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# Spheres, Generalised Geometry and Consistent Truncations

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# Abstract

We introduce generalised geometry in the form given through structure group $O(d,d)$. Then we present a study of consistent truncations on the $S^d$ sphere by employing the $GL^+(d+1,\mathbb{R})$ generalised geometry. We show how a generalised global frame algebra with constant coefficients naturally gives a notion of "generalised parallelisability" to the original manifold. Then a generalised Scherk-Schwarz reduction acts as a consistent truncation and encodes the scalar fields of the reduced theory. We prove that all spheres $S^d$ admit generalised parallelisations and as a consequence, all sphere compactifications, including the consistent truncations, can be viewed as generalised Scherk-Schwarz reductions. Finally we present an application and discussion on the special case of $S^3$.

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# Contents

# Chapter 1

# Introduction

Modern Theoretical Physics is based on the principle that at the core of every fundamental theory there are a priori acceptable symmetries. General Relativity, for example, the first modern theory of gravity, is constructed on the symmetry group of local diffeomorphisms on the four-dimensional space-time manifold. In order to understand the theory, one has to employ the appropriate mathematical framework to make these fundamental symmetries obvious.

Supergravity extends the theory of General Relativity with the requirement that the space-time manifold also preserves Supersymmetry. The properties of Supersymmetry in eleven dimensions for example dictates the content of the Supergravity field multiplet forcing it to include among others a two-form field $B_{\mu\nu}$ called the $B$-field, a scalar field $\phi$ called the dilaton and a $g_{\mu\nu}$ field called the graviton [26]. These compose a sector of the Supergravity called the Neveu-Schwarz or the NSNS sector. Some of these fields, as an example here the $B$-field, exhibit other types of symmetries, such as, in this case, gauge symmetries. In studying Supergravity, building an extension of our mathematical framework that includes both diffeomorphisms and gauge symmetries provides an improved avenue for understanding this fundamental theory better.

Such an extension is provided by Generalised Geometry. It is a new framework that expands the key notions of differential geometry to explore the generalised tangent space, defined as the direct sum of multi-vector tangent spaces and form-field tangent spaces. This allows one to define further generalised objects such as a generalised Lie derivative, generalised Lie bracket, generalised frame or generalised metric and use them in a coherent way that makes obvious the covariance. Beyond this however, Generalised Geometry can be utilised for even more purposes when it comes to Supergravity as will become apparent very shortly.

The main purpose of this work is to study *consistent truncations* of Supergravity theories on round $d$-spheres. A truncation of a Supergravity is a reduction of the theory that only keeps a finite subset of fields from the original un-truncated multiplet. A *consistent truncation* on the other hand is a reduction of the content of the theory with the condition that any solution of the truncated theory is also a solution of the un-truncated one. A consistent truncation hence preserves the same number of supersymmetries as the original un-truncated theory.

There are numerous documented situations of consistent truncations of Supergravity theories, for which [7] presents an appropriate review. A classic example is a local group manifold $M$ with a notion of parallelisability (i.e. it is parallelisable), given by the property that it admits a global frame $\hat{e}_a$ on $M$ and the Lie bracket of the frame satisfies

$$[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c$$

with constant coefficients $f_{ab}{}^c$. If additionally $f_{ab}{}^b = 0$, called the "unimodular condition" [22], then we have a consistent truncation on $M$ [29] [28]. And "if the theory is pure metric, the scalar fields in the truncated theory come from deformations of the internal metric" [22], such as for example through a Scherk-Schwarz reduction given by a rotation of the global frame [29] [28].

When it comes to spheres however, there is a small set of remarkable consistent truncations, apart from which the majority of cases of reductions are not consistent [22], and there is “no known algorithmic prescription” [7] to explain how the special ones arise.

This work will present how by employing Generalised Geometry on the $S^d$

sphere, specifically $GL^+(d+1,\mathbb{R})$ Generalised Geometry, one can arrive at a notion of *generalised parallelisability* by defining a generalised global frame $\{\hat{E}_A\}$ on the generalised tangent space which satisfies [22]

$$L_{\hat{E}_A}\hat{E}_B = X_{AB}{}^C\hat{E}_C \tag{1.1}$$

with constant $X_{AB}{}^C$, making $M$ an analogue of a local group manifold in Generalised Geometry terms [13]. Given the general parallelisation $\{\hat{E}_A\}$, we are lead to the conjecture based on analogy with the regular case that there is a consistent truncation on $M$ that keeps the same number of symmetries and the theory's scalar fields are given through a generalised version of Scherk-Schwarz reductions, i.e. rotations of the generalised frame. [22] We will hence show that "all spheres $S^d$ are generalised parallelisable" [22].

