Master Dissertation

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Can Inflation Be Made Safe?

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Abstract

It is claimed that an accelerated expansion era drove the universe to near flatness, homogeneity and isotropy on large scales and stretched quantum fluctuations to today's large scale structure. Many models have been developed in the past decades and the predictions made by inflation have been validated by cosmological measurements. In this thesis we will begin with its motivation and development. We will see that the theory (class) of inflation suffers from major problems. In the second part we discuss the perturbative non-renormalisability of gravity, the asymptotic safety scenario and its renormalisation group treatment. Finally, we will explore whether and how inflation can be made safe.

Acknowledgements

Hereby, I would like to thank the Imperial Theory Group for the deep and broad knowledge I gained in the Qfff course and also thanks to my fellow peers for exciting discussions and support. Flattening the curve wasn't only important in inflation, we will all remember this year and probably proudly look back. It increased my curiosity in theoretical physics even more. I want to thank Prof Stelle and Prof Litim for their advice in the past months. I am looking forward to proceeding this fascinating research during my PhD.

Herzlichen Dank für die Unterstützung der Studienstiftung des deutschen Volkes.

I would also like to thank my mum and all others that have supported me in the past years to fulfill my dream despite the difficulties outside the academic life and supported my curiosity and meticulosity.

Lastly, I want to thank the science community from yesterday, today and tomorrow. Maths and physics have always been the best medicine .

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Chapter 1

Introduction

> And if inflation is wrong, then God missed a good trick. But, of course, we've come across a lot of other good tricks that nature has decided not to use.
>
> (Jim Peebles — Princeton,1994)

1.1 Background and Aim of This Thesis

It has been assumed and validated that the Universe began in the Big Bang and expanded ever since. In the very early universe *inflation*, an accelerated expansion, is claimed to have happened. Whereas standard Big Bang can't explain the cosmological problems (horizon, flatness, monopoles) the theory of inflation claims to have solved these issues [23]. We will see that the shrinking Hubble sphere during inflation and the enormous expansion in form of repulsive gravity play an important role. Moreover, it provides a mechanism how the early quantum fluctuations turned into the large scale structure today. Flatness, homogeneity and isotropy on large scales with small anisotropies have been confirmed in precise testings such as in the Planck measurements of the CMB. A vast amount of models have been put forward. From false vacuum decay of Guth original idea, over standard slow-roll models to R^2 inflation [75] where the yet unknown scalar field arises because of the additional degree of freedom in the Lagrangian.
However, as much as inflation has been successful, both theoretically and experimentally, we shouldn't forget that it is only a theory or as we will see rather a framework, a class of theories. Criticism has been put forward because of its inconsistency in theory and observations, its need for fine-tuning itself (what it should actually solve), the lack of an explanation for the mechanism and origin of inflation, the measure and unlikeliness problem and the loss of predictivity in the eternal universe/multiverse scenario to name a few [3]. Alternative theories have been developed e.g. by the recent Nobel laureate Penrose [54].
Inflation should take place at very early times and high energies suggesting that along the quest for inflation a theory of *quantum gravity* should be found which is indisputable one of the biggest challenges in theoretical physics today. General relativity is perturbatively non-renormalisable [92][70] if we go to high energies. Its origin lies in

the negative dimensionality of Newton's constant. *Higher derivative gravity* theories were introduced for cosmological reasons and for also these purposes including Stelle's quadratic gravity [79][78] which he showed is renormalisable but non-unitarity. We would like to have a renormalisable, unitary and physical theory of quantum gravity that is predictive at all scales. Such a theory might help to solve the issues with inflation, the universe's (and inflation's) initial conditions (or its need at all).
A promising candidate is *asymptotic safety* which renders gravity non-perturbatively renormalisable if a non-interacting fixed point in the UV with finite number of UV attractive directions exists. From QFTs we know that observables depend o the scale we measure them. First put forward by Weinberg [85] there is now a growing community working on this approach. Functional renormalisation group flow (FRGE) was developed including the exact treatment with the help of the Wetterich equation [87] and applied to gravity in the so-called Einstein Hilbert truncation which depends on the running and dimensionless couplings of the corresponding dimensionful Newton constant G and Λ that accounts for the late time acceleration [64]. Further work was produced with higher derivative truncations including the one introduced by Stelle. Combining perturbative and non-perturbative treatment of those theories might provide a platform to analyse inflation. Additionally, in classical inflation a new scalar field of unknown origin - so far we only know the Higgs scalar - is introduced. It would be a major result if it turns out that inflation is a *quantum gravitational effect* only. This emphasises the search for theories in the asymptotic safety scenario that naturally produce inflation meaning a state of accelerated expansion thereby solving the cosmological and apparent fine-tuning problems and connect the early with the late time behaviour.
In this thesis I will connect results from standard cosmology and general relativity (GR) including different inflation models, higher derivative (HD) theories and asymptotic safety (AS) and its renormalisation group (RG) treatment. I start by motivating the need for inflation including the cosmological principle; the horizon, flatness and monopole problem and the large scale structure in 2. A short introduction to asymptotic safety will be given as well. I proceed by a treatment of the underlying physics which includes the model-independent theory, presentation of different models, observational evidence in terms of perturbations and thereby answering how inflation began and ended and its importance in the universe's evolution 3. I give some extension and alternative theories as well as a connection to dark energy. Then, I will evaluate the problems inflation doesn't solve and discuss the ones that newly arise 4. Finally, I will give an introduction to the AS scenario, its FRG treatment, how it has been applied to inflation so far 5 and conclude 6.
Along I will give some digressions that in my opinion should be emphasised in the context of inflation and AS with my own calculations and detailed explanations of investigations that aren't typically treated during the analysis of inflation. See also 6.2.

Can inflation be made safe?

In the following is a brief overview of the development of inflation and asymptotic safety. In fact, its proper treatment both started around the same time, 40 years ago.

1.2 History of Inflation (and Asymptotic Safety)

1915	Einstein's GR describes the evolution of the universe in physical laws
1917	de Sitter derives an expanding universe driven by vacuum energy
1922	Friedmann's equations describe expanding&collapsing models
1929	Hubble observes the expansion (redshift of galaxies)
1933	Milne postulates the cosmological principle
1946	Gamov describes the initial state as Big Bang followed by nucleosynthesis and predicts the CMB
1961	Dicke says that gravity and EM must be fine-tuned for life to exist (first form of the anthropic principle)
1965	Penzias and Wilson at Bell labs discover the CMB
1969	the horizon and flatness problems are postulated
1976	Weinberg's RG flow for gravity
1979	Starobinsky's first idea of 'inflation' Weinberg's Asymptotic Safety
1977	Stelle proves the renormalisability of quadratic gravity
1981	Guth proposes the idea of old inflation Mukhanov proposes perturbations as seeds for the LSS
1982	Linde and Albrecht&Steinhardt propose new inflation as solution for the exit problem
1982	Nuffield Workshop: perturbations are investigated
1983	Linde proposes chaotic inflation
1986	Linde and Steinhardt develop the (eternal) chaotic universe
1992	COBE satellite gives the nearly scale-invariant spectra T anisotropies
1993	Wetterich equation
1996	Hubble Deep Field proves the cosmological principle
1998	Reuter calculates the flow equations for the gravitational field
2000	AS applications in cosmology
2003	WMAP - CMB and ΛCDM model

- 2009 — Planck release to test anisotropies
- 2012 — Higgs particle detected (Higgs=inflaton?)
- 2014 — BICEP2: B modes detected – false alarm
- 2015 — LIGO: gravitational waves detected
- 2018 — scalar-to-tensor ratio is almost 0
- 2020 — crisis of cosmology[a]

[a] The universe is expanding much faster than thought (see 3.7), some theoretical evidence suggests that it should even slow down. Hubble measurements don't coincide (global/early times vs local/late times give different values). Observations that should help to calculate average distributions of hidden matter/energy forms shows clumps that are almost 10% thinner than predicted.

In the beginning of inflation's developments a lot of simultaneous research was conducted in the US and the Soviet Union. Unfortunately, international communication was not always supported. For a historical introduction into the underlying problems of this thesis, 7.5.

1.3 Conventions

Metric signature $(-,+,+,+)$
$\hbar = c = 1$
$m_p^2 = G^{-1}$
with values of
$c = 299792450\frac{m}{s}$, $G = 6.674 \cdot 10^{-11}\frac{m^3}{kg\,s}$, $\hbar = 1.055 \cdot 10^{-34}$Js, $k_B = 1.38 \cdot 10^{-23}\frac{J}{K}$
$1\text{Mpc} = 3.1 \times 10^{21}\text{m} = 3.3 \cdot 10^{6}$ light years

Many calculations and digressions can be found in the appendix 7 to improve the reading flow, but may be read along the introduction as well.

Chapter 2

Motivation

> The ability to understand something before it's observed is at the heart of scientific thinking.
>
> (C. Rovelli — The Order of Time)

In the following I will present the major cosmological problems that led to the idea of inflation.

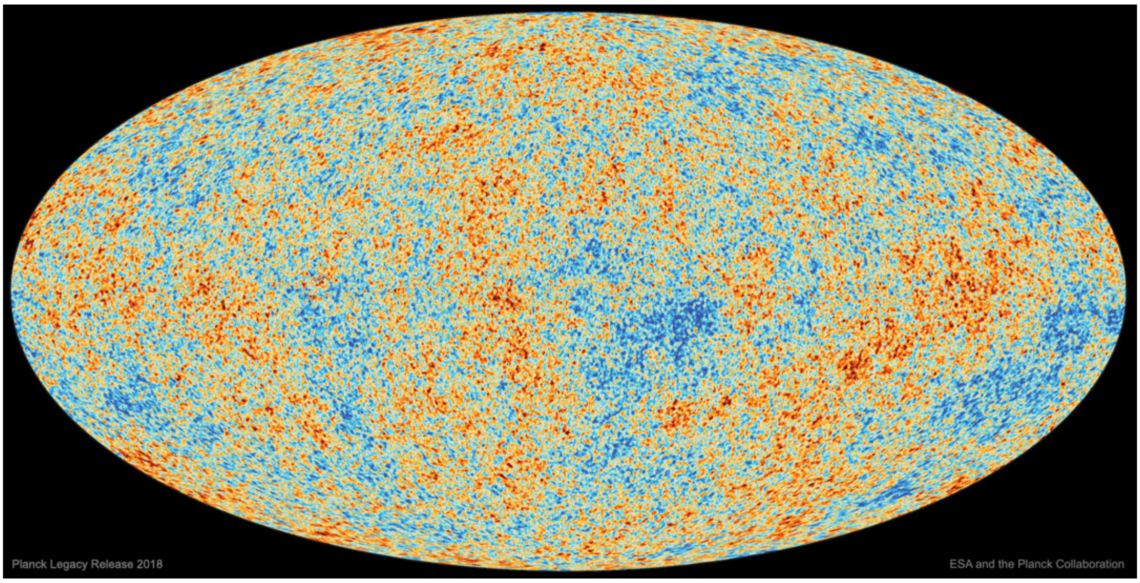

Figure 2.1: CMB 2018 Planck, apod.nasa.gov/apod/ap180722.html

2.1 Cosmological Principle

The Cosmological Principle (Milne, 1933) states that no observer occupies a special place in the universe (Copernican Principle). It is based on two principles of spatial invariance. The universe is *homogeneous and isotropic on large distances*. This means that the universe looks the same at each point (isomorphic under translation $g(\vec{r}) =$

$g(\vec{r}+\vec{r'})$ and in all directions (isomorphic under rotation $g(\vec{r}) = g(|\vec{r}|)$), respectively.[1] Both have been tested and well validated by observations of the cosmic background radiation, the Hubble law and finally inflation. Furthermore, it is based on the hot Big Bang model 7.1 and the Weyl postulate. The former is described in detail along with the cosmological standard model and a brief timeline of the universe in the appendix. According to the *Weyl postulate* the world lines of galaxies, which we will take as test particles on galactic scales, represent a 3-bundle of geodesics that are orthogonal to spacelike hypersurfaces and don't intersect [51]. Hence, geodesics on which galaxies travel cannot intersect and this puts further constraints on the metric. It is also part of the cosmological standard model, 7.3.
Why is the CMB (cosmic microwave background) uniform in all directions in $\frac{1}{10^5}$ parts?

2.2 Horizon Problem

The horizon problem, also called the homogeneity problem, covers the fact that today the universe is homogeneous on large scales. The CMB, that captures the photons that have been travelling since the time of recombination, shows a temperature of $2.7 \pm 10^{-5}\text{K} \sim -270°\text{C}$ [2]. The radiation from opposite sides of the observable universe today is almost the same indicated by the same temperature which can be extrapolated to same fluctuations. However, the different regions weren't able to communicate with each other, there is a much larger portion of the universe observable today than at the time of recombination. How is it possible that causally unconnected regions have the same properties (temperature) if there was no time for thermal equilibrium?

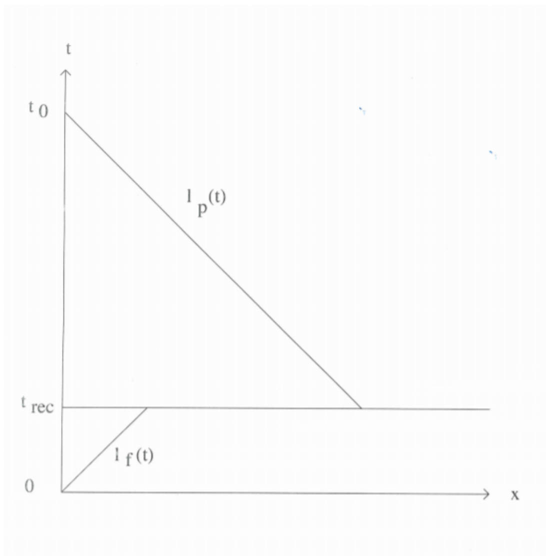

Figure 2.2: Horizon problem [12] past light cone from today and initial light cone from the Big Bang don't meet at recombination.

Precisely, the comoving radius today is much larger than the comoving radius of causally connected parts at the time of recombination:

$$\int_0^{t_{rec}} \frac{dt}{a(t)} \gg \int_{t_{rec}}^{t_0} \frac{dt}{a(t)} \tag{2.1}$$

With the right conditions on matter (radiation dominates after recombination time, see appendix) the former scales as $3t_0^{\frac{2}{3}} t_{rec}^{\frac{1}{3}}$ whereas the latter scales like $3t_0(1 - (\frac{t_{rec}}{t_0})^{\frac{1}{3}})$.

[1]Note the principles don't imply one another. For example, a magnetic field can be homogeneous, but not necessarily isotropic. Spherical symmetry implies isotropy, but not necessarily homogeneity. If, however, isotropy can be found at every point, then homogeneity follows [40].

[2]Compared to Hyde Park lake this would be a ripple of a height of 10^{-2}mm.

Written in that way [12] it is clear that at the time of recombination the comoving radius of the photons doesn't match with the comoving forward light cone. Our past horizon at recombination is larger than the forward light cone originating from the Big Bang. The uniformity must be postulated in the *initial conditions* to eliminate many causally unconnected regions with the same properties (see appendix 7.7.1, 7.7.5).
Along with the horizon problem often the inhomogeneity problem and the issue of entropy is mentioned. Whereas the CMB is quite homogeneous, it does show anisotropies and we are well aware of the fact that galaxy formation was possible. As we will see there seems to be an imposed natural bound on the initial entropy which makes the universe very special with a one in $e^{10^{122}}$.

2.3 Flatness Problem

The flatness problem is another fine-tuning problem. Why is the universe (nearly) flat, especially if one considers that this flatness is not a generic condition? A flat universe is characterised by the fact that it just has the right amount of matter and energy that it continues to expand and doesn't re-collapse - its density is near the critical density ρ_c 7.3. Its ratio of today's density and the critical density Ω_0 has been accurately measured to be almost unity. Considering standard Big Bang theory and the Friedmann equation 7.3 we can show that Ω shifts away from unity during the expansion of the universe.

$$|\Omega - 1| = \left| \frac{\rho(t) - \rho_c}{\rho_c} \right| = \frac{1}{\dot{a}^2(t)} \qquad (2.2)$$

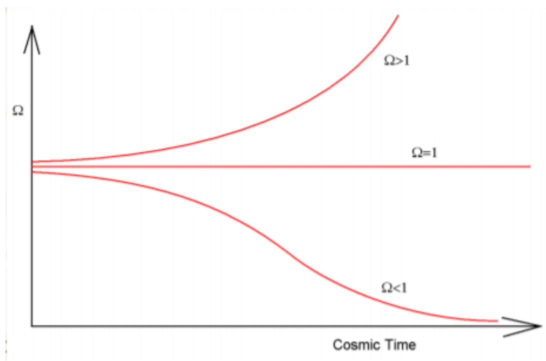

Figure 2.3: Flatness problem, $\Omega = 1$ is very unstable. In order to have spatial flatness today, Ω had to be very close to 1 from the beginning onwards (astro.umd.edu 2015 class23)

At early times it had to be extremely small in the beginning, ρ has to be very close to ρ_c. To be precise, the unity of Ω is unstable. As it is shown in the appendix 7.3 $a^2H^2 \propto t^{-1}$ for radiation and $a^2H^2 \propto t^{-\frac{2}{3}}$ for matter such that for both contents $|\Omega_{tot} - 1|$ scales like t and $t^{\frac{2}{3}}$, increasing functions, respectively. The universe must have been flat from the beginning onwards. The deviation from unity above can also be rewritten as $\frac{\frac{k}{a^2}}{H^2}$ giving basically the ratio of the radius of curvature and the Hubble radius which obviously shows that $\Omega = 1$ is an unstable state in the hot Big Bang model.
To quantify the deviation[3] we can specify $\alpha = \frac{\rho - \rho_c}{\rho_c}$ via its evolution $\dot{\alpha} = \frac{\dot{\rho}}{\rho_c} - \frac{\rho \dot{\rho_c}}{\rho_c^2}$ with $\dot{\rho_c} = \frac{3H\dot{H}}{4\pi G} = \frac{3}{4\pi G}\frac{\dot{a}}{a}(\frac{\ddot{a}}{a} - H^2)$ substituting the second Friedmann equation, using $H^2 = \frac{8\pi G \rho_c}{3}$ and substituting it back into the original equations gives

$$\dot{\alpha} = -\frac{3\dot{a}}{a}(1+w)\frac{\rho}{\rho_c} + \frac{\rho}{\rho_c}\frac{\dot{a}}{a}\left(\frac{\rho}{\rho_c}(1+3w) + 2\right) = \alpha(\alpha+1)\frac{\dot{a}}{a}(1+3w) \qquad (2.3)$$

[3]This was given as an exercise in the R&C class by Magueijo.

Hence, $\dot{\alpha}$ positive[4]. Here I already used the equation of state $p = w\rho$, where $\dot{w} = 0$. For matter it increases with $\alpha(1+\alpha)\frac{\dot{a}}{a}$ and for radiation even double the speed - α is proportional to a^2 for radiation and proportional to a for matter. If $\alpha = 1$ today, then it should be smaller than 10^{-64} at Planck time. Where does this small fine-tuning come from? $\alpha = 0$ clearly is an unstable point.
The entropy problem is often mentioned in addition to the flatness problem as well. How is it possible that such a large entropy and mass was formed?

2.4 Monopole Problem

Sometimes forgotten, but equally important, the monopole problem was the first treated problem in Guth's discovery of inflation and lies on the intersection of cosmology and particle physics. Hence, I will give a deeper introduction here. Monopoles are stable point-like defects in the (Higgs)field[5], just like domain walls are two-dimensional topological defects. The field points radially away from the defect having magnetic field configurations at infinity, far away from its centre the field approaches a specific value that can be calculated via gauge transformation. Simply spoken, they are objects with net magnetic charge[6]. At early times the universe must be described by particle physics. As the universe cools down symmetry breaking is predicted. In all grand unified models there is a gauge group G valid at high energies, at lower energies spontaneous symmetry breaking (SSB) occurs [23] $G \rightarrow H_n \rightarrow ... \rightarrow H_0$ [7]. During this phase transition those monopoles, predicted to be very heavy, formed at the critical temperature $T_c \sim 10^{15}$GeV (Weinberg). This can already be seen by the fact that any GU model predicts quantised electrical charge as the U(1) group is embedded in a simple group that will be spontaneously broken, magnetic monopole production is the consequence [29][57]. The calculated number of monopoles would exceed all other matter forms, but still has never been observed (for a detailed calculation see appendix, 7.7.3). The universe's mean density would exceed its current density of many orders.
Can a theory of inflation solve that problem? Furthermore, monopoles must have been formed when the universe was very hot, so their production should have affected further dynamics, for example, the production of baryons. Even without the experimental evidence of monopoles themselves, which we would never be able to directly observe at such high masses, we should be able to see some kind of traces in the early dynamics and its effect nowadays. For a precise calculation of the mass and number of monopoles please have a look in the appendix.
Further relics occurring in the standard hot Big Bang model and symmetry breaking

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[4]If the strong energy condition is assumed, $1 + 3w > 0$. We will see that inflation eliminates this by postulating $1 + 3w < 0$, see also 7.3.2.

[5]In the literature the Higgs field is often set equal to the inflaton field, which hasn't been proven yet. However, as long as one follows general GUT symmetry breaking the field doesn't need to be specified.

[6]Very massive particles with magnetic charge were already introduced by Dirac in 1931. He tried to establish a symmetry between electric and magnetic charge. He showed that the existence of magnetic monopoles imply quantisation of the electric charge: $g = \frac{n}{2e}$ [14]

[7]Today's favourite is Georgs-Glashow model with $G = SU(5)$ resulting in SU(3)xSU(2)xU(1), the strong and electroweak forces. The standard model of particle physics doesn't include gravity.

are yet unobserved in forms of superheavy domain walls (Zeldovich, 1974). Moreover, according to N=1 supergravity theories the number density of gravitinos should be much higher than observations indicate (Ellis, 1982).

2.5 Large Scale Structure

The Large Scale Structure (LSS) of the universe is often not given as a problem of the standard Big Bang theory. Where does the universe's structure come from? It is well known that galaxies locally cluster on orders of 100Mpc. The clusters themselves arrange in super clusters with large voids in between. The universe is not only homogeneous, but also has *irregularities*. The anisotropies in the CMB observed by the COBE satellite are proof for the seeds of formation at the time of decoupling. It is assumed that those seeds are quantum fluctuations resulting in primordial density perturbations and finally the large scale structure today. However, the Big Bang doesn't provide any explanation for such a mechanism of any initial perturbations. The LSS is one of the most important and observed physics in astronomy. More details can be found in the section on observations 3.5. Worth mentioning here is that the observed parameters are quite fine-tuned such that matter and in the end life is actually possible. Is there a reason why they have their specific values (Hubble constant/expansion rate, critical density etc)?

An overview on the general Big Bang model, its predictions and measurements can be found in 7.1 including a time line 7.1 and an explanation of the CSM 7.3 in detail as well as a short intro to scalar fields in cosmology 7.4.

2.6 The Need for Quantum Gravity (?)

As just mentioned the need for quantum fluctuations at the beginning of the universe as seeds for the macroscopic universe described by general relativity (GR) already suggests that a theory of quantum gravity is needed 7.12. Inflation takes place at very high energies/temperatures, energies higher than we are able to test with the help of accelerators. As it turns out gravity is perturbatively non-renormalisable at high energies. The gravitational constant is dimensionful, it has a negative canonical mass dimension of 2. In the standard model of particle physics we are usually used to dimensionless couplings. We will see that this issue is resolved by introducing a scale dependent running towards a finite value. The problems above go along with the issue of initial conditions that seem to be very carefully chosen such that 'our' universe was possible to form. There is also a large difference of orders between the macro- and micro scale, 7.2.

2.6.1 Asymptotic Safety

As inflation is a magnifying glass of the universe at early times we will see that scale invariance is an important property of inflation. This already suggests using some form of scale invariance as symmetry principle. We will see that AS is based on the non-perturbative study of renormalisation group flows where a non-interacting fixed point (NGFP) exits in the UV making the underlying theory predictive at all scales, even near the Big Bang. The UV completion of the theory such as gravity is done by finding a trajectory that is attracted towards the UV fixed point (from now on simply called FP), the cutoff (both IR and UV as will see) can be safely removed and we end up with finite couplings of a finite dimensional critical hypersurface which means that we also only need to measure a finite number of parameters only. We already know QCD as being asymptotically free at high energies i.e. the couplings vanish in the UV (a so-called Gaussian fixed point, GFP). We expect that gravity is asymptotically safe, attaining the NGFP at HE.

Chapter 3

Physics of Inflation

SPECTACULAR REALIZATION

(Alan Guth — SLAC, Dec. 7 1979)

3.1 Theory

One can quickly see that the idea of an *accelerating expansion* might be able to solve the aforementioned problems. Intuitively, It can smooth out any irregularities in the beginning and flatten the universe by stretching any initial curvature.
Before going into detail, the requirement of $\ddot{a} > 0 \leftrightarrow \frac{d}{dt}\dot{a} > 0 \leftrightarrow \frac{d}{dt}(aH) > 0$ drives Ω_{tot} to unity. Similarly, the expansion can join light cones.
Inflation is the rapid/accelerated expansion in the early universe and should provide an explanation for the initial conditions and today's LSS. Thus, inflation solves the just mentioned cosmological problems. It should explain the homogeneity and isotropy as well as give a platform for primordial fluctuations resulting in today's structure. In the normal cosmological evolution the Hubble radius would grow and so would the distance between objects that should have been in causal contact. Hence, the most obvious solution is that today's Hubble radius is smaller than the one at the time of recombination (whose evidence we have in the CMB). As we will see, if the increase in Hubble radius was behind the speed of expansion, the issue is resolved.
Inflation has developed into a general framework and has been achieved in many different models. In the following, I will first choose a model-independent approach. First, I will assume that such a rapid expansion is possible and investigate how it solves the problems and what constraints are given by the cosmological problems. Then, I will take a deeper look into the possible dynamics and origin of such a behaviour. I will proceed with the theory of perturbations and observational evidence. Finally, some alternative theories and extensions of inflation are given.

3.2 How to Inflate - Constraints on Inflation

In order to inflate a *repulsive form of gravity* is needed. Taking a look at Einstein's equations gives $p = -\rho$ as immediate possibility. Following the weak energy condition $\rho \geqslant 0$ (7.3.2) we find a non-positive pressure. In the following subsections we will investigate the constraints on inflation and how it solves the problems mentioned in the motivational part.

3.2.1 Horizon and Flatness Problem as First Constraints

Taking the horizon and flatness problem as first constraints on inflation we conclude that a *smoothing fluid component* is needed [1]. As previously mentioned the Friedmann equation

$$H^2 = \frac{8\pi}{3m_p^2}\left(\frac{\rho_r}{a^4} + \frac{\rho_m}{a^3} + ...\right) - \frac{k}{a^2} \tag{3.1}$$

can be rewritten as

$$k = 0 \rightarrow 1 = \Omega_{tot}, k = \pm 1 \rightarrow |1 - \Omega_{tot}| = \frac{1}{a^2 H^2} \tag{3.2}$$

This gives a ratio of

$$\frac{|1-\Omega_{tot}|_{today}}{|1-\Omega_{tot}|_{initial}} = \frac{\dot{a}^2_{initial}}{\dot{a}^2_{today}} \gg 1 \tag{3.3}$$

If we add a component to the RHS of the Friedmann equation that dominates it is possible to eliminate this problem: $+\frac{\rho_s}{a^{2\epsilon}}$, where ϵ is the equation of state parameter depending on the matter content $\epsilon = \frac{3}{2}(1+w)$. For $\epsilon < 1$ this is possible.

The constraint on the elimination of the horizon problem gives the amount of inflation needed. The observed horizon at initial time $r_{obs}(t_i) = \frac{r_{obs} t_0 a_i}{a_0}$ can be approximated with $r_{obs} \sim \frac{1}{H_0} \sim t_0, r_{obs}(t_i) \sim \frac{t_0 a_i}{a_0}$. If $a(t) \sim \frac{1}{t^\epsilon}$ the causal horizon at initial time is $r_{causal}(t_i) = a(t_i)\int_0^{t_i} \frac{dt}{a(t)} \sim \frac{\epsilon}{\epsilon - 1} t_i$ Hence, the ratio is

$$\frac{r_{obs}(t_i)}{r_{causal}(t_i)} \sim \frac{\epsilon - 1}{\epsilon}\frac{a_i}{a_0}\frac{t_0}{t_i} \sim \frac{\epsilon - 1}{\epsilon}\frac{a_i}{a_0} \tag{3.4}$$

If $a(t) \sim \frac{1}{T}$ taken again as before we arrive at $\left(\frac{10^{15}\text{GeV}}{10^{13}\text{GeV}}\right)^{\epsilon-1} \sim 10^{28(\epsilon-1)} \sim e^{60(\epsilon-1)}$ Following this method we basically arrive at the slow-roll (SR) conditions, in detail explained in the next subsection. In SR the pressure as difference of kinetic and potential energy is the negative value of the density, the sum of both [2]. Thus, *potential energy dominates over kinetic energy*. The amount of inflation needed is given by the condition

$$\frac{a_f}{a_0}\frac{H_{inf}}{H_0} \leqslant e^N \tag{3.5}$$

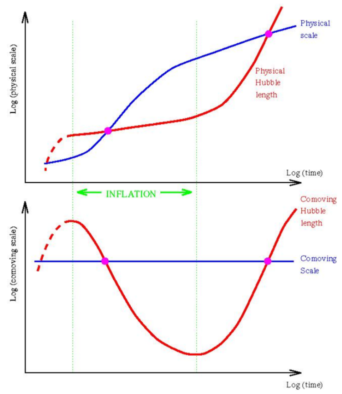


Figure 3.1: Horizon problem, is solved. Upper: temporal evolution of the physical length (blue) vs Hubble radius (red). Lower: temporal evolution of the same, but comoving scales. During Inflation the Hubble radius is behind the growing a, structures initially inside it grow beyond that. Lidde's lectures(ned.ipac.caltech.edu/level5/Liddle)

[1]The following derivation is oriented towards A. Ijjas' talk on *Cosmic Inflation* at Harvard University, 2014.

[2]For a short intro to scalar fields in cosmology see 7.4.

which is the ratio of our horizon and the physical horizon at the end of inflation. Assuming that the temperature behaves antiproportional to a, H behaves like $\sim T^2$. Solving for N gives

$$N \geqslant \ln\left(\frac{T_f}{T_0}\right) \sim 66 \tag{3.6}$$

at an energy of say the GUT scale $\sim 10^{16}$GeV $\sim 10^6$J. This gives a final temperature of $\sim 10^{29}$K (obviously this is rather simplified, a proper calculation will be done in 3.4.1, but as an estimation this is viable). Hence, 66 e-folds would be needed in order for inflation to occur sufficiently long. A further insight is given in the section on slow-roll. In the literature one usually sets the number of e-folds at horizon crossing ($\sim$ end of inflation) to $50 < N* < 65$.
A useful concept is the *comoving Hubble length*, $\frac{1}{aH}$, which is the distance over which communication between two points in space *today* is possible (whereas the particle horizon or *comoving horizon* can identify whether they were in causal contact at some point in time, 7.28). The condition for inflation can be rewritten as

$$0 > \frac{\ddot{a}}{\dot{a}^2} = \frac{d}{dt}\left(\frac{1}{\dot{a}}\right) = \frac{d}{dt}\left(\frac{1}{aH}\right) \tag{3.7}$$

The solution to the horizon problem is then to reduce the comoving Hubble length $(aH)^{-1}$ far below the particle horizon today - different regions cannot be in causal contact today, but were able to communicate at early times. Important to mention is that inflation does not violate relativity since spacetime itself is expanding (i.e. no information is being transferred).
This condition 3.7 also explains the flatness today as it drives $|\Omega - 1|$ to 0.
Before introducing the SR model, we can define inflation via

Condition for inflation-model independent

1. $\ddot{a} > 0$ - accelerating expansion
2. $\left(\frac{i}{aH}\right) < 0$ - decreasing comoving Hubble length
3. $\rho + 3p < 0$ - repulsive form of gravity

3.2.2 Slow-Roll Condition

The slow-roll condition postulates that the potential energy dominates over the kinetic energy. Hence, V dominates over $\dot{\phi}^2$. The Hubble evolution equation further validates this claim. Differentiate the first Friedmann equation 7.11 with respect to time and substitute in the scalar field and perfect fluid expressions for the density and pressure: $-\frac{\dot{H}}{4\pi G} = \rho + p = -\dot{\phi}^2$. Inflation means accelerated expansion, the scale factor scales like $a(t) \propto e^{Ht}$, so $H = \dot{\ln a}$ must be constant. Thus, $\dot{\phi}^2 \to 0$.

$$H^2 = \frac{8\pi G}{3}\left(V(\phi) + \frac{1}{2}\phi\right) \to \boxed{H^2 = \frac{8\pi G}{3}V(\phi)} \tag{3.8}$$

Using the scalar field action described in the appendix and FRW cosmology one obtains the equation of motion (eom):

$$\ddot{\phi} + 3H\dot{\phi} = -V' \rightarrow \boxed{3H\dot{\phi} = -V'} \tag{3.9}$$

The *slow-roll parameters* quantify inflation and are defined as

$$\epsilon = -\frac{\dot{H}}{H^2} \quad \text{and} \quad \eta = \frac{1}{H}\frac{\dot{\epsilon}}{\epsilon} \tag{3.10}$$

independent of the inflaton choice. In terms of a potential it can be rewritten (see appendix, **??**) as

$$\epsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \quad \text{and} \quad \eta = \frac{1}{8\pi G}\frac{V''}{V} \tag{3.11}$$

Inflation takes place as long as $\epsilon < 1$, indicating the end of inflation when ϵ is unity. It can be shown (**??**) $\ddot{a} > 0$ and $a > 0$ iff $\epsilon < 1$.
As already shown previously inflation must be possible in the sense of its beginning and a sufficient amount of it to solve the cosmological problems. As shown in the beginning of the section the horizon problem needs a sufficient amount of e-folds, in the literature about 60. It measures the folds yet to occur and is defined via $dN = -Hdt$, along with $\frac{da}{a} = Hdt$ and $dt = \frac{d\phi}{\dot{\phi}}$.

$$N(\phi) = \log\left(\frac{a(t_f)}{a(t)}\right) = \int_t^{t_f} H(t')dt' = \int_{\phi}^{\phi_f} \frac{H}{\dot{\phi}}d\phi \sim 8\pi \int_{\phi_f}^{\phi} \frac{V}{V'}d\phi \tag{3.12}$$

where the last approximation is done under slow-roll conditions and the boundary condition is given by $\epsilon(\phi_f) = 1$, the final value of the field produces unity in epsilon, inflation stops. ϵ is an important parameter 7.7.4 to investigate whether inflation is possible and long enough (see also observations 3.5). In general, inflation under SR condition is possible for $\epsilon, |\eta| < 1$.
Recall from the motivational part we can now write $\alpha = \epsilon = \frac{3}{2}(w+1) = \frac{\dot{\phi}^2}{2H^2}$ In summary, inflation needs to begin, last long enough and end. Also, $\epsilon < 1$ gives $w < -\frac{1}{3}$ again proving that the kinetic energy is dominated by the potential energy **??** and the 'abnormality' of the matter type (note $w = -1$ is the vacuum energy/dark energy/cosmological constant 3.7).

3.2.3 LSS

Quantum/vacuum fluctuations of the inflaton and gravitational fields at the beginning of inflation are the seeds for density fluctuations and the LSS nowadays. During inflation they became stretched to scales beyond the Hubble size. Clusters can be seen on scales of galaxies, but even galaxies themselves cluster. A detailed explanation is given later 3.5.1. Again, the Hubble radius grows much less than the physical scales which makes it possible for quantum fluctuations to take scales beyond the Hubble size - they become classical perturbations and 'freeze'. With the end of inflation they can again be smaller than the Hubble size and make today's structure.

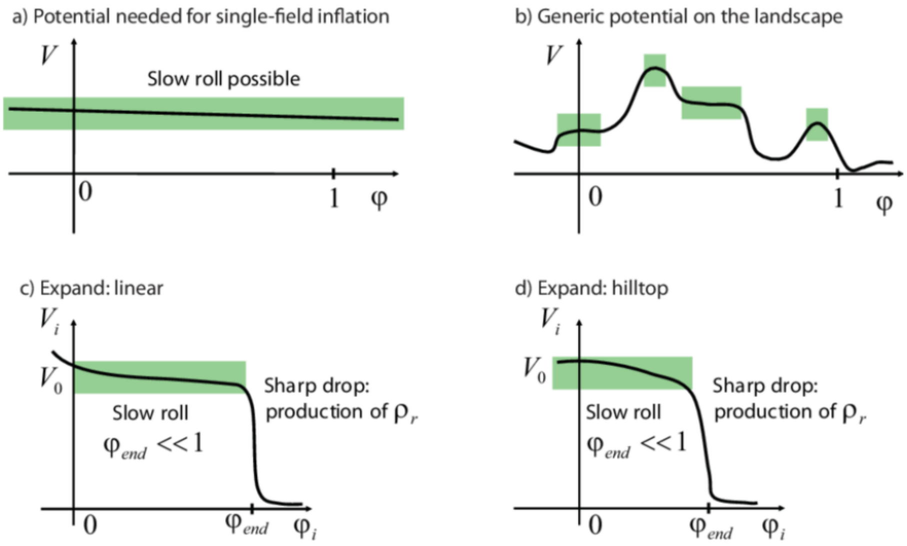


Figure 3.2: Slow-roll potentials that are possible. For SR single field inflation a flat long plateau, linear potential $V_i = V_0 - c\phi_i$ (a), a more generic potential with different regions where on the saddles and maxima SR is possible (b), plateau with potential moving uphill at $\phi < 0$ (c) and hilltop with potential moving down for $\phi < 0$ $V_i = V_0 - \frac{1}{2}m^2\phi_i^2$ (d). 'Multi-field inflation on the landscape', 2009.

3.2.4 Monopole Problem

As it was shown we need an explanation why there is such a discrepancy between prediction and observation. One could either enhance the annihilation process (Fry 1981) or suppress the initial production (Guth and Weinberg 1982), both of which didn't give satisfactory results. Magnetic monopoles might exist, but were produced prior inflation. During the inflationary expansion the density dropped (exponentially) which makes them undetectable today.
Precisely, take $a \sim t^p$ for any $p > 1$, then $H \sim t^{-1} \sim a^{-\frac{1}{p}}$ and the density of the universe behaves like $a^{-\frac{2}{p}}$. Matter density behaves like the scale factor cubed, a^{-3} - the particles undergo a redshift up to a non-detectable state.
Other relics such as gravitinos or domain walls would equally be diluted. Worth mentioning here is that the reheating temperature (discussed in 3.4.1 'end of inflation') needs to be low enough such that no new production can occur. Here inflation obviously can only been seen as a solution if the corresponding theories behind those particle physics predictions are true. In the problem section I will investigate whether the solution of the monopole can been seen as a success of inflation.

3.3 Models of Inflation

The types of models of inflation occupy a huge range. Starting from false vacuum decay, over the simple slow-roll and single scalar field model up to higher derivative (HD) gravity theories. In order to understand why and where theoretical cosmologist are heading nowadays, here is a short overview.

3.3.1 De Sitter model

If one ignores the curvature term (often called the redshifting of the curvature since the exponential expansion makes the term vanishing small[3]) in the first Friedmann equation since the scale factor is rapidly increasing, one arrives at the simple differential equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} \tag{3.13}$$

with a solution of $a(t) \sim e^{Ht}$ where H is constant $H = \sqrt{\frac{8\pi G\rho}{3}}$ as ρ is assumed to be constant during inflation. Here the density is dominated by the vacuum energy, $\rho = \rho_\Lambda$, and the universe would be driven to flatness (see also 7.6). However, its further evolution would be emptiness. Since we live in a universe with matter and radiation this model was disregarded. Inflation must stop at some point such that reheating can take place. Similarly, one can introduce the cosmological constant in the Einstein field equations and produce an accelerated expansion with $\rho = p = 0$ which gives $a(t) \propto e^{\sqrt{\frac{\Lambda}{3}}t}$. Weirdly, substituting this into the metric this gives after a coordinate transformation actually a static universe (invariant under $t \to t' + t_0$ and $x_i = e^{\sqrt{\frac{\Lambda}{3}}t_0}x_i$). However, the static character is only local, globally it is evolving.

3.3.2 Old Inflation

Despite its failure it is important to mention the old inflation scenario as stepping stone for inflationary cosmology. The next step was to reduce the time of rapid expansion such that matter could form. Furthermore, the early universe must be described by some quantum field theory as well. The oldest idea of inflation by Guth [23] is not based on slow-roll conditions, but on a *false vacuum* where the scalar field undergoes a first order phase transition in the early universe combining both cosmological and particle physics. Using a toy model of a potential that has a metastable false vacuum - a temporary state where the field has a high energy density - at $\phi = 0$ and a true vacuum (with the actual lowest energy state) at $\phi = a$ the field finds itself in the true vacuum for $T < T_c$. Classically the false vacuum is stable, but quantum mechanically it can tunnel (locally!) to the true vacuum. The large negative pressure makes it possible $p = -\rho = -\rho c^2$ =energy density. Inflation lasts as long as the decay of the false vacuum lasts. Guth mentions a thought experiment [24] where a piston

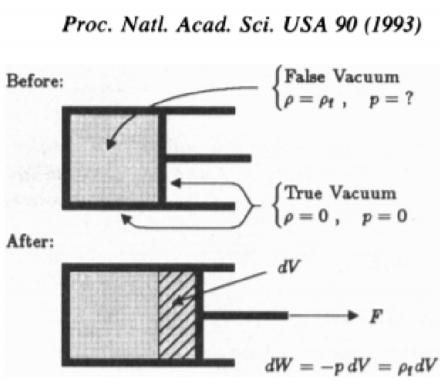


Figure 3.3: Piston explanation, PS6 MIT 'The Early Universe', Guth, 2011.

[3]In fact, it can be shown that the behaviour of the equations with $k = \pm 1$ does not differ too much from the flat case.

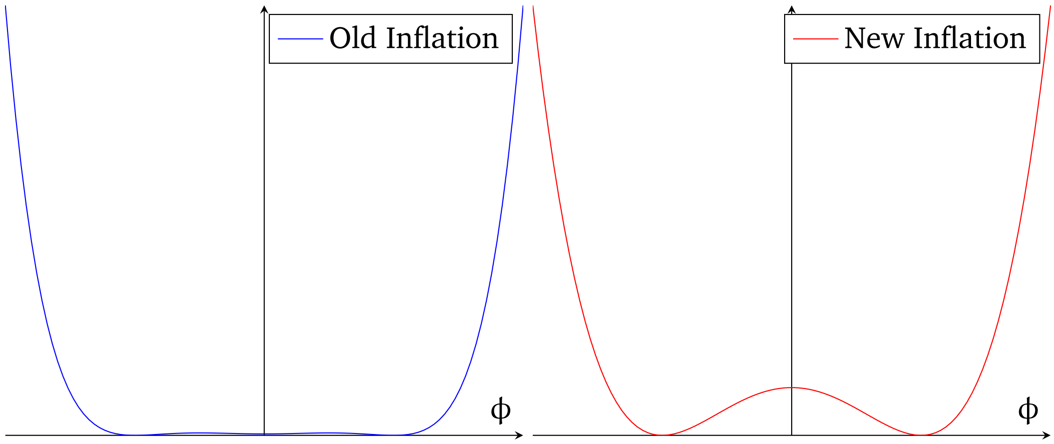


Figure 3.4: Comparison between the old and new inflationary model. Former has a local minimum as false vacuum, latter has a local maximum for which both quantum and classical treatment is metastable.

is filled with the false vacuum with density ρ_f, outside is zero energy vacuum (even the non-zero energy vacuum can be approximated to 0 in comparison to the large pressure). When pulling out the piston a volume δV the density has to stay constant - that is the difference to all other kind of substances, it is an intrinsic property of the spacetime manifold that becomes stretched - such that energy conservation gives $dU = \rho_f dV = dW = -p dV$, one would pull against a suction due to the negative pressure inside. His idea was that, after the decay of the false vacuum, the bubbles when the cooling expansion happened would collide and coalesce to a homogeneous universe (similar to the Coleman-Callan process, 1977). The bubble wall collisions would make reheating possible and standard Big Bang would have followed. Guth points out that due to the supercooling ($\rightarrow$ false vacuum decay) in the early universe the flatness and fine-tuning problem put forward by Dicke would be solved. Early on Dicke claimed that the values of the constants in electromagnetism and gravity are fine-tuned since a slight change wouldn't allow our life to be possible (1961), see also 7.5. We now know that e.g. the small value of the cosmological constant makes LSS formation possible.
The model suffers from the ad hoc construction and the *'graceful exit problem'*. In order for the field to be in the false vacuum in the beginning, finite temperature effects are necessary. This is only possible if the field is in thermal equilibrium with the other fields. The graceful exit problem makes it impossible for inflation to solve the homogeneity problem. The bubbles would nucleate after inflation with a size which would be much smaller than the apparent horizon today (even if they moved at the speed of light). They would cause large inhomogeneities inside the Hubble radius. Percolation wasn't possible due to the exponential expansion. Even if the rate of bubble formation would have been much higher than the rate of expansion, the phase transition would have been to fast and inflation wouldn't have been able to occur.

3.3.3 New Inflation

The solution was independently brought forward by Linde [37] and Albrecht and Steinhardt [5]. The potential takes a more special form, the field is at a local maximum at 0, so classically metastable. It also is a flat plateau such that slow-roll inflation is actually possible. Inflation doesn't end by a tunneling process, but by a second-order phase transition, small fluctuations that build up and push the field down the true vacuum. This happens in small regions $\sim \frac{1}{H}$ over which the fluctuations are uniform and then stretched over a large scale during SR. It is assumed that the inflaton field evolves very slowly from its initial state. Finite temperature effects confine the field to the false vacuum for $T > T_c$, fluctuations then destabilise the field to the true vacuum. In contrast to the old inflation model, the boundary walls are inflated outside the present Hubble radius as the regions of homogeneity are established before inflation. Near the centre the potential might be approximated by $V(\phi) \sim V(0) - \frac{1}{4}\lambda\phi^4$, it turns out that the inflaton has to be weakly coupled $\lambda << 1$. The origin of the very small (fine-tuned) coupling constant is not known. One finds that the perturbations (see later 3.5.1) $\propto \frac{1}{\phi}$ - the potential has to be flat near the origin in order for SR to occur.

3.3.4 Chaotic Inflation

But even here it is not really clear how inflation actually starts. How can a generic universe inflate, a universe without any assumptions on the field's potential? Linde's chaotic inflation [38] doesn't restrict the form of the potential, a simple $V \propto \lambda\phi^4$ in which the scalar field has a completely random state makes a quasi-exponential expansion possible. Where the scalar field has the proper value with large negative pressure $\phi(x) >> 0$, potential energy dominates and it begins to inflate and then dominates the universe's evolution. In some regions SR is possible (in fact, it was proven that there exists a large class of possible inflation theories, where SR is possible under 'natural' conditions).[4] The Hubble damping term in the equation of motion slowly rolls down the scalar field to $\phi = 0$ (similar to a harmonic oscillator). Most of the theories are weakly coupled ones with exponential or polynomial potentials. A detailed calculation can be found in the appendix, 7.7.4. Within the patch, the evolution is independent of the rest[5]. The advantage of this model is that it does not need a phase transition (i.e. no fine-tuning of temperature is needed), the inflaton is displaced from its true vacuum state by some arbitrary mechanism (i.e. can give an explanation for quantum and thermal fluctuations).

[4]This already hints towards the Anthropic Principle - the regions that we need for inflation to occur are favoured.

[5]If we assume the exponential scaling the universe behaves de Sitter-like and one can calculate the event horizon beyond which we cannot communicate. The event horizon, similar to the one of a black hole, is a null hypersurface and can be calculated by looking at the future light cone of the observer's worldine with $t \to \infty$,

$$\int_{t_0}^{\infty} \frac{dt}{a(t)} \sim \int_{t_0}^{\infty} \frac{dt}{e^{Ht}} = \frac{e^{-Ht_0}}{H} < \infty \tag{3.14}$$

for all k in fact. In our entire history we won't ever be in causal contact with regions beyond that such that also nothing outside can affect our evolution.

This scenario still needs mild assumptions (flat potential). As I will show it might explain the further reheating and evolution to the standard Big Bang theory. Inflation ends when the vacuum energy has rolled down into the valley of the potential.

3.3.5 Eternal Inflation

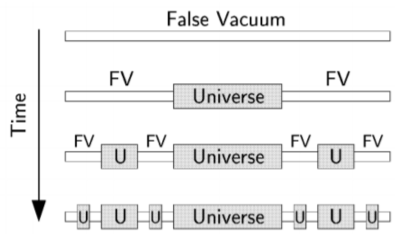


Figure 3.5: Eternal universe, 'Inflationary Models and Connections to Particle Physics', Guth, 2000.

The idea behind eternal inflation (Steinhardt, Vilenkin 1983) is that the false vacuum decays, but never vanishes completely. The exponential expansion of the universe dominates the exponential decay. 3.3.5 is a 1D sketch of the production of local universes where one should imagine that $a(t)$ increases with each bar (here three times the size, the false vacuum is always as big as the first one). The false vacuum randomly decays and each time a local Big Bang produces a pocket universe with completely different properties. This goes on eternally and results in a fractal-like structure of the universe. We find ourselves back in our universe as it exactly provides us the condition to live (Anthropic Principle).

Eternal Chaotic Inflation

Combining both, eternal and chaotic inflation, Linde [39] (1986) explained the production of the pocket universes with the help of a *random walk*[6] of quantum fluctuations (Vilenkin, 1982). The field makes Gaussian jumps in $\delta t = H^{-1}$, $\delta\phi_{\text{quantum}} = \frac{H}{2\pi}$. This quantum fluctuation is superimposed on the classical motion of the field.
The problem of eternal inflation is the loss of predictivity and possibility of probability calculation. All kind of universes are possible: different symmetry breaking types result in different low energy physics, different vacuum states and using higher-dimensional Kaluza-Klein theories[7] even gives different dimensions and they all develop independently. Beyond our universe $r > \frac{1}{H}$ inhomogeneity dominates, we need to be careful to distinguish between the local and *global* description of inflation.
Hawking and Hertog [28] proposed a solution for a smooth exit of eternal inflation. Based on the no-boundary proposal [8], a gauge/gravity duality and Top-Down (3.6.2) probability ('only observe a patch of the universe') they made a well-defined exit possible. The duality between Euclidean AdS (Euclidean deformed S^4) and Lorentzian asymptotic de Sitter with inflation and cosmological constant is derived via the wave-function of the universe approach 3.6.2. A detailed look regarding his last work is beyond the scope of this thesis, but basically they found the complexified solutions

[6] A sample calculation can be found in the chapter on problems, probability and measure problem.

[7] Here domains in a d-dimensional universe can be squeezed or stretched to a tube of different dimensions - independently of its former universe, when the initial energy density reaches its Planck value and its length is bigger of order m_p.

[8] The no-boundary proposal says that prior to the Big Bang time didn't have a direction.

to Einstein's field equations. Eternal inflation would then produce a regular universe, even on larger scales.

Constraints for Eternity

According to Guth almost all inflation models are eternal. I will investigate its constraints on the chaotic model $V = \frac{1}{2}m^2\phi^2$. I follow Guth approach [25] which leaves out the side calculations and is applied to $V = \frac{1}{4}\lambda\phi^4$. The fluctuation is the sum of the classical and quantum fluctuations

$$\delta\theta = \delta\theta_{cl} + \delta\theta_q \tag{3.15}$$

The quantum fluctuation shows a Gaussian distribution of width $\frac{H}{2\pi}$. With the usual approach of calculating correlation functions

$$\langle 0|\, \delta\hat{\phi}^2\, |0\rangle = \int \frac{d^3\vec{k}d^3\vec{k'}}{(2\pi)^3} e^{i(\vec{k}-\vec{k'})\vec{x}} \langle 0|\, \delta\phi_k \delta\phi_{k'}\, |0\rangle$$

$$= \int \frac{dk}{k}\left(\frac{H}{2\pi}\right)^2 = \left(\frac{H}{2\pi}\right)^2 \ln\frac{k_*}{k_i} = \left(\frac{H}{2\pi}\right)^2 N \tag{3.16}$$

with $k = \left|\vec{k}\right|$ and $H\delta t = N$ being the logarithmic ratio of the final and initial value of the comoving wavenumber k (re-entering and entering of horizon crossing). The derivation for the correlation function can be found in 3.5.1 which will be derived later (see also 3.73). With $v_k = a\delta\phi_k$ and in the superhorizon limit

$$\langle 0|\, v_k v_{k'}\, |0\rangle = (2\pi)^3 \frac{1}{2k}\frac{1}{(k\eta)^2}\delta(\vec{k}-\vec{k'}) = a^2 \langle 0|\, \delta\phi_k \delta\phi_{k'}\, |0\rangle \tag{3.17}$$

This gives a deviation for one e-fold of $\frac{H}{2\pi}$. Now, quantum mechanics allows the potential to fluctuate up or down with some probability. During one e-fold in a Hubble time interval $\Delta t = \frac{1}{H}$ it expands exactly by the factor e, so the volume increases by $e^3 \sim 20$. Thus, we have 20 comoving regions. In order to have at least one region further inflating we have a probability of $\frac{1}{20}$. Using a Gaussian distribution with 1σ standard deviation the field's mean value is exceeded by 0.159 such that we have on average $20 \cdot 0.159 \sim 3.18$ regions that oscillate up the potential (= positive value). Inflation will not end in this case. Recall that this calculation is done for one e-fold only, there will always be some regions where SR doesn't stop and inflation continues. Guth then says eternal inflation takes place if the standard deviation of the quantum fluctuation is bigger than $0.61|\delta\phi_{cl}|$ ($e^{-0.5} \sim 0.61$) in order to have $p > \frac{1}{20}$ for the field to fluctuate up the potential. As $< \delta\phi >= 0$ and the classical fluctuation is $\dot{\phi}_{cl}\delta t = \frac{\dot{\phi}_{cl}}{H}$ (see 3.7, the field should be in the green area). This gives with substituting SR condition

$$\sqrt{\frac{2GV}{3\pi}} \sim \frac{H}{2\pi} > 0.61\left|\frac{\dot{\phi}_{cl}}{H}\right| \tag{3.18}$$

This gives in Planck units the conditions

$$\frac{H^2}{|\dot{\phi}_{cl}|} > 3.8 \quad \frac{19}{m_p^3} > \frac{V'}{V^{\frac{3}{2}}} \quad \& \quad \frac{4V}{m_p^4} > \epsilon \tag{3.19}$$

for the quadratic chaotic potential a value of $V > 2 \cdot 10^{-8} m_p^4$. This is far below the Planck scale and hence the condition is quite probable. Similarly, Guth gives a value for $V > 0.079\lambda^{\frac{1}{3}}$, with a approximated coupling of 10^{-10} in order for inflation to occur long enough which is below the Planck scale as well.

3.3.6 Starobinsky Inflation

Already in 1979 Straobinsky had the the first idea of inflation (without using its name)[9]. He proposed a quantum state in the beginning and how it would produce gravitational waves that would be detectable today. Before the classical Friedmann expansion he assumed an exponential expansion ($a = e^{Ht}$), an inflationary de Sitter era. The universe would have been in a maximally symmetric state.
Later, the Starobinsky or R^2 inflation was developed. It is an example for an $f(R)$ theory meaning that the usual Einstein-Hilbert Lagrangian has an additional term in R^2 which should dominate its behaviour in early times (=large curvatures). Hence, its theory is a modification of GR and quantum corrections of the early universe should be described in the additional curvature term. As already shown, domination of curvature can act as an effective source for inflation. The theory's predictions are very close to data with only one free parameter that is defined by the primordial spectrum. The action is given as

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g}(R + BR^2) \tag{3.20}$$

with the parameter $B = -\frac{1}{6m^2}$ and for later reference I include the gravitational constant, where m is the mass of the associated scalar field (see below). Precisely, Starobinsky[75] tried to solve the singularity problem by postulating that the universe spent an infinite time in a de Sitter state without a singularity prior. Here he made use of the exact one loop approximation of quantum corrections to gravity (without matter). Inflation ends with a graceful exit to the usual Friedmann expansion. According to the Starobinsky the initial state was so dense that we should still be able to observe gravitational waves from back then.
The theory gives clear and simple predictions for the spectral index and scalar-to-tensor depending on the number of e-folds only which will be described and derived later[20].

$$n_s = 1 - \frac{2}{N} \quad r = \frac{12}{N^2} \tag{3.21}$$

This gives the motivation of further *higher derivative gravity* research as well as looking at this inflation model in other settings such as asymptotic safety.

From the Lagrangian to the Potential

In the $f(R)$ theory the usual Lagrangian $\sim R$ is substituted by a function of the curvature scalar, $f(R) = R + \alpha_1 R^2 + \alpha_2 R^3$ etc. The reason why I want to derive the following is to emphasise the beauty of higher derivative (HD) theory. Firstly, Einstein-Hilbert is non-renormalisable at higher energies[10] and terms will lead to higher derivative

[9] 'Spectrum of relict gravitational radiation and the early state of the universe', 1979
[10] I will investigate this issue in the chapter on asymptotic safety.

terms in the equation of motion. Secondly, the first ideas on additional terms in the Lagrangian were based on the wish of producing vacuum polarisation effects in a 'natural' way. Einstein's gravity coupled to (quantum) matter fields (i.e. $< T_{\mu\nu} >$ is coupled to gravity, too) produce HD terms when calculating the semi-classical limit.
I want to prove that $L = f(R)$ is conformally equivalent to Einstein-Hilbert plus a scalar field - which we can use as inflaton field. Whereas until now, the scalar field was assumed to just exist in a potential with the right properties it is of major importance to derive a possible origin for this field. The equations are used in later treatments as well.

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) \tag{3.22}$$

Its variation gives

$$\frac{\partial f}{\partial R}R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f + \nabla_i\nabla^i\frac{\partial f}{\partial R}g_{\mu\nu} - \nabla_\mu\nabla_\nu\frac{\partial f}{\partial R} = 0 \tag{3.23}$$

Precisely,

$$\delta S = \int d^4x(\delta(\sqrt{-g})f(R) + \sqrt{-g}\delta(f(R))) \tag{3.24}$$

where the variation of the metric part is derived with the help of $\mathrm{Tr}\ln M = \ln\det M$ with $M = g^{\mu\nu}$, giving $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$. The variation of the curvature function is given by

$$\delta f(R) = \frac{\partial f}{\partial R}\delta R = \frac{\partial f}{\partial R}\delta R_{\mu\nu}g^{\mu\nu} + \frac{\partial f}{\partial R}R_{\mu\nu}\delta g^{\mu\nu} \tag{3.25}$$

The variation of the $R_{\mu\nu}$ is derived with the help of the Palatini equation $\nabla_\alpha\delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu\delta\Gamma^\alpha_{\mu\alpha}$ where I can shift the metric inside the variation assuming the metricity condition. Using symmetry arguments and substituting $\delta\Gamma^\sigma_{\mu\nu}$ the variation can be rewritten (from now on $\frac{\partial f}{\partial R} = f'$):

$$\delta S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left((f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu})\delta g^{\mu\nu} - \frac{1}{2}f'\nabla_\sigma(\nabla^\sigma(\delta g^{\mu\nu}))g_{\mu\nu} - \nabla_\mu(\delta g^{\mu\sigma})\right) \tag{3.26}$$

In order to get the variation with respect to the metric I integrate the second part twice and assume the usual boundary conditions (for a precise treatment one should evaluate the boundary terms). The first part is the known Einstein-Hilbert variation for $f(R) = R$.

$$\frac{\delta S}{\delta g_{\mu\nu}} = f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - \nabla_\mu\nabla_\nu f' + \nabla_\mu\nabla^\mu f'g_{\mu\nu} \tag{3.27}$$

giving 3.25 for the matter-free case. However, I would like to find the corresponding potential when the RHS is given by

$$-\frac{1}{4\pi G\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = \frac{T_{\mu\nu}}{8\pi G} \tag{3.28}$$

Conformal equivalence will give the result

$$\hat{g}_{\mu\nu} = h(R) g_{\mu\nu} \tag{3.29}$$

which should produce Einstein's equations under conformal transformation:

$$\hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{R} = \frac{1}{8\pi G} \hat{T}_{\mu\nu} \tag{3.30}$$

eliminating the undesired part above. The energy tensor is dependent on the scalar field and will give the potential I am looking for.
The approach is similar to Whitt's [88] who investigates the Lagrangian of $R - 2\Lambda + \alpha R^2$ and the conformal factor of $\hat{g}_{\mu\nu} = (1 + 2\phi\alpha) g_{\mu\nu}$. I am using his result (rewritten) on the conformal transformation on the Ricci tensor without a priori knowing the form of h. From now on $\nabla_\alpha =_{;\alpha}$

$$R_{\mu\nu} \rightarrow \frac{R_{\mu\nu}}{h} - \frac{g_{\mu\alpha} h^{\alpha}_{;\nu}}{h^2} + \frac{3 g_{\mu\alpha} h_{;\nu} h^{;\alpha}}{2h^3} - \frac{g_{\mu\nu} h_{;\alpha\beta} h g^{;\alpha\beta}}{2h^2} \tag{3.31}$$

This gives a transformation for the scalar

$$R \rightarrow \frac{R}{h} - \frac{3h_{;\alpha\beta}}{h^2} + \frac{3 h_{;\alpha} h_{;\beta} g^{\alpha\beta}}{2h^3} \tag{3.32}$$

Now I substitute this into the transformed Einstein equation 3.30 with the energy tensor dependent on the scalar field.

