $q_1 > 3/\Lambda > q_0$ so that a single root is imaginary. We now set $q_0 = 0$ so that the Universe nucleates from a vanishing three-geometry meaning the saddle-points of the action are at

$$
N_{sp} = \pm\frac{3}{\Lambda}\left[i \pm \left(\frac{\Lambda}{3}q_1 - 1\right)^{\frac{1}{2}}\right]
\tag{4.33}
$$

each saddle-point has a respective action

$$
S_0^{sp} = \mp\frac{6}{\Lambda}\left[-i \pm \left(\frac{\Lambda}{3}q_1 - 1\right)^{\frac{3}{2}}\right].
\tag{4.34}
$$

In this case the dominant saddle-point for our contour of integration lies at

$$
N_{sp} = \frac{3}{\Lambda}\left[i + \left(\frac{\Lambda}{3}q_1 - 1\right)^{\frac{1}{2}}\right].
\tag{4.35}
$$

This issue with this is that it happens to be a “wrong sign” saddle-point. As was previously mentioned, saddle-points with negative values of N lead to difficulties in recovering quantum field theory in a curved spacetime and lead to fluctuation wave functions that imply unsuppressed fluctuations. More generally, these difficulties arise for any complex saddle-point that corresponds to $Re(\sqrt{g}) < 0$, which we can show is true in our present context.

$$g = -\frac{3}{\Lambda}a^6\left(i + \sqrt{\frac{\Lambda}{3}q_1 - 1}\right) \tag{4.36}$$

so that

$$\sqrt{g} = \sqrt{\frac{3}{\Lambda}}a^3\left(-\sqrt{\frac{\sqrt{\frac{\Lambda}{3}q_1} - \sqrt{\frac{\Lambda}{3}q_1 - 1}}{2}} + i\sqrt{\frac{\sqrt{\frac{\Lambda}{3}q_1} + \sqrt{\frac{\Lambda}{3}q_1 - 1}}{2}}\right) \tag{4.37}$$

$$Re(\sqrt{g}) = -a^3\sqrt{\frac{3}{2}\frac{\sqrt{\frac{\Lambda}{3}q_1} - \sqrt{\frac{\Lambda}{3}q_1 - 1}}{\Lambda}} < 0. \tag{4.38}$$

A contour dominated by a saddle-point with this property fails to provide a sensible physical basis for a predictive framework. Nonetheless, continuing with this, we obtain the no-boundary propagator

$$G[q_1; 0] \approx e^{\frac{i\pi}{4}}\frac{1}{2\left(\frac{\Lambda}{3}q_1 - 3\right)^{\frac{1}{4}}}\exp\left(-\frac{12\pi^2}{\hbar\Lambda} - \frac{4i\pi^2}{\hbar}\sqrt{\frac{\Lambda}{3}}\left(q_1 - \frac{3}{\Lambda}\right)^{\frac{3}{2}}\right). \tag{4.39}$$

The semiclassical weighting obtained by Hawking and Hartle was $\exp\left(\frac{12\pi^2}{\hbar\Lambda}\right)$, precisely the inverse of that obtained here.

Throughout their work, Turok *et al.* refer to this object as a wave function and even treat it as such. A Green's function (which they have calculated) is a completely different type of object altogether and one has no reason to expect that the same procedures that exist for extracting probabilities from wave functions should apply to it. Their belief that the Hamiltonian should not annihilate the object it acts on seems to stem from the worry that one may inadvertently reverse the roles of the initial and final geometry by using the Lie derivative to implement timelike diffeomorphisms [7]. To avoid this they argue it is necessary to restrict the space of diffeomorphisms which is done by imposing the constraint $N > 0$, thus motivating their choice of contour. Therefore, as the group structure of the four-dimensional diffeomorphisms has been disrupted [57], one should expect that the Hamiltonian shall fail to annihilate a causal object such as the propagator. In general, a Green's function $G[a_1; a_0]$ for the Wheeler-DeWitt equation satisfies

$$\hat{H}G[a_1; a_0] = -i\delta(a_1 - a_0) \tag{4.40}$$

where $\hat{H}$ is the operator version of the Hamiltonian constraint (2.10). Put simply, they believe that the delta function that arises is an artefact of maintaining causality. It is the opinion of this author that the argument is unconvincing. While their desire to maintain causality is understandable, the attempt seems misguided. A history of a geometry is defined by the curve it takes in superspace, it is also unclear what it would mean for one three-geometry to be "before" or "after" another [18]. Moreover, there is a point of much greater importance to be made. *Transition amplitudes are not wave functions*. For any subsystem of the Universe, quantum mechanical predictions are derived directly from the associated wave function using a well defined formalism. Turok *et al.* have exempted the Universe itself from this formalism and have not introduced a new framework for computing probabilities with their Green's function. Emphasising this is not a "*rhetorical flourish*", but rather a deeply important point.