The thesis is structured as follows. The first chapter after the Introduction, Chapter 2, reviews key notions from differential geometry needed for the calculations, and then introduces key concepts from Bundle theory that are required to set up any generalised geometry framework. Chapter 3 introduces the $O(d,d)$ Generalised Geometry as a building block for more complicated geometries and all the necessary generalised objects (Dorfman derivative, Courant bracket etc, generalised metric) that will be used for the following cases as well. Chapter 4 introduces the $GL^+(d+1,\mathbb{R})$ Generalised Geometry and its conformal split frame as part of defining a generalised global frame. Chapter 5 sets up the notion of generalised parallelisability on the $S^d$ sphere and then proceeds to define generalised Scherk-Schwarz reductions and analyse the field content after consistent truncations from the standpoint of generalised geometry, proving our conjecture from the Introduction. Finally, Chapter 6 studies an application on the special case of $S^3$ also analysing the link with gauged Supergravity. In the end, we present the Conclusions (7).

# Chapter 2

# Fundamental Concepts for Generalised Geometry

In order to construct the Generalised geometry around any symmetry group, a number of key notions that conceptualize the mathematical framework we aim to generalize need to be defined. Once they are described, the next step is to illustrate the key components that extend them into the language of Generalised geometry. In this Chapter we will follow [23], [12], [18] and [20] to present the most important notions from Differential Forms, Riemannian Geometry and Bundle Theory that will be encountered in the following chapters when describing our Generalised geometries.

## 2.1 Key Notions on Differential Forms

In this section we will very briefly recall several key notions on forms from Differential Geometry [18], [23] that will be used extensively in all subsequent chapters.

**Definition 2.1.1** The *exterior product* of two forms $\lambda, \rho$ is defined using the Cartan wedge product on the basis covectors as

$$\begin{array}{cccc} \wedge : & \Lambda^{r_1}(M) \times \Lambda^{r_2}(M) & \to & \Lambda^{r_1+r_2}(M) \\ & \lambda, \qquad \rho & \to & \lambda \wedge \rho \end{array} \tag{2.1}$$

given by

$$\lambda \wedge \rho \equiv \frac{1}{r_1!r_2!}\lambda_{\alpha_1\dots\alpha_{r_1}}\rho_{\beta_1\dots\beta_{r_2}}dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_{r_1}}\wedge dx^{\beta_1}\wedge\cdots\wedge dx^{\beta_{r_2}} \tag{2.2}$$

The exterior product satisfies graded commutativity and associativity.

**Definition 2.1.2** The *exterior derivative* of a form field $\omega = \frac{1}{r!}\omega_{\alpha_1\dots\alpha_r}dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_r}$ is the natural notion of differentiation, denoted $d$:

$$\begin{aligned} d: \quad \Lambda^r(M) &\rightarrow \lambda^{r+1}(M) \\ \omega &\rightarrow d\omega \end{aligned} \tag{2.3}$$

where

$$d\omega = \frac{1}{r!}\left(\frac{\partial}{\partial x^\mu}\omega_{\alpha_1\dots\alpha_r}\right)dx^\mu\wedge dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_r} \tag{2.4}$$

$d$ satisfies the graded Leibnitz rule, with $\lambda\in\Lambda^{r_1}(M)$, $\rho\in\Lambda^{r_2}(M)$:

$$d(\lambda\wedge\rho) = (d\lambda)\wedge\rho + (-1)^{r_2}\lambda\wedge(d\rho) \tag{2.5}$$

and is nilpotent $d^2 = 0$ due to symmetry of the partial derivative. This is an expression of the principle that "the boundary of a boundary is zero", since the action of the exterior derivative $d$ corresponds to taking the boundary of the surface associated with a differential $r$-form.

**Definition 2.1.3** The *interior product* provided by a vector field $X$ is a natural map that reduces the degree of a form. Denoted $i_X$,

$$\begin{aligned} i_x: \quad \Lambda^r(M) &\rightarrow \Lambda^{r-1}(M) \\ \omega &\rightarrow i_X\omega \end{aligned} \tag{2.6}$$

it is given by

$$(i_X\omega) \equiv \omega(X) \tag{2.7}$$

or in coordinates,

$$i_X\omega = \frac{1}{(r-1)!}X^\nu \omega_{\nu\mu_1\ldots\mu_{r-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{r-1}} \tag{2.8}$$

From the definition it is also nilpotent $i_X^2 = 0$.

**Definition 2.1.4** Let $M$ be an $m$-dimensional manifold and $p$, $p' \in M$ two points. Given a vector field $V$, we may differentiate a vector field $U$ with respect to $V$ to obtain another vector field. This is given by the *Lie derivative*, denoted $\mathcal{L}_V$, defined as in [18] by

$$\mathcal{L}_V U \equiv \lim_{\epsilon \to 0} \frac{\sigma_V(-\epsilon)_* U|_{p'} - U|_p}{\epsilon} \in T_p M \tag{2.9}$$

where $\sigma_V(\epsilon) : M \to M$ is the integral curve through $p$ with parameter $\epsilon \in \mathbb{R}$ such that $p' = \sigma_V(\epsilon)p$.