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{3}{2h^2} \left(g_{\mu\alpha} h^{;\alpha} h_\nu - \frac{g_{\mu\nu} h_{;\alpha} h_\beta g^{\alpha\beta}}{2} \right) - \frac{1}{h} \Big(g_{\mu\alpha} h^{;\alpha}_{\mu} + \frac{1}{2} g_{\mu\nu} h_{;\alpha\beta} g^{\alpha\beta} \\ + \frac{3}{2} h_{;\alpha\beta} g^{\alpha\beta} \Big) = 8\pi G \left(g_{\mu\alpha} \phi^{;\alpha} \phi_{;\nu} - \frac{1}{2} g_{\mu\nu} \phi_{;\alpha} \phi_{;\beta} g^{\alpha\beta} + h V g_{\mu\nu} \right) \end{aligned} \tag{3.33}$$

Inspection gives $\phi = \sqrt{\frac{1}{16\pi G}} \int \frac{h_{;}}{h} = \sqrt{\frac{1}{16\pi G}} \ln h$ to make it similar to the desired Einstein equations by eliminating the kinetic terms.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{h} \left(g_{\mu\nu} h_{\alpha\beta} g^{\alpha\beta} - g_{\mu\alpha} h^{;\alpha}_{\nu} \right) = \ldots$$

If the function h is now taken to be $h = \frac{\partial f}{\partial R}$ the solution is straight-forward. Substituting this back gives a general formula for the potential:

$$V(\phi) = \frac{1}{16\pi G} \frac{f - f' R}{f'^2} \tag{3.34}$$

The curvature scalar R can be substituted by a function dependent on ϕ when the formula for h is substituted into the formula for ϕ above and then rearranged to give $R = g(\phi)$. This might not always be possible. We also see that we have just introduced a new degree of freedom (dof)!
Back to Starobinsky's Lagrangian 3.20. Here,

$$h = 1 - \frac{R}{3m^2} \quad \phi = \sqrt{\frac{3}{16\pi G}} \ln \left(1 - \frac{R}{3m^2} \right) \tag{3.35}$$

Solving for ϕ this gives the potential we are looking for:

$$V = \frac{1}{16\pi G}\frac{R^2}{6m^2}\left(1-\frac{R}{3m^2}\right)^2 = \frac{3m^2}{32\pi G}\left(1-e^{-\sqrt{\frac{16\pi G}{3}}\phi}\right)^2 \tag{3.36}$$

As the figure shows, this is a desired SR potential 3.6. SR is possible from large ϕ values with a flat plateau. When the field has rolled down to the minimum it starts oscillating and reheating takes place 3.4.1. Note that the field's potential value is desirably small. The amplitude is $V_0 = \frac{3m^2}{32\pi G} \sim 0.03 \cdot m^2 m_p^2 \sim 10^{-7} m_p^4$.
Its perturbations will be analysed in the section on cosmological perturbations along

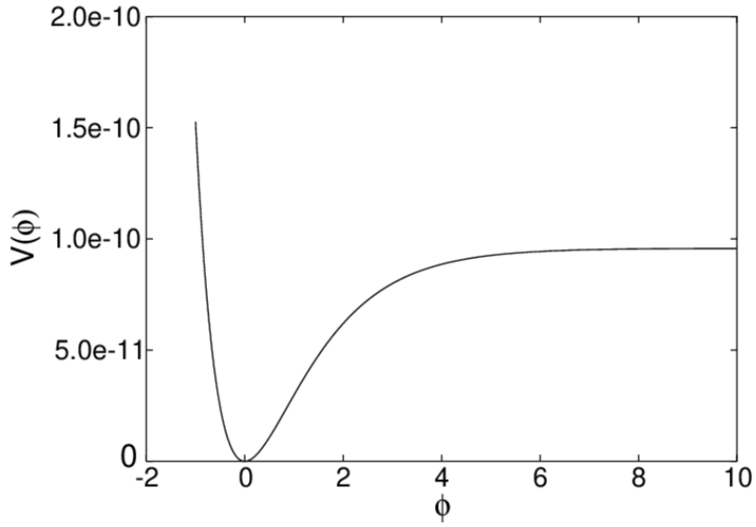


Figure 3.6: Starobinsky potential. The potential has a minimum at $0(V''(0) > 0)$, saturates as $\phi \to \infty$ at $V_0 = \frac{3m^2}{32\pi G}$ and diverges for $\phi \to -\infty$.The RHS is dominated by the R^2 behaviour, the LHS the usual Einstein-Hilbert term. 'Semiclassical analysis of the tensor power spectrum in the Starobinsky inflationary model', 2020.

with a prediction of the scalar field's mass.
As mentioned, Whitt treated a similar approach in which he showed [88] that $\mathscr{L} = R - 2\Lambda + \alpha R^2$ is equivalent to Einstein gravity with a massive scalar field. He then applied his result to black holes which have no hair for the $\Lambda = 0$ case.[11]
In the appendix there is a table with different inflation models 7.3.1, their potential and expansion factor and some further investigations. The number of inflation models increased in the past four decades and also goes beyond simple SR: models with

[11] He calculated the eom, took the trace and found that the conformal transformation, $\hat{g}_{\mu\nu} = (1 + 2\alpha\phi)g_{\mu\nu}$ and identified $\phi = R$. As long as the cosmological constant is non-negative DEC is satisfied and the field is non-tachyonic with $m = \frac{1}{\sqrt{6\alpha}}$. He further investigates he effect of R^2 on FRW cosmology indicated by the eom and the substitution of SR approximation

$$6\alpha(1+2\alpha\phi)\Box\phi - 12\alpha^2\phi_\mu\phi^\mu = \phi - 4\Lambda \tag{3.37}$$

$$6\alpha(1+2\alpha\phi)(-\ddot{\phi} - 3H\dot{\phi}) + 12\alpha^2\dot{\phi}^2 = \phi - 4\Lambda \tag{3.38}$$

If ϕ is large it decreases like $\frac{1}{24H\alpha}$ giving a Hubble parameter of $H^2 = \frac{1}{24\alpha}$ and flat curvature. Hence, it is suitable for Planck era inflation.

multiple fields and beyond 'potential'-driven models such as k-inflation ('kinetically' driven) and many more which we won't discuss here.

3.4 How to End Inflation

It is not known when inflation exactly starts or ends. It is assumed that it starts 10^{-36}s and lasts until 10^{-35}s to 10^{-32}s. This is obviously a huge difference when we recall that the universe expanded exponentially. The size of the universe at the end of inflation is many orders above the size a Planck time less. Even if its initial size is $l_p \sim 10^{-33}$cm, it would be now $10^{10^{12}}$cm (again dependent on the model), more commonly given is an observable size of 10^{28}cm 7.7.5.
Inflation was constructed to solve the aforementioned problems. They control the end of how the state of inflation should be. This depends on the kind of inflation model used. I showed that about 60 e-folds are needed.

Using particle physics, independent from the models described, the universe was driven by an unknown form of matter. In the old inflation inflation ends with the tunneling to the true vacuum, in the new inflation when the rolling is in the same state so when the symmetry breaking/ phase transition is finished. Similarly, chaotic inflation ends when the field is in the minimum of the potential. The expanding universe cools down to the reheating temperature which should make the standard Big Bang scenario 7.1 possible, but shouldn't be too high (Planck scale) that the 'undesired' relics like magnetic monopoles cannot form again. During new and chaotic inflation the field can transfer by slowly rolling its potential energy to kinetic energy. When it begins to oscillate about the true minimum of the potential, i.e. its state decays, particles are produced as the field is coupled to not only itself but also other fields. Basically, the inflaton decays into the quanta of all known elementary particles of today. The electromagnetic radiation produced dominates the era after.

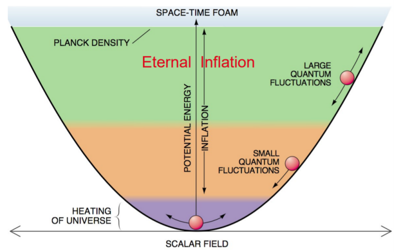


Figure 3.7: Eternal chaotic universe: quantum fluctuations are superimposed on the classical motion, here for $V = \frac{1}{2}m^2\phi^2$. For values greater than the Planck density quantum fluctuations dominate that we cannot describe without a theory of quantum gravity. Below quantum fluctuations of the scalar field may dominate and lead to eternal inflation. For values below $\frac{1}{\sqrt{m}}$ the classical behaviour dominates and the field rolls down the potential (SR). It tehn starts oscillating around the minimum - reheating the universe. Inflationary Cosmology, Linde.

3.4.1 Reheating

Inflation is a phase of supercooling. Dependent on the model, the temperature decreases about five orders of magnitude (10^{27}K to 10^{22}K). The volume of the universes increases by about $e^{3^{60}} \sim 1.15^{78}$ such that the density of any

other matter should decrease to almost 0. After the end of inflation reheating takes place, leading to particle production. It is important to determine the temperature at the end of inflation/the reheating phase to understand mechanism behind and the universe's evolution. How can one calculate the temperature at the end of a model dependent inflation where we don't know when it started, at what temperature it exactly started and what kind of matter it powers?
To get an estimation I choose chaotic inflation with a potential of $V = \frac{1}{2}m^2\phi^2$ as here the reheating process fits the usual explanations. I assume that the scalar particles oscillate at the same frequency and the energy transfer from them onto the other fields is the decay process and is followed by reheating. As SR ends with the end of inflation the equations are given by the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \tag{3.39}$$

and the Friedmann equation

$$H^2 = \frac{4\pi}{3m_p^2}\left(\dot{\phi}^2 + m\phi^2\right) = \alpha\left(\dot{\phi}^2 + m\phi^2\right), \quad \alpha = \frac{4\pi}{3m_p^2} \tag{3.40}$$

I would like to solve those equations with $H(t)$ and $\phi(t)$. As the coupled equations without explicit dependence on time aren't easy to solve, I make an intuitive ansatz that I would like to have oscillations with a frequency proportional to the mass of the scalar field as well as a Hubble parameter behaving like $\frac{2}{3t}$ for late times since reheating is followed by the radiation era. The second equation is solved by[12]

$$\begin{aligned} \phi &= \frac{H}{m\sqrt{\alpha}}\cos\theta \\ \dot{\phi} &= \frac{H}{\sqrt{\alpha}\sin\theta} \end{aligned} \tag{3.41}$$

Substituting this back into the first equation and using $\frac{\dot{H}}{H} = \frac{\sin\theta(1+\dot{\theta}}{\cos\theta})$ and $\ddot{\theta} = \frac{\dot{H}\sin\theta}{\sqrt{\alpha}} + \frac{H\cos\alpha\dot{\theta}}{\sqrt{\alpha}}$ gives

$$\begin{aligned} \frac{\dot{H}}{3H^2} &= -\sin\theta^2 \\ -\dot{\phi} &= m + \frac{3}{4}H\sin\theta\cos\theta \end{aligned} \tag{3.42}$$

confirming for $\left|\dot{\theta}\right| = m$. The first equation then gives

$$H = \frac{2}{3t}\left(1 - \frac{\sin 2mt}{2mt}\right)^{-1} \tag{3.43}$$

$$a \sim e^{\int\left(\frac{2}{3t} + \frac{1}{3t^2 m}\sin 2mt\right)dt} \sim t^{\frac{2}{3}}\left(1 - \frac{1}{24m^2t^2} + \frac{\cos 2mt}{6m^2t^2}\right) \tag{3.44}$$

[12]In another coupled equation system this trick was used in the cosmology course at UCL.

For late times this behaves like a radiation dominated Hubble parameter. Going back to the oscillation ansatz, [13]

$$\phi(t) = \frac{2}{3\sqrt{\alpha}}\frac{\cos mt}{mt}\left(1 - \frac{\sin 2mt}{2mt}\right)^{-1} \sim \frac{m_p}{\sqrt{3\pi}}\frac{\cos mt}{mt}\left(1 + \frac{\sin 2mt}{2mt}\right) \tag{3.48}$$

Now following [8] for example the friction term gets placed for $3H\dot{\phi} \to 3H\dot{\phi} + \Gamma\dot{\phi}$, including the decay rate of the scalar inflaton field.
At the end of inflation the field oscillates around the minimum of the potential

$$\dot{\rho} + 3H\rho = 0 \tag{3.49}$$

The energy of the inflaton field decays $3H\rho \to 3H + \Gamma$ where Γ is the coupling parameter. The derivation can be found in [8].
The scalar particles all oscillate at the same frequency and transfer energy to the other fields. This decay process is the reheating era. The Klein-Gordon equation is changed to

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + V' = 0 \tag{3.50}$$

Using standard up to quadratic Lagrangian with the scalar (inflaton) field being coupled to itself and say another scalar and a fermionic field

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m_\chi^2\chi^2 + g_1^2\phi\chi^2$$
$$+\overline{\Psi}(i\gamma^\mu\partial_\mu - m_\Psi)\Psi - g_2\overline{\Psi}\Psi\phi \tag{3.51}$$

Following symmetry breaking we would need to shift the potential to $\frac{1}{2}m_\phi^2\phi^2 \to \frac{1}{2}m_\phi^2(\phi - a)^2$. The part I am interested in is the interaction Lagrangian.

$$\mathscr{L}_{\text{int}} = -g_1\phi\chi^2 - g_2\overline{\Psi}\Psi\phi \tag{3.52}$$

with no tachyons, $|g_1\phi| < m_\chi^2$. In order for quantum effects to be vanishing small, I

[13] Algebraic steps are left out. A general treatment of $\ddot{\phi} + 3H\dot{\phi} = -V'$ along with $H^2 = \frac{8\pi G}{3}(\frac{1}{2}\dot{\phi}^2 + V)$, $\dot{H} = -4\pi G\dot{\phi}^2$ gives under substitutions $\ln a = y, u = \frac{d\phi}{dy}$,

$$\frac{du}{d\phi} = -\left(1 - \frac{4\pi G u^2}{3}\right)\left(3 + \frac{V'}{\frac{8\pi G}{3}Vu}\right) \tag{3.45}$$

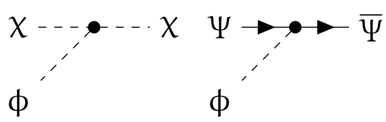
Then proceed by $u = \sqrt{\frac{3}{4\pi G}}\sin\theta, \theta = r\cos\theta$, substitute V, $t' = mt$,

$$\sin 2\theta\frac{d^2\theta}{dt'^2} - 2\left(\frac{d\theta}{dt'} + 1\right)\left(\cos^2\theta\frac{d\theta}{dt'} + \sin^2\theta\right) = 0 \tag{3.46}$$

and check stability of $\theta = -mt$, $\frac{d\theta}{dt'} = x$ gives $x' = -1 + \gamma, \gamma << 1$

$$\frac{-1+\gamma}{\gamma}d\gamma = (\gamma\cot\theta + \tan\theta - \cot\theta)d\theta \tag{3.47}$$

will be driven to 1.



assume $m_\chi, m_\Psi << m$. Reheating should end when $H < \Gamma$, so after about $t \sim \frac{2}{3\Gamma}$ The temperature can be calculated via[8]

$$T_{reh} = \left(\frac{90}{8\pi^3 g_* G}\right)^{\frac{1}{4}} \sim 0.2\left(\frac{100}{g_*}\right)^{\frac{1}{4}} \sqrt{\Gamma m_p} \tag{3.53}$$

where the total decay rate is the sum of the decay rates to scalar particles and fermions and g_* the total number of degrees of freedom, expected to be between 100 and 1000. In the literature this is found to be[8]

$$\Gamma = \Gamma_{\phi\to\chi\chi} + \Gamma_{\phi\to\Psi\overline{\Psi}} = \frac{g_1^2}{8\pi m} + \frac{g_2^2 m}{8\pi} \tag{3.54}$$

Again, the couplings are assumed to be small, giving upper bounds[14] of $g_1 \leqslant 5m$ and $g_2^2 \leqslant \frac{8\pi m}{m_p}$. If the scalar field is of mass $10^{16} m_p$ this gives an upper bound of $\Gamma \sim 10^{16} m_p$ (decay only to light scalar particles) and a lower bound of $\Gamma \sim 10^{-12} m_p$ (decay only to fermions). A short check on dimensional analysis gives the right dimension of mass $[\Gamma] = T^{-1} = M$. We can disregard the decay to itself $\sim \frac{m^3}{m_p^2} \sim 10^{-18} m_p$, but should actually evaluate the probability that the decayed particles can recombine as long as the temperature is still high enough. This is beyond the scope of this work and we work with a supercooling phase anyway.
We can now substitute this back into equation 3.53 to get an estimation for the temperature. For the upper bound we find a temperature of $T_{reh} \sim 2 \cdot 10^{15}$GeV and a lower bound of $T_{reh} \sim 2 \cdot 10^{12}$GeV. With three orders of magnitude this is quite a difference. Hence, in my opinion,the reheating process should be investigated more thoroughly. The upper limit should also coincide with the Hubble parameter at that time. Hence, giving the density at the time of reheating. The derivation of 3.53 by [66] is based on the assumption that the width of the decay rate is equal to the Hubble parameter (giving an instantaneous conversion of the energy of the inflaton field into radiation) and the fact that the potential energy is much smaller than the kinetic one. The author uses a similar oscillation approach as above, giving a formula of

$$\phi_0 = \phi_i \left(\frac{a_i}{a}\right)^{\frac{3}{2}} \cos m(t - t_i) \tag{3.55}$$

where i is at the beginning of the oscillations and 0 the oscillating part around a minimum at a. He then sets the energy density stored in the potential energy of the inflaton and the kinetic energy it gains by having rolled down and now oscillating equal where he also averages over many oscillations.

$$< \dot{\phi}_0^2 >=< V(\phi_0) >= \frac{1}{2} < \rho_\phi > \tag{3.56}$$

[14] To get a rough estimate I used the upper bounds given in 'The Leptonic Higgs Portal', 2016.

To find a formula for the density he rewrites the density's evolution with 3.56 and the KGE without oscillation:

$$< \dot{\rho}_\phi >= \left\langle \frac{d}{dt}\left(\frac{1}{2}\dot{\phi}_0^2 + V\right)\right\rangle = \langle \dot{\phi}_0\ddot{\phi} + V'\dot{\phi}\rangle =< -3H\dot{\phi}_0^2 >= -3H < \rho_\phi > \tag{3.57}$$

This gives indeed a formula for the density which can then be substituted into the SR condition of the Hubble parameter and equated to the decay rate of the inflaton.

$$H^2 = \frac{8\pi\rho_{\phi i}}{3m_p^2}\left(\frac{a_i}{a}\right)^3 = \Gamma^2 \tag{3.58}$$

The next step is disputable as he equates the energy density of the inflaton field with the radiation energy density which can be given in a form proportional to $\rho_r \sim T_{reh}^4$ and depending on the effective number of dof[15]. Solving for the temperature gives the formula 3.53. It is rather nonphysical that the energy transfer is given in an instantaneous form, same for the density transform from matter-like to radiation.
It is not known how the reheating mechanism takes place in detail as well as it is dependent on the model chosen. There is also no systematic analysis of the process (sometimes adding four point interaction terms that seem to dominate, sometimes leaving them out etc.). The reheating temperature should be used as an effective parameter to estimate when and what energy the radiation era started. However, in most derivations it is the other way around.

3.5 Observations

3.5.1 Cosmological Perturbations

The only accelerator so far that gets 'near' the GUT or Planck scales is the observation of the early universe via the LSS and CMB. The idea behind cosmological perturbations is that in the beginning the field and metric fluctuates and produce small irregularities. As we have seen previously, for former inflation would then end at slightly different times i.e different temperatures. Those initial quantum fluctuations were the seeds for the observed LSS such as galaxies today.
How are quantum and density fluctuations related? During inflation wavelengths of vacuum fluctuations of ϕ grow exponentially, if they become greater than H^{-1} the fluctuations stop and the amplitude freezes ($\leftrightarrow$ large friction). One can show that the scalar field perturbation (a more detailed approach can be found below) is of order

$$|\delta\phi(x)| \sim \frac{H}{2\pi} = \sqrt{\frac{2GV}{3\pi}} \tag{3.59}$$

[15] It is summed over all species using Fermi-Dirac and Einstein-Bose statistics giving $\rho_r = \frac{\pi^2}{30} g_* T_{reh}^4$ where g is the effective number of dof. Today's SM gives a total number of dof of 118 ($6\cdot 2\cdot 3\cdot 2 + 3\cdot 2\cdot 1\cdot 2 + 3\cdot 2\cdot 1\cdot 1 + 1\cdot 1\cdot 8\cdot 2 + 1\cdot 1\cdot 1\cdot 2 + 2\cdot 2\cdot 1\cdot 2 + 1\cdot 1\cdot 1\cdot 1 + 1\cdot 1\cdot 1\cdot 1$ where each elementary particle quarks, charged leptons, neutrinos, gluons, photons, $W^\pm$,Z_0, Higgs boson, respectively, is given by flavour·particle/antiparticle·colour·spin). At high energies (as in the early universe) the dof of fermions contribute less $\frac{7}{8}$ than the ones of bosons, giving a smaller effective number of dof. We don't know the effective number of dofs at the time of reheating.

in a time interval of H^{-1} and assuming SR. Assuming adiabatic density perturbation (=fluctuation in energy density) $\delta\rho \sim \delta\phi V'(\phi)$ this can be substituted into the density perturbations which we know are of order $\frac{\delta\rho}{\rho} \sim 5 \cdot 10^{-5}$. For certain potentials one can then calculate the ranges of parameters they can take in order to fit observations.

LSS

The large scale structure is the structure of the macroscopic universe, the pattern of galaxies and their even larger scales of clusters and clusters of clusters. As our solar system has irregularities in form of planets, galaxies tend to group in clusters and in between those there are vast voids. Measurements of the density parameter have proven that the universe is nearly spatially flat, the WMAP satellite, for example, measured a density parameter Ω to almost unity. To measure the flatness both observations are necessary, LSS quantities such as galaxy clustering, velocity measurements and gravitational lensing studies and the CMB 3.5.1 (WMAP, Komatsu et al., 2008).
More about the Big Bang theory and its observational evidence can be found in the appendix 7.1.

CMB

The cosmic microwave background was first observed in 1965 by Penzias and R. Wilson at Bell Labs (first measurements suggested a tempertaure of 3.5K) and predicted by Dicke, Peebles and Wilkinson at nearby Princeton. The steady-state theory in which the universe satisfies the cosmological principle but has and will be expanding in a way that the average density stays the same had to make space for the Big Bang theory. It is the radiation emitted by the last scattering surface 7.1, hence, the oldest electromagnetic radiation that we can use today to probe early cosmology. The Sachs-Wolf effect[16] postulates that the density perturbations induce CMB anisotropies with a corresponding spectrum - as we will see, *scale invariant* on large angular scales. Photons of the CMB are redshifted when they move up the gravitational potential and blueshifted when they move down. The first measurements showed an almost perfect black body at a temperature of $T \sim 2.7$K, almost uniform in all directions 3.8. As the radiation is basically the redshifted picture of the universe at recombination the standard Big Bang assumption was proven. The initial state of the universe must have been isotropic and homogeneous. COBE (Cosmic Background Explorer, 1989 to 1993) was the first satellite that probed the CMB on homogeneity and isotropy as well as small anisotropies. It was followed by WMAP (Wilkinson Microwave Anisotropy Probe, 2001 to 2010) and the Planck spacecrafts (2009-2013) that improved the results further. Further, primordial irregularities resulting in *anisotropies*, were detected in $\frac{\delta T}{T} \sim 10^{-5}$, actually $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{\frac{1}{2}} \sim 10^{-5}$. This is the value after subtracting the dipole caused by the motion of the satellite (e.g. COBE) as radiation is blueshifted at high temperatures in one direction and redshifted in the other. The peaks after the plateau (Sachs-Wolf) in

[16] They proved how photons in the CMB are gravitationally redshifted and produce the fluctuations measured (on large scales) [91]. How are we able to observe the fluctuations in brightness differences? The (inflationary) expansion stretches the wavelength emitted at the time of recombination and hence lowers the photon density causing a decrease in intensity.

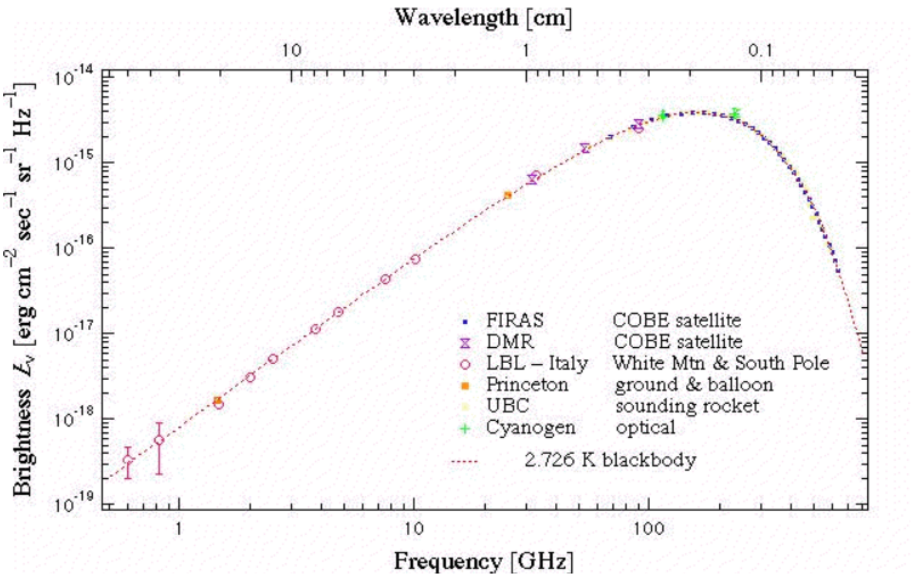


Figure 3.8: FIRAS measurement of the CMB, intensity vs frequency, consistent with a blackbody at 2.7K diagram. (aether.lbl.gov/www/projects/cobe)

the plot of the temperature fluctuations of the power spectrum that will be described later are due to acoustic oscillations in the photon-baryon mixture fluid. The pressure of the photons works against the gravitational compression in the well. The higher the peak (= temperature fluctuations) the more baryons are produced. It follows the dissipation, the tail of the spectrum proceeded by matter-domination. The form of the power spectrum would be shifted if the universe was spherical or hyperbolic to the left or right, respectively, proving a flat universe.
The main features of the power spectrum are encoded in the spectral index n_s and tensor-to-scalar ratio r which will be discussed in detail later. The power spectrum of scalar and tensor perturbations is one of the most analysed in inflationary cosmology (for the derivations see 3.5.1).
The simplest models predict an adiabatic, Gaussian and nearly scale-invariant power spectrum [17], meaning

- equal fluctuations in all forms of energy producing perturbations in spacetime curvature. It predicts 100% *adiabatic* and 0% isocurvature, where latter are the only other possible perturbations via entropy. It means that the 'amount' of perturbations is evenly distributed (baryonic matter, radiation, dark matter). The expansion is physically reversible as the entropy stays constant.

- its statistical distribution is a bell-curve *Gaussian* and any joint distribution of density contrasts at different points is a again multivariate normal.

- a *nearly scale-invariant* statistical measurement of the curve, an almost straight line ($n_s \sim 1$, but not exactly unity). It also means that the measurements not dependent on the size of the area, in all scales it should measure the same strength. They are expected to have slightly larger magnitudes on large scales.

[17] A. Albrecht and P. Steinhardt, 'Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking' (1982)

respectively. In addition to that, it should have

- an *upper limit* for the maximum temperature after inflation of definitely $T < 10^{19}$GeV
- *superhorizon* fluctuations i.e. fluctuations on scales larger than light could have travelled since the Big Bang (inflation stretches quantum fluctuations and they should be observed on scales larger than the cosmic horizon).
- *almost perfect flatness*
- primordial gravitational waves (in form of *B-modes*)

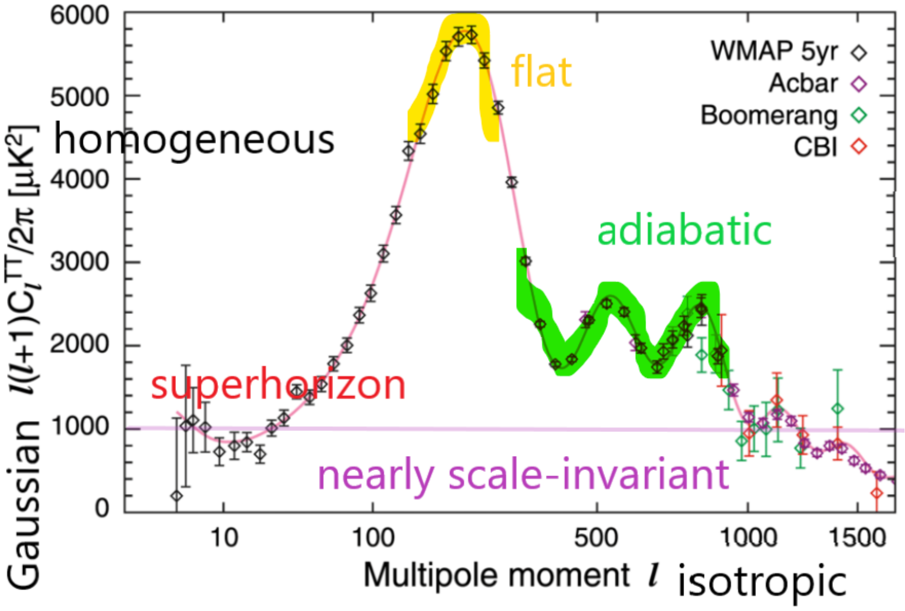


Figure 3.9: WMAP 5-year data along with other measurements give the temperature power spectrum with my comments showing clearly the expected oscillations with the properties I describe. It is given in multipole moments (many single maps) vs anisotropy power. It shows that the temperature measured at different angular scales (R 90°, spike at 1° - about multipole 360°/l). The observed data fit very well with the theoretical predictions (red line). ('Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation', 2008)

Primordial Gravitational Waves

The last point hasn't been measured until now. Today major efforts are taken to find imprints of tensor modes in the polarisation of the CMB. Those should have been generated through the scattering of the anisotropic radiation at the free electrons right before recombination since gravitational waves deformed spacetime. A special polarisation pattern is produced, so-called B-modes. This should be measurable in the amplitude of the CMB, deformed in one direction (also called curled pattern). It cannot be created by scalar fluctuations, so it can only be due to tensor perturbations

- gravitational waves[18]. Satellites like Planck, but also balloon and experiments on Earth like BICEP (Background Imaging of Cosmic Extragalactic Polarization, 2002 until now) look for those B-modes.[19]. Their formation from the metric perturbation can be found below and for a specific treatment of the Starobinsky model see 7.7.8.

Mathematical Derivation Cosmological Perturbations

The idea of cosmological perturbations goes probably back to Zeldovich's and Sakharov's work, but is tackled with another approach today. Latter assumed that the effective theory of the gravitational field is quantum, it should be possible to derive its form from the quantum effect of matter fields.[20] An explicit derivation of cosmological perturbations is beyond this work. I will give a short derivation to show its origin and its result in the power spectrum [8][82][9].
Initially, densities are produced due to quantum/vacuum fluctuations possible via Heisenberg's uncertainty relation (spontaneously created particle-antiparticle pairs) that became separated before they could annihilate. Quantum vacuum is not empty.[21].
Particles n and antiparticles $\overline{n}$ depend on time and their mass, I assume zero curvature and no cosmological constant in a radiation dominated era. Without any interactions the evolution in comoving coordinates is be described by

$$\frac{d}{dt}(na^3) = \frac{\partial n}{\partial t} + 3Hn = 0 \tag{3.60}$$

Adding interaction the RHS should have a term proportional to their interaction rate i.e. the number densities of both particles and antiparticles and their mean product of cross section σ and velocity v (basically think of a tube) and a temporal term depending on the universe's expansion and interaction $P(t)$,

$$\frac{\partial n}{\partial t} + 3Hn = -n\overline{n} < \sigma v > +P(t) \tag{3.61}$$

The evolution of the antiparticles simply substitutes the LHS with $\overline{n}$ (RHS is the same due to symmetry) and we also know that $\frac{d}{dt}(na^3) = \frac{d}{dt}(\overline{n}a^3) \leftrightarrow (n - \overline{n})a^3 = \text{const}$. If we then assume particle anti-particle symmetry, const$= 0$ and $P(t) \propto -n_{eq}\overline{n}_{eq}$, we find

$$\frac{d(na^3)}{dt} =< \sigma v > (\overline{n}_{eq}^2 - n^2)a^3 \tag{3.62}$$

Now I proceed with the change of variables, $Y \propto \frac{n}{T^3}$ and $X \propto \frac{m}{T}$. Along with $t \propto a^2 \propto T^{-2}$ and $T = m$ 3.62 can be rewritten in terms of $Y = Y_0 n a^3$

$$\frac{dY}{dt} =< \sigma v > (T)(n_{eq}^2 - n^2)Y_0 a^3 =< \sigma v > (T)(Y_{eq}^2 - Y^2)g(T), \quad g(T) = \frac{1}{Y_0 a^3} \tag{3.63}$$

[18]Recall that massless vector fields are conformally invariant. If we introduced mass perturbations would be largely suppressed or we would need to break invariance by a further coupling.

[19]The properties of the predicted B-modes can also be produced in other cosmic events. Cosmic dust produced artefacts such that primordial gravitational waves were falsley reported by Bicep2 (Ade et al, 'Detection of B mode polarisation')

[20]'Gravitational instability: An approximate theory for large density perturbations' (1969) and 'Vacuum quantum fluctuations in curved space and the theory of gravitation' (1968), respectively.

[21]The following calculation follows the problem asked on relic baryon density, Cambridge University, PS 4 on conditions of inflation.

Also, for $t = t_m X^2$ with $t = t_m$ for $m = 1$ gives

$$\frac{dY}{dt} = \frac{dY}{2t_m dX} = \frac{1}{HX}\frac{dY}{dX}, \quad H = H(T = m) \tag{3.64}$$

Setting both equal and rearranging gives

$$\frac{dY}{dX} = -\frac{\lambda}{X^2}(Y^2 - Y_{eq}^2), \quad \lambda = \frac{m^3 < \sigma\nu >}{H} \tag{3.65}$$

A number density at equilibrium is then $\propto (mT)^{\frac{3}{2}} e^{-\frac{m}{T}}$ i.e particle anti-particle annihilation as expected. Obviously, we live in a universe dominated by matter (or anti-matter depending on your definition). The energy of the CMB comes in fact from such an annihilation process.
The aforementioned satellites can show the primordial power spectrum in form of perturbations of the metric and field produced during inflation. The CMB gives *scalar and tensor perturbations* - the scalar dependence of the power spectrum of scalar fluctuations is measured with the scalar spectral index n_s. The tensor-to-scalar ratio r measures the suppression of tensor perturbation with respect to scalar perturbations. Perturbations (the metric $g_{\mu\nu}$, field ϕ itself, the density ρ, the pressure p etc.) are represented for any quantity $Q(x) = Q(t, \vec{x})$ in $Q(x) = \overline{Q(t)} + \delta Q(x)$ where the first term on the RHS is the homogeneous background.
The metric perturbations are decomposed into scalar, tensor and vector parts

$$\begin{aligned} ds^2 = &-(1+2A)dt^2 + 2a(\partial_i B - S_i)dx^i dt + a^2((1-2\Psi)\delta_{ij} + 2E\delta_{ij} \\ &+ 2\delta_{(i}F_{j)} + h_{ij})dx^i dx^j \end{aligned} \tag{3.66}$$

where the lapse A (sometimes already set to Φ), the shift B $\partial_i B$, $\Psi\delta_{ij}$ and the shear E $\delta_{ij}E$ are the scalar, S_i and $\delta_{(i}F_{j)}$ vector and h_{ij} the tensor perturbations, respectively. The vector parts are both constrained, the tensor perturbations are symmetric and constrained ($h_i^i = \partial_i h_j^i = 0$). This gives a total dof of 5 ($\delta\phi$ A,B ,E , Ψ) + (6-2) (S_i, F_i) + (6-4) (h_{ij}) = 11. An important remark is to use gauge invariant quantities (such as $\phi - \frac{1}{a}(a(B - E')'$ for a small transformation $x^\mu \to x^\mu + \zeta^\mu$) to analyse the inhomogeneities, for example the (comoving) curvature perturbation Ψ measured in the spatial curvature :

$$\mathcal{R} = \frac{4}{a^2}\nabla^2\Psi = \Psi - \frac{H}{\overline{\rho} + \overline{p}}\delta q \tag{3.67}$$

where q is the 3-momentum, $\partial_i \delta q = T_i^0$. Hence, for inflation we arrive at

$$\mathcal{R} = \Psi + \frac{H}{\dot{\overline{\phi}}}\delta\phi \tag{3.68}$$

as $\partial_i \delta q = -\dot{\overline{\phi}}\delta\phi$ and $\overline{\rho} + \overline{p} = \dot{\overline{\phi}}^2$. A common gauge choice is the comoving gauge, $E = \delta\phi = 0$ (see also Mukhanov, 7.1.1+, 'Physical Foundations of Cosmology'). Using the action 7.4 and substituting the metric we will be able to quantise this. The aim is to get a formula for the power spectrum resulting from the quantum/vacuum fluctuations.

$$S \sim \int d^4x \left(\frac{\dot{\phi}^2 \dot{\mathcal{R}}^2 a^3}{2H^2} - \frac{a(\partial_i \mathcal{R})^2}{2} \right) \tag{3.69}$$

keeping only terms up to second order [48] with five degrees of freedom. This is then redefined $\nu = z\mathcal{R}$ with $z^2 = \frac{a^2\dot{\phi}^2}{H^2} = 2a^2\epsilon$ and using conformal time $\partial_{\bar{\eta}} = \frac{\partial_t}{a}$ gives

$$S = \frac{1}{2} \int d\eta d^3x \left((\nu')^2 - (\partial_i \nu)^2 + \frac{z'' \nu^2}{z} \right) \tag{3.70}$$

Introducing Fourier modes $\nu(\bar{\eta}, \underline{x}) = \int \frac{d^3k}{(2\pi)^3} \nu_{\underline{k}} e^{i\underline{kx}}$ now gives a form of the Mukhanov-Sasaki (Mukhanov (1988), Sasaki (1984)) equation, a simple harmonic oscillator with a time dependent frequency
$w_k = \pm\sqrt{k^2 + m_{\text{eff}}^2}$:

$$\boxed{\nu_k'' + \left(k^2 - \frac{z''}{z} \right) \nu_k = 0} \tag{3.71}$$

This can be found after variation (assuming proper boundary conditions) $-\nu'' + \partial_i \partial_j \nu + \frac{z''\nu}{z} = 0$. Also, ν_k is a special kind of vacuum called *Bunch Davies vacuum* found in the de Sitter case, $\nu_k(\bar{\eta}) = \frac{e^{-ik\bar{\eta}}}{\sqrt{2k}} \left(1 - \frac{i}{k\bar{\eta}} \right)$. The frequency has an effective mass of $\frac{z''}{z} \sim -\frac{2}{\bar{\eta}^2}$ [22]. At early times, $\bar{\eta} \to \infty$ (subhorizon scale) it behaves oscillatory and curvature independent, $\propto e^{\pm ik\bar{\eta}}$ and growing and curvature dependent $\frac{1}{\bar{\eta}}$ at late times (superhorizon), $k\bar{\eta} \to 0$. When the horizon is crossed a comoving momentum scale starting at subhorizon can become superhorizon 3.10. Hence, at early times there is a preferred mode (subhorizon) which provide a time independent vacuum state ($\omega_k^2 = k^2 - \frac{2}{\eta} \to k^2$), a unique vacuum state $\sim$ Minkowski space.
I will quantise this result in order to find the vacuum fluctuations. $\hat{\nu}_k = \nu_k \hat{a}_k + \nu_k^* \hat{a}_{-k}^\dagger$ which as usually gives

$$\langle 0 | \hat{\nu}_k \hat{\nu}_{k'} | 0 \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k'}) |\nu_k|^2 \tag{3.72}$$

Substituting $|\nu_k|^2 = \frac{1}{2k\bar{\eta}} \left(\frac{1}{k^2\bar{\eta}^2} + 1 \right)$ and back $\nu = \frac{a\dot{\phi}\mathcal{R}}{H}$ and $a\bar{\eta} = \frac{1}{H}$ gives the curvature perturbation vacuum two-point function:

$$|\langle 0 | \mathcal{R}_k \mathcal{R}_{k'} | 0 \rangle| = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k'}) \frac{H^4}{2k^3\dot{\phi}^2} (1 + k^2\bar{\eta}^2) \tag{3.73}$$

The superhorizon $|k\bar{\eta}| \to 0$ gives a limit of $(2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k'}) \frac{H^4}{2k^3\dot{\phi}^2}$ at the crossing of horizon value $\bar{\eta}_H$ (it is momentum dependent via $H = \frac{k}{a(\bar{\eta}_H}$. The *power spectrum* is defined by

$$|\langle 0 | \mathcal{R}_k \mathcal{R}_{k'} | 0 \rangle| = (2\pi)^3 \delta^{(3)}(\underline{k} - \underline{k'}) P(k)_{\mathcal{R}} \tag{3.74}$$

giving a formula of

$$P(k)_{\mathcal{R}} = \left. \frac{H^4}{2k^3\dot{\phi}^2} \right|_H \tag{3.75}$$

[22] $\frac{z''}{z} = \frac{\nu^2 - 0.25}{\bar{\eta}}$ with $\nu \sim \frac{3}{2} + 3\epsilon - \eta$ [8].

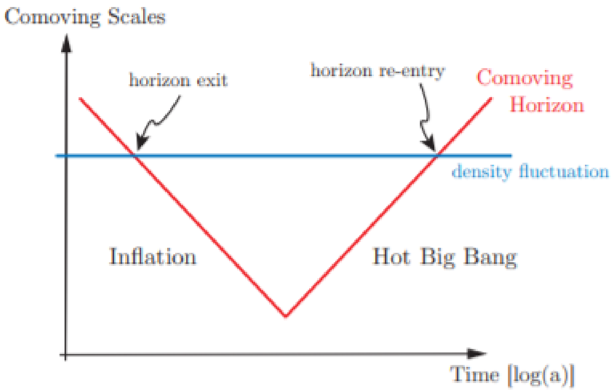


Figure 3.10: TASI Lectures on Inflation, D. Baumann, 2012. Temporal evolution of density fluctuations k^{-1} and comoving Hubble radius $(aH)^{-1}$ From left to right. 1. Subhorizon with 0-pt fluctuations $\mathcal{R}_k$ 2. inflation starts, superhorizon when crossing the horizon the comoving Hubble radius decreases and perturbations freeze until they re-enter 3. transition function, CMB recombination, perturbations are in the anisotropies of the CMB and perturbations in LSS today $\mathcal{C}_l$.

This proves the scalar perturbations from 0-point fluctuations. Often it is the dimensionless quantity $\Delta_s^2 = \frac{k^3 P_{\mathcal{R}}}{2\pi^2} = \frac{H^4}{(2\pi)^2\dot{\phi}^2}\Big|_H$. Similarly, the tensor perturbations (see Baumann [9]) can be calculated $\Delta_t^2(k) = \frac{16H^2}{\pi m_p^2}\Big|_H$. Both are evaluated at the cross-point.
The last step is to correlate this to the CMB's observations. The angular power spectrum

$$\mathcal{C}_l = \frac{2}{\pi}\int dk k^2 P_{\mathcal{R}}(k)\Delta_T(k)^2 \tag{3.76}$$

[9] where Δ_T is the transfer function which encodes all major effects that lead to perturbation growth[23], $\Delta_X(k)\Delta_Y(k)$, representing the anisotropies of the CMB.
Further, we should relate those with the known SR parameters

$$\begin{aligned} \epsilon &= \frac{\dot{\phi}^2}{2H^2} \\ \eta &= -\frac{\ddot{\phi}}{H\dot{\phi}} \end{aligned} \tag{3.77}$$

The above mentioned *scalar spectral index* is then given by

$$n_s = 1 + \frac{d\ln\Delta_s^2}{d\ln k} = 1 + 2\eta - 4\epsilon\Big|_H \tag{3.78}$$

the corresponding tensor spectral index

$$n_t = \frac{d\ln\Delta_t^2}{d\ln k} = -2\epsilon\Big|_H \tag{3.79}$$

[23] Acoustic oscillations, damping on small scales, radiative drags and the Meszaros effect.

and the *tensor-to-scalar ratio*

$$r = 1 + \frac{\Delta_t^2}{\Delta_s^2} = 16\epsilon\Big|_H \tag{3.80}$$

which have been evaluated with $k = \frac{dk}{d\ln k}$, differentiating wrt N[24]. Note that the last two are correlated via $-8n_t = r$. In most experiments n_s and r are measured, latter being very tiny such that only upper bounds have been given so far. Using **??** $\epsilon \sim \epsilon_V$ and $\eta \sim \eta_V - \epsilon_V$ the parameters are easily rewritten

$$\begin{aligned} n_s &= 1 - 6\epsilon_V + 2\eta_V\Big|_H \\ n_t &= -2\epsilon\Big|_H \\ r &= 16\epsilon_V\Big|_H \end{aligned} \tag{3.81}$$

An example can be found in the appendix and I will use the equations in the asymptotic safety section. Lyth (1996) calculated a lower bound for the variation of the inflaton during inflation depending on r. Primordial gravitation waves are here strongly suppressed. As it was shown r can be written in terms of the expressions of tensor to curvature power amplitude.

$$r = \frac{P_t}{P_{\mathscr{R}}} = \frac{\frac{16H^2}{m_p^2\pi}}{\frac{H^2}{m_p^2\pi\epsilon}} = 16\epsilon \tag{3.82}$$

with $\epsilon = \frac{m_p^2}{4\pi}\left(\frac{H'}{H}\right)^2$. We also know

$$\frac{d\phi}{dN} = \frac{m_p}{2\sqrt{\pi}\sqrt{\epsilon}} \tag{3.83}$$

Substituting 3.82 into this equation leads to

$$\Delta\phi = \frac{m_p}{8\sqrt{\pi}\sqrt{r}|\Delta N|} \geqslant m_p\frac{r}{4\pi} \tag{3.84}$$

during inflation. Linde criticised that the density can be sub-Planckian, but ϕ values can/must be higher. The formula is also only valid for single fields.
A solution to 3.78 and 3.79 is a simple power law form

$$\begin{aligned} \mathscr{P}_s &= \Delta_s^2(k) = \mathscr{A}_s(k_0)\left(\frac{k}{k_0}\right)^{n_s(k_0)-1} \\ \mathscr{P}_t &= \Delta_t^2(k) = \mathscr{A}_t(k_0)\left(\frac{k}{k_0}\right)^{n_t(k_0)} \end{aligned} \tag{3.85}$$

[24]For example, with $\frac{dN}{d\ln k} = 1 + \frac{d\ln H}{N}$ and $\frac{d\ln\Delta_s^2}{dN} = 2\frac{d\ln H}{dN} - \frac{d\ln\epsilon}{dN}$ as $\epsilon = -\frac{d\ln H}{dN}$ gives $\sim (-2\epsilon - 2(\epsilon-\eta))(1+\epsilon) \sim -4\epsilon + 2\eta$.

where $\mathscr{A}$ are the scalar and tensor power spectrum amplitudes and k_0 is a reference scale. A general formula for $f(R)$ can be found in 7.7.8.
Note that one could have also related the field perturbation and curvature perturbation in the beginning of the derivation via the expansion of the scalar field perturbation into comoving wavenumbers and then quantising the linearised classical equation. The boundary condition is sufficient flatness for $k >> aH$ and the asymptotic value for $k << aH$ can then be calculated [19].

$$\begin{aligned}\mathscr{R} &= \frac{H\delta\phi}{\dot{\phi}} \\ <|\delta\phi_k|^2> &= \frac{H^2}{2k^3}\end{aligned} \tag{3.86}$$

Worth mentioning is the running of the spectral index

$$\alpha_s = \frac{dn_s}{d\ln k} \tag{3.87}$$

and also the running of its running. Similar models can be compared by different runnings of the scalar spectrum. Recent Planck measurements suggest no scale dependence of the running with a vanishing running and zero running of the running. For SR and $\epsilon \sim \frac{1}{N}$[25] the running should be of order $\sim -\frac{dn_s}{dN} \sim \frac{1}{N^2}$.
Finally, one might also derive an estimation for the matter perturbations (different than the metric perturbations derived above) from classical (cosmological) fluid dynamics, with the density ρ, pressure p, local fluid velocity $\vec{u}$ and entropy density s. Energy, momentum and entropy conservation in order:

$$\frac{\partial\rho}{\partial t} + \nabla(p\vec{u}) = 0 \tag{3.88}$$

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u}\right) + \nabla p + \rho\nabla\phi = 0 \tag{3.89}$$

$$\frac{\partial s}{\partial t} + (\nabla s)\vec{u} + s\nabla\vec{u} = 0 \tag{3.90}$$

To match it with the gravitational field we add $\nabla^2\phi = 4\pi G\rho$. Now let us perturb around the background as we did with Q earlier for $\vec{u}, \rho, p, s, \phi$ with background values Q_0 and assume homogeneity and isotropy $\vec{u}_0 = H(t)\vec{r}$. Substituting into the conservation equations gives (checked with Brandenberger)

$$D_t\delta\rho + 3H\delta\rho + \rho_0\nabla\delta\vec{u} = 0 \tag{3.91}$$

$$\rho_0(D_t\delta\vec{u} + H\delta\vec{u}) + \nabla\delta p + \rho_0\nabla\delta\phi = 0 \tag{3.92}$$

$$D_t s = 0 \tag{3.93}$$

$$\nabla^2\delta\phi = 4\pi G\delta\rho \tag{3.94}$$

with $D_t = \frac{\partial}{\partial t} + \vec{u}_0\nabla$. The equations aren't independent as D_t (1)+∇ (2) implies (3). The equation of state, $p = p(s,\rho)$, gives the speed of sound, $c_s^2 = \frac{\partial p}{\partial \rho}|_s$. Similarly,

[25] $N = H\delta t, |\dot{H}|\delta t \sim H \to N \sim \frac{H^2}{|\dot{H}|} \to \epsilon \sim \frac{1}{N}$

$\zeta = \frac{\partial p}{\partial s}|_\rho$. I introduce comoving coordinates to simulate the expansion of the universe, $\vec{r} = a(t)\vec{r'}$ and Fourier transform, $\frac{\delta\rho}{\rho_0} = \delta_k(t)e^{i\vec{k}.\vec{r'}}$. This gives the so-called Master equation[26]

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \left(\frac{c_s^2 k^2}{a^2} - 4\pi G\rho_0\right)\delta_k = \frac{\zeta}{\rho_0}\delta s \tag{3.95}$$

We have adiabatic perturbations if δs_k vanishes. For $k^2 = k_*^2 = \frac{4\pi G\rho_0 a^2}{c_s^2}$ the last term of the LHS vanishes, for $k >> k_* a$, δ_k is of the form

$$\delta_k \sim \frac{1}{\sqrt{a}}e^{ikc_s\int\frac{dt'}{a(t')}} + \frac{1}{\sqrt{a}}e^{-ikc_s\int\frac{dt'}{a(t')}} \tag{3.96}$$

For large k we observe damped oscillations with frequency $\sim c_s k$. For k much smaller we have exponential growth. Indeed for small k we have $\delta_k \sim t^p, p = \frac{2}{3}, -1$. For the former the perturbation is then proportional to the scale factor, for the latter decreasing in $\sim a^{-\frac{3}{2}}$. With this result we could, for example, compare how different matter types produce perturbations i.e. measure the perturbations and then conclude which matter is the dominating form.

Starobinsky Perturbations

In 1982, during the Nuffield Symposium similar conclusion with respect to the new inflationary universe scenario were drawn. Finally, Mukhanov developed the general theory of inflationary perturbations of the metric, which is valid for a wide class of models including chaotic inflation. In the following I will present the perturbations in the aforementioned R^2 inflation. My aim is to get an estimation for the mass of the scalar field that is introduced since the coefficient in front of the R^2 term will play another role in the AS section.
In fact, Starobinsky gave the first model of primordial perturbations of inflation driven by curvature R^2 (i.e. by a modification of Einstein gravity) in 1980. $f(R)$ theories have attracted much research in the past years (probably because of their simplicity and the lack of 'bad' ghosts).
For general $f(R)$ theory it was derived (Mukhanov, Feldman, Brandenberger (1992)) for $\epsilon = -\frac{\dot{H}}{H^2}$, $' = \frac{d}{dR}$

$$\frac{f'}{H^2}(1-\epsilon) + 6f''(4\epsilon + \frac{\dot{\epsilon}}{H} - 2\epsilon^2) = \frac{f}{6H^4} \tag{3.97}$$

If I substitute $f(R) = \frac{m_p^2}{16\pi} + \frac{R^2}{b_0}$ (for future reference I use this parameter, we will see later that b is running) into the equation and use $N \sim \frac{1}{2\epsilon}$[27] the Hubble parameter is given by

$$H \sim H_0 - \frac{b_0}{576\pi}m_p^2(t-t_0) \tag{3.98}$$

[26] $3H = \nabla\vec{u}_0, H_0 = \sqrt{\frac{8\pi G}{3}\rho_0\phi_0} = \frac{2\rho_0}{3}\vec{r}^2, \nabla(\vec{r}) \to \frac{\nabla}{a}(\vec{r'}), D_t(\vec{r}) \to \frac{\partial}{\partial r'}$.

[27] under SR approximation $\epsilon << 1$ during inflation, $\epsilon = -\frac{\dot{H}}{H^2} \sim \frac{m_p^2 b}{36\cdot 16\pi H^2}, N \sim \frac{288\pi}{m_p^2 b}H^2 \sim \frac{1}{2N}$.

The scalar power spectrum has the general form of

$$\mathscr{P}_s = \frac{1}{48\pi^2}\frac{H^2}{\epsilon^2 f'} \tag{3.99}$$

which I calculated in the appendix, I derived a more accurate form where I also differ between Jordan and Einstein frame (which agree in the result, but not in the form) 7.7.8. Using those results with $N \sim \frac{1}{2\epsilon}$ and $f' \sim \frac{24H^2}{b_0}$ we have

$$\mathscr{P}_s \sim \frac{N^2 b_0}{288\pi^2} \tag{3.100}$$

$$\mathscr{P}_t \sim \frac{b_0}{24\pi^2} \tag{3.101}$$

which gives a value of $b_0 \sim 1.737 \cdot 10^{-9}$ for 60 e-folds according to WMAP data ($\mathscr{A}_s \sim 2.2 \cdot 10^{-9}, n_s \sim 0.965$ rendering $k' = \frac{k}{k_0} \sim 1$, checked with Planck data $10^{10}\mathscr{P}_s = e^{3.089}$ at $k_0 = 0.002\text{Mpc}^{-1}$ gives a scalar power spectrum of $\mathscr{P}_s \sim 10^{-9}$). Indeed r is given by $r \sim 48\epsilon^2 \sim \frac{12}{N^2}$. $\mathscr{P}_t$ hasn't been observed so far and should be for this calculated b_0 of range $7.33 \cdot 10^{-12}$. WMAP data (Komatsu et al., 2011) predicts a spectrum bounded by $\mathscr{P}_t \leqslant 0.2\mathscr{P}_s$ which agrees with this result.
Combining both treatments the fluctuations are commonly given in the form (Mukhanov and others)

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \frac{c_s^2 k^2}{a^2}\delta\phi = 0 \tag{3.102}$$

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} + \frac{c_t^2 k^2}{a^2} h_{ij} = 0 \tag{3.103}$$

for the scalar and tensor modes (under TT gauge), respectively. For an example, see 7.7.8.

Latest Data

The latest published and reviewed data [20] set constraints on inflation models. Firstly, it is again a very good match to the standard model of cosmology, the ΛCDM model. The 2018 Planck CMB anisotropy measurements have further improved the uncertainties. The scalar spectral index is given at $n_s = 0.9649 \pm 0.0042$ at 68%CL i.e. with less than 1% uncertainty and there is no evidence for total scale dependence. Near spatial flatness was measured at 0.4% precision at 95%CL (Planck and BAO data), $\Omega_k = -0.011\Big|^{+0.013}_{-0.012}$. Planck gives a tensor-to-scalar ratio $r_{k*} < 0.1$[28]. In standard inflation, SR models are favoured with $V'' < 0$. The authors conclude that '...based on two different methods for reconstructing the inflaton potential, we find no evidence for dynamics beyond slow-roll'[29]. Parameter constraints for different models (including R^2, power law, natural and hilltop potentials) can be found in there. Starobinsky's

[28] $k* = 0.002\text{Mpc}^{-1}$ in order to simplify the comparison with previous tensor-mode constraints. BICEP2 sets a upper bound of $r_{k*} < 0.056$. Other data is taken at the pivot scale of 0.05MeV^{-1}

[29] In addition to the usual SR approximation they 1. use a Taylor expansion of the inflaton potential and 2. a 'free-form reconstruction' of the potential.

model fits very well with the data, giving $n_s - 1 \sim -\frac{2}{N}$ and $r \sim \frac{12}{N^2}$, $49 < N < 59$. Similarly, other potentials not mentioned and beyond this work are good fits. Linde's ϕ^p, $p > 1$ and hybrid potentials are rather disfavoured.
Current fluctuation observations give a maximum of 0.1% corresponding to 10^{16}GeV i.e. around the expected GUT scale and definitely below Planck scale. Topological defects indeed haven't had the possibility to form again. Predicted reheating temperatures are consistent with the corresponding models and range between 10^9GeV and 10^{13}GeV which is a rather large range. Early fluctuations are observed to be at least at 98.7% adiabatic which can be in agreement with the 100%, but further theoretical and experimental research is going on whether isocurvature fluctuations are possible. Planck and WMAP have observed superhorizon fluctuations which can only be explained by some kind of inflation theory.
Further, constraints on primordial gravitational waves are given. They investigated a possible running and a running of the running of the spectral index, so a wavelength dependence, 3.87 where the former is very unlikely and the latter 0. The power spectrum of scalar perturbations is modified to include those possible runnings. It was also looked into whether there is a minority of isocurvature perturbations (so not 100% adiabatic ones).
In my opinion, it is important to find model-independent constraints as otherwise one might conclude that in the vast sea of possible inflation models one looks for the data in the hope that a class or a certain potential fits the data. This was indeed done with the result that the power spectrum would be given as measured.
Today's Hubble parameter doesn't fit the theory. Planck gives a value of 67.27 ± 0.60km/s/Mpc being in tension with other large scale measurements whereas local measurements give a value of 73.52 ± 1.62km/s/Mpc. Until now it is not known what the cause for the discrepancy is.
The least model-dependent constraints on B-modes are given by Ade et al. (2018) from the BICEP2 measurements. If we account for some running of the spectral index, the constraint on r lowers following Planck date. In fact, a negative running would allow an increase in r as the scalar spectrum would decrease on large scales.

3.6 Inflation's Extensions and Possible Alternatives

> There ought to be something very special about the boundary conditions of the universe and what can be more special than that there is no boundary?[...] There should be no boundaries to human endeavor. However bad life may seem, there is always hope.
>
> (S.W. Hawking)

Before discussing the problems of inflation and how asymptotic safety might be able to reduce them I would like to mention alternative and 'extension' solutions to inflation.

3.6.1 Variable Speed of Light

I mention this alternative solution as it is also based on the assumption that fundamental constants aren't constant. In asymptotic safety couplings are set on the run as we will see.
The varying speed of light (VSL) theory seems to be the most obvious solution to the horizon problem, but the lack of a consistent mathematical theory and its change of fundamental physics don't convince conservative physics. As I will investigate later, in contrast to inflation, it would be falsifiable by observations in high energy cosmic rays, the acceleration of the universe and the WMAP data (Magueijo). In [7] Friedmann equations are solved with varying c and G to get constraints on the running of c in order to solve the cosmological problems. It is based on the Albrecht-Magueijo model [47][30] including the assumption that the geometry of the universe is unaffected by varying c. One cannot just put in Einstein equations, because the conservation equation would imply G =constant. Scalar-tensor gravity theories of the Jordan-Brans-Dicke type tackle this question or one would need to allow the energy-momentum tensor to be not divergence-free anymore, but rather $G(x)c(x)^{-4}T_{\mu\nu}$ should be conserved (as it is shown in the appendix 7.8 there are terms $\propto \frac{\dot{c}}{c}$ in the conservation equation). $c(t)$ is not Lorentz invariant, we have to choose a favoured gauge, the authors choose the cosmological frame. Concerning c only we only arrive at a SR effect i.e. we are in the local Lorentzian frames.
The authors solve the cosmological problems by constraints on the falling of c as the universe expands without the need to change the matter content (see 7.8. Here they modelled the change in c as power law whereas Albrecht and Magueijo propose a sudden fall in c (as in a phase transition). Intuitively, the solution for the horizon problem can then be explained that causal contact is possible if the ratio of the speed of light in the early universe and today's value is much bigger than today's comoving time to the one at the critical point

$$\frac{c_{\text{early}}}{c_{\text{today}}} >> \frac{\eta_0}{\eta_x} \tag{3.104}$$

This corresponds to a comoving radius much less than the one at the critical point, $c_{\text{today}}(\eta_0 - \eta_x) = r \ll r_x = c_{\text{early}}\eta_x$. Similarly, the flatness problem can be solved when calculating α according to 2.3 giving an extra term of $2\frac{\dot{c}}{c}\alpha$. If $\frac{\dot{c}}{c} < 0$ and $\left|\frac{\dot{c}}{a}\right| \gg \frac{\dot{a}}{a}$,

[30] Further investigations on the creation of perturbations and entropy can be found here as well.

then it can drive α to 0.
They also discuss the question how the cosmological constant arises, how to solve its problem (3.7) and further questions on perturbations.
Further, Moffat predicts the spectral index n_s and tensor fluctuations to $n_s = 0.96$ and $n_t = -0.04$ ('Variable speed of light cosmology, primordial fluctuations and gravitational waves') which fits the data.
Importantly, actually only constants without dimension should be used, especially concerning possible experiments (variations in dimensional constants can be transformed away by a suitable choice of coordinate frame). An example for the time-variation of c could be measured by the fine structure constant [7]. Imagine an atomic clock with period of $T \propto \frac{\hbar}{E}$ (Heisenberg's uncertainty relation with the energy being the Rydberg energy) and a length proportional to the Bohr radius, $l \propto a_0$, the speed of light is given by

$$c = \frac{\frac{a_0}{\frac{\hbar}{E}}}{\alpha} 8\pi\epsilon_0 \tag{3.105}$$

3.6.2 Global Structure

As I have already hinted efforts have also been made into the analysis of the global structure of the universe(s) and inflation. This raises also the question for a theory of quantum gravity.

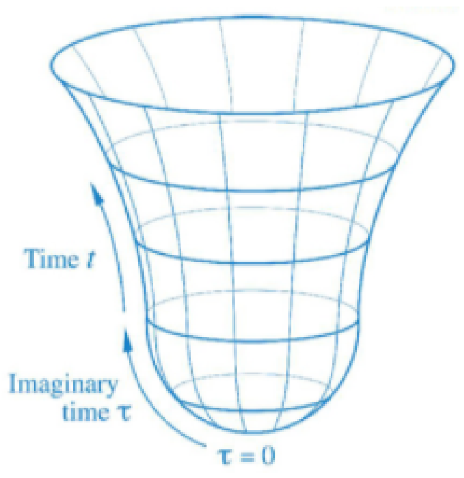


Figure 3.11: Inflation from no boundary. Imaginary turns into real time. http://www.ctc.cam.ac.uk/footer/glossary.php

The *quantum tunneling from nothing* was proposed by Vilenkin in 1982 and Linde in 1983. As geometry should be described by quantum theory in the early universe the geometry of space itself can undergo a quantum transition. Consequently, nothing would turn into something, a small universe, as an initial state for inflation. The question of the origin is self-contradictory. At least, for such a description we need a theory of quantum gravity.
Tyron proposed in 1973 that the universe was created from a *single quantum fluctuation*. The equality of matter and antimatter would imply the conservation of discrete charges such as the electric charge. Continuous quantities such as the energy of all matter (constant, $E = mc^2$) would cancel with the negative gravitational energy. The Heisenberg uncertainty principle makes the spontaneous appearance of vacuum fluctuations possible,

$$\Delta E \Delta t \sim \hbar \tag{3.106}$$

where the energy can be 'borrowed' for a very short time. The universe would be embedded in another space (that he doesn't define) and concludes 'I offer the modest proposal that our Universe is simply one of those things which happen from time to time.' along with the anthropic principle. The theory lacks of explanation, especially for the accelerated expansion and the universe's size today.
The wave function[31] of the universe proposed by Hawking and Hartle [26] along with the *no-boundary proposal* makes the question for 'what came before' meaningless according to Hawking.
Different from Vilenkin's idea where spacetime tunnels into place with a wavefunction of high potential value the universe is in the lowest state, the ground state. The wave function describes the entire past, present and future at once. Likely states of the universe should be given by the sum of all possible ways that it might have smoothly expanded to - similar to the path integral over all possible histories of a quantum system, we sum over all possible toy universes (or rather expansion histories of those toy universes) with euclidean metric and no boundary - except where you evaluate the wave function.
Possible are all kind of shapes, sizes and all kind of properties. First there is no notion of time (imaginary), then spacetime becomes Lorentzian when it open up and classical physics emerges. The theory should explain the state for inflation, but a wave function's amplitude increase is needed. It basically selects a universe with rather long inflation. Important to emphasise is that according to their advocates this eliminates initial fine-tuning and a notion of probability (as in usual quantum theory) arises, the conditional probability as we have to take into account that we observe the universe from here - this is called the Top-Down approach, what we can observe now is used as late time boundary constraint.

$$\Psi(h_{ij}, \phi) = \int_{\text{no boundary}} \delta g_{\mu\nu} \delta\phi e^{-\frac{I_E(g_{\mu\nu},\phi)}{\hbar}} \tag{3.107}$$

where the integral with the no boundary condition describes the initial condition and the exponential the dynamics of the theory. h_{ij} is the 3-metric on the boundary. The wavefunction as superposition of quantum states is expected to peak around a class of oscillatory universes with sufficiently long inflation.
The issue of probability calculation in inflation will be tackled in the next chapter. Furthermore, global irregularities are due to small different amplitudes in the wavefunction. Their prediction of eternal universes, bouncing or multiverses might be differentiated via observations, for example during the bounce the arrow of time would reverse such that in this model communication to the mirror universe wouldn't be possible.
Calculations are done in the so-called '*minisuperspace*' (=here the set of all universes with a single energy field that makes inflation possible, generally the configuration space of all universes where we constrain to a certain space of 3-geometries etc.). We wish to get a high probability for our flat, homogeneous and isotropic universe with ideally constants as in our universe. The equations are hard to solve, Halliwell

[31] One might assume as the Schroedinger equation in atomic physics saved the problem of classical physics that predicts the spiralling of electrons into the nucleus it would solve the singularity problem on larger scales.

and Hartle tackled the issue of contour integration (1990). Major efforts have been invested to find a better way to define the path integral as until now it is only an approximation. A more detailed calculation and explanation can be found in the appendix 7.9.

