Imperial College London

Theoretical Physics Group
Blackett Laboratory

Quantum informational perspectives on quantum gravity and the gauge/gravity duality

Author:
Dario Picozzi

Supervisor:
Professor Jonathan J. Halliwell

Submitted in partial fulfillment of the requirements
for the award of the degree of

Master of Science in Quantum
Fields and Fundamental Forces

Contents

Acknowledgements

I would like to thank the following people for their help and support over the past months: Professor J. J. Halliwell for providing thoughtful guidance and feedback throughout the writing of this work; Professor M. J. Duff, Professor D. Waldram, Professor F. Dowker, Professor T. Wiseman and Dr L. Borsten for the time they have taken to discuss the current literature with me and to answer my questions, as well as for giving me valuable advice; Professor A. Rajantie for being an incredibly supportive presence during my time at Imperial; as well as everyone else in the Theory Group at Blackett; I would not be forgiven for failing to mention my mother, as well as Chiara, Ado, and Tim for their continued support and for displaying exceptional patience when acting (more or less willingly) as my guinea pigs in my efforts to explain theoretical physics to a non-technical audience.

1

Introduction

One of the most surprising and significant developments in theoretical physics over the last two decades has been the gauge/gravity duality, often also referred to as the holographic duality or, in a more restrictive sense, as the anti-de Sitter/conformal field theory (AdS/CFT) duality.

Although some of the foundational ideas of holography had been originally proposed by G. 't Hooft [44] and L. Susskind [41], the first actual realisation of the principle was given by J. Maldacena in 1997 [27]. Maldacena proposed a duality between a gravitational theory that lives on a five-dimensional anti-de Sitter space, type IIB string theory on $AdS_5 \times S^5$, and a conformal gauge theory, a $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. Then the proposal was further developed and explicitly linked to the idea holography by E. Witten the next year. [27] and [49] are now the two most cited papers on the hep-th arXiv.

The duality is a strong/weak coupling duality: when the gauge theory is strongly coupled and we cannot use perturbation theory, it allows us to study the weakly coupled gravitational theory instead. Although at least initially most of the research has been done in that direction, the opposite direction also holds, and we can study properties of the gravitational theory by studying the gauge theory of the boundary.

Although it was originally formulated in string theoretical terms, [30] showed that the gauge/gravity duality can equivalently be formulated independently of string theory. Furthermore, many of the duality's implications for other areas of physics that have been discovered over the last two decades hold independently of AdS/CFT.

The implications for quantum gravity have been surprising: physicists have for decades endeavoured to reconcile particle physics, which is described by quantum gauge theories, and gravity, described by general relativity. The gauge/gravity duality tells us that these two apparently irreconcilable physical frameworks are in a certain sense effectively the same. In the words of [21]: “Hidden within every non-Abelian gauge theory, even within the weak and strong nuclear interactions, is a theory of quantum gravity”.

However, the gauge/gravity duality has also been at the forefront of more immediate applications than the search for a theory of everything: an example is AdS/QCD, the study the strongly couple dynamics of quark-gluon plasma, where the duality allows one to study the strong coupled dynamics of decoupled quarks by studying the black holes in the corresponding AdS theory. Another example is AdS/CMT, where the duality has been used to describe phase transitions in condensed matter.

The duality has given us a series of unexpected correspondences between concepts in the $(d+1)$-dimensional gravitational theory that lives in the ‘bulk’ and the d-dimensional gauge theory that lives on its boundary: perhaps the most central of these has been the Ryu-Takayanagi proposal of the entanglement entropy, which identifies the geometrical properties of surfaces on the gravitational theory with the amount of entropy contained in regions of the gauge theory. The newly restored centre-stage position taken by informational concepts, and the influence of holography on the way physicists now view their subject has been such that many including J. Bekenstein and L. Susskind have argued that we should see the universe not as made of energy and matter as its fundamental building blocks, but out of information.

In this review the focus will be on a conceptual introduction to the key concepts as well as to some of the most recent developments in the areas at the intersection of quantum information and the gauge/gravity duality. Most of the arguments given follow the treatment in the original literature, however they have been at times simplified or expanded for clarity. An effort has been made to keep the presentation as accessible and simple as possible, with the goal of being suitable to both theoretical physicists and those with an quantum information or other related background.

In §2.1 (p. 10) we will see some features of the geometry of general AdS and Schwarzschild-AdS spacetime, including an explicit derivation of the oscillatory trajectories of photons and massive particles. In §2.2 (p. 16) we will define the notion of a conformal field theory and show that the algebra corresponding to the symmetry group of a conformal field theory is in a one-to-one relationship to the that of the anti-de Sitter theory one dimension higher, a first clue of the AdS/CFT duality. Section §2.3 (p. 18) will present some of the main entries in the AdS/CFT dictionary that links the two theories together, as well as an outline of Maldacena’s original derivation.

In §3.1 (p. 22) we introduce the notions of von Neumann entropy and entropy of entanglement, setting the stage for the rest of the exposition.

Historically, the first hint for the gauge/gravity duality came from the study of black holes, a story we briefly summarise in §3.2 (p. 25). Black holes are found to have a finite temperature, the Hawking temperature, and a corresponding entropy, the Bekenstein-Hawking entropy. We see that this entropy does not increase with the black hole's volume, as one would first expect, but with its area.

In §4.1 (p. 30) we introduce the generalisation of the Bekenstein-Hawking entropy called the holographic entanglement entropy and first proposed by Ryu and Takayanagi. In §4.2 (p. 33) we show that this formula obeys the property of subadditivty that we expect for an entropy and in §4.3 (p. 34) we show that it also obey the property of monogamy. In §4.4 (p. 35) we give an outline of how one might go about to give a partial derivation of the Ryu-Takayanagi holographic entropy formula from the AdS/CFT duality.

In §5.1 (p. 38) we give an account of the black hole information problem, with a focus on the current reincarnation of the problem, and in §5.2 (p. 40) we give a review of the proposed solutions, focusing in particular on the $ER = EPR$ proposal by J. Maldacena and L. Susskind and highlighting the importance of the Ryu-Takayanagi formula in their claim.

In §6.1 (p. 44) we give the basics of quantum erasure correction, with the example of a code that uses three qutrits to encode the state of one qutrit, and allows us to reconstruct the original state when one of the three qutrits is lost. In §6.2 (p. 45) we review a proposal that uses the Ryu-Takayanagi formula to claim that fields on the bulk can be reconstructed from local operators defined only on regions of the conformal boundary (and not the whole boundary) in a way that is analogous to how the quantum erasure correction code protects against the erasure of one qutrit. In §6.3 (p. 47) we will see how this proposal has led to research into modelling the gauge/gravity duality in terms of discrete tensor networks that live on the bulk space.

Finally, in §7.1 (p. 51) we define the notion of the quantum complexity of an operator and identify it with the entropy of the time evolution of a corresponding classical system, leading to a second law of quantum complexity. In §7.2 (p. 56) we examine the consequences this has for the physics of black holes.

2

Fundamentals of AdS/CFT

2.1. AdS spacetime

Given a flat spacetime with two timelike directions $\mathbb{R}^{2,1}$:

$$ds^2 = -dZ^2 - dX^2 + dY^2 \tag{2.1}$$

The metric tensor gives us a notion of a (non positive-definite) distance between two points. The group of transformations that preserve the distance of any point from the origin is the indefinite orthogonal group $O(2,1)$.
The set of points at a given distance $-L^2$ from the origin (the orbit of a point under the group action) is given by the surface

$$-Z^2 - X^2 + Y^2 = -L^2 \tag{2.2}$$

Which is the equation of a hyperboloid.
One can define two-dimensional anti-de Sitter spacetime AdS_2 as the surface in Eq. (2.2) with the metric given in Eq. (2.1).
This is analgous to how one can define the sphere S^2 as a surface in the usual Euclidean space $\mathbb{R}^3$ with metric:

$$ds^2 = dX^2 + dY^2 + dZ^2 \tag{2.3}$$

Defined by:

$$X^2 + Y^2 + Z^2 = R^2 \tag{2.4}$$

In general, one can define the $p+2$-dimensional anti-de Sitter spacetime AdS_{p+2} as the embedded hypersurface in $\mathbb{R}^{2,p-1}$ satifying:

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + dX_1^2 + dX_2^2 + \ldots + dX_{p+1}^2 \tag{2.5}$$

And:

$$-X_0^2 - X_{p+2}^2 + X_1^2 + X_2^2 + \ldots + X_{p+1}^2 = -L^2 \tag{2.6}$$

This provides a solution of the Einstein field equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.7}$$

Where:

$$\Lambda = \frac{-p(p-1)}{2\text{Ł}^2} \tag{2.8}$$

And corresponds to a Lagrangain density:

$$\mathcal{L} = \frac{1}{16\pi G_{p+1}}(R - 2\Lambda) \tag{2.9}$$

One can then use the coordinate system:

$$\begin{aligned} X_0 &= L\cosh\rho\cos\tilde{t}, \\ X_{p+2} &= L\cosh\rho\sin\tilde{t}, \\ X_i &= L\sinh\rho\omega_i \end{aligned}$$

With:

$$\omega_1^2 + \ldots + \omega_{p+1}^2 = 1 \tag{2.10}$$

So that the metric can be written in global coordinates as:

$$\frac{ds^2}{L^2} = -\cosh^2\rho d\tilde{t}^2 + d\rho^2 + \sinh^2\rho d\Omega_p^2 \tag{2.11}$$

Where $d\Omega_p^2$ is the metric on the p-sphere S^p.
From the definition of global coordinates we have that the timelike dimension $\tilde{t}$ is periodic with period 2π, and this is problematic as one can then have closed timelike curves.
One can instead then take AdS_{p+2} to be the covering space with $-\infty < \tilde{t} < \infty$, so that what was visible from the initial choice of coordinates is only a region of the full spacetime (hence the name global coordinates).

By making the coordinate change:

$$\tilde{r} = \sinh \rho \tag{2.12}$$

The metric can also be rewritten in static coordinates:

$$\frac{ds^2}{L^2} = -\left(\tilde{r}^2 + 1\right) d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 + 1} + \tilde{r}^2 d\Omega_p^2 \tag{2.13}$$

From Eq. (2.11) by letting instead:

$$\tan \theta = \sinh \rho \tag{2.14}$$

One can rewrite the metric in conformal coordinates:

$$\frac{ds^2}{L^2} = \frac{1}{\cos^2 \theta} \left(-d\tilde{t}^2 + d\theta^2 + \sin^2 \theta d\Omega_p^2\right) \tag{2.15}$$

That is:

$$\frac{ds^2}{L^2} = \frac{1}{\cos^2 \theta} \left(-d\tilde{t}^2 + d\Omega_{p+1}^2\right) \tag{2.16}$$

From the conformal coordinates one can see that the metric of AdS_{p+2} is conformally equivalent to the Einstein static universe $\mathbb{R} \times S^{p+1}$, a spacetime where the spatial dimensions form a $(p+1)$-sphere.
We can also see that there is a spatial boundary at $\theta = \frac{\pi}{2}$, corresponding to $\rho = \infty$.
We can also see that a photon ($ds^2 = 0$) starting from the centre $\theta = 0$ will be able to travel to the boundary at $\theta = \frac{\pi}{2}$ in a finite coordinate time:

$$\begin{aligned} ds^2 &= \frac{L^2}{\cos^2 \theta} \left(-d\tilde{t}^2 + d\theta^2\right) = 0 \\ \frac{d\theta}{d\tilde{t}} &= 1 \\ \Delta \tilde{t} &= \int_0^{\pi/2} d\theta = \frac{\pi}{2} \end{aligned} \tag{2.17}$$

We then need to impose a boundary condition on the boundary. For instance, assume that the photon will be reflected back to the origin, the photon oscillates with a period of 2π.
We can also see that a massive inertial particle starting its motion at the origin will never reach the boundary and return to the origin in a finite proper time (performing an oscillatory motion).
In static coordinates (Eq. (2.13)) there is a conserved quantity E (corresponding to

the energy per mass, as it is given by the $\tilde{t}$-independence of the metric):

$$E = -g_{\tilde{t}\tilde{t}} \frac{d\tilde{t}}{d\tau} \tag{2.18}$$

We find that:

$$\begin{aligned} \left(\frac{ds}{d\tau}\right)^2 = g_{\tilde{t}\tilde{t}} \left(\frac{d\tilde{t}}{d\tau}\right)^2 + g_{\tilde{r}\tilde{r}} \left(\frac{d\tilde{r}}{d\tau}\right)^2 &= -1 \\ g^{\tilde{t}\tilde{t}} E^2 + g_{\tilde{r}\tilde{r}} \left(\frac{d\tilde{r}}{d\tau}\right)^2 &= -1 \\ -\frac{E^2}{L^2(\tilde{r}^2+1)} + \frac{L^2}{\tilde{r}^2+1} \left(\frac{d\tilde{r}}{d\tau}\right)^2 &= -1 \\ \frac{d\tilde{r}}{d\tau} &= \pm \frac{1}{L} \sqrt{((E/L)^2 - 1) - \tilde{r}^2} \end{aligned} \tag{2.19}$$

If the particle starts from the origin at $\tilde{t} = 0$, then we need to choose the positive sign as $\tilde{r} \geq 0$. At $\tilde{r}_* = \sqrt{(E/L)^2 - 1}$ we have $\frac{d\tilde{r}}{d\tau} = 0$ and so the velocity $\frac{d\tilde{r}}{d\tau}$ must change sign. Thus a massive particle does not reach the boundary, but only as far as $\tilde{r} = \tilde{r}_*$ in a proper time:

$$\begin{aligned} \Delta\tau &= \int_0^{\tilde{r}_*} \frac{L}{\sqrt{((E/L)^2 - 1) - \tilde{r}^2}} d\tilde{r} \\ &= L \int_0^{\frac{\pi}{2}} du \\ &= \frac{\pi L}{2} \end{aligned} \tag{2.20}$$

(where $\tilde{r} = \sqrt{(E/L)^2 - 1} \sin u$)
Before falling back towards the origin, oscillating with a period of $2\pi L$, where L is the AdS radius of curvature.
Using Eq. (2.18) to find that $\frac{d\tilde{t}}{d\tau} = \frac{E}{L^2(\tilde{r}^2+1)}$ we conclude that the observer at the origin sees the same journey to r_* happen in a coordinate time given by:

$$\begin{aligned} \Delta\tilde{t} &= \int_0^{\tilde{r}_*} \frac{E}{L(\tilde{r}^2+1)\sqrt{((E/L)^2 - 1) - \tilde{r}^2}} d\tilde{r} \\ &= \frac{E}{L} \int_0^{\frac{\pi}{2}} \frac{du}{((E/L)^2 - 1)\sin^2 u + 1} \\ &= \frac{E}{L} \int_0^{\infty} \frac{dv}{(E/L)^2 + v^2} \\ &= \int_0^{\infty} \frac{ds}{1+s^2} \\ &= \frac{\pi}{2} \end{aligned} \tag{2.21}$$