As the right hand side of (4.40) is purely imaginary, it has been pointed out that the real part of $G[a_1; 0]$ is a genuine solution to the Wheeler-DeWitt equation and therefore has the potential to be treated as a wave function. The contour of which corresponds to taking the integral of the lapse over the entire real line but above the origin along a small circle of radius ϵ, as to avoid the essential singularity [7].

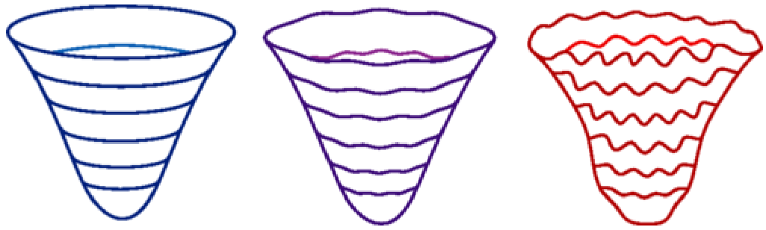

Figure 4.1: This graphic shows the no-boundary background with varying degrees of density fluctuations. On the left-hand side there are no fluctuations, in the middle there are small fluctuations and on the right-hand side the fluctuations are unsuppressed. We would like the theory to predict a homogenous universe with small density fluctuations to allow for the emergence of large scale structure such as galaxies. The graphic has been taken from [6].

If we are to follow Turok *et al.* in treating the propagator itself as a wave function, upon extending the analysis to include perturbations, an inverse Gaussian distribution is obtained [6]

$$G[q_1; \Phi_{1,l}; 0] \propto e^{\frac{l(l+1)(l+2)}{2\hbar H^2}\Phi^2_{1,l}} \tag{4.41}$$

where l is the principal quantum number and $H = \sqrt{\Lambda/3}$. Distributions of this type are problematic as they lead to unsuppressed fluctuations.

In response to these claims, Halliwell and collaborators set off on their own investigation into Lorentzian quantum cosmology where they pushback on the new ideas of Turok *et al.* [15]. Although their approaches have considerable overlap, the key difference is that they have chosen different contours of integration. While Turok and his group are unbothered by their Green's function solution to the Wheeler-DeWitt equation, the same is not true of Halliwell and his group who seek genuine solutions. For this reason the latter choose a contour of integration $\mathcal{C} = (-\infty, \infty)$, that dips below the origin along a small circle of radius ϵ as to avoid the essential singularity at that point. This contour is referred to this as a "*Lorentzian contour*" throughout, but as Turok *et al.* have made clear the inclusion of this small circle means that it is incorrect to describe it as such. Halliwell *et al.* responded to this by arguing that the