Minus or Plus?

The realisation of the no boundary proposal in the path integral has been a major challenge. Especially the negative sign in the exponential has caused some debate (opposite the sign that is predicted by Vilenkin's tunnelling proposal). There have been two approaches, the gravitational path integral in the Euclidean and Lorentzian setting. Former gives after the Wick rotation a well defined integral that can(not) converge $\sim e^{-\frac{S_E}{\hbar}}$ but loses the notion of causality. On the other hand, latter $\sim e^{\frac{iS_L}{\hbar}}$ doesn't give an obvious criterium for convergence. Thus, it is quite interesting to have a look at recent investigations by Lehners et al. They use the action under the metric introduced by Halliwell and Louko (1989) $-\frac{N^2}{q}dt^2 + q d\Omega^2$ in form of

$$S = 2\pi^2 \int dt \left(-\frac{3\dot{q}^2}{4N} + 3N(1 - \frac{\Lambda}{3q}) \right), \quad 3H^2 = \Lambda \tag{3.108}$$

where N is the lapse function, q is the 'scale factor'^{2} depending on t (we assume $\sim$ FRW) and $d\Omega$ is the 3-sphere. This quadratic action should give a viable action.

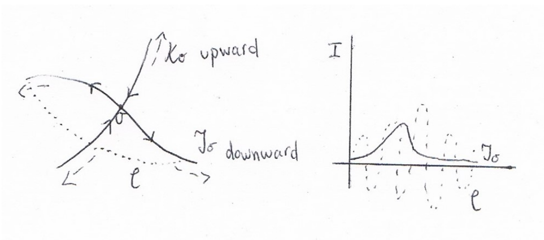


Figure 3.12: $\mathcal{J}_\sigma$ steepest descent (thimble), $\mathcal{K}_\sigma$ steepest ascent. We go away from our original contour $\mathscr{C}$ via those thimbles.

We will apply the following to Einstein Hilbert (in contrast to 3.107), setting $8\pi G = 1$,

$$\Psi = \int_{\mathscr{C}} \delta N \delta q e^{\frac{iS}{\hbar}} \tag{3.109}$$

The idea ([31] and previous works cited in) is to take the original integrand $\frac{iS[x(t)]}{\hbar}$ as a holomorphic function, $x \in$ and look for a integration contour that converges with the help of Picard-Lefschetz theory[32] (we will do a 'Wick' rotation of the field rather than the coordinates which in the end is the physical entity we are interested in). It will help us to get an absolutely convergent integral from a conditionally convergent integral. Starting point is $\Psi = \int_{\mathscr{C}} \delta x e^{\frac{iS}{\hbar}}$ with $iS = h + iH, h, H \epsilon \mathbb{R}$, h is the real part of the integrand (the so-called Morse function). Proceed with the following steps 1. Find the points that are extremal in $h(\leftrightarrow$ extremal in the full integrand) 2. from there find the steepest descent of h

[32]PL theory in maths is a method to analyse topological properties of complex manifolds at the critical points of some holomorphic function on the manifold. We are allowed to deform the integral contours because of Cauchy's Theorem, PL tells us now how exactly to deform it.

lines ($H = \text{const}$).[33] The complexified flow equations (transformed via $g(c, \bar{c}) = \frac{1}{2}\text{diag}(1,1), c = \text{Re}(x) + i\text{Im}(x)$ are given by the usual $\frac{dc}{d\sigma} = -\frac{\delta I}{\delta \bar{c}}, \frac{d\bar{c}}{d\sigma} = -\frac{\partial I}{\partial c}$ with I being the total integrand.[34] In Picard-Lefschetz theory we go away from the contour $\mathscr{C}$ by those steepest descent paths, so-called 'thimbles', $\mathscr{C} = \sum n_\sigma \mathscr{J}_\sigma$ such that the integral is now given by

$$\sum_\sigma n_\sigma e^{i\text{Im}(I_\sigma)} \int_{\mathscr{J}_\sigma} e^h \tag{3.110}$$

Whereas the path integral around $\mathscr{C}$ is rather oscillatory we get a more defined one via $\mathscr{J}_\sigma$. Applying PL to 3.109 with the boundary conditions at $t_0 = 0$ and $t_f = 1$ (since we use the lapse function), using 3.108 and split q into the background[35] and fluctuation $q = \hat{q} + q' \leftrightarrow q(0) = 0, q(1) = q_*$ gives a final path integral of (Halliwell, 1990) (over fluctuations!)

$$\Psi = \int_{0_+}^{\infty} dN \int_{q(0)}^{q(1)} \delta q' e^{\frac{2\pi^2 i}{\hbar}(S_0 + S')} = \frac{3\pi i}{2\hbar} \int \frac{dN}{\sqrt{N}} e^{\frac{2\pi^2 i S_0}{\hbar}} \tag{3.111}$$

since the second term is Gaussian, $q = a^2$, we only need to integrate over $N(> 0)$, S_0 is the classical background action, S' is the classical action for the perturbation. Substituting the action gives an exponent of $\sim \frac{N^3\Lambda^2}{36} + N(3 - \frac{\Lambda}{2}q_* - \frac{3q_*^2}{4N})$ which gives the (complex) critical points we are looking for (the crossing of steepest descent and ascent), $\frac{3}{\Lambda}(\pm\sqrt{\frac{\Lambda q_*}{3} - 1} \pm i)$. The wave function with the boundary condition of no boundary in the past but q_* as final state is calculated, PL tells us to choose the saddle point in quadrant I where the real part of the exponent is negative. This is the only relevant one as only here the steepest ascent contour intersects the original contour $\mathscr{C}$ and obviously the classical action is real along the real line,

$$\Psi \to e^{\frac{-12\pi^2 - 4\pi^2 \frac{\sqrt{\Lambda}}{\sqrt{3}}(q_* - \frac{3}{\Lambda})^{\frac{3}{2}}}{\hbar}} \tag{3.112}$$

Hence, the result in the exponent is minus the one given by Hawking and Hartle 7.9 (i.e. the one given by Vilenkin), it also has a constant amplitude plus a phase (as the universe expands it can become classical due to quantum geometric dynamics only, similar to a decoherence effect). FLT [31] proceed by calculating the effect of perturbations and conclude that the probability of large perturbations of some scalar field is quite high, $|\Psi| \sim e^{\frac{\phi_*^2 (l+2)(l+1)l}{2\hbar H^2}}$, ϕ_* is the amplitude of a frozen, dimensionless tensor perturbation and l is the principal quantum number. Amplified perturbations wouldn't indicate a smooth beginning of the universe. Nonetheless, in their treatment the seemingly different results of H&H and Vilenkin are actually the same, but we simply have to choose Vilenkin's saddle point. They also emphasise that their result impacts inflation as well. The suggested uncontrolled behaviour in the beginning would have an effect on the evolution and topology of the universe. A universe with only inflationary

[33]They define upward flow $\mathscr{J}_\sigma$ with $\frac{dx^i}{d\sigma} = g^{ij}\frac{\partial h}{\partial x^j} \leftrightarrow \frac{dh}{d\sigma} \geqslant 0$ (and downward $\mathscr{K}_\tau$ with minus sign and $\leqslant$), σ is a parameter along the lines. They cross at $\sigma_{\sigma\tau}$. Along the steepest descend the Morse function decreases monotonically, along the steepest ascent it increases.

[34]Note that $H = \frac{1}{2i}(I - \bar{I})$ which is constant along the lines of flow.

[35]$\hat{q} = \frac{\Lambda}{3}H^2 t(t-1) + q_* t$ from the eom $\ddot{q} = \frac{2\Lambda N^2}{3}$

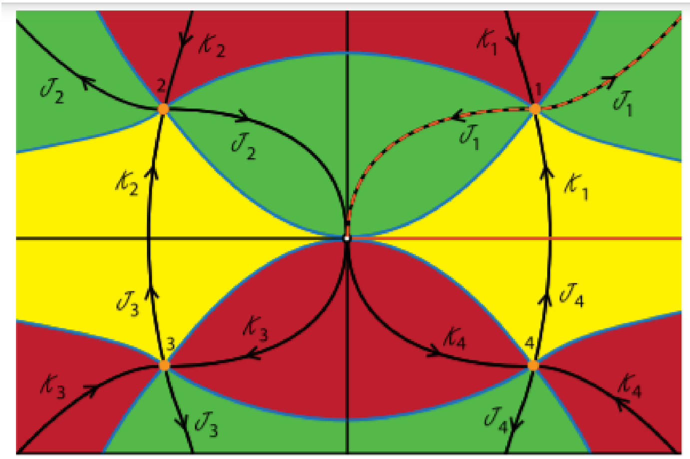


Figure 3.13: Here one can see the four saddle points where the one in quadrant I is chosen by PL theory. The solid orange line is the original contour $\mathscr{C}$, the dashed orange line is the corresponding deformed contour that is given by PL. We flow along the $\mathscr{J}_1$ surpassing the origin along the positive real line (needs to be finite at $\to 0_+$ and parallelly up $\mathscr{K}_1$. The integral is convergent for $N \to 0, \lim_{N \to i\epsilon} \to \int dN e^{-\frac{i}{N}}$ and $N \to \infty, \lim_{N \to N + i\epsilon} \to \int dN e^{iN^3}$. In the red areas the Morse function tends to infinity, in the green areas to minus infinity. Note that the saddles 3 and 4 are the ones that Hawking and Hartle calculated. Unknowingly they integrated over $N_E = iN$ since they start with a Wick rotation from beginning onwards. PL tells us to use the Lorentzian integral and NOT the Euclidean one. Interestingly, if we want to keep the H&H solution we have to add non-perturbative factors that give unsuppressed fluctuations (they also prove that there is no contour to actually avoid that) [31].

potential energy is excluded in their calculations. Some earlier phase (such as an radiation dominated one) is predicted. The issue of our universe being an unstable saddle point was recently tackled by using Robin boundary condition instead of the usual Dirichlet one. They found a well defined path integral, but the 3-geometries don't start from zero size, their origin is in a fuzzy configuration, $\Delta q_{*0} \Delta p_* \sim \hbar$ where p is the conjugate initial momentum. It remains a problem to deal whether the integral should be calculated in the Euclidean or Lorentzian setting. It is important to keep in mind that if we want universality the universe should be described by some quantum state. Obviously, the question of initial conditions also remains open in this treatment.

3.6.3 Inflation in Loop Quantum Gravity

LQG is a tentative, non-perturbative and background-independent quantisation of GR whose foundation relies on Ashtekar variables[36], the quantum corrections are captured in the Hamiltonian of LQG. Spacetime becomes discretised. Here the Big Bang

[36]LQG is based on holonomies, so the exponentials of connections. Ashtekar rewrote Einstein's GR as a canonical theory with the variables of a selfdual spin connection and its conjugate momentum - analogue to Yang Mills gauge theory. Smolin then showed that the Wilson loops of the connection are solution of the (Wheeler-deWitt) WdW equation. Quantum states are the intersections of those loop (Rovelli, Smolin). Along with Penrose's combinatorics formulation they later developed the theory of spin networks and spin foam.

becomes a big bounce due to large repulsive quantum-gravitational (= quantum-geometrical) effects. In some models LQG inflation (called super-inflation) is then followed by the standard SR inflation type. The origin of this bounce can be seen in the Klein-Gordon equation, the friction term becomes anti-friction during the pre-bounce stage, where the universe is contracting ($H < 0$). This forces the field to massively oscillate. Inflation is then a natural process [10] [73]. According to LQC (loop quantum cosmology, the application of LQG towards the universe as a whole) there is no infinite density, but an upper limit.
Already mentioned in VSL where the speed of light is an emergent concept, in LQG Lorentz invariance is, so there would be a natural connection. Mielczarek claims that the modification due to the quantisation of spacetime results in a density-dependence of the speed of light. With increasing density, the speed of light decreases, at $\rho = 0.5\rho_c$, the speed of light vanishes such that different points would become causally disconnected. Interestingly, speeds larger than $0.5\rho_c$ become imaginary numbers. There would be no more notion of time, time turns into space (spacetime becomes Euclidean) [49].
Bouncing from inflation or instead of inflation has been proposed in various other theories as well. Some argue that the Big Bang and the multiverse can be eliminated with this. When contraction starts the universe is large i.e. classical GR dominates and the universe bounces before it can actually shrink to size where quantum effects become important. Further explanation can be found in the next chapter.

3.7 Dark Energy and Inflation

Finally, I would like to briefly mention the issue of dark energy and compare it to inflation (as far as it can be compared as both aren't understood).
Dark energy is yet an unknown, but most dominant, form of energy today. It causes the universe to expand at a constant/nowadays accelerating rate. Supernova measurements and galaxy observations have given sufficient evidence. Experiments coincide with about 70% dark energy content of the universe, measured in $\Omega_{\Lambda 0}$. The expansion goes also along with measurements of the Hubble parameter which disagree by almost 15% which is also due to technical difficulties since it isn't that simple to measure the distance of far galaxies and cepheids aren't that common to find. The density parameter is the fraction $\Omega_{i0}(t) = \frac{8\pi G \rho_{i0}}{3H_0^2}$, so the dimensionless ratio of its density and the critical density of the universe (which is time dependent).

$$\Omega_{m0} + \Omega_{r0} + \Omega_{\Lambda 0} + \Omega_{k0} = 1 \tag{3.113}$$

where the matter content (proven by the power spectrum of the spatial distribution of galaxies) is 0.25 (with a part of 0.2 for dark matter), a tiny fraction of 10^{-4} for the radiation content (accounting for the CMB and neutrinos) and a vanishing curvature part $\Omega_{k0} = -\frac{k}{a_0^2 H_0^2}$.
Introducing the redshift factor $1 + z = \frac{a(t_0)}{a(t_{em}}$ (hence, using $\rho_m \sim a^{-3} \sim (1+z)^3$ etc),

the evolution equation and acceleration equation can be rewritten as

$$
\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= H_0^2(\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{k0} + \Omega_{k0}(1+z)^2) \\
\frac{\ddot{a}}{a} &= -H_0^2(\frac{1}{2}\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 - \Omega_{\Lambda 0)})
\end{aligned}
\tag{3.114}
$$

Both, inflation and dark energy, hence explain the universe's expansion. The former describes the rapid expansion in the early times and the latter today's late times and might give a prediction for the fate of our universe, whether it will expand forever or eventually recollapse. The cosmic acceleration is often described by a positive cosmological constant (but as I will discuss it shouldn't be called equivalent).
However, it is not known why it is so small and why it dominates the universe's content (or rather why, on macroscopic scale, it recently started to dominate). The value of Λ is $3H_0^2\Omega_{\Lambda 0}$ and using Planck2018 data equal to $1.1057 \cdot 10^{-52}m^{-2}$ giving a vacuum density of $\rho_{vac} = \Lambda(\frac{8\pi G}{c^2})^{-1} \sim 5.924 \cdot 10^{-27}\frac{kg}{m^3}$ ([20] gives 5.96). I have already used the cosmological constant when describing the de Sitter state $a \sim e^{H_\Lambda t}$ and when using the SR approximation which gives a classical cosmological constant for $\rho = -p$. If one compares the density to the Planck density (see appendix 7.2) there are 122 orders of magnitude difference. The cosmological constant problem is the massive disagreement of the observed value and the theoretical prediction of the zero-point density [37] between 120 and according to recent calculations 60 orders (taking Lorentz invariance into account. The Casimir force can be calculated and measured between two mirrors in only vacuum due to the pressure of vacuum fluctuation.). Rewritten in GeV the density would be about $\sim 10^{-47}$GeV, many orders above the vacuum fluctuation contributions, $\sim 2 \cdot 10^{71}\text{GeV}^4$ (Weinberg, 1986).
In Einstein's field equations it acts as an additional source for curvature (the associated energy has a dimension of $[L]^{-2}$), hence, its effect on scales is $\sim \frac{1}{\sqrt{\Lambda}}$. $\Lambda > 0$ is a *repulsive* form of gravity which we recall was the idea behind inflation. The value of the dark energy is also important in validating inflation and the Big Bang theory as it provides us with the apparent missing matter density that would rule an almost flat universe.
Back to the issue of dark energy. There are three proposed solution[61] (which might be the same or different and dark energy is the sum of them).

1. A cosmological constant as zero-point radiation of space i.e. the vacuum energy with w=-1.
2. A scalar field similar to slow-roll with a changing equation of state.
3. A non-zero vacuum energy due to vacuum fluctuations of quantum fields which should be explained by usual QFT.

The first option was described above. The second one, also called quintessence and intensively investigated in Susy, predicts a scalar field rolling down a potential in a very

[37]The vacuum fluctuation density is calculated similar to the Casimir energy by summing over all ground states of the fields, their (self)interactions up to a cutoff frequency, $\sim \Lambda < m$ where m is the mass of the field. For one field, say, summing over all normal modes gives $<\rho>= \int_0^\Lambda \frac{dk4\pi k^2}{(2\pi)^3}\sqrt{k^2+m^2} \sim \frac{\Lambda^4}{16\pi^2} \sim 2 \cdot 10^{-10}\pi^{-4}$.

similar way to the one of cosmic inflation. Other than for the classical cosmological constant the equation of state is time dependent, $w(t) = \frac{P}{\rho} > -1$. This might predict a different evolution that could be tested. On the other hand the evolution might be so slow that it cannot be distinguished from the cosmological constant.
There are two types, the 'freezing' (decreasing $w(t)$ that approaches -1 at low z) and 'thawing' ($w \to -1$ at high z with possible deviations at low z) one. The latter one is quite similar to inflation with a graceful exit. It was also shown that potentials associated to those models give an attractor mechanism[38] (some details can be found in the appendix, 7.2, 7.11. Zel'dovich (1967) first suggested a connection between the vacuum energy density and the cosmological constant. It is important to distinguish between dark energy in form of a cosmological constant added on the 'spacetime' part of the Einstein equations and an additional dark energy density on the matter energy side.

$$G_{\mu\nu} + \Lambda_{\mu\nu} = 8\pi T_{\mu\nu} \quad \leftrightarrow \quad G_{\mu\nu} = 8\pi(T_{\mu\nu} - \rho_{DE} g_{\mu\nu}) \tag{3.115}$$

Is it coincidence that there are two eras of accelerated expansion? Some research has been going on whether the nature of inflation and today's expansion has its origin in the same source. The cosmological constant didn't dominate in the early universe as matter and earlier radiation were increasing whereas the cosmological constant density is assumed to have stayed constant. Nonetheless, it has been proposed that the cosmological constant has been time dependent (in the AS chapter it will be shown that G and Λ are both scale dependent, hence, at early times the dimensionless running coupling of Λ might indeed cause inflation as well).
The authors of [18] proposed that an almost massless scalar field, $m < H_0$, causes the accelerated expansion today (quintessence type). This field should have experienced quantum fluctuations in the early universe during inflation and then froze until it recently began to dominate the energy content again, with the condition that its mass is smaller than the Hubble constant today. The fluctuations can be used to allow all kind of values for the value of dark energy and using the anthropic principle we measure exactly the low value today. Similar to before they write the fluctuations of the field

$$\phi(\vec{x}, t) = \phi(t) + \delta\phi(\vec{x}, t) \tag{3.116}$$

as

$$|\delta\phi_{\vec{k}}|^2 \sim \frac{H_{inf}^2}{2k^3} \left(\frac{k}{aH_{inf}} \right)^{\frac{2m^2}{3H_{inf}^2}} \tag{3.117}$$

with $\vec{k}$ being the comoving wavevector. Taking into account the Fourier modes after the Hubble exit a sufficiently long inflation is given by

$$< \delta\phi^2 > \sim \int_{a_i H_{inf}}^{aH} \frac{d^3k}{(2\pi)^3} |\delta\phi_{\vec{k}}|^2 \sim \frac{3H_{inf}^4}{8\pi^2 m^2} \tag{3.118}$$

The resulting potential is independent of the scalar field's mass

$$V(\phi) \sim \frac{1}{2} m^2 \langle \delta\phi^2 \rangle \sim \frac{3H_{inf}^4}{16\pi^2} \tag{3.119}$$

[38] A system, here the universe, evolves to a certain state or set of states for a large range of initial conditions. If values get slightly disturbed near the attractor they remain close to it.

Dark energy is given for $V \sim \Omega_{\Lambda 0}\frac{3m_p^2 H_0^2}{8\pi} \leftrightarrow H_{inf} \sim \Omega_{\Lambda 0}^{\frac{1}{4}}(\sqrt{2\pi}H_0)^{\frac{1}{2}}$, with Planck18 data and converting to eV $H_0 \sim 2.8 \cdot 10^{-33}$eV gives a value of $6 \cdot 10^{-3}$eV. The authors conclude by calculating the corresponding energy during inflation $E_{inf} = \rho_{inf}^{\frac{1}{4}} = (\frac{3m_p^2}{8\pi}H_{inf}^2)^{\frac{1}{4}} \sim 5$TeV which is about the scale where the EW symmetry breaking should have occurred. They numerically solved the equations and compared it to cosmological data.

A major difference to the common inflation scenario and dark energy is that latter still has a positive energy density. It violates SEC as inflation does as well 7.3.2.

Further, using a scalar field as comparison 7.4, we can take a look at the deceleration parameter, $q_0 = \frac{\ddot{a}_0}{a_0 H_0^2}$ (negative value of q: accelerating expansion, positive: decelerating expansion). Adding the acceleration equation of Friedmann with the cosmological constant to the first Friedmann equation with cosmological constant and curvature and using $\Omega_\Lambda + \Omega_k + \Omega_m = 1$ at all times (where we neglect the radiation part since we know that it is vanishing small and won't increase), gives

$$\frac{1}{2}\Omega_{m0} - \Omega_{\Lambda 0} = q_0 \tag{3.120}$$

This gives an approximation for when the universe's expansion is accelerating, $\frac{1}{2}\Omega_{m0} < \Omega_{\Lambda 0}$, which is valid today.

We can rewrite the Friedmann equation in a form where the energy form becomes more obvious. Setting $H_0 t = T$ and $Y(T) = \frac{a(t(T))}{a(t(T))}$ we have the dimensionless equation ($\rho_m \sim a^{-3}$, $\frac{dY}{dT} = \frac{\dot{a(t)}}{a_0 H_0}$):

$$\left(\frac{dY}{dT}\right)^2 + \hat{V} = \Omega_{k0} \tag{3.121}$$

with the 'potential' $\hat{V} = -\frac{\Omega_{m0}}{Y} - \Omega_{\Lambda 0}Y^2$. The 'total energy' is given by the curvature density parameter near 0, $\Omega_{\Lambda 0} > 0$. This gives a maximum of the potential at $Y = \frac{\Omega_{m0}}{2\Omega_{\Lambda 0}}$. If the field has rolled beyond that point it will expand with an ever increasing rate forever, which would be today's state.

Weinberg first proposed the idea that the small value of the cosmological constant is simply a random choice if one assumes a multiverse where each universe has its own vacuum and its own value for Λ. It follows to argue with the anthropic principle, that without this value galaxy formation wouldn't be possible (either big rip or big crunch) and we wouldn't be able to measure this value. This brings us to the discussion of inflation's problems.

Chapter 4

Problems of Inflation

> There are only certain intervals of time when life of any sort is possible in an expanding universe and we can practise astronomy only during that habitable time interval in cosmic history.
>
> (John D. Barrow)

[1] As it was shown in the previous section inflation was introduced to solve the cosmological problems. It can explain the homogeneity and isotropy on large scales, the irregularities due to cosmological perturbations and the absence of magnetic monopoles. Predictions of inflation have been confirmed in experiments and observations. On one hand inflation has been almost accepted as a confirmed theory, on the other hand there has been increasing critique in the past years. I will investigate major problems in the following. As I go along I will also present counter arguments against the typical criticism brought forward by some physicists.

4.1 Fine Tuning Again

[2] Inflation itself is criticised for producing again constraints on *initial conditions* or the need for *fine-tuning*. In the simple SR setting we used the SR parameters ($\epsilon, |\eta| \leqslant 1$ with the end of inflation given by $\epsilon = 1$) and the number of e-folds (at least ~ 60) to put constraints on the potential (such as the mass or coupling), 7.7.4. It turns out, depending on the model, they are rather small. For example, the potential $V \sim \lambda\phi^4$ gives $60 \sim N = \frac{\pi}{m_p^2}(\phi_i^2 - \phi_f^2)$ with ϵ giving $\phi_f = \frac{m_p}{4\pi}$, the initial value can be calculated and with the constraint $V \leqslant m_p^4$ (beyond quantum gravity effects become dominant) $\lambda \leqslant 0.03$. Also, the density perturbations give a constraint. Estimating (see 3.5.1) $\frac{\delta\rho}{\rho} \sim 10^{-5} \sim \frac{H^2}{\dot{\phi}} \sim \frac{V^{\frac{3}{2}}}{V'}$ where the last approximation is done with SR condition gives

[1] Sadly, Prof Barrow passed away in September. He addressed many foundational issues in theoretical physics, especially in cosmology and inflation, but also in maths and philosophy. In his opinion those problems may seem troublesome 'fundamental problems', because we are simply thinking the wrong way.

[2] The *anthropic principle* has two version. The weak version says that the universe has to provide certain properties since otherwise our existence would simply be not possible. The strong version says that our existence CAUSES the universe's nature.

a constraint on $\lambda \sim 10^{-9}$, a rather small coupling. An accurate measurement of r would set further bounds. As I have shown chaotic/eternal inflation doesn't put as much constraint on the values. A sensitivity to initial conditions might be caused by inhomogeneous initial conditions of the scalar field itself since it is basically possible that they froze in during the expansion and will effect the universe at some point in the future.

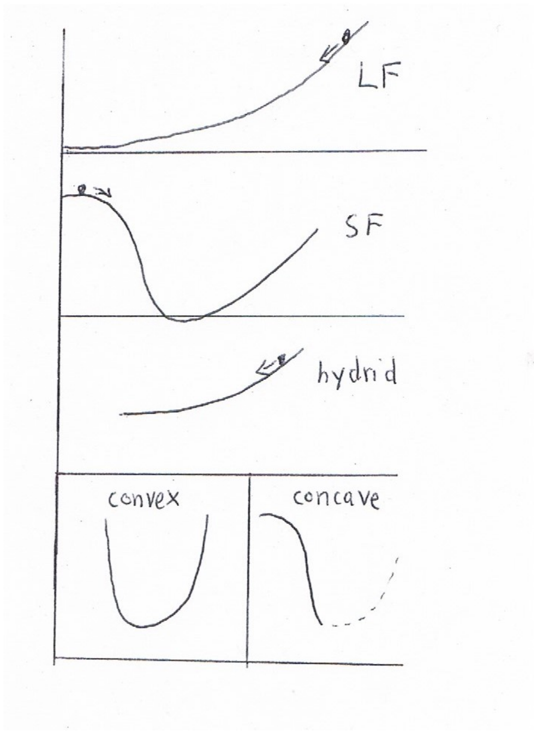


Figure 4.1: Large field (LF) model where the field rolls down to its potential minimum from a high value and small field (SF) model where the scalar field is in an unstable maximum and forced to run down. A hybrid model has a vacuum value different to zero. Convex potential with $V'' > 0$ and concave potential with $V'' < 0$. See also 7.3.1, 7.1.

On the other hand the SM of particle physics suffers from small coupling constants as well, this shouldn't necessarily be seen as critique of inflation. Moreover, initial conditions do matter in classical physics, it isn't entirely proven whether initial conditions shouldn't matter here as well.
Secondly, it is rather unlikely to find the field at rest at a high potential. The constraints are unnatural, but necessary for SR to occur (V must be constant near $\phi = 0$, small parameters, the curvature of the potential near the critical value needs to be large enough s.t. it oscillates at high frequencies after inflation). If one wants to not allow eternal inflation, fine.tuning needs to occur such that quantum fluctuations don't set in (shown in eternal inflation that it is rather probable that eternal inflation does set in). We seem to classify observables that don't produce natural numbers 'naturally' as fine-tuned. If a theory's parameters are all of the same order (ideally unity) we call it a natural theory. This is already troublesome for a theory of quantum gravity that would intuitively connect the small and large scale, see also 7.5. However, physicists don't fully agree on that notion, in the end, those value might be coincidence only and not a problem of physics, but a problem inherited by us.
Ijjas, Steinhardt and Loeb (2013) (ISL) criticised inflation for a similar reason. It *cannot be tested* since its predictions can be changed by different initial conditions or small changes to the potential (see also point on falsifiability below). They also put forward that inflation leads to a multiverse (for eternal/chaotic inflation) that in the end cannot provide a predictive theory and needs fine-tuned conditions that our universe has actually developed. However, if inflation starts in a chaotic state, there is a nonzero probability that there exists a smooth patch what will inflate and dominate which we

can observe.
Linde argues in his self-reproducing eternal inflation model that a large field value can lead to large quantum fluctuations that again may locally increase the value of ϕ in at least some domains of the universe which can then expand at a greater rate and produce new inflationary regions forever. We can only observe our universe since the different universes (where all types are possible) are separated. Inflation continues for separations larger H^{-1} and we can only see inside our horizon. And we live in exactly this universe as it provides the properties we need. Additionally, ISL actually cannot provide this argument since they also put forward the issue of the measure problem such that a (low) probability cannot be calculated.
Lastly, full *numerical general relativity* [33] has provided computer simulations of the

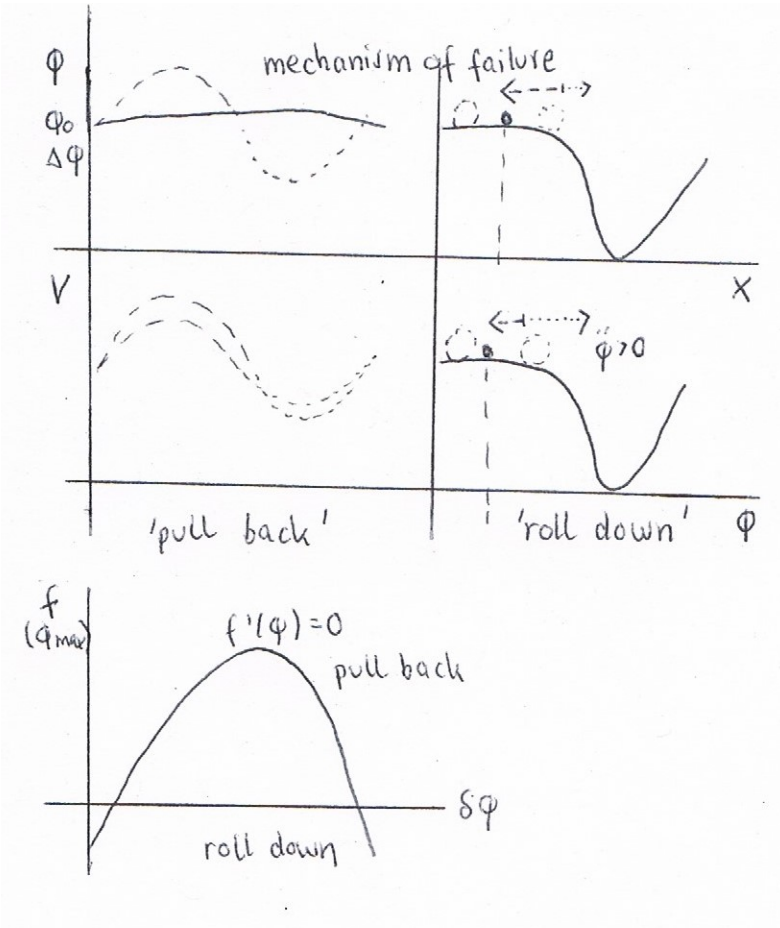


Figure 4.2: Lim, talk 06/20. Convex potentials cannot fail against inhomogeneities.

actual dynamics of spacetime and showed that inflation can start without constraints, even in rather unsmooth regions[3]. Lim et al. also name three options for the apparent fine-tuning problem of initial conditions: either is is indeed fine-tuned or there exists some dynamical process that makes the IC irrelevant (inflation) or there is some unknown theory that defines an ensemble and chooses the IC i.e. there were indeed IC,

[3]talk June 2020, Imperial College based on [33]

but it shouldn't be a 'problem' and then there wouldn't be a need for inflation - an alternative explanation for the LSS and perturbation observations etc. is not given. We would define the cosmological problems as 'problems' since we would expect another state to occur. The problem is that we don't have a proper definition for a sample space as there is only one universe that we can/have observe/d so far. The analysis is based on the Komatsu-Tesileanu Bayes argument,

$$P(\text{model}|\text{data}) = \frac{P(\text{data}|\text{model})P(\text{model})}{P(\text{data})} \tag{4.1}$$

We want to infer the LHS, it is assumed that $P(\text{model})$ exists and $P(\text{data}|\text{model})$ can be evaluated from the data and constraints. The aim is to find a joint measure of the space of IC (initial values of the variables of the theory space) and the model space (different models/dynamics of inflation) whose clear distinction hasn't been well emphasised in the probability problem of inflation 4.4. Numerical relativity then chooses a model with certain IC and they investigated what would happen if inhomogeneities existed. The power spectra of various models and the reheating temperature were calculated [50].
ModeCode (http://modecode.org/) is a numerical solver for perturbation equations in inflation not necessarily depending on the SR approximation and test the robustness against inhomogeneities (for single fields, further simulations for more complex models are in their development - however, even the simplest treatments take already a lot of computational work). Convex potentials are favoured, concave potentials[4] are more robust if they vary on super-Planckian scales rather than on sub-Planckian ones. Planck18 favoured concave potentials. Hence, the varying should then be in the super-Planckian regime. Any assumption of pre-homogeneity are dropped, Lim proves 4.2 that convex potentials cannot fail against such inhomogeneties, but concave can fail. This shouldn't actually be seen as a failure of concave potentials but rather as a selection principle for which concave models inflation still successfully occurs. His ansatz of field perturbation is $\phi = \phi_0 + \delta\phi e^{ikx}$ into the KGE

$$\ddot{\phi} + 3H\dot{\phi} + k^2\delta\phi = -V', \quad k = 2\pi n H \tag{4.2}$$

At ϕ_{max} we find

$$\ddot{\phi} + k^2\delta\phi = -V'(\phi_{max}) = -f(\phi_{max}) \tag{4.3}$$

Negative f values give a roll-down behaviour (failed inflation), positive values give a pull-back behaviour (success).
The maximum of the function is given in the pull-back (=concave) regime.

$$\frac{\partial f(\phi_{max})}{\partial \delta\phi} = 0 = k^2 + V'' \tag{4.4}$$

where the first term is always positive and the second derivative of the potential needs to be positive in order to be nonzero (=robust), whereas a negative value can set the function to zero and hence fails inflation. He tested different potentials and found

[4]Convex potentials satisfy $V'' > 0$ - curving upwards- such as in chaotic inflation. Concave inflation is given by $V'' < 0$ such as in Starobinsky inflation.

constraints on initial conditions.
MultiModeCode is a code that provides Monte Carlo samples of probabilities for different models, their parameters and initial conditions, also for multi-field models. They use a similar probability approach as Lim,

$$P(\Theta|D, M) = \frac{P(D|\Theta, M)P(\Theta, M)}{P(D|M)}, \quad P(D|M) = \int P(D|\Theta, M)P(\Theta|M)d\Theta \quad (4.5)$$

where M is the model, D the data measured and Θ is the set of parameters (power spectrum, late-time parameters and the parameters for the experiment).
Let's also investigate the Starobinsky model that fits so nicely with the Planck data. If we want to get a constraint on the coefficient of R^2 we see that it is rather large, $\sim 10^{10}$ giving a mass of $m \sim 10^{-5}m_p$ which is rather small. This will be explored later in detail.

4.1.1 Penrose's Alternative

Penrose suggested that initial conditions are 'caused' by the second law of thermodynamics [54], better in order for the second law to function the initial conditions were necessary. Specifically, he connects the homogeneity of the universe with thermodynamics and introduces the concept of gravitational entropy. The second law of thermodynamics postulates an increase of total entropy for an isolated system, here the universe, $\Delta S \geqslant 0$. Penrose suggests a coarse-graining of phase space with volume V ($\sim$number of microstates)[5], $S = k_B \log V$. Introducing the notion of gravitational entropy the universe was in a state of low entropy by accounting for the dof of the gravitational field. One might wonder how to count gravitational dof. In electromagnetism one has the Maxwell tensor $F_{\mu\nu}$ and its source in the current $\vec{J}$. An analogy can be drawn to $F_{\mu\nu} \leftrightarrow C_{\mu\nu\alpha\beta}$, so the trace-free part of the curvature measure $R_{\mu\nu\alpha\beta}$, and the Einstein tensor $\sim T_{\mu\nu} \leftrightarrow \vec{J}$. He postulates the *Weyl curvature hypothesis*: The universe is constrained by the effective vanishing of Weyl curvature at any initial spacetime singularity (later sometimes also corrected to a finite value only). The low gravitational entropy state increases with the universe's evolution and increasing Weyl curvature which makes the second law of thermodynamics possible and induces the arrow of time. The constraint must have been in the spacetime geometry. The tidal distortion gives a measure for it:

$$D^2 q^\beta = R^\beta_{\mu\nu\alpha} t^\mu q^\nu t^\alpha \quad (4.6)$$

which is the geodesic deviation equation with the unit timelike t^μ and connecting and orthogonal vector q^μ that measures the displacement of neighbouring particles. It can be rewritten in a form where the importance of the Ricci tensor as volume change measurer becomes more obvious.

$$D^2 \Delta = R_{\mu\nu} t^\mu t^\nu \quad (4.7)$$

Δ is the 3-volume induced by three independent qs. Now, tidal distortion is measured by the remaining curvature if the Ricci tensor vanishes. The Riemann tensor can be

[5]Obviously, it is difficult to define what is macroscopically indistinguishable, but this shouldn't matter too much when calculating ratios.

written as the trace-free and other part $R_{\mu\nu\alpha\beta} = W_{\mu\nu\alpha\beta} + Q_{\mu\nu\alpha\beta}$ where the former is the Weyl curvature tensor that depends on pure gravity only whereas the latter depends on the Ricci tensor and hence the energy-momentum tensor hence matter (simply have a look at the Einstein field equations). Penrose concludes that the Ricci tensor as matter distribution basically measures the volume change and the Weyl tensor presenting gravity only measures the tidal distortion.

Gravitational entropy[6] is now dominated by the amount of Weyl curvature. Since there is no proper measure for spacetime entropy other than the formula for black holes $S_{BH} = \frac{kc^3 A}{4G\hbar}$ he uses this and estimations on matter to calculate the ratio of the entropy at the Big Bang and today or a possible Big Crunch scenario (he assumes a closed universe).

Unfortunately, he doesn't explain where the numbers come from [54]. In an expanding FRW universe (=sequence of maximal expanded regions) he assumes a number of at least 10^{80} baryons[7] and today's CMB contributes 10^8 entropy per baryon giving a total of 10^{88}, this entropy increases when black holes and LSS structures are taken into account. Using the formula for black hole entropy he gives a maximum entropy of 10^{123} (all black holes 'end' the Big Crunch) and today's value somewhere in between of 10^{101} with some massive black holes in the centres of the galaxies (all taken in natural units). Where the entropy can be rewritten in terms of the mass (for a spherically symmetric black hole) $S_{BH} = \frac{2k_B m^2 \pi G}{\hbar c}$ per baryon adding about 35 orders for black holes only and a typical mass is $m \sim 4 \cdot 10^6 M_\odot$, 10^{21} per baryon entropy.

Thus, there was a very big constraint on the universe at the beginning. Why was the Big Bang of such low entropy, if a possible big crunch would be so chaotic? The number of states is the exponential (for such high numbers it doesn't matter to take 10 instead of e, exponential), our universe if randomly chosen would have been constrained to one in $10^{10^{123}}$. Note that if one follows the Big Bang theory the matter entropy should have been high as a thermal equilibrium is assumed. Further, he assumes the origin of this constraint in some new physics, possibly a theory of quantum gravity? The strong version of the Weyl curvature hypothesis says that the spacetime metric can be rescaled to extend the conformal structure of the Big Bang where the Weyl curvature vanishes on the surface. The hypothesis plays an important role in Penrose's CCC (cyclic conformal cosmlogy) where the conformal geometry[8] removes the singularity, each futurelike infinity is a Big Bang for the next universe (i.e. not a multiverse, but the 'same' universe recollapsing and expanding)[9]. The universe had

[6] The Weyl scalar as quantification for entropy was proposed, but disregarded earlier (Goode et al 1982, Rothman et el 1997).
Later $S = \frac{C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}}{R^{\mu\nu}R_{\mu\nu}}$ was proposed - following Penrose this would be the ratio of tidal distortion per volume change, but also criticised. The Bel-Robinson tensor that consists of the Weyl and its dual tensor was suggested as well (Pelavas 2006).

[7] This agrees with Pailla's et al (2017) estimation where he uses data from Planck. He multiplies the total density by the fraction of the baryons' density and the volume of the universe in order to get the total mass of baryons. Then, he divides it by the mass of one baryon.

[8] Conformal geometry defines a measure for angles, but not distances or lengths. The metric can measure angles, but measurements of angles don't predict the metric uniquely. Still, at any point the ratio of two lengths coming from different directions is known.

[9] A rigorous mathematical derivation can be found by P. Tod (2010, 2013). He proves that the 3dim Big Bang surface is a smooth boundary for the spacetime in the past which is a conformal manifold. He actually claims that the Weyl tensor is constant. FRW symmetry isn't assumed anymore, but emerges.

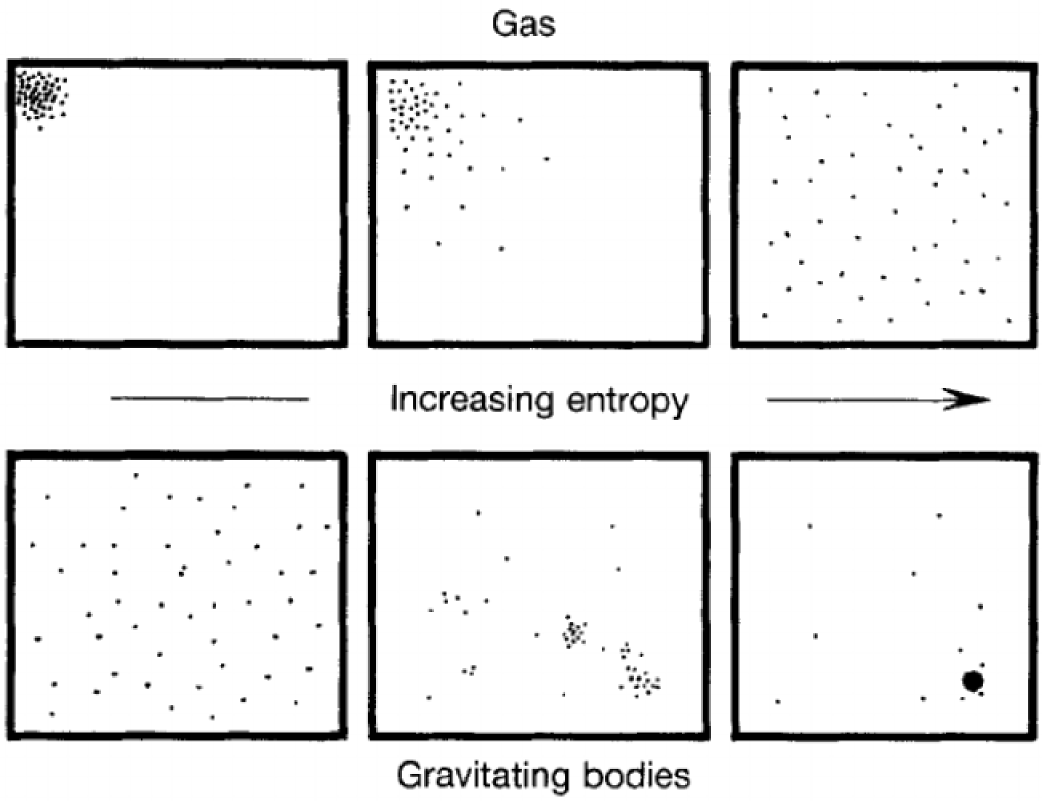


Figure 4.3: Gravitational entropy. Ordinary gas shows a behaviour of increasing entropy corresponding to an increasing uniformity in the distribution. However, gravitational entropy is associated to an increasing clumping of matter, with the maximum entropy being a black hole.Along with that the Weyl tensor increases. Our universe starts in a tiny region of low gravitational entropy and goes through phase space until the largest volume and thermal equilibrium is reached. The corresponding equilibrium in gravity is difficult to define. Taken from [54].

its initial condition for the second law to arise (which leads in some sense to the anthropic principle). In my opinion one should investigate where the past (low) entropy was located. Or rather, what was out of equilibrium in the early universe? What are the degrees of freedom and how 'much' entropy can a dof produce. Along with that we need to further study the thermal history of the universe since there is undoubtedly a connection to statistical thermodynamics. We will see that entropy might arise from AS' renormalisation group flow only. One should also review Penrose's argument on the level of coarse-graining. The act of defining a coarse-graining itself defines what is probable and what isn't. Entropy on macroscopic and microscopic levels should be defined differently. Further treatment can be found in 7.7.7 where I prove that his constraint that the Weyl measurement should also give the black hole entropy is only valid for a five dimensional spacetime.

4.2 'Multimess'

> In an eternally inflating universe, anything that can happen will happen; in fact, it will happen an infinite number of times. Thus, the question of what is possible becomes trivial—anything is possible [...]
>
> (A. Guth — Eternal inflation and its implication, 2007)

As Guth claims *almost all inflation models lead to a multiverse*, but so far there is no proof (as it needs to be looked at for each model or model class). One should be careful with the definition of *multiverse*. Tegmark [80] classifies four levels. The level 1 multiverse is well-accepted in cosmology and describes the idea of eternal inflation that produces separate mini/pocket universes with the same laws of physics. We don't have any observational evidence since causal contact is impossible (at least for now). The level II multiverse mainly put forward by Vilenkin refers to physical universes that are continuously produced to far away to be observed with same fundamental equations of physics but perhaps different constants and particles and unobserved forever. Level 3 (Everett's multiverse - all possible histories of the universe do happen and our universe exists in all those histories, a quantum multiverse assuming cosmic unitarity) and level 4 (the physical universe *is* mathematics and can describe all possible structures available in the mathematical theory, hence different fundamental physics is possible) aren't of much relevance for this thesis. For this thesis it is sufficient to distinguish between 1. multiverse by eternal inflation 2. the many-worlds interpretation of quantum mechanics and 3. the string theory landscape and focus on the first.

The issue of eternal inflation is that it results in a multiverse where due to the randomness of the quantum fluctuations in the early universe all kind of universes are possible. Steinhardt et al. call it the 'Multimess' as it firstly makes probability calculations impossible (see below) and secondly makes inflation in some sense redundant as it would be a theory of all possibilities and couldn't predict anything.
On the other side, the multiverse gives a natural explanation for the anthropic principle. All the observed parameters, ranging from the fine structure constant to the value of dark energy today, have their precise values as in the infinite randomness of the multiverse there is a non-zero probability and we as we can only live under such conditions. The separate mini universes might have different laws, different low energy physics, different values of energy densities etc.[10]
The Hamiltonian evolution in an infinite phase space (which is the case for the multiverse in the eternal inflation setting) in contrast to a finite one without eternal inflation can lead to fine-tuning of parameter if one follows Liouville's theorem[11] If the probability for an event is rare in a finite space it isn't probable that its probability increases, it remains a rare event. In fact, in eternal inflation one can give an example for such an evolution that shows that arbitrary fine-tuning is possible (L. Guth): The time-independent and bounded (from below) Hamiltonian $H = \tan^{-1}(-pq)$ gives the

[10] Weinberg (1982) claimed that we live in a universe governed by SU(3)xSU(2)xU(1) since 'our' inflation puts us in that minimum.

[11] The phase space distribution function remains constant along any trajectory, $\frac{dp}{dt} = 0$ where p is the phase space distribution depending on time, canonical value and conjugate momenta.

equation of motions

$$\begin{aligned}\dot{p} &= \frac{p}{p^2q^2+1} \\ \dot{q} &= \frac{q}{p^2q^2+1}\end{aligned} \tag{4.8}$$

where p^2q^2 is fixed. When p grows exponentially, q decreases. For any normalised initial probability distribution with $\epsilon, \delta > 0$, there exists a time t' such that for $t > t' \quad p(|q| < \delta) > 1 - \epsilon$ meaning we can arbitrary fine-tune which then gives a generic result.

ISL claim that a solution would be a *bouncing cosmology*, as it was proposed by Steinhardt et al. in 2001. The Big Bang is not the beginning, but a bounce of a previous contraction to an expansion phase. Therefore, the flattening of the universe is before 'our' Big Bang. Gravitational waves produced by quantum fluctuations and hopefully visible in the CMB one day would have a smaller amplitude than the ones predicted by inflation (because it doesn't predict such high energies). There is no multiverse and as the contraction starts the universe is large i.e. classical (GR) and bounces back before it can shrink down to a size where quantum effects would dominate there is also no Big Bang in the sense of a singularity which eliminates the problem of quantum gravity, the transition from 'quantum' to 'classical'. The contraction phase is ultra-slow (ekpyrotic) and the parameters, such as H, ρ and T oscillate periodically. Whereas $H(t)$ oscillate between large positive and negative exponential values, $a(t)$'s expansion is longer and its contraction shorter. This then produces over many cycles a de Sitter like universe (Ijjas, Steinhardt 2019) 4.4. They argue that this can solve the coincidence problem[12]. As I explained in the motivational part the smoothing property leads to inflation, ISL describe a universe that undergoes *supersmoothing* through slow contraction. They distinguish four smoothing types that are necessary for today's state [83].

1. classical smoothing - classical inhomogeneities and anisotropies must be suppressed
2. quantum smoothing - quantum fluctuation are suppressed
3. robust smoothing - evolution is independent of initial conditions
4. rapid smoothing - sufficient inflation before phase exit

where the first two have been well established, mainly by perturbative treatment and the latter two are analysed with numerical gravity and non-perturbative theory. The first can again be written in mathematically form as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\left(\frac{\rho_m}{a^3} + \frac{\rho_r}{a^4}\right) - \frac{k}{a^2} + \frac{\sigma^2}{a^6} + \frac{\rho_\phi}{3a^{2\epsilon}}, \quad \epsilon = \frac{3}{2}\left(1 + \frac{p}{\rho}\right) \tag{4.9}$$

with the matter and radiation content, curvature, shear and scalar field matter, respectively. For the domination for decreasing a, $\epsilon > 3$ (slow contraction) and for

[12]The coincidence problem isn't clearly defined, but captures the fine-tuning of parameters of dark energy and matter content that we observe today.

inflationary expansion as usual $\epsilon < 1$. For quantum smoothing the former is substituted by $\epsilon > 2$.
Whether we can ever test the multiverse is not known. For the chaotic model it was proposed that we might observe imprints when our universe collided with another. Further, tests of the topology of universe might also rule out the chaotic scenario, for example, if it is closed (like a torus), so finite, eternally chaotic inflation would be ruled out. Such observations would be repeating patterns in the CMB.

4.3 Physics/Mechanism not Clearly Understood

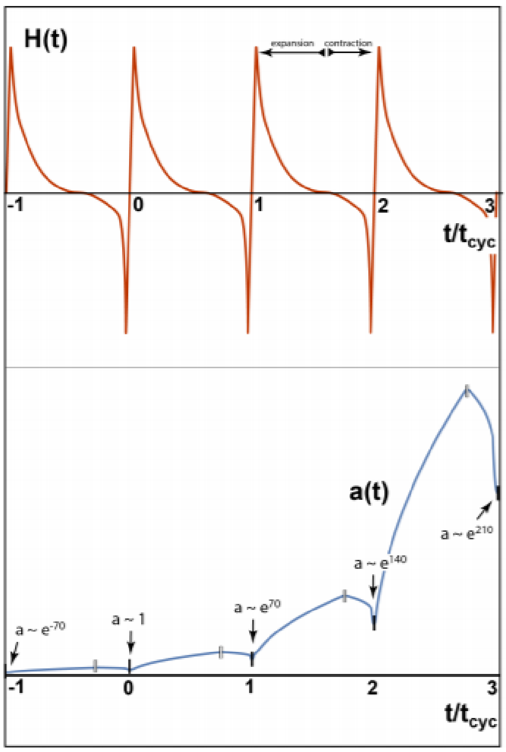


Figure 4.4: Ijjas and Steinhardt cyclic universe (2019), 'A new kind of cyclic universe' showing the oscillatory evolution of the Hubble parameter with long expansion and short contraction. The scale factor increases non-periodically oscillatory.

As it was shown the exact mechanism behind inflation is not known. If an *inflaton field* exits it is not know what scalar field it is. Until now we have only observed one scalar field, the Higgs field. Some research is dedicated to whether the Higgs and inflaton field are the same, but until now without sufficient evidence. Along with that it leaves the question what kind of energy source causes inflation. The usual proposed scalar field that can create negative pressure can maybe only be found at high energies. However, we know that accelerated expansion is definitely possible as we measure it today and call it dark energy - whose physics we also don't understand. The Higgs field has been proposed as scalar field for inflation (Calmet, 2016) where it is non-minimally coupled to gravity. The standard Higgs potential is not flat.

$$V = \lambda(H\dagger H - \nu^2)^2 \tag{4.10}$$

Coupling the field to curvature gives a Lagrangian of

$$\mathcal{L} \sim \frac{R}{16\pi G} - \eta H\dagger H R + \mathcal{L}_{int} \tag{4.11}$$

In the Einstein frame this can be rewritten (similar to Starobinsky inflation, look for $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ with $\Omega = \sqrt{1 + \frac{h^2}{8\pi m_p^2}}$) as a Lagrangian of

$$\mathcal{L} \sim -\frac{m_p}{16\pi}\hat{R} + \frac{1}{2}\delta_\mu \chi \delta^\mu \chi - \hat{V} \tag{4.12}$$

and

$$\frac{d\chi}{dh} = \sqrt{\frac{\Omega^2 + \frac{48\pi\eta^2 h^2}{m_p^2}}{\Omega^4}} \tag{4.13}$$

in the unitary gauge, $H\dagger H = h^2$, with the potential $\hat{V} = \frac{\lambda m_p^4}{4(8\pi\eta)^2}\left(1 + e^{-\frac{2\sqrt{8\pi}\chi}{\sqrt{6}m_p}}\right)^{-2}$. It is flat for $\chi \gg m_p$ and SR inflation follows and the density perturbation constraint gives a non-minimal coupling of $\eta \sim 10^4$ which ensures that λ is sufficiently low and the mass of the field corresponds to the Higgs mass $m_H = 125\text{GeV} \sim \sqrt{\lambda}\mu$ (Steinwachs, 2018). For large fields, $h \gg \frac{m_p}{\sqrt{\eta}}$ the potential $\hat{V}$ is flat. The problem is that at high energies quantum loop corrections of the SM particles need to be taken into account, there is a difference of about 14 orders between the energy scale at inflation and μ, the running needs to be investigated as well, λ is rather small in the early universe. Especially, the quantum corrections of the effective potential put a dependence on the spectral indices. In fact, the renormalisation group flow drives λ to negative values at the electroweak scale at high energies (Degrassi, 2012) which would result in a negative global minimum somewhere below the inflation energy scale and the further evolution would be very different from today's universe as it would destabilise the EW vacuum. Higgs inflation is similar to the Starobinsky one, but the latter doesn't suffer from this problem. There is also the claim that inflation might be due to the R^2 term which itself is triggered by the large non-minimal couplings of the Higgs to the curvature. On the other side one could look for a way to stabilise the RG improved Higgs potential or prevent the decay of the EW vacuum.

So far no model has been as convincing as it would have ruled out all others, but rather inflation has turned into a *framework* or a large class of hundreds of different models that all make the exponential expansion in some sense possible. The evolution of the universe is dependent on the model chosen. Hence, can inflation be classified as a theory if it provides such a large number of possible evolutions?

It is not clear at what energy scale inflation took place 7.7.2 or how long it lasted. In some models it is also not clear how inflation would have ended. In chaotic inflation, for example, there is a probability (due to large quantum fluctuations) that inflation will never end and starts dominating the universe.

Inflation (at least the classical type) doesn't claim to solve the singularity problem, but still is embedded in FRW which has a singularity in classical GR.

As I have shown in 3.4.1 the reheating mechanism is dependent on the model and not fully understood. For example, it is not clear how the inflation field is coupled to normal matter. One requires a realistic theory of elementary particles.

Where did the energy for inflation come from? Depending on the model we need a scalar field in a potential with sufficient high potential energy. If we suppose that the total energy of matter and gravity is conserved equals 0. The total energy during the Big Bang of radiation must have been more than today $> 10^{53}$g, where the number of photons shouldn't have changed much, but the energy did (depending on the temperature), with a Planck temperature of $\sim 10^{32}$, $10^{53} \cdot 10^{32}\text{g} = 10^{85}\text{g}$ at the beginning of inflation. During the inflationary period the energy density stayed constant, the volume $\sim a^3$ increased as $\sim e^{3Ht}$, such that the energy scaled as $\sim e^{3Ht}$. If we then assume that the decay of the inflaton field means the transformation of all energy

into energy per mass of particles and as mentioned, $E = 0$ meaning $E_{mat} \sim e^{3Ht}$ and $E_{grav} \sim -e^{3Ht}$.
The energy available in the beginning, following Heisenberg's uncertainty principle $\Delta t \sim m_p^{-1}$, was of Planck scale.The density of (dimensional analysis) for the false vacuum is assumed to be at the GUT scale, $\rho_f \sim \frac{E_{GUT}^4}{\hbar^3 c^3} \sim 2.3 \cdot 10^{81} \frac{g}{cm^3}$.
During inflation the density (and pressure assuming scalar field inflation and fluid cosmology in FRW) stays constant while the volume is increasing, there is a massive increase in the total matter-energy. Guth describes it as 'free lunch', it violates the strong energy condition, 7.3.2.

4.3.1 ISL Criticism (Again)

Ijjas, Steinhardt and Loeb [3] claim that Planck data disfavours the 'simplest' inflation models (where the definition simple is not really put forward)- In their opinion, inflation has become a *flexible* idea that can produce any model that fits to the measured curvature, amount of matter, power spectrum (red/cold hotspots in the CMB) etc. Inflation explains what we see, but not necessarily why and how it was formed. The data of 2013 disfavours according to them simple models, SR is only possible under fine-tuning. The favoured models are rather plateau-like potentials. This results in the initial conditions problem and the *unlikeliness* problem, along with the already mentioned unpredictability problem in the multiverse scenario. The favoured potentials are 'exponentially unlikely according to the logic of the inflationary paradigm itself'. As properties such as non-Gaussianity are small, a large class of models is already disfavoured. The comparison is done for single scalar field models by comparing n_s and r which indeed favours inflation models with a plateau in the potential. Such a potential exists in symmetry breaking models as in new inflation, some natural inflation models[13], R^2 and hilltop models. For latter the field is near the maximum of the potential and usually predicts a mass of far below the Planck mass. Here one needs more fine-tuning in order to solve the cosmological problems than for a 'simpler' model such as powerlaw inflation. It is then a paradox that Planck data favours a model that is unlikely. Reason is the following. In most models the potential is plateau-like on one side and powerlaw-like on the other (for example, see 3.20). Hence, the minimum can be reached in two different ways via slow rolling from the steep powerlaw side or the plateau one. There is no reason why the scalar field should roll down the more unlikely plateau. Nonetheless, this is favoured by the observational data. If I want to compare simple powerlaw, $V \sim \lambda\phi^4$ with $\Delta \leqslant \frac{m_p}{\lambda^{\frac{1}{4}}}$, and say plateau-type of Higgs inflation[14], $V = \lambda(\phi^2 - \phi_0^2)^2$ with $\Delta\phi \leqslant \phi_0 \sim m_p$ (it has a plateau for $|\phi| << \phi_0$), along the three constraints: amount of inflation ($N \sim 60$), right scale of density fluctuations ($\frac{\delta\rho}{\rho} \sim 10^{-5}$) and a graceful exit and the formulae

$$\frac{\delta\rho}{\rho} \sim 10^{-5} \sim \frac{V^{\frac{3}{2}}}{V'} \tag{4.14}$$

[13] periodic models such as $V(\phi) = \Lambda^4(1 - \cos\frac{\phi}{f})$

[14] I would rather classify this as hilltop inflation. When a model has a long plateau the IC for SR are satisfied as the potential energy will dominate over the kinetic one for a very long time.

and

$$N_{max} \sim \frac{8\pi}{m_p^2} \int_{\phi_i}^{\phi_f} \frac{V}{V'} d\phi \tag{4.15}$$

ISL conclude that there is a larger range of $\Delta\phi$ for the powerlaw part of the potential and also a bigger maximum number of e-folds. Thus, powerlaw inflation is exponentially more likely. N_{max} is given by $\sim \frac{8\pi\phi_0^2}{m_p^2}$ for the plateau and $N_{max} \sim \frac{8\pi}{m_p^2}\max(\phi_i^2 - \phi_f^2)$. Comparing those and as the range for power law (PL) is much bigger than for the plateau (P) potential (as I have previously calculated, in order for inflation to occur- density perturbation constraint, the parameter has to be of order $\lambda \sim 10^{-15}$) $\Delta_{PL} >> \Delta_P$,

$$N_{maxPL} \sim \frac{N_{maxP}}{\sqrt{\lambda}} >> N_{maxP} \tag{4.16}$$

On the other hand they explain that it is not possible to calculate any probabilities, hence, they shouldn't be able to compare the likelihood of different models (see also next section on the measure problem). The authors agree that the plateau and powerlaw inflation models lead to a multiverse in which normal counting or a definition of probability due to infinite sample spaces isn't possible. Their arguments are self-contradictory. They don't account the prior probability for the models themselves but rather for the initial values and tuning of parameters. Moreover, this argument is only valid for single field models and without other processes such as quantum tunneling. The anthropic principle cannot be used as solution since it would also favour more likely models which again is not a convincing argument.
They propose to have a plateau at large ϕ and no powerlaw behaviour such as R^2 or natural inflation[15] where the form is periodic and power law is forbidden. However, such a potential should be the only option which obviously isn't the case. It simple means least amount of parameters the R^2 inflation should be classified as simple and is consistent with Planck data. One should also investigate more what the fine-tuning conditions are for plateau potentials as SR is automatically possible, but should have the right behaviours for large field values (in Taylor expansion the potentials needs certain cancellations).