Writing the vector field $U$ in a coordinate basis, using its transformation laws and noting the relation between the coordinates for the two points, $U$ can be pulled back from $p'$ to $p$. Then, Taylor expanding to linear order in $\epsilon$, it can be shown that the Lie derivative of a vector field is given by

$$\mathcal{L}_V U = (v^\beta \partial_\beta u^\alpha - u^\beta \partial_\beta v^\alpha)\partial_\alpha \tag{2.10}$$

In a similar manner it can be shown that the Lie derivative of a covector field is given by

$$\mathcal{L}_V \lambda = (v^\beta \partial_\beta \lambda^\alpha + \lambda^\beta \partial_\beta v^\alpha)\partial_\alpha \tag{2.11}$$

with $\lambda \in T_p^* M$.

The Lie derivative gives the **Lie bracket** [12] as

$$\mathcal{L}_V U = [V, U] \tag{2.12}$$

a bilinear, skew-symmetric map that also satisfies the Jacobi identity,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \tag{2.13}$$

It can also be shown that for *general forms*, the Lie derivative can be written [18] in terms of the interior product and exterior derivative as

$$\mathcal{L}_V \omega = (di_X + i_X d)\omega \tag{2.14}$$

where $\omega$ is a general $r$-form on $M$.

## 2.2 Key Notions from Riemannian Geometry

Riemannian geometry is the structural framework in which Einstein's theory of gravitation is formulated. Its core aspect is that a differential manifold $M$ can carry an additional structure if it is equipped with a globally defined metric tensor $g$, that will act as a natural generalization of the inner product between two vectors in $\mathbb{R}_n$ to an arbitrary manifold.

**Definition 2.2.1** Given a manifold $M$, A *Riemannian metric* is a (0, 2)-tensor field $g$ that is symmetric and positive, i.e. at any point $p \in M$, for any vectors $V, W \in T_pM$,

$$\begin{aligned} g(V, W) &= g(W, V) \\ g(V, V) &\geq 0 \end{aligned} \tag{2.15}$$

with equality in the second line only if $V = 0$.

A *Riemannian manifold* is then defined as the pair $(M, g)$ of differential manifold $M$ and metric $g$ [18].

**Definition 2.2.2** A *pseudo-Riemannian metric* is a symmetric (0, 2)-tensor field $g$, but where instead of being positive, the second line above is replaced by the property that at a point $p \in M$,

$$\text{if } g(V,W) = 0,\ \forall\, V \in T_pM \Rightarrow W = 0 \tag{2.16}$$

A *pseudo-Riemannian manifold* is then defined as the pair $(M, g)$ of differential manifold $M$ and metric $g$ with the weakened condition 2.16.

The metric tensor $g$ defines the ***non-degenerate inner product*** at the point $p \in M$,

$$\begin{aligned} g: \quad T_pM \times T_pM \quad &\to \quad \mathbb{R} \\ V, \quad W \quad &\to \quad g(V,W) \end{aligned} \tag{2.17}$$

as well as a ***canonical volume form*** $Vol_g$, provided that the manifold $M$ is orientable, given by

$$Vol_g \equiv \sqrt{|\det g_{\mu\nu}|}\, dx^1 \wedge \cdots \wedge dx^m \tag{2.18}$$

which, although is independent of the coordinate chart, is also a pseudo-tensor, as the sign of the volume element flips if it is defined on a manifold with the opposite orientation, hence the requirement for the manifold to be *orientable* [18].

**Definition 2.2.3** The *Hodge star* $\star_g : \Lambda^r M \to \Lambda^{m-r} M$ is a natural isomorphism between the space of $r$-forms and the space of $(m-r)$-forms, where $m$ is the dimension of $M$ and the map, induced by the metric $g$, is given for a general $r$-form $\omega = \frac{1}{r!}\omega_{\mu_1\dots\mu_r} dx^{\mu_1} \wedge \dots dx^{\mu_r}$ by

$$\star_g\, \omega = \frac{\sqrt{|g|}}{r!(m-r)!} \omega_{\mu_1\dots\mu_r} \epsilon^{\mu_1\dots\mu_r}{}_{\mu_{r+1}\dots\mu_m} dx^{\mu_{r+1}} \wedge \cdots \wedge dx^{\mu_m} \tag{2.19}$$

where $\epsilon^{\mu_1\cdots\mu_m}$ is the totally antisymmetric symbol, with components taking the values $\pm 1$. Note that by contracting the original $r$-form defined on a $r$-dimensional submanifold with the $\epsilon$ symbol, we are implicitly using the metric, hidden inside the mixed signature of $\epsilon$. This leads to a notion of "perpendicularity", making the $(m-r)$-form Hodge dual the local orthogonal complement of the initial $r$-form.