(using the same substitution as before for u, and letting $v = \cot u$, $v = \frac{E}{L}s$)
Comparing Eq. (2.21) with Eq. (2.20) we see that this is the same amount of coordinate time it takes for a photon to reach the boundary. From the point of view of the observer at $\tilde{r} = 0$ both massless and massive particles oscillate with period 2π.
Finally one can use so-called Poincaré coordinates defined in terms of the original embedding:

$$\begin{aligned} X_0 &= \frac{Lr}{2}\left(x_1^2 + \ldots + x_p^2 - t^2 + \frac{1}{r^2} + 1\right) \\ X_{p+2} &= Lrt \\ X_i &= Lrx_i \quad (i = 1, \ldots, p) \\ X_{p+1} &= \frac{Lr}{2}\left(x_1^2 + \ldots + x_p^2 - t^2 + \frac{1}{r^2} - 1\right) \end{aligned} \tag{2.22}$$

One can rewrite the metric as:

$$\frac{ds^2}{L^2} = r^2\left(-dt^2 + dx_1^2 + \ldots + dx_p^2\right) + \frac{dr^2}{r^2} \tag{2.23}$$

Alternatively letting:

$$u = 1/r \tag{2.24}$$

We can rewrite this as:

$$\frac{ds^2}{L^2} = \frac{1}{u^2}\left(-dt^2 + dx_1^2 + \ldots + dx_p^2 + du^2\right) \tag{2.25}$$

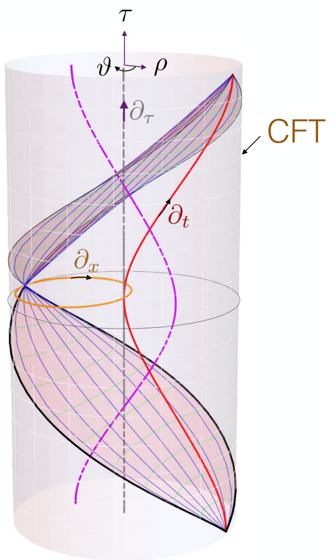
Showing that the metric is also conformally equivalent to flat Minkowski spacetime.
In Poincaré coordinates it can be seen that the metric enjoys a Poincaré group $ISO(1,p)$ invariance on the t and x_i coordinates, as well as a scale invariance given by the dilation:

$$t \to at, \quad x_i \to ax_i, \quad u \to au \tag{2.26}$$

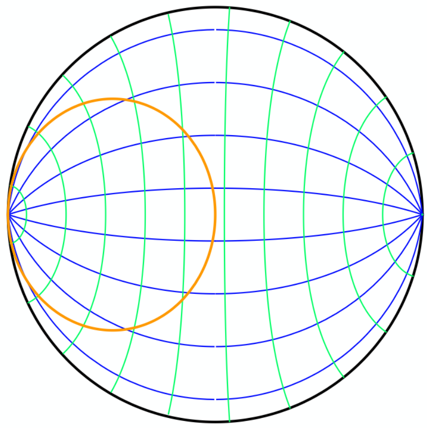
These are actually part of the larger $SO(2, p+1)$ invariance that was our starting point for the definition of Eq. (2.6). Although the symmetries of the space are easier to see in Poincaré coordinates, these only describe a region of the full AdS_{p+2} space described by the global coordinates (see Fig. (2.1)).

More generally, an asymptotically AdS spacetime is one that has a conformal boundary and near that has the same geometry as AdS spacetime.
An example is the $SAdS_5$ (Schwarzschild anti-de Sitter) black hole, given by the metric:

$$\begin{aligned} ds_5^2 &= -\left(\frac{r}{L}\right)^2 h(r)dt^2 + \frac{dr^2}{\left(\frac{r}{L}\right)^2 h(r)} + \left(\frac{r}{L}\right)^2\left(dx^2 + dy^2 + dz^2\right) \\ h(r) &= 1 - \left(\frac{r_0}{r}\right)^4 \end{aligned} \tag{2.27}$$



(a) A plot of AdS_3 spacetime, showing the Poincaré patch with the two horizons. Using global coordinates in Eq (2.13) the vertical component is $\tilde{t}$ (τ in the figure) and the space is compactified so that the radial component is $\arctan \tilde{r}$ ($\arctan \rho$ in the figure). The Poincaré horizons are given by the two null surfaces in red and the curves given are: blue, null geodesics: green, spacelike geodesics: red, a curve with $u = L$ and $x = 0$ (given by the orbit of the Killing field ∂_t); orange, a curve with $u = L$ and $t = 0$ (given by the orbit of the Killing field ∂_x); grey, timelike geodesic (given by the orbit of $\partial_{\tilde{t}}$): purple, which illustrates that a massive particle oscillates without reaching the boundary.



(b) A spatial slice (constant $\tilde{t}$ slice) of AdS_3 with spacelike geodesics in green and projections of null geodesics in blue.

Figure 2.1: Figure reproduced from [23].

This spacetime is dual to a gauge theory at finite temperature.

2.2. Conformal field theories

A conformal transformation is a coordinate transformation that preserves the angle between curves. More formally, a conformal transformation is a coordinate transformation between two metrics g and g' that can be written in terms of a positive scaling function $\Omega^2(x)$ as:

$$g'(x') = \Omega^2(x)g(x) \tag{2.28}$$

Two metrics g and g' for which such a transformation exists are said to be conformally equivalent.
The infinitesimal generators of the conformal group can be found to satisfy the condition:

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \omega(x)\eta_{\mu\nu} \tag{2.29}$$

One finds that (for a space of dimension greater than 2) the generators are:

$$\begin{aligned} M_{\mu\nu} &= i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \\ P_\mu &= -i\partial_\mu \\ D &= -ix_\mu\partial^\mu \\ K_\mu &= i\left(x^2\partial_\mu - 2x_\mu x_\nu\partial^\nu\right) \end{aligned} \tag{2.30}$$

Where P_μ are the generators of the group of translations which together with the generators of angular momentum $M_{\mu\nu}$ form the generators of the Lorentz group. The generator D corresponds to uniform scaling ($x'^\mu = \lambda x^\mu$), and together they give the Poincaré group. This is expected because for the Poincaré group we have $\Omega^2(x) = 1$. The generator D corresponds to a uniform scaling ($x'^\mu = \lambda x^\mu$ and thus $\Omega^2(x) = \lambda$). The generators K_μ are less obvious, and correspond to special conformal transformations:

$$x'^\mu = \frac{x^\mu + a^\mu x^2}{1 + 2x_\nu a^\nu + a^2x^2} \tag{2.31}$$

An inversion is a transformation where each point outside the unit ball is mapped to a point outside the unit ball, and point close to the unit sphere are mapped to themselves:

$$x'^\mu = \frac{x^\mu}{x^2} \tag{2.32}$$

A special conformal transformations is the composition of an inversion, a translation by the vector a^μ ($x'^\mu = x^\mu - a^\mu$) and then another inversion.

The commutation relations are given by:

$$\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i\left(\eta_{\mu\nu}D - M_{\mu\nu}\right) \\
[K_\rho, M_{\mu\nu}] &= i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right) \\
[P_\rho, M_{\mu\nu}] &= i\left(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\right) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\right)
\end{aligned} \tag{2.33}$$

The conformal group of $\mathbb{R}^{m,n}$ is isomorphic to $SO(m+1, n+1)$, which gives the basis for the duality between conformal field theories and anti de Sitter spaces.
To see this, recast the generators as [38]:

$$\begin{aligned}
J_{\mu,\nu} &\equiv M_{\mu\nu} \\
J_{-1,\mu} &\equiv \frac{1}{2}\left(P_\mu - K_\mu\right) \\
J_{0,\mu} &\equiv \frac{1}{2}\left(P_\mu + K_\mu\right) \\
J_{-1,0} &\equiv D
\end{aligned} \tag{2.34}$$

So that the commutation relations in Eq. (2.33) simplifies to:

$$[J_{mn}, J_{pq}] = i\left(\eta_{mq}J_{np} + \eta_{np}J_{mq} - \eta_{mp}J_{nq} - \eta_{nq}J_{mp}\right) \tag{2.35}$$

The same as the commutation relations of the Lie algebra of $SO(m+1, n+1)$. This is the first clue for a correspondence between an AdS theory and a CFT that lives that lives on a space of lower dimensionality.
As a simple example, a massless scalar field theory for a field $\varphi(\boldsymbol{x}, t)$ with equation of motion given by:

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = 0 \tag{2.36}$$

Is invariant under a rescaling $\boldsymbol{x} \to \lambda\boldsymbol{x}$ and $t \to \lambda t$ and is thus a conformal field theory.
Another simple example is a φ^4 theory in Minkowski $\mathbb{R}^{1,3}$ space with equation of motion given by:

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi + g\varphi^3 = 0 \tag{2.37}$$

Where g is a dimensionless constant, which is invariant under a rescaling where $\boldsymbol{x} \to \lambda\boldsymbol{x}$, $t \to \lambda t$ and $\varphi \to \frac{\varphi}{\lambda}$.
In any conformal field theory, there are operators called primary operators, which transform simply under conformal transformations. In particular under a scaling they

transform as:

$$\mathcal{O}'(x') = \lambda^{-\Delta}\mathcal{O}(x) \tag{2.38}$$

Where the number Δ is called the conformal dimension of $\mathcal{O}$.
For a primary scalar operator $\mathcal{O}$ one has $\Delta \geq \frac{d-2}{2}$ where d is the dimension of the space (in the example of the φ^4 theory we had $d = 4$ and $\Delta = 1$).
Primary operators have simple correlation functions, for example a scalar primary operator $\mathcal{O}$ of dimension Δ has two-point function:

$$\langle\Omega|T\mathcal{O}(x,t)\mathcal{O}(0,0)|\Omega\rangle = \frac{1}{(|x|^2 - t^2 + i\epsilon)^\Delta} \tag{2.39}$$

More specifically, primary operators are those operators that are annihilated by the action of the generator K_μ
Furthermore, given a CFT on a cylinder $\mathbb{R} \times S^{d-1}$ with metric:

$$ds^2 = -d\tau^2 + d\Omega^2_{d-1} \tag{2.40}$$

By means of the transformation $\rho = e^\tau$ one can get a conformally flat metric on $\mathbb{R}^d$:

$$ds^2 = \frac{1}{\rho^2}(-d\rho^2 + \rho^2 d\Omega^2_{d-1}) \tag{2.41}$$

There is a bijection between eigenstates $|\psi\rangle$ of the Hamiltonian H that generates t translations (on $\mathbb{R}\times S^{d-1}$) and field configurations which correspond to local operators $\mathcal{O}_\psi(0)$ at the origin (on $\mathbb{R}^d$). This is called the state-operator correspondence.

2.3. The AdS/CFT dictionary and Maldacena's derivation

The gauge/gravity duality or AdS/CFT duality is then the statement that every relativistic conformal field theory on the cylinder $\mathbb{R} \times S^{d-1}$ (that is, with the metric given in Eq. (2.40)) is equivalent to a theory of quantum gravity in an asymptotically $AdS_{d+1} \times M$ spacetime (where M is a compact manifold).
One refers to the conformal field theory/gauge theory as the boundary and the (asymptotically) AdS/gravitational spacetime as the bulk.
The duality is often phrased as an equivalence between the field theory partition function Z_{CFT} (which is a generating function for the the correlation functions of the operators $\langle\mathcal{O}(x)\cdots\mathcal{O}(y)\rangle$, found by taking functional derivatives with respect to source terms) and the gravitational theory correlation function Z_{AdS}:

$$Z_{\text{CFT}} = Z_{\text{AdS}} \tag{2.42}$$

This equivalence is referred to as the GKPW relation.
A more detailed map between the two theories is provided by a dictionary, which relates quantities on the two sides of the correspondence:

- By definition the Hilbert space of physical states is the same for the CFT and the AdS side;
- The generators of the symmetry group $SO(2, d)$ in the CFT are identified with the corresponding generators on the AdS side (as seen before, this was a first clue for the existence of the correspondence);
- The Hamiltonian is the same on both sides;
- Quantities that only depend on the space of states and the Hamiltonian (such as the thermal partition function) are then the same on both sides;
- Scalar primary operators $\mathcal{O}(t, \Omega)$ on the CFT side are associated to scalar fields $\varphi(t, r, \Omega)$ on the AdS side by taking:

  $$\lim_{r\to\infty} r^{\Delta}\varphi(t, r, \Omega) = \mathcal{O}(t, \Omega) \tag{2.43}$$

  One can use this to find the expectation value of products of operators in the CFT from that of the scalar fields in AdS space; for a massive field, the mass of the field gives the conformal dimension fo the operator.
- In the same vein, fermionic operators on the CFT are associated to Dirac fields on the AdS side;
- The CFT stress tensor $T_{\mu\nu}$ is associated to the AdS metric tensor $g_{\mu\nu}$:
- The conserved currents J_μ in the CFT that are given by Noether's theorem are dual to the Maxwell fields A_μ in the AdS space;
- The the entanglement entropy of the CFT is dual to geometrical properties of the AdS spacetime: this is the Ryu–Takayanagi conjecture that we will be turning to in the next chapter.