4.4 Probability Problem

We have already encountered the probability problem in Lim's treatment. In the first place one might wonder what is the probability that inflation occurs? This was first tackled by Gibbons, Hawking and Stewart (1986) who defined a measure on the FRW models with scalar field inflation. The corresponding measures of two Cauchy slices are the same and it is a natural choice (pull back of the symplectic form for this system)[16]. They concluded that most of the models would produce inflation that solves

[15] Freese, 'Natural inflation with pseudo Nambu-Goldstone bosons', 1990.

[16] Basically this was the first kind of volume measure: The universe is a $2n \rightarrow 2n-1$dim constrained Hamiltonian system. An intersection of a hypersurface and a bundle of phase trajectories is $2n-2$ dim. By the symplectic form's pull back a $2n-2$ form, a volume form, is induced. This volume is completely

the horizon and flatness problem. Unfortunately, the measures were infinite (Hawking and Page, 1987) for both the inflation and non-inflation results, same was shown for R^2 inflation later.
Another issue is the question how to define *probabilities* if we accept that there was some kind of inflation era. Steinhardt [76] criticises that there have been improbable conditions at the beginning of inflation (SR parameters) or would have caused eternal inflation[17] i.e. infinite universes with all kind of properties as the random quantum fluctuations give rise to all kind of possibilities. Some of them might be with high curvature or inhomogeneous. This leads to the impossibility of making predictions or computation of probability which should be possible in all kind of physical theories.
Further, he claims that bad inflation is more probable than good one (not corresponding to our observations) where the only solution might be the use of the anthropic principle. Even worse, Penrose showed that it is more probable even when no inflation occurs, just from simple thermodynamics, to have initial conditions that cause flatness and homogeneity.
In my opinion, the probability problem in inflation also consists of the fact that in order to calculate, say the probability for a potential, we first need to know the fundamental degrees of freedom as well as the UV completion of the theory which we both don't know. A further investigation of inflation in quantum gravity setting is necessary.
Without the possibility of calculating probabilities the theory cannot give any *predictions* and would basically be useless (this will be further explored in the AS chapter).

4.4.1 Measure Problem

For the multiverse arising from eternal inflation our sample space is infinite as an eternally inflating universe produces an infinite number of pocket universes and so on. In order to still calculate probabilities we might find a truncation to order. A *probability measure* is needed - we have to assign a likelihood to each event in order to measure how probable it is to live in our universe. To find a measurement it is common to introduce a cutoff similar to a regularisation. Several options have been proposed and disregarded giving different results. Hence, solutions depend on the cutoff chosen which shouldn't be the case.
In order to compare probabilities, we may write

$$\frac{p_a}{p_b} = \frac{N_a}{N_b} \tag{4.17}$$

where the ratio of the probabilities of two events (or two properties such as different temperatures at time of recombination) is equal to the ratio of the number of instances. In a multiverse both are infinite, so it cannot be calculated in that form.
A favoured cutoff was the proper-time cutoff measure where events are collected only

independent of the choice of hypersurface. First, they used a cutoff in the Hubble parameter where the ratio was indeed well-behaved and finite, $H \to \infty$. However, later Gibbons and Turok used a curvature dependent cutoff, $\frac{k}{a^2}$ again concluding that their results were independent of the cutoff chosen. The result severely differ.

[17] Guth himself said that most inflation models would lead to eternal inflation and thus a multiverse. However, he doesn't give a proof or quantitative evidence.

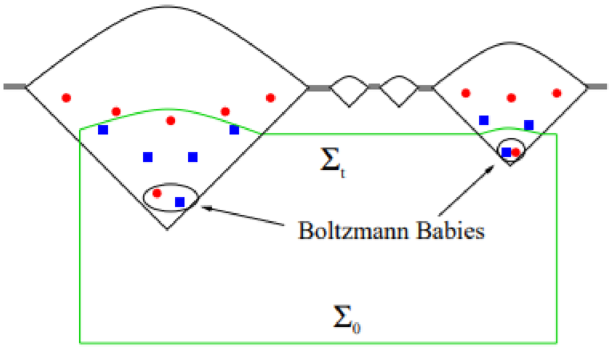


Figure 4.5: Youngness Paradox Two observations of CMB temperatures (red and blue). The Youngness paradox says that Boltzmann babies dominate the counting. Most observations made prior to t are made in bubbles that were formed recently. 'Boltzmann babies in the proper time measure', 2008.

until that time t_c, so only a finite number of events happened. The probability is finite if it is well-defined and one can remove the cutoff to infinity. However, the measure doesn't provide right solutions, it suffers from the *Youngness paradox* [4]. One needs to account for the exponential volume change of Ve^{3H}. The probability of being in younger/hotter universes is here exponentially higher than our older/colder universe. The difference is large [4], the ratio of our universe being at 3K to the actual CMB temperature is

$$\frac{N_{TCMB=3K}}{N_{TCMB=2.725K}} \sim 10^{10^{59}} \tag{4.18}$$

Similarly, compare our 13.7Gyr to a possible age of 13Gyr. If we assume that the multiverse is dominated by the largest $H*$, so inflation takes place in $\Delta t \sim \frac{1}{H*}$ and ask how many universes are at least at the age and compare their probability by their ratios.

$$\sim e^{\frac{3 \cdot 0.76 Gyr}{H*^{-1}}} \tag{4.19}$$

13 Gyr bubbles which is a rather large number. Hence, we need to somehow rescale. Furthermore, models depending on time intervals/equal-time cutoffs in general suffer from the fact that they depend on the definition of time itself (choice of the time coordinate).

Along with this measure the volume measure was introduced which suffers from a similar problem: a large volume causes more observers and is more likely. Often one also encounters the Boltzmann brain problem that also would make our universe quite unlikely and non-typical. Here the major part of observers would be Boltzmann brains 4.5 that are imprints of quantum fluctuations in the late stage. This is even the case when one waits long enough for a stationary state. A pocket-weighted measure was proposed to solve the Youngness problem.

Another choice is the stationary measure [6] embedded in the string landscape which doesn't suffer from the Youngness problem. It is counted from the beginning of the stationary state not the beginning of the evolution. The authors start with an equation that encaptures the evolution of the volume distribution for the pocket universes numbered $j = 1, ..., N$ that stay for different observations:

$$\frac{dV_j}{dt} = V_j(3H_j^\beta - \sum_i \Gamma_{j\to i}) + \sum_i \Gamma_{i\to j} V_j \tag{4.20}$$

where β takes into account the different time coordinate choices and Γ is the transition proportional to the rate and local Hubble parameter. The transition is the time derivative of the incoming flux of new pocket universes, $\frac{d}{dt}Q_j(t) = \Gamma_{i\to j}$. We look at the global structure with an asymptotic behaviour of the volume distribution, $V_j(t) \sim V_j^{(0)} e^{3H_{max}^\beta t}$ which then gives a flux of

$$Q_j(t) \sim \frac{e^{3H_{max} t}}{3H_{max}} \sum_i \Gamma_{i\to j} V_i^{(0)} \tag{4.21}$$

Now we want to calculate the probability of living in state j wrt to the number of observations corresponding to that state N_j. The ratio is finite only when a stationary regime is reached which is at different times for different pocket universes, Δt_i. Hence, local properties don't depend on the time of origin, same as with the probability distribution, but get stationary at some time (stationary era), at least in some models of inflation. To solve this issue the stationary measure resets the clock in every vacuum. The different vacua (universes) are then compared at the same time from where the stationary era begins. In order to find a cutoff a physical condition is needed (beginning of stationary evolution/reheating...). To reset we need to cancel the factor of $e^{3H_{max}^\beta(\Delta_j - \Delta_i)}$. Instead of calculating the usual $\lim_{t\to\infty} \frac{N_i(t)}{N_j(t)}$ the ratio of probabilities is calculated as

$$\frac{P_i}{P_j} = \lim_{t\to\infty} \frac{N_i(t+\Delta t_i)}{N_j(t+\Delta t_j)} = \lim_{t\to\infty} \frac{N_i(t)}{N_j(t)} e^{3H_{max^\beta}(\Delta t_i - \Delta t_j)} \tag{4.22}$$

This difference in time is basically the duration for SR.
A more recent measure is the reheating volume measure [90]. The spacetime during inflation is rather empty, observers can only exist after reheating meaning it is constrained by current experimental knowledge. The average number of observers in a spatial domain of reheating is a function of the cosmological parameters in that domain. Now one might say that each 3-volume produces an infinite 3-volume of reheating in an eternally inflating universe and the reheating surface is rather inhomogeneous without any symmetries such that is not possible to mathematically define a random point. We need a volume cutoff on the reheating surface that makes the volume finite in a well-defined way. We are interested in the distribution $p(Q|V)Vdq$ where $Q|V$ needs to be regularised and q is a specific observable. The infinite reheating region is reduced to a finite subdomain, i.e. a finite volume, giving the probability $p(q) = \lim_{V\to\infty} p(q|V)$. The problem is that this depends on the cutoff chosen. As explained they use the reheating volume cutoff which defines the probability for a cosmic parameter q as

$$p(q) = \lim_{\nu\to\infty} p(q = q_R, V) = \lim_{\nu\to\infty} \frac{< V_{q_r|V} >}{V} \tag{4.23}$$

since we want to have the portion V_{q_r} of V where the parameter $q_r = q$. The volume of the corresponding parameter regularised via the reheating surface is

$$< V_{q_r|V} >= \frac{\int p(V, V_{q_r}; q_0, \phi_0) V_{q_r} dV_{q_r}}{p(V; q_0, \phi_0)} \tag{4.24}$$

with the joint finite probability distribution produced by the reheating in the numerator and ϕ_0 and q_0 are the values of the initial Hubble region. We get the probability for a random q in the interval $[q_r, q_r + dq]$. It is claimed that the result is independent of the initial values since the universes 'forgets' them after self-reproduction. I quote the main result, the ratio of two probabilities is

$$\frac{p(2)}{p(1)} =\sim e^{\frac{3\pi^2}{\sqrt{2}H_0^2}(\phi_2 - \phi_1)^2} \tag{4.25}$$

with $H_0 = \sqrt{\frac{8\pi G V_0}{3}}$ which is gauge invariant and without spacetime coordinates. The comparison is well defined since we deal with two different SR channels. The authors follow a similar stochastic approach as in the SR case. They also only use the information given by the intrinsic geometry of the reheating surface. When one has the finite reheating volume $V(R)$ with a certain probability one considers the joint finite distribution of the reheating volume and portion $V_q(R)$ of this volume where the property is $q = q(R)$, $p(V, V_q(R), \phi_0, q_0)$. There is no Youngness problem. Comparing two domains gives two different types of reheated domains ('two possible slow-roll channels').

Other recent choices were Garriga's immortal and imaginary 'watcher' who can remarkably take a random walk through the multiverse and count events, the frequencies of events give the probabilities (Garriga, Vilenkin, 2013). The measure doesn't need a cutoff and is independent on initial conditions. However, it assumes that the observer is on a timelike geodesic that undertakes infinitely many crunches through spacetime and the vacuum landscape is irreducible[18].

In contrast, Bousso (2007) developed a 'local' measure where he only takes into account a finite patch of the multiverse (causal diamond as finite 4-V subset) as largest region that is ever accessible for a single observer. The cut is given by the intersection of the future light cone from A and the past light cone of the point B where A is the point where an imaginary observer crosses the reheating surface and B is the point where the observer leaves that vacuum. This results in finite regions that are large and rare with many observers and small and typical ones with few observers. It follows the thought that all we can ever measure is indeed all there is such that a measure problem in classical sense doesn't exist.

Another perspective on the probability problem in inflation is the top-down cosmology by Hawking and Hartle as explained in the last chapter and the appendix. One shouldn't ask about our origin or evolution in the sense of a single unique trajectory, but make use of the universe that we observe along with the assumption that it arose from nothing. Every history is possible and can be encoded in a quantum superposition integral with the boundary condition nothing to now. Their sum gives the most

[18]Meaning that every vacuum can be reached from any other. The counting of events takes place in space and is a finite domain that is measured with the cross-section with the watcher's geodesic

probable, ideally our universe.
There is also the opinion that one should only use data from today's observations for future calculations of conditional probabilities i.e. predict the probabilities of future measurements[19].
In my opinion, one should not forget that the probability in cosmology has the same origin as the quantum mechanical probability, indeed one might state that anything that can happen will happen - but not with equal probability. Regarding the measures one should use comoving coordinates, which are not expanding during inflation.
It seems as if the right measure is searched for in order to finally be able to give the right probabilities, but the amount of (again) different models that all predict other results and the rather ad hoc construction are not convincing.

4.4.2 Fokker-Planck treatment

Already in 1989, Nambu tried treating the stochastic evolution of the inflaton field with the Fokker-Planck equation[20]. He calculated a probability distribution of a scalar field for a certain physical volume and found a normalisable stationary solution. The universes are fractal-like distributed. The conditions for eternal inflation was also treated, numerically and analytically solved and applied to different potentials[21]. Again, the fluctuations are the sum of classical and quantum perturbation

$$\delta\phi = -\frac{V'}{3H}\delta t + \delta\phi_q(\delta t) \tag{4.27}$$

with a behaviour of $\delta\phi_q(\delta t) \sim \mathcal{N}(0, \frac{H^3\delta t}{(2\pi)^2})$. Once crossed the horizon the quantum fluctuation becomes classical and be described by a Gaussian noise

$$3H\dot{\phi} + V' = N(t) \tag{4.28}$$

The Fokker-Planck equation is then given for a set of fields i

$$P[\dot{\Phi}, t] = \frac{1}{2}\frac{H^3}{4\pi^2}\partial_i\partial^i P[\Phi, t] + \frac{1}{3H}\partial_i(\partial^i V P[\Phi, t]) \tag{4.29}$$

which is valid at the stage of SR and needs an initial condition at $t = t_0$. The Hubble parameter is assumed to be constant, $H^2 = \frac{V_0 8\pi}{3m_p^2}$, and the potential is of the form $V(\phi) = V_0 + \sum_{i=1}^N V_i(\phi_i)$. The probability distribution takes the multivariate Gaussian form $P[\Phi, t] = \sum_{i=1}^N P_i[\phi_i, t]$ (4.1),

$$P_i = \frac{1}{\sqrt{2\pi}\sigma_i(t)} e^{-\frac{(\phi_i - \mu_i(t))^2}{2\sigma_i(t)^2}} \tag{4.30}$$

[19] A. Linde, 'Sinks in the Landscape, Boltzmann Brains, and the Cosmological Constant Problem'
[20] This is a PDE for the temporal evolution of a probability density function where particles undergo Brownian motion. If there is no diffusion term it simplifies to Liouville's equation. For a probability density $P(x, t)$

$$\dot{P}(x, t) = -\frac{\partial}{\partial x}[P(x, t)\mu(x, t)] + \frac{\partial^2}{\partial x^2}[P(x, t)\frac{\sigma^2(X_t, t)}{2}] \tag{4.26}$$

where μ is the drift and σ the diffusion. satisfies the usual SDE such as $dX_t = \mu(X_t, t)dt + \sigma(X_t)dB_t$ with X_t as the stochastic process and B being the Wiener process. For more than one dimension the formula sums over all particles.
[21] T. Rudelius, poster session during 'Quantum Spacetime and the Renormalization Group', 10/20

potential	$\mu(t)$	$\sigma^2(t)$
constant V_0	0	$\frac{H^3}{4\pi^2}t$
linear $V_0 - \alpha\phi$	$\frac{\alpha t}{3H}$	$\frac{H^3}{4\pi^2}t$
free massive $V_0 + \frac{1}{2}m^2\phi^2$	0	$\frac{3H^4}{8\pi^2 m^2}\left(1 - e^{-\frac{2m^2 t}{3H}}\right)$

Table 4.1: stochastic evolution of various potentials

The author analyses different single-field potentials 4.1.
For the Hubble duration $t \sim H^{-1}$ the constant and linear potential give the usual Hubble size quantum fluctuation of $\frac{H}{2\pi}$. The condition for eternal inflation is as usual SR $\epsilon < 1$ which sets constraints on the parameters of the potential, but also the expansion of the universe needs to be taken into account. The volume space that is still inflating after a certain time, so for a certain value of the scalar field, is the product of its probability (which is decreasing in time) and the increasing volume due to the exponential expansion, $V(t) = V_0 e^{3Ht}$,

$$V(\phi > \phi_c, t) = \text{Prob}(\phi > \phi_c, t)V(t) \tag{4.31}$$

Hence, we need a slower decrease of probability than increase in H.

4.5 Observations

As we have seen inflation provides an explanation for the cosmic perturbations and quantum fluctuations as seeds for the LSS today.
It is clear that it is not possible to test inflation in the lab as we would need energies near the Planck or at least GUT scale[22]. Hence, analysing the past and present of our universe is the ideal (and maybe only) lab. As I have elaborated in the 3.5 part inflation has been confirmed by observations to high accuracy. With the help of current observations we can either test the theoretical status, identify problems and modify the theory according to them and test again or we can predict future observations from the theory only.
Interestingly, there were quite a lot of predictions that didn't fit to the observations and were disregarded, but should also be mentioned in order to quantify the success of inflation models. For example, the amplitude of density perturbations induced by quantum fluctuations in the CMB measured by COBE were smaller than predicted or rather small parameters are needed (Ellis, Bruni, 1989). Additionally, the power spectrum, for example, is not only dependent on the inflation model, but also other assumptions such as dark matter, the density of atoms etc (where at least for the former we don't know much about).
In the vast number of models one differs mainly by comparing the scalar spectral index and tensor-to-scalar ratio (often in the so-called $n_s - r$ plane). The model that fits the observations best is the R^2 (Starobinsky) inflation 4.6. The best model of inflation is the one with a potential that fits the data and needs the least amount of fine-tuning. The simple comparison of the spectral indices seems not very convincing.
The described E-modes are grad-like and created by scalar and tensor perturbations,

[22] Between GUT scale and the LHC accessible energies there are 12 orders of magnitude.

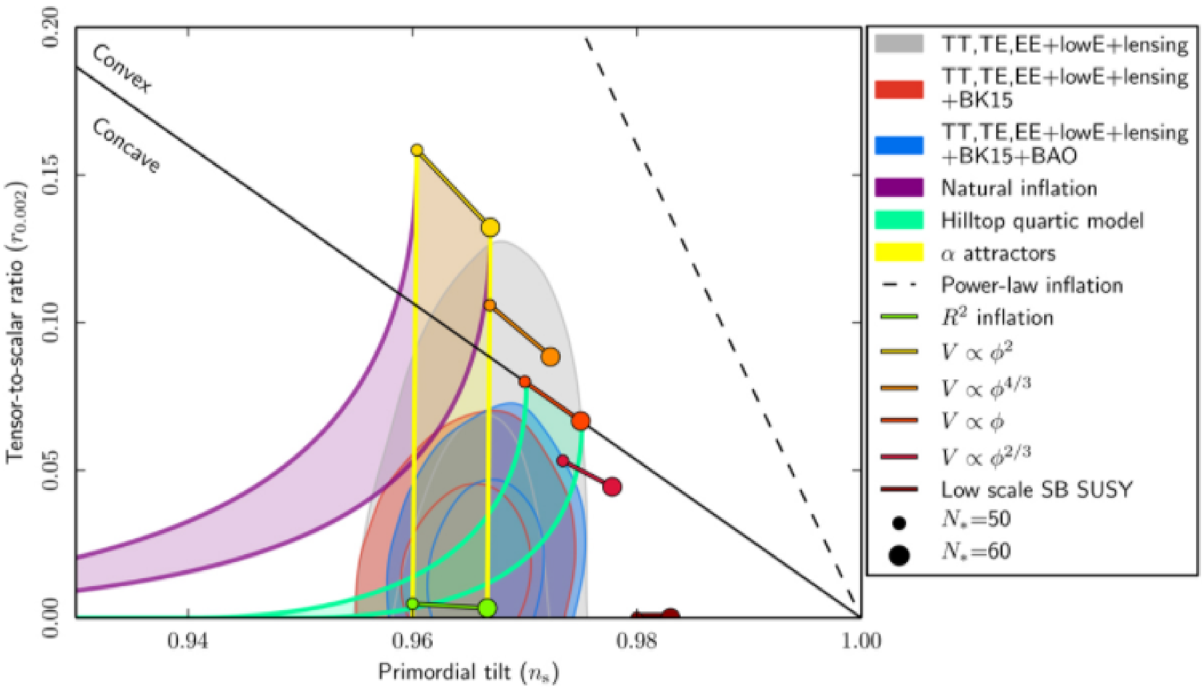


Figure 4.6: Planck2018 vs different inflation models [20] in the n_s-r-plane. We should clearly favour concave potentials. The best fit inside the data is the R^2 model, monomials such as ϕ^2 are rather disfavoured, natural and powerlaw inflation are in the intermediate regime and hilltop and R^2 inflation is favoured. The SR parameters are evaluated for 50 to 60 e-folds at the horizon crossing with reference scales as described in 3.5. Planck's TT, TE, EE+low E+lensing is added by BAO and BK15 experiments (angular power spectrum, temperature polarization cross power spectrum, polarisation only, lensing induced & galaxy surveys & B modes, respectively).

whereas the yet to be measured B-modes are curl-like and created by tensor perturbations only. Hence, E-modes can be seen radial around cold spots ($E < 0$) and tangent around hot spots ($E > 0$) [20].
As already mentioned the bouncing cosmology model would produce gravitational waves as well, but with smaller amplitude. Whereas inflation predicts the scalar-to-tensor ration of order $r = 0.01$, the bouncing model predicts $r \sim 10^{-5}$ which is rather not detectable. B-modes haven't been observed yet which favours the cyclic model, but could rule it out in the future (ISL, 2014).
Another possible observation, but ruled out by inflation, is large scale rotation since any pre-rotationary state should be reduced to near 0 by the exponential expansion. They can also not form during inflation since they would be vector perturbations that don't arise in the cosmic perturbation theory (other than scalar (density) perturbations and tensor perturbations (gravitational waves)) and the scalar field is irrotational. Findings of large scale rotation in the CMB e.g. in some kind of shear anisotropies in the CMB would rule out the theory (or ask for some modification).
Some papers also claim that former bubble collisions (as in false-vacuum eternal inflation models) might have left imprints in the CMB (Aguirre et al., 2007 - detectable with high probability, late-time observers should see a 'nearly-isotropic distribution of bubbles with tiny angular scales'). Based on that a joint collaboration of UCL and Imperial [68][69] found out that bubble collisions would be detectable as small temperature differences in azimuthal patches (projections onto the 2dim last scattering surface). They created an algorithm to analyse these temperature fluctuation. 1. search for areas with azimuthal symmetry[23] 2. search for edges with temperature steps[24] 3. find the best parameters to reproduce those results where they basically created 'fake' CMB fluctuations that correspond to bubble collision imprints signals.
Eternal inflation should produce a probability distribution of five parameters (the collision centre described by two angular values and the amplitude of the temperature modulation there plus value of the causal boundary and its temperature discontinuity) that describe the collision and the number of events. After evaluating the results they used the algorithm for the WMAP 7-year data. In the end, they found several possible regions, four that are larger than expected from false measurements, including the 'cold spot'[25]. However, they found a rather small probability for any circular temperature discontinuities and also didn't find any the WMAP data. If it were detected it would be a a clear evidence for eternal inflation and the multiverse.
What if other theories predict the same results that we can observe as inflation? For example, inflation predicts nearly scale-invariant density perturbations that has been confirmed by cosmic observations. Other conditions can produce such a a spectrum as well (e.g. Geshnizjani et a. 2011). In an expanding universe with a variable speed of sound *only* one of the following conditions need to be the case '1. accelerating expansion (inflation), 2. a speed of sound faster than the speed of light, 3. super-

[23]due to the $SO(2, 1)$ spacetime symmetry during the collision of two bubbles arising from the hyperbolic $SO(3, 1)$ symmetry of the single bubbles (rotations + boosts) - symmetry around a straight line, rotations + boosts transverse to the collision axis only.

[24]The imprint can only be in the future lightcone of the possible collision which is a ring in the CMB, this boundary needs to be smoothly connected to the outer region. The signal is stretched due to inflation.

[25]The cold spot is a rather large region in the CMB map that is colder than the predicted anisotropies.

Planckian energy density'. I will briefly describe the second case. Modes freeze out when crossing the horizon, $\lambda \sim |y|$ where $dy = c_s d\eta$ (as before in the BD vacuum with $\frac{q''}{q} \sim \frac{2}{y^2}$). For the horizon crossing we know that $\lambda_i \geqslant 1000\lambda_f$ and a superhorizon exists for $\lambda_f(\eta_f) > r(\eta_f)$ where r is the Hubble horizon. No other assumptions are made. Those conditions can be rewritten as $|y_i| \geqslant 1000|y_f|$ and $|y_f| > r(\eta_f)$, i.e. $y_f - y_i > 1000r(\eta_f)$. The continuity equation can be rewritten as $\frac{\dot{\rho}}{\rho} = -2\epsilon H$. Hence, integrating after $t \to \eta$, $Hdt \to \frac{d\eta}{r(\eta)}$ gives

$$\ln\frac{\rho_i}{\rho_f} = 2\int_{t_i}^{t_f} \frac{\epsilon}{r(\eta)} d\eta > \frac{2}{r(\eta_f)}\int_{\eta_i}^{\eta_f} \epsilon d\eta > \frac{2}{r(\eta_f)}\epsilon_{min}(\eta_f - \eta_i) \tag{4.32}$$

Thus, without the assumption of any accelerating expansion (hence the second last steps). We also know

$$y_f - y_i = \int_{\eta_i}^{\eta_f} c_s d\eta = < c_s > (\eta_f - \eta_i) \tag{4.33}$$

Therefore, for $\epsilon_{min} > 1$, we conclude

$$\frac{2000}{< c_s >} < \ln\frac{\rho_i}{\rho_f} \tag{4.34}$$

and assuming the initial density of Planck size, m_p^4, and the final at horizon crossing, $(100\text{MeV})^4$ (sub-Planckian), the average speed of sound is above 10.
Also Magueijo's VSL theory predicts such a spectrum.
Following Feynman (1964) the scientific method is a process of thought based on integrating previous knowledge, measurements and observations as well as logical reasoning. In the next chapter of this thesis I will investigate this in the asymptotic safety setting.
Interestingly, it has also been claimed that we cannot classify the nature of fluctuations, whether classical or quantum, by observations. Two point correlations are also classically produced by density perturbations and classical randomness. Three-point, n-point correlation functions, however, differ in the classical and quantum scenario such that it has been suggested to look for observational evidence, currently being done by ESA's Euclid satellite (Green & Porto (2020)). This follows the proposed primordial non-Gaussianity. We have seen that as long as the potential is sufficiently flat the step size in the random walk of Hubble sized regions is Gaussian and so are the quantum fluctuations predicted to be Gaussian. However, more models, especially ones in the String Theory sector and the ones that describe inflation at lower energies, predict non-Gaussianity but also simple models e.g. by isocurvature fluctuations (Langlois,2012) or non-linearity in the dynamics (Dunik, 2013). Measurements haven't been sufficiently sensitive until now.

4.6 Falsifiable?

Guth himself claimed that inflation 'is too flexible to be falsifiable'[26] (Feb 2014) due to the large amount of models inflation has produced.

[26] But he also added that 'there are many different models, just as there are many different gauge theories that we won't be able to test'

Firstly, one may wonder whether inflation is the only theory (if one agrees to that term) that explains the problems described in the motivational part earlier.
Penrose differs between internal and external problems of cosmology. Former are for example the monopoles, they are self-made as they depend on a theory. It might equally be true that our theory of GUT and symmetry breaking is wrong (although this is very unlikely as it has been very successful in testings of the SM). The flatness and horizon problem are external since they are directly correlated to the observable universe. Despite major efforts in the past years it might still be the case that our observations are non-typical.
The solution of the monopole problem is not necessarily a success in favour of inflation, the absence of monopoles might still be disproven and if so it would be an evidence against GUTs. If our theories are wrong, we wouldn't predict any monopoles in the early universe and then there is no need for inflation (at least for this problem). One would need to show that GUTS are definitely true and that there is no other way to dilute the monopoles. As described previously, if one follows symmetry breaking of the GUT, monopoles would have formed during that phase transition. A lower bound on the number of monopoles can be found by causality, $n \sim \zeta^{-3}$. In most models inflation takes place below the GUT scale, such that they could only be produced prior inflation.
However, there exist also other theories how and when those monopoles were created which then changes the further dynamics. [60] investigated the production via thermal fluctuations of the gauge fields in the early universe. The Kibble mechanism (see 7.7.3) can only be applied to a global symmetry that is broken[27] and cannot be used as an explanation. Both, the number density and the spatial distribution of the monopoles differ. They also arise as freeze-out of long wavelength dof, but with a positive correlation length at small scales (whereas Kibble's mechanism predicts negative ζ, a calculation can be found in the appendix 7.7.3). At the transition point (which is rather a smooth region) the correlation length doesn't diverge. There have been claims that in spin ices effective quasiparticle excitations with magnetic charge with similar properties to magnetic monopoles were found[28].
When we want to test inflation the question is whether one can test the entire (inflation) scenario or only specific models or how much we test is model-independent.
Another problem is that inflation can also predict other outcomes that we might test one day or were predicted before observations and then obviously disregarded. Some are in order, all carefully checked and validated. Linde showed that any value of Ω is possible (1998), the power spectrum could be different from the one depicted in the last chapter with, for example, n_s being not scale invariant (Salopek, Bond & Bardeen, 1989) or non-Gaussian and not adiabatic perturbations (Demozzi, Linde & Mukhanov, 2011). Further, Steinhardt (1990) argued that any value above $\Omega \geqslant 0.5$ is possible while still solving the horizon problem. However, Hu, Turner and Weinberg (1994) claim that as long as inflation is constructed to solve the horizon problem Ω is automatically near 1. This makes inflation less convincing. Bucher, Goldhaber and Turok

[27] Elitzur's theorem (1975) says that local gauge symmetries cannot be spontaneously broken. Note one can still have a local gauge symmetry within the global symmetry that undergoes SSB.
[28] Castelnovo, Moessner & Sondhi, 2008, 'Magnetic monopoles in spin ice'

(1995)[29] proved that inflation is the only way to produce an open and homogeneous universe. However, our observations don't indicate that we live in an open universe. The list goes on.
If one defines falsifiable not according to Popper, but simply along other physical theories, whether it predicts the right properties we observe, we can argue: Certain models can be disapproved, but it is difficult to falsify the class of inflation models where basically new models can be added with the right properties. It would be interesting to investigate whether it is possible to always find some model that produces any given list of properties. The observations so far have given sufficient evidence for specific models. Following Popper, however, a theory that is not falsifiable cannot be scientific. Hence, if we won't ever be able to test/observe the multiverse, inflation shouldn't be called a theory or eternal inflation and the multiverse is a mere prediction of the inflationary theory.
The testability of a theory actually doesn't need predictions that are independent of the parameter choice as ISL claim. As already mentioned, the SM in particle physics contains many parameters and has been very well confirmed in the past years. If a theory gives falsifiable predictions this doesn't imply that it is a scientific theory. In fact, if a theory can be expanded to give predictions that are needed one can debate whether this is still a scientific theory. Many models had to be abandoned because of the data not fitting the observations. Usually, one has the same models with say different parameters that need to be 'tuned' according to the experiments. However, falsifiability is a necessary (but not sufficient) requirement.
Still, one can conclude that inflation solves many problems all at once. In my opinion, inflation was constructed to solve the cosmological problems at first, then further predictions such as perturbations arose (or were rediscovered) which was tested later or is still being tested. Barrow and Liddle (1997) distinguish between inflation as theory of initial conditions and as a theory of the origin of structure. Latter has been well confirmed (especially in the power spectrum) whereas the first highly depends on the models or models could be added to describe today's observations of the early universe (such as the observation of a possible open universe) and the fact that one indeed classifies the cosmological problems as problems.

4.7 Need for Quantum Gravity

Matter is described in terms of quantum fields, in modern QFT in flat spacetime. Inflation needs a non-vanishing potential. This already gives the need for the cosmological theory of inflation that takes place in classical GR FRW to embed it in some theory of quantum gravity. Moreover, in the early universe matter was highly compressed and hot (→ QFT) and the curvature was very large (cosmology). For $\phi > m_p$ corrections need to be accounted for[30], such as the gravitational interactions between vacuum fluctuations. As we have seen quantum fluctuations have been stretched to

[29] An epoch of old inflation (false vacuum) is followed by a new inflation model (SR). First the horizon problem is solved, then $k < 0$ universe arises by bubble nucleation that drives $\Omega \to 0$ and along its evolution increases again, but stays (well) below unity.

[30] For energy densities above m_p^4 quantum gravity effects dominate as quantum fluctuations of the metric dominate over the classical metric values.

today's LSS. One may expect that inflation comes from a theory of quantum gravity or modification of general relativity, a UV completion of quantum corrections of the Einstein-Hilbert theory at high energies.
Quantum gravity approaches predict other forms of inflation, substitute or extend inflation. Thus, it is important to compare the results that arise in different approaches. LQG [67] (see also 3.6.3), as already mentioned, predicts a *big bounce*. ϕ functions as clock and gives a probability amplitude for various spacetime geometries (there is no preferred time coordinate, 'emergent time'). First one chooses a classical trajectory with a constant of motion $P(\phi) = P^*(\Phi)$ and a point (ν^*, ϕ^*). GR FRW is valid on any dynamical trajectory where the spacetime curvature and density are sufficiently low (at late times). Now construct a wave packet at the internal time $\phi = \phi^*$, $\mu = \mu^*, P(\phi) = P^*(\phi)$ and evolve this backwards. Investigate whether the wave remains peaked at the classical trajectory or whether quantum effects start to dominate. A UV completion at high curvature is aimed for, the evolution is dominated by the Hamiltonian in LQC.
Near the Planck scale quantum effects produce an effective repulsive force that is greater than the classical gravitational attraction, the contracting universe bounces and a new 'Big Bang' happens. To explain the contraction from the expansion phase, Friedmann's equation is slightly changed for small $a(t)$ (without cosmological constant):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3}\left(1 - \frac{\rho}{\rho_c}\right) \quad \rho_c = \frac{\sqrt{3}}{32\pi^2\gamma^3 G^2 \hbar} \tag{4.35}$$

where ρ_c is the critical density which depends on the Barbero-Immirzi parameter[31] γ. The critical density is then about 0.4 of the Planckian density (in numerical simulations of the order of the matter density). The RHS can become negative which corresponds to a contracting universe. The singularity problem is resolved (see also Ashtekar, 2009) and the initial conditions for inflation (in the standard SR sense) are met during the contraction and bounce back. This bounce back arises out of the quantum properties of the gravitational field itself. SR takes place after the so-called super-inflation. For large $a(t)$ the evolution coincides with the classical description. It also predicts gravitational waves (B-modes), but more suppressed than standard inflation theory predicts (Bojowald, 2008). Similar predictions are made by ISL and Penrose, but of different nature as we have seen.

GR is a deterministic theory that describes the smooth and dynamical spacetime based on the Riemannian metric. Standard quantum mechanics is set in a fixed, non-dynamical spacetime with an external time variable governed by probability rules. On small scales objects become quantised. How can we implement quantum properties into spacetime? Conventional quantum field theory gives the physical content via n-point functions. Perturbative QFT uses regularisation and renormalisation methods. In gravity the usual renormalisation procedure doesn't work and the gravitational field which is spacetime should become a quantum operator. On the approaches for a quantisation of gravity I made a brief summary in 7.12 and this will also be discussed in the following chapter. It will deal with the asymptotic safety approach (AS) and the

[31] This parameter can be fixed (!) via the semi-classical black hole entropy, $\gamma \sim 0.24$. Its origin is not known and until now the only free parameter in LQG.

renormalisation group flow technique and how it might help to solve the problems that the theory of inflation faces.

It is beyond the scope of this thesis to analyse the need for quantum gravity. It is obvious that in the quest to a physically and mathematically consistent theory of inflation it is absolutely necessary to further look for a theory of *quantum gravity*.
Quantum cosmology might give us a theory of *initial conditions*. We should also investigate how to treat the beginning of inflation. For example, we have seen that we usually treat the background classically and 'add' quantum fluctuations. However, it is not clear how to couple the classical dof to the quantum dof and this treatment should only be seen as an approximation. If Inflation started from such a small Hubble patch, surely we should treat the system quantum mechanically only. 3.6.2 calculation has also shown that we might need to go beyond standard cosmology methods. It would be interesting to analyse their result in the presence of complexified SR potentials. The small digression also showed us that it isn't a priori clear that the universe would transform from some quantum to a classical object. We have also encountered one of the issues in QG approaches whether to treat the gravitational path integral Euclidean or Lorentzian. We have seen that the Wick rotation in the no-boundary proposal by H&H is problematic in GR. They rotate the time coordinate and the kinetic term has then the 'wrong' sign (this is the so-called conformal factor problem).
Why quantum gravity[32]/quantum cosmology+ (application of QG to the universe) in brief:

1. classical and quantum concepts seem to rule out each other
2. QFTs usually treated on fixed, non-dynamically background - in GR we have a dynamical curved spacetime which is the gravitational field itself
3. how to transition from quantum to classical?
4. lack of prediction beyond the Planck scale i.e. very high energies, large curvature, small distances
5. cosmological issues such as black holes and the Big Bang (curvature and spacetime singularities) are not explained by GR
6. the SM of particle physics has some intrinsic energy scale above which our QFTs aren't defined anymore
7. hierarchy problems (cosmological coincidence, gauge hierarchy)
8. the SM misses to include GR
9. fine-tuning problems (Λ, α_S, m_{Higgs}..., spacetime dimension?)

[32]One might also take the following Gedankenexperiment: Imagine you want to analyse a quantum box localised at a point. You will need to go to smaller and smaller scales, increasing the momentum uncertainty according to HUP which increases the energy i.e. mass of the box, so much that an event horizon will form with $r \sim m$ as we know from GR. Where did the box go?

10. we are faced with nonphysical divergences in both GR and QFTs (Landau poles...) and naive attempts of the quantisation of gravity
11. unification of all interactions and physical and mathematical theories (4.7)
12. explanation for 'fine-tuned' parameters in the universe
13. how why did space and time arise? (Does there exist a minimum length, is space-time fundamentally discrete? Is causality an emergent phenomenon? 'Problem of time'...)
14. absence of a fixed background, background independence
15. what where the initial conditions of our universe?
16. to keep theoretical physicists busy.
17. interpretation of QM
18. ...

The list goes on.
It would be a very satisfying result if inflation is purely a quantum gravitational effect.

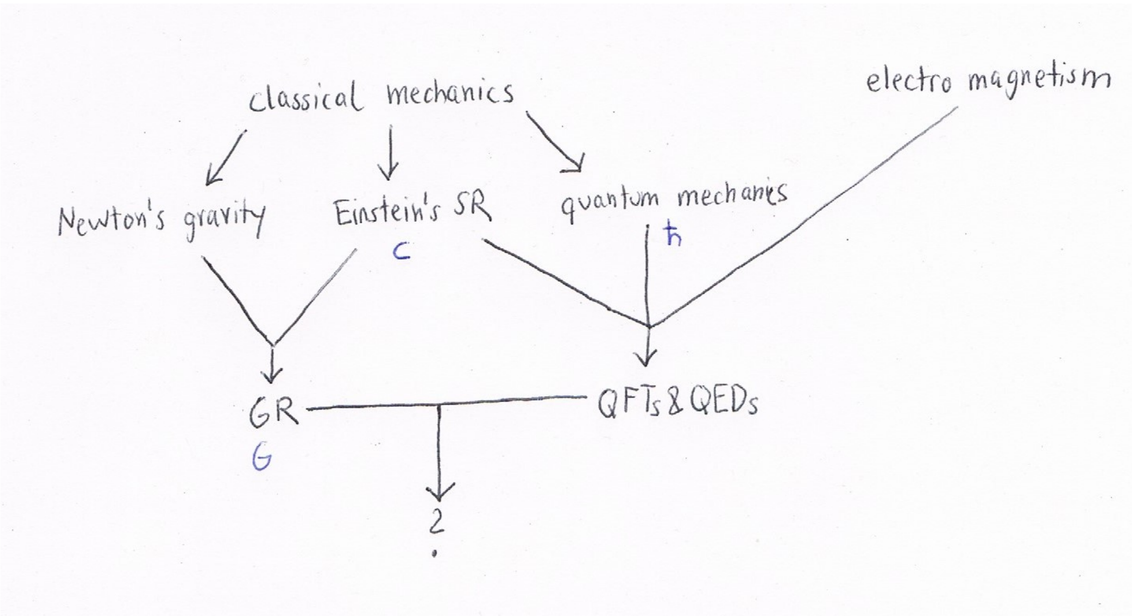


Figure 4.7: Sketchy Unification of Theories

Chapter 5

Asymptotic Safety and Inflation

Scientific theories cannot be deduced by purely mathematical reasoning.

(S. Weinberg — To Explain the World)

5.1 Asymptotic Safety and Renormalisation Group

Asymptotic safety[1] is a conjecture equipped with a consistent mathematical framework in which a QFT might be well-defined at all energies owing to an ultraviolet fixed point, and without the necessity of being perturbatively renormalisable. Its idea, originally brought up by S. Weinberg [85], is based on the assumption that classical GR might be the effective theory (= a theory at low energies, defined up to a certain scale **??**) of a consistent QFT at all energies. For this to occur, the necessary ingredients are a fixed point of the renormalisation group.[2]

The *coarse-graining* can be pictured with the help of the block spin renormalisation group by Kadanoff. Imagine a lattice of 2d with spin up or down at each point that only interact with their neighbours. He used invariance properties of critical points in statistical physics and asked the key question how the system's description might change if one replaces a block of spins by a single spin (by simple averaging). This comes along with a change of length scale i.e. a decrease of the number of degrees of freedom (new couplings) which is for atomic/molecular behaviour analyses favourable. By coarse-graining, averaging and rescaling he was able to show that new effective values can form that describe the systems just as good as the old ones (later, this will correspond to the fixed point since in this case the corresponding theory will always be mapped to itself). Importantly, the rescaling of the whole system is such that the original lattice size is reached. The so-called block spin transformation makes it possible that we lose

[1]The first subchapter was also used in an exposé of my application for a scholarship (Deutsche Studienstiftung). Some ideas of [59] are used in the following chapter.

[2]Weinberg (1979) first introduced the term AS in 'Ultraviolet divergences in quantum theories of gravitation' in 'General Relativity: An Einstein centenary survey' with the definition of a theory class with a well defined and predictive UV behaviour. As $2+\epsilon$ gravity seems to be asymptotically safe he wondered whether gravity can be made so as well. In order for this to occur, a UV FP has to exist with a finite number of UV attractive directions (=relevant critical exponents).

the notion of a reference scale, it looks the same no matter the resolution.[3] While in statistical physics we often have some initial condition and 'flow on' trajectories, here we start in the UV and flow along the trajectory towards the IR regime. We will see that the macroscopic physics (large distances) can be described by the knowledge of the fundamental interactions in the microscopic regime. The Wilsonian way is similar (with the analogue statistical physics → field theory, lattice spacing → cutoff, thermal/statistical fluctuations → quantum fluctuations). In 1971, Wilson combined the works of Kadanoff's block spin and Gell-Mann et al. RG treatment of β-functions.[4]
The starting point is the Euclidean path integral and an adjustable momentum scale k.

$$Z_k(J) = \int_{p^2>k^2} D\phi e^{-S(\phi)+J\phi} \tag{5.2}$$

where S is the bare or classical action, J is the corresponding current/source of the field ϕ, and the integration runs over all quantum fields ϕ with momenta larger than k. Importantly, the Euclidean signature makes it possible to define the direction of the RG flow. It is assumed, but not proven that the Euclidean FP goes over to the Lorentzian setting. Sending $k \to 0$ leads to the full physical theory. The modern way of introducing the cutoff in practice is by writing 5.2 as

$$Z_k(J) = \int D\phi e^{-S(\phi)-\Delta S_k(\phi)+J\phi} \tag{5.3}$$

where the term $\Delta S_k(\phi) \propto \phi \cdot R_k \cdot \phi$[5] with suitable requirements for the cutoff function $R_k(p^2)$ (that suppresses the fluctuations of fields below the momentum scale k i.e. it is an IR (!) regulator) implements the Wilsonian coarse graining [6]. The *effective average action* (EAA) $\Gamma_k(\phi)$ arises from $Z_k(J)$ via a suitable Legendre transform. The variation of Z_k or Γ_k with the RG scale k describes the integrating-out of quantum fluctuations.
Taking the $k\partial_k$ derivative of the EAA – which is equivalent to an infinitesimal variation

[3]The partition function is given by the same $Z = \mathrm{Tr}_d\, e^{-\mathscr{H}} = \mathrm{Tr}_{d'}\, e^{-\mathscr{H}'}, \mathscr{H} = \frac{H}{k_B T}$ for both spacings d, d' which are related by the projection P s.t. $\sum_{d'} P = 1$. We average over the spins in one block and under block spin transformation we project he critical point in the theory space onto itself $c \to c'$ ('RG transformation' r). The critical point is then given by

$$c_* = r(c_*), \quad c' = r(c), \quad c' \sim r(c_*) + (c - c_*)r'(c_*) + ... = c_* + d^{-\theta}(c - c_*), \theta = \frac{\ln r'(c_*)}{\ln d} \tag{5.1}$$

which we will see is the critical exponent that gives the stability properties of the FP.

[4]'Renormalization Group and Critical Phenomena. II. Phase-Space Cell Analysis of Critical Behavior', Wilson solves the critical behaviour for a generic Ising model by integrating out wave packets with momentum of greater than unity. He formulates the average action in the continuous space instead of Kadanoff's lattice approach.

[5]The cutoff is quadratic in the fields to get a 1-loop exact equation, see 5.4

[6]$R_k(p^2)$ takes finite values for small momenta. For $\frac{p^2}{k^2} \ll 1$, $R_k \to 1$ and exponentially decay for $\frac{p^2}{k^2} \gg 1$. Similarly, for $\partial_t R_k$ that takes finite values for $\ll \frac{p^2}{k^2}$, peaks at order of unity and vanishes for $\gg \frac{p^2}{k^2}$.

of k – one finds an exact functional flow equation for Γ_k,

$$\boxed{\partial_t \Gamma_k = \frac{1}{2} \mathrm{STr} \left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + R_k \right)^{-1} \partial_t R_k}\ ^{7} \tag{5.4}$$

which is the starting point of many calculations and was derived by Wetterich in 1991[**?**]. It is exact, there is no need for assumptions on convergence. We also had to safely remove the UV cutoff ($\Lambda \to \infty$), where the trace is well-defined, to arrive at an expression that is dependent only on k. The EAA connects any given initial action with the full quantum effective action. The importance of this equation will become clearer later.[8]

The supertrace sums over all momenta of all fields ϕ, all particles and particle types, spacetime and internal indices (with an extra minus sign for fermions, a multiple of 2 for complex fields). Sending k from large values in the UV to $k \to 0$ in the IR, the flow equation interpolates between the bare microscopic theory with the action S and the full quantum effective action Γ. The effective action can be expanded in '*theory space*' as

$$\Gamma_k(\phi_a, g_i) = \sum_i g_i(k) O_i(\phi_a) \tag{5.5}$$

where ϕ_a are various fields, O_i denote interaction terms and g_i the corresponding scale-dependent couplings. Basically, the basis of the theory space are the operators and the couplings are the coordinates. Given the exact RG flow of the functional (5.4), the exact beta functions for all couplings in the effective action (7.154) can be calculated. Although this flow equation is exact, in practice, it cannot always be solved exactly. Therefore, the functional differential equation must be 'truncated', meaning it has to be projected onto a small patch of the theory space. Most importantly, all orders are well described by equation 5.4, the resulting flow equations give the exact renormalisation group equations (ERGE), for its derivation see 7.13.2.

$$\xrightarrow[k\to 0 \quad \text{ordinary effective action}]{\Gamma_0} \xrightarrow[\text{effective action at } k]{\Gamma_k} \xrightarrow[\text{bare action,} \quad k\to\infty]{S}$$

The flow equation covers usual perturbation theory, but the advantage of RG flow methods is it to go beyond to answer questions that cannot be answered with perturbative methods.

For the theories of the standard model, it is a straightforward calculation as its couplings are dimensionless, for QED we can measure the electric charge and we are done. QCD is well-known asymptotically free [85] [89] meaning that the strong interaction of quarks becomes weaker at high energies and allows perturbative treatment, but at low energies it is strongly coupled (confinement). In general non-Abelian gauge fields (+fermions and scalars under certain conditions) are asymptotically free.

However, *gravity* is quite different from those theories since its coupling, the gravitational Newton constant G has a canonical mass dimension of -2, hence, goes like Gk^{-2}. If the canonical mass dimension satisfies $[g_i] \geqslant 0$ it is a renormalisable coupling

[7] This is sometimes also shown as circle (full propagator)+ $\otimes$ (derivative of the cutoff)

[8] Wetterich's derivation include both real space and momentum space. In FRG one usually calculates in momentum space.

constant, if it is < 0 it is non-renormalisable (UV divergent). A field theory is perturbatively renormalisable if no coupling constant possesses a negative mass dimension (and if all boson propagators $\frac{1}{p^2}$ behave like fermion propagators $\frac{1}{p}$) such as QCD, QED, ϕ^4 in 4d: g_s, the electric charge e and λ, respectively, are dimensionless. In the renormalisation group flow the constants have to be taken on the run, $G(k)$. For an asymptotically safe theory, it is assumed that an interacting UV fixed point exists, denoted NGFP (*non-Gaussian fixed* point meaning g_* is finite and nonzero). In QCD a Gaussian fixed point (GFP) is found, so $g_* = 0$, the Gaussian critical exponent is the canonical mass dimension, perturbation theory applies (non-negative dimension). Hence, if the theory fails to be asymptotically free, we generalise this method and look for the mentioned NGFP. Here, the fixed point is finite, but the beta functions for all couplings vanish at that point $\partial_t g_i = 0$.
The construction of the theory space and thus of the *dimensionless couplings* is needed.

$$g_i(k) = k^{-d_i} G_i(k) \tag{5.6}$$

where d_i is the mass dimension of the dimensionful coupling G_i. At every point in the coordinate system there is a possible operator O_i, equation 7.154, the couplings represent the possible directions and have a certain value at this point.

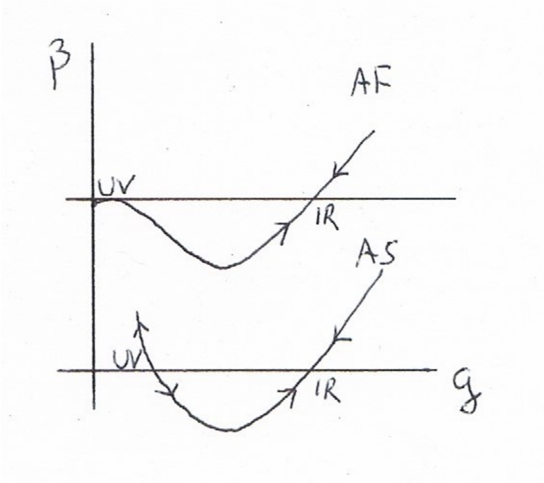


Figure 5.1: The β-function for asymptotic freedom (AF) such as YM and asymptotic safet (AS). The UV attractive FP is non-zero for the latter. The arrows indicate flow to the IR.

The set of these axes is the theory space. A *trajectory* is a line in the theory space that corresponds to all information of exactly this realisation of the theory for all couplings at all scales. Then, it is possible to measure the system at a scale k and make predictions for other scales (one might imagine that at scale k we use a microscope with a resolution of $\frac{1}{k}$). If we then assume that the cutoff k can be safely removed and taken to infinity and the trajectory ends at a NGFP, the theory is asymptotically safe (hence its name). Most importantly, the high momentum regime of the theory is scale-invariant, it looks the same for all resolutions.
A renormalisable quantum field theory can be defined and analysed at the fixed point. The UV fixed point should have a finite number of UV attractive directions as those correspond to the free parameters that have to be fixed by experiment (i.e. we look for a critical manifold of finite dimensionality). Obviously, the less the more predictive the theory. The set of points that flow to the fixed point is called UV critical surface.
The theory space is infinite, different truncations (ansätze) have to be made to organise the couplings and operators, it must be summed over a certain topology and

dimension.
For gravity, *diffeomorphism invariance* (a 1-1 map that moves a point in a manifold M to a new one and fulfils the notion of topological proximity; when a metric is introduced gravity has to be diffeomorphism invariant as we should be able to freely move points around the manifold[9] is the necessary condition with the metric as dynamical field. In GR the notion of active diffeomorphism is important, we cannot distinguish between fields that are related by diffeomorphism as we can always find a coordinate transformation in new coordinates. The second step is the implementation of gauge invariance (redundancy in the system). The background field method, where the metric is split into a background metric and fluctuations is commonly used.
Around the fixed point the linearised flow equation gives insight about its stability.
Until now the *Einstein-Hilbert action* was the focus of many studies (a good summary can be found in [72]). Usually, background field methods are applied, where the metric is split into a background and quantum fluctuation part [64] [36]. In the beginning, fixed points were found with the help of dimensional continuation for $d = 2 + \epsilon$ [22] [17] (the critical dimension for Einstein gravity is $d = 2$ since then Newton's constant is dimensionless). It has recently been explained that a resummed ϵ-expansion remains valid up to $\epsilon = 2$ (meaning $d = 4$) [21]. First indication for a UV fixed point in four and higher dimensions [64], via large N approximation[74], lattice methods [46], in higher dimensions [42], including matter [55], with the help of ERGE [64] [72] [41] and from a perturbative perspective [52] and recent fourth order studies [35]. The RG methods have been applied to a large class of theories including models that are perturbatively non-renormalisable, but were proven to be renormalisable at a nontrivial FP (e.g. Gross-Neveu model, 1985).
Nowadays, renormalisation group flow via *exact renormalisation group equations* and further optimisation methods are used, but other approaches should be recalled as well.
Applications to cosmology have also been investigated, including early cosmological conditions and inflation which I am be interested in here. All of the studies found remarkable *stable fixed points*, many of them are only weakly dependent on the gauge fixing. Fixed points have been found with a finite number of UV attractive eigenvectors (i.e. negative eigenvalues of the stability matrix), namely three.
Summarised, with the help of asymptotic safety we reduce an infinite number of constraints on couplings to a finite number of parameters that can be measured and predict the theory for all scales without the appearance of any nonphysical divergences. In contrast to other effective field theories, asymptotic safety gives extraordinary *predictivity* due to the extra conditions of the existence of a fixed point in a

[9]One distinguishes between active and passive diffeomorphism ϕ. Former is an invertible map on a d-dim manifold, $M \to M$, for say a scalar field A, $A : M \to \mathbb{R}$, a new scalar field $\hat{A}(p) = A(\phi(p))$ exists. This definition doesn't include dependence on coordinates. The function $a = A \circ x^{-1} : \mathbb{R}^d \to \mathbb{R}$ is defined by $a(x) = A(p(x))$ where x is the coordinate system. A passive diffeomorphism is an invertible, differentiable map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ that defines a new coordinate system $\hat{x}$ on M, $x(p) = \phi(x(\hat{p})$ where the function satisfies $\hat{a}(\hat{x}) = a(\phi(\hat{x}))$. Taking the metric as an example, $d(x, y)$ an active diffeomorphism produces a new metric, $\hat{d}(x, y) = d(\phi^{-1}(x), \phi^{-1}(y))$ - the metric forms are distinct, but isometric. The diffeomorphism acts on the space of metrics, for example, $g_{\mu\nu}, e^i_\mu$ the Riemmanian and tetrad formalism. A passive diffeomorphism, however, acts on the space of functions, such as $g_{\mu\nu}(x)$.

UV critical hyper surface of a well-defined RG. Einstein's theory has merely been the starting point for a more fundamental theory. Whilst the dimensionless couplings approach finite values, the dimensional constant, here Newton's constant, vanishes at high energies. It shows an anti-screening effect. An extensive review that I have used in the previous explanations can be found by Niedermaier and Reuter [65].

Asymptotic Safety

1. Assume that a suitable *fixed point* in the UV regime exits i.e. a continuum limit can be taken and it is safe from divergences.
2. The construction must give sufficient *predictive strength*.
3. The UV critical hyper surface develops to a regime where *classical gravity* is the IR approximation.

Asymptotic Freedom	Asymptotic Safety
GFP $g_\star = 0$	NGFP $g_\star \neq 0$
non-interacting	interacting
e.g. QCD	e.g. gravity
canonical power counting	non-canonical power counting

5.2 Renormalisation

When calculating the two point correlation functions of interacting quantum fields[10]

$$\langle\Omega| T\hat{\phi}(x)\hat{\phi}(y) |\Omega\rangle \tag{5.7}$$

To evaluate vevs for time ordered products one uses Wick's theorem which related it to the normal ordered product.

$$T(\phi_1, \phi_2 ... \phi_n) = N(\phi_1, \phi_2 ... \phi_n) + \text{all possible contractions} \tag{5.8}$$

The vevs are then all terms that are completely contracted

$$< \phi_1, \phi_2 ... \phi_n >= \sum \text{all full contractions} \tag{5.9}$$

Those contractions are given by the Feynman propagator $D_F(x-y)$ For $D_F(0) = \int \frac{dp^4}{(2\pi)^4} \frac{-i}{p^2+m^2-i\epsilon}$ corresponding to $x-x$, so loops, we get UV divergences. Whereas the first divergence (for large momenta) can be cured by introducing a vacuum difference, the second part needs further treatment. The trick is to use a resummation and to shift the mass term. The mass in the Lagrangian is infinite, measurements makes it finite. As we have elaborated this in detail in QED, briefly a sketch of explanation for, say $\lambda\phi^4$.

$$D_F + D_F D_F(0) D_F + ... = D_F(1 + D_F(0) + ... + (D_F(0)D_F)^n) = D_F \frac{1}{1 - D_F(0)D_F}$$
$$= \frac{1}{p^2 + m^2 - D_F(0)}$$

[10] Importantly, Ω is the vacuum in the interacting theory and not the same as the free $\langle 0|$ state.

where we can see the shift $m^2 \to m^2 - D_F(0)$. Similarly, along to the mass in all theories the coupling and the wavefunction need to be renormalised.
The main problem in gravity is that the natural unit of Newton's constant [G] is $M^{-2} = L^2$, it is a dimensionful coupling constant and even worse a negative mass dimension. It is important to emphasise Einstein's field equations with G:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{G} T_{\mu\nu} \tag{5.10}$$

One could start with a perturbative treatment.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5.11}$$

Minkowski's flat metric plus some fluctuations, $|h_{\mu\nu}| << 1$. The Einstein-Hilbert action should be dimensionless, in four spacetime dimensions constraints the Lagrangian density to $[M]^4$. The Ricci scalar as second derivative, $\sim \delta_\mu \delta_\nu g_{\alpha\beta}$, has units of $[M]^2$ and

$$\sqrt{-g} R \to (\delta h)^2 + (\delta h)^2 h + ... \tag{5.12}$$

in the action which shows that $[h] = L^{-1}$. In order to account for the dimensions I set $h = \hat{h}\sqrt{G}$. The action becomes[11]

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R \quad \to \quad S = \frac{1}{16\pi} \int d^4x ((\delta\hat{h})^2 + \sqrt{G}(\delta\hat{h})^2 \hat{h} + ...) \tag{5.13}$$

the first part is the free part (2 degrees of freedom - gravitons) and the second part is the interaction with the coupling constant $\sqrt{G}$ with a dimension of length. Increasing to more orders in a perturbative treatment we will then need more factors of momenta in the numerator to keep everything dimensionless. The theory is perturbatively non-renormalisable. It follows that the theory lacks of predictive power.
For a brief overview about the quantisation approaches of gravity see **??**. (a) The metric split into fixed background and fluctuations, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with the Minkowksi metric as 0th order approximation is limited by perturbation theory. (b) The Feynman quantisation of GR was motivated by the fact that spacetime is the gravitational field (other than, for example, the EM field which is embedded in spacetime). Physical events are individual spacetime points. The sum over all histories h between two hypersurfaces σ_1, σ_2 with boundary values f_1, f_2,

$$(f_2, \sigma_2 | \sigma | f_1, \sigma_1) = \frac{1}{N} \sum_h O_h e^{iS_h} \tag{5.14}$$

where $N \sim \sigma_1 \sigma_2$ should make the expression unitary. Due to gauge invariance, however, infinitely many histories are equal to the same physical situation.
Stelle showed in 1977 quadratic terms in the curvature action can renormalise the theory. Unfortunately, the theory would suffer from ghost terms that make it unphysical. From perturbation theory the theory is non-unitary. In fact, it can be restored by infinite many higher derivative operators, but then one would lose again predictivity (Weinberg, Gomis 1996).

[11]For future treatment: $\Gamma \sim \partial h, R^\mu_{\nu\alpha\beta}, R_{\mu\nu}, R \sim (\partial^2 h, \partial h \partial h)$

Donoghue treated GR as *low effective theory*, still leaving the high-energy behaviour untreated. He claimed that the action of a theory of quantum gravity should include all possible interactions that are consistent with the symmetries of the theory at low energies (thus, general covariance and local Lorentz invariance[12]).
To be precise,

1. *general covariance* also called diffeomorphism covariance, GR is invariant under active diffeomorphism (see before), meaning under arbitrary, differentiable coordinate transformations.

2. *Lorentz invariance* (Lorentz covariance for scalar) as property of the given spacetime manifold, if a term/equation is valid in one inertial frame, it is valid in another inertial frame. Locally, GR is described by SR. Lorentz invariance is an exact local symmetry in GR (whereas Poincare symmetry is only approximate) since the tangent space is completely unaffected by any curvature.

SR is ruled by $ds^2 = dx^\mu dx^\nu \eta^{\mu\nu} = ds'^2$ under Lorentz transformation

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{5.15}$$
$$dx'^\mu = \Lambda^\mu_\nu dx^\nu \tag{5.16}$$
$$\eta_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \eta_{\alpha\beta} \tag{5.17}$$

GR on the other hand doesn't specify the coordinate transformation, a general $\Lambda^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$ leaving the line element invariant, $ds^2 = dx^\mu dx^\nu g^{\mu\nu} = ds'^2$. Again, under small coordinate transformation

$$d^4x' = \det \Lambda d^4x \tag{5.18}$$
$$g'_{\alpha\beta} = (\Lambda^{-1})^\mu_\alpha (\Lambda^{-1})^\nu_\beta g_{\mu\nu} \tag{5.19}$$
$$\det g'_{\mu\nu} = \det \Lambda^{-2} \det g_{\mu\nu} \tag{5.20}$$
$$d^4x'\sqrt{-g'} = d^4x\sqrt{-g} \tag{5.21}$$

5.3 Coupling constants

The reason we want finite couplings and a finite number of couplings as free parameters is that couplings correspond to *observables* and those observables should be measurable and finite in order to be physical.
The coupling in electrodynamics is coupled to charge, in gravity it is coupled to stress-energy.
The most studied truncation of Einstein Hilbert contains two coupling 'constants' with canonical mass dimensions,

$$[\Lambda] = 2, \quad [G] = 2 - d \tag{5.22}$$

the cosmological constant and Newton's coupling in d spacetime dimensions.
Now the coupling constants aren't constant but depend on the energy scale. This was

[12] Covariance is defined as the invariance of the form of a physical law under a given transformation.

already proven for the electric coupling whose running can be calculated with the help of the WARD identities from the wave function of the photon (or the electron-photon vertex). Similarly, the QCD coupling can be calculated with the help of Slavnov-Taylor identities from different vertex and propagator diagrams. Famously, QCD is asymptotically free at high energies. Recall also, that we have already seen a running of couplings in 7.8.