Under this mapping, the volume element can be written as $\star_g 1 = Vol_g$ and we can use the Hodge star to define an inner product on the vector space of $r$-forms

$$\begin{aligned} (\cdot,\cdot): \quad \Lambda^r M \times \Lambda^r M \quad &\to \quad \mathbb{R} \\ \lambda, \qquad \rho \quad &\to \quad (\lambda,\rho) \equiv \int_M \lambda \wedge (\star_g \rho) \end{aligned} \tag{2.20}$$

which is symmetric $(\lambda, \rho) = (\rho, \lambda)$ [18].

**Definition 2.2.4** An *affine connection* $\nabla$ is a bilinear map [18]

$$\begin{aligned} \nabla: \quad \Gamma(TM) \times \Gamma(TM) \quad &\to \quad \Gamma(TM) \\ U, \qquad V \quad &\to \quad \nabla_U V \end{aligned} \tag{2.21}$$

where we use the notation $\Gamma(TM)$ for the space of the section of tangent bundle $TM$ (see more on bundles in the next section). For any $U, V \in TM$ and for any $f \in \mathcal{F}(M)$, $\nabla$ must satisfy

$$\begin{aligned} \nabla_{fV} U &= f \nabla_V U \\ \nabla_V (fU) &= V[f]U + f \nabla_V U \end{aligned} \tag{2.22}$$

We can write the affine connection with respect to a basis vector field, due to its linearity, given by

$$\nabla_\mu e_\nu \equiv \Gamma_{\mu\nu}{}^\alpha e_\alpha \tag{2.23}$$

where the $\{\Gamma_{\mu\nu}{}^\alpha\} \subset \mathcal{F}(M)$ are the connection components and $\nabla_\mu := \nabla_{e_\mu}$.

## 2.3 Bundles

As the prerequisite mathematical framework from Differential Geometry has been established, we will now proceed to laying the foundation of Generalised Geometry by defining its building blocks. Following [23], [20] and [30], this section will present key concepts and objects from Bundle Theory that will be used to construct Generalised geometries in the next chapters.

### 2.3.1 Fibre Bundles

**Definition 2.3.1** A (differentiable) *fibre bundle* $(E, \pi, M, F, G)$ is the collection of the following elements [23]:

1. A differentiable manifold $E$ called the *total space* (sometimes simply referred to as 'the bundle'
2. A differentiable manifold $M$ called the *base space*
3. A surjective map $\pi : E \to M$ called the *projection*. In short, the bundle can be written as $E \xrightarrow{\pi} M$
4. A differentiable manifold $F$ called the *fibre*. For a $p \in M$, $F_p \equiv \pi^{-1}(\{p\})$ is the *fibre at* $p$, where $\pi^{-1}$ is the inverse image (preimage) of $\pi$. Then, $\forall p \in M$, $\pi^{-1}(\{p\}) \cong F$ (i.e. $F_p \cong F$)
5. A Lie Group $G$ called the *structure group* that acts on the fibre $F$ on the left. This is the group of transition functions
6. A set of open covering $\{U_i\}$ of $M$ with a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(u_i)$ such that $\pi \circ \phi_i(p, f) = p$, with $f \in F$, $u_i \in U_i$. The map $\phi_i$ is called the *local trivialization*
7. The maps $t_{ij} : F \to F$ on $U_i \cap U_j \neq \emptyset$, which we want to be elements of $G$, given by $t_{ij}(p) \equiv \phi_{i,p}^{-1} \circ \phi_{j,p}$, where we write $\phi_i(p, f) = \phi_{i,p}(f)$. Then $\phi_{i,p} : F \to F_p$ is a diffeomorphism, and $\phi_i$ and $\phi_j$ are related by the smooth map $t_{ij} : U_i \cap U_j \to G$ as $\phi_j(p, f) = \phi_i(p, t_{ij}(p)f)$. The $t_{i,j}$ maps are called the *transition functions* $\in G$

In order for the local pieces of the fibre bundle to be glued consistently, we also require the following conditions on the transition functions [30]:

- $t_{ii}(p) = \mathbb{1}, \; p \in U_i$
- $t_{ij}(p) = t_{ij}(p)^{-1}, \; p \in U_i \cap U_j$

- $t_{ik}(p) = t_{ij}(p)t_{jk}(p), \ p \in U_i \cap U_j \cap U_k$

**Definition 2.3.2** Given fibre bundle $E \xrightarrow{\pi} M$, a *section* of the bundle $\sigma : M \to E$ is a smooth map that satisfies $\pi \circ \sigma = \mathbb{1}_M$. The set of all sections is denoted $\Gamma(M, F)$ and it is clear that $\sigma(p) = \sigma|_p \in F_p = \pi^{-1}(p)$.