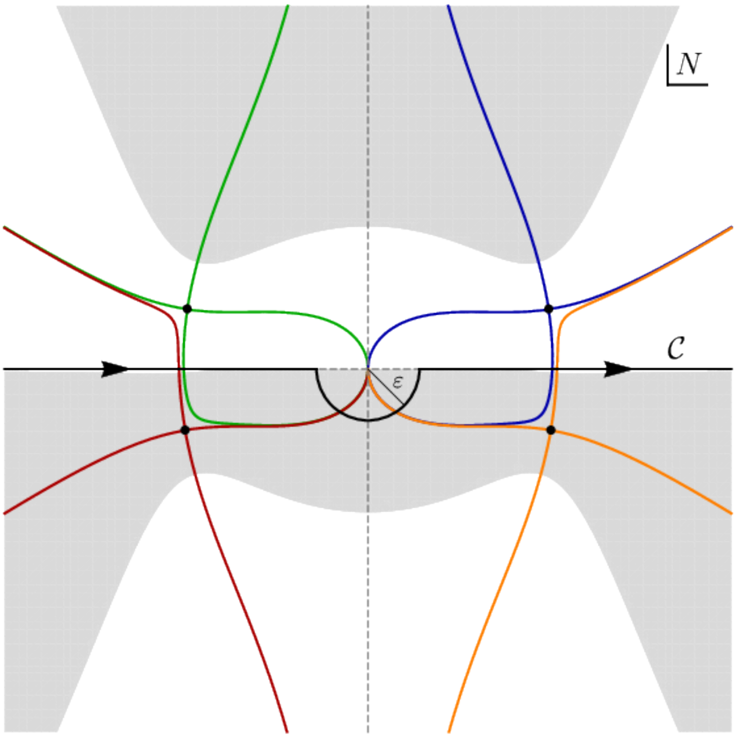

Figure 4.2: The black line is a visual representation of the contour that Halliwell *et al.* integrate over. The graphic has been taken from [15].

terms "Lorentzian" and "Euclidean" are only roughly indicative and the integration is generally over complex contours. The contour can also be made arbitrarily close to an actual Lorenztian contour as we may choose any $\epsilon > 0$. The path of integration is shown in figure 4.2. Although this contour is complex, the authors claim it is the two saddle-points in the lower half plane that contribute and lead to the real wave function

$$\Psi(q_1)_{NB} = \frac{e^{12\pi^2/\hbar\Lambda}}{\left(\frac{\Lambda q_1}{3} - 1\right)^{\frac{1}{4}}} \cos\left[\frac{12\pi^2}{\hbar\Lambda}\left(\frac{\Lambda q_1}{3} - 1\right)^{\frac{3}{2}} + \frac{3\pi}{4}\right] \times \left[1 + \mathcal{O}(\hbar)\right] \tag{4.42}$$

which has a semiclassical weighting in agreement with the Euclidean formulation of Hartle and Hawking. We now check that the fluctuations are suppressed. Halliwell *et al.* consider a model with a scalar field Φ coupled to gravity under the standard Einstein-Hilbert action

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(R - g_{\mu\nu}\partial^\mu\Phi\partial^\nu\Phi - 2V(\Phi)\right) + \int \mathrm{d}^3x\sqrt{h}K \tag{4.43}$$

where $V(\Phi) = \Lambda\cosh\left(\sqrt{\frac{2}{3}}\Phi\right)$. In the case of the no-boundary proposal, the wave function corresponds to the integral

$$\Psi_{NB}(q_1,\Phi_1) = \frac{2\pi}{\hbar}\int_{\mathcal{C}}\frac{\mathrm{d}N}{N}e^{2\pi^2 iS_0/\hbar}. \tag{4.44}$$

Once again, both relevant saddle-points lie in the lower half of the complex plane at

$$N_{\pm} = \frac{1}{H^2}\left(\pm\sqrt{\lambda+F} - i\sqrt{\lambda-F}\right) \tag{4.45}$$

where $F\left(q_1,\Phi_1\right) = \frac{H^2q_1}{2}\cosh\left(\sqrt{\frac{2}{3}}\Phi_1\right) - 1$ and $\lambda\left(q_1\right) = \frac{H^2q_1}{2}$. This leads to an on-shell action

$$\bar{S}_0\left(q_1,\Phi_1\right) = \pm\frac{2}{H^2}\left[\left(F-\lambda\right)^{\frac{3}{2}} \pm \left(F+\lambda\right)^{\frac{3}{2}}\right] \tag{4.46}$$

and finally, in the classical $\lambda \gg 1$, $\Phi_1 \ll \frac{1}{\sqrt{\lambda}}$ domain the wave function takes the form

$$\begin{aligned}\Psi_{NB}\left(q_1,\phi_1\right) \propto \exp&\left[\frac{4\pi^2}{\hbar H^2}\left(1 - \frac{H^2q_1}{4}\phi_1^2 + \mathcal{O}\left(\left(\sqrt{\lambda}\phi_1\right)^4\right)\right)\right]\times\\ \cos&\left[\frac{4\pi^2}{\hbar H^2}\left(H^2q_1 - 1\right)^{3/2}\left(1 + \frac{H^2q_1}{4\left(H^2q_1-1\right)}\phi_1^2 + \mathcal{O}\left(\phi_1^4\right)\right) + \frac{\alpha}{2}\right]\times\left[1+\mathcal{O}(\hbar)\right].\end{aligned} \tag{4.47}$$