1. QED: $\mathscr{L}_{QED} = \overline{\Psi}(i\gamma_\mu\partial^\mu - m)\Psi - e\overline{\Psi}\gamma^\mu\Psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with the coupling e, the charge of the fermion
2. QCD: $\mathscr{L}_{QCD} = \overline{\Psi}(i\gamma_\mu\partial^\mu - m)\Psi - g_i\overline{\Psi}\gamma_\mu T_i\Psi G_i^\mu - \frac{1}{4}G_{i\mu\nu}G_i^{\mu\nu}$ with the couplings $g_i, i = 1, ..., 8$, the strong couplings

Those are the free parameters that need to be measured in the experiment. It was validated that the value of the couplings, from now on $\alpha_{EM} = \frac{e^2}{4\pi}$ and $\alpha_S = \frac{g_S^2}{4\pi}$ is indeed dependent on $\sqrt{Q^2}$, the momentum ($\sim$ energy) at which the measurement takes place. The electric charge increases with smaller distances (= higher momenta), but saturates to $\frac{1}{137}$ at low energies.[13] The coupling strength of the strong coupling, however, shows an opposite behaviour. The colour charge increases with the distance from the bare quark charge (confinement) and becomes asymptotically free (arbitrarily small) at smallest scales (antiscreening).
The gravitational constant should show *antiscreening* behaviour as well. Just like the improvement from Coulomb to Uehling potential.[14]
Weinberg [86] emphasised that the negative mass dimension of the gravitational constant is the root of gravity's non-renormalisability in perturbation theory. A Feynman diagram of order N behaves at large momenta p like $\int p^{A-Nd}dp$ where A is dependent on the process and d is the mass dimension of the coupling. For $d < 0$ such as Newtons constant, $[G] = -2$, this integral diverges for sufficiently high orders. He actually proved the renormalisability for gravity in $2 + \epsilon$ dimensions. However, the problem of 4 dimensions remained first unsolved.

5.4 The Advantage of Asymptotic Safety

First of all, inflation takes place at early times and high energies scales were a theory of quantum gravity is needed[15].
The idea of asymptotic safety is that an infinite number of counterterms combine in a way such that predictions can be made. AS/its RG flow connects large and small scales

[13] Reason is that at smaller distance, bigger Q, the screening behaviour is vanishing small , the coupling can become large.

[14] $V_{Coulomb} = -\frac{e^2}{4\pi r}$ is improved by $e^2 \to e^2(k) = e^2(k_0)(1 - b\ln\frac{k}{k_0})^{-1}, b = \frac{e^2(k_0)}{6\pi^2} \to V_{Uehling} \sim -\frac{e^2(\frac{1}{r_0})}{4\pi r}(1 + b\ln(\frac{r_0}{r})), r_0 = \frac{1}{k_0}$, first treatments by Uehling, 1935, see also 7.13.4. The minus sign indicates screening behaviour. For gravity we intuitively expect a cloud of virtual gravitons that increase the effective mass measured from far away i.e. produce an antiscreening effect for a test particle, we expect something of type $G(k) = G_0(1 + aG_0k^2)^{-1}$.

[15] For a short overview of QG/quantisation approaches, see **??**

and hence, helps to identify the relations of large scale quantities and microscopic ones ($\sim$ LSS, quantum fluctuations). Woodland (2009) gives an example for a higher order Lagrangian[16]. The Lagrangian gets an additional second derivative term

$$\mathscr{L} = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}mw^2q^2 - \frac{1}{2}\frac{gm}{w^2}\ddot{q}^2, \quad 0 < g << 1 \tag{5.23}$$

With the standard boundary condition $q(0), \dot{q(0)}$ this cannot be solved due to the higher derivative (HD) terms in the equation of motion. Substituting the ansatz $q(t) = \sum_{n=0}^{\infty} g^n x_n(t)$ gives via the standard perturbation theory approach for each order[17]

$$\begin{aligned} \ddot{x}_0 + w^2 x_0 &= 0 \\ \ddot{x}_1 + w^2 x_1 &= -\frac{1}{w^2}\frac{d^4x_0}{dt^4} \\ \ddot{x}_2 + w^2 x_2 &= -\frac{1}{w^2}\frac{d^4x_1}{dt^4} \\ &\dots \end{aligned} \tag{5.25}$$

where the sources are always evaluated at the previous order. This can be solved exactly with the ansatz $q(t) = A\cos(k_1 t) + B\sin(k_1 t) + C\cos(k_2 t) + D\sin(k_2 t)$, $q_0(0)$ and $\dot{q(0)}$ and integrating the 0th order etc.

$$q = q_0\cos(kt) + \frac{\dot{q}_0}{kt}\sin(kt), \quad k = w\frac{\sqrt{1 \pm \sqrt{1-4g}}}{\sqrt{2g}} \tag{5.26}$$

The HD term has an effect, it shifts the frequency depending on the coupling constant. Still it gives a predictive theory if we disregard the negative mode since no HD dofs are created. If the negative mode is taken there is a negative dof.[18] Can we use the same approach for quantum gravity? For low energy modes we can use standard perturbation theory, for higher energies the theory isn't predictive anymore.
There are two challenges that one encounters:

1. changes in lower derivative dof
2. new dofs by higher derivative terms (with opposite kinetic energy)

Stelle [79][78] showed that $\frac{1}{16\pi G} + c_1 R^2 + c_2 R^2_{\mu\nu}$ is renormalisable, but contains ghosts. The spectrum of particles contain massive negative energy spin 2 modes. Taking general covariance as constraint, the general effective Lagrangian density is of the type

$$\mathscr{L} = \sqrt{-g}(\lambda + \frac{m_p^2}{16\pi}R + aR_{\mu\nu}R^{\mu\nu} + bR^2 + cR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{d}{m^3}R^3 + e\Box R + \dots) \tag{5.27}$$

[16]HO Lagrangians give higher derivative terms in the equation motion.
[17]The general Euler Lagrange equation is given by

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} \tag{5.24}$$

-same derivation as for the 1-derivative case.
[18]The amplitudes are $A/C = \frac{q_0 k^2_{2/1} + \ddot{q}_0}{k^1_{2/1} - k_{1/2}}^2$ and $B/D = \frac{\dot{q}_0 k^2_{2/1} + \dddot{q}_0}{k_{1/2}(k^1_{2/1} - k_{1/2})^2}$.

where a,b,c,d,e are all dimensionless and m should be the smallest scale (dimensional power counting), the first term is proportional to the cosmological constant, the second is the usual Einstein-Hilbert term. Those terms are not all independent (before investigating the action one should look for total derivatives or terms that vanish when we evaluated the solutions of eoms below that order). In four dimensions the curvature squared terms are related to a topological invariant[19] (→ total derivative),

$$\chi = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R^{\mu\nu}_{\mu\nu} + R^2 \tag{5.28}$$

Hence, up to second curvature order, one of the terms can be substituted, e.g. $(a, b, c) \rightarrow (a - c, b + 4c, 0)$. Stelle investigated the (a, b) case and showed that the additional term modifies gravity at small scales. In fact, the term can also be rewritten in terms of the Weyl squared tensor.

$$C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} \sim R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 \tag{5.29}$$

in 4 dimensions which I will explain more in detail later.

$$S = \int d^4x\sqrt{-g}(-\alpha C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} + \beta R^2 + \gamma R) \tag{5.30}$$

The advantage of asymptotic safety is that it predicts finite parts for all counter terms in such a way that the theory can be used at *ALL* energy scales. Other schemes such as the normal regularisation with a UV cutoff or perturbation theory (infinite coefficients, finite many counter terms can cure divergences order by order) weren't able to solve the non-renormalisability.

5.4.1 Effective Field Theory

We know that the fundamental theory (whatever it may be and we may call it quantum gravity) has to reduce to classical GR at low energies which has been tested and validated. Approaches such as String Theory or LQG suffer from the transition problem (HE to LE), but as we have and will see(n) AS and its functional RG are based on the connection of the UV and IR regime.

An effective field theory is a theory that is valid up to a certain energy scale. Quantum loop corrections don't change predictivity at low energy values. GR can be classified as EFT that doesn't need to be quantised itself, but as low energy theory of a more fundamental theory. So far, GR and QFT, have been tested and validated and are mathematically consistent in their regimes. Together they are well defined at all accessible scales.

The quantum correction for Newton's potential[20] (Donoghue, [15])

$$\hat{V} = -\frac{Gm_1m_2}{r}\left(1 + \frac{3G(m_1 + m_2)}{r} + \text{const}\frac{G\hbar}{r^2c^3} + ...\right) \tag{5.31}$$

[19] For a compact manifold of two dimensions, the Euler number is $\chi(M) = \frac{1}{32\pi^2}\int_M \sqrt{-g}Xd^4x$. This can be extrapolated to higher (even) dimensions.

[20] Basically analogue to QED's Uhling potential, the quantum corrections to the Coulomb potential.

where I re-introduced $\hbar$ and c.[21] The first part is the usual Newtonian potential, the second part is from GR only, the third term is due to quantum gravitational effects, already indicated by the dependence on G and $\hbar$.[22]
He first calculated [16] the gravitational scattering amplitude between two masses to define the potential (to one loop order) in momentum space.

$$\langle f|\, T\, |i\rangle = (2\pi)^4 \delta^{(4)}(p-p')M(q) = -(2\pi)\delta(E-E')\langle f|\, \hat{V}\, |i\rangle \tag{5.32}$$

following usual QFT rules (p incoming, p' outgoing momentum) gives the tree-level contribution

$$M_1(q) = -\frac{4\pi G m_1 m_2}{q^2} \quad \leftrightarrow V_1(r) = -\frac{G m_1 m_2}{r} \tag{5.33}$$

in coordinate space[23] and similar calculations for higher orders graphs resulting in 5.31 where I also added the usual special relativistic contribution via part of the triangle contribution as well as the double seagull and vertex correction and vacuum polarisation (calculations can be found in Donoghue's works, see also 7.12.2).
At the Earth's surface the correction for const.$= \frac{41}{10\pi}$ is of order 10^{-84}, almost vanishing. Nonetheless, we cannot simply ignore its quantum effects.
The quadratic Stelle theory in linearised gravity gives a spherical symmetric static solution in terms of a Newton and two Yukawa potentials (or unboundedness at infinity which has to be eliminated by boundary conditions)[78].

$$V = \frac{-2MG}{r} + \frac{8GM}{3r}\left(e^{-m_2 r} - \frac{1}{4}e^{-m_0 r}\right) \tag{5.35}$$

with m_0 corresponding to the scalar and m_2 to the spin 2 ghost (see later, 7.13.5).
The problem of non-renormalisability becomes an actual problem at high energies. At low energies the scale dependence between the characteristic energy scale of the experiment E and the heavy scale up to the theory is defined M, $\propto \frac{E}{M}$, suppresses the higher order divergent part since at low energies $E << M$. In general EFT we fix the level of precision and sort by $\frac{E}{M}$ at a given order all terms that contribute to the action. As mentioned and below explained in detail we cannot use standard perturbation theory to quantise. The root of the problem lies in the negative dimension of the gravitational constant. In four dimensions we can define the dimensionless coupling

$$g = GM^2 \tag{5.36}$$

where M is the typical energy scale (this plays an important role in AS as well).
The idea behind EFT is to have a well defined and predictive theory up to that scale where the theory cannot make any predictions anymore. First, one has to identify the

[21] There is also another treatment by him which gives an antiscreening effect, 1994. Here, $G(r) = G\left(1 - \frac{\tau G_N \hbar}{r^2}\right) + G\left(1 - \frac{\tau G_N \hbar}{r^2}\left(1 - \frac{\tau G_N \hbar}{r^2}\right)\right) + ... = \frac{G}{1+\frac{\tau G_N \hbar}{r^2}}, \tau = \frac{167}{30\pi}$.

[22] In momentum space the terms correspond to $\sim -\frac{1}{q^2}, -\frac{G\sqrt{q^2}}{q^2}$ and $-\frac{G q^2 \ln q^2}{q^2}$, respectively.

[23]

$$V(x) = \frac{1}{4m_1 m_2}\int \frac{d^3q}{(2\pi)^3} e^{iqx} M(q) \tag{5.34}$$

most general Lagrangian (and the low energy dofs) and order the terms by their energy order. One starts with the renormalisation of parameters at lowest order. Then, one 'connects' the theory to the real world by conducting experiments to measure the parameters. The ones that are left are the predictions of the theory. With a suppression scale of Planck size $\sim 10^{18}$GeV, the inflation energy scale $\sim 10^{16}$GeV, our testing possibilities at the LHC $\sim$ 10TeV and direct testing of gravity ($\sim 60-300\mu$m$\leftrightarrow$ 0.1eV) are well below m_p. So far we seem to have evaluated two different theories, the low energy regime with $k_{\text{late}} = H_0 \sim 10^{-33}$eV and the high energy regime with $k_{\text{inf}} = H_{\text{inf}}$. Additionally, we have also seen that in Starobinsky inflation the R^2 should dominate at early times and EH at low energies. The need for sufficient inflation gives constraints on the coefficient of R^2 and today's measurements give constraints on the cosmological constant and Newton's constant.
Since there is some overlap between an EFT and the RG approach let us summarise how to use EFT in a theory as inflation (Donoghue)

1. *identify* the low energy degrees of freedom and the symmetries of the underlying theory (GR+inflation: needs to reproduce usual EH and on cosmological setting the LSS today, general covariance, local Lorentz invariance and any other symmetries that are constraint by the underlying inflation model)

2. write the most *general effective Lagrangian* (e.g. EH, for Starobinsky inflation as an R^2 term) which should be ordered by local energy expansion

3. calculate the action, starting with the lowest order

4. renormalise

5. *match* your findings (use observational data or measure free parameters in experiments)

6. now you can use the theory to give further *predictions*

To see the direct difference between an EFT and the RG treatment we conclude that an EFT is valid up to a certain energy scale whereas the RG is defined at *all* scales. Both can be applied in truncations (in EFT the ground state would be the EH term + inflaton term) and both cover symmetry principles such as general covariance (and any other symmetry the inflaton brings into the theory).
Take, for example, the multipole expansion of electrostatics, $V(\vec{r}) = \frac{1}{r}\sum_{l,m}\frac{b_{lm}Y_{lm}}{r^l}$ with a spacing of d, $r >> d$, we can rewrite the coefficients into dimensionless $b'_{lm} = \frac{b_{lm}}{a^l}$. We can treat two different scales, the IR on r scale and the UV on d scale. If we measure the b coefficients we can calculate b' (either by distances or momenta, $r \sim \frac{1}{p}$ for the IR and $a \sim \frac{1}{\Lambda}$ for the UV) and estimate the behaviour of the UV from the IR. However, we won't be able to make predictions when our experiment has reached its maximal resolution given by some maximum l. To improve this we can either improve the resolution, more precise measuring in the LE limit or measure at HE. In the RG treatment we don't suffer from this issue when we assume that a FP exists. We have already seen that in the Kadanoff picture, the RG technique is based on scale invariance at the FP, the behaviour is the same independent of the resolution scale. In EFT one often expands around the small parameter $\alpha = \frac{a}{r}$, in the FRGE

there is no small parameter and the Wetterich equation makes the EXACT treatment possible. However, we need to find an identification and RG improvement. A generic EFT treatment of quantum gravity would include all diffeomorphism invariant terms, such as Riem^2, but also the ∇^2 term, there are many different truncations that have been analysed in AS.

5.5 Einstein-Hilbert Truncation

As already shown Einstein's GR is perturbatively non-renormalisable.

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R \tag{5.37}$$

Expanding the metric around Minkowski, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, with $\kappa = 8\pi G_N$, produces infinitely many powers of $h_{\mu\nu}$ which is the spin 2 field (gravitons, 2dof). Each term carries a quadratic term in momentum, such that the overall divergence X in mass units is

$$X = l(d-2) + 2 \tag{5.38}$$

[spacetime dimension]-[momentum]^2+[2 R derivatives], at loop order l and a certain order in fluctuation expansion. It is not possible to shift those infinities into the parameter (which would be G_N here). The biggest divergence occurs if we have loops only - taking diffeomorphism symmetry into account we need to add the cosmological constant (we want a momentum independent diffeomorphism invariant term). It follows in -2 steps, first we add two external momenta (equals 2 derivative term), then 4 etc. We cannot have odd terms as we wouldn't be able to contract them to invariant scalars. The divergent terms that are produced from the Ricci scalar can be put into Newton's constant but in the end we are left with a $\sim$ log divergence at 1-loop order. That is the reason 't Hooft et al. (for one-loop) [92] and later Goroff et al. (higher loops) [70] added curvature squared terms and higher, respectively.[24] It was shown that the 1-loop corrections (in 4 dimensions) produces UV divergences in form of curvature squared terms. In dimensional regularisation their result was

$$\begin{aligned} \Gamma^{(1)} &= \frac{1}{d-4} \int \sqrt{g} \left(\frac{1}{120} R^2 + \frac{7}{20} R_{\mu\nu} R^{\mu\nu}\right) \\ \Gamma^{(2)} &\sim \frac{G}{d-4} \int \sqrt{g} R^{\mu\nu}_{\alpha\beta} R^{\sigma\rho}_{\mu\nu} R^{\alpha\beta}_{\sigma\rho} \end{aligned} \tag{5.39}$$

The Riemann squared term is eliminated by the Gauss Bonnet theorem[25] (it can be re-expressed in terms of R and $R_{\mu\nu}$ an then produces a topological invariant - note

[24] The divergence at two-loop level can be paramterised as $\sim \sqrt{-g} R^{\alpha\beta}_{\mu\nu} R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\alpha\beta}$.

[25] A topological invariant is a property of a topological space that is invariant under homeomorphism. An example is the Euler characteristic χ. On a compact orientable (Riemannian) manifold M in 2 dimensions

$$\int_M K dA + \int_{\partial M} k_g ds = 2\pi\chi(M) \tag{5.40}$$

is valid (Chern, Shen), K is the Gaussian curvature, k_g the geodesic curvature of the boundary ∂M and the integration is over the surface dA of M and the line element ds of the boundary. This result was

that this is not the case in $d \geqslant 6$ which does include Riemann squared terms.) In pure gravity divergences can be put into field renormalisations, but as soon as we allow interaction (we seem to live in a universe with matter - 'matter matters', Eichhorn et al.) divergences occur that cannot be eliminated, not even when we employ the 'improved energy momentum tensor'[26].
With each term new couplings arise whose low energy values should then be evaluated in measurements. However, infinite values need infinitely many experiments and the theory loses its predictivity.
Hence, EH cannot be perturbatively quantised without losing predictivity.
In asymptotic safety we find a NGFP in the UV making gravity well-defined at high energies and the theory is non-perturbatively renormalisable.
An overview on the quantisation of gravity and the literature background can be found in the appendix, 7.12.
In GR, gauge invariance causes some problems as there exist infinite gauge copies in the path integral. Hence, we need to fix the gauge. However, some gauges introduce non-physical dofs. They can be eliminated by the Feynman-DeWitt-Faddeev-Popov trick[27] 7.12.1.

5.5.1 RG Flow

A QFT is entirely defined by the n-point correlation functions $< \phi_1, ...\phi_n >$ that can then give rise to physical observables. In the Wilsonian idea of renormalisation we need to mathematically formulate a coarse graining where dofs are integrated out and make the theory well defined and predictive at all scales. We start with the already mentioned path integral [42][41]

$$\boxed{\int \mathcal{D}\phi e^{-S[\phi]+\Delta S_k[\phi]+J_i\phi^i} = e^{W[K]}} \tag{5.43}$$

with the external sources J for each field ϕ, the usual action S and regulating term $\Delta S_k = \frac{1}{2} R_k^{ij} \phi_i \phi_j$ and W_k the Schwinger functional. The effective average action can be calculated (= generating functional for 1PI correlation functions). Regulator functions should satisfy the following conditions:

1. $\frac{p^2}{k^2} \to 0$, $R_k(p^2) > 0$.

---

proven for higher (even) dimensions as well (Weil, Allendorfer). For $2n = 4$,

$$\chi(M) = \frac{1}{\sqrt{(2\pi)^2}} \int_M Pf(-R) = \frac{1}{32\pi^2} \int_M (R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2 = \frac{1}{4\pi^2} \int_M (\frac{1}{8} C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} + Q \tag{5.41}$$

where Pf is the Pfaffian integral of the associated curvature form of the Levi-Civita connection and Q is dependent on the Ricci scalar and tensor such that one can conclude

$$C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} \sim R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 + \text{vanishing terms} \tag{5.42}$$

Obviously, if this term couples to matter it doesn't vanish anymore.

[26]The Lagangian gets two extra terms $\sim aR\phi^2, bR^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. See also 'A new improved energy momentum tensor', 1969

[27]Exponentiating gauge fields as ghost particles.

2. $\frac{k^2}{p^2} \to 0$, $R_k(p^2) \to 0$.

3. full quantum theory for $k \to 0$

4. microscopic action (classical in perturbative setting) $k \to \infty$

The modified Legendre transform of the Schwinger functional gives the effective average action Γ_k which can then be fed into the Wetterich equation (wrt the RG time $\partial_t = k\partial_k$).
In the Wetterich equation the derivative wrt RG time peaks at $q^2 \sim k^2$ and vanishes for $q^2 \to \infty$. The $\sim (q^2 + r_k(q^2))^{-1}$ takes finite values for $q \to 0$, k^{-2} and is decreasing for higher momenta until it vanishes for $q \to \infty$ as well, 5.2. To write it in some neater form:

$$\partial_t \Gamma_k = \frac{1}{2} \delta_i^k \left(\Gamma^{(2)} + R_k\right)^{-1}_{kj} \partial_t R^{ij} \tag{5.44}$$

To summarise the advantages of renormalisation group flow in this setting I will give the most important properties [65]:

1. the RG flow contains *all information*, if one knows the full effective action, one has the full 'solution' of the problem.

2. finite in the UV and IR, a well defined and predictive theory

3. Wetterich[87] realised that the equation is basically a loop with regulator insertion $\dot{R}_k$

4. one arrives at a PDE with the boundary conditions at a scale $k = \Lambda$ (if the theory was perturbatively renormalisable one would have $\Gamma_{k=\Lambda} \sim S_{cl}$[28]

5. we have a well defined QFT, in the perturbative regime and beyond.

6. the RG flow can undergo well chosen truncations and optimisations. Importantly, the quantum effective action should not depend on the regulator choice.[29]

7. Importantly, the cutoff described includes the IR cutoff (all modes below k get an effective mass) and we have implemented the k-dependence.

[28]It isn't that obvious that it remains finite as we assume that a global flow with well defined boundary conditions exists. This leads us to the need of truncations.

$$\dot{\Gamma}_k = -\frac{1}{2}\partial_t \log F_i^i|_i^i - \frac{1}{2}\dot{\Gamma}_k^{(2)^{ij}} F_{ij} \tag{5.45}$$

with F being the 'loop part'. Naively integrating the rewritten Wetterich equation would lead to divergences for $\Lambda \to \infty$

$$\Gamma_k = \Gamma_\Lambda - \frac{1}{2}\left(\log F_i^i|_\Lambda^k + \int_\Lambda^k dt \dot{\Gamma}^{(2)}_{kij} F^{ij}\right), \quad \lim_{\Lambda\to\infty} \Gamma_k \to \infty \tag{5.46}$$

[29]Common regulator choices are the exponential cutoff $\sim \frac{p^2}{e^{\frac{p^2}{k^2}}-1}$, Litim's optimised cutoff $\sim (k^2 - p^2)\theta(k^2-p^2)$ and the sharp cutoff $\sim \frac{p^2}{\theta(k^2-p^2)} - p^2$[43]. The Litim cutoff is 0 for momenta above k^2 and for momenta below the propagator $\sim \frac{1}{p^2+R_k}$ becomes constant as itself adds a momentum dependent mass term $\sim k^2 - p^2$ such that ALL IR modes are treated with the same weight.

An example can be found in the appendix 7.13.1.
The properties of the physical system we want to analyse are encoded in the critical exponents near the fixed point. The solution we aim for are the flow equations in the manifold $\mathscr{S}$,

$$\dot{\hat{g}}_i = \beta_i(\hat{g}) \tag{5.47}$$

where the β-functions can be interpreted as vector field, $\underline{\beta} = (\beta_i)$. A fixed point is a point in that manifold that vanishes for ALL

$$\boxed{\beta_i(\hat{g}_*) = 0} \tag{5.48}$$

(using the Kadanoff picture, the correlation length at the FP is equal to the scaled length, $\zeta(\hat{g}_*) = \frac{\zeta(\hat{g}_*)}{b}$, the RG time is $t = \ln b$). The FP is fully scale invariant, the critical FP exits for $\zeta(\hat{g}_*) \to \infty$. The set of all points under the RG flow projected on the FP is called the critical surface (IR), under the inverse flow we project onto the commonly used *UV-critical surface* (flow starts at FP and moves away from it). We will see that the dimension of this surface should be finite and so far 3 dimensional[30]. The lower its dimension the better the theory as the dimension corresponds to the number of free parameters that we need to get from experiments.
Back to the stability question. Around the FP we can linearise the flow, $\hat{g}_i = \hat{g}_{i*} + \delta\hat{g}_i$

$$\dot{\hat{g}} = k\partial_k \hat{g}_i = -\beta_i(\hat{g}) = -\sum M_{ij}\delta\hat{g}_j, \quad M_{ij} = \frac{\partial}{\partial\hat{g}_j}\beta_i(\hat{g_*}) \tag{5.49}$$

where M is the stability matrix. The 'perturbation' can be written in terms of fields and coefficients whose rate of change wrt RG time is proportional to itself and the so-called critical exponents.[31]

$$Mv^a = -\theta_a v^a, \quad \delta\hat{g}_j = \sum_a \omega_a v_j^a, \quad \dot{\omega_a} = \theta_a\omega_a \leftrightarrow \omega_a = \omega_a^0 (e^t)^{\theta_a} \tag{5.50}$$

The sign of the real part of the critical exponents θ_a describes the behaviour under the RG flow.

1. $\mathrm{Re}\theta > 0$: relevant (grows under RG flow)
2. $\mathrm{Re}\theta < 0$: irrelevant (decreases under RG flow)
3. $\mathrm{Re}\theta = 0$: marginal (HO terms decide on its behaviour)

[30] Even including HD operator the dimension seems to be robust. Some investigations are currently done with quadratic operators, such as Riemann^2 which seem to give a four dimensional surface.

[31] It isn't proven, but likely assumed that those are *universal* which means that the behaviour near the critical point depends on the critical exponents rather than from other details of the theory. With the critical exponents we could define universality classes i.e. systems of same IR behaviour. We already know from statistical physics that the behaviour of systems near phase transitions is quite similar, the correlation length proportional to the power of the temperature-critical temperature difference $m^{-1} \sim |T - T_c|^{-\theta}$, with the critical exponent given by $\theta = \left(\frac{\partial \ln m_0^2}{\partial \ln m^2}\right)^{-1}$ can be found in specific heat, spontaneous magnetisation and other formulae (see e.g. Stanley, 1971).

The number of positive θ, m, in the n-dimensional parameter space gives the finite (and small) dimension of the UV critical hypersurface (for the RG time $t \to -\infty$ the coefficients $\omega^0_{m+1}, ..., \omega^0_n = 0$ leaving m coefficients of free choice), dim$= m$. Note for the IR critical surface we would have the opposite behaviour, such that its dimension is $n - m$.
Now applying this RG analysis to the FRGE of asymptotic safety with the dimensionless couplings

$$\hat{g}_i = g_i \Lambda^{-[g_i]} = \hat{g}^0_i e^{[g_i]t}, \quad \hat{g}^0_i = \Lambda(\hat{g}_o)^{-[-g_i]} g \tag{5.51}$$

gives the β-functions

$$\dot{\hat{g}}_i = -\beta_i(\hat{g}) = -[g_i]\hat{g}_i \tag{5.52}$$

A FP where all β-functions vanish can be Gaussian if $\hat{g}_{i*} = 0$ for all i (with critical exponents equal to the corresponding mass dimension of the dimensionful coupling, $\theta_i = [g_i]$-with renormalisability if all are relevant or marginal) or non-Gaussian (NGFP) if all β-function vanish, but away from the origin. The calculations are as the ones above for the dimensionless, but running couplings $\hat{g}_i(k) = k^{-d_i} g_i(k)$ which can be an infinite set of couplings.

$$k\partial_k \hat{g}_i(k) = \sum_{\alpha=1}^{\infty} M_{i\alpha}(\hat{g}_\alpha(k) - \hat{g}_{\alpha*}), \quad M_{i\alpha} = \frac{\partial \beta_i}{\hat{g}_\alpha}(g_*) \tag{5.53}$$

again with the critical components that describe the RG behaviour near the FP given by

$$\sum_\alpha M_{i\alpha} v^a_\alpha = -\theta_a v^a_\alpha, \quad \hat{g}_i(k) = \hat{g_{i*}} + \sum_a \omega_a v^a_i \theta_a \tag{5.54}$$

Now sending k to infinity the number of critical exponents with positive real component is the dimension of the UV critical hypersurface (as $\omega_a = 0$ for all a with Reθ_a. An example can be again found in 7.13.3.
Worth mentioning is that the RG flow can be analysed via different approaches of functional RG methods (FRG). The Polchinski equation is a functional equation obeyed by the interaction part of the Wilsonian action (1983).
ERGE (Exact Renormalisation Group Equation) is its Legendre transform giving the effective average action by using the Wetterich equation (1993). Γ_k is a functional of infinitely many couplings, the ERGE contains all their β- functions.
Worth mentioning is also that AS is nothing else than the quantum realisation of *scale symmetry* since we relate different physical scales by the RG flow via the relations of the couplings of the underlying system. Symmetries give the restrictions on possible interaction structures. This helps us to reduce the dofs and only measure a (small) number of free parameters. In QFT quantum fluctuations lead to scale anomaly, the breaking of the classical scale symmetry since they have scale dependence (the value of the interaction strength is proportional to the scale). A priori we do not know whether the theory is well defined at a certain scale if it is valid at another scale. In asymptotic freedom we are fortunate of having vanishing quantum fluctuations, in asymptotic safety we have EXACT scale symmetry at the FP.
If we follow usual EH scale invariance is obviously broken due to the dimension of G.

5.5.2 Quantum Einstein Gravity

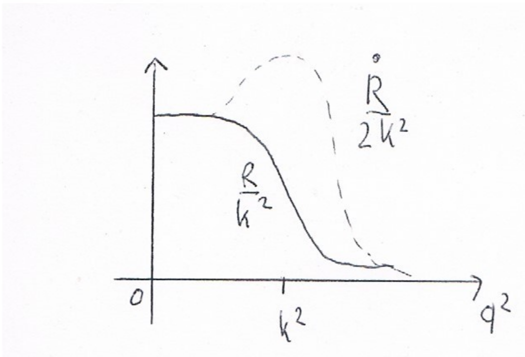


Figure 5.2: Behaviour of the cutoff and its derivative wrt RG time. An exponential cutoff shows such an R behaviour. Litim's cutoff would be linearly decreasing and then vanishing for $k^2 = p^2$.

I want to apply the RG method towards gravity. Reuter analysed the RG flow in $2 + \epsilon$ and 4 dimensions and concluded that the gravitational constant is antiscreening and increasing at large distances (see also [72][32], [64]). The degrees of freedom is the metric $g_{\mu\nu}$[33] which is invariant under diffeomorphism. The Euclidean path integral is given by

$$\int \mathcal{D} g_{\mu\nu} e^{-S[g_{\mu\nu}]} \tag{5.55}$$

The next step is the *background field method* where the metric is split into the non-dynamical background and the fluctuation part. For a detailed treatment of linearised gravity, have a look at 7.12.1.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \mathcal{D} g_{\mu\nu} = \mathcal{D} h_{\mu\nu} \tag{5.56}$$

We further proceed [65] with *gauge fixing* 7.12.1 [34] and the integration over the Faddeev-Popov ghost fields 7.12.1 and the regulator term

$$\mathcal{D} h_{\mu\nu} \mathcal{D} C^\mu \mathcal{D} \bar{C}_\mu e^{-S}, \quad S' = S[\bar{g} + h] + S_{gf}[\bar{g}, h] + S_{gh[C, \bar{C}, \bar{g}, h]} - S_{source} + \delta S_k = e^{W_k} \tag{5.57}$$

$$\Delta S_k = -\frac{1}{2} \int d^4x \sqrt{\bar{g}} h R_k h \tag{5.58}$$

$$S_{source} = -\int d^4x \sqrt{\bar{g}} (j^{\mu\nu} h_{\mu\nu} + \bar{\sigma}_\mu C^\mu + \sigma^\mu \bar{C}_\mu), \tag{5.59}$$

$$< h_{\mu\nu} >= \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta j^{\mu\nu}}, \quad < C^\mu >= \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta \bar{\sigma}_\mu}, \quad < \bar{C}_\mu >= \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta \sigma^\mu} \tag{5.60}$$

The effective average action is then given by[35]

$$\Gamma_k[g, \bar{g}, c^\mu, \bar{c}_\mu] = \Gamma_k[h, \bar{g}, c^\mu, \bar{c}_\mu]|_{h=g-\bar{g}} = \int d^4x (j^{\mu\nu} h_{\mu\nu} + \bar{\sigma}_\mu c^\mu + \omega^\mu \bar{c}_\mu) - W_k[j, \sigma, \bar{\sigma}, \bar{g}]|_{source} - \Delta S_k[h, c, \bar{c}, \bar{g}] \tag{5.61}$$

[32] In 1996, Reuter wrote the exact RG equations for gravity but left the result in a form that wasn't really calculable. Improved cutoff forms helped this issue later on.

[33] There is also research going on in using affine gravity formalism or Einstein-Cartan where the metric is added by the vielbein e^a_μ which leads to the inclusion of torsion and coupling to fermionic fields.

[34] The main two choices are the background transformation, $\mathcal{L}_V h_{\mu\nu} = \delta h_{\mu\nu}$ and $\mathcal{L}_V \bar{g}_{\mu\nu} = \delta \bar{g}_{\mu\nu}$, and the quantum transformation, $\mathcal{L}_V g_{\mu\nu} = \delta h_{\mu\nu}$ and $\mathcal{L}_V g_{\mu\nu} = 0$ under coordinate transformation, $x^\mu \to x^\mu - V^\mu$.

[35] To be precise, W_k and Γ_k also depend on the sources that couple to BRS variations of the fluctuations and the ghost fields. The BRS variation of the total action action only contribute by the cutoff and source terms as modified Ward identities were derived.

which is invariant under field diffeomorphism,$\Gamma_k[\phi] = \Gamma_k[\phi + \mathscr{L}_V \phi]$ and satisfies the boundary condition $\lim_{k\to 0} \Gamma_k \to \Gamma_0$ and $\lim_{k\to\infty} \Gamma_k \to S + \Gamma_{gf} + \Gamma_{gh}$ and satisfies the Wetterich equation.
Using Einstein-Hilbert

$$\boxed{\frac{1}{16\pi G_k} \int d^4x \sqrt{-g}(-R + 2\Lambda_k)} \tag{5.62}$$

The dimensionless couplings are

$$g(k) = k^2 G_k, \quad \lambda(k) = k^{-2} \Lambda_k \tag{5.63}$$

in arbitrary dimensions, $g_k(k) = k^{d-2} G(k), \lambda_k(k) = k^{-2}\Lambda$, with the β-functions

$$\dot{g} = k\partial_k g = \beta_g = (d - 2\eta_N)g, \quad \dot{\lambda} = \beta_\lambda \tag{5.64}$$

where $\eta_N = \frac{g_k B_1(\lambda_k)}{1 - g_k B_2(\lambda_k)}$ is the anomalous dimension ('quantum scaling' dimension, at the FP, -2) and a function of the dimensionless cosmological constant and the cutoff. The β-function of Λ is a bit more involved. We have a GFP, $g(k) = \lambda(k) = 0$, and a NGFP which can be numerically calculated in the upper right quadrant of the $g - \lambda$-space. At large distances the Newton constant goes like k^2, at small distances it thus has a fixed value. Once past the GFP in the EH truncation, the behaviour is given by the canonical scaling.
The (dimensionful) Newton constant is finite for $k \to 0$, G_0, and falls like $\sim \frac{g}{k^2}$ for large momenta, vanishing for $k \to \infty$. The cosmological constant is finite (small) for vanishing momenta, increases with $\sim k^4$ at Planck scale and $\sim k^2$ above. Hence, $g(k) = G(k)k^2$ approaches a fixed value for large k. The product of $g_* \lambda_*$ is constant at and nearly constant near the FP and for pure gravity of order unity.
The spiralling into the FP in the UV is due to the complexity of the critical exponents, only the real part decides on the behaviour near the FP (relevant/ irrelevant). The FP has been numerically calculated and also proven against the change of cutoff functions, the result seems to be stable. Another example of calculating the β-functions and fixed point can be found in the appendix 7.13.1. EH stability coefficients are obviously both positive.
In the Einstein-Hilbert truncation there is a GFP and a NGFP in the upper right quadrant of the $\lambda - g$ diagram (other couplings can be included as well, so we should imagine this embedded in a higher dimensional space).[36] With the help of asymptotic safety a QFT (also a quantum theory of gravity) can be well defined at ALL energy scales WITHOUT being perturbatively renormalisable. The theory is asymptotically safe if it lies on a trajectory of the RG flow that ends at a UV FP.

5.5.3 FRG and Perturbation Theory

When calculating the quantum corrected Newton potential it is interesting to note that the coefficient in front of the non-analytic part indicates a screening behaviour

[36] Note, the dimensionless couplings don't need to coincide with our observations of the values of G and the positive cosmological constant, as long as the it flows to the right values in the low momenta regime. Nonetheless, most truncations have given a similar result.

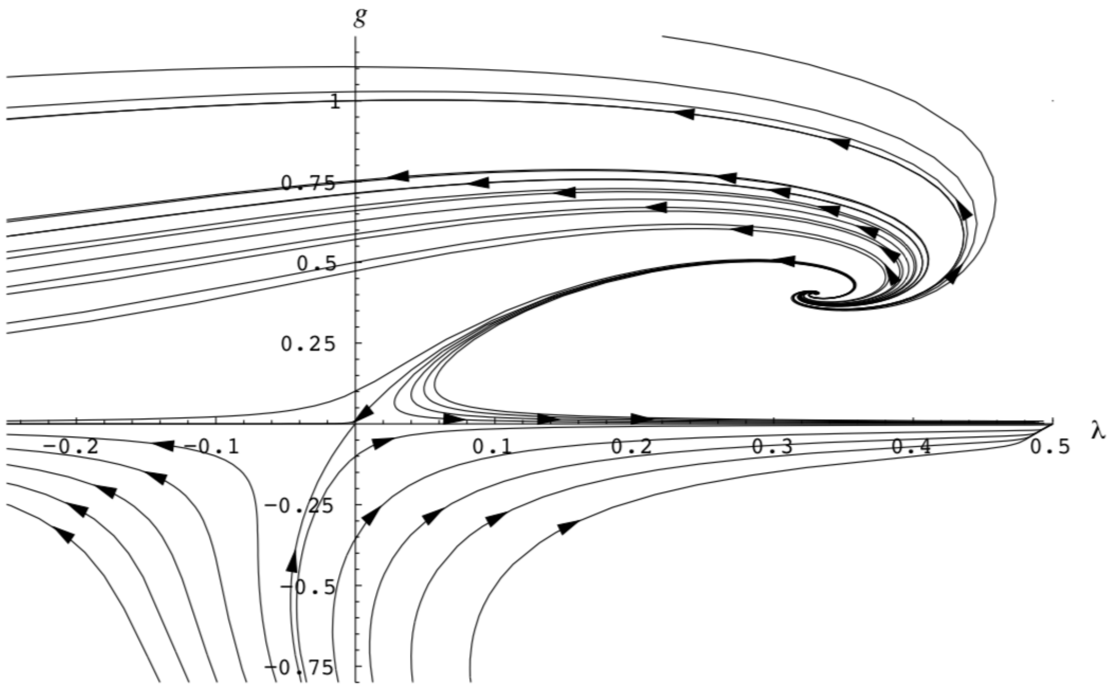


Figure 5.3: Template phase diagram for quantum gravity in the Einstein-Hilbert approximation (Plot taken from Reuter&Saueressig, 2002). In the EH setting a NGFP is found for positive parameters (here the gravitational coupling, cosmological constant - both dimensionless and dependent on k) and a trivial GFP at the origin. The arrows point towards the lower momentum. On the separatrix we find a vanishing renormalised cosmological constant.

of Newton's constant, $G(r) \to G(1+\frac{a\hbar G}{r^2})$. Donoghue calculated $a = \frac{41}{10\pi}$, but in order to be antiscreening a should be negative. We see that those approaches are quite different. Nonetheless, there has also been some work on asymptotic safety in a perturbative setting by Niedermaier. If gravity has an NGFP then it should be somehow visible in PT as well. He re-introduced perturbation theory treatment in the RG method and analysed EH as well as higher derivative terms. With perturbation theory he calculated the 1-loop β-functions with indications of asymptotic safety.
In the following I would like to give a short example using FRG and a perturbation ansatz.[37] The anharmonic oscillator gives an action of

$$S = \int dt \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \frac{1}{24}\lambda x^4\right), \quad \omega^2, \lambda > 0 \tag{5.65}$$

where the dot is wrt $\hat{t}$ - not to be confused with the RG time. In order to emphasise the advantage of the FRG method I assume a large coupling, $\lambda >> 1$. The effective average action is given by

$$\Gamma_k = \int dt(\frac{1}{2}\dot{x}^2 + V_k(x)) \tag{5.66}$$

The variation wrt $x(t_1), x(t_2)$ is

$$\begin{aligned} \Gamma_k^{(2)} = \frac{\delta}{\delta(t_2)} \int d\hat{t}(\dot{x} d\hat{t}\delta(\hat{t}-\hat{t_1}) + V_k'(x)\delta(\hat{t}-\hat{t}_1)) \\ = (-\partial_{t_1}^2 + V_k'')\delta(t_1 - t_2) \end{aligned} \tag{5.67}$$

which gives a $p^2 + V_k''$ term after Fourier transforming ($p^2 = -\partial_{\hat{t}}^2 ...$). Following Reichert's choice of using the Litim regulator $R_k = (k^2-p^2)\theta(k^2-p^2)$, $\delta_t R_k = 2k^2\theta(k^2-p^2) + 2k^2(k^2-p^2)\delta(k^2-p^2) = 2k^2\theta(k^2-p^2)$[38] the propagator can be calculated

$$\frac{1}{\Gamma_k^{(2)}} = \frac{1}{p^2 + (k^2-p^2)\theta(k^2-p^2) + V''} = \begin{cases} \frac{1}{k^2+V''}, & p^2 \leqslant k^2 \\ \frac{1}{p^2+V''}, & p^2 \geqslant k^2 \end{cases} \tag{5.68}$$

This is the nice feature of the cutoff choice. Modes below k are equally weighted. I substitute the propagator and $\dot{R}_k$ into Wetterich's equation integrating over space and momenta which simplifies to

$$\partial_t V_k = \frac{1}{2}\int_p \frac{\partial_t R_k}{p^2 + R_k + V_k''} = \frac{dp}{2\pi}\frac{2k^2\theta(k^2-p^2)}{k^2+V_k''} \tag{5.69}$$

$$= \frac{1}{\pi}\frac{k^3}{k^2+V''} \tag{5.70}$$

$$\frac{d}{dk}V_k = \frac{1}{\pi}\frac{k^2}{k^2+V_k''} \tag{5.71}$$

[37] Idea from M. Reichert.
[38] $x\delta(x) = 0$

Our aim is it to get an expression for the potential energy of the form $V_k = E_k + \frac{1}{2}w_k^2 k^2 + \frac{1}{24}\lambda_k x^4 + ...$ with the boundary condition that at the cutoff the couplings approach the given values in the action, $\omega_\Lambda = \omega, \lambda_\Lambda = \lambda$ as $\Lambda \to \infty$. We note that the ground state energy is depending on the regulator, so we need to 'shift' it back to the true value, $\hat{E}_k|_{\omega=\lambda=0} = 0$ with $\partial_k E_k = \partial_k V_k|_{x=0} = \frac{1}{\pi}\frac{k^2}{k^2+\omega_k^2}$,

$$\partial_k \hat{E}_k = \frac{1}{\pi}\left(\frac{k^2}{k^2+\omega_k^2} - 1\right) \tag{5.72}$$

The β-functions are integrated with the boundary conditions and the values are substituted back into 5.72 and again integrated we get the ground state energy.[39]. The result is

$$E_0 = \frac{\omega}{2} + \frac{3}{4}\left(\frac{\lambda}{24\omega^3}\right)\omega - \frac{3(8\pi^2+29)}{16\pi}\left(\frac{\lambda}{24\omega^3}\right)^2 \omega + ... \tag{5.73}$$

One can also see that for the non-interacting, $\lambda = 0, \omega_k = \omega$, Hamiltonian the ground state is given by

$$E_{0k} = -\frac{1}{\pi}\arctan\left(\frac{k}{\omega}\right) + \frac{\omega}{2} \tag{5.74}$$

which vanishes for $k \to \infty$ and for $k \to 0$ the energy is, as expected, $\to \frac{\omega}{2}$.
Now compare the result to standard perturbation theory of quantum mechanics, I assume the simple Schrödinger equation where the coefficients can be calculated recursively.

$$\left(-\partial^2 + \frac{1}{2}x^2 + \frac{\lambda}{24}x^4\right)\Psi(x) = E(\lambda)\Psi(x) \tag{5.75}$$

which leads to the expansion $E_0 = \sum_n a_n \left(\frac{\lambda}{24\omega^3}\right)^n \omega$ which could either have been guessed having the FRG result already or from the corrections up to second order

$$E_n = E_n^0 + \langle n|\hat{H}_I|n\rangle + \sum_{n\neq m}\frac{|\langle m|\hat{H}_I|n\rangle|^2}{E_n^0 - E_m^0} \tag{5.76}$$

with the interaction Hamiltonian $\frac{\lambda}{24}\hat{x}^4$, $\hat{H}_0 = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$ and the ground state known eigenvalue equation, $\hat{H}_0|n\rangle = E_n^0|n\rangle, E_n^0 = \frac{1}{2}(2n+1)$. I use the standard ladder operators to get a formula for E_n.

$$\hat{a}^{\mp} = \sqrt{\frac{\omega}{2}}\left(\hat{x} \pm \frac{1\hat{p}}{\omega}\right), \quad [\hat{a}^-, \hat{a}^+] = 1 \tag{5.77}$$

$$\hat{n} = \hat{a}^+\hat{a}^-, \quad [\hat{n}, \hat{a}^\pm] = \pm\hat{a}^\pm \tag{5.78}$$

$$\hat{H}_I = \frac{\lambda}{96\omega^2}(\hat{a}^+ + \hat{a}^-)^4 \tag{5.79}$$

$$E_n = \omega\Big(n + \frac{1}{2} + \frac{\lambda}{32\omega^2}(2n^2+2n+1) - \omega\left(\frac{\lambda}{96\omega^3}\right)^2(68n^3 + 120n^2 + 118n + 42)\Big) \tag{5.80}$$

[39] For an accurate result the integration should be done numerically as otherwise we (at least) use perturbation theory to analytically derive the integral, here it should be sufficient to expand in $\omega_k^2 = \omega_{0k}^2 + \omega_{1k}^2\lambda + \omega_{2k}\lambda^2 + ...$ with the conditions $\omega_{0k=\infty} = \omega, \omega_{ik=\infty} = 0$.

This gives a similar ground state energy as 5.73, but the second term's coefficient is $\frac{21}{8}$ so ~ 0.4 of the FRG result.
The 1-loop term is the same, it is universal. The 2-loop term, however, differs. Obviously, the FRG result depends on the cutoff. Comparing both results with 'exact' numerical integration Reichert concludes that the FRG technique gives a good result. Again I should emphasise that as soon as the couplings are large perturbation theory is a rather bad choice. Bender and Wu (1969) showed that the expression isn't convergent for all λ.

5.6 Inflation in ASG

The asymptotic safety approach has been well established in quantum gravity and a NGFP was found in different truncations. The application towards phenomenology was put forward in particle physics, the theory was applied towards black holes and the early universe. I will give a short overview about asymptotically safe inflation in the following and investigate how the renormalisation group flow technique may help to solve the problems of standard inflation.
We have seen that Einstein-Hilbert (EH) is unitary, but non-renormalisable (at least in the perturbative sense). Higher derivative (HD) terms can renormalise the theory but one loses unitarity (at least in the perturbative sense...). Corrections of higher derivative terms should become relevant at high energies/ small scales and hence have been investigated in settings as the resolution of (black hole) singularities or the early universe. It is natural to extend it to inflation and also include the technique of renormalisation group flow and asymptotic safety. With teh help of the RG flow we can investigate the matching of the UV behaviour (early universe, quantum behaviour) and IR behaviour (late time behaviour, LSS) which - as we have seen - have/are both affected by the mechanism of inflation.

5.6.1 RG Improvement

RG improvement is the procedure where we replace the cutoff scale by a physical parameter. Depending on the problem we want to solve there are different choices. Moreover we can classify different types on when the RG improvement is done

Level of RG Improvements

1. at the level of the solution
2. at the level of the equation of motions (a)
3. at the level of the action (b)

The first one is probably the most obvious one, the couplings are replaced with the corresponding running couplings as substituted into the (non-improved) Einstein equations. It is similar to the treatment we used in the VSL 7.8 theory.

For (a) we would change the Einstein equations to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = 8\pi G(k) T_{\mu\nu} \tag{5.81}$$

i.e. substituting the running constant into the equations which we have yet to solve. Clearly, in the vacuum it is the same as substituting it into the final solution.

(b) is done similar to the improvement of the effective potential in scalar field theory (Coleman& Weinberg, 1973, see the example in the appendix 7.13.4). The running coupling is substituted into the action such that we start with a action that already covers the quantum corrections. This is especially helpful in HD theories where we identify the k with the curvature as I will show.
But first let me give a short overview of *identifications* that have been used so far. Prior to the RG improvement we have to decide on the *cutoff identification*. One might start with the RG scale being identified as a function of space or time such as the inverse time or inverse physical length (simply via dimensional analysis) since we want to describe spacetime in the end. However, we cannot simply use any distance, we need to satisfy the underlying symmetries of the theory (at least general covariance and local Lorentz symmetry, if we include some inflation potential this might introduce another symmetry as well). In flat space we could simply use $k = \frac{1}{l}$, but as soon as curvature is turned on we have to be very careful. For example, for static spherically symmetric Schwarzschild black hole solutions we could use $k^{-1} = \sqrt{g_{ij} dx^i dx^j}^{-1}$.
The first step is:

$$\boxed{G \to G(k) \to G(k(X))} \tag{5.82}$$

where X is the chosen identification.
The quantum corrections of the RG improved equations or formulae that describe the universe's properties can be studied in the LE limit. If we think we have a well-defined trajectory it should produce classical behaviour as $k \to 0$ i.e. when we flow to the IR. The couplings should become constant or vanish and quantum corrections should be absent and not measurable anymore. We should find a regime where they are still measurable such that we can compare and test the validity of the quantum corrections to data. Usually, black holes and inflation are the best labs for that.
RG scale proportional to

- the scale factor (at the FP the RG scale freezes and then the flow evolves with $a(t) \to$ inflation) (Bonanno & Reuter (2002) [62], Solá (2014)), possible for homogeneous and isotropic states
- cosmological time and the Hubble scale which is quite attractive as the universe expands in time (Bonanno & Reuter (2002))
- the fourth root of the energy density (Guberina, Horvat & Stefancic (2003), Bonnano, Koch & Platania (2017))
- cosmological event and particle horizons as $k \sim H \sim \frac{1}{d_H}$ (Bauer (2005), Bonanno & Reuter (2006))

- curvature invariants such as $R^{\frac{1}{2}}$ (Falls, Litim & Schroder (2018)), $(R_{\mu\nu}R^{\mu\nu})^{\frac{1}{4}}$ (Moti & Shojai (2018))
- ...

Simple Example and Comparison

Let us start with a simple example where I *improve the Einstein equations*. I assume EH with a NGFP and the behaviour of the couplings derived by Reuter, FRW and perfect fluid treatment 7.3. The cutoff is identified with the inverse of time $k = \frac{k_0}{t}$.

$$\Gamma_k = \frac{1}{16\pi G(k)} \int d^4x \sqrt{g}(2\Lambda(k) - R) \tag{5.83}$$

$$G(k) \sim G_0(1 - \alpha G_0 k^2), \quad \Lambda(k) \sim \beta G_0 k^4 \tag{5.84}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = 8\pi G(k(t)) T_{\mu\nu} - g_{\mu\nu}\Lambda(k(t)) \tag{5.85}$$

Let us find the RG improved first Friedmann equation and the conservation equation.

$$H^2 = \frac{8\pi\rho}{3} G(k(t)) + \frac{\Lambda(k(t))}{3} \tag{5.86}$$

$$\dot{\rho} + 3H\rho(1+w) = -\frac{\rho G(\dot{k}(t))}{G(t)} - \frac{\Lambda(\dot{k}(t))}{8\pi G(k(t))} \tag{5.87}$$

Differentiating the first wrt t and substituting it into the second gives

$$\frac{\dot{H}}{H^2 - \frac{\Lambda}{3}} = -\frac{3}{2}(1+w) = -a \tag{5.88}$$

with $k = \frac{k_0}{t}$. We see, that if we substitute the running coupling $\Lambda(t)$ into the equation, we have a first order nonlinear ODE which cannot be solved directly. Let us first keep the implicit dependence and introduce $H = \frac{1}{\hat{H}}$ such that we can rewrite the equation as

$$\dot{\hat{H}} = -a \left(\frac{\hat{H}^2 \Lambda}{3} - 1 \right) \tag{5.89}$$

$$\hat{H} = \hat{H}_0 + a \left(t - t_0 + \frac{1}{3} \int_t^{t_0} \hat{H}^2 \Lambda dt' \right) \tag{5.90}$$

where $\hat{H}_0$ is $\hat{H}$ at t_0 (today). We can only proceed to treat the late time behaviour for small Λ, with the perturbatively treated couplings

$$G(t) = \left(1 - \frac{\alpha k_0^2 G_0}{t^2}\right), \quad \Lambda(t) = \frac{\beta k_0^4 G_0}{t^4} \tag{5.91}$$

To get an estimation we evaluate 5.90 at the non-corrected part $(a(t - t_0) + \hat{H}_0)$ and re-substitute the result into 5.90 again. The sum of the non-corrected part, the first and second order correction and substituting the t-dependence 5.91 is then given by (after some algebra and only up to order G_0)

$$\begin{aligned}\hat{H} &\sim a(t-t_0) + \hat{H}_0 - G_0\frac{a\beta k_0^4}{3}\Bigg(-\frac{a^2}{t} - \frac{a(\hat{H}_0 - at_0)}{t^2} - \frac{(\hat{H}_0 - at_0)^2}{3t^3} \\ &+ \frac{a^2}{t_0^2}\left(1 + \frac{\hat{H}_0 - at_0}{a}\right) + \frac{(\hat{H}_0 - at_0)^2}{3t_0^3} \\ &= at + a_0 + G_0\frac{a\beta k_0^4}{3t}\left(a^2 + \frac{a_0^2}{3t^2} + \frac{aa_0}{t}\right) + \text{const} \\ a_0 &= \hat{H}_0 - at_0, \quad \text{const} = -G_0\frac{a\beta k_0^4}{3t_0}\left(a^2 + \frac{a_0^2}{3t_0^2} + \frac{aa_0}{t_0}\right)\end{aligned} \tag{5.92}$$

We can now take the inverse to get the quantum corrected Hubble parameter and integrate this to get the quantum corrected scale factor. We could also further proceed and substitute the result into the improved Einstein equation to get the quantum corrected density formula. The Hubble term should be proportional to $\frac{1}{at}$ with the next term being the quantum correction and the scale factor should be $\propto t^{\frac{1}{a'}}$, $a' = a = \frac{3}{2}(1+w)$ ($a \to a'$ to differ from the scale factor a)

$$H_{\text{q corrected}} \sim \frac{1}{at}\left(1 - \frac{c}{at} + \left(\frac{c}{at}\right)^2 - G_0\frac{a\beta k_0^4}{3t^2}\left(a^2 + \frac{3a_0^2}{3t^2} + \frac{aa_0}{t}\right)\right) \tag{5.93}$$

$$a_{\text{q corrected}} \sim At^{\frac{1}{a'}}\left(1 + \frac{c}{a'^2 t} + \frac{c^2(1-a')}{2a'^4 t^2} + G_0\frac{a'\beta k_0^4}{a' t^2}\right), \quad A \text{ s.t. } a(t_0) = 1 \tag{5.94}$$

with const + $a_0 = c$. It turns out that the quantum corrections are of order $\frac{1}{t}$. This doesn't agree with Reuter & Bonanno (2002).

They [62] investigated the modification of FRW cosmology with the help of the exact RG approach and the identification of $k = \frac{\eta}{t}, \eta > 0$, t being cosmic time with the explanation that it should encode homogeneity and isotropy well. With the RG equations (usual FRGE treatment as described before, Reuter 96/98 and using Litim's and the exponential cutoff in the RG flow) of Newton's constant and the cosmological constant they improved the Einstein equations, derived the Bianchi identities and improved the quantities a, ρ, p all dependent on the cutoff identification time. The analysis of the FP gives insight on the early time (with the UV FP itself being at QG scale) and for $t >> t_p$ they are able to analyse FRW cosmology perturbatively. It is reliable for $t \to 0$ as gravity becomes AF $G \propto t^2$ near the UV FP. They claim to not need any fine-tuning as there are no ad hoc structures and $w \leqslant 1/3$ naturally arises. [40]
They also analyse perturbations and density fluctuations with approximated RG equations. During the expansion of the universe (lower k, increase t) fluctuations are

[40] for flat curvature there exist two limiting solutions for all w and a continuous interpolation between $(t = 0, t_{cl})$. Nearby the UV FP where $k \sim t^2$ we have attractor solutions with the same universal behaviour, for positive curvature only $w = \frac{1}{3}$ for the FP and $w = -\frac{1}{3}$ for the classical/perturbative are possible, there is no consistent interpolation. The RG improvement selects the $k = 0$ case and removes the flatness problem. Precisely, there are per se no strong arguments against non-flatness, but if flat curvature is chosen then there is no naturalness problem. Also, the RG improved spacetime has no particle horizon for $w \leqslant \frac{1}{3}$, but still does not provide a direct solution to the horizon problem.

amplified and magnified which can be measured in the LSS of today. The sub Hubble scale modes evolve according to perturbation theory with a (nearly) scale invariant spectrum at the beginning of inflation. They emphasise the need of a dynamical and consistent matching between the HE and LE regime.
Further they analysed a cutoff identification $\frac{1}{a(t)}$ (s.t. $a \propto t^{\alpha}$). $k = \frac{\eta}{a(t)}$ in the improved Friedmann equation determines the constant η near the FP, $G(t) = g'_* a^2$, $\Lambda(t) = \lambda'_* a^{-2}$ with $g' = g\eta^{-2}, \lambda' = \lambda\eta^2$.

$$\dot{\rho} + 3(1+w)\frac{\dot{a}}{a}\rho = 0 \rightarrow \rho(t) = \frac{M}{8\pi a(t)^{3(1+w)}} \tag{5.95}$$

$$\rho(t)a(t)^{3(1+w)} = \frac{M}{8\pi} = \text{const} \tag{5.96}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{\Lambda}{3} + \frac{8\pi G\rho}{3} \tag{5.97}$$

$$\dot{\Lambda} + 8\pi\dot{G}\rho = 0, \quad \dot{\Lambda} = \frac{d\Lambda}{da}\dot{a} \tag{5.98}$$

$$-\frac{d\Lambda}{da} = \frac{M}{a^{3(1+w)}}\frac{dG}{da} \tag{5.99}$$

where we now see why my result differs from this one (even if we look at the $\frac{1}{t}$ solutions), they separately covariantly conserve the energy momentum tensor.
Near the FP gives $ka = \eta = \left(\frac{g_* M}{\lambda_*}\right)^{\frac{1}{4}}$ where M is a constant of integration with canonical dimension $1-3w$, consistent with dimensionlessness when the matter is radiation dominated as it is assumed to be after inflation.
For Both identifications the quantities are characterised by power law behaviour when taking $t \rightarrow 0$.

(b) RG Improvement in the Action

Bonnano [11] assumes that the cutoff is proportional to the square root of R and substitutes this into EH to get an *effective f(R)* theory, he then analyses the behaviour at and near the FP (which should always include to calculate the critical exponents). The effective theory contains a $\cos\log R$ term which produces an infinite number of countable de Sitter solutions. He further analyses their stability some of which are rather unstable. He also tackles the question of inflation and shows that sufficient e-foldings would be possible to solve the cosmological problems with a long enough unstable de Sitter phase. He uses FRGE applied to EH. His emphasis on how we possibly best extract all relevant information that is encoded in the running of the couplings is important to note. How can we identify the energy/momentum scale such that the spacetime properties of the underlying theory are met? We have seen that spacetime is the gravitational field with its properties determined by curvature affected by the matter content and vice versa. Thus, the choice of the cutoff $k^2 = R$ is a good one. The EH flow with the couplings G and Λ is linearised around the NGFP and the effective Lagrangian shows a similar structure to QCD's log type leading term. When applying the action to FRW cosmology, $k = 0$, its stability properties do not depend on the scale (but only on the critical exponents which are believed to be universal).

Moreover, near the FP we have an R^2 behaviour suggesting Starobinsky inflation in the early universe.
Let us compare the results to Hindmarsh and Saltas [71] who use the same cutoff up to a dimensionless constant ρ factor. They find infinitely many de Sitter solutions (the evolution of the universe in the $\lambda - g$ plane (5.4) is the following: the RG flow begins from an outer de Sitter state in the UV giving naturally inflation, then it passes near the GFP to approach another de Sitter state giving accelerated late time expansion). The UV and IR regime is dynamically connected and satisfies the matter type eras. However, the factor should differ between early and late time behaviour, being 1 today and rather large early on as otherwise large fluctuations wouldn't be suppressed. This issue isn't resolved, but possible solutions are suggested. Primordial inflation is only possible for ρ large which then makes the mass of the scalar field that is introduced by the effective R^2 infinite/nonphysical at late times. The authors suggest that either the fluctuations that we can observe today were generated in a later 'inflation' period or to introduce more dofs into the action. A careful analysis of the quantum corrected action also gives a class of eternal inflation universes and a class of graceful exit models depending on the sign of the potential at its minimum. For a positive value it is constrained in SR producing eternal inflation. The calculated spectra indices are in agreement with Planck data. The identification of $k^2 \sim R$ using SR gives a cutoff of $\sim H^2$, hence the Hubble constant sets the cutoff scale which is quite natural (I will this identification to get an estimation on the ghost's mass).