Intuitively, the fibre at the point $p \in M$ is a set of points in $E$ attached to the point $p$ and all the points in the fibre $F_p$ are sent to the point $p$ by the *projection map*. On the other hand, a *section* on the bundle is a map $\sigma$ that takes each point $p \in M$ to a certain point in its fibre $F_p$, while the projection map $\pi$ takes $\sigma(p) \in F_p \subseteq E$ back to point $p \in M$ [30].

**Definition 2.3.3** Given fibre bundle $E \xrightarrow{\pi} M$, a *subbundle* is a bundle $E' \xrightarrow{\pi'} M'$ if $E' \subseteq E, \ M' \subseteq M$ and $\pi' := \pi|_{E'}$.

**Definition 2.3.4** Let $E \xrightarrow{\pi} M$, $E' \xrightarrow{\pi'} M'$ be two fibre bundles. A *bundle map* is a smooth map $\tilde{f} : E' \to E$ that maps each fibre $F'_p$ of $E'$ onto $F_q$ of $E$. This means that $\tilde{f}$ naturally induces a smooth map $f : M' \to M$ such that $f(p) = q$ (see below) [20].

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{f} & M \end{array} \qquad \left( \begin{array}{ccc} u & \xrightarrow{\tilde{f}} & \tilde{f}(u) \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ p & \xrightarrow{f} & q \end{array} \right)$$

Note that $\tilde{f}$ is not necessarily a bundle map if it maps different parts of fibre $F'_p$ onto different fibres on $E$.

**Definition 2.3.5** Given two manifolds $M$ and $N$, the triple $(M \times N, \pi, M)$ is called a *product bundle* if:

$$\begin{aligned} \pi : M \times N &\to M \\ (p, q) &\to p \end{aligned} \tag{2.24}$$

and $\pi$ is a continuous and surjective map. A more extended description is provided in [23].

**Definition 2.3.6** Two bundles $E' \xrightarrow{\pi'} M$ and $E \xrightarrow{\pi} M$ are *equivalent bundles* if there exists a bundle map $\bar{f} : E' \to E$ such that $f : M \to M$ is the identity map and $\bar{f}$ is a diffeomorphism:

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\mathrm{id}_M} & M. \end{array}$$

such that the above diagram commutes.

**Definition 2.3.7** Given two bundles $E' \xrightarrow{\pi'} M'$ and $E \xrightarrow{\pi} M$ and two maps $u : E \to E'$ and $v : M \to M'$, the pair $(u, v)$ is a *bundle morphism* if $\pi' \circ u = v \circ \pi$:

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{v} & M' \end{array}$$

**Definition 2.3.8** Two bundles are *isomorphic* if there exist bundle morphisms $(u, v)$ and $(u^{-1}, v^{-1})$ satisfying:

$$\begin{array}{ccc} E & \underset{u^{-1}}{\overset{u}{\rightleftarrows}} & E' \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ M & \underset{v^{-1}}{\overset{v}{\rightleftarrows}} & M' \end{array}$$

Then $(u, v)$ is a *bundle isomorphism* and we write $E \xrightarrow{\pi} M \underset{bdl}{\cong} E' \xrightarrow{\pi'} M'$ [30].

**Definition 2.3.9** A *vector bundle* $E \xrightarrow{\pi} M$ is a fibre bundle whose fibre is a vector space. The tangent bundle is a vector bundle because according to the definition, its fibre is the vector space $\mathbb{R}^n$. The tangent bundle from Differential Geometry, which is defined as $T(M) \equiv \bigcup_{p \in M} T_p(M)$, can also be written as $(T(M), \pi, M, \mathbb{R}^m, GL(m, \mathbb{R}))$, with $\pi : T(M) \to M$ the projection map, differential manifold $M$ and structure group $GL(m, \mathbb{R})$.

**Definition 2.3.10** A fibre bundle $E \xrightarrow{\pi} M$ is *trivial* if its base space $M$ is contractible to a point.
Another definition: A bundle $E \xrightarrow{\pi} M$ is *trivial* if it is isomorphic to a product bundle.
A cylinder is an example of a trivial bundle (globally and locally), while the Möbius strip is an example of an only locally trivial bundle. [30]

### 2.3.2 Principal and Associated Bundles

This subsection will define several key notions related to Principal Bundles and then Associated Bundles. Speaking very broadly, a *principal fibre bundle* is a bundle whose typical fibre is a Lie group. [30] They are a key concept that will reappear in the following chapters when frame bundles and conformal split frames on the Generalised geometry will be introduced. In the current segment we will be following only [23] and [30]. The rest of the notions on principal bundles not defined here are expanded upon in Appendix A.