Possibly the most well-known example of a gauge/gravity duality, and the first to be proposed by Maldacena in 1997 [27] is the duality between $\mathcal{N} = 4$ supersymmetric Yang–Mills theory (a theory with $SU(N)$ gauge and and type IIB string theory on the product space $AdS_5 \times S^5$.
The argument in Maldacena's original derivation goes as follows. In string theory

we have both open and closed strings: open strings have their endpoints constrained. We start from a number N D3-branes in type IIB string theory, a maximally supersymmetric theory where we have open strings living on the branes as well as closed string that live on a flat 10-dimensional spacetime. and look at the low energy limit: at a weak coupling $g_s N \ll 1$ (where g_s is the string coupling) we end up with a 4-dimensional $SU(N)$ super Yang-Mills gauge theory describing the dynamics of the D3-branes, which are now decoupled from the open strings away. At strong coupling $g_s N \gg 1$ the branes curve the spacetime giving a metric that can be derived from symmetry to be:

$$ds^2 = f(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + f(r)^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right) \tag{2.44}$$

Where x^μ are the 4 coordinates along the D3-brane worldvolume and:

$$f(r) = 1 + \frac{4\pi g_{\mathrm{s}} N \ell_{\mathrm{s}}^4}{r^4} \tag{2.45}$$

Near the branes at the horizon ($r \to 0$) the metric tends to:

$$ds^2 = \frac{r^2}{\ell^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{\ell^2}{r^2}dr^2 + \ell^2 d\Omega_5^2 \tag{2.46}$$

Where $\ell = (4\pi g_{\mathrm{s}} N)^{1/4}\,\ell_{\mathrm{s}}$, which is just the metric for $AdS_5 \times S^5$.
For an observer far away from the branes ($r \to \infty$) even an arbitrary large amount of energy emitted near the horizon is finite because of the gravitational redshift, hence also in this case the open strings are decoupled from the branes.
We then have that for $g_s N \gg 1$ the branes only are described by the $\mathcal{N} = 4$ SYM gauge theory, and for $g_s N \ll 1$ by the AdS theory.
An alternative derivation is given in [21], starting from the guess that the spin-two graviton might arise as two spin-one gauge bosons, something that would be normally ruled out by the Weinberg-Witten theorem [48]. An unstated assumption of the theorem is that the graviton would need to live in the same space as the gauge bosons of which it is made. The holographic principle is the loophole that allows us to evade the no-go theorem. We are then led to make the educated guess that the gauge bosons should live in a conformal space where the energy scale corresponds to the extral spatial direction of the graviton. The spacetime that is consistent with the symmetries of this conformal field theory is an anti-de Sitter spacetime.
An introduction to the duality by Maldacena himself is given in [28]. A very readable introduction is given in [23]. A review of aspects of the correspondence and applications to quantum chromodynamics see [32]. A more advanced review is given by Polchinski

in [9]. For a review to AdS/CFT within a quantum informational background see [13].

3

Black holes and entropy

3.1. The von Neumann entropy and the entropy of entanglement

In quantum mechanics, given a pure state $|\psi\rangle$ the corresponding density matrix is defined as the operator:

$$\rho = |\psi\rangle\langle\psi| \tag{3.1}$$

A mixed state is a probabilistic mixture of pure states. Given a mixed state made up of pure states $|\psi_i\rangle$ with probabilities p_i, the density matrix is then defined as a generalisation of the previous formula:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{3.2}$$

Which makes the density matrix formulation convenient for describing mixtures of pure states.
Crucially, a probabilistic mixture where $\sum_i p_i = 1$ is different to a superposition of pure states $|\psi_i\rangle = \sum_i c_i|\phi_i\rangle$ where $\sum_i |c_i|^2 = 1$, which is still a pure state.
The expectation value of a measurement is then given by:

$$\begin{aligned} \langle A\rangle &= \sum_i p_i \langle\psi_i|A|\psi_i\rangle \\ &= \mathrm{tr}(\rho A) \end{aligned} \tag{3.3}$$

A generic quantum state will be a mixed state. For instance if a quantum system is put in contact with a reservoir at temperature $T = \frac{1}{\beta}$ and is governed by a Hamiltonian H

it will be eventually found in a mixed state described by the Gibbs canonical ensemble:

$$\rho = \frac{e^{-\beta H}}{\mathrm{tr}\left(e^{-\beta H}\right)} = \frac{e^{-\beta H}}{Z} \tag{3.4}$$

Where the partition function $Z = \mathrm{tr}\left(e^{-\beta H}\right)$ arises from the requirement that:

$$\mathrm{tr}\left(\rho\right) = 1 \tag{3.5}$$

Which is given by conservation of probability.
We can determine if a given state is a pure our mixed state by using the fact that the density matrix is idempotent ($\rho^2 = \rho$) if and only if the corresponding state is pure. Alternatively, we can define the von Neumann entropy $S(\rho)$ of a state as:

$$S(\rho) \;=\; -\,\mathrm{tr}(\rho \ln \rho) \tag{3.6}$$

Which is equal to zero if and ony if ρ is a pure state. Written in terms of the eigenvalues λ_i of the density matrix this formula reduces to:

$$S(\rho) = -\sum_j \lambda_i \ln \lambda_i \tag{3.7}$$

Which has the same form as the formula for the Gibbs entropy in thermodynamics:

$$S = -k_B \sum_i p_i \ln p_i \tag{3.8}$$

Or the Shannon entropy $H(X)$ of a random variable X from information theory:

$$H(X) = -\sum_i p(x_i) \ln p(x_i) \tag{3.9}$$

As said before, the von Neumann entropy is equal to zero if and only if ρ is a pure state (just as the Shannon entropy of X is zero if and only if X is a constant random variable).
Furthermore, the von Neumann entropy is maximal if and only if ρ is a maximally mixed state and equal to $\ln N$ where N is the number of states (just as the Shannon entropy of X is maximal if and only if X is a uniformly distributed random variable).
A bipartite system is one for which the Hilbert space $\mathcal{H}$ can be written as the product of two subspaces $\mathcal{H}_A$ and $\mathcal{H}_B$:

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{3.10}$$

One can then take the partial trace of the density matrix with respect to one of the subsystems to obtain the reduced density matrix of the other subsystem:

$$\rho_A = \mathrm{tr}_B\, \rho \tag{3.11}$$

The von Neumann entropy of the reduced density matrix is called the entropy of entanglement:

$$S(\rho_A) = -\,\mathrm{tr}_A\, \rho_A \ln \rho_A \tag{3.12}$$

The entropy of entanglement for the two subsystems $\mathcal{H}_A$ and $\mathcal{H}_B$ satisfies the triangle inequality:

$$|S(\rho_A)\; -\; S(\rho_B)|\; \leq\; S(\rho_{AB})\; \leq\; S(\rho_A)\; +\; S(\rho_B) \tag{3.13}$$

Where the first inequality is called the Araki-Lieb inequality and the second inequality is the property of subadditivity.
The entropy of entanglement for the subsystems $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$ further satisfy strong subadditivity:

$$S(\rho_{ABC})\; +\; S(\rho_B)\; \leq\; S(\rho_{AB})\; +\; S(\rho_{BC}) \tag{3.14}$$

From which subadditivity follows by letting $B = \emptyset$.
For example, given a system in the state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{3.15}$$

One has reduced density matrices of the form:

$$\rho_A = \rho_B = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \tag{3.16}$$

And entropies of entanglement:

$$S(\rho_A) = S(\rho_B) = \ln 2 \tag{3.17}$$

Whilst because the original state $|\psi\rangle$ was a pure state one can see immediately that the von Neumann entropy for the whole system is:

$$S(\rho_{AB}) = 0 \tag{3.18}$$

Satisfying the triangle inequality. Given a random variable X, one can also define a more general notion of entropy, dependent on a parameter α, the Rényi entropy $H_\alpha(X)$ (which reduces to the Shannon entropy $H_\alpha(X)$ in the limit when $\alpha \to 1$, as can be checked in terms of):

$$H_\alpha(X) = \frac{1}{1-\alpha} \ln \left(\sum_i p(x_i)^\alpha \right) \tag{3.19}$$

Which for a quantum density matrix ρ becomes a generalisation of the von Neumann entropy of the form:

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \ln \left(\mathrm{tr} \left(\rho^\alpha \right) \right) \tag{3.20}$$

3.2. The Bekenstein-Hawking entropy and the Hawking temperature

Until around fifty years ago, black holes were assumed to have no (or otherwise infinite) entropy.
In 1971 S. W. Hawking proved [16] under quite general assumptions that the area of a black hole horizon never decreases in time (the area theorem).
Hawking also noticed what he thought was only an accidental similarity with the laws of thermodynamics.
One has for instance a first law of black hole mechanics which for a rotating, charged black hole takes the form:

$$dM = \frac{\kappa}{8\pi G} dA + \Omega dJ + \Phi dQ \tag{3.21}$$

Where κ is the surface gravity of the horizon, A is the area of the horizon, M is the mass of the black hole, Ω is the angular velocity, J is the angular momentum, Φ is the electrostatic potential and Q is the electric charge.
The surface gravity corresponds (for a static black hole) to the gravitational acceleration at the event horizon in the reference frame of a distant observer and is defined in terms of a Killing horizon given by a normalised Killing vector k^μ as the constant κ satisfying the equation at the horizon:

$$k^a \nabla_a k^b = \kappa k^b \tag{3.22}$$

The second law is the aforementioned area theorem:

$$\frac{dA}{dt} \geq 0 \tag{3.23}$$

In both of these equations, the area of the black hole seemed to have the same role as the entropy in classical thermodynamics.
J. A. Wheeler first gave an argument in conversation with his student J. Bekenstein that if one were to throw a cup of hot tea into a black hole, the entropy of the uni-

verse would seem to decrease (violating the second law of thermodynamics). In 1972 Bekenstein went on to show that black holes must have a well-defined entropy, and that this must be proportional (up to a constant) to the area of the black hole horizon [4]: the similarity between black hole mechanics and thermodynamics was not just a coincidence.

In 1974 Hawking, who had set out to prove Bekenstein was wrong, ended up confirming his result [17] and fixed the constant of proportionality to 1/4:

$$S_{\mathrm{BH}} = \frac{k_{\mathrm{B}} A}{4\ell_{\mathrm{P}}^2} \tag{3.24}$$

In Planck units this can be written as:

$$S_{\mathrm{BH}} = \frac{A}{4G_N} \tag{3.25}$$

He also found in 1975 that black holes radiate energy in the form of thermal radiation, the Hawking radiation [18], given by the temperature:

$$T_{\mathrm{H}} = \frac{\hbar c^3}{8\pi G_N M k_{\mathrm{B}}} \tag{3.26}$$

Or in natural units and in terms of the Schwarzschild radius r_{s} (see below):

$$T_{\mathrm{H}} = \frac{1}{4\pi r_{\mathrm{s}}} \tag{3.27}$$

And they can thus decrease in size, violating his earlier area theorem.

The expression for the Hawking radiation can be derived as the thermal radiation measured by an observer faraway given by the Unruh effect: for a uniformly accelerating observer, the ground state of an inertial observer is seen as a thermally excited mixed state.

The simplest black hole solution to the Einstein field equation (and the one Hawking considered) is the Schwarzschild black hole, which Birkhoff showed is the spherically symmetric solution of the vacuum field equations (that is with $R_{\mu\nu} = 0$), and is given by the metric:

$$ds^2 = -\left(1 - \frac{r_{\mathrm{s}}}{r}\right) dt^2 + \left(1 - \frac{r_{\mathrm{s}}}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3.28}$$

Where r_{s} is the Schwarzschild radius of the black hole $r_s = \frac{2GM}{c^2}$.

Famously this was the first exact solution to the Einstein field equations which Karl Schwarzschild first found in 1915 (the same year Einstein first introduced general relativity) in two papers written while serving on the Russian front in the German army during World War I, a few months before his death.

At a position near the horizon $r = r_s + \frac{\rho^2}{4r_s}$ with $\rho \ll 1$ the Schwarzschild metric becomes to first order:

$$\begin{aligned} ds^2 &\simeq -\left(\frac{\rho}{2r_s}\right)^2 dt^2 + d\rho^2 + \dots \\ &\simeq -\rho^2 d\tau^2 + d\rho^2 + \dots \end{aligned} \tag{3.29}$$

Which is the same as the Rindler metric for an accelerating observer in Minkowski space with $\tau = \frac{t}{2r_s}$ and acceleration given by $a = \frac{1}{\rho}$.
By analytic continuation with the Euclidean signature (imaginary time) $\theta = i\tau$ the metric near the horizon becomes:

$$ds^2 \simeq \rho^2 d\theta^2 + d\rho^2 + \dots \tag{3.30}$$

As long as the coordinate θ is periodic $\theta \sim \theta + 2\pi$ that is $it \sim it + 4\pi r_s$, this is the same as a flat metric. Otherwise, the metric has a conical singularity at the horizon $\rho = 0$.
In quantum statistical mechanics, the periodicity β of imaginary (Wick-rotated) time corresponds to a temperature of $\beta = \frac{1}{T}$. This can heuristically be justified as follows: the partition function Z of the canonical ensemble is:

$$\begin{aligned} Z &= \mathrm{tr}\, e^{-\beta H} \\ &= \sum_n \langle \psi_n(0)| e^{-\beta H} |\psi_n(0)\rangle \\ &= \sum_n e^{-\beta E_n} \end{aligned} \tag{3.31}$$

For some complete set of states $\{|\psi_n\rangle\}$.
Their time evolution is given by (in the Schrödinger picture):

$$|\psi(t)_n\rangle = e^{-itH}|\psi_n(0)\rangle \tag{3.32}$$

So that we can be tempted [50] to rewrite:

$$Z = \sum_n \langle \psi_n(0)|\psi_n(-i\beta)\rangle \tag{3.33}$$

Which looks like a vacuum amplitude as long as $it \sim it + i\beta$. We then make the identification that the inverse temperature β is the periodicity of imaginary time.

