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Review of AdS/CFT Correspondence

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Abstract

Here I give a basic introduction on the AdS/CFT correspondence by considering a specific case of the duality between $\mathcal{N} = 4$ super-Yang Mills in 3+1 spacetime dimensions and type IIB superstring theory on $AdS_5 \times S^5$. The duality is considered in its weak form - low energy limit where the string theory is approximated by supergravity. I explain the necessary prerequisites to arrive at the conjectured duality, which includes an introduction on conformal field theory, supersymmetry, AdS spacetime and string theory. The duality is motivated by considering D-branes and their decoupled effective theories. The symmetries and representations are matched on both sides of proposed correspondence and a two point correlation function on the field theory side is calculated using supergravity partition function. The results are compared to those calculated from conformal field theory side.

Contents

Chapter 1

Introduction

AdS/CFT is an exciting new theory that relates two major parts of theoretical physics, which are quantum field theories and gravitational theories (usually string theories). On one hand we have gravity theories that in certain limits reduce to classical theory of relativity. On the other hand we have a quantum theory or should I say quantum framework which, by its nature is entirely different. The notions of classical and quantum theory has always been well separated and who could have thought that these seemingly unrelated theories have a very intimate connection. The AdS/CFT correspondence led to huge popularity [1] in the physics community which in turn led to its applications in different areas of physics, such as strong interactions and condensed matter physics. The reason I find it so exciting is because it gives new insights into a well defined theory of quantum gravity, one of the biggest mysteries of modern theoretical physics.

AdS/CFT correspondance states that a gravity theory on anti-de Sitter spacetime is dual to a conformal field theory [27]. Generally the gravity theories are string theories and CFT's are gauge theories but this is not necessarily true for all cases [15]. Which is why the theory is also referred to as gauge/gravity duality. The main idea of the duality is that AdS spacetime in $d+1$ dimensions has the same symmetry group as a conformal field theory in d-dimensions. So the CFT "lives" on the boundary of AdS spacetime. The correspondence was first conjectured by Juan Maldacena in 1997 [19] when he looked at different perspectives on D-branes in type IIB superstring theory at low energies. But even before that in 1974 Gerard 't Hooft has noticed a few hints that there exists similarities between quantum field theory and string theory calculations when he considered QCD theories with arbitrary number of colour charges [13]. Another clue for the motivation of the duality was the paper by the same author in 1993 where he expanded on the work of Stephen Hawking on black hole thermodynamics. 't Hooft argued that the degrees of freedom of our world on Planck scale can be described on two dimensional lattice evolving in time [35]. This was motivated by the fact that entropy of a black hole is proportional to the surface

$$S = \frac{A_d}{4G} \tag{1.1}$$

where A_d is the area in Plank units and G is the Newtons gravitational constant. This idea was developed later by Leonard Susskind and is now known as the holographic principle [33], which states that the information in a $d+1$ dimensional volume is encoded onto a d dimensional area of the surface of the volume. So it's not just about one paper, physicists have been seeing evidence of this duality for a long time but I think Maldacena was the one who made it concrete since he gave a specific example[2].

The useful aspect of the correspondence is that one can substitute often difficult calculations in one theory, with easier calculations in the theory on the other side of the duality. To compute entropy in a strongly coupled field theory is a very hard computation. However, if this is dual to a gravitational theory, the calculations becomes much more straightforward, since the calculations reduces to computing the entropy of a black hole [21]. A wide area of physics where AdS/CFT correspondence shows much promise is strong coupling theories. There is not one way of dealing with strongly coupled physical systems and perturbation theory works only for weakly coupled ones. usually when a field theory is strongly coupled the gravitational theory is classical and

[1] Maldacena's original paper on AdS/CFT correspondence was cited almost 12,000 times in 2015.

[2] This was a proposed duality between string theory on $AdS_5 \times S^5$ and $\mathcal{N}=4$ super-Yang-Mills theory in four dimensions.

weakly coupled. Hence, using a dual theory to do calculations brings interesting insights into theories with strong coupling [22]. A considerable amount of studies have been carried out on the applications for quantum chromodynamics and particularly for quark-gluon plasma [16]. Strong coupling is also a familiar feature in many interesting systems in condensed matter physics and it is subject to the use of duality principle as well [30]. A standard example is a study of quantum phase transitions [10], which are a consequence of quantum fluctuations rather than changes in temperature. Generally we get these transitions when varying the coupling and in many cases it is not possible to use perturbation theory.

Despite the popularity and many uses of the correspondence, one might be surprised that there is no formal proof of the conjecture. The reason for this is the same reason why this theory is so popular. Mainly for strong coupling. We do not have a method of dealing with these systems. For interacting theories we can only describe them using perturbation theory, but this is valid only for small couplings. We use the correspondence to gain new insights on how we could deal with large couplings, but to actually prove that the conjecture is true we would have to understand theories on both sides. Which we do not. Hence, until that is the case it is not likely we will have a definitive proof anytime soon [23]. Regardless, the conjecture is believed to be true since many non-trivial checks have been made. By tests I mean the computation and comparison of observables, such as correlation functions on both sides of the correspondence [8].

In this thesis I aim to give a general introduction on the AdS/CFT by considering the original example of Maldacena between superstring theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang-Mills. In first chapter I give a basic introduction of conformal field theory, starting with Poincaré symmetry group and extending it to include conformal transformations. At the end of this part I explain how to calculate the correlation functions which are later used for comparison of the ones calculated by a dual theory. In the succeeding chapter I expand the Poincaré symmetry to include fermionic generators and introduce supersymmetry. Then I connect the two extensions of the Poincaré into a superconformal group. Chapter 4 is a very brief overview of anti-de Sitter spacetime followed by a chapter on string theory. This was the part of the thesis I spent most time on, since the topic is very broad and condensing it into a few pages was a challenging task. I started by introducing bosonic string theory then I introduced supersymmetry into the picture and considered its low-energy effective action - supergravity. At the end I gave an overview of D-branes which formed a large part of the discussion of chapter 6. This was a chapter on the correspondence itself and the motivation for it, which stems from two different perspectives on D-branes. I conclude this chapter by talking about the map between operators on the CFT side and fields on AdS and checking that the form of the correlation functions matches on both sides using the results from the chapter on CFT.

Writing this dissertation I mostly followed the book by M. Ammon and J. Erdmenger [1]. This was supplemented by many other reviews such as [22] and [17] for a more basic introduction on the subject together with [8] for a more detailed discussion. The online lecture recordings on CFT by Tobias Osbourne was helpful for the first part of section 2. For the introduction on supersymmetry I mostly followed [3] together with [18]. Section on string theory was mainly complimented by [36]. In general I tried giving a coherent and self contained introduction of the original example of AdS/CFT. Although skipping some (important) parts was inevitable as I felt that the condensed version of particular aspects would only contribute to the confusion rather than clarity of an inexperienced reader.

Chapter 2

Conformal field theory

I think that theoretical physics is mainly based by a reduction principle - trying to look at the problems at smaller and smaller scales. But what happens when a theory is exactly the same on all scales? This is what we are about to explore in this chapter.

Conformal field theory is a field theory that is invariant under scale transformations and something called special conformal transformations in addition to the plain vanilla Poincaré symmetries. It has gained popularity around 1984 due to its role in string theory [11]. It is also an important part of statistical physics for solving critical phenomenon [6] and is a very useful tool to study interactions in field theory. In two-dimensions some of CFT's are solvable exactly. [28]. The recent increase in interest of CFT's is due to the main topic of this thesis - AdS/CFT correspondence.

In this chapter I will give a basic outline of conformal field theory and since it is a very broad field and I will only focus on the main points that are crucial for the AdS/CFT correspondence. For the majority of this section I followed Chapters 3.1 and 3.2 of [1].

Let us start with a review of the Poincaré group of spacetime symmetries and work our way towards expanding it to include conformal transformations. Then we will look at calculating some correlation functions of the theory.

2.1 Poincaré algebra

Poincaré group $ISO(d-1,1)$ is the group of spacetime symmetries that transforms the coordinates by

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{2.1}$$

where $\Lambda^\mu{}_\nu$ is an element of Lorentz group of rotations and boosts satisfying

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma} \tag{2.2}$$

and a^μ is an element of translation subgroup. An infinitesimal Lorentz transformation can be expressed in terms of the generators as $\Lambda^\mu{}_\nu \approx \delta^\mu_\nu - \frac{i}{2}\omega_{\alpha\beta}(J^{\alpha\beta})^\mu{}_\nu$ where $\omega_{\alpha\beta}$ are infinitesimal and $(J^{\alpha\beta})^\mu{}_\nu$ are the elements of the Lie algebra $\mathfrak{o}(d-1,1)$ that generates the Lorentz group $O(d-1,1)$. We may take the explicit representation of the generators to be

$$(J^{\alpha\beta})^\mu{}_\nu = i(\eta^{\mu\alpha}\delta^\beta_\nu - \eta^{\mu\beta}\delta^\alpha_\nu) \tag{2.3}$$

which satisfy the commutation relation

$$[J^{\alpha\beta}, J^{\rho\sigma}]^\mu{}_\nu = -i(\eta^\rho J^{\beta\sigma} - \eta^{\beta\rho} J^{\alpha\sigma} - \eta^{\alpha\sigma} J^{\beta\rho} + \eta^{\beta\sigma} J^{\alpha\rho})^\mu{}_\nu. \tag{2.4}$$

The expression above defines the Lie algebra of the Lorentz group [40]. We can extend this group by including the generators of translations $\mathfrak{t} \subset \mathfrak{iso}(d-1,1)$ which would define the full group of Minkowski isometries. In field theory the generators of Poincaré group are realised as differential operators acting on field space. Let us derive this representation for a scalar field $\phi(x)$. First consider infinitesimal Lorentz transformation which in the active picture can be expressed as

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) = \phi(x) - \frac{i}{2}\omega_{\alpha\beta}(J^{\alpha\beta})^\mu{}_\nu x^\nu \partial_\mu \phi(x) + ... \tag{2.5}$$

Plugging in (2.3) into (2.5) and defining the differential operator as $(J^{\alpha\beta})^{\mu}{}_{\nu} x^{\nu} \partial_{\mu} \equiv J^{\alpha\beta}$ we can show that on a scalar the Lorentz transformation is generated by

$$J^{\alpha\beta} = i(x^{\alpha}\partial^{\beta} - x^{\beta}\partial^{\alpha}). \tag{2.6}$$

Now consider an infinitesimal translation ϵ^{μ}. The field transforms in the active picture as

$$\phi(x) \to \phi'(x) = \phi(x - \epsilon) = \phi(x) - \epsilon^{\mu}\partial_{\mu}\phi(x). \tag{2.7}$$

So we can define the translation generator to be $P_{\mu} = i\partial_{\mu}$. Together with $J^{\alpha\beta}$ they generate the Poincaré group. We can see that these indeed satisfy the Poincaré algebra

$$[J^{\alpha\beta}, J^{\rho\sigma}]^{\mu}{}_{\nu} = -i(\eta^{\rho} J^{\beta\sigma} - \eta^{\beta\rho} J^{\alpha\sigma} - \eta^{\alpha\sigma} J^{\beta\rho} + \eta^{\beta\sigma} J^{\alpha\rho})^{\mu}{}_{\nu}, \tag{2.8}$$

$$[P_{\mu}, P_{\nu}] = 0, \tag{2.9}$$

$$[P^{\mu}, J^{\alpha\beta}] = i(\eta^{\mu\alpha} P^{\beta} - \eta^{\mu\beta} P^{\alpha}). \tag{2.10}$$

2.2 Conformal algebra

We can extend the Poincaré group to include two more generators of conformal transformations that changes the distances between points but preserve angles [12]. Under these transformations the metric transforms as

$$g_{\mu\nu}(x) \to \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x) = \Omega(x) g_{\mu\nu}(x). \tag{2.11}$$

What transformations can we do such that the variation of the metric is proportional to itself? Well, consider a general infinitesimal transformation

$$x^{\mu} \to x^{\mu} + \epsilon(x)^{\mu}. \tag{2.12}$$

Then we can calculate the variation in the metric

$$\delta g(x)_{\mu\nu} = \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} \tag{2.13}$$

and demand that this must be proportional to the metric $\delta g_{\mu\nu}(x) \propto g_{\mu\nu}(x)$. The result is an equation for $\epsilon^{\mu}(x)$

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = g_{\mu\nu}(x)\frac{2}{d}\partial^{\rho}\epsilon_{\rho} \tag{2.14}$$

$$\implies (g_{\mu\nu}(x)\partial_{\rho}\partial^{\rho} + (d-2)\partial_{\mu}\partial_{\nu})\partial^{\sigma}\epsilon_{\sigma} = 0. \tag{2.15}$$

We can see that there is something special when the number of dimensions $d = 2$. Actually in this case the equations simplify to Cauchy-Riemann equations and it is an interesting case [1]. However, I will not discuss this case further and focus only on $d > 2$. The most general form of the infinitesimal transformation can be written as

$$\epsilon_{\mu} = a_{\mu} + \omega_{\mu\nu}x^{\nu} + \lambda x_{\mu} + b_{\mu}x^2 - 2(b \cdot x)x_{\mu}. \tag{2.16}$$

The first two terms in the expression corresponds to translations and Lorentz transformations and the generators of these make up the familiar Poincaré group that we all know and love. But the rest belong to conformal group which is an extension of the Poincaré. Lets first look at

$$x^{\mu} \to x^{\mu} + \lambda x^{\mu}. \tag{2.17}$$

This corresponds to **dilation** and is related to **scale** symmetry. A scalar transforms as

$$\phi(x) \to \phi'(x) = \phi(x - \lambda x) = \phi(x) - \lambda x^{\mu}\partial_{\mu}\phi(x). \tag{2.18}$$

The generator of this transformation is $D = ix^{\mu}\partial_{\mu}$. The second form of infinitesimal transformation is called the **special conformal transformation**

$$x^{\mu} \to x^{\mu} + b^{\mu}x^2 - 2(b \cdot x)x_{\mu}. \tag{2.19}$$

Conforaml transformations			
Transformation	Infinitesimal	Finite	Generator
Translation	$x^\mu + a^\mu$	$x^\mu + a^\mu$	$P_\mu = i\partial_\mu$
Lorentz	$x^\mu + \omega_{\mu\nu} x^\nu$	$\Lambda^\mu{}_\nu x^\nu$	$J^{\alpha\beta} = i(x^\alpha \partial^\beta - x^\beta \partial^\alpha)$
Dilation	$x^\mu + \lambda x^\mu$	λx^μ	$D = i x^\mu \partial_\mu$
Special	$x^\mu + b^\mu x^2 - 2(b \cdot x) x_\mu$	$\frac{x^\mu - x^2 b^\mu}{1 - 2(b\cdot x) + b^2 x^2}$	$K_\mu = i(x^2 \partial_\mu - x_\mu x^\nu \partial_\nu)$

Table 2.1: Summary of conformal transformations [26]

Similarly a scalar transforms under this transformation as

$$\phi(x) \to \phi'(x) = \phi\big(x - bx^2 + 2(b \cdot x)x\big) = \phi(x) - b^\mu (x^2 \partial_\mu - x_\mu x^\nu \partial_\nu).\phi(x) \tag{2.20}$$

So the generator of this transformation is $K_\mu = i(x^2 \partial_\mu - x_\mu x^\nu \partial_\nu)$. The summary of these transformations is given in table 2.1. Together with Poincaré, these generators satisfy the **conformal algebra**

$$\begin{aligned} [J_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu) \\ [D, P_\mu] &= i P_\mu \\ [K_\mu, K_\nu] &= 0 \\ [D, K_\mu] &= -i K_\mu \\ [K_\mu, P_\nu] &= 2i(\eta_{\mu\nu} D - J_{\mu\nu}) \\ [D, J_{\mu\nu}] &= 0 \end{aligned} \tag{2.21}$$

In d spacetime dimensions this group has $(d+2)(d+1)/2$ generators, which is exactly the same number of generators of the group $SO(d,2)$. In fact if we combine these generators in the following way:

$$\begin{aligned} \bar{J}_{\mu\nu} &= J_{\mu\nu} \\ \bar{J}_{\mu d} &= \tfrac{1}{2}(K_\mu - P_\mu) \\ \bar{J}_{\mu(d+1)} &= \tfrac{1}{2}(K_\mu + P_\mu) \\ \bar{J}_{(d+1)d} &= D \end{aligned} \tag{2.22}$$

then we see that these are in fact the generators of the Lie group $SO(d,2)$ [1] satisfying

$$[\bar{J}_{AB}, \bar{J}_{CD}] = i(\eta_{BC} \bar{J}_{AD} - \eta_{AC} \bar{J}_{BD} - \eta_{BD} \bar{J}_{AC} + \eta_{AD} \bar{J}_{BC}). \tag{2.23}$$

This group leaves the metric $\eta_{AB} = diag(- + ... + -)$ invariant. Where $\mu, \nu = 0, 1, ..., d-1$ and $A, B = 0, 1, ..., d+1$.