**Definition 2.3.11** Given a Lie group $G$, a *principal G-bundle* is a *smooth* bundle $(E, \pi, M)$ equipped with a *free* right $G$-action and

$$\begin{array}{ccc} E & & E \\ {\scriptstyle\pi}\big\downarrow & \cong_{\text{bdl}} & \big\downarrow{\scriptstyle\rho} \\ M & & E/G \end{array}$$

where $\rho$ is the quotient map, which sends each $p \in E$ to its equivalence class (i.e. orbit) in the orbit space $E/G$. The condition of *smoothness* for a bundle $(E, \pi, M)$ is just the requirement that $E$ and $M$ are smooth manifolds and the projection map $\pi : E \to M$ is smooth.

**Definition 2.3.12** An *associated fibre bundle* is a fibre bundle that is associated in a very precise manner to a principal $G$-bundle. [30] Given a principal $G$-bundle $(P, \pi, M)$ and a smooth manifold $F$ endowed with a left $G$-action $\triangleright$, we can define

1. $P_F := (P \times F)/\sim_G$, with the equivalence relation $\sim_G$ given by

$$(p,f) \sim_G (p',f') \quad :\Leftrightarrow \quad \exists g \in G \quad \text{s.t.} \quad p' = p \triangleleft g \;\; \text{and} \;\; f' = g^{-1} \triangleright f \tag{2.25}$$

   and we will write the points of $P_F$ as $[p,f]$.

2. The map $\pi_F$ given by

$$\begin{aligned} \pi_F : P_F &\to M \\ [p,f] &\to \pi(p) \end{aligned} \tag{2.26}$$

   If $[p',f'] = [p,f]$, for some $g \in G$ then

$$\pi_F([p',f']) = \pi_F([p \triangleleft g, g^{-1} \triangleright f]) := \pi(p \triangleleft g) = \pi(p) =: \pi_F([p,f]) \tag{2.27}$$

   and so the map $\pi_F$ is well defined. Then we say that the bundle $(P_F, \pi_F, M)$ is the *associated bundle* to $(P, \pi, M)$, $F$ and $\triangleright$.

Associated bundles are related to their principal bundles in a manner that gives the transformation law for components under a change of basis, which can be seen exemplified in Appendix B for the concept of *frame bundles*.

**Definition 2.3.13** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Given a principal $G$-bundle $P$ over a smooth manifold $M$ and the adjoint representation of $G$, $\text{Ad} : G \to \text{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})$, the *adjoint bundle* $\text{ad}P = P \times_{ad} \mathfrak{g}$ is the associated bundle to $P$ with fibre $\mathfrak{g}$ and representation Ad, where $\times_{ad}$ is the projection from the bundle onto the adjoint representation of $\mathfrak{g}$, $\text{Ad}_g$. [23]

In explicit form, elements of the adjoint bundle are equivalence classes of pairs $[p,x]$ for $p \in P$, $x \in \mathfrak{g}$ such that $[p \triangleleft g, x] = [p, \text{Ad}_g x]$, $\forall g \in G$. Because the structure group of the adjoint bundle is composed of Lie algebra automorphisms, then the fibers carry a Lie algebra structure that makes the adjoint bundle into a bundle of Lie algebras over $M$.

# Chapter 3

# $O(d,d)$ generalised Geometry

This chapter provides an introduction to generalised Geometry, covering the original $O(d,d)$ generalised geometry considered by Hitchin and Gualtieri [15] [14]. This particular geometry is later used in Chapter 6 for an application with consistent truncations on the sphere $S^3$. The current chapter starts with a closer inspection of the gauge symmetries of the $B$-field and the resulting patching rules for one-forms that are part of the generalised tangent bundle, introduced in the following section. The chapter continues with the formal description of the $O(d,d)$ generalised geometry and the differential structures that can be defined on the generalised space, along with their symmetries under $B$-field transformations. It concludes with the construction of a generalised Riemannian metric on the generalised tangent space.

## 3.1 $B$-field, Bosonic Symmetries and Patching Rules

The action that describes the bosonic NSNS sector of Type II Supergravity has extended symmetry beyond the local diffeomorphism invariance from General Relativity. As it will become apparent in the course of this chapter, the local gauge transformations of the two-form $B$-field that need to be included will give rise to a larger natural $O(d,d)$ structure on the generalised tangent space. [4] More precisely, the NSNS bosonic symmetry group has a fibre structure that infinitesimally combines both diffeomorphisms and the local gauge symmetry and such a transformation can be expressed as

$$\begin{aligned} \phi' &= \phi + \mathcal{L}_v \phi \\ g' &= g + \mathcal{L}_v g \\ B_i' &= B_i + \mathcal{L}_v B_i - d\lambda_i \end{aligned} \tag{3.1}$$

where $v$ is some vector from $\Gamma(TM)$, $g$ is the graviton field, scalar field $\phi$ is the dilaton and $\lambda_i$ is a one-form field from $\Gamma(T^*M)$. Note that in order to include the dilaton as well, the generalised space needs to be extended to $O(d,d) \times \mathbb{R}^+$, but for the purposes of this chapter we will be focusing only on $O(d,d)$ and the $B$-field. Also note that under the transformation for the $B$-field in 3.1, the action is invariant since it only depends on $H = dB$ and not on $B$ itself, and $d^2B = 0$ since $d$ is nilpotent. [4]