The above shows that small homogenous perturbations around the de Sitter saddle-points are suppressed. If we are now to include spatially varying scalar perturbations in a similar manner to [6], one can show [15]

$$\Psi_{NB}(q_1,\Phi_{l,1}) \propto e^{-\pi^2 l(l+1)(l+2)\Phi_{l,1}^2/\hbar H^2}. \tag{4.48}$$

Once again, due to the Gaussian like behaviour of Ψ_{NB}, perturbations are also sup-

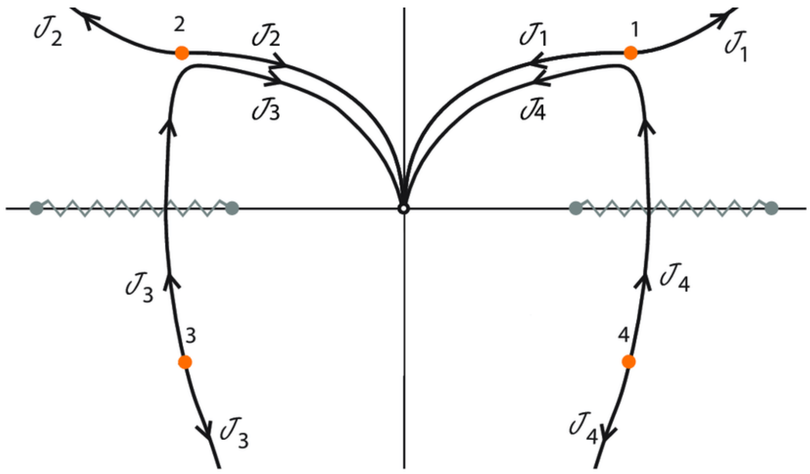

Figure 4.3: Each black line represents the Picard-Lefschetz thimble associated with one of the four saddle-points. The jagged gray lines on the real-axis represent branch cuts introduced by the no-boundary perturbations. This graphic has been taken from [7].

pressed here. The authors have seemingly recovered the predictions of the Euclidean no-boundary proposal. This work failed to convince Turok *et al.* who claim to have proven in [7] that there is no contour of integration that avoids the contributions of the saddle-points in the upper half-plane which lead to unsuppressed perturbations. A sketch of their argument is provided below.

Proof. To maintain the reparameterisation invariance of time, it is necessary for the contour $\mathcal{C}$, for which N is integrated over, to start and end on a point where the Morse function h approaches minus infinity. As we also require our integral to converge, $\mathcal{C}$ must approach the singularities at its ends in such a way that it can be deformed into paths of steepest-descent. Using Picard-Lefschetz theory, it is possible to write any contour that satisfies these two conditions as a sum of Lefschetz thimbles. After introducing small perturbations to break degeneracies between thimbles, in the limit that the perturbations are taken to zero, $\mathcal{J}_1$ and $\mathcal{J}_4$ are equivalent along the path connecting saddle-point 1 to the origin. As the same argument holds true for $\mathcal{J}_2$ and $\mathcal{J}_3$, every thimble includes contributions from the saddle-points in the upper half-plane, both of which lead to unsuppressed perturbations. If we include branch cuts on the real-axis, any contour that includes contributions from a saddle-point in the lower half-planes includes contributions from at least one of the branch cuts which once again, leads to unsuppressed perturbations. □

Halliwell *et al.* pointed out in [18] that the proof appears to be flawed due to an error made by Turok *et al.* in their application of perturbation theory. To arrive at their conclusion, the trio have considered a de Sitter minisuperspace filled with a massless scalar field. The action for such a space is of the form

$$S_{scalar}[Q,\{\varphi_n\};N] = S_{isotropic}[Q;N] + \sum_n \int_0^1 \mathrm{d}\tau N \left(\frac{Q^2}{2N^2}\dot{\varphi_n}^2 - \frac{n\,(n+2)}{2}{\varphi_n}^2\right) \tag{4.49}$$

where

$$S_{isotropic}[Q;N] = \int_0^1 \mathrm{d}\tau N \left(-\frac{3}{4N^2}\dot{Q}^2 + 3 - Q\right). \tag{4.50}$$

To quadratic order, the associated equations of motion are

$$\ddot{Q} - \frac{2N^2}{3} = -\frac{2}{3}\sum_n Q\dot{\varphi_n}^2 \tag{4.51}$$

$$\ddot{\varphi_n} + 2\frac{\dot{Q}}{Q}\dot{\varphi_n} + N^2 n\,(n+2)\,\frac{\varphi_n}{Q^2} = 0. \tag{4.52}$$