2.3 Conformal action on fields

In the previous section we figured out the algebra of the conformal group by considering the action on scalar fields. The algebra does not depend on a particular representation so let us also consider fields that transform non trivially under Lorentz transformations [1]

$$\Phi(x) \to e^{-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}} \Phi(\Lambda^{-1} x) \tag{2.24}$$

where $\mathcal{J}^{\mu\nu}$ forms a representation of the Lorentz algebra and is related to the spin of the field. The point $x = 0$ is invariant under these transformations, so the variation at the origin is

$$\delta\Phi(0) = -\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\Phi(0). \tag{2.25}$$

In quantum field theory the action of the group on fields is determined by the commutator since the quantum field transforms as

$$\Phi(0) \to e^{\frac{-i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}} \Phi(0) e^{\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}} = \Phi(0) - \frac{i}{2}\omega_{\mu\nu}[\hat{J}^{\mu\nu}, \Phi(0)] + ... \tag{2.26}$$

Where $\hat{J}^{\mu\nu}$ is now an operator on Hilbert space. So we see that this field satisfies

$$[\hat{J}^{\mu\nu}, \Phi(0)] = \mathcal{J}^{\mu\nu}\Phi(0). \tag{2.27}$$

Then we can use the translation operator $e^{-i\hat{P}^{\mu}x_{\mu}}$ to translate the field to a general point x. This results in the following commutator

$$[\hat{J}_{\mu\nu}, \Phi(x)] = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi(x) + \mathcal{J}_{\mu\nu}\Phi(x). \tag{2.28}$$

This method of obtaining transformations is called **induced representations** [1]. Following a similar logic we can find the action of the conformal group on a general field $\Phi(x)$. Then (omitting the hats on the operators)

$$\begin{aligned}
[P_\mu, \Phi(x)] &= i\partial_\mu\Phi(x) \\
[D, \Phi(x)] &= -i\Delta\Phi(x) - ix^\mu\partial_\mu\Phi(x) \\
[J_{\mu\nu}, \Phi(x)] &= i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi(x) + \mathcal{J}_{\mu\nu}\Phi(x) \\
[K_\mu, \Phi(x)] &= i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu + 2x_\mu\Delta)\Phi(x) - 2x^\nu\mathcal{J}_{\mu\nu}\Phi(x)
\end{aligned} \tag{2.29}$$

Notice that the commutator $[D, \Phi(x)]$ implies that the field transforms non-trivially under dilations. Specifically, the field transforms as

$$\Phi(x) \to \Phi'(x) = \lambda^\Delta\Phi(\lambda x) \tag{2.30}$$

where Δ is called the **scaling dimension**. And the field $\Phi(x)$ is an eigenstate of this operator with eigenvalue $-i\Delta$. We can also see from the conformal algebra that the operator P_μ raises the scaling dimension by 1 and the operator K_μ lowers the scaling dimension by 1. We can see this from the Jacobi identity. For example

$$\begin{aligned}
&[D, [P_\mu, \Phi(0)]] + [\Phi(0), [D, P_\mu]] + [P_\mu, [\Phi(0), D]] = 0 \\
\implies &[D, [P_\mu, \Phi(0)]] = -i(\Delta + 1)[P_\mu, \Phi(0)].
\end{aligned} \tag{2.31}$$

We define a special kind of field called the conformal **primary field**. It is an eigenfunction of D and has the lower bound on Δ, i.e. it is annihilated by K_μ

$$[K_\mu, \mathcal{O}] = 0, \quad [D, \mathcal{O}] = -i\Delta\mathcal{O}. \tag{2.32}$$

A general field at the origin $\Phi(0)$ is primary since it satisfy $[K_\mu, \Phi(0)] = 0$ and $[D, \Phi(0)] = -i\Delta\Phi(0)$. All the other fields are obtained by the action of P_μ on primary fields and they are called **descendant fields** [1]. Consider a primary field with dimension Δ and spin j. Other spin states are obtain by the action of $J^{\mu\nu}$, which fills the spin representation space. Then acting with P_μ would give us an operator with dimension $\Delta+1$ but this descendant state would be of different spin since it would carry another Lorentz index. Then the other spin states are again obtain by the action of $J^{\mu\nu}$. Hence, the primary operators define the full representation of the algebra which is classified by Δ and j.

2.4 Energy-momentum tensor

According to Noethers theorem we get a conserved current for every continuous symmetry. For spacetime translations the current is the **energy-momentum** tensor $T_{\mu\nu}$. We can use a slightly different definition of the energy momentum tensor which can be written as the functional derivative of the action with respect to metric

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{2.33}$$

This definition leads to same results, however it has a few advantages such as it is automatically symmetric, it can easily be extended to curved spacetimes and for gauge theories it automatically is gauge invariant [36]. Scale symmetry puts an additional constraint on the tensor. To see this consider the variation of the action

$$\delta S = \int d^dx T^{\mu\nu}\delta g_{\mu\nu}. \tag{2.34}$$

Now recall that scale symmetry transformations leaves the metric proportional to itself, hence

$$\delta S \propto \int d^d x T^\mu{}_\mu. \tag{2.35}$$

But the variation of the action must vanish. So we have that the energy-momentum tensor must be traceless $T^\mu{}_\mu = 0$. We would arrive to the same result by considering the conserved currents for the dilation which is given by [1]

$$J_{(D)\mu} = x^\nu T_{\mu\nu}. \tag{2.36}$$

Then since it is conserved it would lead to the same result

$$\partial^\nu J_{(D)\mu} = \partial^\nu (x^\rho T_{\mu\rho}) = (\partial^\nu x^\rho) T_{\mu\rho} + x^\rho (\partial^\nu T_{\mu\rho}) = T^\rho{}_\rho = 0. \tag{2.37}$$

2.5 Correlation functions

Important objects in CFT are correlation functions and they will be relevant for us when checking if the AdS/CFT correspondence holds. Naturally conformal symmetry puts restrictions on the form of the correlators. Consider a two-point correlation function of scalar fields $\langle 0| \phi_1(x)\phi_2(y) |0\rangle$ with definite scaling dimensions Δ_1 and Δ_2. Then under dilation the correlator transforms as

$$\langle 0| \phi(x)\phi(y) |0\rangle \to \lambda^{\Delta_1}\lambda^{\Delta_2} \langle 0| \phi(\lambda x)\phi(\lambda y) |0\rangle . \tag{2.38}$$

Scalar fields are functions of spacetime coordinates and the fact that the correlator is invariant under translations implies that it only depends on the difference of the two coordinate points $x^\mu - y^\mu$. Also the correlator should be Lorentz invariant so it must be a function of $(x-y)^2$ [25]

$$\langle 0| \phi_1(x)\phi_2(y) |0\rangle = f\big((x-y)^2\big). \tag{2.39}$$

By translating to the origin

$$\langle 0| \phi_1(x)\phi_2(0) |0\rangle = f(x^2). \tag{2.40}$$

Another important fact is that the vacuum states are invariant under conformal symmetry [12], which means that the generators of conformal algebra annihilates the vacuum state $|0\rangle$. Then we can use the conformal algebra to see the restriction on the two point correlator

$$\begin{aligned} &\langle 0| \phi_1(x)\phi_2(0) D |0\rangle = 0 \\ \implies &\langle 0| \phi_1(x)[\phi_2(0), D] |0\rangle + \langle 0| \phi_1(x) D \phi_2(0) |0\rangle = 0 \\ \implies &\phi_1(x)[\phi_2(0), D] |0\rangle + \langle 0| [\phi_1(x), D]\phi_2(0) |0\rangle = 0 \\ \implies &i(\Delta_1 + \Delta_2 + x^\mu \partial_\mu) f(x^2) = 0. \end{aligned} \tag{2.41}$$

This differential equation has a solution of the form

$$f(x^2) = \frac{C}{x^{\Delta_1+\Delta_2}} \tag{2.42}$$

where C is an integration constant. Or more generally

$$\langle 0| \phi_a(x)\phi_b(y) |0\rangle = f\big((x-y)^2\big) = \frac{C_{ab}}{(x-y)^{\Delta_a+\Delta_b}} \tag{2.43}$$

Where C_{ab} is symmetric. If the two fields are primary, then C_{ab} vanishes unless $\Delta_a = \Delta_b$. Similarly one would find a three-point correlator to be

$$\langle 0| \phi_a(x)\phi_b(y)\phi_c(z) |0\rangle == \frac{C_{abc}}{(x-y)^{\Delta_a+\Delta_b-\Delta_c}(x-z)^{\Delta_a+\Delta_c-\Delta_b}(y-z)^{\Delta_b+\Delta_c-\Delta_a}} \tag{2.44}$$

The two and three-point correlation functions are completely determined (up to an integration constant) by conformal algebra. The four and more point functions are less constrained due to crossing ratios [1].

Chapter 3

Supersymmetry

The beginning of supersymmetry can be traced back to 1966 when Hiranaro Miyazawa tried relating baryons and mesons [20]. This was based on internal symmetries and the idea was largely ignored. It was later considered in quantum field theory by Volkov and Akulov [38] and was further developed by Julius Wess and Bruno Zumino in 1974 [39] where they considered spacetime supersymmetries and the possible applications in particle physics. In simple words supersymmetry is a connection between bosonic and fermionic degrees of freedom. The supersymmetry generators Q act on quantum states by turning a boson into a fermion and vice versa

$$Q\left|\text{fermion}\right\rangle = \left|\text{boson}\right\rangle , \qquad Q\left|\text{boson}\right\rangle = \left|\text{fermion}\right\rangle . \tag{3.1}$$

This is another non-trivial expansion of the Poincaré algebra, which bypasses the Coleman-Mandula[1] theorem by considering graded Lie algebra with fermionic generators. Although there is a bit of scepticism lurking around supersymmetry and whether it has something to do with reality at all, it solves few of the great mysteries in physics. Some of these are the following [3]:

- *Hierarchy problem.* Generally this is a problem with the scale of some parameters of a theory. Where there is a huge discrepancy between the effective value measured by the experiments and predicted value by the theory. Such as the requirement for fine-tuning for Higgs field to account for quadratic radiative contributions at high energies. In supersymmetry these quadratic contributions gets cancelled by the corrections of the supersymmetric particle.

- *Gauge coupling unification.* Grand unification theorems state that the three gauge couplings of the standard model is expected to meet at some higher energy scale. If one only considers the standard model as it stands now the three coupling only meet approximately. However if one considers a supersymmetric extension of the standard model, the gauge couplings meet exactly.

- *Dark matter.* Supersymmetry provides good candidates for dark matter.

These and many more reasons motivate people to study supersymmetry. If supersymmetry is correct it must be broken by the vacuum, which is why we do not observe it in the energy scales accessible to us. However, there is no indication on which energy scale the supersymmetric particles should exist and whether we can expect to confirm them experimentally in the future.

In this chapter I first introduce the supersymmetry algebra and supersymmetric field theory. Then I define the $\mathcal{N} = 4$ super-Yang-Mills theory, which is a supersymmetric version of Yang-Mills theory. I end the discussion by including conformal invariance and defining superconformal group. Writing this chapter I mostly followed [3] together with chapter 3.4 of [1]. A reader not familiar with supersymmetry and spinor notation is advised to consult Appendix A.

3.1 Supersymmetry algebra

In the previous chapter we gave a review of the Poincaré symmetry and then we extended it to include scaling symmetry. Now we will introduce another non-trivial extension of the Poincaré group

[1] Coleman-Mandula theorem states the only possible symmetries of quantum field theory S-matrix is a combination of Poincaré and internal symmetries. By relaxing some assumptions we are able to add non-trivial extensions to the allowed symmetry transformations. In supersymmetry case we consider fermionic generators.

by considering graded Lie algebra and fermionic generators. Together with Poincaré generators these make up the Supersymmetry algebra.

A **graded Lie algebra** of grade 1

$$L = L_0 \oplus L_1 \tag{3.2}$$

is defined to be a vector space just as an ordinary Lie algebra and it has the product defined by

$$[\quad , \quad \} : L \times L \to L \tag{3.3}$$

whether the product is a commutator or an anticommutator depends on the number of bosonic and fermionic generators. For operators O_a we would have

$$[O_a, O_b\} = O_a O_b - (-1)^{\eta_a \eta_b} O_b O_a \tag{3.4}$$

where $\eta_a = 1$ if it is a fermionic generator and $\eta_a = 0$ if it is bosonic. In supersymmetry L_0 is the Poincaré algebra and L_1 consists of a set of fermionic generators Q^I_α and $\bar{Q}^I_{\dot{\alpha}}$ were $I = 1, ..., \mathcal{N}$ is the number of generators. Together with the Poincaré generators they have the following commutation relations [3]

$$\begin{aligned}
&\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{IJ}, \quad \{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \\
&\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^*, \quad [P_\mu, Q^I_\alpha] = 0 \\
&[P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0, \quad [J_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta \\
&[J_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}}
\end{aligned} \tag{3.5}$$

where Z^{IJ} are called central charges (i.e. they commute with every other generator) and are antisymmetric $Z^{IJ} = -Z^{JI}$. One should mention the significance of the first commutation relation. Two supersymmetry transformations results in a translation, which implies that the theory is independent of the choice of coordinates which is typical for gravitational theories. If one considers local supersymmetry the result would be a theory of gravity called **supergravity** or SUGRA. Together with (2.21) these commutation relations make up the supersymmetry algebra. The values of Z^{IJ} puts constraints on an additional symmetry that leaves the algebra invariant called the **R-symmetry** which transforms the fermionic generators as

$$Q^I_\alpha \to R^I{}_J Q^J_\alpha, \quad \bar{Q}^I_{\dot{\alpha}} \to \bar{Q}_{J\dot{\alpha}} R^{\dagger J}{}_I. \tag{3.6}$$

If the central charges vanish then this symmetry group is $U(\mathcal{N})$. In other case the symmetry is a subgroup of $U(\mathcal{N})$.

3.2 Representations of supersymmetry algebra

The irreducible representations of the Poincaré group are what we call particles. The two operators that commute with all the generators of the algebra (a.k.a **Casimir operators**) are $P_\mu P^\mu$ and $W_\mu W^\mu$ where $W^\mu = 1/2\epsilon^{\mu\nu\rho\sigma} P_\mu J_{\rho\sigma}$. Hence the massive particles can be labelled by their mass and spin and massless by their energy and helicity.

Superparticles are irreducible representations of supersymmetry algebra where Poincaré is just a subalgebra. Here $P_\mu P^\mu$ is still a Casimir and superparticles have definite mass. However, particles within the same multiplet have different spin so $W_\mu W^\mu$ is no longer suitable to label representations.

A very important note that I would like to make is the language commonly used by physicists. Technically speaking a representation ρ of a group G is a homomorphism between an abstract group element and a general linear operator. i.e. $\rho : G \to GL(n, \mathbb{C})$. However, physicists are often sloppy and use imprecise language to describe states or fields being representations. They are not! These objects belong to a representation space or a module that the representation acts on. This is often a source of confusion in representation theory and I felt that it is worth pointing this out.

3.2.1 Massless representations

For massless representations it turns out that the central charges Z^{IJ} vanish [18]. So this slightly simplifies our commutation relations (3.5). Our goal is to figure out how the supersymmetry operators are realized and how they act on a representation space. So let's start by taking a particular frame in which a massless particle has a momentum $P_\mu = (E, 0, 0, E)$, then

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}_{\alpha\dot{\alpha}} \delta^{IJ}. \tag{3.7}$$

Looking at the first row and column entry of the matrix above and using the supersymmetry algebra we can write the following expression for generators Q^I_1 and $\bar{Q}^I_{\dot{1}}$

$$\langle\phi| \{Q^I_1, \bar{Q}^J_{\dot{1}}\} |\phi\rangle = 0. \tag{3.8}$$

But this is just a sum of two norms $||Q^I_1 |\phi\rangle ||^2$ and $||\bar{Q}^J_{\dot{1}} |\phi\rangle ||^2$. We know these are positive definite, hence

$$Q^I_1 = \bar{Q}^I_{\dot{1}} = 0. \tag{3.9}$$

Now from (3.7) we see that Q^I_2 and $\bar{Q}^I_{\dot{2}}$ generators must satisfy

$$\{Q^I_2, \bar{Q}^J_{\dot{2}}\} = 4E\delta^{IJ}, \quad \{Q^I_2, Q^J_2\} = 0, \quad \{\bar{Q}^I_{\dot{2}}, \bar{Q}^J_{\dot{2}}\} = 0. \tag{3.10}$$

These look like our familiar creation and annihilation operators except for the factor of $4E$. So we can define

$$a_I \equiv= \frac{1}{\sqrt{4E}} Q^I_2, \quad a^\dagger_I \equiv= \frac{1}{\sqrt{4E}} \bar{Q}^I_{\dot{2}} \tag{3.11}$$

to get rid of the unnecessary factor and then get the commutation relations for creation and annihilation operators

$$\{a_I, a^\dagger_J\} = \delta^{IJ}, \quad \{a_I, a_J\} = 0, \quad \{a^\dagger_I, a^\dagger_J\} = 0. \tag{3.12}$$

Now to build up the representation space we must define some sort of a vacuum state that is annihilated by the annihilation operator (commonly referred to as the **Clifford vacuum**) and then act on it with the creation operator to build up the rest of the spectrum. But what do these states represent? Well let's look at the following commutators

$$[J_{12}, a_I] = -\frac{1}{2} a_I, \quad [J_{12}, a^\dagger_I] = \frac{1}{2} a^\dagger_I. \tag{3.13}$$

So if we have a state $|p^\mu, \lambda\rangle$ of a massless particle with helicity λ, then $a_I |p^\mu, \lambda\rangle$ lowers the helicity by $\frac{1}{2}$ since

$$J_{12}(a_I |p^\mu, \lambda\rangle) = (a_I J_{12} + [J_{12}, a_I]) |p^\mu, \lambda\rangle = (\lambda - \frac{1}{2}) a_I |p^\mu, \lambda\rangle. \tag{3.14}$$

Similarly $J_{12}(a^\dagger_I |p^\mu, \lambda\rangle) = (\lambda + \frac{1}{2}) a^\dagger_I |p^\mu, \lambda\rangle$. So we see that

$$a_I |p^\mu, \lambda\rangle = |p^\mu, \lambda - 1/2\rangle, \quad a^\dagger_I |p^\mu, \lambda\rangle = |p^\mu, \lambda + 1/2\rangle. \tag{3.15}$$

To get the full representation space lets define a lowest helicity state $|\Omega\rangle$ such that $a_I |\Omega\rangle = 0$, then act on this state with $a^\dagger_I$ to reach maximum helicity. Since these creation and annihilation operators are fermionic $a^2_I = 0$ the number of states are finite. Also CPT invariance dictates that for every state we must also add a state of opposite helicity $|p^\mu, \pm\lambda\rangle$. For example for $\mathcal{N} = 1$ and a minimum helicity of $\lambda = 0$ we would get a $(0, \frac{1}{2})$ multiplet. We must also add the CPT conjugate $(-\frac{1}{2}, 0)$. Then this multiplet corresponds to one fermionic degree of freedom (Weyl fermion) and one complex scalar. Different multiplets can be constructed by considering different numbers of fermionic generators $\mathcal{N}$ and different minimum helicity states. More examples of different multiplets for $\mathcal{N} = 1$ is given in table 3.1.