Similarly to the electromagnetic field, the $B$-field is only locally defined since the only requirement on $H$ is to be a closed form, and so in an overlap of coordinate patches $U_i \cap U_j \neq \emptyset$ the components of the $B$-field can be related by

$$B_i = B_j - d\Lambda_{ij} \tag{3.2}$$

Using 2.14, the Lie derivative of $d\Lambda_{ij}$ can be expressed as

$$\mathcal{L}_v d\Lambda_{ij} = (di_v + i_v d) d\Lambda_{ij} = di_v d\Lambda_{ij} \tag{3.3}$$

since the exterior derivative is nilpotent. Hence the $B$-field symmetry transformation from 3.1 can be re-written to obtain the **patching rules** for the components of the one-form $\lambda_i$ on the overlap

$$\begin{aligned} \mathcal{L}_v B_i - d\lambda_i &= \mathcal{L}_v (B_j - d\Lambda_{ij}) - d\lambda_i \\ &= \mathcal{L}_v B_j + di_v d\Lambda_{ij} - d\lambda_i \end{aligned} \tag{3.4}$$

Hence, one can write

$$d\lambda_j = d\lambda_i + di_v d\Lambda_{ij}$$
$$\lambda_i = \lambda_j - i_v d\Lambda_{ij} \tag{3.5}$$

This specific patching actually describes the components of the generalised tangent bundle $E$, which will be expanded upon in the next section,

$$V_i = v_i + \lambda_i \in \Gamma(TU_i \oplus T^* U_i) \tag{3.6}$$

for a section of $E$ on patch $U_i$, with

$$V_i = e^{d\Lambda_{ij}} V_j \tag{3.7}$$

on the overlap $U_i \cap U_j$. The one-forms $\Lambda_{ij}$ also satisfy [4]

$$\Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki} = d\Lambda_{ijk} \tag{3.8}$$

on the overlap $U_i \cap U_j \cap U_k$, making the $B$-field a "connection structure on a gerbe" [16] (i. e. "describing a hierarchy of successive gauge transformations on intersections" such as the one in 3.8) [3].

## 3.2 Generalised Tangent Bundle and the Natural Metric

As seen from 3.6, the **generalised tangent bundle** $E$ over the entire manifold $M$ is isomorphic to the direct sum between the tangent bundle $TM$ and the cotangent bundle $T^*M$, i.e.

$$E \simeq TM \oplus T^* M \tag{3.9}$$

and we can hence write a generalised vector $V \in \Gamma(E)$ as

$$V = \begin{pmatrix} v \\ \lambda \end{pmatrix} \tag{3.10}$$

or sometimes for simplicity $V = v + \lambda \in \Gamma(E)$, where $V$ is a $(d+d)$-column vector, with $v \in \Gamma(TM)$ and $\lambda \in \Gamma(T^*M)$. [13]

One can use the patching rules from 3.5, knowing that for two different patches $U_i$ and $U_j$, the relation between the vectorial components is $v_i = v_j$, to express the relation between the components of the entire generalised vector on the overlap of the two charts

$$\begin{pmatrix} v_i \\ \lambda_i \end{pmatrix} = \begin{pmatrix} v_j \\ \lambda_j - i_{v_j} d\Lambda_{ij} \end{pmatrix} \tag{3.11}$$

Furthermore, one can define the **natural metric tensor** by the matrix [4]

$$\eta = \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix} \tag{3.12}$$

and given two generalised vectors $V = v + \lambda \in \Gamma(E), U = u + \rho \in \Gamma(TM)$ use it to define a notion of **inner product of generalised vectors**

$$\begin{aligned} \langle \cdot , \cdot \rangle : E \times E \to \Lambda^{d-1} T^* M \\ (V, U) \to \langle V, U \rangle = V^T \eta U \end{aligned} \tag{3.13}$$

and after plugging in 3.11 and 3.12 one can see that the inner product is given by

$$\langle V, U \rangle \equiv \frac{1}{2} (\lambda(u) + \rho(v)) \tag{3.14}$$

The natural metric has $2d$ eigenvalues, with a number of $d$ eigenvalues being equal to $-\frac{1}{2}$ and $d$ eigenvalues equal to $\frac{1}{2}$, i.e. the metric signature is $(-1, 1)$. Thus the group of morphisms that preserves the inner product is isomorphic to the Lie group $O(d,d)$ and so $O(d,d)$ is the structure group of the generalised tangent bundle $E$.