We must require that $\varphi_n \ll 1$ for all values of τ so that the effective field theory approximation that we are using remains vaild. Once again, we are only interested in the leading semiclassical order and so

$$\Psi(N) \approx \int_{\mathcal{C}} \mathrm{d}N\mathcal{P}e^{iS(N)/\hbar} \tag{4.53}$$

where $\mathcal{P}$ is simply a prefactor. Turok *et al.* attempt to proceed analytically by solving the equations of motion for arbitrary $N \in \mathbb{C}$ with the Dirichlet boundary conditions

$$\mathcal{B} : Q(0) = 0, \quad \varphi_n(0) = 0 \tag{4.54}$$

at the centre of the $\overline{B^4}$, and

$$\mathcal{B} : Q(1) = Q, \quad \varphi_n(1) = \varphi_{n,1} \tag{4.55}$$

on the three-surface (an S^3) on which the wave-function is evaluated on. To do this, they make the assumption that it is consistent to neglect the term on the RHS of equation (4.51). This however turns out not to be a valid assumption as it is possible for the term to diverge for certain values of $N \in \mathbb{C}$ while the other terms remain finite. Therefore unless a very special contour has been chosen where all values of N are consistent with the effective field theory approximation, Turok *et al.* will obtain an erroneous on-shell action for many values of $N \in \mathcal{C}$. Figure 4.4 is a graphical illustration showing where their application of perturbation theory is consistent (green regions) and where it is inconsistent (orange and red regions). As much of their analysis is done in the orange and red regions the calculation itself must be inconsistent. The work is therefore insufficient to conclude that any NBWF constructed from a minisuperspace is ill-defined.

Halliwell *et al.* made their own attempt to repeat the calculation of Turok *et al.* and correct for the invalid analysis but found themselves unable to solve the equations of motion in the entire complex plane of N. This meant that they were unable to determine the contour of steepest-descent due to a lack of knowledge of $S(N)$. They were subsequently unable to find a contour on which the phase of $\exp(iS(N)/\hbar)$ is constant. If one does not have a good knowledge of $S(N)$, the calculation cannot be performed.

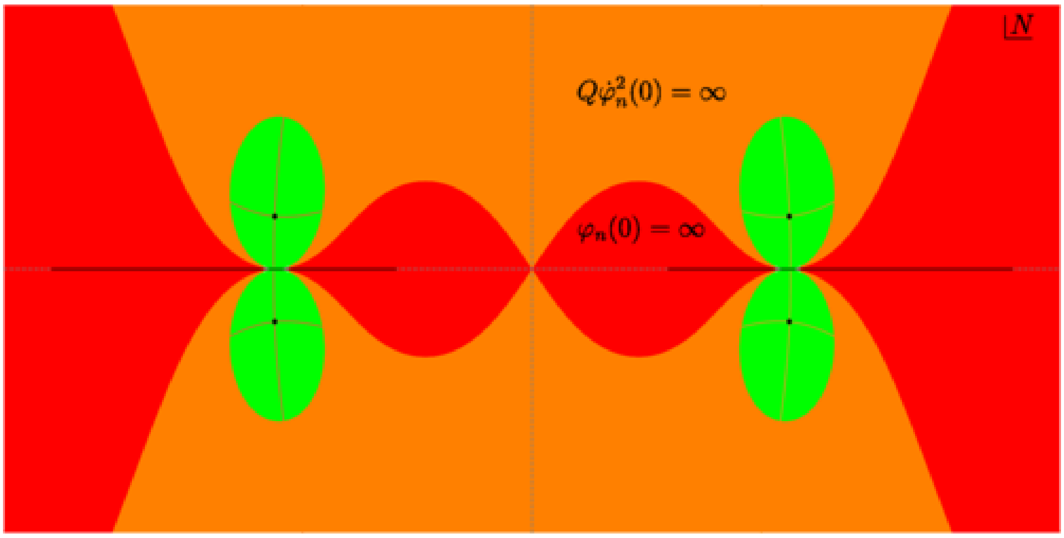

Figure 4.4: Halliwell *et al.* provide this graphic of the complex N-plane in [18]. The orange and red regions of this plane are the regions which correspond to singular solutions where the effective field theory approximation is not valid and/or where one is unable to neglect the term on the RHS of (4.51) . The green region is the region where the analysis of Turok *et al.* is valid and the correct on-shell action has been obtained. The graphic was generated with numerical values $n = 3$, $Q = 100$ and $\Lambda = 1$.