(a) RG Improvement in the EOMS

Platania uses both RG improvement types in [56]. The identification with a physical scale and k is classified as *decoupling mechanism* when it acts as a decoupling scale. Via the RG treatment of the couplings we can add quantum terms to the classical behaviour, here at the level of the action to get the leading order terms of the quantum effective action.
Following (a) is equivalent to setting $\Delta t_{\mu\nu} = 0$ which is the effective energy momentum tensor. So far, we have treated the UV behaviour at/near the FP and the IR behaviour for very low k, but there exits also some intermediate regime in which this tensor becomes important and added on the RHS of the Einstein equations. The modified Bianchi identities is again derived with the assumption that the energy momentum tensor is covariantly conserved,

$$\nabla^\mu(8\pi G_k T_{\mu\nu} - \Lambda_k g_{\mu\nu} + \Delta t_{\mu\nu}) = (8\phi G'_k T_{\mu\nu} - \Lambda'_k g_{\mu\nu})\nabla^\mu k(x) = 0 \qquad (5.100)$$

where $k(x)$ is some intermediate scale. At the level of the action the effective energy momentum cannot be disregarded.
Platania also investigated the backreaction. When we analyse the behaviour of a particle in spacetime we usually ignore its backreaction i.e. how the particle's properties affect spacetime (mass, charge) etc which then reacts back to the particle. She includes the backreaction effects produced by the running of G via an iteration method (the cutoff is constructed by $k_{n+1}(r)$, for its convergence the NGFP is essential) which she also applies to black holes. The modification of the dynamics of the theory by quantum fluctuations becomes clearer. The effect is self-sustaining since a small change in

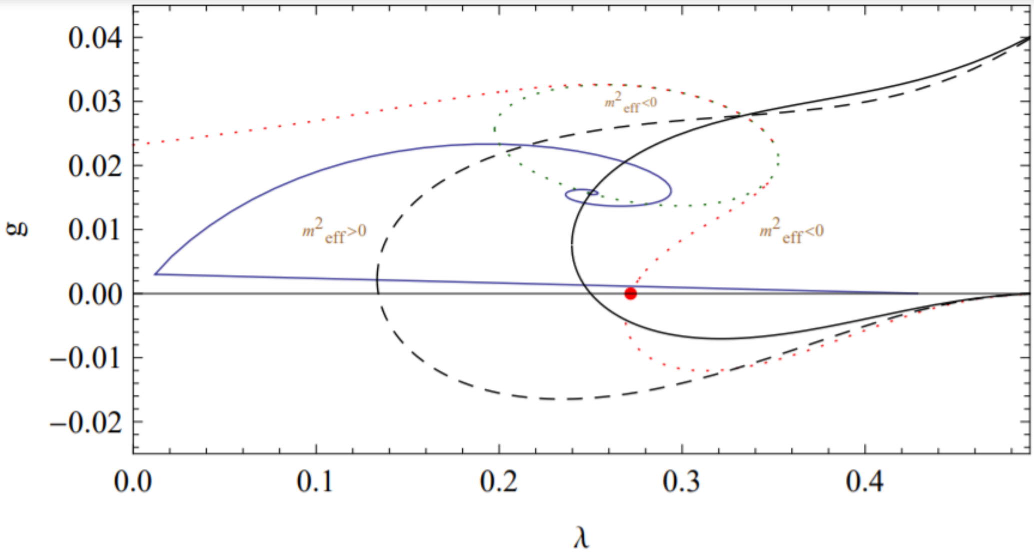


Figure 5.4: Hindmarsh & Saltas. An RG trajectory in the dimensionless couplings $\lambda - g$ plane in EH truncation that could describe our universe (blue) with $\rho = 1$, we see the usual spiralling due to the complex critical exponents into the UV FP and we flow along to later times, lower energies and lower curvature passing away from the GFP such that classical values can be attained. The intersection of the phase curve with the de Sitter line (black) is the described possibility of a de Sitter point, the intersection with the dashed (black) one is where we have $\epsilon_V = 1$ indicating the end of inflation. We also see that the inner UV de Sitter points are protected by the outer ones such that we cannot reach them. m_{eff} is the effective mass of the scalar field in the Jordan frame. With the identification $k^2 = \rho R$ we immediately see that for $\rho << 1$ HD curvature terms become important whereas for $\rho >> 1$ those are negligible but radiative corrections are introduced [71].

the running coupling G introduces a ripple effect which itself changes the running of G. Such a treatment could be analysed in the inflation setting as well[41].
Going back to its application to inflation Platania analyses a toy model of an effective action from the RG improvement of EH that provides SR inflation solutions. Near the UV FP scale invariance is also encoded in the nearly scale invariant scalar power spectrum, depending on the (universal) critical exponents. This kind of treatment should be favoured since the FP itself shouldn't been as universal (or even physical) quantity. The density fluctuations can be interpreted as quantum gravity fluctuations in a pre-inflation era when flowing towards the FP.
(a) evaluated on the sphere have been used to model inflation since they give (Euclidean) de Sitter solutions (Falls, Litim, Nikolakopoulos & Rahmede (2018)).

5.6.2 Asymptotically Safe Starobinsky Inflation

As we have seen, Starobinsky's model fits cosmological data very well. Other HD theories have been investigated as well. However, for R^2, for example, the coefficient needs

[41]Note, simply if we calculate k^2, in the HE energy regime $g_{\mu\nu}$ should be replaced by the $< g_{\mu\nu} >$ fluctuation metric which itself depends on k. In Platania's treatment the classical black hole singularity is dynamically replaced by an de Sitter cone (2019)

to be very large in order for inflation to occur. There is no reasonable explanation for that. It is a straight forward question to investigate those models in the asymptotic safety scenario. We also know that including the Weyl squared term the R^2 action can be made renormalisable.

We analyse $\mathscr{L} \sim \frac{R}{16\pi G} + \frac{R^2}{b} \leftrightarrow \frac{R}{16\pi G} + \frac{1}{2}\partial\phi^2 + V$ under the transformation $\hat{g}_{\mu\nu} = g_{\mu\nu} e^{\sqrt{\frac{16\pi}{3}}\frac{\phi}{m_p}}$ whose potential we derived earlier. For $\sim \frac{B}{16\pi G}$ we know $B = -\frac{1}{6m_2^2}$, $b = 6Gm_2^2$. Under Euclidean setting and G_k, b_k (Machado& Saueressig; Codello, Percacci & Rahmede (2008)) we have the RG flow under

$$\Gamma_k = \int d^4x\sqrt{g}\left(-\frac{R}{16\pi G_k} + \frac{R^2}{b_k}\right) \tag{5.101}$$

$$\hat{G} = Gk^2 \tag{5.102}$$

as b is already dimensionless. I will use Litim optimation and expand in the Planck regime i.e. $k \sim m_p$. Hence, I can use the solution of M&S and CPR that proceed by calculating the ERGE with the Wetterich equation, gauge fixing, Litim cutoff. The EAA is evaluated over Euclidean 4-spheres (which intuitively is associated to inflation=expansion). The β-functions are rather complicated, there is a GFP and two UV FPs, one with a large b value and one with non-perturbative properties at $(\frac{24\pi}{17}, 0)$ with a small b at HE (large B). For the Planck regime, b small, $\hat{G}$ small up to unity, integrating the β-functions we have the runnings

$$\hat{G} \sim \frac{\hat{G}_0 k'^2}{1 + \frac{41}{72\pi}\hat{G}_0 k'^2} \tag{5.103}$$

$$b \sim \frac{b_0}{1 + \frac{41}{72\pi}\hat{G}_0 k'^2} \tag{5.104}$$

around the second NGFP with the dimensionless constant parameters $\hat{G}_0, b_0$ and the scaled $k' = \frac{k}{k_0}$. For $k' \to \infty$ the coupling of the R^2 term vanishes (making B very large) and $\hat{G}$ approaches a finite value as expected, $\hat{G} \sim 5.51695$ (note, the values of teh dimensionless couplings don't have a physical meaning). For $k' \to 0$, however, b approaches a finite value, b_0 and $\hat{G} \to G_0 k'^2$ as expected. I choose the reference scale to be of Planck order, $k_0 \sim m_p$. As we have seen we need to match the results with the physical scale (would need to observe it or do experiments e.g. from the CMB) to get the constant parameters. We know that $\hat{G}_0 \sim 1$ since $k_{us} << k_0$ and we are left with b_0 as scale for inflation where I also assume $k_{inf} < k_0$ (we have seen that it is assumed to be at GUT scale, so at least three orders below the Planck regime - relics shouldn't be able to form again). The important question is: What is the value of the cutoff scale during inflation? The Hubble parameter sets a bound on k during inflation. We saw $b_0 \sim 2 \cdot 10^{-9}$ 3.99 to agree with the CMB data. This result gives a tensor spectrum of the order 10^{-12} and would explain the fact that we still haven't observed primordial gravitational waves, the best resolution measurements give a detection possibility of $\sim 0.01+$ which is clearly to high. In comparison, the EH truncation predicted a tensor power spectrum of nine orders greater which might be detectable one day.[42] The

[42]The number comes from the constraints for an accelerated expansion and sufficient e-foldings to solve the cosmological problems. 5.4 evaluate N in terms of the curvature R and the couplings. For 60

theory would also provide a graceful exit since for R^2 term dominates at early times, but after a sufficient number of e-folds the de Sitter solution has an instability and the exponential expansion ends5.4.
With b_0 we can then use the flow equations to extrapolate the coupling at a specific scale, for example at $b(k = m_p)$. Using Codello et al.β-functions we get $b(m_p) \sim 1.7 \cdot 10^{-9}$ which only differs by less than 2% meaning that the treatment at the UV FP and at the Planck scale indeed shows slow running.
If the quantum correction is taken into account the R^2 term may modifie the Langrangian at HE, $\rightarrow \sim \frac{\alpha R^2}{2(1+\beta \ln \frac{R}{\mu^2})}$ i.e. by $\Delta S \sim \int d^4x \frac{\alpha(\mu) R^2}{2}$ where α is a slow logarithmic running (Demnel, Saueressig & Zanusso (2015)) which is comparable to 7.13.4.

5.6.3 Consistency Conditions

Before I continue with the next section I would like to emphasise the desired consistency conditions for a new fundamental theory. It should include

- *unitarity*, $U^\dagger U = \mathbb{1}$
- *no anomalies*, failure of gauge symmetry on the full quantum level
- *causality*, A causes B or B causes A and information cannot be transferred faster than light - no action at a distance
- *locality*, an object is only directly influenced by its neighbourhood and the large scale behaviour is determined by the small scale
- *Lorentz invariance* (see before) and other symmetry requirements, note in QG Lorentz invariance should be seen as a local (rather than a global) symmetry, general covariance, gauge invariance...

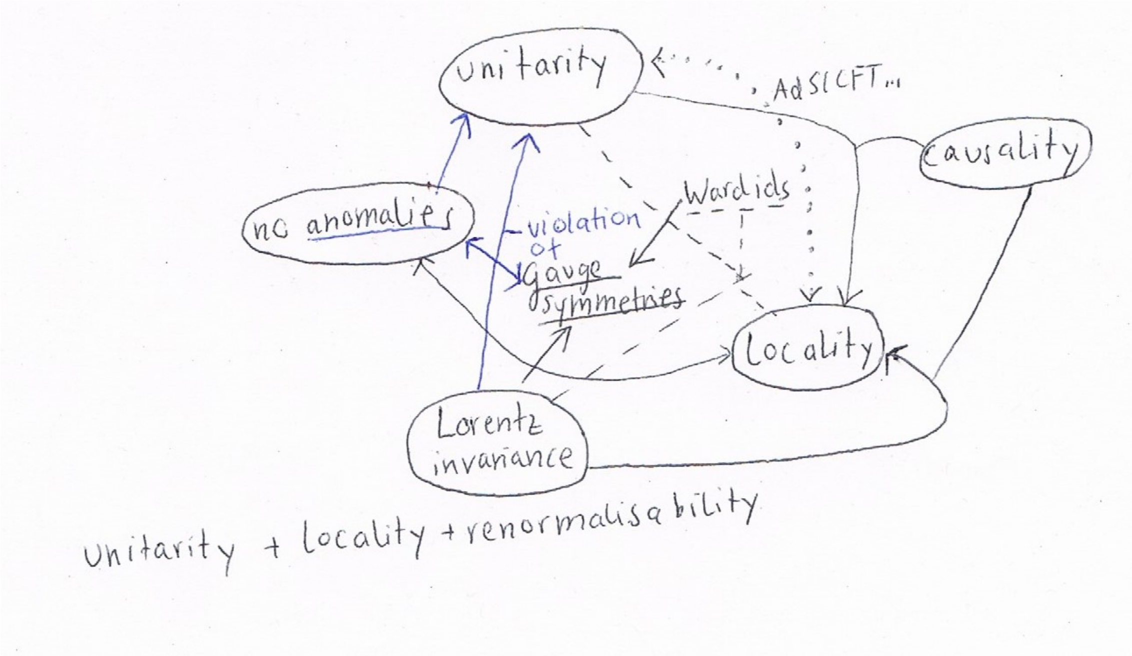
which aren't independent of each other and we may violate certain conditions. Moreover, what we are after is a theory that is

Condition for inflation-model independent

- unitary
- renormalisable
- physical and unique

Unitarity follows from the need of a well-defined notion of probability and logic, there are no negative probabilities. The squared complex amplitudes give probabilities that a certain state is given, $P = u_f^\dagger u_f$ where u_f is a final state vector. The total

e-folds from the UV de Sitter point to the end of inflation, $\epsilon = 1$, (g, λ), $(0.02, 0.27) \rightarrow (0.02, 022)$ they find a value of $\mathcal{P}_s \sim 0.067, \mathcal{P}_t \sim 0.052$ i.e. of same order which clearly is not the case. Cai&Easson (2011) analyse $f(R)$ models and connect them to some Brans-Dicke type theories where EFT treatment breaks down, with RG treatment, however, they find that the quantum fluctuations are of the same order as the background.



probability must add to unity and under a transformation under U the probability is conserved from an initial state to final state.

$$P = u_f^\dagger u_f = (Uu_i)^\dagger U u_i = u_i^\dagger U^\dagger U u_i = \mathbb{1} \tag{5.105}$$

where I assume that the states are normalised. In particle physics we identify U with the scattering S-matrix. In fact, the need for a well-defined probability definition leads to the constraints of a unitary S-matrix and a positive definite Hilbert space where the Hamiltonian is Hermitean. The issue of Stelle's fourth order gravity is that you either have instability since the Hamiltonian is unbounded from below or you have states with negative norm (→ non-unitary). The negative energy of the ghost must in fact be traded for quantum states of positive energy but with a negative metric in the state vector space (Stelle, 1977). If you don't trade them the negative energy would allow vacuum decay into ghosts and normal particles with positive energy which would lead to an infinite phase space. We also have to deal with the possibility that the spin 2 field couples to external matter forces. Independent of the mass of the ghost, classical GR's two dofs are added by further five dofs given by the $\pm 2, \pm 1, 0$-helicity modes. Early on Ostrogradsky realised that HD terms in the actions in classical mechanics (higher than the second time derivative such as in 5.24 & a nondegenerate Lagrangian) generate instabilities at the non-linear level (1850)[43]. Ghosts from HD theories may violate unitarity, but other nonphysical fields can be projected away and don't violate unitarity.

The gauge fixing breaks the classical gauge invariance, 'good' ghosts 7.12.1 are introduced to restore unitarity. We recover gauge invariance back at the quantum level if we follow the 'recipe' of gauge fixing and ghost introduction (it is then invariant

[43] Indeed, we don't have any physical theories in CM that are described by terms like the third derivative wrt time.

under the fermionic BRST transformation (1976) which Stelle also used in his gravity treatment.
We want to have a physical theory that is uniquely defined, we should only need to measure a finite number of independent parameters and no nonphysical fields should arise.

Where Does the Ghost Come From?

It is a good exercise to actually prove that the new dof introduced by R^2 and W^2 (see below) are actually a scalar and spin 2 massive field. We will rewrite the higher derivative theories in terms of a second order canonical theory where the scalar field and the spin 2 field are coupled to gravity.
We learnt that Wigner classified the unitary irreducible representations by mass and spin (say given by the operators M^2, S^2). I follow the Fierz and Pauli [53] and Whitt [88] treatment and use formulae from 3.3.6. First, the condition for $s = 2$[44]

$$M^2 \rightarrow (\partial^2 - M^2)S_{\mu\nu} = 0 \tag{5.106}$$
$$S^2 \rightarrow \partial^\mu S_{\mu\nu} = 0 \quad \eta^{\mu\nu} = 0 \tag{5.107}$$

for a symmetric $S_{\mu\nu}$ tensor. That's also how we can see that the ghost will contribute to five dof. A symmetric rank 2 tensor has ten dof and we need to specify its value and its derivative initially. Along with teh given conditions this reduces to $10 - 5 = 5$ dofs.
First we introduce an auxiliary field which will reduce the higher derivative terms of fourth to second order. We will couple that field to gravity.

Starting with the R^2 term we should add a term of

$$-\frac{1}{6m_0^2}(R - 3m_0^2c_0)^2 \tag{5.108}$$

with a dimensionless coupling c_0, the action can then be rewritten

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R + \frac{R^2}{6m_0^2} - \frac{1}{6m_0^2}\left(R - 3m_0^2c_0\right)^2\right) \\ &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R(1+c_0) - \frac{3}{2}m_0^2c_0^2\right) \end{aligned} \tag{5.109}$$

which reduces to the Starobinsky action on shell, $R = 3m_0^2c_0$. Now we proceed by a conformal transformation (as we did for the Starobinsky action), $\hat{g}_{\mu\nu} = e^{\ln(1+c_0)}$, which we call the field Ψ and its corresponding dimensionful field should be proportional to $\overline{\Psi} \sim \frac{\Psi}{\sqrt{G}}$. Assuming metricity of the transformed metric and also (for now)

[44] Derived by the equation of motion of the action, taking the divergence and tracing and substituting back in. We linearise around flat space.

constant field values the eom is then given by

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-\hat{g}}(\hat{R} - \frac{3}{2}m_0^2(1-e^{-\Psi})^2 - \frac{3}{2}(\hat{\nabla}\Psi)^2) \tag{5.110}$$

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R} + \frac{3}{4}\left((\hat{\nabla}\Psi)^2 - 2\hat{\nabla}_\mu\Psi\hat{\nabla}_\mu\Psi + m_0^2\hat{g}_{\mu\nu}(1-e^{-\psi})\right) = 0 \tag{5.111}$$

$$\hat{R} = Re^{-\Psi} \tag{5.112}$$

We see that the sign in front of the kinetic term in the action is *negative* as expected. A positive mass squared of the scalar field gives a potential that diverges for $\Psi \to -\infty$ and saturates for $+\infty$ with a global minimum at the origin which makes it a likely candidate for inflation as we have already seen. We have introduced a new dof.
The same treatment is now applied to the *Weyl squared* term W^2 using some results of Stelle and Magnano (1978, 1990). We introduce an auxiliary field, a symmetric rank 2 tensor that should satisfy 5.106. First I rewrite the action in terms of R and $R_{\mu\nu}$,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R - \frac{W^2}{2m_2^2}\right) = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R + \frac{R^2}{3m_2^2} - \frac{R_{\mu\nu}R^{\mu\nu}}{m_2^2}\right) \tag{5.113}$$

If we define $T_{\mu\nu} = \frac{1}{m_2^2}(2R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)$ we need $Q_{\mu\nu} = \frac{m_2^2}{2}(T_{\mu\nu} - g_{\mu\nu}(g_{\rho\sigma}T^{\rho\sigma}))$ in order to rewrite the Lagrangian in analogy to the R^2 contribution only.

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R - T^{\mu\nu}Q_{\mu\nu} + \frac{m_2^2}{4}\left(T_{\mu\nu}T^{\mu\nu} - (T_{\mu\nu}g^{\mu\nu})^2\right)\right) \tag{5.114}$$

which indeed gives the eom for Q. We aim to show that T satisfies 5.106 adding new five dof to the already present two dof of the metric. We will see that the equations are exactly the ones given by Fierz and Pauli with $\partial \to \nabla$. Before calculating the eom (which is basically the one I gave for the Starobinsky one) we should only get the term proportional to m_2^2 which is $(T_{\mu\nu} - g_{\mu\nu}T)$ with $T = g_{\mu\nu}T^{\mu\nu}$. This term indeed satisfies 5.106 as taking the divergence vanishes and the KGE is indeed given by $\nabla^\mu\nabla^\nu T' + \frac{3}{2}m_2^2 T$ giving $T = 0$ [45]. A quick look (we want to have it in the form $R_{\mu\nu} + \frac{m_2^2}{4}(T_{\mu\nu}T^{\mu\nu} - T^2)$) gives us the transformation of the metric:

$$\sqrt{-\hat{g}}g_{\mu\nu} = \sqrt{-g}\left((g_{\mu\nu} + \frac{1}{2}Tg_{\mu\nu} - T_{\mu\nu}\right) \tag{5.115}$$

This gives us a transformed $\hat{T}_{\mu\nu}$ and the metric should be substituted by

$$g_{\mu\nu} = (\det\ldots)^{-\frac{1}{2}}\left(\left(1+\frac{1}{2}\hat{T}\right)\delta^\rho_\mu - \hat{T}^\rho_\mu\right)\hat{g}_{\rho\nu} \tag{5.116}$$

where ... is the nonzero determinant of the term above. Finally, we can write the action that clearly shows the problem of the new spin 2 dof:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-\hat{g}}\left(\hat{R} - \hat{g}^{\mu\nu}\Gamma^\sigma_{\mu[\rho}\Gamma^\rho_{\nu]\sigma} + \frac{m_2^2}{4}(\det\ldots)^{-\frac{1}{2}}(\hat{T}_{\mu\nu}\hat{T}^{\mu\nu} - T^2)\right) \tag{5.117}$$

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}(g^{-1})^{\sigma\rho}\left(\hat{\nabla}_{(\mu}g_{\nu)\rho} - \hat{\nabla}_\sigma g_{\mu\nu}\right) \tag{5.118}$$

[45] $T' = m_2^2(T_{\mu\nu} - g_{\mu\nu}T) = R^\rho_{(\mu}(T_{\rho\nu)} - \frac{1}{2}g_{\rho\nu)}T) - \frac{1}{2}g_{\mu\nu}R^{\rho\sigma}(T_{\rho\sigma} - \frac{1}{2}g_{\rho\sigma}T) + \nabla_\mu\nabla_\nu T + \nabla^2 T_{\mu\nu} + g_{\mu\nu}\nabla^\rho\nabla^\sigma T_{\rho\sigma} - g_{\mu\nu}\nabla^2 T - \nabla_{(\mu}\nabla^\rho T_{\rho\nu)}$

Again det $\neq 0$ (for the R^2 we had $c_0 > -1$). Indeed, the spin conditions are satisfied. The kinetic term is $\sim \Gamma\Gamma$ and if we expand around $\hat{T}_{\mu\nu} = 0$ (as Fierz-Pauli treatment, for flat space we get Stelle's result) we get the same result with $\partial \to \nabla$ and indeed the '*wrong*' sign, $\mathscr{L} \sim +\frac{m_2^2}{4}(\hat{T}_{\mu\nu}T^{\hat{\mu}\nu} - T^2)(1+\hat{T})$, a massive spin 2 field (tachyonic for $m^2 < 0$ and normal for $m^2 > 0$).[46]
Combining the results the particle spectrum can be nicely seen in the Lagrangian. We simplify $\hat{T}_{\mu\nu} \sim \hat{g}_{\mu\nu}\hat{T}$

$$\mathscr{L} \sim Re^{\Psi} - \frac{3m_0^2}{2}(e^{\Psi} - 1)^2 - Q_{\mu\nu}T^{\mu\nu} + \frac{m_2^2}{4}(T_{\mu\nu}T^{\mu\nu} - T^2) \tag{5.119}$$

with the scalar field $\Psi = \log(1 + c_0)$. I will come back to this when calculating the mass of the ghost. Thus, we have a total of 5+2+1 dof. Stelle decomposed a symmetric tensor and used the TT decomposition to separate the action into different masses and helicities where clearly the spin 2 field appeared with a relative sign to the other fields.
To put a bound on the masses it was emphasised that astronomical measurements are rather unhelpful. The correction is of order $\sim e^{-mr}$ 5.35. Taking, for example, the Mercury precision of $\sim 10^{-9}$ with a radius of $5 \cdot 10^9$m this gives a lower bound of $4 \cdot 10^{-11}$cm. The application to black holes ([78] and proceeding works) give a 5-parameter family which can be split into 1 (usual Schwarzschild) + 2 + 2 (one of which gives at suitable boundary conditions asymptotic flatness). The sign of the coupling of the scalar field has caused some confusion with either a real mass growing as energy increases (Avramidi & Barvinsky, 1985[47]) or negative/complex which leads to mathematical consistency, but is phenomenologically rather difficult due to its tachyonic instability. In [78] it was already emphasised that the masses are real for positive $f_0^2, f_2^2 > 0$ for $m_0^2 \sim f_0^2$ as otherwise we would have oscillatory $\frac{1}{r}$ terms. Note also that HD applied to low-mass Schwarzschild black holes give rise to Gregory-Laflamme instability for $M \geqslant M_{max}$ with $M_{max} \sim \frac{m_p^2}{m_2}$ (Stelle, e.g. 2017).

5.6.4 Further Higher Derivative AS Inflation

We have already encountered the Starobinsky inflation and Stelle's gravity as HD theories. So far a NGFP has been calculated in a large number of different truncations. Interestingly, the GFP of EH doesn't generalise to an additional R^2 term, whereas the NGFP does. In fact, it is given by almost the same properties.
Let us recall the fourth order Lagrangian,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R^{\mu\nu\rho\sigma}_{\mu\nu\rho\sigma}) \tag{5.120}$$

In this form α is dimensionful, β and γ are dimensionless and the action can be reduced to Einstein Hilbert with $\Lambda = 0$, $\gamma = 2$. We use the fact that in four dimensions the Gauss-Bonnet term vanishes and rewrite the action in terms of R^2 and

[46] A rigorous should also investigate the case if we do not expand around 0 i.e. whether another vacuum state wouldn't produce a ghost.

[47] The Euclidean action was found to be AF in all essential couplings for positivity/negativity of the couplings. Later on it was shown that in the Lorentzian setting we need the R^2 term of different sign in order to exclude tachyonic instability.

$W^2 = C^{\mu\nu\rho\sigma}_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma}_{\mu\nu\rho\sigma} - 2R^{\mu\nu}_{\mu\nu} + \frac{R^2}{3}$ only. Let us also rewrite it such that Starobinsky's inflation and the spin 2 ghost are encapturated.

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R + \frac{R^2}{6m_0^2} - \frac{W^2}{2m_2^2}\right) \tag{5.121}$$

The masses are given by the original $(m_o)^{-2} = 6\alpha + 2\beta + 2\gamma, (m_2)^{-2} = -\beta - 4\gamma$ where Stelle showed that the R^2 term introduces a new scalar field with mass m_0 and W^2 introduces a massive spin 2 ghost of mass m_2 (note that in Stelle's treatment we first proved that the coefficient in front of R is indeed the EH one), $m^2 > 0$ introduces non-tachyonic fields, $m^2 < 0$ tachyonic fields. In recent AS paper (Ohta et al.) we again find the higher derivative action

$$u_0 + u_1 R - \frac{\omega}{3\lambda}R^2 + \frac{1}{2\lambda}W^2 + \frac{\theta}{\lambda}E \tag{5.122}$$

where it turns out that the standard EH action gives $u_1 = -\frac{1}{16\pi G}$ and for $\Lambda \neq 0$ $u_0 = \frac{\Lambda}{8\pi G}$, E is the Gauss Bonnet term that vanishes in 4d $E = R^2 - 4R^{\mu\nu}_{\mu\nu} + R^{\mu\nu\rho\sigma}_{\mu\nu\rho\sigma}$. Following the FRGE the authors find an NGFP. Interestingly, the inclusion of the coupling of the Weyl term stabilises the coefficients at the fixed point. Taking the RG flow into account the mass of the additional fields is not constant, $m_2^2 = \frac{\lambda}{16\pi G}$ grows as k^2, we should evaluate the mass at the point $p^2 \sim k^2$. Depending on the sign of λ at the FP the behaviour is different. For $k^2 = -m^2(k^2)$

1. $\lambda_* > 0$ the effective mass diverges as $k \to \infty$ meaning that it decouples in the UV

2. $\lambda_* < 0$ the mass behaves like $m^2(k^2) \sim ck^2$ depending on the product of the couplings at the fixed point, $c = -g\lambda|_*$

This would indicate a strong coupling of the ghost at low energies.
As Litim et al. and Niedermaier put forward, former with a *bootstrap strategy* based on the hypothesis that canonical power counting should remain a valid principle at and near the FP[48]. Recent work [44] includes the effective actions of $\Gamma_k \sim F_k(X) + RZ_k(X), X = \alpha R^2 + \beta R^{\mu\nu}_{\mu\nu} + \gamma R^{\alpha\beta\gamma\delta}_{\alpha\beta\gamma\delta}$ with F, Z being powers of X where $(\alpha, \beta, \gamma) = (0, 0, 1)$ is put forward as a good action to study inflation on spheres. Here it also seems that a cubic term in curvature becomes relevant changing the dimension of the UV critical surface (recall, the number of relevant directions of the FP which should be finite to satisfy predictivity) which until now stayed the same as adding HD terms does not add more relevant directions.
In previous works the Einstein Hilbert truncation was added by R^2 [36], $R^2 + \text{Riem}^2$ (e.g. Ohta, Percacci, 2014), $R^2 + \text{Ricci}^2 + \text{Riem}^2$ (Falls, Ohta, Percacci, 2020), then generic polynomials of $f(R)$ (Falls, Litim, Schröder, 2019), with the Goroff Sagnotti counterterm (Rechenberger, Saueressig 2012) 5.39... All of which found a FP and some physical interpretation. For further truncations see 5.1.

[48]This 'can be verified a posteriori' since the new quantum scaling dimension by adding a new operator of higher mass dimension should be more irrelevant than the one given at the order prior, but the other scaling dimensions are only slightly shifted [34].

The critical exponents can be real in some HD theories whereas EH gives complex conjugate pairs that cause the spiralling into the NGFP. It is not clear whether this is due to the approximations or whether this has some importance.
The results that were obtained in the past years should be analysed which truncations might explain inflation.

5.6.5 Ghosts Once More

The mass of the ghost is dependent on the couplings and is also given by the pole[49]

$$\frac{1}{m^2}\left(\frac{1}{k^2} - \frac{1}{k^2 \pm m^2}\right) \tag{5.123}$$

where the sign in the propagator is dependent on the sign of the coupling of the Weyl squared term, for a negative coupling we have a normal ghost, for a positive one we have a tachyonic one. In fact, the $\frac{1}{k^4}$ dependence makes the theory renormalisable.
One might also wonder when the issue of ghosts 'physically' becomes a real problem. Following simple kinematics in the rest frame (it still is a relativistic theory) [50] the production of a scalar ghost particle requires at least (assuming the ghost at rest so $(\vec{p}, E) = (\vec{0}, E_g = -m_g)$ $|\vec{p}| \sim \frac{m_g}{2}$ if we assume a heavy ghost with $m_g >> m_i, m_f$ wrt to the initial $(\vec{p}, E_i)$ and final $(-\vec{p}, E_f)$ particles masses.

$$|\underline{p}|^2 = \frac{1}{2m_g 2m_g}(m_g^2 - (m_i + m_f)^2(m_g - (m_i - m_f)^2) \sim \frac{m_g^2}{4} - \frac{1}{2}(m_i^2 + m_f^2) + ... \tag{5.124}$$

as the ghost mass dominates over the initial and final particle, the second term can be ignored. Calculations show that the ghost mass is Super-Planckian. Stelle already concluded in 1977 that HD theories might be used as effective theories in the quest for a quantum theory of gravity since the underlying 'undesired' effects become apparent at scales near/below the Planck scale. Using our previous results we know that the mass in Stelle's theory in 5.122 form is given by

$$m_0^2 = \frac{\lambda}{32\pi G\omega}, \quad m_2^2 = \frac{\lambda}{16\pi G} \tag{5.125}$$

In [77] they argue to take the mass at the FP where we know how the couplings behave. A more rigorous treatment would use the RG equations to calculate the mass away from the FP. In the RG approach we need to find a reference scale μ_0, here

$$\frac{1}{16\pi G} = \frac{\mu_0^2}{g_N(\mu_0)} = m_p^2 \leftrightarrow \mu_0 = \sqrt{g_N(\mu_0)}m_p, m_p = \sqrt{16\pi G} \tag{5.126}$$

[49] After linearising the theory one can add a field s.t. $\int d^4x(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\lambda\partial_\mu\phi\Box\partial^\mu\phi + J\phi$ where λ is the Weyl coupling and then solve for $\phi = \frac{\frac{J}{\lambda}}{k^2(k^2+\frac{1}{\lambda})} = \frac{J}{k^2} - \frac{J}{k^2+\frac{1}{\lambda}}$ using partial fractions in momentum space, massless graviton + massive spin 2 field. Tomboulis (1977), Salam (1978) and others suggested that unitarity can be restored if quantum corrections are taken into account. Those could transform the real (nonphysical) into a complex conjugate pair of poles. A lot of research has been dedicated to the resolution of ghosts in HD theories.

[50] Stelle, talk during 'Quantum Spacetime and the Renormalization Group' 10/2020.

Substituting this into the formula for the mass of the ghost we will see that the scale at which ghosts are produced is above the Planck scale, see also 7.13.5.
Hence, the theories that are renormalisable but contain massive ghosts should remain valid for very HE until ghost are produced. Nonetheless, we cannot simply ignore the issue as we would like to have a theory of quantum gravity that is valid at all energy scales. It has also been proposed that quantum corrections might make unitarity possible via destabilising the massive s=2 ghost (Donoghue, Menezes, 2019).

5.6.6 Finite Action

Barrow and Tipler (1988) [81] claimed that as in quantum mechanics physical solutions are described by a finite (Euclidean) action so should the universe's evolution correspond to a finite total action[51]. Thus, infinities in the 'path integral' of the universe should sort out nonphysical models or put constraints on the ones we are looking for. They assume spatial finite volume that approaches 0 as $t \to 0$ and the time integral at a fixed arbitrary time back to $t \to 0$ has to be finite. With that they actually give very powerful constraints on the properties and topology of the universe such as

- the universe is closed
- initial and final singularities exist, the universe has a finite lifetime
- there are no bouncing or indefinite cyclic universes
- no HD theories are applicable (at least very harsh constraint on quadratic terms)
- certain constraints on possible matter

A recent follow-up added the exclusion of compactifications of flat and open universes, constraints on inhomogeneties and isotropies and further constraints on HD actions.
With that in mind, Stelle and Lehners [77] analysed the effect of 5.122 ($\lambda \to \sigma$ and adding Λ) in 4d with the help of Niedermaier's Euclidean β-functions at 1-loop order where the scale-dependent (μ) couplings are given by

$$g_N = \mu^2\kappa^2, \quad \kappa^2 = 16\pi G, \quad \Lambda = \lambda\mu^4 \tag{5.127}$$

such that g_N, λ are dimensionless (σ and ω are dimensionless by definition). From before we know that the theory is perturbatively renormalisable and adding AS makes it possible to trust the theory at high energies such as near the Big Bang. Knowing the β-functions and the FP we can calculate the mass of the new scalar and ghost fields.[52] They fluctuate around a background for which the kinetic term is $\sim (\partial^2\hat{h})^2$ for a rescaling of $\sqrt{\ln\mu}h = \hat{h}$.

$$m_0^2 \sim \frac{\mu^2}{\ln\mu} \quad m_2^2 \sim \frac{\mu^2}{\ln\mu} \tag{5.129}$$

[51] $S = S_{EH} + S_{matter}$ + boundary terms each of which should be finite on its own.
[52] Niedermeier (2010):

$$\mu\frac{d}{d\mu}c = f_c(g_N, \lambda, \sigma, \omega), c = g_N, \lambda \quad \mu\frac{d}{d\mu}\sigma \sim \sigma^2 \quad \mu\frac{d}{d\mu}\omega = f_\omega(\sigma, \omega) \tag{5.128}$$

giving a FP for finite positive $g_{N*}, \lambda_*, \omega_* = -0.0228, \sigma_* = 0$.

Note that the scalar is tachyonic for $\omega_* < 0$. The FP is dependent on the scheme chosen (whereas the critical exponents are believed to be universal). Repeating the same calculation with Ohta&Percacci's HD treatment (2014) (which is done in arbitrary dimension and on general backgrounds) and Codello&Percacci (2006) the mass behaviour is the one given by S&L. Precisely, we know from the RG treatment of Niedermaier and Stelle's mass formula

$$\mu \sim |\vec{p}| \sim \frac{1}{2} m_2(\mu) \tag{5.130}$$

$$\mu \frac{d\sigma}{\dot{\mu}} = -c\sigma^2, \quad c \sim 0.08 \tag{5.131}$$

$$m_2^2 = \frac{\mu^2 \sigma}{g_N} \sim \frac{\mu^2}{c g_* \ln \frac{\mu}{\mu_0}} \tag{5.132}$$

substituting the second into the third equation. Since we identify the reference scale with $\sqrt{g_N} m_p$ and the flow changes only slowly near the FP, we have $g_N(\mu_0) \sim g_{N*} \sim 0.42$.

$$\leftrightarrow m_2^2 \sim (2\mu)^2 \sim \mu^2 \sim \frac{\mu^2}{c g_{N*} \ln \frac{\mu}{\mu_0}} \to \mu \sim \mu_0 e^{\frac{1}{4 c g_{N*}}} \sim 0.648 e^{7.44} m_p \sim 1.1 \cdot 10^3 m_p \tag{5.133}$$

Hence, the theory should be valid up to very high Super-Planckian scales. An additional thought on that can be found in the appendix, 7.13.5.

They further analysed possible anisotropies and inhomogeneities in the early universe. For former they rewrite the action in terms of the BianchiIX metric[53] and investigate how the HD terms change from the expected evolution of the universe. Other than speeding up the crunch there is no significant change. The only way to get a finite action is to choose the anisotropy parameters $\beta_\pm \to 0$ as $t \to 0$ ($l_i \propto e^{\beta_\pm}$ where α is the spatial volume and β the shape of the spatial slicings) forcing the anisotropies to remain small during the expansion.

Next, they rewrite the action in terms of the LTB metric where the scale factor is replaced by $A = A(t, r)$ and an inhomogeneity measure in terms of $F(r)$ is introduced ($ds^2 = -dt^2 + \frac{A'^2}{F^2} dr^2 + A^2(d\theta^2 + \sin^2\theta d\phi^2)$). Whereas EH doesn't give any constraints on when the integral would diverge as $t \to 0$, the HD theory gives the constraint that only a homogeneous universe is allowed since $F(r) \to 1$ in order for the action to remain finite. Furthermore the scale factor undergoes accelerated expansion, $A \sim t^s, s > 1$ which is the definition of inflation. Thus, the R^2 and Weyl^2 terms are necessary in order for the *selection principle* to set in to allow a primordial homogeneous and isotropic universe that undergoes accelerated expansion in the early phase. Their findings contribute to Penrose's Weyl curvature hypothesis with a homogeneous and isotropic universe with low Weyl curvature (=low entropy) in the beginning which causes the second law of thermodynamics to arise.

It would be interesting to apply the same strategy to other HD theories and investigate which curvature invariants are necessary in order to put selection constraints on the universe (i.e. which truncations make the integral divergent) and also analyse the setting with other metrics.

[53] This is the metric they also used to analyse black holes in HD gravity, $ds_{IX}^2 = -dt^2 + \sum_m (\frac{l_m}{2})^2 \sigma_m^2$ with σ being different forms on the 3-sphere. The metric gives typical oscillatory BKL behaviour.

5.6.7 Scale Invariance

Recall that the homogeneity and isotropy on large scales shows a *scale invariant* behaviour (galaxies cluster in galaxy clusters with large voids in between, primordial fluctuations cause the LSS today etc). The critical scaling around the NGFP depends on the effective Lagrangian (i.e its (modified) gravity terms, matter content etc.) such that for a certain power spectrum, in our case a (nearly) scale invariant one, for example, we have constraints to put on the critical exponents. The FP itself depends on the gauge fixing and the cutoff but still shows some kind of general properties (e.g. g_*, λ_* both in quadrant I), the critical exponents are assumed to be universal. If we want to follow this idea we should calculate the FP and critical exponents for all couplings and then rewrite the running couplings in terms of those, e.g. for EH

$$g_k = g_* + \alpha f_{11} \left(\frac{k}{k_0}\right)^{-\theta_1} + \beta f_{12} \left(\frac{k}{k_0}\right)^{-\theta_2} \tag{5.134}$$

$$\lambda_k = \lambda_* + \gamma f_{21} \left(\frac{k}{k_0}\right)^{-\theta_1} + \delta f_{22} \left(\frac{k}{k_0}\right)^{-\theta_2} \tag{5.135}$$

where we linearised the β-functions around the FP, f are the corresponding eigenfunctions and the $-\theta$ the eigenvalues of the stability matrix. The parameters are given by the identification and RG improvement (we select a trajectory by identifying a scale $k = x$). Adding HD terms we do the same and in the end we analyse the RG improvement under which conditions inflation is possible and under which it produces the power spectrum we measure. It is tempting to associate the scale symmetry of the fluctuations with the FP in the UV and we have indeed seen that although inflation might be well below the Planck scale that the running near the FP and Planck regime is rather slow.
At the FP we have *exact scale symmetry*. An interesting thought has been put forward by Wetterich that scale invariance as dimensionless ratio of scales a priori doesn't tell us what to favour. Precisely, we know that the universe expands by measuring (indirectly) the velocity and distance of galaxies. The ratio of, say an elementary particle, and the distance to galaxies, $\frac{d_{\text{galaxies}}}{d_{\text{particle}}}$ stays the same under the expansion of the universe and the shrinking of the particles' size. The frequency emitted from a particle is proportional to its mass, the wavelength proportional to its size. We could simply follow standard cosmology and alternatively describe a universe with shrinking particles/exponentially growing mass (Wetterich, 2013). This leads to the questions whether an alternative description of observables is indeed viable and what the physical meaning of the dimensionless couplings or the FP is.
There is a nice proof to show that the RG flow produces a nearly scale invariant power spectrum (Mottola et al. (1996)). We know that the effective graviton propagator goes like k^{-4} at the background level i.e. we have a two-point correlator $\log(x-y)^2$. The curvature δR is proportional to $\partial^2 h$ which means $< \delta R_x \delta R_y >$ is proportional to $|x-y|^{-4}$. We also know that the Einstein equations gives us a curvature δR proportional to $\delta\rho$ and correlation function 3.99 $\zeta(x) =< \delta(x)\delta(0) >$ is then proportional to $\frac{1}{x^4}$ If we Fourier transform this we get $|\delta_k|^2 \propto |k|$ i.e. it is completely scale invariant! Here we see that is is necessary to not evaluate directly at the NGFP, but a bit away such that we only get nearly scale invariance.

5.6.8 Entropy Production

We have seen that along the horizon and flatness problem, the formation of LSS and the issue of the enormous entropy today should have been solved by inflation. Bonanno and Reuter investigated the entropy production during inflation. The adiabatic inflationary expansion in the early universe should be described near the NGFP, they analyse a special class of RG trajectories (type IIIa, Reuter et al.) in the EH truncation with a long classical regime and small positive cosmological constant in the IR. λ decreases as the universe expands. This decrease 'pumps' energy into the matter dofs, similar to Platania's backreaction. Entropy is then naturally produced by quantum gravity effects (as we flow from the UV to the IR). A positive cosmological constant can indeed cause accelerated expansion, but as we have seen earlier it is not necessarily assumed that today's expansion is of the same nature as inflation was. Following the IIIa trajectory there is a phase of inflation caused by Λ which then dynamically ends because of the RG flow which drives Λ to near zero. They RG improve the Einstein equations and use the Hubble parameter as cutoff, G and Λ are time-dependent which changes the usual entropy treatment. They then calculate the entropy production rate for different matter types and conclude with interesting results including almost adiabatic expansion given from the RG improved equations. Near the singularity the entropy takes an integration constant S_c which is assumed to be zero i.e. all the entropy produced comes ONLY from the RG running. For further treatment see [63].

5.6.9 Mathematical Tools

Before I conclude I want to mention that the AS and within the RG treatment has also successfully improved the mathematical tools used in a wide range of physics, both perturbative and non-perturbative techniques. Just to give one example, the trace of the Wetterich equation can either be calculated by summing over the eigenvalues of the background ($\sim$ spectral sums, e.g. Benedetti (2012)) or by using heat-kernel techniques. Heat kernel coefficients arise during the computation of differential operator traces such as in the Wetterich equation and are also sometimes needed to separate the β-functions.[54] Kluth and Litim [45] derived the heat coefficients for arbitrary dimension on a sphere for Laplacians that act on scalars, vectors and tensors. In GR we would like to evaluate truncations on fully symmetric spaces and at some point also general manifolds. We don't know the spectral sums of a general manifold.

5.7 Some Remarks

It is a not entirely clear how the renormalisation scale and physical scale are related, but AS started a very promising attempt. How exactly shall we evolve the universe away from the FP? How can we describe the transition from UV (quantum) to IR

[54] When we have successfully calculated $\Gamma^{(2)}$ and found a regulator R_k this can be written as a matrix-valued function, $f(\nabla^2)$. Its trace is given by $\mathrm{Tr}\{f(-\nabla^2)\} = \int_0^\infty dt\,\hat{f}(t)\,\mathrm{Tr}\left\{e^{t\nabla^2}\right\}$ which can then be rewritten in terms of heat coefficients (Kluth, 2020).

(classical) in a clearer way? First of all we need to assume that the NGFP does indeed always exists. A large number of different truncations with different gauge and cutoff choices as well as different RG improvement levels and identification choices have so far all found a viable NGFP, but we still lack a general proof[55], ideally for a generic cutoff.

We have encountered different RG identification and improvement possibilities. Further explorations where we have to make sure that symmetry such as general covariance is satisfied and the physical scale will be matched with some observational data. The choice does effect the predictivity, on the level of solutions it is the simplest calculation, but the improvement in the eoms or even in the action enhances predictivity.

Interestingly, quadratic gravity is perturbatively renormalisable, but non-unitarity. Moreover, the R^2 and Weyl ^{2} terms do affect the dynamics of the evolution and play a major role in the early universe if one considers the finite action approach or the need for low entropy and Penrose's equivalence of Weyl curvature and gravitational entropy. R^2 curvature modifies EH to a Planck era inflation. Thus, one should continue to investigate the behaviour of HD theories. The RG techniques have proven to be an effective mathematical tool. The early universe and inflation are in the HE regime such that the calculation of UV and the analysis of the flow near the FP can give insight in the behaviour of the the theory that would otherwise break down.

Nonetheless, one should continue to match the UV and IR behaviour. Inflation is a good example how the quantum/microscopic can influence the macroscopic (primordial fluctuations, density perturbations, LSS...). The R^2 term is the only HD term that doesn't cause nonphysical results and the Starobinsky model fits Planck data which both suggests a further investigation. It is convincing to analyse truncations beyond EH since it provides results that might fit observational data better. It is also important to further analyse the Starobinsky model in AS and refine the model (start with R^2 or should it arise as quantum correction, what are the effects of the RG improvement and identification choices...?).

We should analyse which curvature terms drive inflation i.e. accelerated expansion, which terms provide a solution to a graceful exit, which ones render gravity unitary perturbatively/non-perturbatively renormalisable i.e. which terms provide a NGFP, further which terms would provide the further evolution (radiation-matter-dark energy domination) as a dynamical flow from the UV to the IR. The f(R) theories seem promising. Theories from classical and modified GR from the past decades should be analysed with the RG. Can all inflation models be realised?

Can we classify HD theories into the ones that make inflation possible? I showed that inflation takes place at HE, but it isn't assumed that it took place at $_p$ and surely not above, is the treatment near the FP really viable? Near the FP we have slow log type behaviour, but one should carefully investigate which HD terms give a FP regime that we can trust for inflation. The choice of cutoff identification doesn't seem to produce significantly different results, whereas the level of RG improvement choice does matter. Still, one should further enlarge the choices, especially the ones that are dependent on curvature since they are given by the intrinsic geometry. I haven't

[55] We can only prove its existence for all possible couplings in the large $\frac{1}{N}$ approximation where N is the number of matter fields coupled to the spacetime metric which is actually quite a viable approximation since we only have one graviton and definitely $\geqslant 17$ matter fields.

come across a treatment that uses the fourth power of the Riemann or Weyl tensor. Both are important in the description of the gravitational field, tidal forces and a proposed quantification of gravitational entropy. Moreover curvature invariants such as $C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$ are by definition accompanied by dimensionless coefficients[56] (if we add to EH the basic truncation ansatz) which already suggests that those should give viable cutoffs. Furthermore, cutoffs like $\sim$ R are rather daunting since it vanishes for the vacuum solution of the Einstein equations.

The disadvantage of the inclusion of the Weyl tensor is the loss of unitarity. However, we still don't have a proper notion of unitarity in the non-perturbative treatment so we shouldn't neglect terms that seem to lead to the break down of the theory. Suggestions are made that the problem of ghosts can be eliminated by quantum corrections (Donoghue et al.). In the RG improvement we are left with parameters that we need to measure. For the R^2 term which is already connected to a certain inflation model we can finds its coupling values at the scale of inflation with the help of cosmic data. For other terms such as the Weyl squared term this is rather difficult since we do not know how to identify (or rather) measure those terms. An explicit reconstruction with the RG equations might improve the bounds. Deeper investigations are needed. We should also be careful when counting the number of dofs. At first glance it isn't obvious that the R^2 term introduces a scalar field into the theory. Are those instabilities true physical or rather artefacts due to the approximations we used? Lastly, we haven't covered 'direct' matter-gravity models which make the treatment more complicated but are obviously necessary if we analyse the universe. Especially, if we want to investigate a possible phase of reheating we need to introduce matter fields.

The quest for a theory of quantum gravity, a consistent theory of inflation and the development of mathematical tools and frameworks such as asymptotic safety go hand in hand.

The running of the cosmological constant might give a natural explanation for its low value today (we are simply on that trajectory) and might have even driven inflation in the early universe (recall our treatment of dark energy vs inflation earlier 3.7).

Finally, the notion of observables in QG is difficult. In fact, in classical GR the nature of a 'physical' spacetime point is lost by diffeomorphism invariance. There are no local observables, but only global ones where we actually should integrate over all spacetime.

When we analyse a theory in the AS scenario we should be able to answer the following questions:

1. does a UV FP exist? (if not we should stop working on that theory)

2. what does the running and the values of the couplings tell us about

   - the possibility of inflation/how 'much' inflation is produced?
   - the quantum fluctuations during inflation?
   - the further evolution towards lower momenta? Does a graceful exit exist?

[56] Similar special treatment goes to the R^2 term which in 4 dimensions is the only power that gives a dimensionless coefficient,
$sim \int d^4x \frac{BR^2}{G}$, $-4+2-(-2)=0$.

3. does the theory produce the right IR behaviour
4. within a model, how does the gauge and cutoff choices change the RG flow behaviour?
5. within a model, how do the identification and RG improvement change the results?
6. which observables can be calculated? (n_s, r...
7. how does the result differs from the classical treatment? Are there new possible observables?

Chapter 6

Conclusion

6.1 Outlook

Inflation as accelerated expansion along with its shrinking Hubble sphere does solve the problems such as the horizon, monopole and flatness problem. It provides an answer for the LSS and the observations we find in the CMB today. However, inflation is nowadays a class of many different models. It is difficult to evaluate it as a whole. We have seen that the slow-roll models introduce an unknown scalar field and suffer from fine-tuning themselves. The chaotic model is less constrained, but gives rise to eternal inflation and a multiverse scenario which takes away predictivity, testability and the notion of probability.

Observations as done by the Planck satellites favour the Starobinsky model which is a modification of Einstein's gravity by adding an additional R^2 curvature term which then introduces a further dof in form of a massive scalar field. In order for inflation to occur the coefficient in front of the term should be rather large. We have discussed other alternative theories such as Penrose's Weyl Curvature Hypothesis and shown that inflation is not the only theory that can give an explanation for the so-called cosmological problems some of which predict a bouncing or cyclic universe which also give other predictions than inflation.

However, the framework of inflation is the only successful theory so far that solves many different problems all at once and made predictions that were validated with rather high precision. Nonetheless, we are still missing the detection of primordial gravitational waves in form of B-modes which would give a further constraint on inflation models. Inflation provides us with an idea of mechanism such as a repulsive form of gravity (false vacuum, high potential energy, ...), but not with a detailed theory of physics that could predict many details about the process itself such as the origin of an ad-hoc introduced scalar field of unknown nature, the energy scale during inflation or the reheating temperature at the end of inflation. Hawking et al. started treating inflation on the 'global' scale and already introduced the need for a theory of quantum gravity.

The high energies at which inflation is assumed to take place as well as the interface of quantum (microscale) and cosmology (macroscale) suggest a simultaneous quest. Moreover, QFTs and GR both suffer from problems within such that the standard theory of inflation which is embedded in both, cannot be safe.

GR is perturbatively non-renormalisable which is solved by asymptotic safety's claim of a non-interacting fixed point in the UV. The issue of the dimensionality of Newton's constant is tackled by setting it on the run. We have indeed seen that with the help of the renormalisation group flow a fixed point has been found in a large number of truncations, starting from Einstein-Hilbert (G and Λ) up to higher derivative theories such as Stelle's perturbatively renormalisable quadratic gravity and even to a consistent analysis of HD terms. The proposed theories have successfully found a FP that controls scaling at HE and no divergences occur, gravity as quantum theory is located in a UV critical hypersurface. AS' power lies in its predictivity and natural matching to the IR regime. Along with the RG identification and improvement on different levels and with the help of various approximation methods inflation has been successfully modelled near/at the fixed point regime by encoding QG effects near the NGFP in the effective Lagrangian or equation of motions. The effective action interpolates in a smooth way between the UV (Planck era inflation) and the IR (late time acceleration) and inflation can naturally arise from the RG flow only (with a possible connection between inflation and dark energy).

This gives a promising suggestion to further introduce different and larger truncations, regulator, cutoff and gauge choices as well as to test the same model under different RG improvement and identification choices.

Both quests should drive each other in a way that improves the analysis of inflation's behaviour at high energies and get more predictions with the help of AS. Similarly, we can use inflation to break ahead towards the QG regime at the FP where the fundamental nature of spacetime might be encoded. Nonetheless, limitations of the RG improvement and FRGE should be critically analysed, perhaps with validations from and cross-checkings with techniques from AS, but also other perhaps complimentary (non-)perturbative sectors such as methods from statistical physics, GR in the vielbein formalism, LQG (which implement a notion of quantum (discrete) geometry or the coarse graining in spin foams) or random lattice techniques (which give a notion of background independence). We have seen that tests and simulations in numerical relativity or techniques from complex analysis were further improved to solve the problems of inflation and initial conditions. Investigations of theories from modified gravity can provide further toy models and truncations. We have also seen that AS goes beyond an EFT approach to quantum gravity since it can give predictions of the Planck scale and beyond and it provides a better predictivity which is both necessary for a deeper analysis of inflation.

Some of the aforementioned points along with the emphasis of the issue on unitarity, the physical meaning of the running, the importance of background effects and other problems such as the notion of observables are also being discussed in [1]. A proper notion of unitarity in the non-perturbative regime hasn't been found and the issue of ghosts seem to rule out HD terms. Hence, we shouldn't rule out any (diffeomorphism invariant) operators per se.

An important notion should be given to the Weyl tensor (squared), one of the terms that would naturally come with a dimensionless coupling since we have seen that it can renormalise a theory, stabilise effects and is a measure for tidal distortion and perhaps for gravitational entropy. Truncations in the Weyl term, but also a cutoff identification of the fourth power of the Weyl tensor ($(C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta})^{\frac{1}{4}}$) seem like a

good continuation to previous and ongoing work. Further, should the terms that may describe inflation be ad-hoc given in a finite truncation or arise from other terms from the RG improved effective Lagrangian? The measure problem has attracted a wide range of possible cutoffs that give different results. A treatment of eternal inflation, a possible multiverse along with the issue of probability and predictivity (which lies in the heart of AS) need to be evaluated in the AS scenario.
Both, AS and inflation, should give predictions for parameters or effects we can measure. How much /less fine-tuning in contrast to alternative theories is needed in AS is a viable question.
Further treatments such as some kind of backreaction effect or a self-sustaining RG improvement and 'selection principles' such as the finite action principle or the no-boundary proposal have given promising results and should be applied to future cases.

Inflation might be right, but it is not the whole picture – just as Newton and Einstein were right when they treated their theories on a certain scale. Today we cannot answer the question Whether inflation can be made safe.

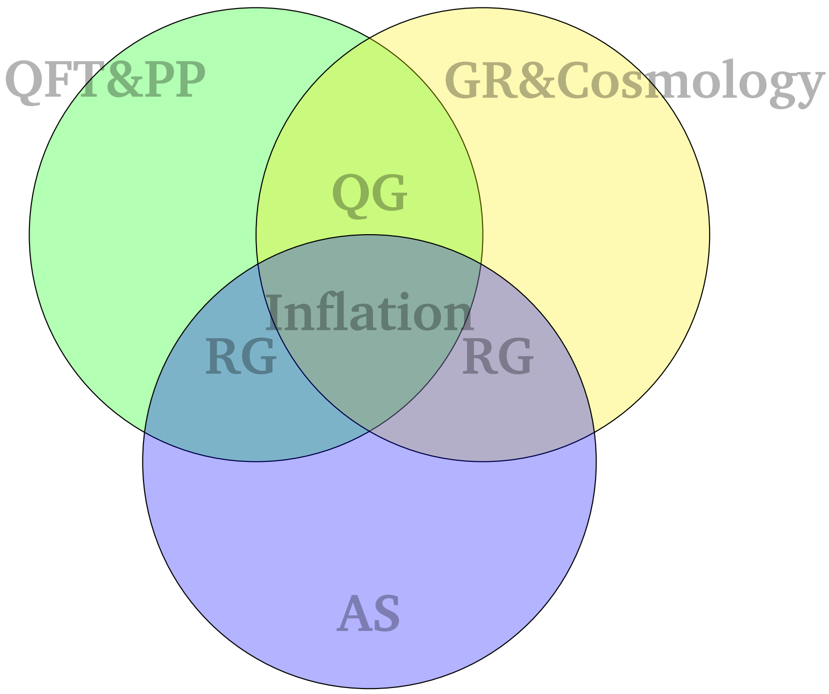
6.2 A Summary of Important Questions

I am looking forward to continue evaluating and maybe answering all those fascinating issues during my PhD.
A list of questions is attached (see also the end of 5), aside of the the issue of HD non-unitarity and the [1] critique within the AS program:

- Have we introduced the right classification of inflation models?
- How can we define a better notion of fine-tuning?
- Can we embed all inflation models in AS? Which ones differ the most from their 'classical' treatment?
- What is the cutoff during inflation?
- Are their new possible observations or predictions due to the AS embedding?
- What are typical observables during inflation/in QG?
- Which terms shall be exactly added such that gravity behaves well at those early times and give inflationary behaviour? Which truncations shouldn't be chosen?
- Which higher order curvature interactions describe phenomena such as inflation?
- Is it possible that inflation emerges as a purely (quantum)gravitational phenomenon?
- What is the energy scale at inflation?
- What other predictions does AS give in specific (and Planck favoured models) e.g. normal Starobinsky and R^2 RG improved?

- How can we extract all the relevant information encoded in the running of G or other couplings that enter the theory?
- Does AS give a notion of probability? What is the eternal inflation setting in AS?
- What is the importance of the C^2 term (and HO), what is its effect on cosmology measurements?

Outline and Review



Chapter 7

Appendix

7.1 A Brief History of the Universe

In order to have a basic overview over the universe's evolution and to bring inflation into temporal context a short overview[1]. Standard hot Big Bang theory predicts an explosive beginning of the universe of an initial singularity of infinite temperature and density. It follows an adiabatic expansion that makes the formation of matter possible. Particle physics and thermodynamics and standard cosmology can describe (most) of the universe's further evolution. Questioning what lies beyond the Planck time cannot be asked. As it is shown in this thesis it is not entirely known when exactly inflation takes place, at what energy scale or temperature (model dependent). I define the Big Bang as the theory that is well tested and proven such that inflation would then take place prior to the Big Bang.

7.2 Scale Overview

To get an overview of the scales we are talking about I list the values of the Planck scales.
The Planck mass is given by dimensional analysis or equating the Compton wavelength with the Schwarzschild radius.

$$m_p = \sqrt{\frac{\hbar c}{G}} \sim 2.2 \cdot 10^{-5}\text{g} \tag{7.1}$$

This is about the mass of a grain of sand!
Unification is expected to take place at the Planck energy

$$E_p = m_p c^2 = \sqrt{\frac{\hbar c^5}{G}} \sim 10^{19}\text{GeV} \sim 1.96 \cdot 10^9\text{J} \sim 10^{19}\text{GeV} \tag{7.2}$$

This rest energy is about the kinetic energy of an airplane. Testings at the LHC are at maximum $\sim$ 14TeV $\sim 1.6 \cdot 10^{-7}$J, so 16 orders below where we might be able to

[1] Brandenberger 'Inflationary cosmology: progress and problems', J. A. Peacock 'Cosmological Physics'

Energy (GeV)	Time (s)	Temperature (K)	relics	
$10^{19} - \infty$	0	$10^{32} - \infty$	?	?
10^{19}	10^{-43}	10^{26}	?Quantum Gravity	black holes?
10^{16}	10^{-35}	10^{26}	GUT symmetry breaking	quarks
300	10^{-12}	10^{15}	EW breaking	leptons
$?10^{15}$	$?10^{-35}$	$?10^{26}$	inflation starts	
?	$?10^{-32}$		Inflation ends	
$2 \cdot 10^5$	$10^{-12} - 10^{-6}$	$2 \cdot 10^{13}$	quark confinement	photons
		10^{-4}	Baryogenesis	neutrinos
10^4	1	10^{10}	proton/neutron ~ 6	p,n
10^3	180	10^9	Nucleosynthesis	electrons
			blackbody	
			radiation dominated	
10^{-9}	$5.6 \cdot 10^4$yr	9000	matter=radiation	
			matter dominated	
10^{-10}	10^5yr	3000	decoupling photons	CMB
	$1.5 * 10^8$	$100 - 18$	re-ionisation	
	10^8yr		star formation	
10^{-11}	10^9yr	100	galaxy formation	
10^{-4}	$13.7 \cdot 10^9$yr	2.725	today	

Table 7.1: Cosmic Overview

measure quantum gravity effects.
The Planck length is about 20 orders below the Fermi scale and 17 orders below the smallest size measured.

$$l_p = \sqrt{\frac{\hbar G}{c^3}} \sim 1.6 \cdot 10^{-15}\text{m} \tag{7.3}$$

The corresponding time is

$$t_p = \sqrt{\frac{\hbar G}{c^5}} \sim 5.4 \cdot 10^{-44}\text{s} \tag{7.4}$$

and the corresponding density can be calculated

$$\rho_p = \frac{c^5}{\hbar G^2} \sim 5.1 \cdot 10^{96} \frac{\text{kg}}{\text{m}^3} \tag{7.5}$$

Finally, the Planck temperature is given by

$$T_p = \sqrt{\frac{\hbar c^5}{G k_B^2}} \sim 1.4 \cdot 10^{32}\text{K} \tag{7.6}$$

where k_B is the Boltzmann constant. The hottest temperature produced during collisions at the LHC where at $\sim 5.5 \cdot 10^{12}$K.
I used that the fundamental constants are given by the dimensions

$$[c] = LT^{-1} \quad [G] = L^3M^{-1}T^2 \quad [\hbar] = L^2MT^{-1} \quad [k_B] = L^2MT^{-2}Q^{-1} \tag{7.7}$$

Furthermore, comparing the Planck values (equals high energy equals early times) to today's values give rather many orders of difference.
With an estimated age of $\sim 8.1 \cdot 10^{60} t_p$, a length of $5.5 \cdot 10^{61} l_p$ and a mass of $10^{60} m_p$ the values are about 60 orders bigger. Similarly, the temperature has decreased to $1.9 \cdot 10^{-32}$K, so 64 orders smaller. The major difference can be seen in the density,$1.94 \cdot 10^{-123} \rho_p$, which emphasises the need for an explanation of the dilution. See also 7.5 for a discussion on those large/small numbers.

The following two sections are oriented along the syllabus taught in the Unification and Relativity and Cosmology modules this year.

7.3 Cosmological Standard Model

Based on general relativity (large scales) and the standard model of particle physics (small scales) the cosmological standard model is set on the following pillars:

1. The *hot Big Bang model* which assumes the expansion of the universe from a singularity of infinite density and temperature. The expansion and cooling down allowed the building of matter. GR breaks down at the initial singularity, or rather at the Planck time (maybe even before) [2]. The ΛCDM model describes today's content of approximately 73% dark energy (with the cosmological constant as favourite candidate), 23% cold dark matter and 4% baryons.