Minimum helicity λ	Multiplet	Name
$\lambda = 0$	$\big(0, +\frac{1}{2}\big) \overset{\text{CPT}}{\oplus} \big(-\frac{1}{2}, 0\big)$	Chiral multiplet
$\lambda = \frac{1}{2}$	$\big(+\frac{1}{2}, +1\big) \overset{\text{CPT}}{\oplus} \big(-1, -\frac{1}{2}\big)$	Gauge multiplet
$\lambda = 1$	$\big(+1, +\frac{3}{2}\big) \overset{\text{CPT}}{\oplus} \big(-\frac{3}{2}, -1\big)$	Spin 3/2 multiplet
$\lambda = \frac{3}{2}$	$\big(+\frac{3}{2}, 2\big) \overset{\text{CPT}}{\oplus} \big(-2, -\frac{3}{2}\big)$	Graviton multiplet

Table 3.1: $\mathcal{N} = 1$ massless supersymmetry multiplets. Each containing two fermionic and two bosonic degrees of freedom.

3.2.2 Massive representations

In the massive case we have more work to do. This is because the central charges Z^{IJ} may not necessarily vanish and secondly because none of the supersymmetry generators vanish. Let us consider a massive particle with mass m and boost to its rest frame in which the momentum is $P^\mu = (m, 0, 0, 0)$. In this case we get the commutator

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{IJ} = \begin{pmatrix} 2m & 0 \\ 0 & 2m \end{pmatrix}_{\alpha\dot{\alpha}} \delta^{IJ}. \tag{3.16}$$

First let's look at the case where the central charges vanish $Z^{IJ} = 0$.

Vanishing central charges

We can define again creation and annihilation operators as before (only now we will have twice more)

$$a^I_\alpha \equiv= \frac{1}{\sqrt{4E}} Q^I_\alpha, \quad a^{I\,\dagger}_{\dot{\alpha}} \equiv= \frac{1}{\sqrt{4E}} \bar{Q}^I_{\dot{\alpha}}. \tag{3.17}$$

Recall that massive states are labeled by their mass m and spin j and a state $|m, j\rangle$ has degeneracy of $2j + 1$ since we can have states with projected spin taking values $j_3 = -j, ..., +j$ (where j_3 is the eigenvalue of the operator J_{12}). We have again a very similar story where we can lower and raise the spin of the particle by acting with creation and annihilation operators to build up the representation space. The commutation relations for these operators are as before

$$\{a^I_\alpha, a^{J\dagger}_{\dot{\alpha}}\} = \delta^{IJ}\delta_{\alpha\dot{\alpha}}, \quad \{a^I_\alpha, a^J_\beta\} = 0, \quad \{a^{I\,\dagger}_{\dot{\alpha}}, a^{J\dagger}_{\dot{\beta}}\} = 0. \tag{3.18}$$

We can again use the commutation relations for J_{12} to deduce that the action on a state with spin j_3 is (ignoring other labels)

$$a^I_1 |j_3\rangle = |j_3 - 1/2\rangle, \quad a^{I\dagger}_1 |j_3\rangle = |j_3 + 1/2\rangle \tag{3.19}$$

when $\alpha, \dot{\alpha} = 1$. However for $\alpha, \dot{\alpha} = 2$ we have

$$a^I_2 |j_3\rangle = |j_3 + 1/2\rangle, \quad a^{I\dagger}_2 |j_3\rangle = |j_3 - 1/2\rangle. \tag{3.20}$$

Again define a Clifford vacuum with some spin j such that

$$a^I_\alpha |\Omega\rangle = 0. \tag{3.21}$$

Acting on this state with the creation operators is analogous to adding two spin quantum systems of spin-j and spin-$\frac{1}{2}$ [18]. Looking at the j and j_3 labels of the Clifford vacuum $|\Omega\rangle = |j, j_3\rangle$ we have

$$\begin{aligned} a^{I\dagger}_1 |j, j_3\rangle &= c_1 |j + 1/2, j_3 + 1/2\rangle + c_2 |j - 1/2, j_3 + 1/2\rangle, \\ a^{I\dagger}_2 |j, j_3\rangle &= c_3 |j + 1/2, j_3 - 1/2\rangle + c_4 |j - 1/2, j_3 - 1/2\rangle. \end{aligned} \tag{3.22}$$

Where c_i are the Clebsch-Gordon coefficients. so we have a superposition of different total spins since $j \otimes \frac{1}{2} = (j - \frac{1}{2}) \oplus (j + \frac{1}{2})$. By acting with other raising operators we can build the rest of the representation space. Note that here we no longer have to add a CPT conjugate since these states are automatically CPT invariant because of the values of projected spin can take. For example $N = 1$ and $j = 0$ the multiplet we construct is $(-\frac{1}{2}, 0, 0, \frac{1}{2})$ with the degrees of freedom of one massive complex scalar and one massive Weyl fermion and is CPT invariant.

Non-vanishing central charges

Now consider the case where $Z^{IJ} \neq 0$. making use of the R-symmetry mentioned earlier it is possible to rotate the generators Q^I_α and $\bar{Q}^J_{\dot{\alpha}}$ so that the central charge matrix takes the following form [4]

$$Z^{IJ} = \begin{pmatrix} 0 & Z_1 & 0 & 0 & \cdots \\ -Z_1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & Z_2 & \\ 0 & 0 & -Z_2 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}. \tag{3.23}$$

Then we want to define the creation and annihilation operators in such a way that we get our familiar harmonic oscillator algebra. This is achieved with

$$\begin{aligned} a^1_\alpha &= \frac{1}{\sqrt{2}}(Q^1_\alpha + \epsilon_{\alpha\beta} Q^2_\beta{}^\dagger) \\ b^1_\alpha &= \frac{1}{\sqrt{2}}(Q^1_\alpha - \epsilon_{\alpha\beta} Q^2_\beta{}^\dagger) \\ a^2_\alpha &= \frac{1}{\sqrt{2}}(Q^3_\alpha + \epsilon_{\alpha\beta} Q^4_\beta{}^\dagger) \\ b^2_\alpha &= \frac{1}{\sqrt{2}}(Q^3_\alpha - \epsilon_{\alpha\beta} Q^4_\beta{}^\dagger) \\ &\vdots \end{aligned} \tag{3.24}$$

Then one can check that these indeed give us the desired commutation relations

$$\{a^i_\alpha, a^j_\beta{}^\dagger\} = (2m + Z_i)\delta_{ij}\delta_{\alpha\beta}, \quad \{b^i_\alpha, b^j_\beta{}^\dagger\} = (2m - Z_i)\delta_{ij}\delta_{\alpha\beta},$$
$$\{a^i_\alpha, b^j_\beta{}^\dagger\} = \{a^i_\alpha, a^j_\beta\} = \cdots = 0. \tag{3.25}$$

Given a particular Clifford vacuum state one can use these operators to construct the representation space. Note that in order to avoid negative norm states we must have $|Z_i| \leq 2m$. We can see that if some $|Z_i|$'s are equal to $2m$ then the corresponding multiplets would be shorter and are called **short multiplets**[2]. If its an equality for all i's then the multiplets are said to be **ultrashort.**

3.3 Supersymmetric field theory

In field theory we have fields $\Phi(x)$ that transforms in some irreducible representation of the Poincaré group. In this section we will try to generalize this to fields that transform under an irreducible representation of the super Poincaré group (Poincaré + supersymmetry). From these fields we want to construct an invariant Lagrangian $\mathcal{L}$ under supersymmetry transformations. We could start by doing something similar as before by defining a field ϕ that acts as a Clifford vacuum and demand that $[\bar{Q}_{\dot{\alpha}}, \phi] = 0$. Then build the rest of the representation space by acting on it with Q_α. However, constructing supersymmetric field theories this way is quite difficult in general. To greatly simplify things we can introduce the notion of **superspace** [1].

3.3.1 Superspace and superfields

In the Poincaré symmetry case we exponentiate the elements of the algebra to get the group element g

$$g = e^{i(\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} + a_\mu P^\mu)}. \tag{3.26}$$

Is there an analogous way of getting the supersymmetry group elements? For that we need to introduce an extension of the ordinary Minkowski coordinates with (for $\mathcal{N} = 1$ case) four anticommuting coordinates θ_α and $\bar{\theta}_{\dot{\alpha}}$. Then fields (or more precisely superfields) are functions of superspace, a total of eight variables $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. The supersymmetry group elements can be written as

$$g_{SUSY} = e^{i(\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} + a_\mu P^\mu + \theta^\alpha Q_\alpha + \bar{\theta}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}})}. \tag{3.27}$$

[2]Sometimes called BPS states after Bogomonlyi-Prasad-Sommerfeld. If n of the $|Z_i|$'s are equal to $2m$ then the multiplet is called $1/2^n$ BPS multiplet.

The new coordinates are fermionic and their commutation relations are trivial along with every other generator

$$\{\theta^\alpha, \theta^\beta\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}_{\dot{\beta}}\} = 0. \tag{3.28}$$

Then the anti-commutators between the Q's can be realized as a commutator

$$[Q\theta, \bar{Q}\bar{\theta}] = 2\theta\sigma^\mu\bar{\theta}P_\mu\delta^{IJ}. \tag{3.29}$$

Where $\theta Q \equiv \theta^\alpha Q_\alpha$. Anti-commuting numbers (also called Grassmann variables) such as the one we have introduced have interesting albeit weird properties that are explained in appendix B. We know that any terms of third order in θ or $\bar{\theta}$ will vanish $\theta_\alpha\theta_\beta\theta_\rho = 0$ since the indices only take the values $\alpha = 1, 2$. So we can write a general expansion of the superfield in terms of these coordinates [3]

$$Y(x,\theta,\bar{\theta}) = \varphi(x)+\theta\psi(x)+\bar{\theta}\bar{\chi}+\theta\theta m(x)+\bar{\theta}\bar{\theta}n(x)+\theta\sigma^\mu v_\mu(x)+\theta\theta\bar{\theta}\bar{\lambda}(x)+\theta\theta\bar{\theta}\rho(x)+\theta\theta\bar{\theta}\bar{\theta}d(x) \tag{3.30}$$

where $\varphi(x)$, $m(x)$, $n(x)$, $d(x)$ are scalars, $\psi(x)$, $\bar{\chi}(x)$, $\bar{\lambda}(x)$, $\rho(x)$ are Weyl spinors and $v_\mu(x)$ is a vector field. Under translations the adjoint action of the group on this field is

$$Y(x+\delta, \theta+\delta\theta, \bar{\theta}+\delta\bar{\theta}) = e^{-i(\epsilon\theta+\bar{\epsilon}\bar{\theta})}Y(x,\theta,\bar{\theta})e^{i(\epsilon\theta+\bar{\epsilon}\bar{\theta})}. \tag{3.31}$$

Note that we did include the spacetime translation operator $e^{-iP_\mu a^\mu}$. This is a consequence of the supersymmetry algebra (3.5). For two subsequent supersymmetry transformations we get a translation. The explicit form of the translations are

$$\delta\theta^\alpha = \epsilon^\alpha, \quad \delta\bar{\theta}^{\dot{\alpha}} = \bar{\epsilon}^{\dot{\alpha}}, \quad \delta x^\mu = i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}. \tag{3.32}$$

We can Taylor expand the field and get

$$Y(x+\delta, \theta+\delta\theta, \bar{\theta}+\delta\bar{\theta}) = Y(x,\theta,\bar{\theta}) + (\epsilon^\alpha\partial_\alpha + \bar{\epsilon}^{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}} + i(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta})\partial_\mu + ...)Y(x,\theta,\bar{\theta}). \tag{3.33}$$

On the other hand we can expand 3.31 and get

$$Y(x+\delta, \theta+\delta\theta, \bar{\theta}+\delta\bar{\theta}) = Y(x,\theta,\bar{\theta}) + -i\epsilon^\alpha[Q_\alpha, Y] + i\bar{\epsilon}^{\dot{\alpha}}[\bar{Q}_{\dot{\alpha}}, Y] + ... \tag{3.34}$$

We see by comparing (3.33) and (3.34) that the operators Q, $\bar{Q}$ can be represented as differential operators

$$\mathcal{Q}_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \tag{3.35}$$

$$\bar{\mathcal{Q}}_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu. \tag{3.36}$$

One can check that indeed these operators are consistent with the supersymmetry algebra (3.5).

3.3.2 Supersymmetric action

We can now construct a Lagrangian that is invariant under super Poincaré transformations (up to a total derivative). And using the formalism of superspace and superfields it is a relatively trouble free task. This is because of the properties of the Grassmann variables we have that the integration of a superfield over superspace is automatically invariant under supersymmetry transformations. Under these transformations the variation of the field is [3]

$$\delta_{\epsilon\bar{\epsilon}}Y(x,\theta,\bar{\theta}) = (\epsilon^\alpha\partial_\alpha + \bar{\epsilon}^{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}} + i\partial_\mu(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta}))Y. \tag{3.37}$$

The first two terms vanish after integration and we are only left with a total derivative. Hence the variation of the action is zero

$$\delta_{\epsilon\bar{\epsilon}}\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta}\delta_{\epsilon\bar{\epsilon}}Y(x,\theta,\bar{\theta}) = 0. \tag{3.38}$$

However, the general form of a superfield given by (3.30) does not transform in an irreducible representation of the group. Hence we need to put additional constraints on $Y(x,\theta,\bar{\theta})$ such that it remains a superfield and transforms in a nice way. Here are some of these fields

1. Chiral superfield Φ satisfying $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$. Where $\bar{\mathcal{D}}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - i\theta^{\beta}\sigma^{\mu}_{\beta\dot{\alpha}}\partial_{\mu}$.

2. Anti-chiral superfield $\bar{\Phi}$ satisfying $\mathcal{D}_{\alpha}\bar{\Phi} = 0$. Where $\mathcal{D}_{\alpha} = \partial_{\alpha} + i\sigma^{\mu}_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu}$.