We then have that a faraway observer sees the temperature to be:

$$T_{\mathrm{H}} = \frac{1}{4\pi r_s} \tag{3.34}$$

The same expression as in Eq. ((3.27)).
The near-horizon observer sees a temperature redshifted as:

$$\begin{aligned} T &= T_{\mathrm{H}} \sqrt{\frac{g_{00}(\infty)}{g_{00}(r_s + \frac{\rho^2}{4r_s})}} \\ &= \frac{2T_{\mathrm{H}}}{\rho} \\ &= \frac{1}{2\pi\rho} \\ &= \frac{a}{2\pi} \end{aligned} \tag{3.35}$$

That is, the Unruh temperature.
In more general cases a black hole metric can still be written in the form:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + \ldots \tag{3.36}$$

for some function of the radial coordinate $f(r)$.
The horizon is given by $r = r_0$ where $f(r_0) = 0$, as for a null ray we have that its trajectory freezes in coordinate time upon approaching the horizon:

$$\begin{aligned} 0 &= -f(r)\dot{t}^2 + \frac{1}{f(r)}\dot{r}^2 \\ \frac{\dot{r}}{\dot{t}} &= \frac{dr}{dt} = f(t) \end{aligned} \tag{3.37}$$

By analytic continuation with the Euclidean signature (imaginary time) $t_{\mathrm{E}} = it$ the metric becomes:

$$ds_{\mathrm{E}}^2 = +f(r)dt_{\mathrm{E}}^2 + \frac{dr^2}{f(r)} + \cdots \tag{3.38}$$

Near the horizon $r \simeq r_0$ one can approximate the function $f(r)$ as (as long as $f'(r_0) \neq 0$):

$$f(r) \simeq f'(r_0)(r - r_0) \tag{3.39}$$

So that the metric near the horizon is approximatively:

$$ds_{\mathrm{E}}^2 \simeq \frac{dr^2}{f'(r_0)(r - r_0)} + f'(r_0)(r - r_0)\, dt_{\mathrm{E}}^2 \tag{3.40}$$

Rewriting the radial coordinate as:

$$\rho := 2\sqrt{\left(r - r_0\right) / f'\left(r_0\right)} \tag{3.41}$$

And the Euclidean time as:

$$\theta = \frac{f'\left(r_0\right)}{2} t_{\mathrm{E}} \tag{3.42}$$

The metric near the horizon can be written as:

$$ds_{\mathrm{E}}^2 \simeq d\rho^2 + \rho^2 d\theta^2 \tag{3.43}$$

Just like before, in order to avoid a singularity at the horizon $\rho = 0$ we require $t_{\mathrm{E}} \sim t_{\mathrm{E}} + \frac{4\pi}{f'(r_0)}$.
The periodicity β of t_{E} gives us a generalisation of the Hawking temperature:

$$T = \frac{1}{\beta} = \frac{f'\left(r_0\right)}{4\pi} \tag{3.44}$$

We can as a sanity check see that for the Schwarzschild radius we have:

$$\begin{aligned} f(r) &= 1 - \frac{r_{\mathrm{s}}}{r} \\ f'(r) &= \frac{r_{\mathrm{s}}}{r^2} \end{aligned} \tag{3.45}$$

And thus:

$$T = \frac{1}{4\pi r_{\mathrm{s}}} \tag{3.46}$$

Which is again the result from Eq. (3.27).
We can also go back to the Rindler time experienced by an observer near the horizon $i\theta = \tau$ and rewrite it using Eq. (3.42) and Eq. (3.44) as:

$$\begin{aligned} \tau &= \frac{f'\left(r_0\right)}{2} t \\ &= 2\pi T t \end{aligned} \tag{3.47}$$

Something that we are going to use later on, in §7.2.

4

The Ryu–Takayanagi conjecture

4.1. The holographic entanglement entropy

In 2006 S. Ryu and T. Takayanagi first proposed [39] a holographic generalisation of the Bekenstein-Hawking entropy, which has since been known as the Ryu–Takayanagi conjecture.

The conjecture is that the entropy of a d-dimensional region A on the CFT_{d+1} gauge theory is equal (up to a constant) to the area of the d-dimensional minimal surface γ_A on the asymptotic AdS_{d+2} gravitational theory whose boundary is a $(d-1)$-dimensional manifold $\partial\gamma_A$ that coincides with the boundary ∂A of the region A:

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_{d+2}} \tag{4.1}$$

In [25] Hubeny, Rangamani, and Takayanagi generalised the conjecture to an explicitly covariant form, so that the conjecture is now often referred to as the HRT conjecture. This holographic entanglement entropy S_A measures the amount of information hidden inside the region B, when the full space is divided into two complementary regions A and B such that an observer in the region A cannot receive any signal.

This is a generalisation of the situation one has with an event horizon, where the horizon γ_A partitions the space into two regions. Furthermore, one can claim that it is a generalisation of the Bekenstein-Hawking entropy because in the presence of a black hole the minimal surface will often be the same as the event horizon or wrap around it (such as in the case of a SAdS horizon).

For free theories at least, one can show that for quantum field theories on d dimensions ($d \geq 3$) the entanglement entropy S_A is divergent, but that the leading term is inversely proportional to the ultraviolet cutoff a (which is proportional to the lattice spacing) to the power of $d-1$ and directly proportional to the area of the boundary of the region ∂A:

$$S_A = \gamma \cdot \frac{\text{Area}(\partial A)}{a^{d-1}} + O\left(a^{2-d}\right) \tag{4.2}$$

This makes sense because the entanglement between two regions A and B occurs most strongly at the boundary $\partial A = \partial B$ between the two.
One exception to the area law Eq (4.2) is given by a conformal field theory of dimension $d=2$ where for a system of total infinite length $L \to \infty$ one has the law for a subsystem A of length l:

$$S_A = \frac{c}{3} \log \frac{l}{a} \tag{4.3}$$

In this case the CFT_2 central charge c can be written in terms of the radius of the AdS_3 theory as:

$$c = \frac{3R}{2G_N} \tag{4.4}$$

A relatively simple example is then given by the AdS_3/CFT_2 duality.
Given the metric in Eq. (2.25) for the case of AdS_3 (two space-like dimensions):

$$\frac{ds^2}{L^2} = \frac{1}{u^2}\left(-dt^2 + dx^2 + du^2\right) \tag{4.5}$$

One finds that the one-dimensional surface γ_A on a slice of constant time ($dt = 0$) that minimises the distance will satisfy the geodesic equations (for an affine parameter):

$$\begin{aligned} \ddot{u} + \frac{1}{u}\left(-\dot{u}^2 + \dot{x}^2\right) &= 0 \\ \ddot{x} - \frac{2}{u}\dot{u}\dot{x} &= 0 \end{aligned} \tag{4.6}$$

This is given by a semi-circumference of radius $\frac{l}{2}$:

$$x = \sqrt{\left(\frac{l}{2}\right)^2 - z^2} \tag{4.7}$$

The induced metric on the geodesic is given by:

$$ds^2 = \frac{L^2 l^2}{4u^2(\frac{l}{2}^2 - u^2)} du^2 \tag{4.8}$$

And the Ryu–Takayanagi formula gives us the holographic entanglement entropy as:

$$\begin{aligned} S_A &= \frac{2}{4G_N}\int_a^{\frac{l}{2}} ds \\ &= \frac{1}{2G_N}\int_a^{\frac{l}{2}} \frac{Ll}{4u\sqrt{\frac{l}{2}^2 - u^2}} \end{aligned} \tag{4.9}$$

Which in the limit of small a gives:

$$S_A = \frac{L}{2G_N}\ln\frac{l}{a} \tag{4.10}$$

Which using Eq. (4.4) is the same as the standard result stated in Eq. (4.3).
The last result assumed zero temperature or at a ground state. A finite temperature in the CFT is dual to a black hole in the AdS space. In this respect the gauge/gravity duality gives a deeper insight for what was originally regarded as a mysterious coincidence, that black holes follow laws similar to the ones of thermodynamics (see Eq. (3.21) and (3.23).
At a finite temperature $T = \beta^{-1}$ the gravity dual of the conformal field theory becomes the Bañados-Teitelboim-Zanelli (BTZ) black hole [3]:

$$ds^2 = -\left(r^2 - r_+^2\right)dt^2 + \frac{R^2}{r^2 - r_+^2}dr^2 + r^2 d\varphi^2 \tag{4.11}$$

By letting $r \to \infty$ and rescaling the coordinates one sees that the BTZ black hole metric is asymptotically the same as AdS_3 (in the form of Eq. (2.23) with $\varphi = x$ and $r = 1/u$). Making the usual Euclidean time substitution $t_{\mathrm{E}} = it$ and requiring periodicity we find:

$$\frac{\beta}{L} = \frac{R}{r_+} \tag{4.12}$$

Where L is the total circumference of the CFT circle.
If we now look at the region on the boundary A defined as $0 \le \varphi \le \frac{2\pi l}{L}$, one can find that the geodesic distance satisfies [33]:

$$\cosh\left(\frac{\text{Length}\,(\gamma_A)}{R}\right) = 1 + \frac{2r_0^2}{r_+^2}\sinh^2\left(\frac{\pi l}{\beta}\right) \tag{4.13}$$

The relation between the CFT cut off a and the AdS one r_0 is given by $\frac{r_0}{r_+} = \frac{\beta}{a}$, so that the entanglement entropy is given by the Ryu–Takayanagi formula as:

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right) \tag{4.14}$$

(a) $S(\rho_{AB}) + S(\rho_{BC}) \geq S(\rho_{ABC}) + S(\rho_B)$

(b) $S(\rho_{AB}) + S(\rho_{BC}) \geq S(\rho_A) + S(\rho_C)$

Figure 4.1: A geometrical proof of strong subadditivity of the holographic entanglement entropy for CFT_2/AdS_3, which shows clearly the idea of the proof in higher dimensions

A result that agrees with direct calculations on the CFT (see below).
In plain AdS we have for two complementary regions A and B that $\gamma_A = \gamma_B$ so that the entanglement entropies always match:

$$S(\rho_A) = S(\rho_B) \tag{4.15}$$

Which is not extensive: if A is a small region on the boundary, B is a large region on the boundary.
However, the minimal surfaces γ_A and γ_B will not be equal in the presence of a black hole (and hence in the thermal CFT we won't have that $S_A = S_B$). In this case, the topology is not trivial and we make the requirement that the minimal surface in the holographic entanglement entropy γ_A is homologous to the region A.
As long as the region A is small, the minimal curve γ_A for the BTZ black hole looks similar to the corresponding minimal surface in plain AdS space (as the BTZ black hole is asymptotically AdS). As A gets larger the surfaces γ_A and γ_B wrap around different parts of the black hole, until the minimal surface γ_A is given by a disconnected surface: one component coincides with the black hole horizon, whilst the other is a small curve near the boundary which coincides with γ_B. This gives that in general we have that Eq. (4.15) generalises to:

$$S(\rho_A) = S(\rho_B) + S_{\text{BH}} \tag{4.16}$$

Where S_{BH} is the Bekenstein-Hawking entropy for the black hole.
This saturates the Araki-Lieb inequality in Eq. (3.13) as the total entropy equals the Bekenstein-Hawking entropy S_{BH}.

4.2. Subadditivity in the Ryu–Takayanagi formula

A check for the Ryu–Takayanagi formula is that satisfies the properties of the entanglement entropy stated previously. In particular strong subadditivity can be proven with a straightforward geometrical argument shown in Fig. (4.1a) and (4.1b) for the case of CFT_2/AdS_3: if they intersect (in the case of CFT_2/AdS_3 if they intersect they intersect at a point), we can break apart the minimal surfaces γ_{AB} and γ_{BC} into two surfaces each, and regroup them into two surfaces γ'_{AC} and γ'_{B} (Fig. (4.1a), so that:

$$\text{Area}\,(\gamma_{AB}) + \text{Area}\,(\gamma_{BC}) = \text{Area}\,(\gamma'_{AC}) + \text{Area}\,(\gamma'_{B}) \tag{4.17}$$

Now, the new surfaces γ'_{AC} and γ'_{B} are not necessarily minimal, so in terms of the minimal surfaces γ_{AC} and γ_{B} we have:

$$\begin{aligned} \text{Area}\,(\gamma'_{AC}) &\geq \text{Area}\,(\gamma_{AC}) \\ \text{Area}\,(\gamma'_{B}) &\geq \text{Area}\,(\gamma_{B}) \end{aligned} \tag{4.18}$$

Which implies:

$$\text{Area}\,(\gamma'_{AC}) + \text{Area}\,(\gamma'_{B}) \geq \text{Area}\,(\gamma_{AC}) + \text{Area}\,(\gamma_{B}) \tag{4.19}$$

And hence:

$$\text{Area}\,(\gamma_{AB}) + \text{Area}\,(\gamma_{BC}) \;\geq \text{Area}\,(\gamma_{AC}) + \text{Area}\,(\gamma_{B}) \tag{4.20}$$

Which by dividing both sides by a factor of $4G_{d+2}$ gives:

$$S(\rho_{AB}) \;+\; S(\rho_{BC}) \;\geq\; S(\rho_{ABC}) \;+\; S(\rho_{B}) \tag{4.21}$$

Similarly one can prove the property (Fig. (4.1b):

$$S(\rho_{AB}) \;+\; S(\rho_{BC}) \;\geq\; S(\rho_{A}) \;+\; S(\rho_{C}) \tag{4.22}$$

Which shows that the strong subadditivity of entanglement entropy in a boundary theory is connected to the geometrical properties of minimal surfaces in the corresponding bulk theory.

4.3. Monogamy from geometry

Subadditivity also allows us to define another quantity of interest from information theory, the mutual information, which for disjoint separated regions A and B is given by:

$$I(A:B) \equiv S_A + S_B - S_{A+B} \tag{4.23}$$

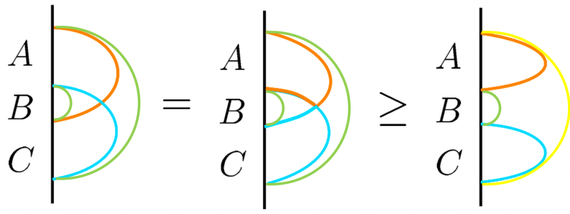


Figure 4.2: A geometrical proof of monogamy $I(A:B:C) \leq 0$ for AdS_3 showing that:
$S(\rho_{AB}) + S(\rho_{BC}) + S(\rho_{AC}) \geq S(\rho_A) + S(\rho_B) + S(\rho_C) + S(\rho_{ABC})$

Which is zero for unentangled regions A and B. We have that in general:

$$I(A:B) \geq 0 \tag{4.24}$$

Mutual information gives a measure of the total amount of correlation (both classical and quantum) between the regions A and B as shown in [11].
One can also define the more general tripartite information, defined as:

$$\begin{aligned} I(A:B:C) &\equiv S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC} \\ &= I(A:B) + I(A:C) - I(A:BC) \end{aligned} \tag{4.25}$$

One can then show that the tripartite information also obeys:

$$I(A:B:C) \leq 0 \tag{4.26}$$

Called the monogamy of entanglement.
Similarly to the proof of subadditivity in the previous section (see Fig. (4.2)), the area of the surfaces γ_{AB}, γ_{BC} and γ_{AC} can be broken up into twelve pieces, which can then be rearranged to show that:

$$\gamma_{AB} + \gamma_{BC} + \gamma_{AC} \geq \gamma_A + \gamma_B + \gamma_C + \gamma_{ABC} \tag{4.27}$$

Which gives the stated result (see [19] for the full derivation).