After this "proof" was published, Halliwell *et al.* succeeded in constructing a well defined no-boundary state for a biaxial Bianchi IX minisuperspace model [16], thus providing an explicit counter-example to the claims of Turok *et al.*.

4.3 Biaxial Bianchi IX Minisuperspace

A biaxial Bianchi IX (BB9) minisuperspace model is a homogenous but generally anisotropic spacetime containing a spacelike section with a squashed three-sphere geometry. The wave function $\Psi(p, q)$ of such a space is a function of two coordinates $p, q \geq 0$, which appear in the homogeneous metric as

$$2\pi^2 \mathrm{d}s^2 = -\frac{N(t)^2}{q(t)}\mathrm{d}t^2 + \frac{p(t)}{4}(\sigma_1^2 + \sigma_2^2) + \frac{q(t)}{4}\sigma_3^2. \tag{4.56}$$

Here, we have introduced $\sigma_{1,2,3}$ which are the left-invariant one-forms [1]

$$\sigma_1 = -\sin\psi \mathrm{d}\theta + \cos\psi \sin\theta \mathrm{d}\phi, \quad \sigma_2 = \cos\psi \mathrm{d}\theta + \sin\psi \sin\theta \mathrm{d}\phi, \quad \sigma_3 = \mathrm{d}\psi + \cos\theta \mathrm{d}\phi \tag{4.57}$$

with $0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$ and $0 \leq \psi < 4\pi, \psi \cong \psi + 4\pi$. Note that for the special case $p(t) = q(t)$ the three-metric is proportional to $\delta^{ij}\sigma_i\sigma_j/4$ which is the metric of the unit three-sphere. The amount of squashing can be expressed by the quantity

$$\alpha = \frac{p}{q} - 1, \tag{4.58}$$

where $\alpha = 0$ corresponds to the three-sphere geometry, $\alpha > 0$ to an oblate spheroid and $\alpha < 0$ to a prolate spheroid.

To compute our wave function on this anisotropic space in the gauge $\dot{N} = 0$, we are to evaluate the integral

$$\Psi(p,q) = \sum_{\mathcal{M}} \int_{\mathcal{C}} \mathrm{d}N \int_{\mathcal{B}}^{x(1)=(p,q)} \mathcal{D}x^\alpha \mathcal{D}\Pi_\alpha e^{iS[x,\Pi;N]/\hbar} \tag{4.59}$$

with $\mathcal{B}$ denoting our boundary conditions. We first need to deal with the sum of manifolds $\mathcal{M}$. As we are considering the no-boundary proposal we are summing over a set of four-manifolds with a single boundary which admit everywhere a regular saddle-point solution to the Einstein equations. In this case there are three such manifolds, $\mathbb{C}\mathrm{P}^2 \backslash B^4, \mathbb{R}\mathrm{P}^4 \backslash B^4$ and the one which provides the dominant contribution, $\overline{B^4}$. Halliwell *et al.* focus on the latter of these claiming *"including the other topologies does not significantly change our results"*.

Clearly, to have any hope of evaluating this path integral we must figure out how

[1]Let G be a Lie group and $L_g : G \to G$ denote the function of left multiplication by an element $g \in G$. A differential from σ is said to be left-invariant on G if for any choice of $g \in G$, it is invariant under the pullback of L_g. That is, $(L_g)^* \sigma = \sigma \; \forall \, g \in G$.

we can implement the no-boundary proposal on this space. So that the boundary conditions are consistent with quantum mechanics, it is not possible to specify the value of both a coordinate and its conjugate momentum at $\tau = 0$. Instead the following boundary conditions are given which they claim are the only possible ones that lead to a "*well-defined and normalisable NBWF in Bianchi IX minisuperspace*"

$$\mathcal{B} : p(0) = 0, \quad \Pi_q(0) = -i. \tag{4.60}$$

Using these boundary conditions, it can be shown that

$$\int_{\mathcal{B}}^{x(1)=(p,q)} \mathcal{D}x^\alpha D\Pi_\alpha e^{iS|x,\Pi;N|/\hbar} \propto \frac{\sqrt{P}}{N^2} e^{iS_0/\hbar} \tag{4.61}$$

where

$$S_0 = -\frac{i\Lambda}{3}N^2 + \left(4 - \frac{\Lambda p}{3}\right) N + iq - \frac{pq}{N}. \tag{4.62}$$