2. The universe is homogeneous and isotropic on sufficiently large scales (cosmological principle).

3. The universe can be described as a hydrodynamic model. Often the Weyl Postulate is also assumed (geodesic lines don't meet in more than one single singular point meaning that there is a bundle of hypersurfaces that is orthogonal to the geodesics).

The cosmological SM was tested in high precision.

1. Abundances of light elements (hydrogen, deuterium, helium and lithium) exist in a certain ratio of abundances (nucleosynthesis),

2. Cosmic microwave background temperature anisotropy (CMB), ('Five-Year Wilkinson Microwave Anisotropy Probe Observations: Cosmological Interpretation') and blackbody nature, (A. Raghavan, 'Tracing the Evolution of Our Universe through Blackbody Photon Dynamics')

3. galaxy cluster observations, e.g. the Sloan Digital Sky Survey, ('The 3D power spectrum of galaxies from the SDSS')

[2] via dimension analysis $t_p = \sqrt{\frac{\hbar G}{c^5}} \sim 10^{43}$sec., 7.2

4. Hubble's law[3] and accelerating expansion, e.g. from distant type Ia supernovae, ('The Supernova Legacy Survey: Measurement of Ω_M, Ω_{Lambda} and w from the First Year Data Set1').

Perfect Fluid and Friedmann Equations

We follow the usual assumption of the universe being a perfect fluid and the equation of state $p = w\rho$. Hence, the energy-momentum tensor can be written as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu} \tag{7.8}$$

with $u^\mu = (1, 0, 0, 0)$. The equation of motion of Einstein's field equations is given by the *Friedmann equations*. Using the cosmological principle as starting point, place the universe in FRW [4] setting to calculate the Einstein tensor and treat the energy-matter tensor in terms of cosmological fluids, gives ten equations from which two are non-zero.

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{7.10}$$

the tt-component gives the first Friedmann equation:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}} \tag{7.11}$$

where the first term of the RHS is the matter content with ρ the energy density and the second term corresponds to the spatial geometry with $k = 1, 0, -1$ spherical, flat and hyperbolic curvature respectively. The Hubble parameter indicates that this equation can be seen as the expansion evolution. The critical density mentioned in the flatness problem 2.3 is $\rho_c = \frac{3H_0^2}{8\pi}$. Note that the critical density is a function of time.[5] Furthermore, the acceleration of the expansion is given as the sum of the tt- and rr-component:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p)} \tag{7.12}$$

---

[3]= the redshift-distance relation on large scales. In order for the expansion to preserve homogeneity we need the mean rate of change of galaxy separations to follow the Hubble law, $v = Hr$. Standard candles (same properties everywhere with period-luminosity relation in cepheid distances) are often used e.g. by the Hubble space telescope.). The universe must be expanding, a direct result of CP.

[4]Spatial homogeneity and isotropy imply a special foliation with a manifold where the cosmic time component does take a 'special' place in. The usual FRW metric of $ds^2 = -dt^2 + a^2(t)^2\gamma_{ij}dx^i dx^j$ with the expansion factor a(t) and the comoving coordinates is often written in the following two forms:

$$ds^2 = -dt^2 + a^2(t)(d\eta^2 + \left\{ \begin{array}{l} = \sin^2\eta \\ = \eta^2 \\ = \sinh^2\eta \end{array} \right\} d\Omega^2) \tag{7.9}$$

where $d\Omega$ is the volume element of the 2-sphere followed by the related terms for curvature $k = 1, 0, -1$ respectively, and in areal coordinates: $ds^2 = -dt^2 + a^2(t)(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2)$.

[5]For the current Hubble 'constant' - another big mystery - of $H_0 = 100h\text{km/sec/Mpc} = h(9.78\text{Gyr})^{-1}$ and WMAP's data for $h = 0.7$ this give $\sim 1.88 \cdot 10^{-32} h^2 \frac{\text{kg}}{\text{cm}^3}$ giving $\sim 9.21 \cdot 10^{-27} \frac{\text{kg}}{\text{cm}^3}$.

Those can be combined to give the equation of conservation

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(p + \rho) = 0} \tag{7.13}$$

This gives indeed the energy conservation since assuming $p = \omega\rho$ and the first law of thermodynamics $dE = -pdV$, this can be rewritten

$$d(\rho a^3) = -pd(a^3) \rightarrow \rho d(a^3) + a^3 d\rho = -\omega\rho d(a^3) \tag{7.14}$$

Integration leads to the same result as above

$$\int \frac{d\rho}{\rho} = -(1+\omega)\int \frac{d(a^3)}{a^3} \rightarrow \rho \propto a^{-3(1+\omega)} \tag{7.15}$$

[6] Similarly, $\nabla^\mu T_{\mu\nu} = 0$ results in the previous equation. Thus, we arrive at two independent equations and the dynamical $a(t), \rho(t), p(t)$ where the density and the pressure are correlated via the corresponding matter content. After integrating the conservation equation, it is helpful to collect all relations dependent on the form of matter. This can again trace back to the history of the universe. The radiation epoch dominates as $a \rightarrow 0$, then the matter epoch dominates and finally the cosmological constant. The relations are given for $k = 0$.

	radiation	matter	vacuum
ω	$\frac{1}{3}$	0	-1
ρ	$\sim a^{-4} \sim t^{-2}$	$\sim a^{-3} \sim t^{-2}$	const.
p	$\sim \frac{1}{3}\rho$	0	$-\rho$
$a(t)$	$\sim t^{\frac{1}{2}}$	$\sim t^{\frac{2}{3}}$	$\sim e^{\sqrt{\frac{\Lambda}{3}}t}, e^{H_0 t}$
H	$\frac{1}{2t}$	$\frac{2}{3t}$	$\sqrt{\frac{\Lambda}{3}}$=const.
horizon	$\frac{2t}{a} = 2t$	$\frac{3t}{a} = 3t$	

A particle from $w = \frac{1}{3}$ to 0 goes from relativistic to non-relativistic. A scalar field with 0 potential would have a value of 1. Intuitively, degenerate gas might have a higher value than $\frac{1}{3}$, but so far there is no experimental or theoretical evidence. Values below $-\frac{1}{3}$ give exponential expansion, with -1 being today's dark energy (assumed to be)[7] that makes the accelerating late time expansion possible and inflation having a value of ~ -1. Values even below that are called phantom energy and would cause a Big Rip of the universe.

[6] The density is intuitively for matter anti-proportional to the expansion factor cubed as the it presents the expanding volume of the universe. For radiation the additional redshifting makes its dependence fourth order.

[7] An alternative would be quintessence where ω is dynamic, the ratio of p and ρ of a hypothetical scalar field 3.7.

7.3.1 SR

It is useful to calculate the general form of a for a given potential that depends on the scalar field ϕ. SR gives

$$
\begin{aligned}
H = \frac{d \ln a}{dt} \sim \sqrt{\frac{8\pi V}{3}} \\
3H\dot{\phi} =\sim -V'
\end{aligned}
\tag{7.16}
$$

which can be combined to

$$
\dot{\phi}\frac{d \ln a}{d\phi} \sim -\frac{V'}{3H}\frac{d \ln a}{d\phi}, \quad -d \ln a \frac{8\pi V}{V'} d\phi \tag{7.17}
$$

The scale factor can then be written in terms of the potential which itself is dependent on ϕ:

$$
a(\phi) \sim a_i e^{8\pi \int_{\phi_i}^{\phi} \frac{V}{V'} d\phi'} \tag{7.18}
$$

potential	V	a	n_s	r
polynomial	$\sim \phi^n$	$\sim exp(\phi^{\frac{n}{2}} kt)$		
- monomial	$\lambda m_p^4 \left(\frac{\phi}{m_p}\right)^n$		$1 - 2\epsilon\left(\frac{n+2}{2}\right) = 1 - \frac{n+2}{2N*}$	$\frac{-8n}{n+2}(n_s - 1) = \frac{4n}{N*}$
- chaotic, $n = 2$	$\frac{1}{2} m^2 \phi^2$	$exp(k\phi t)$	$1 - \frac{2}{N*} = 0.967$	$\frac{8}{N*} = 0.133$
$n = 4$	$\lambda \phi^4$	$exp(k\phi^2 t)$	$1 - \frac{3}{N*} = 0.95$	$\frac{16}{N*} = 0.267$
powerlaw	$exp(k\sqrt{\frac{2}{3n}} \frac{\phi}{m_p}), n > 1$	t^n	$1 - \frac{2}{n}$	$\frac{16}{n}$
$n = 40$			0.95	0.4
natural	$\Lambda^4(1 + \cos\frac{\phi}{f}), f > m_p$			
hilltop	$\Lambda^4(1 - \left(\frac{\phi}{\mu}\right)^p + ...)$		< 1	~ 0[8]
hybrid	$\Lambda^4(1 + \left(\frac{\phi}{\mu}\right)^p + ...)$		$1 - \frac{1}{N*} = 0.983$	
Starobinsky	$\Lambda^4(1 - e^{-\sqrt{\frac{2}{3}} \frac{\phi}{m_p}})^2$		$1 - \frac{2}{N*} = 0.967$	$\frac{12}{N*^2} = 0.003$
Higgs	$\lambda(\phi^2 - m^2)^2$		$1 - \frac{8(4N*+9)}{(4N+3)^2} = 0.966$	$\frac{192}{(4N*+3)^2} = 0.003$
intermediate	$\phi^{\mathscr{F}}, \mathscr{F} = 4(\frac{1}{n} - 1)$	e^{t^n}		

Table 7.2: Different potentials and their predictions. $N* = 60$, between horizon crossing and end of inflation and $k = \sqrt{\frac{8\pi}{3}}$

Further restriction on the parameters can be calculated with help of the density perturbation constraint, $\frac{\delta\rho}{\rho} \sim 10^{-5}$
One can also differ between large field (LF) and small field (SF) potentials. Former has a positive value for the second derivative wrt ϕ whereas latter can change the sign. They predict different evolutions and for example different $n_s - r$ values. For example, the hilltop potential for $\phi < \mu$ is SF if $\phi < \lambda << m_p$ and LF if $\lambda \sim m_p$. SF predicts $0 < \epsilon << \eta$, a scalar spectrum index of smaller one and r almost vanishing whereas LF predicts $r \leqslant -\frac{8}{3}(n_s - 1)$ (and weirdly $\eta < 0$).
Initially, LF models are away from the stable minimum, but evolve to the state during the end of inflation. SF models, however, simply evolve away from an unstable maximum. Hybrid models' end state is a minimum (vacuum) other than 0.

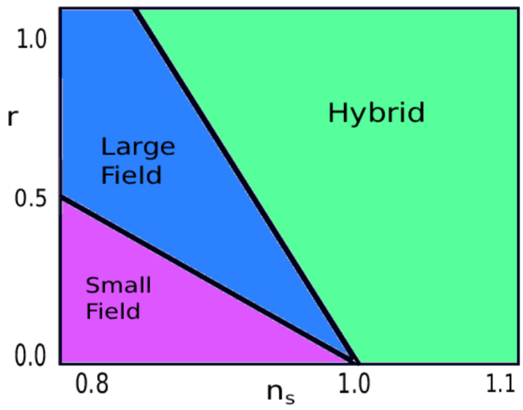


Figure 7.1: SF and LF potentials predict different value ranges of n_s and r. Taken from 'Inflationary Cosmology: From Theory to Observations', 2018. Today's data favours the area around where SF/LF meet.

7.3.2 Energy Conditions

There is a hierarchy of conditions.

1. strong energy condition: $(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)V^\mu V^\nu \geqslant 0$.
2. dominant energy condition: $T_{\mu\nu}V^\mu \leqslant 0 +$ WEC.
3. weak energy condition: $T_{\mu\nu}V^\mu V^\nu \geqslant 0$.

where V^μ are future-pointing timelike for the first and third and timelike or null for the second. Note that the WEC part is nothing else than the total mass-energy density and the DEC gives minus the pressure. The SEC part is proportional to the Raychaudhuri scalar. There is also the null energy condition, $T_{\mu\nu}V^\mu V^\nu \geqslant 0$, for every future-pointing null vector field V.
Obviously, dark energy 3.7 and inflation violate the SEC.
Postulating that the universe behaves like a perfect fluid gives

1. NEC: $\rho + p \geqslant 0$
2. WEC: NEC $+ \rho \geqslant 0$
3. DEC: $\rho \geqslant |p|$
4. SEC: NEC $+ \rho + 3p \geqslant 0$

meaning dominant implies weak, weak implies null and strong implies null as well. Again it is obvious that the false vacuum during inflation violated the SEC.
I can also rewrite this in terms of the FRW metric (including curvature),

$$\begin{aligned} \rho &= \frac{3}{8\pi G}\left(\left(\frac{\dot{a}}{a}\right) + \frac{k}{a^2}\right) \\ p &= -\frac{1}{8\pi G}\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) \end{aligned} \tag{7.19}$$

to

1. NEC: $-\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \geqslant 0$
2. WEC: $\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \geqslant 0$
3. DEC: $-2\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) \leqslant \frac{\ddot{a}}{a} \leqslant \left(\frac{\dot{a}^2}{a^2}\right) + \frac{k}{a^2}$
4. SEC: $\frac{\ddot{a}}{a} \leqslant 0$

again shows that SEC is violated by inflation (simply by its definition).

Furthermore, when calculating the expansion factor during inflation curvature was assumed to be negligible.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} \;\rightarrow\; a(t) = a_0 e^{Ht} \text{ with } H = \sqrt{\frac{8\pi G\rho}{3}} \tag{7.20}$$

where a possible negative exponent would decay quickly. Note for arbitrary k this gives

$$dt = \left(\frac{8\pi G\rho a^2}{3} - k\right)^{-\frac{1}{2}} da \tag{7.21}$$

$$t = \frac{1}{H}\cosh^{-1}(Ha) \quad k = +1 \quad a(t) = \frac{1}{H}\cosh(Ht) \tag{7.22}$$

$$t = \frac{1}{H}\sinh^{-1}(Ha) \quad k = -1 \quad a(t) = \frac{1}{H}\sinh(Ht) \tag{7.23}$$

both of which tend to the flat space's result as $t \rightarrow \infty$.

7.4 Scalar Fields in Cosmology

The cosmological principle states that a possible scalar field has only temporal dependence. The most general Lagrangian would be

$$\mathscr{L}_{\prec} = \frac{1}{2} g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - V \tag{7.24}$$

where ϕ is the scalar field and V is the potential. This should be added to the normal Einstein-Hilbert action.

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + \frac{1}{2}g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - V\right) \tag{7.25}$$

Following the conditions and FRW metric and using Noether's theorem this gives

$$T_{\mu\nu} = -\frac{-2}{\sqrt{-g}}\frac{\partial S_\phi}{\partial g_{\mu\nu}} = g_{\mu\nu}\mathscr{L} + \partial_\mu\phi\partial_\nu\phi \tag{7.26}$$

giving $T_{00} = \rho = \frac{\dot{\phi}^2}{2} + V$ and $T_{ij} = 3p = a^2\gamma_{ij}(\frac{\dot{\phi}^2}{2} - V)$. The equation of state is then obviously bounded $w = \frac{1-\frac{2V}{\dot{\phi}^2}}{1+\frac{2V}{\dot{\phi}^2}}$ with $-1 \leqslant w \geqslant 1$. The slow-roll condition implies that the kinetic energy $\dot{\phi}^2$ is smaller than the potential energy $V(\phi)$ and in the equation of motion of this Lagrangian (via Euler-Lagrange)

$$\frac{\delta S_\phi}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) + V' = 0 \;\leftrightarrow\; \ddot{\phi} + 3H\dot{\phi} = -V'(\phi) \tag{7.27}$$

$\ddot{\phi}$ is vanishing small.[9] The spatial part was ignored, $\phi(x) = \phi(t, \vec{x}) = \phi(t)$, due to the homogeneity and isotropy conditions.

Worth mentioning here is that physical gradients are related to comoving ones by the scale factor

$$\nabla_{\text{comoving}} = a(t)\nabla_{\text{physical}} \tag{7.28}$$

Inhomogeneities are then redshifted during inflation at the same rate as the scale factor increases during inflation.

7.5 Dicke's Fine-Tuning

Would Guth come up with the 'spectacular realization' of inflation without Dicke's talk on the issue of fine-tuning?

Dicke analysed the size of the constant in nature and their dimensionless ratios, $\frac{Gm_p^2}{\hbar c} \sim 5 \cdot 10^{-39}$ (m_p is a mass of an elementary particle, here the proton) is a very small dimensionless ratio. Why is the (dimensionless) gravitational constant compared to other coupling constants so small? And why are there apparent relations between the numbers?[10] In [13] he puts Eddington's opinion (e.g. 1936) that such dimensionless constants should be regarded as mathematical expressions and Dirac's opinion (e.g. 1938)[11] to somehow relate such numbers into contrast. Dirac indeed suggested that large/small numbers vary in time, $(10^{40})^n$ where the number 40 comes from quantities like the dimensionless age of the universe ($T \sim \frac{1}{H}$), $\frac{Tm_pc}{\hbar}$ or the observable size of the universe $\frac{M}{m_p} \sim (10^{40})^2$. Other ratios that are more common in classical physics such as the fine structure constant or the ratio of the masses of elementary particles are compared to that of order unity. If we simply follow statistical arguments it is rather unlikely that such numbers occur. If we however add biological requirements encoded in physical quantities such as the age of the universe/ sufficient time for galaxies to

[9] Note that this equation is similar to a damping oscillation motion with H dominating the friction behaviour.

[10] gravitational constant = (Hubble age)$^{-1}$ = (total particle number)$^{-\frac{1}{2}}$.

[11] Dirac compared the dimensionless constants $\frac{T_0}{\frac{e^2}{m_e c^3}} \sim 10^{39}$, $\frac{\rho(\frac{c}{H_0})^3}{m_p} \sim 10^{78}$, $\frac{e^2}{Gm_pm_e} \sim 10^{38}$ with $T_0 = H_0^{-1}$, ρ as mean density of all matter and m_e electron, m_p proton mass.

have formed etc which we then can observe - otherwise we wouldn't exist and then we couldn't observe it - Dirac's argument can then be taken seriously according to Dicke. For specific calculations see [13]. He continues that the smallness of the gravitational constant and the apparent relations between the numbers can be explained by Mach's Principle which says that $\frac{GM}{c^3 T}$ stays constant of order unity i.e. in this version it is varying according to the total mass of the universe (the spacetime metric is determined by the mass of the universe) and hence time, $G \sim \frac{1}{t}$. Because of its enormous size the gravitational constant is so small.
Obviously, this cannot be correct, as $\leftrightarrow \frac{\dot{G}}{G} \sim -\frac{1}{t}$ and according to Dirac $\frac{\dot{G}}{G}|_{\text{today}} = -3H_0$. This would mean that our universe is rather young, $t \sim \frac{T_0}{3} \sim 10^8$yrs, even younger than the Earth. Nonetheless, it was still an important thought.

7.6 Cosmological No-Hair Conjecture

The cosmological no-hair conjecture was proven for small perturbations is assumed to be true for large perturbations as well (Hawking and Moss, 1982). It states:

If $p = -\rho c^2$ and $\rho > 0$, then any system will evolve to locally resemble a flat exponential expanding spacetime.
This means that for initial conditions (not necessarily homogeneity and isotropy) the region approaches de Sitter. It is similar to the no-hair theorem for black holes as generic conditions lead to the same end state and the initial state cannot be extrapolated from the final state.

7.7 Further Calculations

7.7.1 How Many Causally Unconnected Regions Were There?

Without inflation one can estimate the number of causally unconnected regions. Taking the Planck time as starting point (before that we can only make further assumptions) $t_p \sim \frac{1}{m_p} \sim 10^{-43}$s corresponds to a length of $l_p = ct_p \sim 10^{-33}$cm. Today's universe is of the size 10^{28}cm. Now, if we assume an adiabatic hot universe that $aT =$ constant with the Planck temperature of $T_p \sim 10^{32}$K and today's temperature $T_0 \sim 2.7$K the size at Planck time was 10^{-4}cm (so 10^{30} times the 'initial' size). Dividing by the Planck length as measurement for the causally connected regions gives 10^{90} different regions. That's a huge amount! [40] In fact, there must be an infinite number of horizons inside the singularity.
Let us also calculate the observable size of the universe depending on the curvature. The comoving distance to the cosmological horizon from is given by

$$\int_{t_e}^{t^0} \frac{dt'}{a(t')} = \int_0^{r_e} \frac{dr'}{\sqrt{1-kr'^2}} = \begin{cases} \arcsin r_e, k = 1 \\ r_e, k = 0 \\ \text{arcsinh}\, r_e, k = -1 \end{cases} \tag{7.29}$$

assuming the simple BB model where the universe expands in FRW 7.1. We can measure the redshift z, and the dimensionless values of today's density parameters, on large scales, we can use $\Omega = 1$ in total 3.113 and rewrite the Hubble parameter in terms of the matter content

$$H(t) = \frac{\dot{a}(t)}{a(t)}, dt = \frac{dt}{dz}dz, 1+z = \frac{a_0}{a(t)} \tag{7.30}$$

$$\Omega_{k0} = 1 - \Omega_{r0} - \Omega_{m0} - \Omega_{\Lambda 0} = -\frac{k}{(H_0 a_0)^2} \tag{7.31}$$

$$H(z) = H_0(\Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{k0} + \Omega_{\Lambda 0}) \tag{7.32}$$

The cosmological horizon $a_0 r_e$ is then given by

$$\begin{cases} (H_0\sqrt{\Omega_{k0}}) \sinh\left(I\sqrt{\Omega_{k0}}\right), & \Omega_{k0} > 0 \\ \frac{I}{H_0}, & \Omega_{k0} = 0 \\ (H_0\sqrt{-\Omega_{k0}}) \sin\left(I\sqrt{-\Omega_{k0}}\right), & \Omega_{k0} < 0 \end{cases} \tag{7.33}$$

where the integral can be numerically solved

$$I = \int_0^{z_e} \frac{dz}{\sqrt{\Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{k0} + \Omega_{\Lambda 0}}} \tag{7.34}$$

Using Planck data [20] this gives for 0 curvature about $4 \cdot 10^{10}$ lightyears which is quite accurate with a proposed diameter of $9.3 \cdot 10^{10}$ lightyears.

7.7.2 Energy Scale at Inflation

It is not known what the energy scale during inflation is. It is assumed to take place at E_{GUT} i.e. the energy at which symmetry breaking in the Grand Unified Theory occurs. From data measured at particle accelerators at accessible energies we can extrapolate where the interactions of the standard model meet. It is expected to be at $E_{GUT} \sim 2 \cdot 10^{16}$GeV[12], so three magnitudes smaller than the Planck energy. It has been tested up to 100GeV. Dimensional analysis then gives a false vacuum density of $\rho_f \sim \frac{E_{GUT}^4}{\hbar^3 c^5} \sim 2.3 \cdot 10^{81} \frac{g}{cm^3}$.
(to be continued)

7.7.3 Magnetic Monopoles in Cosmology

Following Preskill [58] we can estimate the minimum *size* of monopoles by extremising the sum of the energy stored in the core and the magnetostatic field energy

$$r \sim \frac{d}{dr}\left(4\pi M_{GUT}^2 r + \frac{4\pi g^2}{r}\right) = 0 \text{ giving } r \sim \frac{M_{GUT}}{g} \text{ with } g = g_D = \frac{1}{2e} \tag{7.35}$$

Hence, the radius is of order $r \sim \frac{1}{eM_{GUT}} \sim \frac{1}{M_X} \sim 10^{-28}$cm for a typical heavy gauge boson mass and assuming the desert hypothesis.
The *mass* can be approximated via

$$m \sim \frac{4\pi}{e^2} M_X \sim 10^{16}\text{GeV} \sim 10^{-8}\text{g} \tag{7.36}$$

[12] See 'GUT Physics in the Era of the LHC'(2019), with the help of RG flow the crossing point of the coupling constants of the interactions is calculated. Note that the energy isn't that big, but for a region of subatomar size it is.

[13] This is a classical object, with a mass much higher than any observed particle today and a size greater than its Compton wavelength. Why is it stable despite its high mass? The lightest magnetic monopole must be stable as it would be the lightest magnetically charged particle and magnetic charge must be conserved.
The *number* of monopoles is dependent on the mechanism [58]. We need to calculate the primordial production number of monopoles and the dilution via the expansion (without inflation) afterwards. If the universe cooled down in a second order phase transition[14] large fluctuations in the Higgs field result in frozen defects during the expansion (Kibble). Uncorrelated vacua meet and form the monopoles. The initial density can be approximated via $n_{\text{initial}} \sim p x_i^{-3}$ where $_i$ is the correlation length and p the probability of forming, so antiproportional to the volume. An upper bound was calculated by Guth and Tye (1980) [2] with the correlation length (the maximal distance over which the field at point A in space is correlated with the field at point B) smaller than the particle horizon (furthest possible distance information could have travelled in a finite amount of time of the transition) $x_i < d_H \sim \frac{Cm_p}{T^2}$, where C is the specific heat $C = 0.6N^{-0.5}$, N the effective number of massless spin degrees of freedom at T. Calculations beyond the scope of this thesis give $C \sim \frac{1}{20}$ at the critical temperature of $T_c \sim 10^{15}$GeV, a Planck mass of $m_p \sim 10^{19}$GeV and an estimated probability of 10%.

$$\left(\frac{n}{T^3}\right)_{\text{initial}} \gtrsim p\left(\frac{T_c}{Cm_p}\right)^3 = 10^{-10} \tag{7.37}$$

If the universe underwent a first order phase transition i.e. a supercooling phase bubbles nucleated for $T < T_c$, expanded, collided and coalesced. Monopoles are formed when the homogeneity within the uncorrelated bubbles is disturbed by the collision. The field attempts to smoothly match over the boundary, hence topological defects form. The density is proportional to the density of collisions $n > p\, d_H^{-3}$. Hence, the previous bound holds here as well.
Now, the annihilation, after the production the expansion of the universe dominates, has to be estimated. Preskill assumes equal densities of monopoles and anti monopoles and describes the density's evolution with $\frac{dn}{dt} = -D(T)n^2 - 3\frac{\dot{a}}{a}n$ where the first part describes the annihilation process and the second part the expansion. In the following he proves that the density doesn't change much. Setting the mean free path not bigger than the characteristic distance he arrives at $\frac{n}{T^3} \sim \frac{N_c}{g^2}\frac{T}{Cm_p} \sim \frac{1}{N_c g^6}\frac{m}{Cm_p} \sim 10^{-9}$. The expansion dominates so much that the probability of annihilation decreases and the initial and current density shouldn't change that much. We expect a density of $n \sim T^3 10^{-10}$. This might be comparable with baryons, but recall that monopoles are expected to have 10^{16} times the mass.
As a side note, as mentioned in the footnote in the monopole section, the *Dirac quantisation condition* can be derived with the help of simple electromagnetism [30]. Re-

[13] Note if I would have calculated the energy of the monopole as I would calculate the energy of a particle in a standard field it would go like $E = 4\pi \int r^2 |\nabla H|^2$ for a sphere of radius r, far away the Higgs field goes like $\frac{1}{r'}$, so the energy would diverge. The Higgs field must couple to other fields, it contains magnetic charge. Again, those defects of the Higgs field create the monopoles.

[14] A first order transition is characterised by a jump in the β- energy diagram due to a discontinuity in β to log Z map, giving an infinite second derivative of log Z (basically the specific heat) with respect to β at the phase transition. A second order phase transition is the continuous counter part.

garding the sketch the radius $r = \int \frac{S\rho}{c^2} dV$ where $S = E \wedge B$ and $\rho = R \sin\beta$ can be calculated by substituting the electric field $E = \frac{1}{4\pi\epsilon_0}\frac{e}{d^2}$ and the magnetic field $B = \frac{1}{4\pi\mu_0}\frac{g}{R^2}$ and the geometric equations $\frac{\sin\alpha}{\sin\beta} = \frac{1}{d}$ and $d^2 = R^2 + x^2 + 2Rx\cos\beta$ one arrives at the formula for the radius of $r = \frac{eg}{4\pi}$. Using quantum mechanics (recall, the production takes place at high energies) we need to equate the size with $\frac{eg}{4\pi} = \frac{n\hbar}{2} \rightarrow eg = nh$. To be clear this is the so called Dirac monopole which should be distinguished from the cosmic ('t Hooft type) ones. Latter can be created/ are solution of the field equations in all gauge theories with the electromagnetic U(1) as subgroup of a larger group.

Magnetic monopoles in gauge theory phase transition

The mechanism explained above is called the Kibble mechanism. The number density at the freeze out is here proportional to $\frac{1}{\eta^3}$, where η as mentioned above is the radius of uncorrelated ϕ domains. If we want to introduce local symmetries the Lagrangian is changed with $\delta_\mu \rightarrow D_\mu - igA_\mu$, hence covariant derivative instead of partial derivative and adding the term $\frac{1}{ig}[D_\mu, D_\nu]$. Importantly, the transition is now not a point, but rather a smooth connection between the phases. The monopole production are suppressed by a magnetic Coulomb interaction. Assuming the ratio of $(\frac{\lambda}{g^2} \sim 1)$ to have a smooth phase transition gives the number of monopoles at the freeze out (hat) is

$$n_{\hat{M}} \sim \frac{g^2 T}{\hat{\zeta}^2} \tag{7.38}$$

In the broken phase, so $m_M > T$ (Bais, Ruddas 1980) the equilibrium monopole density is given by

$$n_M^{eq} \sim \tag{7.39}$$

which is similar to the Debye–Hueckel approximation for the electric- field screening[15] with $m_M \sim \frac{\phi}{g} \sim \sqrt{\frac{-m^2}{\lambda g^2}}$. The screening length for the magnetic field is given by $\eta \sim \frac{1}{m_M}$.

Furthermore, in the Kibble mechanism there is a strong negative correlation between the monopoles at short distance, whereas here it is positive. (to be continued)

7.7.4 Slow-Roll Parameters

$$H^2 = \frac{8\pi G}{3} V(\phi) \tag{7.40}$$

$$3H\dot{\phi} = -V' \tag{7.41}$$

Claim: $\ddot{a} > 0$ and $a > 0$ iff $\epsilon < 1$

Proof:

$$\rightarrow \quad \dot{H} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \quad \text{with } \ddot{a} > 0 \text{ and } a > 0 \;\; \dot{H} + H^2 > 0 \leftrightarrow -\frac{\dot{H}}{H^2} < 1 \leftrightarrow \epsilon < 1 \tag{7.42}$$

[15] The Debye length is the distance in which measurable charge separation can occur. Electric-field screening is similar to the screening described here.

Now proving $\epsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2$ as

$$\text{Differentiate 7.40 } 2H\dot{H} = \frac{8\pi G}{3}\frac{d\phi}{dt}\frac{dV}{d\phi} \text{ using 7.41 } \leftrightarrow \quad -\frac{\dot{H}}{H^2} = \frac{4\pi G}{H^2}\dot{\phi}^2 \tag{7.43}$$

$$\text{substituting both equations } \leftrightarrow \epsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \tag{7.44}$$

$$\leftarrow \epsilon < 1 \quad \leftrightarrow \quad -\dot{H} < H^2 \quad \leftrightarrow -\frac{\ddot{a}}{a} < 2\left(\frac{\dot{a}}{a}\right)^2 \tag{7.45}$$

which is only possible if $\ddot{a} > 0$ and $a > 0$, hence the SR conditions. Note that the SR condition is a sufficient but not necessary condition for inflation.

Simple Massive Scalar Potential

To get an intuition for the initial values needed I will calculate two examples. The textbook example is given by a potential of $V = \frac{1}{2}m^2\phi^2$. This gives $\epsilon = \eta = \frac{1}{4\pi G\phi^2}$. Inflation occurs for $\phi >> \sqrt{\frac{1}{4\pi G}}$, so a field value of much higher than the Planck value and for a number of e-folds of 60 would need an initial field value of $\sim 15m_p$, again a very fine-tuned number ($N = 60 = 2\pi G(\phi_i^2 - \phi_f^2)$ with $\phi_f = \sqrt{\frac{1}{4\pi G}}$).
Using 3.81 for $\epsilon = \eta = \frac{1}{120}$ we get a spectral index of $n_s \sim 0.967$ and a tensor-to-scalar ratio of $r \sim 0.133$.

Powerlaw Inflation

For a potential of $V = V_0 \exp\left\{(-\sqrt{\frac{16\pi G}{p}}\phi)\right\}$ type[16] the parameters are easily derived $\epsilon = \frac{1}{p}$ and $\eta = \frac{2}{p}$. Calculating ϵ and η for a power law inflation gives constant values. Hence, inflation is satisfied for $p < 1$. However, this inflation would never end.

Chaotic Inflation

Whilst Linde shows this for $V = \lambda\phi^4$, I will apply the scenario to the usual $V = \frac{1}{2}m^2\phi^2$. We also assume that the field is classical and homogeneous over its domain (not necessarily over H^{-1}, such that the Friedmann equation in SR can be applied.

$$H = \sqrt{\frac{8\pi VG}{3}} = \sqrt{\frac{4\pi G}{3}}m\phi \tag{7.46}$$

$$3H\dot{\phi} = -V' = -m^2\phi \tag{7.47}$$

giving $\phi(t) = \phi_0 - \frac{m}{2\sqrt{3\pi G}}t$ (for the quadratic potential he arrives at an exponential

[16] This gives $a(t) \propto t^p$ when substituting the potential into the SR equations and solving for H in the first equation and substituting it in the equation of motion and integrating gives $\phi \propto \sqrt{p}\ln\left(\frac{16\pi G V_0}{(3p-1)p}t\right)$. Back in 7.40 this gives $H^2 = \frac{p^2}{t^2} \leftrightarrow a(t) \propto t^p$.

field) and substituting back in and rearranging gives $a(t) = a_0 \exp(2\phi G(\phi_0^2 - \phi^2))$. The characteristic time (where the field remains constant i.e. flat rolling) is then $\delta t \sim \frac{\phi G}{m}$ under which we have $a(\delta t) \sim a_0 e^{H\delta t} = \exp^{2\sqrt{\frac{\pi}{3}}\phi_0^2}$. For 60 e-folds ϕ must exceed $3m_p$ (which he also calculates for the quadratic potential). This is more reasonable assumption than several orders higher predicted by the old and new inflation scenario. For values below the field oscillates and reheating occurs.
Worth mentioning is that chaotic inflation does need fine-tuned parameters. The $V = \lambda\phi^4$ potential needs for 60 e-folds a parameter of $\lambda^{-\frac{1}{2}} >> 60$. Using perturbation estimations $10^{-5} \sim \left|\frac{\delta\rho}{\rho}\right| \sim \left|\frac{H^2}{\dot{\phi}} \sim \frac{V^{\frac{3}{2}}}{V'}\right|$ even give $\lambda \sim 10^{-10}$, again a rather small coupling. In order for quantum gravity effects to be small it is postulated that the energy-matter tensor is smaller than m_p^4 (classical description is only possible for the field values being smaller than the Planck mass) sets a constraint of $\lambda < 10^{-2}$ (for the massive scalar potential this would constraint the initial field value to $\frac{m_p^2}{m}$. Nonetheless, this potential gives $n_s \sim 0.95$ and $r \sim 0.3$, so not far away from observational results using the formulae derived in 3.5.1.
Interestingly, the quadratic choatic model gives the same prediction as the Starobinsky model with $\epsilon = \eta = \frac{m_p^2}{4\pi\phi^2}$, $N \sim 2\pi\frac{\phi_*}{m_p^2}$ giving $n_s \sim 1 - \frac{2}{N}$ which is favoured by Planck.

7.7.5 Size of the Universe

To get an estimate of the increase of the universe's size during inflation, I assume that it entered inflation at $t_i = 10^{-36}$s and ended at either $t_f = 5 \cdot 10^{-34}$s or $t_f = 10^{-34}$s. This will show that only small changes in those unknown values have a major impact. I assume radiation domination and constant Hubble constant.

$$H_i = \frac{1}{2t} = 5 \cdot 10^{35} s^{-1} \text{ substituting in } r_i = \frac{2c}{H} = 1.2 \cdot 10^{-27} m \tag{7.48}$$

Now assuming exponential expansion for the duration

$$\frac{a_f}{a_i} = e^{H\delta t} = \begin{cases} 2.272 \cdot 10^{108} & \text{for } t_f = 5 \cdot 10^{-34} s \\ 3.145 \cdot 10^{21} & \text{for } t_f = 10^{-34} s \end{cases} \tag{7.49}$$

giving a final size of $r_f = 2.7264 \cdot 10^{81}$m and $r_f = 3.774 \cdot 10^{-6}$m, respectively. This is obviously a huge difference and differs from 7.7.1.

7.7.6 CMB Temperatures

A simple way to calculate today's CMB temperature is with the help the Boltzmann law (by integrating the Planck spectrum over all wavelengths)

$$E_\gamma = \frac{\rho_{\gamma_t}}{n_\gamma} = \frac{\frac{g}{30}\pi^2 T^4}{\frac{g_* \zeta(3)}{\pi^2} T^3} \tag{7.50}$$

which gives for photons (bosons and two spin states), $g = g_* = 2$ and introducing k_B for dimensional reasons

$$\frac{\pi^4 k_B T}{30\zeta(3)} \sim 2.701 k_B T \tag{7.51}$$

where I reintroduced the Boltzman constant. I also calculate an estimation for the temperature at the second Hubble crossing, to compare it to the calculated reheating temperature and today's temperature.
I assume instantaneous radiation domination and $h = 0.7$. The horizon size is given by $2t = H^{-1}$, SR gives $\frac{8\pi}{3m_p}\rho = H^2 = (2t)^{-2}$ and again using the energy density formula for radiation $\rho = \frac{g\pi^2}{30}T^4$[17]. Equating with constants for dimensional reasons gives a temperature of

$$T_\gamma = \frac{1}{k_B\sqrt{t}}\left(\frac{45\hbar^3c^5m_p}{16\pi^3 g}\right)^{\frac{1}{4}} \tag{7.52}$$

As at the Hubble crossing $\lambda(t) = cH^{-1}$, $\frac{\lambda(t)}{\lambda_0} = \frac{a(t)}{a_0}$ and aT is approximately constant, the time of crossing is

$$t = \left(\frac{\pi^3 gG}{45\hbar^3c^5}\right)^{\frac{1}{2}}\left(\frac{k_B T_{\gamma 0}}{c}\right)^2\lambda^2 \tag{7.53}$$

A wavelength of the size of the Andromeda galaxy ($\sim 2.2\cdot 10^6$ ltyr) gives $t \sim 2.06$ yr. A time of 50,000 years corresponds to a wavelength of $3.8925\cdot 10^8$ ltyr (the observable universe's size is about $93\cdot 10^9$ a good approximation).

7.7.7 Weyl Curvature

Further treatment of 4.1.1.
Firstly, the FRW metric $(-1, a(t)^2\gamma_{ij})$ gives non-zero Christoffel symbols[18]

$$\Gamma^t_{ij} = a\dot{a}\gamma_{ij} \quad \Gamma^i_{jt} = \frac{\dot{a}}{a}\delta^i_j \quad \Gamma^i_{jk} \tag{7.54}$$

giving only three (+the terms that can be calculated from the symmetry properties) non-vanishing Riemann components

$$R^t_{itj} = a\ddot{a}\gamma_{ij} \quad R^i_{ttj} = \frac{\ddot{a}}{a}\delta^i_j \quad R^i_{jkm} =^{(3)} R^i_{jkm} + \dot{a}^2(\gamma_{jm}\delta^i_k - \gamma_{jk}\delta^i_m) \tag{7.55}$$

where ${}^{(3)}R^i_{jkm}$ is the spatial Riemann tensor, for a maximally symmetric spacetime ${}^{(3)}R_{ijkm}k(\gamma_{ik}\gamma_{jm} - \gamma_{im}\gamma_{jk}$. Similarly,

$$R_{tt} = -\frac{3\ddot{a}}{a} \quad R_{ij} =^{(3)} R_{ij} + \gamma_{ij}(2\dot{a}^2 + a\ddot{a}) \tag{7.56}$$

where ${}^{(3)}R_{ij} = 2k\gamma_{ij}$ and finally

$$R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) \tag{7.57}$$

[17] I calculate the degrees of freedom for photons and neutrinos. $g = g_\gamma + g_\nu = 2 + \frac{21}{4}(\frac{4}{11})^{\frac{4}{3}} \sim 2.36$ where fermions contribute $\frac{7}{8}$ of the $3\cdot 2\cdot 1$ (flavours, particle and antiparticle, spin) and the factor accounts for the number density (-temperature ratio) between photons and neutrinos, $\left(\frac{T_\nu}{T_\gamma}\right)^4$.

[18] calculated during the R&C course.

The Weyl curvature tensor vanishes for FRW. One can either substitute in the Riemann and Ricci components, in 4dim,

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{1}{2}(R_{\mu\beta})g_{\nu\alpha} - R_{\mu\alpha}R_{\nu\beta} + R_{\nu\alpha}g_{\mu\beta} - R_{\nu\beta}g_{\mu\alpha}) + \frac{R}{6}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) \tag{7.58}$$

which cancel for all. For example, $C_{titj} = -a\ddot{a}\gamma_{ij} + \frac{1}{2}(\frac{3\ddot{a}}{a}a^2\gamma_{ij} + \gamma_{ij}(2k + 2\dot{a}^2 + a\ddot{a})) + \frac{R}{6}a^2\gamma_{ij} = 0$ Similarly, it is obvious that the metric is conformally flat for $k = 0$ if one rewrites the metric in conformal and spherical coordinates

$$ds^2 = a(\eta)^2(-d\eta^2 + dr^2 + r^2 d\Omega^2) \tag{7.59}$$

For $k \pm 1$ this can be proven using the vielbein formalism.
Thus, the Weyl tensor vanishes for the FRW metric.
Now, I want to investigate the constraint on the Weyl curvature as measurement for gravitational entropy, i.e. that it equals Hawking's black hole entropy formula. The Schwarzschild metric in four dimensions gives a Weyl scalar, due to the vanishing of $R_{\mu\nu}$ and R of

$$C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{12r_s^2}{r^6} \tag{7.60}$$

with a Schwarzschild radius of $r_s = 2GM$. Note that the metric is not defined at the curvature singularity $r = 0$, but for now I am interested in large scales. The Hawking entropy is given by

$$S = \frac{A}{4G} = \frac{\pi r_s^2}{G} \tag{7.61}$$

Hence, the Weyl scalar is proportional to the entropy.
However, one should integrate the Weyl scalar over the proper volume to get the entropy. Simple dimensional counting for

$$S = \int C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} dV \tag{7.62}$$

$[C_{\mu\nu\alpha\beta}] = 2$ in natural units (second derivative), the volume element of an n-dim manifold is $[dV_{n-1}] = -n + 1$, the entropy should be dimensionless in natural units. So this only works for $4 - n + 1 = 0$ a spacetime with five dimensions. The n-dim metric is given by[19]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{n-2}^2 \tag{7.63}$$

with $f(r) = 1 - \left(\frac{r_{sn}}{r}\right)^{n-3}$ and the Schwarzschild radius $r_{sn} = \sqrt{\frac{16\pi GM}{n-2}}$. For n=5 the Weyl scalar gives an additional $6\left(\frac{r_s}{r}\right)^2$ factor, so

$$C^2 = \frac{72r_{s5}^4}{r^8} \quad r_{s5} = \sqrt{\frac{8GM}{3\pi}} \tag{7.64}$$

[19] Emparan, Reall, 2008, 'Black Holes in Higher Dimensions'

and a volume element of $dV = r^3 \left(\left| 1 - \frac{r_{s5}^2}{r^2} \right| \right)^{-\frac{1}{2}} 2\pi^2 dr$, gives the result

$$S = \frac{144\pi^2 (8GM)^2}{9\pi^2} \int_{r \to 0}^{r_{s5}} \frac{dr}{r^5 \sqrt{-f(r)}} \int_{r_{s5}}^{\infty} \frac{dr}{r^5 \sqrt{f(r)}} \tag{7.65}$$

where I took into account that we have constant time hypersurfaces that are spacelike for $r > r_{s5}$ ($\frac{\delta}{\delta t}$ is timelike) and timelike for $r < r_{s5}$ ($\frac{\delta}{\delta t}$ is spacelike). The integral is not defined for $r = 0$, quantum gravity effects should be below $r \sim G^{\frac{1}{3}}$ (dimensional analysis, other G). This results in

$$\begin{aligned} S &= 1024 G^2 M^2 \left(\frac{\sqrt{\frac{r_{s5}^2}{r^2} - 1}(r_{s5}^2 + 2r^2)}{3 r_{s5}^4 r^2} \Bigg|_{G^{\frac{1}{3}}}^{r_{s5}} + \frac{\sqrt{-\frac{r_{s5}^2}{r^2} + 1}(r_{s5}^2 + 2r^2)}{3 r_{s5}^4 r^2} \Bigg|_{r_{s5}}^{\infty} \right) \\ &= 96\pi^2 \left(1 - \sqrt{\frac{8MG^{\frac{1}{3}}}{3\pi} - 1} \left(1 + \frac{4MG^{\frac{1}{3}}}{3\pi} \right) \right) \end{aligned} \tag{7.66}$$

For $\frac{8MG^{\frac{1}{3}}}{3\pi} >> 1$ (Schwarzschild radius bigger than Planck scale) this approaches $48\pi^2 \left(\frac{8MG^{\frac{1}{3}}}{3\pi} \right)^{\frac{2}{3}} = 48\pi^2 \frac{r_{s5}^3}{G}$. Calculating the entropy via Hawking's formula gives $S_{BH} = 4 \left(\frac{\pi^{\frac{1}{3}} G^{\frac{1}{3}} M}{3} \right)^{\frac{2}{3}} = \frac{\pi^2 r_{s5}^3}{2G}$. Thus, $S_{BH} = \frac{1}{96} S$.
Thus, Penrose's constraint that the Weyl measurement should give the black hole entropy is only valid for a five dimensional spacetime (at least if one literally takes the Weyl scalar, modifications might be possible).

7.7.8 Starobinsky Powerspectrum

The Jordan frame gives the Starobinsky model in the form of modified gravity i.e. $\sim R^2$, the Einstein frame is the model after conformal transformation and the introduction of the scalar field ϕ which I derived in 3.20, 3.3.6. I follow the treatment of 3.99 which derivations I checked, but won't reproduce here.

Jordan Frame

Starting with a general $f(R)$ theory, $\mathscr{L} = \frac{\sqrt{-g}}{16\pi G}$ the eom is given by (Mukhanov, also see similar derivation in 3.20)

$$f' R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f' = 0 \tag{7.67}$$

see also 3.3.6, which gives for the background and $H = \frac{\dot{a}}{a}$ in FRW cosmology

$$0 = 3(\dot{H} + H^2) f' - \frac{1}{2} f - 3H \dot{f}' \tag{7.68}$$

$$0 = (\dot{H} + 3H^2) f' - \frac{1}{2} f - \ddot{f}' - 2H \dot{f}' \tag{7.69}$$

where f, f' is evaluated at $R = R^{\mu}_{\mu} = 6\dot{H} + 12H^2$ which is time dependent only. The perturbed metric is given by

$$g_{ij} = a^2 e^{2\zeta + h} \tag{7.70}$$

ζ being the scalar perturbation and h_{ij} the tensor perturbation (TT gauge, $h_{ii} = h_{ij,i} = 0$. Homogeneity and isotropy 3.99 (see also treatment in 3.5.1) gives the perturbations both dependent on time and space

$$\zeta(x) = 2\sqrt{\pi G} \int \frac{d^3k}{(2\pi)^3} \left(\nu(t, \vec{k}) e^{i\vec{k}.\vec{x}} \alpha_s(\vec{k}) + \nu * (t, \vec{k}) e^{-i\vec{k}.\vec{x}} \alpha_s^{\dagger} \right) \tag{7.71}$$

$$h_{ij}(x) = 4\sqrt{2\pi G} \int \frac{d^3k}{(2\pi)^3} \left(u(t, \vec{k}) e^{i\vec{k}.\vec{x}} \alpha_t(\vec{k}) \epsilon_{ij} + u * (t, \vec{k}) e^{-i\vec{k}.\vec{x}} \alpha_t^{\dagger} \epsilon_{ij}^{*} \right) \tag{7.72}$$

The polarisation is as in flat space, $k_i \epsilon_{ij} = \epsilon_{ii} = 0$, $\epsilon_{ij}(\lambda_1)\epsilon_{ij}^{*}(\lambda_2) = \delta_{12}$ and summed over $(\vec{k}, \lambda), \lambda = \pm$. α satisfies the usual commutation relations, $[\alpha_s, \alpha_s] = (2\pi)^3 \delta^{(3)}(k_1 - k_2)$ and for the tensor relation with an additional polarisation δ_{12}. The power spectra is now given by $\langle \Omega | \, \zeta(t, \vec{x}) \zeta(t, \vec{0}) \, | \Omega \rangle$ evaluated over $\frac{k^3}{2\pi^2} \int d^3x e^{-i\vec{k}\vec{x}}$ and similar for the tensor perturbations. Hence, we arrive at a late time formula depending on the norm of ν and u and k only meaning momentum and time dependent (compare with 3.85).

$$\Delta_s^2(t, k) = \frac{2k^3 G}{\pi} |\nu(t, k)|^2 \tag{7.73}$$

$$\Delta_t^2(t, k) = \frac{32k^3 G}{\pi} |u(t, k)|^2 \tag{7.74}$$

Those can be evaluated from substituting the modes into the eom and quantising canonically, giving the formulae 3.99.

$$\nu\dot{\nu}^{*} - \dot{\nu}\nu^{*} = \frac{i}{f' a^3 C}, \quad C = \frac{\frac{3(f'' \partial_t (6\dot{H} + 12H^2))^2}{(2f'H)^2}}{\left(\frac{f'' \partial_t (6\dot{H} + 12H^2}{2f'H} + 1 \right)^2} \tag{7.75}$$

$$\ddot{\nu} + \dot{\nu} \left(3H + \frac{f'' \partial_t (6\dot{H} + 12H^2)}{f'} + \frac{\dot{C}}{C} \right) + \left(\frac{k}{a} \right)^2 \nu = 0 \tag{7.76}$$

where f is again evaluated at the background $6\dot{H} + 12H^2$. Similarly, for the tensor mode. This can be evaluated up to a factor which is given by the Bunch Davies IC which can then be substituted back into the power spectra formulae.

Einstein Frame

The treatment is similar to the one above with the eom given by the transformed Lagrangian where we identify $\phi = f'$ (3.3.6). The hat indicates the evaluation in the Einstein frame.

$$\hat{\mathscr{L}} = \sqrt{-\hat{g}} \left(\frac{\hat{R}}{16\pi G} - \frac{1}{2} \partial_{\mu} \partial^{\nu} \phi - V \right) \tag{7.77}$$

$$V = \frac{1}{16\pi G} e^{-2\sqrt{\frac{16\pi G}{3}} \hat{\phi}} \hat{V} \left(e^{\sqrt{\frac{16\pi G}{3}} \hat{\phi}} \right) \tag{7.78}$$

$$U = \phi \hat{R} - f(\hat{R}), \quad \hat{\phi} = \sqrt{\frac{3}{16\pi G}} \ln \phi \tag{7.79}$$

I proceed by calculating the background equation, gauge fixing as before (note new variables), calculating $\hat{g}_{ij}, \hat{h}_{ij}$ and the power spectra. Again by getting $\hat{u}, \hat{v}$. $\zeta = \hat{\zeta}, h_{ij} = \hat{h_{ij}}$ which should then also give the same numerical values. For the power spectra we actually arrive at another form,

$$\Delta_s^2 \sim \frac{g\hat{H}^2}{\pi\hat{\epsilon}}, \quad \hat{\epsilon}(\hat{t}) = \partial_t \frac{1}{\hat{H}} \tag{7.80}$$

$$\Delta_t^2 = \frac{16G\hat{H}^2}{\pi} \tag{7.81}$$

compared to

$$\Delta_s^2 = \frac{H^2}{48\pi^2 f' \epsilon^2} \tag{7.82}$$

$$\Delta_t^2 = \frac{H^2}{\pi^2 f'} \tag{7.83}$$

We can already see that this is the case as the expansion factor is related by $\hat{a}(\hat{t}) = \sqrt{f'}a(t), \frac{d\hat{t}}{dt} = \sqrt{f'}$. In the Einstein Frame we have indeed $r \sim 16\hat{\epsilon}$ whereas the Jordan Frame gives $\sim 48\epsilon^2$!.
The application to R^2 can be found in 3.20.

7.8 VSL Constraints

Digression from 3.6.1. The Friedmann equations with Newton's constant and the speed of light time-varying [7]

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi\rho G(t)}{3} - \frac{kc(t)^2}{a^2} \\ \ddot{a} &= -\frac{4\pi G(t)}{3}\left(\rho + \frac{3pa}{c(t)^2}\right) \end{aligned} \tag{7.84}$$

It is actually very useful to keep this in mind as often due to $c = G = 1$ dimensions are forgotten.
$[c] = LT, [G] = L^3T^{-2}M^{-1}$, $[H] = [\frac{\dot{a}}{a}] = T^{-1}$, $[\rho] = ML^{-3}$, $[a] = 0, [\dot{a}] = T^{-1}$
Differentiating the first and substituting this and the equation itself in gives the new conservation equation.

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\frac{p}{c^2} + \rho\right) = -\rho\frac{\dot{G}}{G} + \frac{3kc\dot{c}}{4\pi G a^2} \tag{7.85}$$

Assuming an equation of state $p = (\gamma - 1)\rho c^2$ (giving a GR result $\dot{G} = \dot{c} = 0$ of $\rho \sim a^{3\gamma}$). Looking at Friedmann's equations 7.3 the flatness and horizon problem are solved for $0 \leqslant \gamma < \frac{2}{3}$ as long as this is sufficiently long. A power law in $c(t) = c_0 a^n$ without an explicit form for G gives

$$\frac{\dot{\rho}}{\rho} + 3\frac{\dot{a}}{a}\left(\frac{p}{\rho c^2} + 1\right) = \frac{\dot{\overline{\rho a^{3\gamma}}}}{\rho a^{3\gamma}} \tag{7.86}$$

Putting G on the other side it can be integrated to give

$$\begin{aligned} G &= \frac{3kc_0^2 n a^{2n-2}}{4\pi(2n-2+3\gamma)\rho} + \frac{C}{\rho a^{3\gamma}} \quad 2n+3\gamma \neq 2 \\ G &= \frac{3kc_0^2 n \ln a}{4\pi\rho a^{3\gamma}} + \frac{C}{\rho a^{3\gamma}} \quad 2n+3\gamma = 2 \end{aligned} \tag{7.87}$$

In order for the curvature term to dominate we need $n \leqslant \frac{1}{2}(2-3\gamma)$. This solves both the flatness and horizon problem. Further, if an explicit $G(t)$ is introduced the cosmological constant comes along, $\rho \to \rho + \rho_\Lambda$ with the usual $p_\Lambda = -\rho_\Lambda c^2$ (further is explained in 3.7. Interestingly, in the simplest asymptotic safety approach for gravity we will make use of the runnings of G and Λ as well. Note, here the cosmological constant is assumed to be constant. Solving this gives again the need of a decrease of c, even faster than to solve the flatness and horizon problem alone, $n < -\frac{3}{2}\gamma$. With $w = \gamma - 1$ in matter and radiation dominated era the problems would be solved for ($n \leqslant -0.5$ for matter, $n \leqslant -1$ for radiation without cosmological constant $n \leqslant -1.5$ for matter, $n \leqslant -2$ for radiation and with cosmological constant).

7.9 The Wavefunction of the Universe

Where does the wavefunction come from? I will follow Hawking's and Hertog's work [27].
The wavefunction satisfies a second order differential equation, the Wheeler de Witt equation which is informally speaking the Schrödinger equation of the universe. The equation is solved on an infinite dimensional hyperbolic manifold (superspace), the space of all 3-metrics and matter field configurations[20].
In the easiest setting it might be visualised as follows:

$$\frac{\partial\Psi}{\partial t} = -iHt = -i\left(-\frac{\partial^2}{\partial x^2} + V\right)\Psi = -i\int\left(-\frac{\delta^2}{(\delta\phi)^2} + m^2\phi^2\right)d^3x\Psi(x) \tag{7.88}$$

which is the wave function of the Wheeler de Witt equation

$$0 = H\Psi \tag{7.89}$$

where H is the Hamiltonian, for example, given by

$$-G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} - \sqrt{h}\left(R + 2\Lambda + \frac{8\pi T_{ab}}{m_p^2}\right) \tag{7.90}$$

where G is the metric on the superspace, the wavefunction Ψ dependent on the 3-metric h_{ij} and the scalar field ϕ, the 3-curvature dependent on h.
As until now it hasn't been possible to solve it, the space is restricted to a finite subspace (minisuperspace) already explained in the thesis 3.6.2. It follows a semi-classical approximation of the path integral.

$$\sim N_0 = \sum_i A_i e^{-B_i} \tag{7.91}$$

[20]Hence, if one knows Ψ for small 3-surfaces, i.e. early times, we can solve the equation as Cauchy problem for larger surfaces, i.e. late times.

where B_i are the classical actions and the coefficients are the determinants of the small fluctuations of the classical solutions.
The Euclidean action given by Hawking and Hartle for FRW with cosmological constant $\Lambda = \frac{8\pi V}{m_p^2}$ is

$$I_E = \frac{3\pi m_p}{4} \int d\eta \left(\left(\frac{da}{d\eta} \right)^2 - a^2 + \frac{\Lambda a^4}{3} \right) \tag{7.92}$$

with the boundary condition $a(0) = a$ and $\phi(0) = \phi$. A solution for the scale factor $a(\tau) = \frac{\cos H\tau}{H(\phi)}$ gives the wavefunction of

$$\Psi_0 \sim e^{\frac{3m_p^4}{16V(\phi)}} \tag{7.93}$$

The probability of a certain ϕ =const. and corresponding $a = \sqrt{\frac{3m_p^2}{8\pi V}}$ is given by the amplitude of the wavefunction.

$$P \sim |\Psi_0|^2 \sim e^{\frac{3m_p^4}{8V}} \tag{7.94}$$

It has a maximum at $V \to 0$ giving a probability for a universe in a state with large ϕ and a long inflation period isn't very likely. It has been argued that there could be a change to a minus sign (due to the Wick rotation from Lorentzian to Euclidean the minus sign disappears to $iI \to -I$). One would then have a probability of

$$P \sim e^{-2|I_E|} = e^{-\frac{3m_p^4}{8V}} \tag{7.95}$$

Hawking concludes that there are only two positive definite types of metrics: compact metrics and matter fields that are regular on them or non-compact asymptotic metrics of maximally symmetric spaces and matter fields that are asymptotically 0 (the latter is the vacuum state which we don't look for).
For an FRW universe, with a first assumed conformally invariant scalar field constant on 3-spheres, gives a solution of[27]

$$\frac{1}{2} \left(\frac{1}{a^p} \frac{\partial}{\partial a} \left(a^p \frac{\partial}{\partial a} \right) - a^2 - \frac{\partial^2}{\partial \chi^2} + \chi^2 \right) \Psi(a, \chi) = 0 \tag{7.96}$$

where $\chi = \sqrt{\frac{3}{\pi}} \frac{m_p}{a\phi}$ and N is the lapse function with the metric in conformal coordinates

$$ds^2 = \frac{2}{3\pi m_p^2} \left(N^2(\eta) d\eta^2 + a^2(\eta) d\Omega_s^2 \right) \tag{7.97}$$

representing a spatially closed universe of radius a which is regular at $a \to 0$ Due to conformal invariance we could substitute in $\Psi(a, \chi) = C(a) f(\chi)$ which gives a harmonic oscillator type eigenvalue equation ($n = 0$ is the ground state).

$$\frac{1}{2} \left(-\frac{d^2}{d\chi^2} \right) f = \left(n + \frac{1}{2} \right) f \tag{7.98}$$

However, the $C(a)$ part would give an almost matter free universe. They realised that one needs a non-conformally invariant field. The simplest model (later they also add effective R^2 terms by quantum corrections) is given by

$$\frac{1}{2}\left(\frac{1}{a^p}\frac{\partial}{\partial a}\left(a^p\frac{\partial}{\partial a}\right) - a^2 - \frac{\partial^2}{a^2\partial\phi^2} + a^4 m^2\phi^2\right)\Psi(a,\phi) = 0 \tag{7.99}$$

In the (a,ϕ) plane the wavefunction can be analysed by rewriting it to

$$x = a\sinh\phi \quad y = a\cosh\phi \tag{7.100}$$

giving the light cone at $a = 0, \phi = \pm\infty$. When we know the solution of this equation we can solve the wavefunction with the boundary condition for large ϕ (semi-classical approximation). The positive energy density of the potential in the action behaves like a positive cosmological constant. The solutions give large ϕ and a long inflation period as well as oscillations of expansions and contractions of the universe.
Interestingly, in [32] they argue that the probability has to be multiplied by e^{3N} to account for the e-foldings in slow roll inflation. This is similar to the solution ansatz of volume weighting in eternal inflation's measure problem. They also combine their result with the landscape scenario in string theory giving an arena for the many possible universes with, for example, different values for the cosmological constant[21]. The no-boundary proposal and wavefunction of the universe together with slow-roll inflation can explain why we are in the universe we observe since it acts as a selection principle. Here the authors the calculate the probability on our universe's past light cone with the probability of all information in the Hubble volume e^{3N} on a surface that fits our observations (spacelike at the present time). The intersections of the Hubble volume and this surface give the right terms, because those spacetimes would have lasted long enough for inflation to occur. Basically, as observers we are part of the universe we observe and we need to take into account the probability that we have evolved through the Hubble volume.

7.10 The Top-Down Approach

The wavefunction together with the no-boundary proposal gives the probabilities of entire classical history trajectories. However, we need to account for the fact that we can only receive information dependent on a part of our past light cone. The so-called top-down probabilities are the sum of the ones that represent our information at least once and the ones that are the possible locations of our light cone in them.
Top-down along with anthropic reasoning basically changes conditional probability of a certain history of the universe U given a Hamiltonian H and wavefunction Ψ with a

[21] The landscapes in modern string theory is the set of all possible false vacua as there are many choices for possible Calabi-Yau spaces. The number arises due to the geometry of the hidden dimensions. In higher dimensional GR we find multiple solutions that all give one vacuum state. It is expected to be a very large number. Weinberg used this to explain the low value of the cosmological constant we observe (Polchinski, 'Dualities of Fields and Strings')

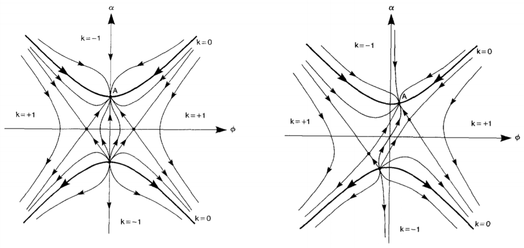


Fig 1 Fig 2

Figure 7.2: Halliwell, 'Scalar fields in cosmology with an exponential potential', 1986. The $\alpha\phi$ plane shows the autonomous system of the equation of motions and constraints of the potential $V = V_0 e^{\frac{2\sqrt{\pi}\lambda}{\sqrt{3}m_p}\phi}$ and metric $ds^2 = \frac{3m_p^2}{4\pi V_0}(-N(t)^2dt^2 + e^{2\alpha(t)}d\Omega^2(k))$. Halliwell presented the solutions for various $\lambda > 0$. $\lambda = 0$ (left) is the massless ϕ and cosmological constant state (on the α-axis de Sitter). A is the attractor lying on flat space, $k = 0$. Inflation is guaranteed. Deviating to $0 < \lambda < \sqrt{12}$ gives a smooth distortion, but still the same kind of evolution, now powerlaw inflation. He goes further and present larger values, where the the horizon and later the flatness problem cannot be solved anymore, hence, don't give inflationary solutions.

set of constraints C (such as flatness),

$$p(U|C, H\Psi) = \frac{p(UC|H\Psi)}{p(C|H, \Psi)} \tag{7.101}$$

to $p(U|L, H, \Psi)$ where L are all the conditions for making life possible.
(to be continued)

7.11 Dark Energy - Attractor in Quintessence

As mentioned in 3.7 the density of a possible scalar field can slowly decrease and might dominate again the universe's energy content after radiation/matter dominated era.[61] gives a toy example of a potential of type $V \sim \frac{8\pi G}{\phi^p}$ giving a power law inflation of $a \sim t^n$ and a field of the form (substituting into the Friedmann equation t^q gives $q = \frac{2}{2+p}$ to cancel the variable) $\phi \sim t^{\frac{2}{2+p}}$.
The ratio of the field's and today's density would then be $\frac{\rho_\phi}{\rho} \sim t^{\frac{4}{2+p}}$. The cosmological constant corresponds to $p \to 0$ ($\rho \sim t^{-2}$). A positive p increases the field value and either leads to $\rho_\phi \to 0$ (classical Minkowski space), $\rho_\phi \to const.$ (de Sitter) or $\rho_\phi < 0$ (Big Crunch). It is also possible to construct attractor solutions (Luchin & Matarreste 1985, Halliwell 1986) where small deviations from the initial conditions still let the system tend to the same state near the attractor 7.2.