4.4. Sketch of a derivation of the holographic entropy formula

In quantum field theories the entanglement entropy is usually calculated using the so-called replica trick [7]:

$$\begin{aligned} S_A &= -\frac{\partial}{\partial n} \operatorname{tr}_A \rho_A^n \Big|_{n=1} \\ &= -\frac{\partial}{\partial n} \ln \operatorname{tr}_A \rho_A^n \Big|_{n=1} \end{aligned} \tag{4.28}$$

We can see that this formula when applied to the example of a canonical ensemble given by Eq. (3.4) $\rho = \frac{e^{-\beta H}}{Z}$ with $Z = \mathrm{tr}\left(e^{-\beta H}\right)$ gives the classical thermodynamic entropy S_{thermal} as we have:

$$\begin{aligned} S(\rho) &= -\left.\frac{\partial}{\partial n}\ln\left(\mathrm{tr}\left(\rho^n\right)\right)\right|_{n=1} \\ &= -\frac{\partial}{\partial n}\left(\ln\left(\mathrm{tr}\left(e^{-\beta n H}\right)\right) - n\cdot\log Z\right) \\ &= \beta\langle H\rangle + \log Z \\ &= \beta(E-F) = S_{\text{thermal}} \end{aligned} \tag{4.29}$$

By the definition of the Helmholtz free energy $F = E - TS$.
In order to use the formula in the general case we need to evaluate $\mathrm{tr}_A\,\rho_A^n$.
In the simpler example of a $(1+1)$-dimensional QFT the ground state wave functional $\Psi\left(\phi_0(x)\right)$ is found by doing the Euclidean path integral in the interval $-\infty < t_E < 0$

$$\Psi\left(\phi_0(x)\right) = \int_{t_E=-\infty}^{\phi(t_E=0,x)=\phi_0(x)} D\phi e^{-S(\phi)} \tag{4.30}$$

The corresponding density matrix ρ is given by

$$[\rho]_{\phi_0\phi_0'} = \Psi\left(\phi_0\right)\bar{\Psi}\left(\phi_0'\right) \tag{4.31}$$

Where the complex conjugate of the ground state $\bar{\Psi}\left(\phi_0(x)\right)$ is obtained by performing the integral in the interval $0 < t_E < \infty$.
The reduced density matrix is then obtained from an expression of the form:

$$\begin{aligned} \mathrm{tr}_A\,\rho_A^n &= \frac{1}{\left(Z_1\right)^n}\int_{(t_E,x)\in\mathcal{R}_n} D\phi e^{-S(\phi)} \\ &\equiv \frac{Z_n}{\left(Z_1\right)^n} \end{aligned} \tag{4.32}$$

Where $\mathcal{R}_n$ is an n-sheeted Riemann surface obtained by 'gluing together' copies of ρ_A in the form $\left[\rho_A\right]_{\phi_{1+}\phi_{1-}}\left[\rho_A\right]_{\phi_{2+}\phi_{2-}}\cdots\left[\rho_A\right]_{\phi_{n+}\phi_{n-}}$, where each term is evaluated as:

$$\left[\rho_A\right]_{\phi_+\phi_-} = \frac{1}{Z_1}\int_{t_E=-\infty}^{t_E=\infty} D\phi e^{-S(\phi)}\prod_{x\in A}\delta\left(\phi(+0,x)-\phi_+(x)\right)\delta\left(\phi(-0,x)-\phi_-(x)\right) \tag{4.33}$$

This is the way Eq. (4.3) can be derived without using holography, but directly from the field theory. This approach can be generalised to an arbitrary number of dimensions using twisted vector operators (although such calculations are in the general case not well understood).

Now, the n-sheeted Riemann surface $\mathcal{R}_n$ is characterised by a deficit angle $\delta = 2\pi(1-n)$ on the surface ∂A. This eventually leads one to assume that the Ricci scalar R can be written in terms of a codimension two surface γ_A as a delta function:

$$R = 4\pi(1-n)\delta\left(\gamma_A\right) + R^{(0)} \tag{4.34}$$

Where $\delta\left(\gamma_A\right)$ is a delta function localised on γ_A.
Plugging this into the Hilbert-Einstein action for the AdS theory:

$$\begin{aligned} S_{AdS} &= -\frac{1}{16\pi G_{d+2}}\int_M dx^{d+2}\sqrt{g}(R+\Lambda) + \cdots \\ &= -\frac{1}{4G_{d+2}}\int_M dx^{d+2}\sqrt{g}\delta\left(\gamma_A\right) + \cdots \\ &= -\frac{(1-n)\,\mathrm{Area}\left(\gamma_A\right)}{4G_{d+2}} + \cdots \end{aligned} \tag{4.35}$$

Where the other terms cancel out in the following computation.
Using the AdS/CFT correspondence we can equate the partition functions (the GKPW relation of Eq. (2.42)) we can calculate this as:

$$\begin{aligned} S_A &= -\frac{\partial}{\partial n}\ln \mathrm{tr}\,\rho_A^n\bigg|_{n=1} \\ &= -\frac{\partial}{\partial n}\ln Z_{CFT}\bigg|_{n=1} \\ &= -\frac{\partial}{\partial n}\ln e^{-S_{AdS}}\bigg|_{n=1} \\ &= \frac{\partial S_{AdS}}{\partial n}\bigg|_{n=1} \\ &= \frac{\partial}{\partial n}\left[\frac{(n-1)\,\mathrm{Area}\left(\gamma_A\right)}{4G_{d+2}}\right]_{n=1} \\ &= \frac{\mathrm{Area}\left(\gamma_A\right)}{4G_{d+2}} \end{aligned} \tag{4.36}$$

Which is the result given by the Ryu–Takayanagi conjecture (Eq. 4.1). This however is not a full derivation: the conjecture still does not have the status of an established theorem, although derivations in specific settings have been given for instance in [10] and [14].

5

The black hole information problem

5.1. The black hole information problem

The radiation emitted by a black hole according to Hawking's calculation in [18] is, like black body radiation, random. However in quantum mechanics unitary evolution means that information is conserved. This problem is widely known as the black hole information paradox or black hole information problem.

A more detailed argument for this goes as follows [37] : consider starting with a large number of EPR pairs (maximally entangled pairs of particles) which together are in a pure state $|\psi\rangle$, so that by definition the von Neumann entropy is $S = 0$; throw one of each pair inside the black hole, ending up with an entropy for the region outside the black hole of $S_{\text{outside}} = n \ln 2$ (and inside the black hole also $S_{\text{inside}} = n \ln 2$).

At a later time, after the black hole has completely evaporated, we are left with half of each pair with a von Neumann entropy given by:

$$S = S_{\text{outside}} = n \ln 2 \tag{5.1}$$

Which is the entropy characterising a highly mixed state for the remaining n particles. As the entropy for the region outside the black hole (which is now the whole space) cannot have decreased because the two regions are causally separated.

However, the time-evolution given by a Schrödinger-like evolution is unitary:

$$i\partial_t |\psi\rangle = H|\psi\rangle \tag{5.2}$$

(in terms of quantum field theory this translates to the requirement of unitarity of the S-matrix [47]: $SS^\dagger = 1$)

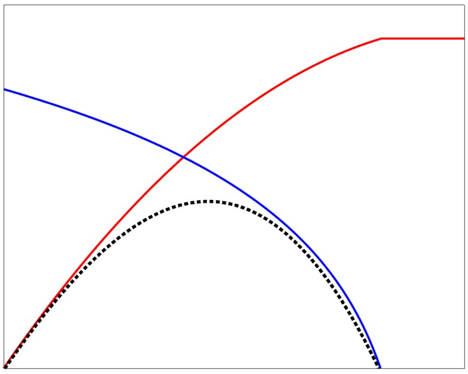


Figure 5.1: Qualitative diagram for the time evolution of the von Neumann entropy of the Hawking radiation: time t shown on the horizontal axis, on the vertical axis the von Neumann entropy of the emitted radiation $S_{\mathrm{N}}(t)$ (red curve), the Bekenstein-Hawking entropy $S_{\mathrm{BH}}(t)$ proportional to the area of the black hole (blue curve), and Page's proposal for the shape of the true entropy $S_{\mathrm{P}}(t)$ (dotted black curve).

And unitary evolution preserves the purity of the state $|\psi\rangle$ of the system.
Following [34] we consider three different measures of entropy outside the black hole in the hope of reinstating unitarity once the black hole has evaporated:

1. The von Neumann entropy S_{N} of the Hawking radiation, which by the argument before must be increasing in time;

2. The Bekenstein-Hawking entropy S_{BH} which is proportional to the event horizon area (Eq. (3.24)) and can be understood as a coarse-graining of the von Neumann entropy: as the black hole evaporates, this decreases in time;

3. Page's proposal for the time-evolution of the entropy S_{P} of the outside region, which to a good approximation should follow the smaller of the other two quantities (which meet at around the midpoint of the evaporation process, the Page time).

If the entropy follows Page's proposal, once the black hole has fully evaporated To see why Page's proposal brings about further trouble, let us consider an observer crossing the event horizon at time $t = t'$ and let ρ_A be the density matrix of the Hawking radiation emitted before $t = t'$, B be the next outgoing mode leaving the black hole horizon, and C be the in-falling partner mode trapped inside the horizon.
We have a set of five conditions that taken together are contradictory:

1. The next outgoing mode and its partner ρ_{BC} are entangled with each other, but if they are not be entangled with the radiation emitted before we have that ρ_{BC} is a pure state and thus:

$$S(\rho_{BC}) = 0 \tag{5.3}$$

2. Like [18], we assume that the emitted radiation ρ_B is thermal, and hence itself in a mixed state:

$$S(\rho_B) > 0 \tag{5.4}$$

3. Radiation that has been previously emitted can interact with the outside world, so will in general also not be a pure state. However if ρ_A and ρ_{BC} are not entangled, and taking into account Eq. (5.3), the entropy of the system ρ_{ABC} will be simply the entropy of ρ_A:

$$S(\rho_{ABC}) = S(\rho_A) \tag{5.5}$$

4. If t' is after the Page time, the emission of the new mode will decrease the entropy:

$$S(\rho_A) > S(\rho_{AB}) \tag{5.6}$$

5. The entropy of a tripartite system satisfies a condition called the strong subadditivity (or monogamy) (see Eq. 3.14) given by:

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}) \tag{5.7}$$

To see the contradiction, start from 5. and 1. to get:

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) \tag{5.8}$$

From 3. we get:

$$S(\rho_A) + S(\rho_B) \leq S(\rho_{AB}) \tag{5.9}$$

4. gives:

$$S(\rho_A) + S(\rho_B) \leq S(\rho_A) \tag{5.10}$$

And finally we get:

$$S(\rho_B) \leq 0 \tag{5.11}$$

Contradicting 2.

5.2. ER = EPR?

Black Hole Complementarity (so called in analogy to Heisenberg Complementarity in quantum mechanics) is the view that the apparent inconsistencies we have shown would never manifest themselves in real experiments, and thus do not actually violate the laws of physics. The argument given above seems to make at least naive form of Black Hole Complementarity

The gauge/gravity duality brought a partial resolution to the problem: if bulk gravitational theories with black holes can fully be described by some thermally excited quantum field theory without gravity on the boundary, then at least there does not seem to be room for any special gravitational mechanism to violate unitarity. Although the duality was originally formulated in string theoretical terms, [30] showed that it can equivalently be formulated independently of string theory, giving further strength to the argument.

It looks then like we have to give something up: if we give up 4 then we are back at the start and might have to abandon the unitarity of quantum mechanics altogether and modify quantum mechanics at its core: some work has been done in this direction, with proposals to introduce final state boundary condition at black hole singularities [22] to alter the way in which probabilities are calculated from the density matrix, or even non-linear dynamics [35] (Hawking himself famously originally proposed that black holes might violate unitarity, whilst others including Susskind and Preskill disagreed: however, in view of arguments based on the gauge/gravity duality, Hawking eventually had to concede).

But if we give up 1, then there is still something crucial missing in Hawking's original semiclassical calculation: one proposal is that [2] after Page time, the quantum fields near the horizon of a black hole are not well described by a vacuum, instead being highly excited. This is the AMPS firewall hypothesis.

However the firewall hypothesis is prima facie in conflict with another basic tenet of physics, the equivalence principle (in the information problem literature often amusingly referred to as the no-drama postulate): the observer crossing at time $t = t'$ should not see anything special as she crosses the horizon. This is because the horizon for, for instance, a Schwarzschild black hole is a coordinate singularity in Schwarzschild coordinates, but not a true singularity, as one can see by changing coordinates to Kruskal–Szekeres coordinates.

In the maximal analytical extension of the Schwarzschild geometry or the AdS-Schwarzschild geometry Eq. (2.27)), there are two regions outside the black hole (commonly referred to as region I and region III).

In [45] showed that a maximally extended AdS-Schwarzschild black hole is dual to a pair of maximally entangled thermal conformal field theories on the CFT boundary.

Starting from two identical non-interacting copies of the CFT A and B, we can define a state $|\psi\rangle$:

$$|\psi\rangle = \sum_i e^{\frac{-\beta E_i}{2}} |E_i\rangle_A \otimes |E_i\rangle_B \tag{5.12}$$

Where the state $|E_i\rangle_A$ is the ith eigenstate for the CFT A and similarly for B.
Taking the trace with respect to B we see that ρ_A is:

$$\begin{aligned}\rho_A &= \mathrm{tr}_B(|\psi\rangle\langle\psi|) \\ &= \sum_i e^{-\beta E_i} |E_i\rangle_A \langle E_i|_A\end{aligned} \tag{5.13}$$

Which is a thermal density matrix, which the gauge/gravity dictionary tells us corresponds to a black hole in the bulk theory. It is tempting to interpret this as the two outside regions in the eternal AdS-Schwarzschild black hole.
If this interpretation is correct, it implies that a quantum superposition of disconnected spacetimes can be identified with a connected spacetime.
As we have seen, given two complementary regions A and B on a CFT (say the two hemispheres on a d-sphere S^d), the Ryu-Takayanagi formula of Eq. (4.1) says that as the entanglement entropy $S_A = S_B$ decreases the area of the region corresponding to the boundary on the bulk: we can interpret this as saying that as the entanglement increases, the two regions pinch off from each other.
For any two CFT operators $\mathcal{O}_C$ and $\mathcal{O}_D$ acting on the regions $C \subset A$ and $D \subset B$, the mutual information $I(C,D)$ defined in Eq. (4.23) obeys:

$$I(C,D) \geq \frac{\left(\langle \mathcal{O}_C \mathcal{O}_D \rangle - \langle \mathcal{O}_C \rangle \langle \mathcal{O}_D \rangle\right)^2}{2\left|\mathcal{O}_C\right|^2 \left|\mathcal{O}_D\right|^2} \tag{5.14}$$

Furthermore, certain two-point correlators of local operators give a measure of the distance between the spacetime points on which they act:

$$\langle \mathcal{O}_C(x_C)\, \mathcal{O}_D(x_D) \rangle \sim e^{-mL} \tag{5.15}$$

(where m is the mass and L is the length of the shortest geodesic between x_C and x_D)
We see that as the entanglement decreases, the bulk regions corresponding to C and D become more physically separated: the two regions of spacetime pull apart and pinch off from each other. This idea can be summarised as: entanglement generates geometry.
Taking this idea seriously, [29] postulate that we should interpret the two spacetime regions I and III not as distinct universes, but as spatially separated parts of the same spacetime. In this view the Einstein-Rosen bridge between two black holes is caused by the entanglement between the microstates of two spatially separated black holes.
They take the claim even further by stating that every entangled system is connected by some quantum version of an Einstein-Rosen bridge.
According to [29], an observer in region III can, by performing computations on her

black hole using a quantum supercomputer, send messages inside the horizon, when they are inaccessible to the observer in region I, but only as long as he does not fall into the horizon. These messages can include a high energy wave which, if sent at the right time, would create an AMPS firewall. However, [15] gives possibly the first instance of an argument from quantum computation complexity theory in a black hole paper, showing that whether an observer is able to determine if an outgoing radiation mode is entangled to the previously emitted radiation depends on an unsolved problem in quantum computer science, whether $SZK \subseteq BQP$. They use this to claim that firewalls might not be necessary after all.