To obtain the wave function of this Bianchi IX minisuperspace, we must decide on a contour $\mathcal{C}$ to integrate the lapse over for the manifold with topology $\overline{B^4}$. The authors advocate a complex circular contour enclosing the origin, so that the remaining integral is now

$$\Psi(p,q;\overline{B^4}) = \sqrt{p} \oint \mathrm{d}N \frac{1}{N^2} \exp\left\{ \frac{i}{\hbar}\left[-\frac{i\Lambda}{3}N^2 + \left(4 - \frac{\Lambda p}{3}\right) N + iq - \frac{pq}{N} \right] \right\} \tag{4.63}$$

which satisfies the Wheeler-DeWitt equation. The action S_0 has three saddle-points in the complex plane, which are solutions of

$$\frac{2\Lambda}{3} i N^3 + \left(\frac{\Lambda p}{3} - 4\right) N^2 - pq = 0. \tag{4.64}$$

So that we have a chance of predicting that the Universe behaves classically whilst

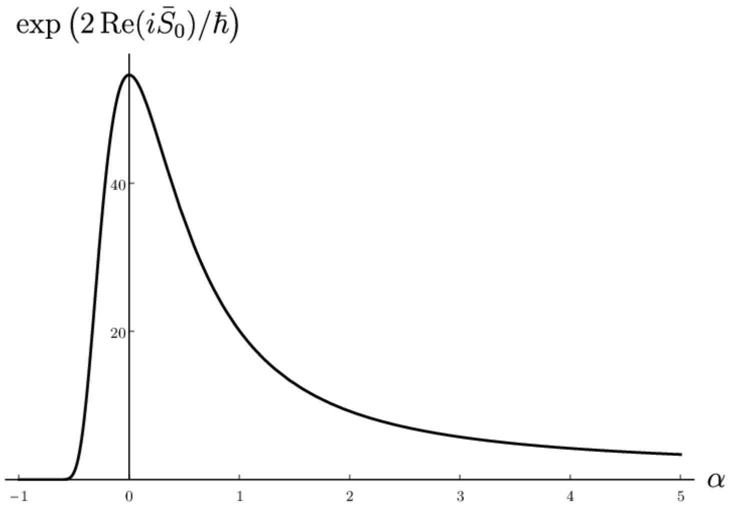

Figure 4.5: The leading order probability distribution obtained from the NBWF shows that it is heavily biased towards isotropic Universes. The values $\hbar = 1, \Lambda = 3$ were used in this plot. The plot itself has been taken from [16].

large, we need complex saddle-point solutions. Making a substitution $M = iN$, allows us to rewrite (4.64) so that the coefficients are real

$$M^3 + \left(\frac{p}{2} - \frac{6}{\Lambda}\right) M^2 + \frac{3pq}{2\Lambda} = 0. \tag{4.65}$$

The sign of the discriminant Δ, for a cubic equation with real coefficients determines the nature of its solutions. If $\Delta > 0$, the equation has three real solutions. For $\Delta < 0$, two of the solutions form a complex conjugate pair whilst the third solution is purely real. For reasons mentioned above, we shall restrict to the latter case which reduces to the condition $-4\left(\frac{p}{2} - \frac{6}{\Lambda}\right)^3 - \frac{81pq}{2\Lambda} < 0$. Or equivalently

$$\Lambda q > \frac{(\Lambda p)^2}{81}\left(\frac{12}{\Lambda p} - 1\right)^3. \tag{4.66}$$