3. Vector field V satisfying $V = V^{\dagger}$.

Then we can construct a Lagrangian using these definitions. For example the most general renormalizable theory that one can build from a set of (anti-) chiral superfields is [3]

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\Phi^i\bar{\Phi}_i + \int d^2\theta W(\phi^i) + \int d^2\bar{\theta}\bar{W}(\bar{\Phi}_i) \tag{3.39}$$

with W being a **superpotential** with an expression

$$W(\Phi^i) = a_i\Phi^i + \frac{1}{2}m_{ij}\Phi^i\Phi^j + \frac{1}{3}g_{ijk}\Phi^i\Phi^j\Phi^k. \tag{3.40}$$

For vector superfield let us first consider the case where $\mathcal{N} = 1$. Imposing the condition $V = V^{\dagger}$ would give us eight bosonic and eight fermionic degrees of freedom. However a massless vector multiplet should have only two of each as we discussed previously. So we have a freedom to choose a gauge to reduce the number of degrees of freedom[3]. The gauge we choose is called the Wess-Zumino gauge and the explicit expression for the vector superfield is

$$V_{WZ} = \theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x), \tag{3.41}$$

where v_{μ} is a vector field, λ is a Weyl fermion (gaugino) and D is an auxiliary field that we can later integrate out. For an abelian gauge group $U(1)$ the role of the field strength in this theory is played by

$$W_{\alpha} = -\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_{\alpha}V, \quad \bar{W}_{\dot{\alpha}} = \frac{1}{4}\mathcal{D}\mathcal{D}\bar{\mathcal{D}}_{\dot{\alpha}}V. \tag{3.42}$$

Since the algebra commutes with $\mathcal{D}$'s these are still superfields, in fact they are (anti-) chiral. For non-Abelian theories these are modified to

$$W_{\alpha} = -\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}(e^{-V}\mathcal{D}_{\alpha}e^{V}), \quad \bar{W}_{\dot{\alpha}} = \frac{1}{4}\mathcal{D}\mathcal{D}(e^{V}\bar{\mathcal{D}}_{\dot{\alpha}}e^{-V}). \tag{3.43}$$

Explicitly in $y^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\bar{\theta}$ coordinates

$$W_{\alpha} = -i\lambda_{\alpha}(y) + \theta_{\alpha}D(y) + i(\sigma^{\mu\nu}\theta)_{\alpha}F_{\mu\nu} + \theta\theta\sigma^{\mu}(\partial_{\mu}\bar{\lambda}_{\alpha}(y) - \frac{i}{2}[v_{\mu}, \bar{\lambda}_{\alpha}(y)]) \tag{3.44}$$

where $F_{\mu\nu} = \partial_{\mu}v_{\nu} - \partial_{\nu}v_{\mu} - \frac{i}{2}[v_{\nu}, v_{\mu}]$. Then the action of super Yang-Mills is

$$\mathcal{S} = \frac{1}{4g^2_{YM}}\int d^4x\Big(\int d^2\theta\,\mathrm{Tr}(W^{\alpha}W_{\alpha}) + \int d^2\bar{\theta}\,\mathrm{Tr}\big(\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\big)\Big) \tag{3.45}$$

This can be slightly simplified by introducing complex coupling $\tau = \frac{\vartheta}{2\pi} + i\frac{4\pi}{g^2_{YM}}$

$$\mathcal{S} = \frac{1}{8\pi^2}\int d^4x\,\mathrm{Im}\,\mathrm{Tr}\left(\tau\int d^2\theta\,\mathrm{Tr}\,W^{\alpha}W_{\alpha}\right) \tag{3.46}$$

3.3.3 $\mathcal{N} = 4$ Super Yang-Mills theory in $d = 4$

In particular we will be interested in the extension of (3.46) to $\mathcal{N} = 4$ supersymmetry. In four spacetime dimensions this is the maximally extended supersymmetric field theory that includes particles with spin ≤ 1 and thus, does not include gravity. The $\mathcal{N} = 4$ super Yang-Mills (SYM) action can be derived in two ways. One by using $\mathcal{N} = 1$ superspace formalism. Second by a dimensional reduction of $\mathcal{N} = 1$ SYM theory in 10 dimensions. In the former case we require three

[3]Choosing a gauge reduces the degrees of freedom to four bosonic and four fermionic, then going on shell reduces the number to two bosonic and two fermionic.

chiral superfields Φ_i $(i = 1, 2, 3)$ and one vector superfield V. Without going into too much details we can write down a particular action (see chpter 3.3 of [1] for more detailed explanation)

$$\mathcal{S} = \int d^4x \,\mathrm{Tr} \Bigg(\int d^2\theta d^2\bar{\theta} \Phi^{i\dagger} e^V \Phi_i e^{-V} + \frac{1}{8\pi^2} \mathrm{Im} \Big(\tau \int d^2\theta W^\alpha W_\alpha \Big) + \Big(i g_{YM} \frac{\sqrt{2}}{3!} \int d^2\theta \epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] + h.c. \Big) \Bigg). \tag{3.47}$$

Then writing this out in terms of component fields we get the action of $\mathcal{N} = 4$ super Yang-Mills in four spacetime dimensions with the Lagrangian

$$\mathcal{L} = \mathrm{Tr} \Big(- \frac{1}{2g_{YM}^2} F_{\mu\nu} F^{\mu\nu} + \frac{\vartheta}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - i \bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu \phi^i D^\mu \phi_i + g_{YM} \sum_{a,b,i} C_i^{ab} \lambda_a [\phi^i, \lambda_b] + g_{YM} \sum_{a,b,i} \bar{C}_{iab} \bar{\lambda}^a [\phi^i, \bar{\lambda}^b] + \frac{g_{YM}^2}{2} \sum_{i,j} [\psi^i, \phi^j]^2 \Big) \tag{3.48}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$, D_μ acts on the fields as $D_\mu \cdot = \partial_\mu \cdot + i[v_\mu, \cdot]$ and C_i^{ab} are the Clebsch-Gordon coefficients. This Lagrangian consists of a massless supersymmetry gauge multiplet that contains six real scalars ϕ_i $(i = 1, ..., 6)$, four Weyl fermions λ_α^a $(a = 1, ..., 4)$ and a gauge vector field v_μ. It is also invariant under supersymmetry transformations that act on these fields as [8]

$$\begin{aligned}
\delta_\epsilon \phi^i &= [\epsilon_a^\alpha Q_\alpha^a, \phi^i] = \epsilon_a^\alpha C^{iab} \lambda_{\alpha b} \\
\delta_\epsilon \lambda_{\beta b} &= [\epsilon_a^\alpha Q_\alpha^a, \lambda_{\beta b}] = \frac{1}{2} (F_{\mu\nu} + \tilde{F}^{\mu\nu}) \epsilon_{\alpha b} (\sigma^{\mu\nu})^\alpha{}_\beta + [\phi^i, \phi^j] \epsilon_{\beta a} (C_{ij})^a{}_b \\
\delta_\epsilon \bar{\lambda}_{\dot{\beta}}^b &= [\epsilon_a^\alpha Q_\alpha^a, \bar{\lambda}_{\dot{\beta}}^b] = C_i^{ab} \epsilon_a^\alpha \bar{\sigma}_{\alpha\dot{\beta}}^\mu D_\mu \phi^i \\
\delta_\epsilon v_\mu &= [\epsilon_a^\alpha Q_\alpha^a, v_\mu] = \epsilon_a^\alpha (\sigma_\mu)_\alpha{}^\beta \bar{\lambda}_{\dot{\beta}}^a
\end{aligned} \tag{3.49}$$

where $(C_{ij})^a{}_b$ are related to bilinears in Clifford Dirac matrices of $SO(6)_R \sim SU(4)_R$. This theory is also invariant under the R-symmetry group $SU(4)_R$ where the vector field is in a singlet representation, the fermions form a fundamental representation and the scalars are in the 6-dimensional anti-symmetric representation of the group. An important property of this theory is that it is scale invariant since the fields are massless and the coupling constant is dimensionless.

Next we can look at the properties of a theory that is invariant under both supersymmetry and conformal transformations.

3.4 Superconformal symmetry

We saw that the $\mathcal{N} = 4$ SYM theory is invariant under super Poincaré group. Not only that the theory is also scale invariant due to dimensionless coupling g_{YM}. The Poincaré and conformal symmetries combine to form the group $SO(4, 2) \sim SU(2, 2)$ as we already discussed. Is there a larger group that also includes the supersymmetric generators Q and $\bar{Q}$? Yes, but for that we need to introduce additional fermionic generators to close the algebra. Let us review what symmetries we have for SYM [8]:

- Conformal symmetry $SO(2, 4) \sim SU(2, 2)$. Of which Poincaré is a subgroup generated by Lorentz transformations, translations. Together with scale and special conformal transformations generated by D and K^μ respectively.
- Poincaré supersymmetry. Generated by fermionic generators Q_α^I and $\bar{Q}_{I\dot{\alpha}}$. These are fermionic superpartners of the translations generator P^μ.
- R-symmetry $SU(4)_R \sim SO(6)_R$. Generated by T^A, $A = 1, ..., 15$.

We include the conformal and supersymmetry into a one larger group **superconformal group** $SU(2, 2|4)$ (for $\mathcal{N} = 4$). But we need to introduce more fermionic generators. So the theory should also be invariant under

- Conformal supersymmetry. Generated by fermionic generators $S_{I\alpha}$ and $\bar{S}^I_{\dot{\alpha}}$ to ensure closure of the algebra. Similar to translation generator P^μ, these are the fermionic super partners of K^μ.

The superconformal algebra is defined by the following commutation relations (obviously including the commutators of (2.21) and (3.5)) [1]:

$$
\begin{aligned}
&[D, Q^I_\alpha] = -\frac{i}{2}Q^I_\alpha, \quad [D, \bar{Q}_{\dot{\alpha}}] = -\frac{i}{2}\bar{Q}_{\dot{\alpha}}, \quad [D, S_{I\alpha}] = \frac{i}{2}S_{I\alpha}, \\
&[D, \bar{S}^I_{\dot{\alpha}}] = \frac{i}{2}\bar{S}^I_{\dot{\alpha}}, \quad [Q^I_\alpha, K^\mu] = -i\sigma^\mu_{\alpha\dot{\alpha}}\bar{S}^{I\dot{\alpha}}, \quad [\bar{Q}_{I\dot{\alpha}}, K^\mu] = i\epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\beta}\alpha}S_{I\alpha}, \\
&[S_{I\alpha}, K^\mu] = 0, \quad [\bar{S}^I_{\dot{\alpha}}, K^\mu] = 0, \quad [S_{I\alpha}, P^\mu] = i\sigma^\mu_{\alpha\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{Q}_{I\dot{\beta}}, \\
&[\bar{S}^I_{\dot{\alpha}}, P^\mu] = -i\epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\beta}\alpha}Q^I_\alpha, \quad [S_{I\alpha}, J^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha{}^\beta S_{I\beta}, \\
&[\bar{S}^I_{\dot{\alpha}}, J^{\mu\nu}] = (\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\bar{S}^I_{\dot{\beta}}, \quad \{S_{I\alpha}, \bar{S}^J_{\dot{\beta}}\} = 2\delta^I_J(\sigma^\mu)_{\alpha\dot{\beta}}K_\mu, \\
&\{Q^I_\alpha, \bar{S}^J_{\dot{\beta}}\} = \{\bar{Q}_{I\dot{\alpha}}, S_{I\beta}\} = 0, \quad \{S_{I\alpha}, S_{J\beta}\} = \{\bar{S}^I_{\dot{\alpha}}, \bar{S}^J_{\dot{\beta}}\} = 0, \\
&\{Q^I_\alpha, S_{J\beta}\} = 2\epsilon_{\alpha\beta}\delta^I_J D - i(\sigma^{\mu\nu})_\alpha{}^\gamma\epsilon_{\gamma\beta}J_{\mu\nu}\delta^I_J - 4i\epsilon_{\alpha\beta}(\delta^I_J T + B^{iI}_J T^i), \\
&\{\bar{Q}_{I\dot{\alpha}}, S^J_{\dot{\beta}}\} = 2\epsilon_{\dot{\alpha}\dot{\beta}}\delta^J_I D - i(\sigma^{\mu\nu})^{\dot{\gamma}}{}_{\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\gamma}}J_{\mu\nu}\delta^J_I - 4i\epsilon_{\dot{\alpha}\dot{\beta}}(\delta^J_I T + B^{i\,J}_I T^i).
\end{aligned}
\tag{3.50}
$$

Where $B^{i\,J}_I$ are defined by the commutation relations between generators of the R-symmetry and the supersymmetry charges which we did not include here.

Next we need to consider the representations of this algebra.

3.4.1 Representations of the superconformal algebra

Recall from before that in conformal field theory we call a conformal operator primary if it is annihilated by the generator K_μ. Consider an operator $\mathcal{O}$ with a definite scaling dimension $[D, \mathcal{O}] = -i\Delta\mathcal{O}$. Now looking at the commutation relations we can deduce that the fermionic operators S and $\bar{S}$ lower the dimension Δ by $\frac{1}{2}$. This can be seen from the Jacobi identity

$$
\begin{aligned}
[D, [S_{I\alpha}, \mathcal{O}]] &= -[S_{I\alpha}, [\mathcal{O}, D]] - [\mathcal{O}, [D, S_{I\alpha}]] \\
&= -i(\Delta - \frac{1}{2})[S_{I\alpha}, \mathcal{O}].
\end{aligned}
\tag{3.51}
$$

We then define **superconformal primary operators** $\mathcal{O}$ to be the lowest dimension operators belonging to a superconformal multiplet of $\mathfrak{su}(2, 2|\mathcal{N})$ such that

$$[S^I_\alpha, \mathcal{O}\} = 0, \quad [\bar{S}_{I\dot{\alpha}}, \mathcal{O}\} = 0. \tag{3.52}$$

In these cases we use the commutator if the operator is bosonic and anti-commutator if it is fermionic. To construct descendant operators we can act with other operators from the superconformal algebra (3.50). Then these descendants together with the primary operator transform in an irreducible representation of the superconformal algebra.
Similarly to (3.51) one can show that the fermionic superpartners of P_μ raises the dimension of an operator by $\frac{1}{2}$. Descendants of this type are called **superdescendants** and are defined by

$$\mathcal{O}' = [Q, \mathcal{O}\}. \tag{3.53}$$

These operators are conformal primary operators since they are annihilated by K_μ

$$[K_\mu, [Q, \mathcal{O}\}] = 0. \tag{3.54}$$

Each superdescendant operator defines a conformal multiplet called **Verma module** and these modules are related by a supersymmetry transformation [1].

Let us return to the theory of interest - $\mathcal{N} = 4$ super Yang-Mills. How can we realise these operators in terms of fields of our theory? In a gauge theory we only consider gauge invariant local operators $\mathcal{O}(x)$. And we know that all the fields: scalars ϕ^i, Weyl fermions λ^a and the field strength constructed from vector field v_μ all transform covariantly. Hence we can construct gauge invariant operators by taking the trace. Important examples are **single-trace operators** obtained from the trace of the scalars

$$\mathcal{O}(x) = Str(\phi^{i_1}(x)...\phi^{i_n}(x)) = \mathrm{Tr}\Big(\phi^{(i_1}(x)...\phi^{i_n)}(x)\Big) \tag{3.55}$$

The Str stands for **symmetrised trace**. In four dimensions the dimension of the fields ϕ^i is one. Hence the dimension of the operator (3.55) is $\Delta = n$, i.e. the number of scalar fields. Finally, how do we label these operators? We need to find a maximally commutative bosonic subalgebra of $\mathfrak{su}(2,2|4)$. This is just the Lorentz algebra $\mathfrak{so}(1,3)$ so spin j is a good label, then we have dilation subgroup $\mathfrak{so}(1,1)$ with label Δ and we have the $\mathfrak{su}(4)_R$ R-symmetry subalgebra. These are labelled by something called Dynkin labels $[r_1, r_2, r_3]$ which determine the dimension of the representation [8].

Chapter 4

Anti-de Sitter spacetime

The theory of special and general relativity is formulated on Minkowski spacetime. This is a flat space and it is the simplest case one can consider. De-Sitter and anti-de Sitter spacetimes are closest relatives of Minkowski as they are the simplest non-flat spacetimes, since they have a constant curvature everywhere. There are a few ways to define anti-de Sitter spacetime, but I found the following way the most intuitive [17]. Consider a flat $d+2$-dimensional spacetime which has a metric that can be written as

$$ds^2 = -dX_0{}^2 + \sum_{i=1}^{d} dX_i^2 - dX_{d+1}^2. \tag{4.1}$$

We can embed a hypersurface in this spacetime defined by

$$X_0^2 - \sum_{i=1}^{d} X_i^2 - X_{d+1}^2 = -R^2 \tag{4.2}$$

where R is the radius of curvature of this surface. The result is the **anti-de Sitter** or **AdS** spacetime.

In this chapter we will briefly look at different ways of parametrizing this spacetime, see what symmetries it has and what is the the conformal boundary.