6

Holographic quantum erasure correction

6.1. Quantum erasure correction

A simple example of a quantum correction code that uses three-state qutrits (although in quantum information we usually work with binary state qubits, the two formulations are equivalent) is given by encoding the state (which requires one qutrit):

$$|\psi\rangle = \sum_{i=0}^{2} a_i |i\rangle \tag{6.1}$$

As a state (which requires three qutrits):

$$|\widetilde{\psi}\rangle = \sum_{i=0}^{2} a_i |\widetilde{i}\rangle \tag{6.2}$$

Where

$$\begin{aligned}
|\widetilde{0}\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle) \\
|\widetilde{1}\rangle &= \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle) \\
|\widetilde{2}\rangle &= \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle)
\end{aligned} \tag{6.3}$$

This is referred to in the literature as a (2, 3) threshold scheme, because knowing any two of the three qutrits (say the first and the second) allows us to reconstruct the full state $|\widetilde{\psi}\rangle$ and hence the original state $|\psi\rangle$. It can equivalently be seen as an erasure correction code, as if we know that one of the qutrits has been erased (say the third), we are still able to reconstruct the original state.

Formally the reconstruction can be taken as acting on the by a unitary transformation

$U_{12} \otimes I_3$ (where U_{12} acts on the first two qutrits and I_3 on the last one) such that:

$$(U_{12} \otimes I_3) \, |\widetilde{i}\rangle = |i\rangle \otimes \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \tag{6.4}$$

Where the operator U_{12} is explicitly written down as a permutation that sends $|00\rangle \to |00\rangle, |11\rangle \to |01\rangle, |22\rangle \to |02\rangle, \ldots$

If O is an operator that acts on the one-qutrit space as:

$$O|i\rangle = \sum_j (O)_{ji}|j\rangle \tag{6.5}$$

We can always find a non-unique operator $\widetilde{O}$ on the three-qutrit space that implements the same transformation:

$$\widetilde{O}|\tilde{i}\rangle = \sum_j (O)_{ji}|\tilde{j}\rangle \tag{6.6}$$

In general $\widetilde{O}$ will act on all three qutrits. But in the example before one can take:

$$\widetilde{O}_{12} \equiv U_{12}^{\dagger} O U_{12} \tag{6.7}$$

So that the support of $\widetilde{O}_{12}$ is on the first two qutrits only. Similarly one can define operators $\widetilde{O}_{13}$ and $\widetilde{O}_{23}$. This gives an example of three distinct operators with support on different qutrits that have the same action on the code subspace (the space spanned by $|\widetilde{0}\rangle$, $|\widetilde{1}\rangle$ and $|\widetilde{2}\rangle$).

6.2. Bulk reconstruction

Given a gravitational theory on a asymptotically AdS space M we can reconstruct the operators $\mathcal{O}$ on the CFT boundary ∂M from fields φ defined on the AdS theory so that they satisfy Eq. (2.43). This can be done in perturbation theory by requiring that:

$$\varphi(x) = \int_{\partial M} K(x; y)\mathcal{O}(y)dy \tag{6.8}$$

Where the kernel $K(x; y)$ is a suitably defined smearing function, which obeys the bulk equations of motion in the x index and obeys Eq. (2.43) when x is taken to the boundary.

The operators $\mathcal{O}$ only satisfy the expected commutation relationship for the AdS theory to low order in the perturbation theory.

The smearing function $K(x; y)$ can be taken to have support only for x and y spacelike separated (that is, y can be taken to be from the set of points on the boundary that

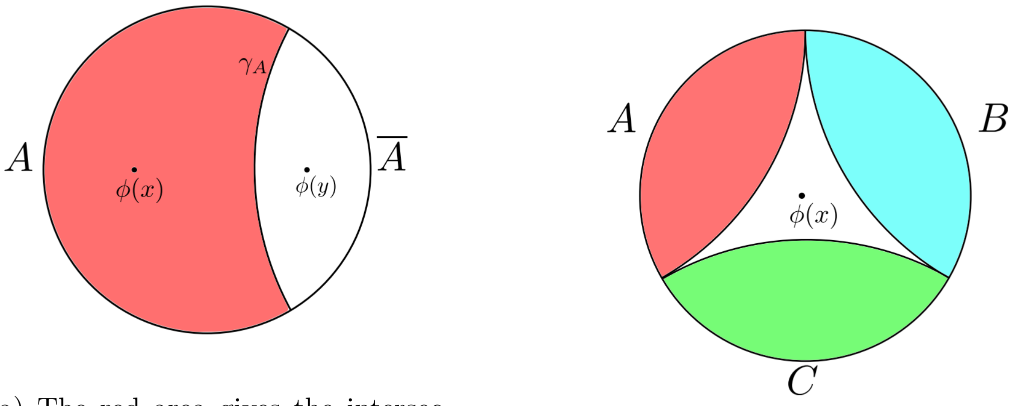
are outside the future or past light cone of x).
The casual wedge $\mathcal{W}_C[A]$ of a region A of a Cauchy surface Σ (for instance the surface given by $t = 0$) in the boundary is a region defined in [24] as:

$$\mathcal{W}_C[A] \equiv \mathcal{J}^+[D[A]] \cap \mathcal{J}^-[D[A]] \tag{6.9}$$

Where $D[A]$ is the boundary domain of dependence of A defined as the set of points on the boundary such that every inextendible causal curve that touches any of them must also touch A.
The causal surface χ_A is the null surface that bounds the causal wedge within the bulk (that is not on ∂M).
The notion of the casual wedge is motivated by the fact that classical bulk field equations are causal in the AdS radial direction [12], and data on the boundary is enough to determine a bulk operator $\varphi(x)$ at point x if x lies within the past light cone of x.
An example is the AdS-Rindler wedge given by the simple case where the geometry is that of pure AdS_{p+2}, the Cauchy surface Σ corresponds to $t = 0$, and A is a p-dimensional hemisphere on the boundary. The metric on the AdS-Rindler wedge can be written as:

$$ds^2 = -\left(\rho^2 - 1\right) d\tau^2 + \frac{d\rho^2}{\rho^2 - 1} + \rho^2 \left(dx^2 + \sinh^2 x d\Omega_{p-1}^2\right) \tag{6.10}$$

With coordinate ranges $\rho > 1, x \geq 0, -\infty < \tau < \infty$.
By acting on the AdS-Rindler wedge with the isometries of the AdS theory one can obtain the causal wedge for any disc in Σ: in the example of AdS_3 any interval can be obtained from a causal wedge.
The support of the function $K(x; y)$ in Eq. (6.8) can be taken to a more restrictive set than stated before: for $x \in \mathcal{W}_C[A]$ one can take y to be in the domain of dependence $D[A]$. This means that for a point x close to the boundary, only a small region A is needed to be able to reconstruct the field φ in terms of operators acting on A.
Alternatively, it was shown in [20] that one can define a entanglement wedge of A, defined in terms of the minimal Hubeny-Rangamani-Takayanagi surface in Eq. (4.1), and we can take the support for $K(x; y)$ to be on the boundary of this wedge instead. The entanglement is equal or larger than the the causal wedge, in particular it can be much larger in the case where A is a disconnected region, meaning that bulk operators far outside the causal wedge can still be reconstructed in this case.
In [1] the authors consider what happens when we try reconstructing some bulk field operator $\varphi(x)$ which lies on the causal wedges of distinct regions in the boundary, say A, B and C.
This seems to be problematic as we can take a local operator $\mathcal{O}(y)$ defined at a point



(a) The red area gives the intersection of entanglement wedge $\mathcal{E}_A$ of region A and a $t = 0$ slice in AdS_3. Point x lies in $\mathcal{E}_A$, so $\varphi(x)$ can be reconstructed on A, point y lies in $\mathcal{E}_{\bar{A}}$, so $\varphi(y)$ can be reconstructed on $\bar{A}$.

(b) The bulk field $\varphi(x)$ cannot be reconstructed on A, B, or C, however it can be reconstructed on AB, AC, or BC.

Figure 6.1: Figure reproduced from [14]

y on the complement of the region A on the boundary, and locality would imply that the field $\varphi(x)$ reconstructed on the wedge given by A commutes with $\mathcal{O}(y)$. By choosing different regions, we could make our reconstructed $\varphi(x)$ commute with all local operators on the slice Σ. But it is a consequence of Schur's lemma applied to the algebra of operators of the CFT that no nontrivial operator in the CFT can commute with all local operators, hence we must have that the reconstructed operators $\varphi_A(x)$ and $\varphi_B(x)$ on two different regions A and B are not necessarily the same operator.

6.3. Hyperbolic tilings of tensor networks

The situation described at the end of §6.2 displays an apparent similarity with the quantum correction code described in Fig. 6.1b: knowledge of one of the three qutrits does not allow us to reconstruct the original system, but knowledge of two of them does; this mirrors how the bulk field $\varphi(x)$ in Fig. 6.1b cannot be reconstructed on any of the regions A, B, or C, but it can be reconstructed on the union of any two of them.

In [1] a code subspace $\mathcal{H}_{\mathcal{C}}$ of the Hamiltonian for the bulk theory $\mathcal{H}$ is proposed, given by the span of a finite set of bulk field operators $\varphi_i(x)$ acting on the vacuum on a finite number of points x in the bulk:

$$|\Omega\rangle, \phi_i(x)|\Omega\rangle, \phi_i\left(x_1\right) \phi_j\left(x_2\right)|\Omega\rangle, \ldots \tag{6.11}$$

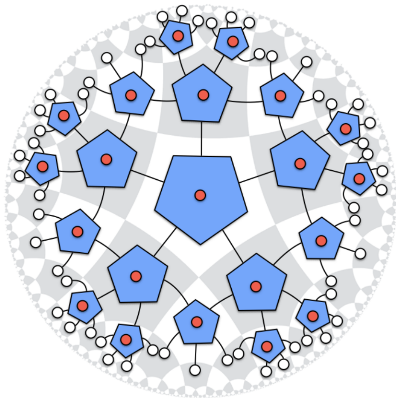


Figure 6.2: Diagram for the holographic pentagon-tiling code, with white dots representing physical indices on the boundary and red dots representing the logical indices in the bulk. The gauge/gravity duality gives an isometric map from white dots to red dots. Figure reproduced from [36].

And it is claimed that this subspace corresponds in a perturbation theory sense to the code subspace of a quantum correcting code. In [31] a different approach is taken, and it is claimed that quantum error correction appears automatically as a consequence of the $O(N)$ gauge invariance on the boundary CFT theory, without the need to introduce a distinct code subspace as in [1].

Building up on a model originally proposed in the multi-scale entanglement renormalization ansatz proposal in [46], a holographic code is realised in [36] by a discretisation of the bulk space at a time slice into a tiling, from which they construct a tensor networks made up of perfect tensors.

In information theory, the probability amplitudes a pure quantum state of m particles with v-dimensional spins (in the usual case of qubits $v = 2$) can be represented as a tensor with m indices, each ranging over v values:

$$|\psi\rangle = \sum_{a_1, a_2, \ldots, a_m} T_{a_1 a_2 \ldots a_m} |a_1 a_2 \ldots a_m\rangle \tag{6.12}$$

A perfect tensor T is such a $2n$-index tensor such that for any bipartition of its indices into A and $\bar{A}$ (where $|A| \leq |\bar{A}|$) it is proportional to an isometry (a transformation preserving the inner product) from the Hilbert spaces of A into $\bar{A}$ (that is, T is maximally entangled for any cut).

Perfect tensors can be taken to represent a quantum error correcting code that encodes one logical spin into $2n-1$ physical spins and protects against the erasure of $n-1$ (less than half) of the physical spins. For instance the tensor $\sum_{a_1,\ldots a_6} T_{a_1 \ldots a_6} |a_3 a_4 a_5 a_6\rangle \langle a_1 a_2|$ is an isometry from a 2-dimensional to a 4-dimension space and corrects at most one erasure.