If the third solution is purely real in M, then it is purely imaginary in N. It is for this reason that we must choose a contour of integration for which this is not the dominant saddle-point. From (4.63) we can see that in such a situation, the semiclassical exponential factor would be real so that we would fail to predict that the Universe will evolve to behave classically. In the regime where the Universe is large $p \gg 1/\Lambda$, and α is finite

$$\Psi\left(p, \alpha, \overline{B^4}\right) \propto \sqrt{\hbar}\Lambda\left(\frac{1+\alpha}{\Lambda p}\right)^{\frac{3}{4}} \exp\left[\frac{6\left(1+2\alpha\right)}{\hbar\Lambda\left(1+\alpha\right)^2}\right] \cos\left[\frac{6}{\hbar\Lambda\sqrt{1+\alpha}}\left(\frac{\Lambda p}{3}\right)^{\frac{3}{2}} - \frac{3\pi}{4}\right] \tag{4.67}$$

to leading order in $1/\Lambda p$. To its credit, the no-boundary wave function predicts that deviations from isotropy are suppressed. The probability distribution is peaked around the three-sphere geometry where $\alpha = 0$, and quickly decays for any deviation from this. Thus, it seems that the no-boundary wave function is able to account for the observed isotropy of the Universe to a high probability.

Turok *et al.* responded to this paper by once again attacking the choice of contour that was used, stating "*it has no geometrical interpretation as it involves metrics with complex proper time*" [8]. Halliwell *et al.* however do not believe that one should ascribe a physical meaning to the contour itself in a particular minisuperspace model. They believe that since it has been shown in [58] that the same contour can lead to different results depending on which variables are retained and how the metric is parameterised, the physical motivation for a choice should simply be that it is compatible with the criteria outlined in [13].

Clearly, much of the discussion in this section has revolved around contours of integration. While I believe that the arguments of Halliwell *et al.* that have been presented so far in this chapter are adequate responses to the claims of Turok *et al.*, in my opinion the most convincing and simplest argument was made in [17]. When the no-boundary wave function was first discussed in section 3.2, a recently developed method for specifying its semiclassical approximation from a collection of appropriate saddle-points was given. This method is logically independent of any functional integral representation, so to legitimately refute the wave function you cannot rely on properties of the associated functional integral [17]. This extremely consequential claim makes no attempt to dispute the claims of Turok *et al.*, it simply evades them altogether. There is no longer a contour of integration that can be subject to criticism as the choice of contour has been replaced with a choice of saddle-points. In my view, this paper is in some sense the final word. While the no-boundary proposal was originally cast in terms of a functional integral, it would be a mistake to think of this representation as being somehow fundamental. It is simply a method for computing a wave function. A response to these claims was never made and an explicit example of this new method in action was provided in [18]. Encouragingly, other authors were also able to arrive at similar conclusions independently [59]. It seems to me that Hartle and Hawking's proposal remains a viable candidate for the boundary conditions we are in search of. Perhaps the no-boundary wave function is the quantum state necessary for the final theory of our Universe.

Chapter 5

Conclusion

We began by introducing the basic principles of quantum cosmology. The Wheeler-DeWitt equation was obtained by Dirac quantising the Hamiltonian constraint of an Einstein-Hilbert action coupled to matter. Several issues such as those associated with operator ordering and time were briefly discussed along the way. As it is not known how to solve the Wheeler-DeWitt equation using modern techniques, we introduced simplification in the form of a minisuperspace approximation. The tunnelling proposal of Vilenkin and no-boundary proposal of Hartle and Hawking were discussed as possible candidates for the boundary conditions of the Universe. It was shown that while the tunnelling proposal leads to a Universe that undergoes sufficient inflation in a direct manner, the no-boundary proposal requires that we first introduce a volume-weighting. Introducing such a volume-weighting leads to eternal inflation resulting in an infinite fractal-like multiverse. One is then faced with the well known Measure Problem of cosmology.

The literature concerning the recent criticisms of the no-boundary proposal from Turok *et al.* and the subsequent defence from Halliwell *et al.* was reviewed. In an attempt to maintain causality, Turok *et al.* do not compute solutions to the Wheeler-DeWitt equations, but rather Green's functions. Quantum mechanical predictions are derived directly from a wave function using a well-defined formalism. As these Green's functions do not satisfy the Wheeler-DeWitt equation, they cannot be considered to be valid wave functions and should not be treated as such. Many of their criticisms of the no-boundary proposal are based on the functional integral representation of the NBWF. During the course of the dispute Halliwell *et al.* were able to advance a method for obtaining the NBWF of the Universe directly in terms of a collection of saddle-points of the dynamical theory. This method is logically independent of the NBWF's integral representation so that properties of the associated functional integral cannot be used to refute it. The arguments of Turok *et al.* rely on these properties and are therefore insufficient. It is my opinion that the defence of Hartle and Hawking's NBWF was successful.

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