4.1 Parametrization

There are different coordinate systems for AdS spacetime. Let's look at a few that are most widely used. The **global coordinates** of a $d+1$-dimensional AdS is defined by [17]

$$\begin{aligned} X^0 &= R\cosh\rho\cos\tau \\ X^i &= R\Omega_i \sinh\rho \\ X^{d+1} &= R\cosh\rho\sin\tau \end{aligned} \tag{4.3}$$

where $i \in \{1, ..., d\}$, $\rho \in \mathbb{R}_+$, $\tau \in [0, 2\pi]$ and Ω_i is the usual parametrization of a $d-1$ sphere. In these coordinates the metric takes the form

$$ds^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}^2). \tag{4.4}$$

With the metric $d\Omega_{d-1}^2$ of a unit sphere S^{d-1}. The metric is independent of τ, so ∂_τ can be taken to be a time coordinate. However, since this coordinate is periodic with a period of 2π the spacetime has closed timelike curves. One can deal with this by letting $\tau \in \mathbb{R}$ and "unwrapping" the circle, thus obtaining something called a **universal cover** of AdS [1]. When we study the boundary of AdS it will be useful to introduce a new parameter by defining $\tan\theta = \sinh\rho$ which changes the metric to

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2). \tag{4.5}$$

Another useful parametrization is called the **Poincaré patch** with the following parametrization [17]

$$\begin{aligned}
X^0 &= \frac{R^2}{2r}\left(1+\frac{r^2}{R^4}\big(\vec{x}^2-t^2+R^2\big)\right)\\
X^i &= \frac{rx^i}{R} \qquad (4.6)\\
X^d &= \frac{R^2}{2r}\left(1+\frac{r^2}{R^4}\big(\vec{x}^2-t^2-R^2\big)\right)\\
X^{d+1} &= \frac{rt}{R} \qquad (4.7)
\end{aligned}$$

where $i \in \{1, ..., d-1\}$, $t \in \mathbb{R}$, $\vec{x} = \mathbb{R}^{d-1}$ and $r \in \mathbb{R}_+$. In these coordinates the metric takes the form

$$ds^2 = \frac{R^2}{r^2}dr^2 + \frac{r^2}{R^2}\big(-dt^2 + d\vec{x}^2\big). \qquad (4.8)$$

This parametrization only covers half of AdS since $z > 0$ and the boundary is now at $z = 0$ but we will talk more about that later. It is also useful to define $z = R^2/r$, in this case the metric is

$$ds^2 = \frac{R^2}{z^2}\big(dz^2 - dt^2 + d\vec{x}^2\big). \qquad (4.9)$$

It is clear that at fixed value of z the we get a flat Minkowski spacetime. Also an important property here is that this metric is invariant under a conformal transformation

$$(z, t, \vec{x}) \to \lambda(z, t, \vec{x}). \qquad (4.10)$$

For AdS we can show that the Ricci scalar is [1]

$$R = -\frac{d(d+1)}{R^2} \qquad (4.11)$$

and it satisfies the Einstein's vacuum field equations with a negative cosmological constant

$$\Lambda = -\frac{d(d-1)}{2R^2}. \qquad (4.12)$$

4.2 Symmetries

The symmetries of a spacetime can be calculated by an infinitesimal shift in coordinates in the direction of some vector field k^μ. I.e. $x^\mu \to x^\mu + \epsilon k^\mu$ which changes the metric as

$$g_{\mu\nu}(x) \to \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}g_{\rho\sigma}(x'). \qquad (4.13)$$

Expanding in orders of ϵ

$$\begin{aligned}
\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}g_{\rho\sigma}(x') &= (\delta^\rho_\mu + \epsilon k^\rho{}_{,\mu})(\delta^\sigma_\nu + \epsilon k^\sigma{}_{,\nu})(g_{\rho\sigma}(x) + \epsilon k^\gamma g_{\rho\sigma,\gamma}(x) + ...)\\
&= g_{\mu\nu}(x) + \epsilon(g_{\mu\sigma}(x)k^\sigma{}_{,\nu} + g_{\rho\nu}(x)k^\rho{}_{,\mu} + k^\gamma g_{\mu\nu,\gamma}(x)) + ...\\
&= g_{\mu\nu}(x) + \epsilon\mathcal{L}_k g_{\mu\nu}(x) + ... \qquad (4.14)
\end{aligned}$$

then demanding that the metric is left unchanged we get the **Killing equation**

$$\mathcal{L}_k g_{\mu\nu}(x) = 0. \qquad (4.15)$$

This equation gives the isometries of spacetime (i.e. coordinate transformations that leave distances unchanged) [7]. For example d-dimensional Minkowski spacetime has d translations and $d(d-1)/2$ rotations and boosts. A total of $d(d+1)/2$ isometries. It turns out that is the maximum number of isometries a spacetime can have. These spacetimes are called **maximally symmetric** and have constant curvature everywhere. Note that the hypersuraface defined by (4.2) is invariant under $SO(2, d)$[1] transformations acting on the coordinates. So a $d+1$-dimensional AdS has $(d+1)(d+2)/2$ Killing generators. Hence AdS is a maximally symmetric spacetime which can also be described as a coset space $SO(d, 2)/SO(d, 1)$ [1].

[1]This is the same symmetry group as a d-dimensional conformal group, which is an important fact for AdS/CFT correspondence.

4.3 Boundary of AdS

We can take the global AdS metric defined by (4.5) and multiply by a Weyl factor $\cos\theta/R^2$. This way we deform the spacetime, but still retain the causal structure. We can see that at the spacial infinity (boundary of AdS), when $\theta \to \pi/2$ we have a cylindrical conformal boundary $\mathbb{R}\times S^{d-1}$with a metric

$$ds^2 = -d\tau^2 + d\Omega_{d-1}^2. \tag{4.16}$$

Note that this is an important result since it tells us that the boundary is actually a Lorentzian spacetime. If it were Euclidean then a quantum field theory on the boundary would have to be defined in a null spacetime. Which is one of the reasons we have AdS in the AdS/CFT correspondence.

Now consider the Poincaré AdS metric defined by (4.9). The boundary is located at $z = 0$ since this corresponds to $\theta = \pi/2$. We can also Multiply the metric by a Weyl factor z^2/R^2. In this case the conformal boundary is a flat Minkowski spacetime

$$ds^2 = -dt^2 + d\vec{x}^2. \tag{4.17}$$

Notice that we get a different boundary then before. This is related to the fact that the Poincaré coordinates does not cover the whole manifold. In fact we can have infinitely many possibilities which are related to flat spacetime by a Weyl transformation. This can be done by approaching different points of the boundary at different rates [17].

Chapter 5

String theory

In 1960's physicists were trying to figure out a new model of strong interactions. Experimental data suggested that hadrons were not fundamental since their spin was proportional to mass squared of the particle. These are known as **Regge trajectories**. In 1968 Gabriele Veneziano proposed a model based on the S-matrix approach [37] which explained this behaviour. This marked the beginning of the string theory, since it led to the idea of hadrons behaving like strings. The first correct generalization of Gabriele's work was done by Nambu and Goto (independatly) that described relativistic string [32]. However, string theory had many problems as a theory of strong interactions and the development of QCD led to its temporary demise. One of the problems the theory had was an appearance of spin-2 particle. It took scientists until mid-1970's to consider the string theory to be a theory of quantum gravity and the spin-2 particle to be identified with the graviton [31]. At the moment string theory is extremely popular in the physics community because it brings much promise not just in physics but also gives interesting insights in mathematics.

I start this chapter with a general approach to the subject by first introducing bosonic string theory. After that I give a very short description on the quantization of the string before discussing superstring theory and the low-energy effective action - supergravity. Finally, I end up by talking about D-branes which is very relevant for the next chapter.

String theory is a vast subject, so I will focus on giving a greatly simplified introduction, often skipping many steps along the way. However one should note that the subject gets very involved very quickly. I have mainly followed [36] for the first half of this chapter and chapter 4 of [1] for the second half.

5.1 Bosonic string theory

Until the development of string theory, we viewed point particles as the fundamental objects. In string theory these objects are one dimensional strings. I believe it would be wise to start by remembering a few things about the treatment of point particles in general relativity and draw analogies to it when we look at strings later on. The action of a massive relativistic particle is given by

$$\mathcal{S} = -m \int d\tau \sqrt{-\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\eta_{\mu\nu}} \tag{5.1}$$

where the indices $\mu, \nu = 0, ..., d-1$ and τ is a parameter that labels points of a particle on the worldline[1] [7]. We can interpret this action as simply being the proper time along a worldline. From (5.1) we could say that the action has d degrees of freedom. However we know that is not true since time is not really a dynamical degree of freedom. Instead a particle must move in time. This is actually just a redundancy in the description (a gauge symmetry) and actually the action is invariant under reparametrization $\tau' = \tau'(\tau)$. So one of the coordinates is a fake degree of freedom. The reason why we write the action like this is because the Poincaré symmetry is manifest.

A string on the other hand traces out a **worldsheet** on spacetime and it is a $1+1$-dimensional sheet embedded in d-dimensional Minkowski space. So we will need another parameter to describe the shape of this surface. We choose the other parameter to be $\sigma \in [0, \sigma_0]$. For closed strings

[1] Worldline is a path that a particle traces in spacetime. Naively these look like long strings in a d-dimensional Minkowski space.

σ_0 is taken to be 2π. So the embedding of the surface on the target space is given by $X^\mu(\tau,\sigma)$. And the two coordinates are commonly denoted $\sigma^\alpha = (\tau,\sigma)$ with $\alpha \in \{1,2\}$. For convenience we define $f(\tau,\sigma) \equiv f(\sigma)$. How should the action of this theory look like? Firstly we would like it to be invariant under Poincaré. Secondly we want the action to be independent of our choice of parameters, i.e. it should be invariant under reparametrization. Looking back at the action for a particle we see that it is proportional to the length of the worldline. It turns out that the action for a string is proportional to the area of the worldsheet [36] and reads

$$\mathcal{S} = -T \int d^2\sigma \sqrt{-\det \gamma} \tag{5.2}$$

where

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu} \tag{5.3}$$

is the induced metric on the worldsheet. Or for people familiar with differential geometry it is the flat metric pull-back on the target space.The parameter T is actually the tension of the string[2]. The action (5.2) is called the **Nambu-Goto** action. This action has a square root in it and it makes quantization very difficult [36]. So we have an equivalent way of writing the action for a string:

$$\mathcal{S} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{5.4}$$

where we have exchanged the square root for an additional auxiliary field $h_{\alpha\beta}$ which is the metric of the worldsheet. Using equations of motion for this field we would obtain the on-shell action that is just the Nambu-Goto action (5.2). This is known as the **Polyakov** action who was the first one to use it to quantize the string. The symmetries of this action are not just the familiar Poincaré and reparametrization symmetries, but also there is a new symmetry called the **Weyl invariance** [41] with the following transformations

$$X^\mu(\sigma) \to X^\mu(\sigma), \quad h_{\alpha\beta}(\sigma) \to \Omega^2(\sigma)h_{\alpha\beta}(\sigma) = e^{2\omega(\sigma)}h_{\alpha\beta}(\sigma). \tag{5.5}$$

Which is just the scale invariance which preserve angles we've seen in chapter 2. Using these symmetries we can choose a convenient gauge called the **conforaml gauge**:

$$g_{\alpha\beta}(\sigma) = e^{2\omega(\sigma)}\eta_{\alpha\beta} \tag{5.6}$$

with $\eta_{\alpha\beta}$ being a $1+1$-dimensional Minkowski metric. In this gauge the Polyakov action simplifies to[36]

$$\mathcal{S} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X^\nu \eta_{\mu\nu}. \tag{5.7}$$

Varying this action we get the equations of motion for $X^\mu(\sigma)$

$$\partial_\alpha\partial^\alpha X^\mu(\sigma) = \partial_+\partial_- X^\mu(\sigma) = 0 \tag{5.8}$$

where we have introduced partial derivatives with respect to light cone coordinates $\sigma^\pm = \tau \pm \sigma$. These equations of motion are subject to boundary conditions and something called **Virasoro constraints** which are obtained as a consequence of the vanishing energy-momentum tensor of the worldsheet [1]. The boundary conditions simply arises from integration by parts when we vary the action. These of course should vanish. Explicitly we must satisfy the following condition

$$\partial_\sigma X^\mu \delta X_\mu|_0^{\sigma_0} = 0. \tag{5.9}$$

Considering for a moment open strings, there are two ways we can satisfy this condition:

1. Neumann boundary conditions
$$\partial_\sigma X^\mu(\sigma) = 0 \tag{5.10}$$
which is evaluated at the boundary $\sigma = 0$ and $\sigma = \sigma_0$ (for convenience we may take $\sigma_0 = \pi$).

[2]For historical reasons T is commonly expressed as $T = \frac{1}{2\pi\alpha'}$, where α' is the Regge slope that relates the spin of a particle with it's mass squared and is related to the string length by $\alpha' = l_s^2$.

2. Dirichlet boundary condition
$$\delta X^\mu(\sigma) = 0 \tag{5.11}$$
also evaluated at $\sigma = \{0, \pi\}$.

Intuitively, Neumann boundary condition allows the string to move freely (this condition must be imposed for $\mu = 0$), while the Dirichlet restricts the motion of the string in some directions. We can impose different boundary conditions for different coordinates for example if we have

$$\begin{aligned} &\partial_\sigma X^a(\sigma) = 0 \quad \text{for} \quad a = 0, ..., p \\ &X^i(\sigma) = \text{const.} \quad \text{for} \quad i = p+1, ..., d-1. \end{aligned} \tag{5.12}$$

Which is a mixture of both conditions, then the string end-points are constrained to live on on a $(p+1)$-dimensional hypersurface called **D-brane** (Dirichlet brane).

The Virasoro constraints arises when we calculate the energy-momentum tensor

$$T_{\alpha\beta} = \frac{4\pi\alpha'}{\sqrt{-h}} \frac{\delta \mathcal{S}}{\delta h^{\alpha\beta}}. \tag{5.13}$$

But this vanishes since we must satisfy the equations of motion for $h_{\alpha\beta}$ i.e. $\frac{\delta \mathcal{S}}{\delta h^{\alpha\beta}} = 0$. Explicitly in the light-cone coordinates these constraints are [1]

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0, \quad T_{--} = \partial_- X^\mu \partial_- X_\mu = 0, \quad T_{+-} = T_{-+} = 0. \tag{5.14}$$

5.1.1 Classical solutions

The solutions of the equations of motion (5.8) can be decomposed into two modes: left moving modes $X_L^\mu(\sigma_+)$ and right moving modes $X_R^\mu(\sigma_-)$ which are functions of σ_+ and σ_- respectively. Then the general solutions takes the form [1]

$$X^\mu(\sigma) = X_L^\mu(\sigma_+) + X_R^\mu(\sigma_-). \tag{5.15}$$

These modes have Fourier expansions

$$\begin{aligned} X_L^\mu(\sigma_+) &= \frac{\tilde{X}_0^\mu}{2} + \frac{\alpha'}{2}\tilde{p}^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in\sigma^+}, \\ X_R^\mu(\sigma_-) &= \frac{X_0^\mu}{2} + \frac{\alpha'}{2} p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\sigma^-}. \end{aligned} \tag{5.16}$$

Where the center of mass and the center of momentum is expressed by

$$X_C^\mu = \frac{\tilde{X}_0^\mu + X_0^\mu}{2}, \quad p_C^\mu = \frac{\tilde{p}^\mu + p^\mu}{2}. \tag{5.17}$$

Taking particular choices of these values we can satisfy the boundary conditions for the string. For a closed string, the solution has to be periodic with a period of σ_0 (which can be taken to be 2π) and the solutions must satisfy $p^\mu = \tilde{p}^\mu$. For open string we have a bit more freedom. We can have different boundary conditions for each end. For example if both ends satisfy Dirichlet boundary conditions then it is referred to as DD condition. If both are Neumann then it is NN (free moving string). We can have ND or DN as well.

The equations (5.16) can also be expanded in oscillator modes and Virasoro constraints put restrictions on the values of α_n^μ and $\tilde{\alpha}_n^\mu$.

5.1.2 String quantization

We mentioned earlier that this theory has a gauge symmetry. And quantizing a gauge theory gives rise to problems such as nonphysical states like we have in QED. Where we overcome these by fixing a gauge. Mainly by choosing to work in a particular gauge either use Gupta-Bleuler formalism in Lorentz gauge or only quantize the physical states of classical solutions in Coulomb gauge [25]. Similarly here we can use the diffeomorphism and Weyl symmetries to fix a gauge which lead us to Virasoro constraints [2].

Lets first start with defining the canonical momentum[3]

$$\Pi^\mu(\sigma) = \frac{\partial_\tau X^\mu(\sigma)}{2\pi\alpha'}. \tag{5.18}$$

Then we impose the canonical commutation relations

$$\begin{aligned} &[X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma'), \\ &[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = [\Pi^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')] = 0. \end{aligned} \tag{5.19}$$

From this we can deduce that the Fourier modes of the expansion (5.16) has the following non-vanishing commutation relations [1]. For open strings satisfying NN boundary conditions we have

$$[X_0^\mu, p^\mu] = i\eta^{\mu\nu}, \quad [\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0}. \tag{5.20}$$

The first one is just the commutation relation for the operators of position and momentum of center of mass. The other one looks like the familiar creation and annihilation commutation relations, we just need to tweak it a little by defining

$$a_n^\mu = \frac{1}{\sqrt{n}}\alpha_n^\mu, \quad a^{\dagger}{}_n^\mu = \frac{1}{\sqrt{n}}\alpha_{-n}^\mu \quad \forall n > 0. \tag{5.21}$$

Which yields $[a_n^\mu, a^{\dagger}{}_m^\nu] = \delta_{m,n}\eta^{\mu\nu}$. These give rise to the Fock space of harmonic oscillators[4]. However, there is a problem with this space. The commutator $[a_n^0, a^{\dagger}{}_m^0] = -1$ gives rise to negative norm states. But as discussed we use the Virasoro constraints to decouple these states from the theory. So ignoring the unphysical solutions we can define a vacuum that satisfies

$$p^\mu |0,k\rangle = k^\mu |0,k\rangle, \quad a_n^i |0,k\rangle = 0. \tag{5.22}$$

Where $i = 1, ..., d-2$[5]. Then we can build up the rest of the space by acting with creation operators. A general state then takes the form [1]

$$|N,k\rangle = \Big(\prod_{i=1}^{d-2}\prod_{n=1}^{\infty}\frac{(a_n^{i\,\dagger})^{N_{in}}}{\sqrt{N_{in}!}}\Big)|0,k\rangle \tag{5.23}$$

N_{in} is defined by

$$a_n^i a_n^{i\,\dagger} |N,k\rangle = N_{in} |N,k\rangle. \tag{5.24}$$

We can work out what is the mass of the state after doing a bit of work and using the ζ-function renormalization (which I will not include). This turns out to be

$$M^2 = \frac{1}{\alpha'}\Big(N + \frac{2-d}{24}\Big). \tag{5.25}$$

For the vacuum $N = 0$ this state has negative mass if $d > 2$, hence it is unstable. But this is solved in superstring theory. For the first excited state $N = 1$

$$M^2 = \frac{1}{\alpha'}\frac{d-26}{24}. \tag{5.26}$$

The state transforms under $SO(d-2)$ which implies that the state must be massless, hence the theory is only consistent in $d = 26$ dimensions [36].