In the examples given in [36], the bulk space is taken to be AdS_3 and the hyperbolic geometry of time slices $t = 0$ is discretised into a uniform hyperbolic tiling (for instance hexagonal or pentagonal tiling). Physical spins are associated to the boundary (corresponding to the CFT theory).
The number of logical v-dimensional spins is given by the number of tilings considered N_{bulk}, and the number of physical spins is given by the number of uncontracted boundary indices at the boundary N_{boundary}. The tiling can be read as a circuit, with input coming qubits coming from the outer layers into the central layers. In the example of a pentagonal tiling given in Fig. (6.2), each tile has at most two input legs, corresponding to a perfect tensor with three inputs (where the bulk logical index gives the third leg). The whole circuit, the holographic code, gives an isometry from the logical indices in the bulk to the physical indices in the boundary. For large N_{bulk} with a pentagonal tiling one has that the ratio $\frac{N_{\text{bulk}}}{N_{\text{boundary}}}$ approaches $\frac{1}{\sqrt{5}}$ (the rate of the code). This minimal curve γ_A of the Ryu-Takayanagi conjecture (Eq. (4.1)) is represented by a cut through the tensor networks which partitions it in two, and the notion of length to be minimised is given by the number of legs γ_A cuts through.
A holographic state $|\psi\rangle$ corresponds to a holographic code with no bulk indices. Given a cut γ_A which partitions the boundary into regions A and $\bar{A}$ the holographic state can be written in terms of the tensors $|P_i\rangle$ which represents the state on A and $|Q_i\rangle$ which represents the state on $\bar{A}$ (up to normalisation) defined so that:

$$\begin{aligned} |\psi\rangle &= \sum_{a,b,i} |ab\rangle P_{a,i} Q_{b,i} \\ &\equiv \sum_{i} |P_i\rangle_A \otimes |Q_i\rangle_{\bar{A}} \end{aligned} \tag{6.13}$$

Where a runs over the basis for the region A and b over the basis for region $\bar{A}$ (with dimensions v^{N_A} and $v^{N_{\bar{A}}}$, where N_A is the number of physical spins on A and $N_{\bar{A}} = N_{\text{bulk}} - N_A$ is the number of physical bits on $\bar{A}$) and i runs through the values of the indices contracted along γ_A (where $0 \leq i < N_{\text{cuts}}$ and N_{cuts} is the number of legs γ_A cuts through).
The reduced density matrix for A is found by tracing out $\bar{A}$:

$$\rho_A = \sum_{i,i'} \langle Q_{i'} \mid Q_i\rangle \, |P_i\rangle \, \langle P_{i'}| \tag{6.14}$$

One finds that the entropy is bounded from above by the length of γ_A:

$$S_A \leq |\gamma_A| \cdot \ln v \tag{6.15}$$

And for a holographic state on a tiling with nonpositive curvature the tensors P and Q are isometries and the sets $\{|P_i\rangle\}$ and $\{|Q_i\rangle\}$ are isometries and we have:

$$S_A = |\gamma_A| \cdot \ln v \tag{6.16}$$

In agreement with (the lattice version of) Eq. (4.1).
In [36] a simple greedy algorithm is also proposed to find an estimate γ_A^* for the lattice version of the Ryu-Takayanagi curve γ_A. In information science, greedy algorithms are iterative optimisation algorithms which make the locally optimal choice at each stage in the optimisation (in the example from [36], this means adding an extra tile to the region P). Such algorithms often result in a globally sub-optimal solution. Indeed there are circumstances (for instance cases when A is a disconnected region) under which using the greedy algorithm described we have $\gamma_A^* \neq \gamma_A$.
In spite of these limitations, it does not seem to be the case that alternatives to the greedy algorithm have been put forward in recent literature, with much of the literature building up on the greedy algorithm proposal. Investigations of alternative optimisations methods for tensor network cuts might be a fruitful future area of research.

7

Circuit complexity and black holes

7.1. Auxiliary entropy and the second law of quantum complexity

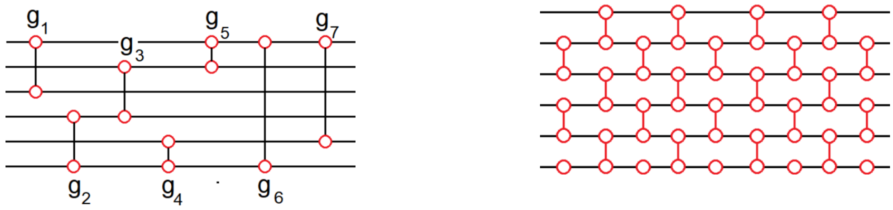
When considering measures of the size of the space of a quantum system of K qubits in order to have a notion of how distant two states are we can define two different metrics. One is the traditional inner product metric given by the Fubini-Study metric given for two vectors $|A\rangle$ and $|B\rangle$ by their inner product:

$$d_{\mathrm{FS}}(A,B) = \arccos |\langle A \mid B\rangle| \tag{7.1}$$

Which takes values from 0 (when $|A\rangle = |B\rangle$) to $\frac{\pi}{2}$ (when $|A\rangle$ and $|B\rangle$ are orthogonal). For the unitary operators that give the time evolution of states in the Schrödinger picture, the inner product can be equivalently defined as (for a Hilbert space operators that can be represented as $2^K \times 2^K$ matrices):

$$d_{\mathrm{FS}}(U,V) = \arccos |\frac{1}{2^K}\operatorname{tr} U^\dagger V| \tag{7.2}$$

Where $\frac{1}{2^K}$ is a normalisation factor so that the distance between U and V is well-defined and $d_{\mathrm{FS}}(U,U) = 0$.
The shortcoming of the Furbini-Study metric [42] is that given two systems with states $|A\rangle$ and $|B\rangle$ which are identical in every respect except for one electron having spin up in system $|A\rangle$ and down in system $|B\rangle$, the Furbini-Study metric gives a distance of 0, the same as the distance between $|A\rangle$ and a completely different system $|C\rangle$.
The second metric, which better captures the notion of what it means for two states to be different, is called the relative complexity $\mathcal{C}(U,V)$ which we introduce below.
In order to do so, we will need to define some notions from the theory of quantum



(a) Example of a 2-local all-to-all circuit.

(b) Example of a 2-local circuit that is also spatially local.

Figure 7.1: Figures reproduced from [42].

computer circuits: local k-gate is qubit unitary operator from the space of k-qubits (the incoming qubits) to itself (the outcoming qubits), where $k \leq K$ and K is again the total number of qubits in our system), to gives some allowed universal gate set. If g is in the set, we want its inverse $g^\dagger$ to also be in the set.
In an actual quantum computer circuit, we expect the qubits to be arranged on a 2D lattice, and gate to be allowed to act only on neighbouring qubits, giving spatial locality as well as k-locality. Lattices of higher dimensions might be useful in modelling condensed matter systems.
However, to define relative complexity we do not need this constraint and we can define k-local all-to-all circuits, where a k-gate can act on any two input qubits.
The relative complexity $\mathcal{C}(U,V)$ of operators U and V is then defined as the minimum number n of gates g_i from the allowed set of gates $g_i \in G$ satisfying:

$$U = g_n g_{n-1} \dots g_1 V \tag{7.3}$$

Where we identify operators that are within a radius of ϵ (the tolerance) from each other as measured by the inner product (we are coarse-graining the space $SU(2^K)$).
We can now define the complexity $\mathcal{C}(U)$ of the operator U as the operator relative complexity with the identity:

$$\mathcal{C}(U) \equiv \mathcal{C}(U, I) \tag{7.4}$$

So that it is simply the minimum number of gates such that we can write U as:

$$U = g_n g_{n-1} \dots g_1 \tag{7.5}$$

Equivalently for two states $|A\rangle$ and $|B\rangle$ we can define their relative complexity as the minimum number of gates such that:

$$|A\rangle = g_n g_{n-1} \dots g_1 |B\rangle \tag{7.6}$$

The relative complexity satisfies the axioms of a metric: the identity of indiscernibles ($\mathcal{C}(U,V) \geq 0$ if and only if $U = V$), symmetry ($\mathcal{C}(U,V) = \mathcal{C}(V,U)$) and the triangle inequality ($\mathcal{C}(U,V) \leq \mathcal{C}(U,W) + \mathcal{C}(W,V)$). Furthermore it is right-invariant:

$$U = g_n g_{n-1} \dots g_1 V \tag{7.7}$$

Gives:

$$UW = g_n g_{n-1} \dots g_1 VW \tag{7.8}$$

So that:

$$\mathcal{C}(U,V) = \mathcal{C}(UW,VW) \tag{7.9}$$

However it is not necessarily true that $\mathcal{C}(U,V) = \mathcal{C}(WU,WV)$ (we don't have left-invariance).
Another property is that for many choices of G, the maximum value of the complexity grows as e^K whilst the number of operators which differ from each other by more than ϵ grows as e^{2K}, so that the volume of the space grows exponentially in the diameter K. This gives a clue that the geometry should be negatively curved: in positively curved geometries the volume grows polynomially, and even more slowly in positively curved spaces.
The idea of complexity geometry is then to find a smooth metric on $SU(2^K)$ that satisfies these properties. For instance, the property of right invariance means that there is a symmetric matrix $\mathcal{I}_{IJ}$ such that the metric is given by:

$$ds^2 = d\Omega_I \mathcal{I}_{IJ} d\Omega_J \tag{7.10}$$

Where:

$$d\Omega_I = i \operatorname{tr} dU^\dagger \sigma_I U \tag{7.11}$$

And where σ_I denote the Pauli operator on the space of K qubits (so with the index I running over $4^K - 1$ values).
The choice of the matrix $\mathcal{I}_{IJ}$ is equivalent to the choice of the set of allowed gates G. Preparing a unitary operator U by applying n gates to the identity can be seen as a discrete curve in the auxiliary classical space $\mathcal{A}$ where operators U are seen as classical particles with $4^K - 1$ degrees of freedom. We endow $\mathcal{A}$ with a metric which corresponds to the distance in Eq. (7.2), and up to a constant is given by:

$$dl^2 = \operatorname{tr} dU^\dagger dU \tag{7.12}$$

Geometrically, the evolution in time of the operator $U(t)$ can be seen as tracing out a geodesic in this space.

We can see this starting from the Schrödinger equation for the time evolution operator (a first order equation):

$$\begin{aligned} i\dot{U} &= HU \\ H &= i\dot{U}U^\dagger \end{aligned} \tag{7.13}$$

And differentiating with respect to time to obtain a second-order equation geodesic equation that does not depend on H:

$$\ddot{U} - \dot{U}U^\dagger\dot{U} = 0 \tag{7.14}$$

Which is of the familiar form:

$$\ddot{X}^M = -\Gamma^M_{AB}\dot{X}^A\dot{X}^B \tag{7.15}$$

Where Γ^M_{AB} are the Christoffel symbols for the standard metric on $SU(2^K)$.
Geodesics are curves that minimise the distance between points only locally: in curved geometries a geodesic might well not be the same as the shortest curve between two far away points. A cut locus is defined as a point p' where the geodesic is no longer the (only) shortest curve: travelling along a geodesic, a non-geodesic curve appears at p' that is shorter or equally long as the geodesic one. An simple example is a closed path on great circle on the sphere S^2: starting at p and following a great circle, we find ourselves at the antipodal point p', and changing direction and following any other great circle now will take us back to p as quickly as the geodesic path would (in this simple example the length of the non-geodesic curves that appear at p' is the same as that of the geodesic, but changing slightly the example to that of a torus $S^1 \times S^1$ the non geodesic paths are of shorter length).
Now, as its time evolution goes on, the length of geodesic traced out by the unitary operator $U(t)$ initially grows as:

$$\mathcal{C}(t) = Kt \tag{7.16}$$

But as time grows to the order of e^K it is increasingly likely to hit a cut locus. As the length of the shortest path in the metric of Eq. 7.10 corresponds to the complexity of $\mathcal{C}(U)$, this means that there is a maximum in the complexity $\mathcal{C}_*$ at a time $t = t_* \sim e^K$. As a consequence of the linear evolution up to t_* in Eq. (7.16) its value also goes as:

$$\mathcal{C}_* \sim e^K \tag{7.17}$$

We can say that the system has reached complexity equilibrium.
An accessible argument justifying this behaviour that does not use differential geometry, but is instead based on a discretisation of the auxiliary system and on graph

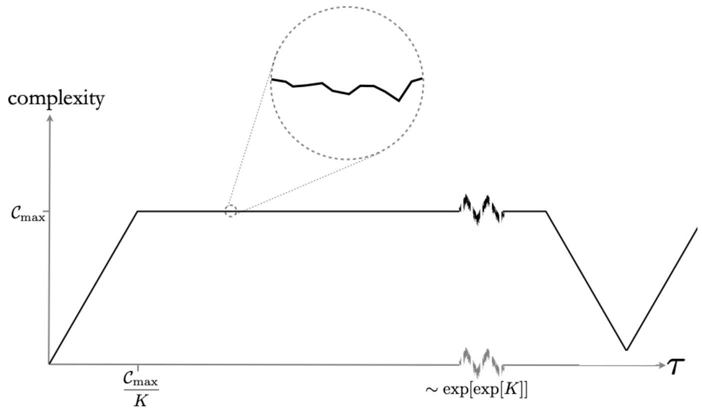


Figure 7.2: Time evolution of the complexity $\mathcal{C}$ of the operator $U(t)$ for an ensemble of K qubits. The growth is linear in time up to a time of the order $t_* = \frac{\mathcal{C}}{K} \sim e^K$. A decrease in complexity occurs only after a time $t \sim e^{e^K}$. Figure reproduced from [6].

theory, is given in [42].

The original treatment of the properties of complexity geometry as a Riemannian manifold is found in [8].

Quantum complexity can be seen as an entropy: not the entropy of the original system of K bits (which can be at most equal to $K \ln 2$ and unlike quantum complexity increases linearly with the number of qubits), but the entropy in the auxiliary system where operators U are seen as classical particles evolving in Brownian motion from the origin (corresponding to the identity operator).

We have:

$$S_{\mathcal{A}} \approx C \log K \tag{7.18}$$

To see that this is the case consider again the discrete model [42]. At each iteration a particle U in the auxilary space can propagate in d directions of the order:

$$d \sim \frac{K!}{(K/2)!} \sim \left(\frac{2K}{e}\right)^{\frac{K}{2}} \tag{7.19}$$

Then the entropy $S_{\mathcal{A}}$ of goes as the volume as the particles expand:

$$S_{\mathcal{A}} \sim n \log d \sim \frac{nK}{2} \tag{7.20}$$

At each iteration the maximum number of 2-local gates that can be applied to the operator is also $C = \frac{K}{2}$. We can then identify quantum complexity with the entropy of the auxiliary system.

Eq. (7.16) and Eq. Eq. (7.17) are then the quantum complexity equivalent of the second law of thermodynamics, the second law of quantum complexity.