7.12 Quantum Gravity in Brief

One can basically distinguish between the covariant, canonical and sum-over-histories approaches to quantise gravity [67].
The latter uses the Feynman functional integral from quantum field theory. Euclidean

quantum geometry began with Misner, Wheeler and Hawking in the 50/60s.
In the *canonical* treatment there is no background dependence and gravity is treated as a quantum theory where the Hamiltonian carries the representations of the operators. As non-perturbative theory it should be well defined at all scales. It probably started with Dirac's treatment in the 50s, followed by Deser, Misner and Ashtekar in the 60s-80s, especially with the introduction of the new variables (see 3.6.3). The Wheeler de Witt equation (where no background metric was assumed, merely a spacetime manifold) seemed promising, but was ill-defined and suffered from its failure of transition to the low energy behaviour. LQG arose in the 80s. 't Hooft and Veltman introduced a proper treatment of the background field method that retains the symmetries of the path integral in gravity.
Lastly, the *covariant* approach treats gravity as quantum field theory of fluctuations that are added upon a fixed (flat) background metric. It goes back to Fierz and Pauli in the 30s, the Feynman rules for GR by de Witt and Feynman in the 60s. The perturbatively non-renormalisable theory of GR was proven by t'Hooft, Veltmann [92], Deser et al in the 70s. Gauge fixing and introduction of ghost fields in GR were tackled. (Super)string theory and modifications of GR (such as HD gravity) followed. Analogue to QED only fluctuations are quantised on a fixed background where the operators are defined, it is background dependent and perturbative. Unfortunately, diffeomorphism invariance is violated and due to its non-renormalisability it cannot be a fundamental theory .
in order:

year	authors	importance
1930	Rosenfeld	diffeomorphism invariance
1938	Heisenberg	dimension of G is problematic
1952	(Rosenfeld, Fierz, Pauli) Gupta	flat-space quantisation,
.	.	gravitational self-energy γ, e^-
1957	(Feynman) Misner	Feynman rules for GR, H principle
1959	Arnowit, Deser, Misner	ADM(=Hamiltonian) formalism
1967	de Witt	Wheeler-DeWitt equation
1970	Zumino	GR as LE limit
1971/73	't Hooft, Veltmann	Yang Mills is renormalisable, GR has non-renormalisable divergencies
1974	Kadanoff, Wilson	UV FP in scalar field theory
1976	Weinberg	Asymptotic safety as QG approach
1977	Stelle	quadratic Langrangian is renormalisable, but non-unitary
1979	Weinberg	nontrivial FP in gravity for $d = 2 + \epsilon$
1983	Hawking, Hartle	Wavefunction of the universe
1986	Goroff, Sagnotti	two-loop GR divergencies
	Ashtekar	new variables based on connection
1993	Wetterich	Wetterich equation as exact formula for FRGE
1994/96	Wetterich, Reuter	FRG for general gauge theories
1998-today		matter, HD terms... FP found

Table 7.3: QG's beginning

1. 'Zur Quantelung der Wellenfelder'

2. 'Die Grenzen der Anwendbarkeit der bisherigen Quantentheorie'

3. 'Quantization of Einstein's gravitational field: general treatment': a fictitious space, the flat space, is introduced such that fluctuations are the difference of the metric and the flat Minkowski metric, $h_{\mu\nu} = g_{\mu\nu} - \eta\mu\nu$ (covariant approach). As for the EM field problems arise due to gauge invariance. (a)

4. 'Feynman quantization of general relativity': the integration is over all field histories and the Einstein action in the exponential. It is emphasised to account for

gauge invariance in the integral and the vanishing of the quantum Hamiltonian. (b)

5. 'Dynamical Structure and Definition of Energy in General Relativity'

6. 'Quantum theory of gravity. I. The canonical theory': (canonical approach) Laterd developed by Misner.

7. 'Effective Lagrangians And Broken Symmetries'

8. 'Regularization and renormalization of gauge fields', 'Renormalizable Lagrangians for massive Yang-Mills fields', 'One-loop divergencies in the theory of gravitation'

9. 'The renormalization group and the ϵ expansion'

10. 'Critical Phenomena for Field Theorists'

11. 'Renormalization of higher-derivative quantum gravity': The Lagrangian $\mathscr{L} = \alpha R + \beta R^2 + \gamma R^{\mu\nu} R_{\mu\nu}$ is renormalisable for certain values, but for those also not well-defined and unitary. Negative energy modes destabilise.

12. 'Ultraviolet divergences in quantum theories of gravitation'

13. 'Wave function of the universe', see also 7.9.

14. 'The ultraviolet behaviour of Einstein gravity'

15. 'New variables for classical and quantum gravity'

16. 'Exact evolution equation for the effective potential'

17. 'Effective average action for gauge theories and exact evolution equations', 'Non-perturbative evolution equation for quantum gravity'

One may also classify three groups according to the 'quantisation approach':

1. quantise only matter fields and leave the (curved) background classical, also called semi-classical approach

2. quantise matter fields and the gravitational field

3. quantise something else which induces gravity and matter (and also unify all theories)

7.12.1 Linearised Gravity

Side calculations of **??**.
The metric is split into the background (here Minkowksi,more general any background metric, general $\overline{g_{\mu\nu}}$) and the fluctuation part

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \quad g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu \tag{7.102}$$

The inverse contains infinite many power terms in κ and h.

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\rho} h_\rho^\nu + ... \tag{7.103}$$

which gives

$$\begin{aligned} \sqrt{-g} &= 1 + \frac{h_\rho^\rho}{2} - \frac{h_\sigma^\rho h_\rho^\sigma}{4} + \frac{(h_\rho^\rho)^2}{8} + ... \\ R_{\mu\nu\alpha\beta} &= \frac{\kappa}{2}(\partial_{\mu\alpha} h_{\nu\beta} - \partial_{\nu\beta} h^{\mu\alpha} - \partial_{\nu\alpha} h_{\mu\beta} + \partial_{\nu\beta} h_{\mu\alpha}) \\ R_{\mu\nu} &= \frac{\kappa}{2}(\partial_{\rho\mu} h_\nu^\rho + \partial_{\rho\nu} h_\mu^\rho - \partial_{\mu\nu} h_\rho^\rho - \eta^{\mu\nu}\partial_\mu\partial_\nu h_{\mu\nu}) \\ R &= \kappa(\partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h_\rho^\rho) \end{aligned} \tag{7.104}$$

Putting this into the usual conservation equation gives

$$\begin{aligned} 0 = \partial^\mu(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R) = \frac{\kappa}{2}(\partial_\rho \Box h_\nu^\rho + \partial_{\rho\nu\mu} h^{\mu\rho} - \Box\partial_\mu h_\nu^\mu - \Box\partial_\nu h_\rho^\rho \\ - \partial_{\nu\rho\sigma} h^{\rho\sigma} + \partial_\mu \Box h_\rho^\rho) = \frac{\kappa^2}{4}\partial^\mu T_{\mu\nu} \end{aligned} \tag{7.105}$$

Collecting terms, $X_{\mu\nu\alpha\beta}h^{\alpha\beta} = \frac{\kappa}{2}T_{\mu\nu}$. Adding gauge invariance

$$\hat{x}^\mu = x^\mu + \kappa\zeta^\mu(x), \tag{7.106}$$

$$\hat{g} = \eta + \kappa\hat{h}, \quad \hat{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu \tag{7.107}$$

The Ricci scalar is invariant. When we choose the harmonic gauge we arrive at the same result we arrived in the GR module, $\partial_\mu h_\nu^\mu - \frac{1}{2}\partial_\nu h_\rho^\rho = 0$.

$$\Box h_{\mu\nu} - \frac{1}{2}\Box h_\rho^\rho \eta_{\mu\nu} = \frac{-\kappa}{2}T_{\mu\nu} \tag{7.108}$$

a wave equation. A static point mass $T_{\mu\nu} = \text{diag}(M, 0, 0, 0)\delta(x)$ gives a fluctuation metric of $\sim \frac{\kappa M}{32\pi}\text{diag}(1, 1, 1, 1)\delta(x)$ and gravitational waves for $T_{\mu\nu} = 0$ are of the form $\sim (\epsilon_{\mu\nu}e^{-ipx} + \epsilon^*_{\mu\nu}e^{ipx}$.
Linearising around a background metric $\hat{g}_{\mu\nu}$ under infinitesimal transformation $x'^\mu = x^\mu - \zeta^\mu$, covariant derivative ';',

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} + \hat{g}_{(\mu\alpha}\zeta^\alpha_{;\nu)} + \kappa h_{\nu\nu} + \kappa h_{\mu\nu;\alpha}\zeta^\alpha + \kappa h_{(\mu\alpha}\zeta^\alpha_{;\nu)} \tag{7.109}$$

with the gauge transform of the fluctuation metric

$$h_{\mu\nu} \to h_{\mu\nu} + \left(\frac{1}{\kappa}\hat{g}_{(\mu\alpha} + h_{(\mu\alpha)}\zeta^\alpha_{\hat{;}\nu)}\right) + \zeta^\alpha h_{\mu\nu\hat{;}\alpha} \tag{7.110}$$

and the background transformation itself is given by

$$\hat{g}_{\mu\nu} \to \hat{g}_{\mu\nu} + \hat{g}_{(\mu\alpha}\zeta^\alpha_{\hat{;}\nu)} \tag{7.111}$$

$$h_{\mu\nu} \to h_{\mu\nu}h_{(\mu\alpha}\zeta^\alpha_{\hat{;}\nu)} + h_{\mu\nu\hat{;}\alpha}\zeta^\alpha \tag{7.112}$$

I define $h^\rho_\rho = h$ and linearise the action of EH without the cosmological constant and a simple scalar field which could be an inflaton field (see also Donoghue, 1995).

$$S = \int d^4x\sqrt{-g}\left(\frac{2R}{\kappa^2} + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right) \tag{7.113}$$

where anywhere else we raise and lower with the background metric $\hat{g}$. Let us substitute the results from above and order the action according to the constant background part, the usual unperturbed term that vanishes on-shell, the quadratic part and the gravitational interaction (proportional to κ). The massive scalar is $\phi = \hat{\phi} + \phi'$, again background plus fluctuation.

$$S \to \int d^4x\sqrt{-\hat{g}}\left(\frac{2}{\kappa}\hat{R} + \frac{1}{2}g^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} - \frac{1}{2}m^2\hat{\phi}^2\right) \tag{7.114}$$

$$- 2h_{\mu\nu}\left(\hat{R}^{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}^{\mu\nu}\right) - \frac{1}{2}h_{\mu\nu}\left(\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} - \frac{1}{2}\hat{g}^{\mu\nu}\partial_\rho\hat{\phi}\hat{\phi}^\rho\right) + \left(\hat{\phi}_{;\hat{\nu}^{;\nu}+m^2\hat{\phi}}\right)\phi' \tag{7.115}$$

$$+ 2\left(\partial_\rho h_{\mu\nu}\partial_\gamma h_{\alpha\beta}f_1 + h_{\mu\nu}f_2 + \frac{1}{2}\phi'_{\hat{;}\mu}\phi'_{\hat{;}\nu} - \frac{1}{2}m^2\phi'^2 + h_{\mu\nu}\phi'_{\hat{;}\rho}f_3\right) \tag{7.116}$$

$$+ \ldots \tag{7.117}$$

with f_1 and f_2, tensors $T_1^{\mu\nu\alpha\beta\rho\gamma}(\hat{g},\hat{\phi})$ and $T_2^{\mu\nu\alpha\beta}(\hat{g},\hat{\phi})$, $T_3^{\rho\mu\nu}$. Following Donoghue we proceed with gauge fixing and need to introduce Faddeev Popov ghosts.

$$\mathscr{L}_{\text{gf}} = \frac{1}{2}\sqrt{-\hat{g}}C_\mu C^\mu, \quad C_\mu = h^{\hat{;}\nu}_{\mu\nu} - \frac{1}{2}h^\rho_{\rho\hat{;}\mu} - \partial_\mu\hat{\phi}\phi' \tag{7.118}$$

$$\mathscr{L}_{\text{ghost}} = \sqrt{-g}\eta^\mu(g_{\mu\nu} - R_{\mu\nu} - \partial_\mu\phi\partial_\nu\phi)\eta^\nu \tag{7.119}$$

where η is the fermionic field. The final Lagrangian is the sum of all.

Faddeev Popov Ghosts

The background field method introduced by DeWitt (1967) and improved further by 't Hooft and others follows as mentioned, including the introduction of gauge fixing and ghosts[22]. Taking scalar QED as an example we arrive at the issue that we integrate over all configurations, also the ones that are related by a gauge transformation.

$$\mathscr{L} = \phi_{;\mu}(\phi^{;\mu})^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad ; = D_\mu = \partial_\mu + ieA_\mu \tag{7.120}$$

$$A_\mu \to A_\mu + \partial_\mu\theta = A^{(\theta)}_\mu \tag{7.121}$$

Hence, we would integrate over an infinite number of configuration copies. We need to account for the gauge fixing. With the Faddeev Popov trick we fix the correct integration measure. The trick is to insert the following two identity integrals into Z.

[22]Already early on Feynman proposed the idea of artificial particles to make the optical theorem valid in GR. The optical theorem is an implication of unitarity.

$f(A_\mu) = F(x)$ is some gauge fixing condition

$$\mathbb{1} = \int \mathcal{D}\theta \delta_p(f(A_\mu^{(\theta)}) - F)\Delta(A) \tag{7.122}$$

$$\Delta(A) = \det \frac{\partial f}{\partial \theta} \tag{7.123}$$

$$1 = N(\zeta) \int \mathcal{D}F e^{-\frac{i}{2\zeta}\int d^4x F(x)^2} \tag{7.124}$$

Now we perform integration over θ and $F(x)$,

$$Z = \frac{1}{N} \int \mathcal{A}_{\preceq} \Delta(A) e^{iS - \frac{i}{2\zeta}\int d^4x f(A_\mu)^2} \tag{7.125}$$

with the Faddeev Popov determinant $\Delta(A)$ and the action minus the gauge fixing term in the exponent. We need to introduce ghost fields which are artificial and no physical states and then we can split the integration over physically distinct configurations and over gauge orbits. In the case of ED we could actually drop the term as $\frac{\partial f}{\partial \theta}$ is independent of A_μ, but they are necessary in non-abelian theories and obviously in GR. We introduce the ghost fields $C, \overline{C}$ by

$$\det \frac{\partial f}{\partial \theta} = \int \mathcal{D}C \mathcal{D}\overline{C} e^{i\int d^4x \overline{C} \frac{\partial f}{\partial \theta} C} \tag{7.126}$$

with the gauge fixing term $\mathcal{L}_{gf} = \frac{1}{2} C_\mu C^\mu$. For example, in the de Donder gauge $\sim (h^{\hat{;}\mu}_{\mu\nu} - \frac{1}{2}h_{\hat{;}\nu})^2$, where it is transformed as $C_\mu \to C_\mu + \hat{g}_{\mu\nu}\zeta^{\nu\hat{;}\rho}_{\hat{;}\rho} + \hat{R}_{\mu\nu}\zeta^\nu$. Since we have a gauge fixing vector we need fermionic vectors η in GR (fermionic scalars for usual gauge theories such as Yang Mills) as Faddeev Popov fields. Reason is that we need to cancel the bosonic vector field's loop diagrams[23]. Now we can rewrite 7.125 with that transformation[15]

$$\det \frac{\partial C_\nu}{\partial \zeta_\mu} = \int \mathcal{D}\eta_\alpha \mathcal{D}\hat{\eta}_\beta e^{i\int d^4x \sqrt{-g} e \hat{t} a^\mu (\hat{g}_{\mu\nu}\hat{D}^2 + \hat{R}_{\mu\nu})\eta^\nu} \tag{7.127}$$

$$Z = \int \mathcal{D}h_{\mu\nu} \mathcal{D}\eta_\alpha \mathcal{D}\hat{\eta}_\beta \mathcal{D}\phi (\mathcal{L}(h) + \mathcal{L}_{gf}(h) + \mathcal{L}_{ghost}(\eta, \hat{\eta}, h) + \mathcal{L}_{mat}) \tag{7.128}$$

[23]The determinant can be rewritten in terms of a Grassmann integration, $\int e^{-\theta^T M \eta} d\theta d\eta = \det M$ for an $n x n$ matrix where θ, η should be interpreted as anti-commuting complex ghost fields. Ghost as fermions have the advantage of having the opposite sign in the loops such that they cancel the redundancies. Note that the result of a Grassmann integral is invariant leads to the fact that the Grassmann integration measure transforms as the inverse (!) of the Jacobian (for normal numbers it is simply the Jacobian). Simply put, bosonic Gaussian integral $\sim (\det A)^{-\frac{1}{2}}$, fermionic integral $\sim \det A$ (without 0 modes).

$$\int \mathcal{D}\overline{\psi}\mathcal{D}\Psi e^{-S} = \int \prod_i d\overline{a}_i da_i e^{-\sum_{i,j}\int dt \overline{\eta}_j \overline{a}_j A \zeta_i a_i} = \int \prod_i d\overline{a}_i da_i e^{-\sum_i \lambda_i \overline{a}_i a_i} = \sum_i \lambda_i = \det A$$

for the modes $\Psi = \sum_i a_i \zeta_i, \overline{\Psi} = \sum_i \overline{a}_i \overline{\eta}_i, A\zeta_i = \lambda \eta_i$

and the Grassmann integration $\int d\theta\theta = 1, \int d\theta 1 = 0$

if we introduce matter fields as well. The ghosts violate the spin-statistics theorem (in normal gauge theories they have fermionic statistics but are spinless), but as they are not physical the theory is still well-defined.

7.12.2 Quantum Corrected Newton Potential

Useful integrals from real space to momentum space are

$$\begin{aligned}\int \frac{d^3q}{(2\pi)^3}\frac{e^{iqr}}{|q|^2} &= \frac{1}{4\pi r}\\ \int \frac{d^3q}{(2\pi)^3}\frac{e^{iqr}}{|q|} &= \frac{1}{2\pi^2 r^2}\\ \int \frac{d^3q}{(2\pi)^3}e^{iqr}\ln q^2 &= -\frac{1}{2\pi r^3}\end{aligned} \tag{7.129}$$

The Feynman rules for gravity are 7.3 with the massless graviton propagator given by

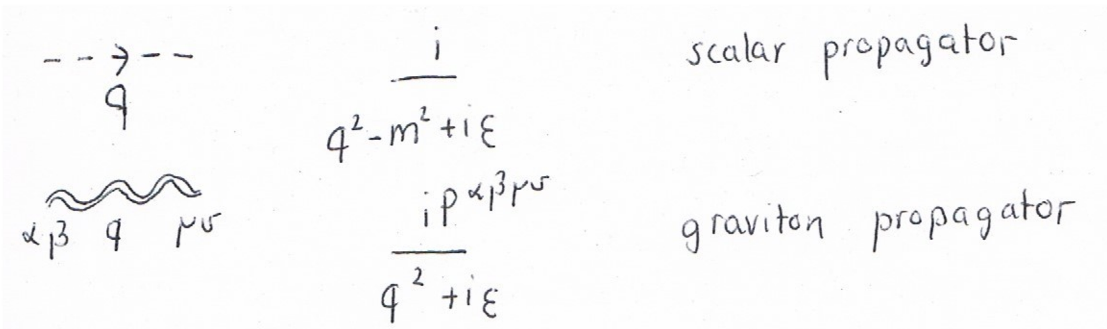


Figure 7.3: Feynman rules for gravity

$P^{\alpha\beta\mu\nu} = \frac{1}{2}(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\beta\mu}\eta^{\alpha\nu} - \eta^{\alpha\beta}\eta^{\mu\nu})$. Donoghue linearises gravity around flat Minkowski and uses the harmonic gauge. $\kappa^2 = 32\pi G$,

$$\tau_1^{\mu\nu}(k,k',m) = -\frac{i\kappa}{2}\left(k^\mu k'^\nu + k^\nu k'^\mu - \eta^{\mu\nu}(k.k' - m^2)\right) \tag{7.130}$$

$$\tau_2^{\sigma\rho\mu\nu}(k,k',m) = i\kappa^2((I^{\sigma\rho\alpha\delta}I^{\mu\nu\beta}_{\delta} - \frac{1}{4}(\eta^{\sigma\rho}I^{\mu\nu\alpha\beta} + \eta^{\mu\nu}I^{\sigma\rho\alpha\beta}))(k_\alpha k'_\beta + k'_\alpha k_\beta) \tag{7.131}$$

$$-\frac{1}{2}(I^{\sigma\rho\mu\nu} - \frac{1}{2}\eta^{\sigma\rho}\eta^{\mu\nu}(k.k' - m^2)) \tag{7.132}$$

$$I_{\alpha\beta\gamma\delta} = \frac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma}) \tag{7.133}$$

[24] A useful identity for below is

$$P_{\gamma\delta\sigma\rho}P_{\alpha\beta\mu\nu}\tau^{\sigma\rho\mu\nu}(k,k',m) = \tau_{\gamma\delta\alpha\beta}(k,k',m) \tag{7.134}$$

[24] See also 'Gravitational interaction to one loop in effective quantum gravity', 1996.

The tree level (0) is given by

$$iM_1(q) = \tau_1^{\mu\nu}(k_1, k_2, m_1)\frac{iP_{\mu\nu\alpha\beta}}{q^2}\tau_2^{\alpha\beta}(k_3, k_4, m_2), \quad q = k_1 - k_2 = k_4 - k_3 \tag{7.135}$$

which can be solved with 7.129 to give

$$M_1(q) = -\frac{4\pi G m_1 m_2}{q^2} \tag{7.136}$$

He continues with calculating the non-analytic part of the corrected potential given by the diagrams (1) 7.4, using the Feynman rules and integrating over the momenta k', k'' and the four propagators (mathematically complex to calculate, see [15]), taking the non-relativistic limit and again Fourier transforming 7.129, they contribute $-\frac{47}{3}\frac{m_1 m_2 G^2}{\pi r^3}$. Further, he treats triangle diagrams ((2) and the momenta swapped) over three propagators whose result adds to the non-analytic part and the relativistic term, $-\frac{4G^2 m_1 m_2(m_1+m_2)}{3\pi r^2} + \frac{28 m_1 m_2 G^2}{\pi r^3}$. I will give an example for calculating the 'seagull' (3) diagram. The symmetry factor is 2 as we can exchange the two virtual particles and get the same diagram.

$$iM_{\text{seagull}} = \frac{1}{2}\int \frac{d^4k'}{(2\pi)^4}\tau_2^{\alpha\beta\gamma\delta}(k_1, k_2, m_1)\frac{iP_{\alpha\beta\mu\nu}}{(k'+q)^2}\tau_2^{\sigma\rho\mu\nu}(k_3, k_4, m_2)\frac{iP_{\gamma\delta\sigma\rho}}{k'^2} \tag{7.137}$$

Using 7.134 this can be rewritten

$$-\frac{1}{2}\int \frac{d^4k'}{(2\pi)^4}\tau_2^{\alpha\beta\gamma\delta}(k_1, k_2, m_1)\frac{1}{(k'+q)^2 k'^2}\tau_{2\gamma\delta\alpha\beta}(k_3, k_4, m_2) \tag{7.138}$$

substituting the formulae for τ and integrating $\int \frac{d^4k'}{(2\pi)^4}\frac{1}{(k'+q)^2 k'^2} = \frac{-2i\ln q^2}{32\pi^2}$ gives

$$M = 44 G^2 m_1 m_2 \ln q^2 \tag{7.139}$$

The seagull only adds to the non-analytic part, $-\frac{22 m_1 m_2 G^2}{\pi r^3}$, after Fourier transforming again. Then, he proceeds with vertex corrections such as (4) that contribute again to both correction terms, $\frac{G^2 m_1 m_2(m_1+m_2)}{r^2} + \frac{7 m_1 m_2 G^2}{\pi r^3}$ and the two vacuum polarisation diagrams (5) (see also Duff, 1974), $-\frac{43 m_1 m_2 G^2}{30\pi r^3}$ [25]
Adding all terms gives the quantum corrected potential

$$V(r) = -\frac{G m_1 m_2}{r}\left(1 + \frac{3G(m_1+m_2)}{r} + \frac{41 G\hbar}{10\pi r^2}\right) \tag{7.140}$$

7.13 ASG

7.13.1 Example

I will derive the effective action for a simple action such as $S_0 = \frac{1}{2}\int d^4x(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2$ in four dimensions and first assuming non-Euclidean.

[25] The vacuum polarisation tensor is given by the effective Lagrangian $\sim \log q^2(aR^2 + bR^{\mu\nu}_{\mu\nu}$ derived by 't Hooft and Veltman [29].

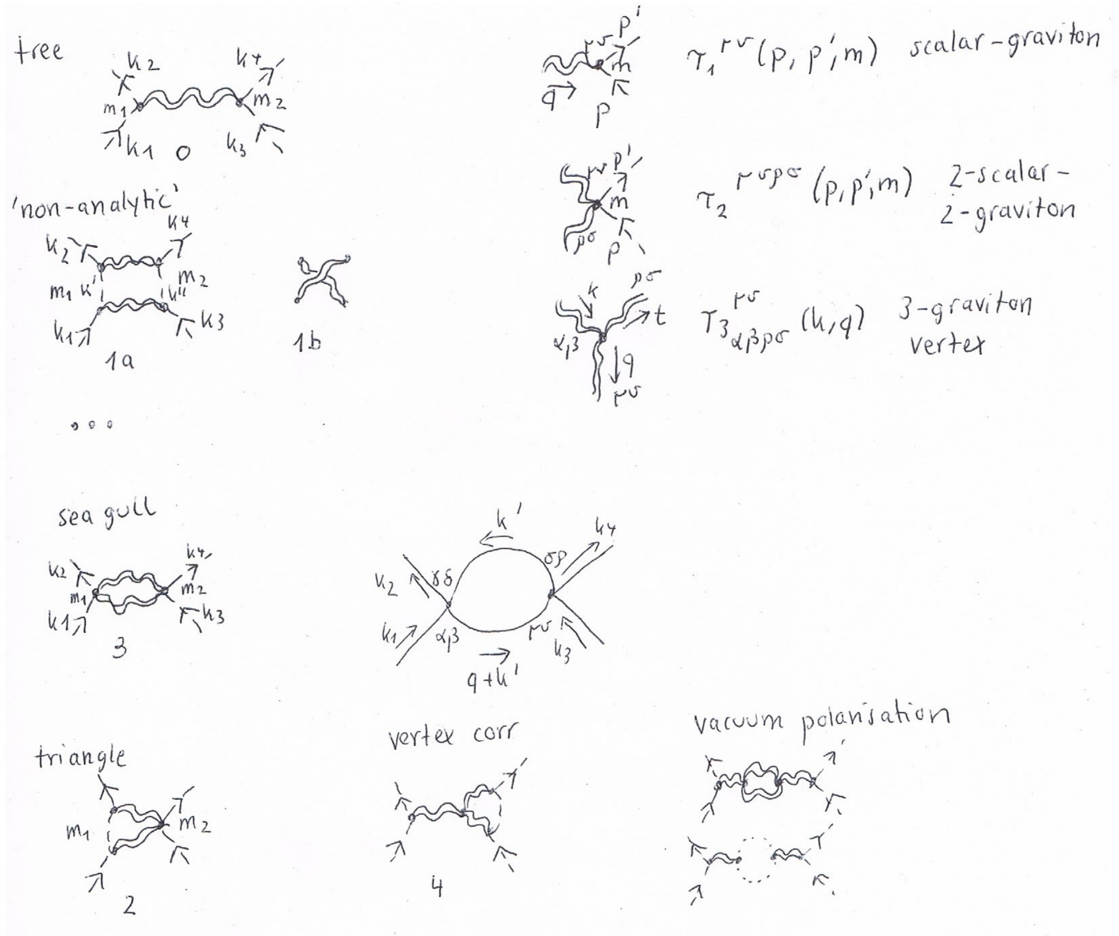


Figure 7.4: gravitational Feynman diagrams following Donoghue's treatment

The S-matrix is related to the n-point Green function via LSZ

$$G_n(x_1, ..., x_n) = \langle 0| T\hat{\phi}(x_1)...\hat{\phi}(x_n)|0\rangle = (-i)^n \frac{\delta}{\delta J(x_1)} ... \frac{\delta}{\delta J(x_n)} Z[J]|_{J=0} \quad (7.141)$$

Hence,

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1...d^4x_n J(x_1)...J(x_n) G_n \quad (7.142)$$

We also know that Z can be calculated from the path integral, where $\mathscr{D}$ is the integration over all functionals

$$Z[J] \sim \int \mathscr{D}\phi e^{iS[\phi]+i\int d^4x J(x)\phi(x)}, \quad Z_0 = e^{-\frac{i}{2}\int d^4x d^4x' J(x) D_F(x-x') J(y)} \quad (7.143)$$

The Schwinger functional can be calculated

$$Z[J] = e^{iW[J]} \quad \rightarrow \quad W[J] = -i \ln Z[J] \quad (7.144)$$

that generates all connected diagrams

$$G_n(x_1, ..., x_n)^{\text{connected}} = i(-i)^n \frac{\delta}{\delta J(x_1)} ... \frac{\delta}{\delta J(x_n)} W[J]|_{J=0} \quad (7.145)$$

Similarly, from the connected diagrams we can infer W. We proceed with the Legendre transform of W to arrive at the effective average action.

$$\frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\,\hat{\phi}(x)\,|0\rangle}{<0\,|0\rangle>} \tag{7.146}$$

which we define as $\overline{\phi}$. Obviously, one can also derive it directly from $-\frac{i}{Z[J]}\frac{\delta Z[J]}{\delta J(x)}$. From $\overline{\phi} = \overline{\phi}[J]$ we get the expression for the source current, $J = J[\overline{\phi}]$, which leads us to the effective action

$$\Gamma[\overline{\phi}] = W[J[\overline{\phi}]] - \int d^4x J(x)\overline{\phi}(x) \tag{7.147}$$

which is the classical action (tree level) plus the corrections (loops), $\Gamma = S + \Gamma_1 + ...$ It is the generating functional of 1PI Green's functions. We have 'solved' the theory. Note the equations of motion can be calculated by $\frac{\delta\Gamma}{\delta\overline{\phi}=-J}$. In asymptotic safety we mostly calculate in the Euclidean setting $S \to S_E$, $t \to -i\tau$ (Wick rotation), where the computations and the interpretation as probability density become clearer. The steps are similar.

$$\begin{aligned} Z[J] = e^{W[J]} &= \int \mathscr{D}\phi e^{-S_E[\phi]+\int dx\phi(x)J(x)} \\ G_n^E(x_1,...,x_n) &= \frac{\delta}{\delta J(x_1)}....Z[J]|_{J=0} \\ \frac{\delta W}{\delta J(x)} &= \overline{\phi}(x), \quad J = J(\overline{\phi}) \\ \Gamma &= \int d^4x\overline{\phi}(x)J[\overline{\phi}(x)] - W[J[\overline{\phi}]] \end{aligned} \tag{7.148}$$

Dimensional analysis gives $[x] = -1$, $[d^4x] = -4$, $[\partial_\mu] = 1$ a mass dimension of $[\phi] = 1$.
A typical truncation could be, for example, $\mathbb{Z}$ symmetry, $\phi \to -\phi$.

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + c_1\phi^2 + c_2\phi^4 + ... + d_1\phi\Box\Box\phi + ...\right) \tag{7.149}$$

which is perturbatively renormalisable for dimensionless c_2 ($[c_1] = 2$) and up to that term only. In higher dimensions, $d > 4$, we would have $[\phi] = \frac{1}{2}(d-2)$ and couplings of dimension $[c_1] = d - 2(\frac{d}{2}-1)$, $[c_2]d - 4(\frac{d}{2}-1)...d - n(\frac{d}{2}-1)$ for ϕ^n and $[d_1] = d - 2(\frac{d}{2}-1) - 4....$ In four dimensions, however, we cannot add higher than quadratic terms. The truncation can be written in the form

$$S = \sum_i g_i O_i(\phi) \tag{7.150}$$

where each operator (here under the reflection symmetry constraint) comes with a coupling. Importantly, in ASG those couplings are redefined to become running ones and thus, dimensionless.

7.13.2 'Derivation' ERGE

As the Wetterich equation [87] is used in almost all calculations in asymptotic safety I will derive it in the following and point out the most important properties (I will leave out the gauge and ghost fixing part).
First, have a look at the usual Legendre transform. In a Euclidean setting[26] for a simple scalar field ϕ we should first calculate the generating functional of connected Green functions $W[J]$, *defined* from the path integral

$$e^{-W[J]} = \int \mathcal{D}\phi e^{-S[\phi]-\int J\phi} \tag{7.151}$$

The Legendre transform which generates the 1PI Green functions is given by

$$\Gamma = W[J] - \int J\overline{\phi}, \quad \overline{\phi} =< \phi >= \frac{\delta W}{\delta J} \tag{7.152}$$

In order to get the renormalisation group flow we take

$$\begin{aligned} &W \to W_k \\ &S \to S + \Delta S_k, \quad \Delta S_k = \frac{1}{2}\int d^d p \phi R_k(p^2)\phi \\ &\Gamma \to \Gamma_k = W_k - \int \overline{\phi} J - \Delta S_k \end{aligned} \tag{7.153}$$

with a suitable chosen cutoff function and J is given by $\frac{\delta W_k}{\delta \phi} \leftrightarrow J = J(\phi)$. We also know that the effective average action can be written as the sum of couplings and operators,

$$\Gamma_k = \sum g_i O_i \tag{7.154}$$

where the couplings depend on the momentum and the operators are terms of the field which should be the tree level and all loops.

$$\Gamma^{(1)} = S + \frac{1}{2}\mathrm{Tr}\left\{\log \frac{\delta^2 S}{\delta\phi\delta\phi}\right\} \leftrightarrow \Gamma_k^{(1)} = S + \frac{1}{2}\mathrm{Tr}\left\{\log \frac{\delta^2 S'}{\delta\phi\delta\phi}\right\} \tag{7.155}$$

where $S' = S + \Delta S_k$. In the expression itself the cutoff term cancels, but needs to be evaluated in the derivative.

$$\Gamma_k^{(1)} = S + \frac{1}{2}\mathrm{Tr}\left\{\log\left(\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k\right)\right\} \tag{7.156}$$

The Wetterich equation is now motivated by taking the derivative wrt the RG time (defined by $\delta_t = k\delta_k$) of the effective 1-loop action

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k}\right)\dot{R}_k \tag{7.157}$$

[26] Based on AQFT notes.

Along with 7.154, $\dot{\Gamma}_k = \sum_i \beta_i O_i$, where we define the β-functions as

$$\beta(g_i) = \dot{g}_i \tag{7.158}$$

we can set both expressions equal and find ALL 1-loop β-functions that already tell us a lot about teh theory's behaviour. However, we still haven't derives the Wetterich expression which is an exact equation in Γ_k only. It turns out that we can replace $S \to \Gamma_k$.

$$\dot{\Gamma}_k = \dot{W}_k - \Delta \dot{S}_k = \frac{1}{2}\mathrm{Tr}\{< \phi\phi >\}\dot{R}_k - \frac{1}{2}\mathrm{Tr}\{< \phi >< \phi >\}\dot{R}_k \tag{7.159}$$

which can be rewritten as

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\left\{\frac{1}{\frac{\delta^2\Gamma_k}{\delta\overline{\phi}\delta\overline{\phi}} + R_k}\right\} \tag{7.160}$$

where I again used $\overline{\phi} = \frac{\delta W_k}{\delta J}$ and

$$\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\overline{\phi}\delta\overline{\phi}} = -\left(\frac{\delta^2 W_k}{\delta J \delta J}\right)^2 \tag{7.161}$$

and obviously $J = -\frac{\delta(\Gamma_k + \Delta S_k)}{\delta\overline{\phi}}$.
This is the beauty of the Wetterich equation. It is exact (although it cannot be solved exactly) and only contains the effective average action.

7.13.3 Example β-functions

Take a simple Lagrangian of $\mathscr{L} = \frac{1}{2}\delta_\mu\phi\delta_\mu\phi + V_k(\phi^2)$, giving the effective average action, $\Gamma_k = \int d^d x \mathscr{L}$. The second variation wrt the field (I assume a constant scalar field) and suitable boundary conditions plus the cutoff is given by

$$\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k = -\delta^2 + 2V_k' + 4\phi^2 V_k'' + R_k(-\delta^2) \tag{7.162}$$

and the derivative wrt RG time using the Wetterich equation

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\left\{\frac{\dot{R}_k(-\delta^2)}{-\delta^2 + R_k(-\delta^2) + 2V_k' + 4\phi^2 V_k''}\right\} \tag{7.163}$$

The trace is the integration over all space and momenta $\int d^d x \int \frac{d^d p}{(2\pi)^d}$. Using a spherical transformation (idea taken from Percacci, same approach as in QED regularisation problems), $r = |p|^2$, the expression inside the trace, call it $T(-\delta^2)$ can be rewritten as

$$A_d \int d^d x Q_{\frac{d}{2}}(T) \tag{7.164}$$

$$A_d = \frac{1}{2^{d+1}\pi^d} V_0(\mathbb{S}^{d-1})\Gamma\left(\frac{d}{2}\right) = \frac{1}{2}\frac{2}{(2\pi)^d}\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\Gamma\left(\frac{d}{2}\right) = (4\pi)^{-\frac{d}{2}} \tag{7.165}$$

$$Q_{\frac{d}{2}} = \frac{1}{\Gamma(\frac{d}{2})}\int_0^\infty dr r^{n-1} T(r), r = |p|^2 \tag{7.166}$$

where n is the order of the expansion, $V = \sum_{n=1}^{N} g_{2n}\phi^{2n}$. Both sides now have the same volume element and the LHS reduces to the RG time derivative of the potential only.

$$\dot{V}_k = \frac{1}{2} A_d Q_{\frac{d}{2}} \left(\frac{\dot{R}_k}{-\delta^2 + R_k(-\delta^2) + 2V_k' + 4\phi^2 V_k''} \right) \tag{7.167}$$

The β-functions of the couplings (here easily computed from the form of the potential, $g_{2n} = \frac{1}{n!}\frac{\partial^n V}{\partial(\phi^2)^n}|_{\phi=0}$) are given by

$$\beta_{2n} = \dot{g_{2n}} = \frac{1}{n!}\frac{\partial^n}{\partial(\phi^2)^n}\dot{V}_k\Big|_0 \tag{7.168}$$

with $\dot{V}_k$ given by 7.167. To simplify the calculations I choose the potential with the first two terms only, so $V_k = g_2\phi^2 + g_4\phi^4$. Also, recall that the critical exponents and the fixed point are calculated in the space of dimensionless couplings, here

$$\hat{g}_{2n} = k^{-d+n(d-2)} \tag{7.169}$$

to account for the usual mass dimensions to cancel the d spacetime dimensions and for each order the ϕ terms. Hence, the β-functions are of the form

$$\hat{g}_{2n} = (-d + n(d-2))\hat{g}_{2n} - k^{-d+n(d-2)}\beta_{2n} \tag{7.170}$$

Since it has been calculated for 3 dimensions (Wilson-Fisher FP) I will calculate it for 4 dimensions.

$$g_2 = \frac{\partial V}{\partial(\phi^2)}|_0,\ g_4 = \frac{\partial^2 V}{\partial(\phi^2)^2} \tag{7.171}$$

$$\beta_2 = \dot{g_2} = \frac{\partial}{\partial(\phi^2)}\dot{V}_k|_0 = -\frac{12\cdot A_d Q_{\frac{d}{2}} g_4}{2}\left(\frac{\dot{R}_k}{(-\delta^2 + R_k(-\delta^2) + 2g_2)^2}\right) \tag{7.172}$$

$$\beta_4 = \dot{g_4} = \frac{\partial^2}{\partial(\phi^2)^2}\dot{V}_k|_0 = \frac{144\cdot A_d Q_{\frac{d}{2}} g_4^2}{2}\left(\frac{\dot{R}_k}{(-\delta^2 + R_k(-\delta^2) + 2g_2)^3}\right) \tag{7.173}$$

(other term vanishes as the couplings for g_6... vanish. Now we substitute those into 7.170 with $d = 4$

$$\begin{aligned}\dot{\hat{g}}_2 &= -2\hat{g}_2 - 6A_4Q_2g_4\frac{1}{k^2}\left(\frac{\dot{R}_k}{(-\delta^2 + R_k(-\delta^2) + 2g_2)^2}\right)\\ \dot{\hat{g}}_4 &= 0 - \beta_4 = 72A_4Q_2g_4^2\left(\frac{\dot{R}_k}{(-\delta^2 + R_k(-\delta^2) + 2g_2)^3}\right)\end{aligned} \tag{7.174}$$

The couplings can be expressed in terms of their dimensionless couplings, $\hat{g}_2 = k^{-2}g_2$ and $\hat{g}_4 = g_4$.[27] which is in the case of 4 dimensions $g_4 \to \hat{g}_4$. Again following Percacci, Litim's cutoff is used, $R_k = (k^2 - r)\theta(k^2 - r)$, recall $r = |p|^2$ such that $\dot{R}_k = 2k^2\theta(k^2 - r)$

[27] Note for 3 dimensions we have $\hat{g}_2 = k^{-2}g_2$, $\hat{g}_4 = k^{-1}g_4$.

and $-\delta^2 + R_k(-\delta^2) \to k^2$. Using the formulae 7.164 and with the cutoff constraint at k^2 and $\Gamma(n) = (n-1)!$. For Percacci $\Gamma(\frac{5}{2}) = (\frac{5}{2}-1)! = \frac{3}{2}! = \frac{4\sqrt{\pi}}{4^2 \cdot 2}$ results in

$$\begin{aligned} \dot{\hat{g}}_2 &= -2\hat{g}_2 - \frac{2\hat{g}_4}{\pi^2(1+2\hat{g}_2)^2} \\ \dot{\hat{g}}_4 &= -\hat{g}_4 + \frac{24\hat{g}_4}{\pi^2(1+2\hat{g}_2)^3} \end{aligned} \tag{7.175}$$

which give a GFP and a NGFP at $\hat{g}_2 = -\frac{1}{26}$, $\hat{g}_4 = \frac{72\pi^2}{2197}$. If we calculate the same in four dimensions we arrive at the problem of the following coupled system of equations

$$\begin{aligned} \dot{\hat{g}}_2 &= -2\hat{g}_2 - \frac{3\hat{g}_4}{8\pi^2(1+2\hat{g}_2)^2} \\ \dot{\hat{g}}_4 &= \frac{9\hat{g}_4}{2\pi^2(1+2\hat{g}_2)^3} \end{aligned} \tag{7.176}$$

A GFP is given for $\hat{g}_2 = \hat{g}_4 = 0$ but we cannot find another FP.[28]
In order to analyse its stability we calculate the stability matrix around the fixed point $\hat{g}_*$,

$$M_* = \left(\frac{\partial \hat{\beta}_i}{\partial \hat{g}_j}\right)|_* = \begin{pmatrix} -2 + \frac{4\hat{g}_4}{\pi^2(1+2\hat{g}_2)^3} & -\frac{2}{\pi^2(1+2\hat{g}_2)^2} \\ -\frac{144\hat{g}_4^2}{\pi^2(1+2\hat{g}_2)^4} & -1 + \frac{48\hat{g}_4}{\pi^2(1+2\hat{g}_2)^3} \end{pmatrix}|_* = \begin{pmatrix} -\frac{5}{3} & -\frac{36\pi^2}{169} \\ -\frac{169}{72\pi^2} & 1 \end{pmatrix} \tag{7.177}$$

Diagonalising gives the critical exponents (the eigenvalues times -1), $\theta_1 = 1.8425$ and $\theta_2 = -1.1759$, hence one is relevant and one is irrelevant and the dimension of the critical surface is 1.

7.13.4 Effective potential

I stated that the RG improvement is similar to the treatment of the effective potential in scalar field theory. Here is a short explanation of the Weinberg-Coleman potential type[29]. Take the Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{a}{2}\partial_\mu\phi^2 - \frac{b}{2}\phi^2 - \frac{c}{4!}\phi^4 \tag{7.178}$$

which is symmetric under $\phi \to -\phi$. The importance of the last three terms will come into a play in a bit and the coefficients will be determined. For $\lambda > 0$ we know that for $m^2 < 0$ spontaneous symmetry breaking occurs since there isn't only one (classical) vacuum state at 0 anymore.
The effective potential which is the classical potential (unbroken) plus some quantum corrected term is calculated from the partition function $Z = e^{iW}$. We will see that the quantum fluctuation encoded in the additional term arises from the vacuum. Let us consider the case of $m = 0$ i.e. right between the unbroken and broken case. We

[28] See also Critical exponents in 3.99 dimensions, Wilson& Fisher.
[29] Following Zee's 'QFT in a Nutshell' and [84]

use the same calculations as for the FRGE in Lorentzian setting (no gauge fixings or ghosts here).

$$Z = e^{iW} = \int \mathscr{D}\phi e^{i\int d^4x\mathscr{L}+J\phi} \tag{7.179}$$

$$\frac{\delta W[J]}{\delta J} =< \phi > \tag{7.180}$$

$$\Gamma[< \phi >] = W - \int d^4xJ < \phi > \Big|_x \tag{7.181}$$

$$J = -\frac{\delta\Gamma}{\delta < \phi >}\Big|_{x'} \tag{7.182}$$

$$\tag{7.183}$$

The functional derivative wrt the expectation value of the scalar field is identified with the derivative of the leading term of the effective action (derivative terms as the expectation value should be constant), $\Gamma \sim \int d^4x - v(< \phi >) + ...$, $J(x') = v'$ which is equivalent for $J \to 0$ to the classical $V' = 0|_{\text{minimum}}$. Weinberg concluded that the potential can be written as sum of the classical potential plus the quantum correction of order $O(\hbar)$ (recall in teh exponent we have $\frac{i}{\hbar}W$ by evaluating the loop correction. For this we calculate W at leading order at $< \phi >$.

$$W = S(< \phi >) + \int d^4xJ < \phi > + \frac{i\hbar}{2}\text{Tr}\{\log(\partial^2 + V'')\}|_{<\phi>} \tag{7.184}$$

where the trace is again over all momenta and position space, in 4d
$\int d^4x \int \frac{d^4k}{(2\pi)^4}\log(-k^2 + V'')|_{<\phi>}$. We find the correction term of order $\hbar$ of

$$-\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\log\left(1 - \frac{V''}{k^2}\right) \tag{7.185}$$

which I called previously the leading log type correction.
Substituting the action 7.178 after renormalisation via a cutoff, $\Lambda^2 = k^2$, and $< \phi > \to \phi$ with the condition (recall we have $m^2 = 0$), $\frac{d^2V_{\text{eff}}}{d\phi^2}|_0 = 0$ and we assume (the only left dependence), $V''''_{\text{eff}}|_E = \lambda_E$. We arrive at the result

$$V_{\text{eff}} = \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2}{256\pi^2}\phi^4\left(\log\left(\frac{\phi}{E}\right)^2 - \frac{25}{6}\right) \tag{7.186}$$

where the coupling is evaluated at the energy scale E. Under $\lambda_E \to \lambda$ the correction to the classical potential $\frac{\lambda}{4!}\phi^4$ is independent of the cutoff and of $\lambda\log(\frac{\phi}{E})$ order.

7.13.5 Ghost mass

Taking Niedermaier's and A&B β-functions agree (but not with Codello&Percacci's FRGE treatment (2006)). For the action $\frac{1}{2\lambda}C^2 - \frac{\omega}{3\lambda}R^2 + \frac{\theta}{\lambda}E$ with E being the Gauss-

Bonnet term, the β-functions read

$$(4\pi)^2 k \frac{d\lambda}{dk} = (4\pi)^2 \beta_\lambda = -\frac{133}{10}\lambda^2 \tag{7.187}$$

$$(4\pi)^2 \beta_\omega = \frac{-25 + 1098\omega + 200\omega^2}{60}\lambda \tag{7.188}$$

$$(4\pi)\beta_\theta = \frac{1}{45}(56 - 171\theta)\lambda \tag{7.189}$$

and a stable FP exists for $\omega_1, \omega_2 = -5.47, -0.0229$ we will use the second one,$\theta_* = 0.327$. Percacci gives a factor of $\frac{7}{90}$ in front of the θ β-function, the treatment (gauge and cutoff choice) does matter for the FP, but not much. The mass of the ghost is given by $m_2^2 = \frac{\lambda}{16\pi G}$. The result differs quite a lot from Benedetti, Machado & Saueressig (2009) and their FRGE treatment. Presicesly, let us finally evaluate the Lagrangian

$$\int d^4x\sqrt{g}\left(-\frac{k^2}{g}\left(2\lambda k^2 - R\right) - \frac{C^2}{2\sigma} + \frac{w}{3\sigma}\right) \tag{7.190}$$

with corresponding β-functions

$$(4\pi)^2 k \frac{d\sigma}{dk} = (4\pi)^2 \beta_\sigma = -\frac{133}{10}^2 (4\pi)^2 \beta_\omega = \frac{-25 + 1098\omega + 200\omega^2}{60} \tag{7.191}$$

I want to investigate the theory away from the FP with $\omega_* = -0.0229$. In this notation we know that the mass is $m_2^2 \frac{\sigma k^2}{g}$. I need to identify some scale, the Hubble scale is a viable choice. We know that under $k \to 0$ the first two terms should approach $\sim \frac{m_p^2}{16\pi}(2\Lambda - R)$ and for constant H during inflation, $a(t) = e^{Ht}$, we can find a solution of the eoms as in [86] and with the equations for the dimensionless couplings, of order $H^2 = \frac{\lambda}{3}k^2$ (which is true, since $H \sim k$ at the FP), hence, we can choose $\frac{m_2^2}{H^2} \sim \frac{\sigma}{g\lambda}$. We should evaluate the upper bound for σ which will depend on the observations, inflation produces perturbations, $H'(t) = H + \delta H(t)$ and following Weinberg's analysis of the de Sitter eom with $e^{\zeta H t}$ instabilities in de Sitter are given, we get $\zeta^2 + 3\zeta = A << 1$ (for inflation we need $\sim \frac{1}{20}$) with the positive value giving the instability. Following some algebra with the couplings we get the constraint $\frac{1}{20} \sim (8\pi)^{-1} s \frac{m_p^2}{H^2}$ (see also Weinberg, ζ corresponds to the slowly growing perturbation which ends the exponential expansion after $\frac{3}{A} \sim 60$ e-foldings). Thus, we conclude $m_2^2 \sim \frac{H^2(-2\omega_*)}{N} \sim 0.0276$ The energy scale at inflation is bigger than the ghost's mass. We also seen that a naive treatment of the running of the C^2 term is AF and we have seen that $s \sim 10^{-10}$ during inflation. See also 5.6.5.

(to be continued)

Bibliography

[1] H. Gies J.M. Pawlowski & R. Percacci A. Bonanno, A. Eichhorn. Critical reflections on asymptotically safe gravity. *Front. Phys.*, 8, 2020.

[2] S.H. Tye A. Guth. Phase transitions and magnetic monopole production in the very early universe. 44(631), 1980.

[3] A. Loeb A. Ijjas, P. Steinhardt. Inflationary paradigm in trouble after planck2013. *Phys. Lett. B*, 723, 2013.

[4] D. Linde & A. Mezhlumian A. Linde. From the big bang theory to the theory of a stationary universe. *Phys. Rev. D*, 49(4), 1983.

[5] A. Albrecht and P. Steinhardt. Cosmology for grand unified theories with radiatively induced symmetry breaking. 48(17), 1982.

[6] J. Garcia-Bellido & A.Linde. Stationarity of inflation and predictions of quantum cosmology. *Phys. Rev. D*, 51(2), 1995.

[7] J. Barrow. Cosmologies with varying light-speed. *Phys. Rev. D*, 59, 1998.

[8] B. Bassett. Inflation dynamics and reheating. *Phys. Rev. Mod*, 78, 2005.

[9] D. Baumann. Tasi lectures on inflation. 2009.

[10] M. Bojowald. Quantum cosmology- a fundamental description of the universe. *Lecture Notes in Physics 835*, 59, 2011.

[11] A. Bonanno. An effective action for asymptotically safe gravity. *Phys. Rev. D*, 85, 2012.

[12] R. Brandenberger. Inflation and the theory of cosmological perturbations. 1997.

[13] R.H. Dicke. Dirac's cosmology and mach's principle. *Nature*, 192, 1961.

[14] P. Dirac. Quantised singularities in the electromagnetic field. 133(821), 1931.

[15] J. Donoghue. General relativity as an effective field theory: The leading quantum corrections. *Phys. Rev. D*, 50, 1994.

[16] J. Donoghue. Introduction to the effective field theory description of gravity. *Advanced School on Effective Theories*, 1995.

[17] S.M. Christensen & M. J. Duff. Quantum gravity in two + epsilon dimensions. *Phys. Lett. B*, 79(123), 1978.

[18] C. Ringeval et al. Stationarity of inflation and predictions of quantum cosmology. *Phys. Rev. Lett.*, 105, 2010.

[19] J. E. Lidsey et al. Reconstructing the inflaton potential—an overview. *Rev. Mod. Phys.*, 69(2), 1997.

[20] Y. Akrami et al. Planck 2018 results. x. constraints on inflation. *Astron.Astrophys.*, 641, 2020.

[21] K. Falls. Physical renormalization schemes and asymptotic safety in quantum gravity. *Phys. Rev. D*, 96(12), 2017.

[22] R. Gastmans. Quantum gravity near two-dimensions. *Nucl. Phys. B*, 133(3), 1978.

[23] A. Guth. Inflationary universe: A possible solution to the horizon and flatness problems. *Phys. Rev. D*, 23(2), 1981.

[24] A. Guth. Inflation. *Proc. Natl. Acad. Sci*, 90, 1993.

[25] A. Guth. Eternal inflation and its implications. *J. Phys. A*, 40, 2007.

[26] J. B. Hartle & S. W. Hawking. Wave function of the universe. *Phys. Rev. D*, 28(12), 1983.

[27] S.W. Hawking. The quantum state of the universe. *Nuclear Physics B*, 239(1), 1984.

[28] S. H. Hawking & T. Hertog. A smooth exit from eternal inflation? *Journal of High Energy Physics*, 147(4), 2017.

[29] G.'t Hooft. Magnetic monopoles in unified gauge theories. 79(2), 1974.

[30] J. Pietenpol I. Lapidus. Classical interaction of an electric charge with a magnetic monopole. 28(17), 1960.

[31] J.L. Lehners & N. Turok J. Feldbrugge. No rescue for the no boundary proposal: Pointers to the future of quantum cosmology. *Phys. Rev. D*, 97, 2018.

[32] S.W. Hawking & T. Hertog J. Hartle. The no-boundary measure of the universe. *Phys. Rev. Lett*, 2008.

[33] R. Flauger E. A. Lim J.C. Aurrekoetxeaa, K. Clough. The effects of potential shape on inhomogeneous inflation. *Annales de L'Institut Henri Poincare Section Physique Theorique*, 2020.

[34] D.F. Litim K. Nikolakopoulos & C. Rahmede K. Falls, C.R. King. Asymptotic safety of quantum gravity beyond ricci scalars. *Phys. Rev. D*, 97, 2018.

[35] N. Ohta & R. Percacci K. Falls. Towards the determination of the dimension of the critical surface in asymptotically safe gravity. 2020.

[36] M. Reuter & O. Lauscher. Flow equation of quantum einstein gravity in a higher-derivative. *Phys. Rev. D*, 66(2), 2002.

[37] A. Linde. A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems. 108(6), 1982.

[38] A. Linde. Chaotic inflating universe. 38(3), 1983.

[39] A. Linde. Eternal chaotic inflation. 1(2), 1986.

[40] A. Linde. Introduction to cosmology. *Contemp.Concepts Phys. 5*, 2005.

[41] D. Litim. Fixed points of quantum gravity. *Phys Rev. Lett.*, 92(20), 2004.

[42] D. Litim. On fixed points of quantum gravity. *AIP Conf.Proc.*, 841, 2006.

[43] D.F. Litim. Optimised renormalisation group flows. *Phys. Rev. D*, 64, 2001.

[44] Y. Kluth & D.F. Litim. Asymptotic freedom in higher derivative quantum gravity. 2020.

[45] Y. Kluth & D.F. Litim. Heat kernel coefficients on the sphere in any dimension. *Eur. Phys. J.C*, 80, 2020.

[46] J. Ambjorn & R. Loll. Reconstructing the universe. *Phys. Rev. D*, 72(6), 2005.

[47] A. Albrecht & J. Magueijo. A time varying speed of light as a solution to cosmological puzzles. *Phys. Rev. D*, 59, 1999.

[48] J. Maldacena. Non-gaussian features of primordial fluctuations in single field inflationary models. *Journal of High Energy Physics*, 5, 2003.

[49] J. Mielczarek. Asymptotic silence in loop quantum cosmology. *talk presented at the Multiverse and Fundamental Cosmology Conference*, 2012.

[50] H. V. Peiris & R. Easther M.J. Mortonson. Bayesian analysis of inflation: Parameter estimation for single field models. *Phys. Rev. D*, 83, 2011.

[51] J. V. Narlikar. An introduction to cosmology. 2008.

[52] M.R. Niedermaier. Gravitational fixed points from perturbation theory. *Nucl. Phys. B*, 833, 2010.

[53] M. Fierz & W. Pauli. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. *Royal Society*, 173(953), 1939.

[54] R. Penrose. Difficulties with inflationary cosmology. *Annals of the New York Academy of Sciences*, 571(1), 2006.

[55] D. Dou & R. Percacci. The running gravitational couplings. *Class. Quant. Grav*, 15, 1998.

[56] A. Platania. The inflationary mechanism in asymptotically safe gravity. *Universe*, 5(8), 2019.

[57] A. Polyakov. Particle spectrum in the quantum field theory. 20, 1974.

[58] J. Preskill. Monopoles in the very early universe. 1982.

[59] R. Gambini & J. Pullin. A first course in loop quantum gravity. 2013.

[60] A. Rajantie. Magnetic monopoles from gauge theory phase transitions. *Phys. Rev. D*, 63, 2003.

[61] P. Peebles & B. Ratra. The cosmological constant and dark energy. *Rev. Mod. Phys.*, 75, 2002.

[62] A. Bonanno & M. Reuter. Cosmology of the planck era from a renormalization group for quantum gravity. *Phys. Rev. D*, 65, 2002.

[63] A. Bonanno & M. Reuter. Entropy production during asymptotically safe inflation. *Entropy*, 13(1), 2011.

[64] M. Reuter. Nonperturbative evolution equation for quantum gravity. *Phys. Rev. D*, 37(2), 1998.

[65] M. Niedermaier & M. Reuter. The asymptotic safety scenario in quantum gravity. *Living Reviews in Relativity*, 9(5), 2006.

[66] A. Riotto. Inflation and the theory of cosmological perturbations. *ICTP Lect.Notes*, 2002.

[67] C. Rovelli. Quantum gravity. *Cambridge monographs on mathematical physics*, 2004.

[68] D. Mortlock H. V. Peiris S. Feeney, M. Johnson. First observational tests of eternal inflation. *Phys. Rev. Lett.*, 107, 2010.

[69] D. Mortlock H. V. Peiris S. Feeney, M. Johnson. First observational tests of eternal inflation: Analysis methods and wmap 7-year results. *Phys. Rev. D*, 84, 2010.

[70] M. Goroff & A. Sagnotti. Quantum gravity at two loops. *Phys. Lett. B*, 160, 1985.

[71] M. Hindmarsh & I.D. Saltas. f(r) gravity from the renormalisation group. *Phys. Rev. D*, 86, 2012.

[72] M. Reuter & F. Saueressig. Renormalization group flow of quantum gravity in the einstein-hilbert truncation. *Phys Rev. D*, 65, 2001.

[73] A. Ashtekar & D. Sloan. Loop quantum cosmology and slow roll inflation. *Phys. Lett. B*, 649, 2010.

[74] L. Smolin. A fixed point for quantum gravity. *Nucl. Phys. B*, 208(3), 1982.

[75] A. Starobinsky. A new type of isotropic cosmological models without singularity. *Physics Letters B*, 91(1), 1980.

[76] P. Steinhardt. Kosmische inflation auf dem prüfstand. *Spektrum der Wissenschaft*, 11(8), 2011.

[77] J.L. Lehners & K.S. Stelle. A safe beginning for the universe? *Phys. Rev. D*, 2019.

[78] K. S. Stelle. Renormalization of higher-derivative quantum gravity. *Phys. Rev. D*, 16(4), 1977.

[79] K. S. Stelle. Classical gravity with higher derivatives. *Gen. Rel. Grav.*, 9(353), 1978.

[80] M. Tegmark. Our mathematical universe: My quest for the ultimate nature of reality. 2014.

[81] J.D. Barrow & F.J. Tipler. Action principles in nature. *Nature*, 331, 1988.

[82] H.A. Feldmanc & R.H. Brandenberger V.F. Mukhanov. Theory of cosmological perturbations. *Physics Reports 215*, 5, 1992.

[83] A. Ijjas F. Pretorius & P. Steinhardt W. Cook, I. Glushchenko. Supersmoothing through slow contraction. *Phys. Lett. B*, 808, 2020.

[84] E.J. Weinberg. Vacuum decay in theories with symmetry breaking by radiative corrections. *Phys. Rev. D*, 47, 1993.

[85] S. Weinberg. Critical phenomena for field theorists. *Understanding the Fundamental Constituents of Matter*, 1976.

[86] S. Weinberg. Asymptotically safe inflation. *Phys. Rev. D*, 81, 2009.

[87] C. Wetterich. Exact evolution equation for the effective potential. *Phys. Lett. B*, 301, 1993.

[88] B. Whitt. Fourth-order gravity as general relativity plus matter. *Physics Letters B*, 145(3), 1984.

[89] D.J. Gross & F. Wilczek. Ultraviolet behavior of non-abelian gauge theories. *Phys. Rev. Lett.*, 30, 1973.

[90] S. Winitzki. Reheating-volume measure for random-walk inflation. *Phys. Rev. D*, 78, 2008.

[91] R. Sachs & A. Wolfe. Perturbations of a cosmological model and angular variations of the cosmic microwave background. *Astrophysical Journal*, 73, 1967.

[92] G. 't Hooft & M. Veltman. One-loop divergencies in the theory of gravitation. *Annales de L'Institut Henri Poincare Section Physique Theorique*, 20, 1974.