For closed strings the only difference is that we would have two modes - left and right moving. We can define the other set of operators to be $\tilde{a}_n^i$ and $\tilde{a}_n^{i\,\dagger}$. Then a general closed string state can be expressed as [1]

$$\Big|N,\tilde{N},k\Big\rangle = \Big(\prod_{i=1}^{D-2}\prod_{n=1}^{\infty}\frac{(a_n^{i\,\dagger})^{N_{in}}(\tilde{a}_n^{i\,\dagger})^{N_{in}}}{\sqrt{N_{in}!\tilde{N}_{in}!}}\Big)|0,0,k\rangle. \tag{5.27}$$

Where the occupation numbers must satisfy $N = \tilde{N}$. Similarly here the mass of the first excited state is also tachyonic and for the first excited state we require $d = 26$ to get a $M^2 = 0$. These states are massless rank two tensors. The symmetric traceless part of this tensor is identified with the graviton. The scalar part is identified with the dilaton and the anti-symmetric part is the Kalb-Ramond field.

[3] The total momentum is then just the integral $p^\mu = \int_0^{\sigma_0} d\sigma \Pi^\mu(\sigma)$

[4] For a closed string there will be two spaces since we have two modes: left and right moving.

[5] The reason for this is because when quantizing we define light cone coordinates $X^\pm = X^0 \pm X^{d-2}$ to solve the Virasoro constrains and get rid of unphysical states. The dynamical degrees of freedom in terms of creation and annihilation operators are just a_m^i and $a_n^{i\,\dagger}$.

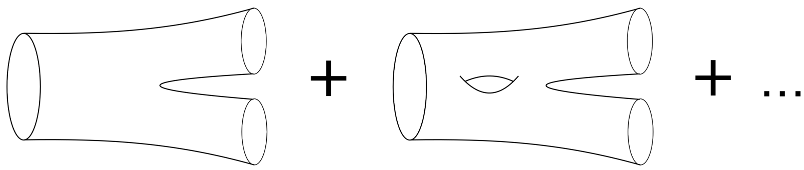

Figure 5.1: Lowest order string interaction expansion of a string splitting into two closed strings (from left to right).

5.1.3 Interactions

The discussion above was only for non-interacting strings with worldsheet topologies of a sheet or a cylinder. However in interacting theories the worldseets are more complicated. For example open strings can join and form a single string or the two endpoints of an open string can joint to make a closed string and so on. But it turns out that the story is actually similar to what we have in perturbation theory of QFT where we sum over different Feynmann diagrams with different numbers of interaction vertices. In string theory perturbative expansion we need to sum over different topologies of the worldsheet. For closed string the expansion is characterized by the number of "handles" or genus h which is related to Euler characteristic see figure 5.1. But to calculate these are extremely difficult so I will not talk about this in more detail since it is not very important for AdS/CFT. For a detailed explanation see chapter 6 of [36].

5.2 Superstring theory

The theory we considered so far has a few issues with it. One we already mentioned is that vacuum states have negative mass. The other issue we have not discussed is the fermionic degrees of freedom. So far there have been none. But fermions arises naturally if we introduce supersymmetry in the picture [2]. Supersymmetry not only gives us fermions, but also solves the tachyon problem. Here the spacetime coordinates (bosons[6]) $X^\mu(\sigma)$ are related by supersymmetry to their fermionic partners $\Psi^\mu(\sigma)$ which are two-component spinors

$$\Psi^\mu(\sigma) = \begin{pmatrix} \psi_-^\mu(\sigma) \\ \psi_+^\mu(\sigma) \end{pmatrix}. \tag{5.28}$$

In conformal gauge (5.6) the supersymmetric Polyakov action in d-dimensional flat spacetime $\eta_{\mu\nu}$ is given by [1]

$$\mathcal{S} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \eta^{\alpha\beta} (\partial_\alpha X^\mu \partial_\beta X^\nu + i\bar{\Psi}^\mu \gamma_\alpha \partial_\beta \Psi^\nu) g_{\mu\nu} \tag{5.29}$$

with γ_α being the Dirac matrices of the world sheet. The action (5.29) is invariant under the following supersymmetry transformations

$$\delta_\epsilon X^\mu = \bar{\epsilon}\Psi^\mu, \quad \delta_\epsilon \Psi^\mu = \gamma_\alpha \partial_\alpha X^\mu \epsilon. \tag{5.30}$$

Let's just look at the fermionic part of this action which can be rewritten in the lightcone coordinates $\sigma^\pm$ as

$$\mathcal{S}_f = \frac{i}{2\pi\alpha'} \int d^2\sigma (\psi_-^\mu \partial_+ \psi_{\mu-} + \psi_+^\mu \partial_- \psi_{\mu+}). \tag{5.31}$$

As before the equations of motion describe left and right-moving modes

$$\partial_\pm \psi_\mp^\mu = 0 \tag{5.32}$$

and we get boundary terms from integration by parts which should vanish

$$(\psi_-^\mu \delta\psi_{\mu-} - \psi_+^\mu \delta\psi_{\mu+})|_0^{\sigma_0} = 0. \tag{5.33}$$

First consider open strings:
To satisfy the boundary conditions for open strings we can choose $\psi_+^\mu(\tau, 0) = \psi_-^\mu(\tau, 0)$ and then we are left with two choices

$$\psi_+^\mu(\tau, \sigma_0) = \psi_-^\mu(\tau, \sigma_0) \tag{5.34}$$

[6]If we look at the Polyakov action (5.4), from the perspective of the worldsheet it just looks like d scalars coupled to $2d$ gravity.

or

$$\psi_+^\mu(\tau,\sigma_0) = -\psi_-^\mu(\tau,\sigma_0). \tag{5.35}$$

These boundary conditions correspond to two different sectors of the theory. Solutions that satisfy the (5.34) condition belong to **Neveu-Schwarz** (NS) sector. The ones that satisfy (5.35) belong to **Ramond** (R) sector [41]. We can Fourier expand the solutions that satisfy these boundary conditions. For Ramond sector the expansion reads

$$\psi_\pm^\mu(\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma_\pm} \tag{5.36}$$

and for Neveu-Schwarz sector we have

$$\psi_\pm^\mu(\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}-\frac{1}{2}} b_n^\mu e^{-in\sigma_\pm}. \tag{5.37}$$

The fermionic strings are then quantized by promoting the coefficients d_n^μ and b_n^μ to operators and imposing anti-commutation relations

$$\{d_n^\mu, d_m^\nu\} = \eta^{\mu\nu}\delta_{n,-m}, \quad \{b_n^\mu, b_m^\nu\} = \eta^{\mu\nu}\delta_{n,-m} \tag{5.38}$$

with the rest vanishing [1]. Again, one can see that the time component of these commutation relations gives rise to negative norm states. But again we can use (super-)Virasoro constraints to decouple those from the theory. Similarly as for the bosonic case we can define a vacuum state that is annihilated by the annihilation operator. Then we can build the rest of the representation space by acting on it with creation operators[7]. We can now calculate the mass spectrum of the states. The ground state of the NS sector is still tachyonic since $M^2 = -\frac{1}{2\alpha'}$. We can deal with this by consistently truncating the spectrum of the states based on the whether the state has an odd or an even number of creation operators applied on the vacuum. This prescription is called **Gliozzi, Schrek, Olive projection** (GSO). We only keep the odd or even number of operators applied to the vacuum and these are referred to as having positive or negative G-parity [41]. The first excited state transforms under the group $SO(d-2)$ as a vector, so again this implies that the state must be massless. Because the mass for this state is given by

$$M^2 = \frac{1}{\alpha'}\Big(\frac{1}{2} - \frac{d-2}{16}\Big) \tag{5.39}$$

we require that $d = 10$ for superstring theory. In the R sector the ground and excited states transform as massless spinors. In general, open strings are classified by their representation of the $SO(8)$ group.

Now consider closed strings:
States of closed strings are constructed from left and right moving modes. Each mode can have two different choices for satisfying the boundary conditions. Hence, we have four different sectors: R-R, NS-NS, R-NS and NS-R. The former two sectors are spacetime bosons and the latter two are spacetime fermions. The lowest closed string states are obtained from two open string states and depend on their GSO projection. The theories that are relevant for AdS/CFT are the **type IIA** and **type IIB**. The sectors contained in each of these theories are [1]

- Type IIA: (NS+,NS+), (R+,NS+), (NS+,R-), (R+,R-).
- Type IIB: (NS+,NS+), (R+,MS+), (NS+,R+), (R+,R+).

Where the $\pm$ denotes the G-parity or the fermion number from the GSO projection. In 10 dimensions the ground state of NS+ sector transforms in the fundamental representation of SO(8) - $\mathbf{8_V}$. The R+ and R- ground states transforms in spinorial representations with different chirality $\mathbf{8}$ and $\mathbf{8'}$. The NS- ground state transforms under the singlet representation and is tachyonic.

Using Young tableau we can decompose the closed string sectors into irreducible representations of $SO(8)$. These are summarized in table 5.1. Where $\mathbf{28}$ represents a two-form, $\mathbf{56_t}$ is a three-form, $\mathbf{35_+}$ a four-form. The $\mathbf{35}$ is a symmetric rank two tensor with vanishing trace. And the $\mathbf{56}$ and $\mathbf{56'}$ are vector spinors that we identify with gravitinos, the superpartners of a gravitons [8]. So the two theories consists of fields that transforms in these representations. For example the type IIB string theory contains the following representations

$$\mathbf{1}^2 \oplus \mathbf{28}^2 \oplus \mathbf{35} \oplus \mathbf{35}_+ \oplus \mathbf{8'}^2 \oplus \mathbf{56}^2 \tag{5.40}$$

[7] Creation operators are d_n^μ and b_m^μ with $n, m < 0$

Sector	$SO(8)$ representations
NS+ $\otimes$ NS+	$\mathbf{8_V} \otimes \mathbf{8_V} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$
NS+ $\otimes$ R-	$\mathbf{8_V} \otimes \mathbf{8'} = \mathbf{8} \oplus \mathbf{56'}$
NS+ $\otimes$ R+	$\mathbf{8_V} \otimes \mathbf{8} = \mathbf{8} \oplus \mathbf{56}$
R+ $\otimes$ R-	$\mathbf{8} \otimes \mathbf{8'} = \mathbf{8_V} \oplus \mathbf{56_t}$
R+ $\otimes$ R+	$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_+}$

Table 5.1: Irreducible representations of different sectors of superstring theory of closed strings [1].

5.2.1 Supergravity action

Lets look at low-energy action we can write down using these massless closed string states. This is the action of **supergravity**. One way of writing the action of supergravity is just considering the bosonic part of the full action[8]. Which reads [1]

$$\mathcal{S}_{IIB} = \frac{1}{2\tilde{\kappa}_{10}^2}\Big(\int d^{10}X\sqrt{-g}\big(e^{-2\phi}\big(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_{(3)}|^2\big) - \frac{1}{2}|F_{(1)}|^2 - \frac{1}{2}|\tilde{F}_{(3)}|^2 - \frac{1}{4}|\tilde{F}_{(5)}|^2\big) - \frac{1}{2}\int C_{(4)} \wedge H_{(3)} \wedge F_{(3)}\Big) \tag{5.41}$$

where we have

$$2\tilde{\kappa}_{10}^2 = (2\pi)^7\alpha', \tag{5.42}$$

$$\int d^{10}X\sqrt{-g}|F_{(p)}|^2 = \frac{1}{p!}\int d^{10}X\sqrt{-g}g_{\mu_1\nu_1}...g_{\mu_p\nu_p}\bar{F}^{\mu_1...\mu_p}F^{\nu_1...\nu_p} \tag{5.43}$$

and we have the following definitions for field strength tensors:

$$F_{(p)} = dC_{(p-1)}, \quad H_{(3)} = dB_{(2)}, \quad \tilde{F}_{(3)} = F_{(3)} - C_{(0)}H_{(3)},$$
$$\tilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge F_{(3)}. \tag{5.44}$$

The field content is summarized in table 5.2.

Field	$SO(8)$ representations	
$g_{\mu\nu}$	$\mathbf{35}$	Graviton
$C_{(0)} + ie^{-\phi}$	$\mathbf{1}^2$	Axion-dilaton
$B_{(2)}$, $C_{(2)}$	$\mathbf{28}^2$	Two-form
$C_{(4)}$	$\mathbf{35}_+$	Four-form
$\Psi_{I\alpha}^{\mu}$	$\mathbf{56}'^2$	Majorana-Weyl gravitinos
$\lambda_{I\alpha}$	$\mathbf{8}'^2$	Majorana-Weyl dilatinos

Table 5.2: Type IIB supergravity fields and their representations

5.3 *D*-branes

Let us return to the discussion of boundary conditions for open strings. Recall that if the endpoint of a string satisfies Neumann boundary conditions in a particular direction, then it is free to move in that direction. If it satisfies Dirichlet, then it is fixed in that spacial direction. Thus, the endpoints of open strings are constrained to move on hyersurfaces called Dp-branes. Where p stands for the spacial dimensions of the hypersurface. Note that this brakes Lorentz invariance $SO(1, d-1) \to SO(1, p) \times SO(d-p-1)$.

How should we look at these objects? It turns out that D-branes are themselves dynamical objects. And we should be able to write down the action for them. Which is just the generalized Nambu-Goto action [36]

$$\mathcal{S} = -T_p \int d^{p+1}\xi\sqrt{-\det\gamma_{ab}} \tag{5.45}$$

[8] Another way to construct supergravity action is to consider local supersymmetry.

where T_p is the tension of the brane and γ again is the pull back

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \eta_{\mu\nu} \tag{5.46}$$

and ξ^a $(a = 0, ..., p)$ are the coordinates of a $p+1$ dimensional worldvolume that the D-brane sweeps in spacetime. This is the **Dirac action** which describes the transverse fluctuations of the brane that gives rise to massless scalar fields on it φ^I, $I = p+1, ..., D-1$. But this is not the full story. Open strings are able to deform the D-brane which leads to gauge fields on the brane [24]. So the endpoints of a string are charged under these fields. However this is not present in (5.46). The more general form we can write down is the Dirac-Born-Infeld action[9] (DBI) which is the bosonic part of a Dp-brane action:

$$\mathcal{S}_{DBI} = -\tau_p \int d^{p+1} e^{-\phi} \xi \sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab})} \tag{5.47}$$

Where F_{ab} is the field strength of the $U(1)$ gauge field A_a that lives on the brane. We can consider constant dilaton field $e^\phi = g_s$, which can be identified with the string coupling constant. Then we see that it is related to the tension by $T_p = \frac{\tau_p}{g_s}$[10]. The above action includes the dynamics of the transverse fluctuations of the brane and the dynamics of the gauge field. Note that since $\mu, \nu = 0, ..., d-1$ it looks like we have d degrees of freedom. But we should only have $d-p-1$ physical degrees of freedom that correspond to transverse fluctuations. However, we can use the reparametrization invariance of (5.47) to eliminate the unnecessary degrees of freedom which actually correspond to longitudinal fluctuations [36]. We can expand the DBI action in powers of α'[11] to get

$$\mathcal{S}_{DBI} = -(2\pi\alpha')^2 \frac{\tau_p}{g_s} \int d^{p+1}\xi \left(\frac{1}{4} F^{ab} F_{ab} + \frac{1}{2} \partial_a \varphi^I \partial^a \varphi_I + ...\right) \tag{5.48}$$

with $\varphi^I = \frac{X^I}{2\pi\alpha'}$. This is just the Maxwell action coupled to scalar fields. If we focus on the first part of (5.48) which is the Yang-Mills theory with $U(1)$ gauge group we can read off the coupling constant

$$g_{YM}^2 = \frac{g_s}{\tau_p(2\pi\alpha')} \tag{5.49}$$

5.3.1 Coincident D-branes

Let us expand on the previous ideas and consider N coincident D-branes. Then strings are able to stretch from one D-brane to another. This is characterized by **Chan-Paton** factors λ_{ij} which label strings stretching from i'th brane to j'th brane. The factors make up a Hermitian matrix of $U(N)$ Lie algebra, where N is the number of coincident D-branes [1]. The gauge field can then be expressed as $(A_a)^i{}_j$ which describes a $U(N)$ gauge symmetry. The scalars $(\varphi^I)^i{}_j$ then transform in the adjoint representation of $U(N)$. We can write down the non-Abelian action that describes the dynamics of N D-branes [36]

$$\mathcal{S} = -(2\pi\alpha')^2 \frac{\tau_p}{g_s} \int d^{p+1}\xi \,\mathrm{Tr} \left(\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} \mathcal{D}_a \varphi^I \mathcal{D}^a \varphi_I - \frac{1}{4} \sum_{I \neq J} [\varphi^I, \varphi^J]^2\right) \tag{5.50}$$

with covariant derivative

$$\mathcal{D}_a \varphi^I = \partial_a \varphi^I + i[A_a, \varphi^I] \tag{5.51}$$

and the field strength tensor

$$F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b] \tag{5.52}$$

In Type II string theory not all Dp-branes are stable. We have that

- Type IIA has stable branes for even p's.
- Type IIB has stable branes for odd p's.