As expected for the behaviour of a statistical ensemble, the unitary operator U only returns in the neighbourhood of the identity (causing the complexity to return close to 0) only in a prohibitively long time, of the order of a double exponential with the number of qubits $t \sim e^{e^K}$ (see Fig. (7.2)).
If the Hamiltonian is time-independent, the evolution of the particle is costrained to live on a submanifold of the original space $SU(2^K)$ given by the $2^K 1$ energy levels $|E_n\rangle$:

$$\begin{aligned} U(t) &= e^{iHt} \\ &= \sum_{n=1}^{2^K} |E_n\rangle \langle E_n| \, e^{-iE_n t} \end{aligned} \tag{7.21}$$

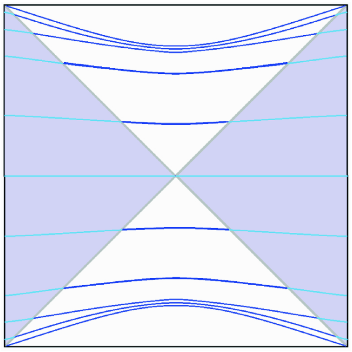
Geometrically the trajectory is given by 2^K phases $\theta_n(t) = E_n t$, that is the motion is restricted to a torus defined by 2^K-dimensional torus embedded in the $(4^K - 1)$-dimensional space $SU(2^K)$.
If the system is chaotic the energy levels are incommensurate, and we have ergodicity (U visits every region of the torus in the long run). In this case [42] hypotheses an even more direct relationship between the auxiliary entropy and the complexity of the form:

$$S_{\mathcal{A}} \approx C \tag{7.22}$$

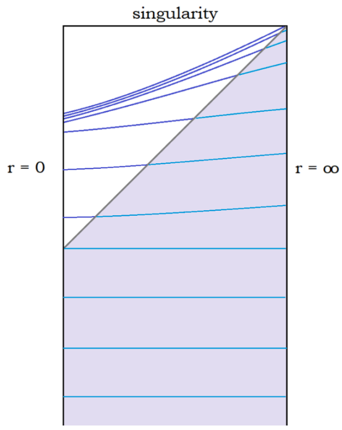
Following Schrödinger's notion of less-than-maximal entropy (negentropy) as a physical resource [40], we can then argue that having less-than-maximal complexity (uncomplexity) is a physical resource [6] that can be expended to perform directed quantum computation.
However complexity, corresponding to the entropy of the larger auxiliary system $SU(2^K)$ behaves in some respects dramatically differently to classical entropy. In particular adding one single qubit to the system doubles the maximal complexity of the system. This puzzling behaviour has again an unexpected connection with the theory of quantum computation, where in the presence of a highly mixed state of K qubits, adding a single clean qubit of known state [26] can allow one to make computations which have no known efficient classical algorithms.

7.2. The Black Hole/Quantum Circuit Correspondence

The Penrose diagram for the Schwarzschild-AdS black hole with the metric given in Eq. (2.27) is shown in Fig. 7.3). Following the ER = EPR proposal outlined in §5.2 we can identify this as two spacetime regions (regions I and III) connected by two entangled black holes/white holes (regions II/IV).
We see that the Rosen-Einstein bridge (wormhole) volume, the volume of the inside



(a) Penrose diagram for an eternal Schwarzschild-AdS black hole



(b) Penrose diagram for a Schwarzschild-AdS black hole created by gravitational collapse

Figure 7.3: In both diagrams, blue lines represent a foliation in maximal slices. Figures reproduced from [42].

region of the black hole given by the maximal foliation for the eternal Schwarzschild-AdS black hole in Fig. (7.3a,) decreases for the white hole and then increases again for the black hole.

We expect white holes to be unphysical: in the more physical scenario of a black hole created by gravitational collapse shown in Fig. (7.3b) the corresponding Rosen-Einstein bridge volume (where in this case the bridge does not go anywhere [43]) increases over time. In both cases it reaches a maximum limit.

One would be tempted to say that there is an increasing quantity like entropy that increases over the anchoring time t and favours a scenario like Fig. (7.3b) over the one in Fig. (7.3a). However the classical entropy is fixed in both cases as the black hole is in thermodynamic equilibrium with its surroundings throughout its evolution.

The growth of the Rosen-Einstein bridge volume $V_{\mathrm{RB}}(t)$ satisfies:

$$\frac{dV_{\mathrm{RB}}}{dt} \sim ALT \tag{7.23}$$

Where L is the anti-de Sitter radius in §2.1, A is the black hole area, and T is its temperature, so that in terms of the Bekenstein-Hawking entropy (Eq. (3.24) we can also write:

$$\frac{dV_{\mathrm{RB}}}{dt} \sim S_{\mathrm{BH}}LT \tag{7.24}$$

The near horizon geometry is given by the Rindler metric of Eq. (3.29) and the relationship between Rindler time for a near-horizon observer τ and the coordinate time of a faraway observer t was given back in Eq. (3.47) as:

$$\tau = 2\pi T t \tag{7.25}$$

Consider a quantum circuit covering the horizon of the black hole. For subexponential time the number of gates in the circuit corresponds to the complexity of the unitary operator prepared by the circuit (we have seen that for exponential time the complexity plateaus), and if at every time-step we have the maximum number $K/2$ of 2-local gates, then

$$\frac{d\mathcal{C}}{\partial \tau} = \frac{K}{2} \tag{7.26}$$

Assuming that the number of qubits is proportional to the size of the circuit (the area of the black hole) we have $K \sim A$:

$$\frac{d\mathcal{C}}{\partial \tau} \sim A \tag{7.27}$$

So that an observer at infinity sees by Eq. (7.25):

$$\frac{d\mathcal{C}}{\partial t} \sim AT \tag{7.28}$$

Hence finally:

$$\frac{d\mathcal{C}}{\partial t} \sim \frac{dV_{\mathrm{RB}}}{\partial t} \tag{7.29}$$

And the two quantities are the same up to constants.

We are then tempted to say the black hole behaves like a quantum circuit that encloses its horizon, with circuit time identified with Rindler time and complexity identified with the volume of the Einstein-Rosen bridge. This is the black hole/quantum circuit correspondence.

This relationship is further explored and refined in [5].

Epilogue

We have seen that the gauge/gravity duality has led us to uncover a series of unexpected connections between quantum gravity and the theory of quantum information and of quantum computation. We have given a hopefully clear and accessible presentation of some of the most central concepts (for example, the holographic entanglement entropy of the Ryu-Takayanagi conjecture introduced in §4.1 has had a pivotal role in everything that has followed in our discussion) as well as an introduction to some of the most recent developments in the literature. The deep connections between information and geometry unveiled by the duality have been a recurring theme in our journey, following M. Van Raamsdonk's proposal [45] that the intrinsically quantum phenomenon of entanglement leads to the emergence of classical spacetime geometry. As we remarked briefly in the introduction, and in spite of its string theoretical origins, the duality is not a purely theoretical tool. It also has a wealth of applications to other areas of physics such as AdS/QCD and AdS/CMT, as well as, importantly, proposed experimental tests. One of the unexpected paths the duality has led us on has been the growing importance of the relativistic physics of black holes: originally considered an aberration and later a curiosity in the history of the subject, and later on of interest mainly to string theorists and other theoreticians, has in light of the correspondence now become more central to the study of some areas of applied physics.

At the same time, the current research programmes in the more theoretical direction have shown the promise to give us completely new perspectives and understanding of the fundamental aspects of physics.

We note in passing that the cross-contamination of theoretical physics and information theory might have also contributed to change how physics present their subject, with current literature in high-energy physics turning again to the Gedankenexperimente and a renewed focus on conceptual aspects.

One of the most exciting aspect of AdS/CFT is in our view precisely that it interconnects so many previously disconnected ideas.

Bibliography

[1] Ahmed Almheiri, Xi Dong, and Daniel Harlow. Bulk locality and quantum error correction in AdS/CFT. *Journal of High Energy Physics*, 2015(4), November 2015.

[2] Ahmed Almheiri, Donald Marolf, Joseph Polchinski, and James Sully. Black holes: Complementarity or firewalls? *Journal of High Energy Physics*, 2013(2), July 2013.

[3] Máximo Bañados, Claudio Teitelboim, and Jorge Zanelli. Black hole in three-dimensional spacetime. *Physical Review Letters*, 69(13):1849–1851, April 1992.

[4] Jacob D. Bekenstein. Black Holes and Entropy. *Physical Review D*, 7(8):2333–2346, April 1973.

[5] Adam R. Brown, Daniel A. Roberts, Leonard Susskind, Brian Swingle, and Ying Zhao. Holographic Complexity Equals Bulk Action? *Physical Review Letters*, 116(19), September 2016.

[6] Adam R. Brown and Leonard Susskind. Second law of quantum complexity. *Physical Review D*, 97(8), January 2018.

[7] Pasquale Calabrese and John Cardy. Entanglement entropy and quantum field theory. *Journal of Statistical Mechanics: Theory and Experiment*, (6), May 2004.

[8] Mark R. Dowling and Michael A. Nielsen. The geometry of quantum computation. *Quantum Information and Computation*, 8(10):0861–0899, December 2008.

[9] Johanna Erdmenger. Introduction to gauge/gravity duality, oct 2012.

[10] Dmitri V. Fursaev. Proof of the holographic formula for entanglement entropy. *Journal of High Energy Physics*, 2006(9), June 2006.

[11] Berry Groisman, Sandu Popescu, and Andreas Winter. Quantum, classical, and total amount of correlations in a quantum state. *Physical Review A - Atomic, Molecular, and Optical Physics*, 72(3), oct 2005.

[12] Alex Hamilton, Daniel Kabat, Gilad Lifschytz, and David A. Lowe. Holographic representation of local bulk operators. *Physical Review D - Particles, Fields, Gravitation and Cosmology*, 74(6), June 2006.

[13] Daniel Harlow. Jerusalem lectures on black holes and quantum information. *Reviews of Modern Physics*, 88(1), September 2016.

[14] Daniel Harlow. The Ryu–Takayanagi Formula from Quantum Error Correction. *Communications in Mathematical Physics*, 354(3):865–912, July 2017.

[15] Daniel Harlow and Patrick Hayden. Quantum computation vs. Firewalls. *Journal of High Energy Physics*, 2013(6), January 2013.

[16] S. W. Hawking. Gravitational radiation from colliding black holes. *Physical Review Letters*, 26(21):1344–1346, May 1971.

[17] Stephen William Hawking. Black hole explosions? *Nature*, 248(5443):30–31, March 1974.

[18] Stephen William Hawking. Particle creation by black holes. *Communications in Mathematical Physics*, 43(3):199–220, 1975.

[19] Patrick Hayden, Matthew Headrick, and Alexander Maloney. Holographic mutual information is monogamous. *Physical Review D - Particles, Fields, Gravitation and Cosmology*, 87(4), July 2013.

[20] Matthew Headrick, Veronika E. Hubeny, Albion Lawrence, and Mukund Rangamani. Causality & holographic entanglement entropy. *Journal of High Energy Physics*, 2014(12), August 2014.

[21] G. Horowitz and J. Polchinski. Gauge/gravity duality. *Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter*, 9780521860:169–186, February 2009.

[22] Gary T. Horowitz and Juan Maldacena. The black hole final state. *Journal of High Energy Physics*, 8(2):197–211, oct 2004.

[23] Veronika E. Hubeny. The AdS/CFT correspondence. *Classical and Quantum Gravity*, 32(12), December 2015.

[24] Veronika E. Hubeny and Mukund Rangamani. Causal holographic information. *Journal of High Energy Physics*, 2012(6), April 2012.

[25] Veronika E. Hubeny, Mukund Rangamani, and Tadashi Takayanagi. A covariant holographic entanglement entropy proposal. *Journal of High Energy Physics*, 2007(7), May 2007.

[26] E. Knill and R. Laflamme. Power of one bit of quantum information. *Physical Review Letters*, 81(25):5672–5675, February 1998.

[27] Juan Maldacena. The large-N limit of superconformal field theories and supergravity. *International Journal of Theoretical Physics*, 38(4):1113–1133, November 1999.

[28] Juan Maldacena. The gauge/gravity duality. In *Black Holes in Higher Dimensions*, volume 9781107013, pages 325–347. June 2012.

[29] Juan Maldacena and Leonard Susskind. Cool horizons for entangled black holes. *Fortschritte der Physik*, 61(9):781–811, June 2013.

[30] Donald Marolf. Unitarity and holography in gravitational physics. *Physical Review D - Particles, Fields, Gravitation and Cosmology*, 79(4), August 2009.

[31] Eric Mintun, Joseph Polchinski, and Vladimir Rosenhaus. Bulk-Boundary Duality, Gauge Invariance, and Quantum Error Corrections. Technical Report 15, 2015.

[32] Makoto Natsuume. *AdS/CFT Duality User Guide*, volume 903 of *Lecture Notes in Physics*. Springer Japan, Tokyo, September 2015.

[33] Tatsuma Nishioka, Shinsei Ryu, and Tadashi Takayanagi. Holographic entanglement entropy: An overview. *Journal of Physics A: Mathematical and Theoretical*, 42(50), May 2009.

[34] Don N. Page. Information in black hole radiation. *Physical Review Letters*, 71(23):3743–3746, June 1993.

[35] Kyriakos Papadodimas and Suvrat Raju. An infalling observer in AdS/CFT. *Journal of High Energy Physics*, 2013(10), November 2013.

[36] Fernando Pastawski, Beni Yoshida, Daniel Harlow, and John Preskill. Holographic quantum error-correcting codes: toy models for the bulk/boundary correspondence. *Journal of High Energy Physics*, 2015(6), March 2015.

[37] Joseph Polchinski. The Black Hole Information Problem. pages 353–397, September 2017.

[38] Joshua D. Qualls. Lectures on Conformal Field Theory, November 2015.

[39] Shinsei Ryu and Tadashi Takayanagi. Aspects of holographic entanglement entropy. *Journal of High Energy Physics*, 2006(8), May 2006.

[40] Erwin Schrodinger and Roger Penrose. *What is Life?* Cambridge University Press, Cambridge, 2012.

[41] Leonard Susskind. The world as a hologram. *Journal of Mathematical Physics*, 36(11):6377–6399, September 1995.

[42] Leonard Susskind. *Three Lectures on Complexity and Black Holes*. SpringerBriefs in Physics. Springer International Publishing, Cham, oct 2020.

[43] Leonard Susskind and Ying Zhao. Switchbacks and the Bridge to Nowhere. August 2014.

[44] Gerard 't Hooft. Dimensional reduction in quantum gravity. *Conf. Proc. C*, 930308:284–296, 1993.

[45] Mark Van Raamsdonk. Building up space-time with quantum entanglement. *International Journal of Modern Physics D*, 19(14):2429–2435, May 2010.

[46] Guifre Vidal. Class of quantum Many-Body states that can be efficiently simulated. *Physical Review Letters*, 101(11), oct 2008.

[47] Steven Weinberg. *The Quantum Theory of Fields*. Cambridge University Press, June 1995.

[48] Steven Weinberg and Edward Witten. Limits on massless particles. *"Physics Letters, Section B: Nuclear, Elementary Particle and High-Energy Physics"*, 96(1-2):59–62, oct 1980.

[49] Edward Witten. Anti de sitter space and holography. *Advances in Theoretical and Mathematical Physics*, 2(2):253–290, February 1998.

[50] Anthony Zee. *Einstein Gravity in a Nutshell*. Princeton University Press, 2013.