[9]The DBI action should also include the pullback of the bulk Kalb-Ramond field, but for simplicity we can assume it vanishes.

[10]$\tau_p = (2\pi)^{-p} \alpha'^{-(p+1)/2}$

[11]Using $\det(1+M) = 1 - \frac{1}{2}\mathrm{Tr}(M^2)$.

In flat spacetime Type IIB superstring theory is invariant under thirty-two supercharges in total. D-brane solutions are invariant under half of these supersymmetry generators. Theory with such number of generators is the maximally symmetric super Yang-Mills theory, where (5.50) is just the bosonic part of the action. The low-energy limit of $D3$-branes has one vector field A_μ, six scalars φ^I together with four Weyl fermions. The theory that describes N of these branes is the $\mathcal{N} = 4$ super Yang-Mills theory with a gauge group $U(N)$ [1].

Chapter 6

AdS/CFT correspondence

Finally, we have worked through the main topics required to understand the first concrete example of the AdS/CFT correspondence that was given by Maldacena in his original paper [19]. The example we will work on is the duality between $\mathcal{N} = 4$ super-Yang-Mills in 3+1 dimensions with gauge group $SU(N)$ and type IIB string theory on $AdS_5 \times S^5$ with the following matching of the constants

$$g_{YM}^2 = 2\pi g_s, \quad 2g_{YM}^2 N = \frac{L^4}{\alpha'^2} \tag{6.1}$$

where L is the radius of curvature on string side, g_s is the string coupling constant, α' is related to the string length by $l_s^2 = \alpha'$, and g_{YM} is the coupling constant on the gauge side.

The statement above is referred to as the **strongest form** of the correspondence, i.e. valid for any values of the constants. However, calculations for this form are very difficult to perform. So we will only focus on the **weak form** of the duality by taking some limits that will simplify things a great deal. Firstly, we will work in the limit $g_s \ll 1$ since currently perturbation is the best way to understand string theory. On the gauge side this corresponds to $g_{YM} \ll 1$. Gauge theories simplify when taken large N limit ($N \to \infty$) as was noted by Gerard 't Hooft in 1974 [34]. This is referred to as the **'t Hooft limit** and actually AdS/CFT is a realization of his insight that by expanding a field theory in powers of $\frac{1}{N}$ it can be mapped to topological expansion of string worldsheet with a coupling $g_s \propto \frac{1}{N}$. A free parameter on the gauge side is the 't Hooft coupling $\lambda = g_{YM}^2 N$ which in the weak form of the correspondence is taken to be large and corresponds to strongly coupled field theories. This implies that $\frac{\sqrt{\alpha'}}{L} \to 0$ and the string length becomes very small which can be approximated as a point particle. The resulting theory is type IIB supergravity on $AdS_5 \times S^5$. Even though we take these limits in the following discussion the correspondence is believed to hold for any values of these constants.

In this chapter we motivate the correspondence by looking at two different faces of D-branes. Then we reinforce the arguments by performing some checks. Explicitly we check that the symmetries and representations match on both sides and we calculate correlation functions and compare them to those obtained in chapter 2. For the basic outline of this part of the dissertation I referred to chapter 5 of [1]. For the section on symmetry I mainly used [8]. For the discussion on representation mapping I followed the discussion of [9].

6.1 Different perspectives on $D3$-branes

The AdS/CFT correspondence can be motivated by looking at $D3$-branes in different ways depending on the value of $g_s N$ [19]. In the low energy regime these are commonly referred to as **closed** or **open** string perspectives.

- Open string: This is the view we have used so far. In this perspective, the $D3$-branes are viewed as hyperplanes where open strings end. This perspective is valid for small coupling $g_s N \ll 1$. Where N is the number of coincident branes or the number of the gauge group $U(N)$.
- Closed string: Here $D3$-branes are viewed as solutions of supergravity (superstring theory in the limit of low energy). They are viewed as massive objects that curve spacetime. This perspective is valid for $g_s N \gg 1$.

6.1.1 Weak coupling

First lets look at the open string perspective of type IIB superstring theory with N coincident $D3$-branes embedded in flat ten-dimensional Minkowski spacetime. In perturbative regime the theory consists of open strings that corresponds to the excitations of $D3$-branes, and closed strings that are identified with excitations of the whole ten-dimensional space. In this low energy limit we are effectively considering massless excitations. The total action can be written as the sum of actions for closed and open strings and their interaction [1]

$$\mathcal{S} = \mathcal{S}_{closed} + \mathcal{S}_{open} + \mathcal{S}_{int}. \tag{6.2}$$

Writing the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{6.3}$$

where $h_{\mu\nu}$ is a perturbation of the metric and $2\kappa^2 = (2\pi)^7\alpha'^2 g_s^2$. Then the closed part of the action can be obtained by expanding the supergravity action. Schematically the action of the metric fluctuations to lowest order can be written as

$$\mathcal{S}_{closed} \sim -\frac{1}{2}\int d^{10}x\partial_\mu h\partial^\mu h + \mathcal{O}(\kappa). \tag{6.4}$$

The open string and interaction parts of the action (6.2) can be obtained by expanding the DBI action (5.47). The open part we already saw before

$$\mathcal{S}_{open} = -\frac{1}{2\pi g_s}\int d^{p+1}\xi\, \mathrm{Tr}\Big(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\mathcal{D}_a\varphi^I\mathcal{D}^a\varphi_I - \frac{1}{4}\sum_{I\neq J}[\varphi^I,\varphi^J]^2 + O(\alpha')\Big). \tag{6.5}$$

For the interaction part we also need to expand the dilaton field $e^{-\phi}$. Then to leading order in α' the interaction part is

$$\mathcal{S}_{int} = -\frac{1}{2\pi g_s}\int d^{p+1}\xi\, \mathrm{Tr}\big(\frac{1}{4}F_{ab}F^{ab}\phi + \dots\big). \tag{6.6}$$

Important thing to note is that $\mathcal{S}_{int}$[1] is of order κ since we would need to rescale the dilaton field for canonical normalization [1]. Now lets take the limit $\alpha' \to 0$ ($\implies$ $\kappa \to 0$)[2]. Then we see that the interaction part of full action vanishes (since it is of order κ) and the theory decouples. Closed part of the action describes ten-dimensional supergravity. And as we mentioned before the free part of the action is just the bosonic part of $\mathcal{N} = 4$ super-Yang-Mills action.

6.1.2 Strong coupling

Now let's look at the closed string perspective ($g_sN \gg 1$). Here the $D3$-branes are viewed as massive objects that curve spacetime. In type IIB theory the closed strings propagate in the curved geometry that is sourced by the branes. The metric that solves equations of motion of supergravity has the following form [1]

$$ds^2 = H(r)^{-\frac{1}{2}}\eta_{ab}dx^a dx^b + H(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) \tag{6.7}$$

where η_{ab} $a = 0, ..., 3$ is the metric along worldvolume of the $D3$-brane and $d\Omega_5^2$ is the metric of a unit 5-sphere S^5. Radial coordinate r is defined by $r^2 = \sum_{i=4}^{9} x_i^2$ and

$$H(r) = 1 + \frac{L^4}{r^4}. \tag{6.8}$$

The solution also comes with a self-dual five-form. Using the arguments from string theory that the flux of this five-form on S^5 is quantized [9], we can show that

$$L^4 = 4\pi g_s N\alpha'^2. \tag{6.9}$$

[1]The action implies in the lowest order non-trivial interaction a dilaton decays into to two bosons.

[2]More precisely we should really take the limit while holding $\frac{r}{\alpha'}$ constant. Where r is related to the position of the branes. This is called the **Maldacena limit**.

There are two limiting cases we can take: $r \gg L$ or $r \ll L$. In the former case $H(r) \sim 1$ so the metric (6.7) simplifies to a ten-dimensional Minkowski metric. In the other case $H(r) \sim \frac{L^4}{r^4}$ and the metric is

$$ds^2 = \frac{L^2}{z^2}(\eta_{ab}dx^a dx^b + dz^2) + L^2 d\Omega_5^2 \tag{6.10}$$

with $z = \frac{L^2}{r}$. The region $r \ll L$ is referred to as the **near-horizon** region or **throat**. The first term in (6.10) is just AdS_5 so the spacetime is $AdS_5 \times S^5$. We can show that taking the low energy limit the two theories decouple as before. On one side we have type IIB supergravity in ten-dimensions and on the other we have type IIB supergravity on $AdS_5 \times S^5$.

6.2 Maldacena's argument

In the previous section we looked at two perspectives on the $D3$-branes. In the low energy limit we saw that in both cases we get two decoupled effective theories. This is summarized in table 6.1. But this is just two different perspectives of the same physical theory. And since on both sides

Open string perspective $g_s N \ll 1$	Closed string perspective $g_s N \gg 1$
Type IIB supergravity in $\mathbb{R}^{9,1}$	Type IIB supergrnavity in $\mathbb{R}^{9,1}$
$\mathcal{N}=4$ super-Yang-Mills in $\mathbb{R}^{3,1}$	Type IIB supergravity in $AdS_5 \times S^5$

Table 6.1: Decoupled theories in open and closed string perspectives in low energy limit.

we have a type IIB supergravity in flat ten-dimensional spacetime, Maldacena proposed that the two remaining theories are dual to each other. This should be true even at higher energies. More precisely, the $\mathcal{N}=4$ super-Yang-Mills and type IIB string theory on $AdS_5 \times S^5$ are equivalent and describe the same physics from different points of view [9].

6.3 Checks of the correspondence

There is no rigorous proof of the correspondence and it will most likely remain this way for a while because we still lack the understanding of strongly coupled theories. However, it is widely believed that the conjecture holds regardless. Let us see what motivates us to believe that the correspondence should hold.

6.3.1 Symmetries

First, the most trivial thing we can ask is whether the symmetries of the two theories agree? We already discussed the symmetries of $\mathcal{N}=4$ super-Yang-Mills. The theory is invariant under supergroup $PSU(2,2|4)$. The bosonic subgroups of this supergroup are the conformal group $SO(4,2) \sim SU(2,2)$ and the $SO(6)_R \sim SU(4)_R$ R-symmetry group. The theory also has $\mathcal{N}=4$ supersymmetry generated by superchonformal and Poincaré supercharges S_α's and Q_α's which generate the fermionic subgroup of $PSU(2,2|4)$.
The isometry group of AdS_5 space is $SO(4,2)$ and for S^5 it is $SO(6)$. So we immediately see that the bosonic subgroups matches. The remaining fermionic symmetries also match because in type IIB string theory the $D3$-brane solutions preserves 16 Poincaré supercharges (it is a 1/2 BPS solution that preserves half of the Poincaré supercharges). Also in the AdS limit there are additional 16 conformal supersymmetries that are broken by the geometry of $D3$-branes [8]. Hence the symmetries of both theories agree as both sides are invariant uder the full supergoup $PSU(2,2|4)$.

6.3.2 Representation mapping

Since the symmetry groups of both theories match, we can talk about their representations. We should expect that the representations of the symmetry group should also coincide if the two theories are equivalent. Not only that there must be a map between the representations. This is commonly referred to as the **field-operator map** where the operators of super-Yang-Mills theory are identified with fields in type IIB supergravity on $AdS_5 \times S^5$ spacetime which transform in the same representation of the symmetry group. We already discussed the representations of super-Yang-Mills in chapter 3. The important class of operators are the single-traced operators since these correspond to single-particle states in AdS [8]. To find the irreducible representations on the gravity side we perform something called the **Kaluza-Klein** compactification of $AdS_5 \times S^5$ onto S^5. The fields $\varphi(\vec{x}, z, \Omega_5)$ living on the full space can be decomposed into spherical harmonics $Y_l(\Omega_5)$ living on S^5 and fields $\varphi_l(\vec{x}, z)$ living in AdS_5 spacetime. The decomposition reads

$$\varphi(\vec{x}, z, \Omega_5) = \sum_{l=0}^{\infty} \varphi_l(\vec{x}, z) Y_l(\Omega_5) \tag{6.11}$$

where $\vec{x}, z$ are the coordinates of AdS_5 and Ω_5 are the coordinates of S^5 [9]. The fields φ_l are Kaluza-Klein modes that have a dual operator in the gauge theory. For example, recall the expression for a single-trace operator

$$\mathcal{O}_\Delta(x) = \mathrm{STr}(\phi^{i_1}(x)...\phi^{i_\Delta}(x)). \tag{6.12}$$

This operator is dual to a scalar field $s^l(\vec{x}, z)$ that can be constructed from the metric and the five-form of the supergravity Kaluza-Klein modes [1]. These fields satisfy

$$\Box_{AdS_5} s^l(\vec{x}, z) = -\frac{1}{L^2} l(l-4) s^l(\vec{x}, z). \tag{6.13}$$

$\mathcal{O}_\Delta(x)$ and $s^l(\vec{x}, z)$ are in the same representation provided $l = \Delta$. We can insert the expansion (6.11) into the equations of motions to determine the relation between the mass and the scaling dimension. The relationship is summarized for various supergravity fields in table 6.2. Similarly

Field	Relation
Scalars	$m^2L^2 = \Delta(\Delta - 4)$
Spin 1/2, 3/2	$\lvert m\rvert L = \Delta - 2$
p-form	$m^2L^2 = (\Delta - p)(\Delta + p - 4)$
Massive spin 2	$m^2L^2 = \Delta(\Delta - 4)$
Massless spin 2	$m^2L^2 = 0$
Rank s symmetric traceless tensor	$m^2L^2 = (\Delta + s - 2)(\Delta - s - 2)$

Table 6.2: The relationship between mass and scaling dimensions of different field types [8]

the map exists not just for primary operator but also for their descendants.
Another important point to note is that the supergravity fields at the boundary can be interpreted as the source for the dual operators on the field theory side [1]. As an example consider again the action for a scalar field in AdS

$$\mathcal{S} = -\frac{C}{2} \int dz d^d x \sqrt{-g}(g^{mn}\partial_m \phi \partial_n \phi + m^2\phi^2) \tag{6.14}$$

with the metric expressed in Poincaré coordinates

$$g_{mn}dx^m dx^n = \frac{L^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu). \tag{6.15}$$

Choosing to decompose the field into plane wave modes $\phi(\vec{x}, z) = e^{ip^\mu x_\mu}\phi_p(z)$ we get two independent solutions for $\phi_p(z)$ at the boundary $z \to 0$. Which are $\phi_p(z) \sim z^{\Delta_+}$ and $\phi_p(z) \sim z^{\Delta_-}$. With $\Delta_\pm$ being the roots of the relation between mass of the field and the conformal dimension

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2}. \tag{6.16}$$

The solution z^{Δ_+} is normalizable, i.e. the action evaluated on the solution is finite. And the other solution is non-normalizable. The full solution near the boundary can be approximated as

$$\phi(\vec{x}, z) \sim \phi_{(0)} z^{\Delta_-} + \phi_{(+)} z^{\Delta_+} + ... \tag{6.17}$$

The normalizable mode $\phi_{(+)}$ is identified with the vacuum expectation value of the dual operator $\mathcal{O}_\Delta$ and the other mode $\phi_{(0)}$ is identified with the source of the dual operator [1].

6.3.3 Correlation functions

Having discussed the relationship between the fields and operators of both theories we can ask whether the correlation functions also agree on both sides. First let us recall how one can calculate correlations functions in quantum field theory for a field ϕ. A formal method in calculating correlators is to introduce an object called **generating functional**

$$Z[J] = \int \mathcal{D}\phi e^{iS[\phi]+i\int d^d x J(x)\phi(x)}. \tag{6.18}$$

Where we perturbed the action in the path integral by a source $J(x)$. We can functionally differentiate the generating functional with respect to source to obtain correlation functions [25]

$$\langle T\phi(x_1)...\phi(x_n)\rangle = \frac{1}{Z[0]}\frac{(-i)^n\delta^n Z[J]}{\delta J(x_1)...J(x_n)}\Big|_{J=0}. \tag{6.19}$$

However in this case we would generate all correlation functions (Connected+disconnected+vacuum bubbles). So we define

$$\frac{Z[J]}{Z[0]} = e^{iW[J]} = \langle e^{i\int d^d x J(x)\phi(x)}\rangle \tag{6.20}$$

where the $W[J]$ generates only the connected Feynman diagrams

$$\langle T\phi(x_1)...\phi(x_n)\rangle_{connected} = \frac{(-i)^n\delta^n W[J]}{\delta J(x_1)...J(x_n)}\Big|_{J=0}. \tag{6.21}$$

Recall from the previous section that the bulk field near the boundary $\phi_{(0)}$ has the interpretation of the source for a dual operator $\mathcal{O}$. So we can write the generating functional for the boundary quantum field theory as

$$\frac{Z_{CFT}[\phi_{(0)}]}{Z_{CFT}[0]} = e^{-W_{CFT}[\phi_{(0)}]} = \langle e^{\int d^d x \phi_{(0)}(x)\mathcal{O}(x)}\rangle_{CFT}. \tag{6.22}$$

Note that we defined (6.22) in Euclidean signature. The AdS/CFT correspondance precisely states that the generating functionals on both sides are equal

$$Z_{CFT}[\phi_{(0)}] = Z_{string}[\phi]\big|_{z\to 0}. \tag{6.23}$$

The generating functional on the gravity side can be approximated as a saddle point of the superstring partition function [1] which is given by

$$Z_{string}[\phi_{(0)}] \sim e^{-S_{SUGRA}}\big|_{z\to 0} \tag{6.24}$$

which implies that the generating functional of the connected correlation functions in four dimensions is identified with the supergravity action on AdS_5 where the fields are taken on the boundary $z \to 0$

$$W_{CFT}[\phi_{(0)}] = S_{SUGRA}[\phi]\big|_{z\to 0}. \tag{6.25}$$

We can calculate the correlation functions of the operators $\mathcal{O}_i$ by

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)...\mathcal{O}_n(x_n)\rangle = -\frac{\delta^n W[\phi^i_{(0)}]}{\delta\phi^1_{(0)}(x_1)\delta\phi^2_{(0)}(x_2)...\delta\phi^n_{(0)}(x_n)}\Big|_{\phi^i_{(0)}=0} \tag{6.26}$$

Which is equivalent to, as the correspondence states, to calculating tree level diagrams on the gravity side. There exists a set of rules to compute these diagrams, which are reminiscent of Feynman rules in QFT. The diagrams themselves are called the **Witten diagrams** and the rules to compute them can be summarized by the following points [1]

- The sources $\phi_{(0)}$ are represented by a circle (the boundary of AdS). And the AdS spacetime is the interior of that circle.
- **Boundary-to-boundary** propagator connects two points on the boundary of the circle.
- **Bulk-to-boundary** propagator connects points on the boundary of the circle with an interaction vertex in the bulk[3].

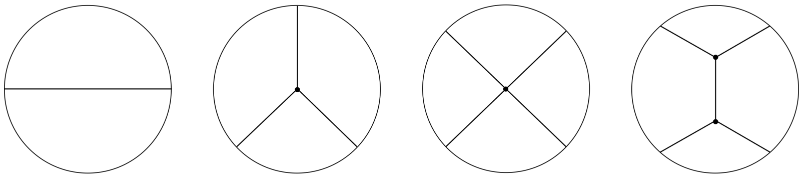

Figure 6.1: Examples of Witten diagrams. From left to right: two-point boundary-to boundary propagator; three-point bulk-to-boundary propagators; four-point bulk-to-boundary propagators; four-point bulk-to-boundary propagators and a bulk-to-bulk propagator.

- **Bulk-to-bulk** propagator connects two interaction vertices in the bulk.

Example diagrams are given in figure 6.1 The explicit expressions for the propagators of scalar field (dual to operator of dimension Δ) may be derived from operator $\Box_g - m^2$ on AdS [8]. The bulk-to-bulk propagator G_Δ is defined by

$$(\Box_g - m^2)G_\Delta(\vec{x}, z; \vec{y}, w) = \frac{\delta(z-w)\delta^d(\vec{x}-\vec{y})}{\sqrt{g}} \tag{6.27}$$

where $\vec{x}, \vec{y}$ denote the coordinates on the boundary and z, w denotes the coordinates of the bulk. The solution to (6.27) is rather messy and can be expressed in hypergiometric functions as

$$G_\Delta(\vec{x}, z; \vec{y}, w) = \frac{C_\Delta}{2^\Delta(2\Delta - d)}\xi^\Delta \cdot {}_2F_1\Big(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta - \frac{d}{2} + 1, \xi^2\Big) \tag{6.28}$$

where

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - \frac{d}{2})}, \quad \xi = \frac{2zw}{z^2 + w^2 + (\vec{x}+\vec{y})^2}. \tag{6.29}$$

The bulk-to-boundary propagator K_Δ is defined by taking the boundary limit of one of its coordinates

$$K(\vec{x}, z; \vec{y}) = \lim_{w\to 0} \frac{2\Delta - d}{w^\Delta} G_\Delta(\vec{x}, z; \vec{y}, w). \tag{6.30}$$

The explicit expression is

$$K(\vec{x}, z; \vec{y}) = C_\Delta\Big(\frac{z}{z^2 + (\vec{x}-\vec{y})^2}\Big)^\Delta. \tag{6.31}$$

Now the boundary-to-boundary propagator can be obtained from the boundary behaviour of the bulk-to-boundary propagator

$$B(\vec{x}; \vec{y}) = \lim_{z\to 0} z^{-\Delta} K_\Delta(\vec{x}, z; \vec{y}) \sim \frac{1}{(\vec{x}-\vec{y})^2\Delta}. \tag{6.32}$$

But the boundary-to-boundary propagator is just a two-point function of the operators in the field theory $\langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle$. We already calculated this two-point function back in chapter 2 and of course this has the same form as what we had in (2.43). The two-point function can also be computed using (6.26) or equivalently using the gravity action $\mathcal{S}$

$$\langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle = -\frac{\delta^2\mathcal{S}|_{z\to 0}}{\delta\phi_{(0)}(\vec{x})\delta\phi_{(0)}(\vec{y})}\Big|_{\phi_{(0)}=0}. \tag{6.33}$$

Since we do not care about interactions in this case we can write down the relevant part of $\mathcal{S}[\phi]$ in Euclidean signature as

$$\mathcal{S}[\phi] = \frac{C}{2}\int dz d^d x\sqrt{g}(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2). \tag{6.34}$$

We can integrate this action by parts and use the equations of motion to get the on-shell action which simplifies to

$$\mathcal{S}[\phi] = \frac{C}{2}\int dz d^d x\sqrt{g}(g^{zz}\phi(\vec{x}, z)\partial_z\phi(\vec{x}, z))\Big|_{z=\epsilon}^{z=\infty}. \tag{6.35}$$

[3]The interaction terms are given by the supergravity action.

The integral evaluated at infinity vanishes but it diverges on $z \to 0$ hence the reason why we evaluate at $z = \epsilon$. The field solution to ϕ can be written in terms of the source as [1]

$$\phi(\vec{x}, z) = \int d^d y K_\Delta(\vec{x}, z; \vec{y})\phi_{(0)}(\vec{y}) \tag{6.36}$$

then inserting this into (6.35) we get

$$\mathcal{S}[\phi] = -\frac{CL^{d-1}}{2\epsilon^{d-1}} \int d^d x d^d y d^d y' K_\Delta(\vec{x}, z; \vec{y})\partial_z K_\Delta(\vec{x}, z; \vec{y}')\phi_{(0)}(\vec{y})\phi_{(0)}(\vec{y}')\Big|_{z=\epsilon}. \tag{6.37}$$

Taking the derivative of the propagator, evaluating the result at $z \to 0$ and taking the double functional derivative as per (6.33) we get the expression for the two-point function

$$\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle = CL^{d-1}\frac{\Gamma(\Delta)}{\Gamma(\Delta - \frac{d}{2})}\frac{2\Delta - d}{\pi^{d/2}|\vec{x} - \vec{y}|^{2\Delta}}. \tag{6.38}$$

Which of course agrees with our previous results. Only here we used the supergravity action to compute the correlation function of a field theory.

We have only considered the weak form of the correspondence, however this can be extended for the the strongest form of the correspondence where we do not use the saddle point approximation of the partition function of string theory. However, we do not know the explicit form of Z_{string}. Presently it is not possible to give a rigorous proof of the AdS/CFT correspondence since we only understand quantum string theory using perturbation. Ragrdless of that, some very non trivial tests of the correspondence has been performed and the calculations on both sides always agrees [23].

Chapter 7

Conclusion

I have already mentioned in the introduction a few of the applications of the AdS/CFT correspondence. An excellent example is the study of confinement in QCD. Quarks behave as if they were connected by flux tubes and their energy scales with length. This flux can be described as a sting in a dual theory, which gives familiar characteristics of confinement in certain geometries [5]. Another popular area of physics where the correspondence is realized is the study of strongly coupled systems in condensed matter theory. this is useful in two ways. The first is obviously that we can do simpler calculations in a dual theory, which has weak coupling. The other reason is that experiments of such systems are very accessible, so we can design experiments to test certain properties of the correspondence [14]. For me the most interesting development of the AdS/CFT correspondence is the study of the connection between the entanglement entropy and the entropy of a black hole, which is the generalization of Bekenstein-Hawking formula for the entropy of a black hole. This was done by Ryu and Takayanagi [29]. Their work gives a very explicit connection between quantum entanglement and the geometry of the bulk space in one higher dimension. Given more time I believe that this is would be an interesting area to explore.

In this thesis I gave a basic introduction to AdS/CFT correspondence, which focus on a particular example. Namely the one Maldacena proposed in his paper [19]. We focused on the weak form of the duality which states that $\mathcal{N} = 4$ *super-Yang-Mills in 3+1 spacetime dimensions and gauge group* $SU(N)$ *is dual to type IIB supergravity on* $AdS_5 \times S^5$. By dual I mean that they are equivalent and describe the same physics but from different perspectives.

I took a general approach to arrive at this statement that is characteristic to many reviews on AdS/CFT [8][9][24][14][27]. For the majority of the thesis I followed [1]. This was supplemented by various other reviews on more specific subjects such as [3] for chapter on supersymmetry and [36] for string theory.

I started with a chapter on conformal symmetry where I extended the Poincaré symmetry group to include conformal transformations and introduced a field theory that is invariant under these transformations. In the next chapter another non-trivial extension of Poincaré was introduced by considering graded Lie algebra and fermionic generators which form the supersymmetry algebra. Then I defined the superconformal group that includes both of these extensions. In this chapter I went into slightly more detail by also introducing superspace and superfield formalism and briefly explained how one arrives at the $\mathcal{N} = 4$ super-Yang-Mills Lagrangian which is invariant under the superconformal group. Moving on to the gravity side, I first gave a short introduction on anti-de Sitter spacetime and various parametrizations and its symmetries. The symmetries gave a first hint of the correspondence since both AdS_{d+1} spacetime and the d-dimensional conformal group is invariant under $SO(d, 2)$ transformations. The introduction to AdS was useful for the next chapter where the goal was to arrive at the type IIB superstring theory and the low energy solution - supergravity. I felt that it was necessary to give the reader a vague idea on where it comes from, which included a very short review of the development of string theory in general. Furthermore, I wanted to give reasoning for a few popular aspects of string theory such as why we need so many dimensions and the need for supersymmetry. After I arrived at superstring theory I introduced a low-energy effective action of supergravity and talked about D-branes. The last part was important since it was used to motivate the example of AdS/CFT correspondence given above in the next chapter. This was done by looking at D-branes from two different perspectives. Finally, I made a few easy checks of the correspondence such as making sure that the symmetries on both sides agree, matching the representations (field-operator map), using the supergravity action to calculate the two point correlation function on field theory side and comparing to results obtained in chapter 2 on CFT's.

The duality is a major step forward in understanding quantum gravity and string theory. It is an important tool in studying quantum field theories in strong coupling regime where perturbation theory is no longer valid. The principles of this idea is applicable in many different fields such as condensed matter physics, nuclear physics or cosmology. Despite the fact that the duality remains a conjecture it is and increasingly popular field of study.

Appendix A

Spinor notation

Here I will give a short introduction on spinor notation that is used in supersymmetry. For more detailed introduction see [3]. Let us define some relationships between a few Lie groups that will help us deal with representations of the Lorentz group. First, the Lorentz group can be expressed as

$$SO(1,3) \sim SU(2) \times SU(2)^*. \tag{A.1}$$

Also Lorentz group can be expressed as a double cover of the group $SL(2,\mathbb{C})$, i.e.

$$SO(1,3) \sim SL(2,\mathbb{C})/\mathbb{Z}_2. \tag{A.2}$$

We can arrange the representation of the Lorentz group by the representation of the $SU(2)$ labelled by spins. But first let us define a spinor that carries a representation of $SL(2,\mathbb{C})$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{A.3}$$

With the following transformation property under $M \in SL(2,\mathbb{C})$

$$\psi_\alpha \longrightarrow M_\alpha{}^\beta \psi_\beta \tag{A.4}$$

where $\alpha, \beta \in \{1,2\}$. The complex conjugate representation is defined by

$$\bar{\psi}_{\dot{\alpha}} \longrightarrow M^*{}_{\dot{\alpha}}{}^{\dot{\beta}} \psi_{\dot{\beta}} \tag{A.5}$$

and $\dot{\alpha}, \dot{\beta} \in \{1,2\}$. These are known as **Weyl spinors**. Note that $M \neq CM^*C^{-1}$ for some C, so the fundamental and the conjugate representations are not equivalent. ψ and $\bar{\psi}$ both transform in an irreducible representation of the Lorentz group and they can be labeled by their $SU(2)$ transformations due to the isomorphism (A.1)

$$\psi_\alpha = \left(\frac{1}{2}, 0\right), \quad \text{Left-handed spinor.} \tag{A.6}$$

$$\bar{\psi}^{\dot{\alpha}} = \left(0, \frac{1}{2}\right), \quad \text{Right-handed spinor.} \tag{A.7}$$

The invariant $SU(2)$ tensors are

$$\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.8}$$

It is used to raise and lower indices of ψ

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta. \tag{A.9}$$

The invariant tensor of the conjugate representation is

$$\epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.10}$$

which is used to lower and raise indices of $\bar{\psi}$

$$\bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}, \quad \bar{\psi}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}. \tag{A.11}$$

The convention of contracting two indices are from upper left to lower left for undotted indices $\psi^{\alpha}\chi_{\alpha} \equiv \psi\chi$. For dotted it is the other way round $\bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \equiv \bar{\psi}\bar{\chi}$. Spinors are Grassmann (anti-commuting) so we have the following relations (note the convention of contraction of indices!)

$$\psi\chi = \chi\psi, \quad \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}. \tag{A.12}$$

We also have objects with mixed indices of $SO(3,1)$ and $SL(2,\mathbb{C})$

$$(\sigma^{\mu})_{\alpha\dot{\alpha}}, \quad \sigma^{\mu} = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \tag{A.13}$$

where the σ's are the Pauli matrices. These transform as

$$(\sigma^{\mu})_{\alpha\dot{\alpha}} \longrightarrow M_{\alpha}{}^{\beta}(\sigma^{\nu})_{\beta\dot{\beta}}(\Lambda^{-1})^{\mu}{}_{\nu}M^{*}{}_{\dot{\alpha}}{}^{\dot{\beta}} \tag{A.14}$$

where $\Lambda \in SO(3,1)$. This is used to define a product of a dotted and undotted spinor

$$\psi\sigma^{\mu}\bar{\chi} \equiv \psi^{\alpha}(\sigma^{\mu})_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}. \tag{A.15}$$

For a finite Lorentz transformation we define the generators of the group as

$$(\sigma^{\mu\nu})_{\alpha}{}^{\beta} = \frac{i}{4}(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu})_{\alpha}{}^{\beta}, \tag{A.16}$$

$$(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}} = \frac{i}{4}(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu})_{\dot{\alpha}}{}^{\dot{\beta}}. \tag{A.17}$$

Where $(\bar{\sigma}^{\mu})^{\alpha\dot{\alpha}} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^{\mu})_{\beta\dot{\beta}} = (\sigma_0, -\vec{\sigma})$. Then a finite Lorentz transformation is

$$\psi_{\alpha} \longrightarrow \left(e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}\right)_{\alpha}{}^{\beta}\psi_{\beta} \tag{A.18}$$

$$\bar{\psi}^{\dot{\alpha}} \longrightarrow \left(e^{-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}} \tag{A.19}$$

The **Dirac spinor** notation is also widely used so it is reasonable to mention that one can express a Dirac spinor as

$$\Psi = \begin{pmatrix} \psi_{\alpha} \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{A.20}$$

Which implies that Dirac spinors transforms in a reducible representation of the Lorentz group.

Appendix B

Grassmann variables

The simplest Grassmann number θ is defined such that

$$\theta^2 = 0. \tag{B.1}$$

This stems from the fact that they are anti-commuting variables $\{\theta, \theta\} = 0$. A function of a Grassmann number has the following expansion

$$f(\theta) = \sum_{n=0}^{\infty} f_n \theta^n = f_0 + f_1\theta + f_2\theta^2 + ... = f_0 + f_1\theta. \tag{B.2}$$

So it is linear. The derivatives with respect to θ are

$$\frac{\partial f}{\partial \theta} = f_1, \quad \frac{\partial^2 f}{\partial \theta^2} = 0. \tag{B.3}$$

We define the integral with respect to θ such that it preserves the following property

$$\int d\theta \frac{\partial f}{\partial \theta} = 0 \tag{B.4}$$

Then we define

$$\int \theta d\theta = 1 \tag{B.5}$$

which implies that θ acts as a delta function $\delta(\theta) = \theta$. Then we have

$$\int f(\theta) d\theta = f_1 \tag{B.6}$$

Hence we get that the integral of $f(\theta)$ is the same as the derivative

$$\int d\theta = \frac{\partial}{\partial \theta} \tag{B.7}$$

This can be expanded for multiple Grassmann numbers. For a more general discussion see for example [4].

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