One can note that the dual generalised tangent bundle is also defined, since the tangent and cotangent bundles are dual to each other, and it is given by $E^* = T^*M \oplus TM$. This means that given a generalised vector $V \in \Gamma(E)$, written in component form using generalised indices, the natural metric defines a *lowering map* $\eta_{MN} : E \to E^*$ given by

$$\eta_{MN} V^M = V_N \tag{3.15}$$

with $V_N \in \Gamma(E^*)$, and similarly the inverse natural metric gives a *raising map*.

Lastly, one can use the anti-symmetry of the interior product acting on forms, $i_v i_u \lambda = v^\mu u^\nu \lambda_{\mu\nu} = -v^\mu u^\nu \lambda_{\nu\mu} = -i_u i_v \lambda$, to check the independence of the inner product 3.14 on the coordinate system

$$\begin{aligned}
\langle V_i, U_i \rangle &= \frac{1}{2}(\lambda_i(u_i) + \rho_i(v_i)) \\
&= \frac{1}{2}(i_{u_i}\lambda_i + i_{v_i}\rho_i) \\
&= \frac{1}{2}(i_{u_j}(\lambda_j - i_{v_j} d\Lambda_{ij}) + i_{v_j}(\rho_j - i_{u_j} d\Lambda_{ij})) \\
&= \langle V_j, U_j \rangle - \frac{1}{2}(i_{u_j} i_{v_j} \Lambda_{ij} + i_{v_j} i_{u_j} \Lambda_{ij}) \\
&= \langle V_j, U_j \rangle
\end{aligned} \tag{3.16}$$

## 3.3 Dorfman Derivative and the Courant Bracket

Similarly to the notion of a Lie derivative from Differential Geometry as presented in 2.1.4, one can define a generalization that combines the action of infinitesimal diffeomorphisms generated by $v$ and local gauge transformations of the $B$-field, generated by $\lambda$ on the generalised tangent bundle, given a generalised vector $V = v + \lambda \in \Gamma(E)$ [4]. This indeed gives an operator $\mathcal{L}_V$ acting on another generalised vector $U = u + \rho \in \Gamma(E)$, called the **Dorfman derivative** or the generalised Lie derivative, and is defined as [13]

$$\mathcal{L}_V U \equiv \begin{pmatrix} \mathcal{L}_v u \\ \mathcal{L}_v \rho - i_u d\lambda \end{pmatrix} \tag{3.17}$$

Note that if the point $p$ in which generalised vector $U$ is considered is the origin (i.e. $U \in \Gamma(E_0)$), then $\mathcal{L}_V$ is simply the *Dorfman bracket* treated extensively in [9].

To be able to write $\mathcal{L}_V U$ in a $O(d,d)$ covariant manner, one can write its terms in a coordinate basis on a chart $\{U_i\}$ of manifold $M$. The vectorial components are then $v = v^\mu \partial_\mu$, $u = u^\nu \partial_\nu$, and we can use 2.10 to obtain the vectorial part (index $M \in [1,d]$) of the Dorfman derivative

$$\mathcal{L}_V U^M = \mathcal{L}_v u = (v^\beta \partial_\beta u^\alpha - u^\beta \partial_\beta v^\alpha)\partial_\alpha \tag{3.18}$$

One can similarly use 2.11 to write the form part of the Dorfman derivative (index $M \in [d,2d]$), with $\lambda = \lambda_\mu dx^\mu$, by first noting that

$$i_u d\lambda = i_u((\partial_\nu \lambda_\mu - \partial_\mu \lambda_\nu) dx^\nu \wedge dx^\mu) = u^\nu (\partial_\nu \lambda_\mu - \partial_\mu \lambda_\nu) dx^\mu \tag{3.19}$$

where 2.1.2 and 2.1.3 were used. Then one can immediately plug in 3.19 and 2.11 to rewrite the form part of the Dorfman derivative as

$$\mathcal{L}_V U^M = \mathcal{L}_v \rho - i_u d\lambda = (v^\nu \partial_\nu \rho_\mu + \rho_\nu \partial_\mu v^\nu) dx^\mu - u^\nu (\partial_\nu \lambda_\mu - \partial_\mu \lambda_\nu) dx^\mu \tag{3.20}$$

Collecting all terms in 3.20 and combining with 3.18, one can make use of the raising and lowering maps of the natural metric to raise or lower generalised indices to rewrite the Dorfman derivative using only the generalised vectors and a generalised partial derivative operator defined as [4]

$$\partial_M = \begin{cases} \partial_\mu, & M \in [1,d] \\ 0 & \text{otherwise} \end{cases} \tag{3.21}$$

Hence the Dorfman derivative can be written in terms of generalised objects (as in [31], [32], [17]):

$$\mathcal{L}_V U^M = V^N \partial_N U^M + (\partial^M V^N - \partial^N V^M) U_N \tag{3.22}$$

It can also be shown that the Dorfman derivative satisfies the Leibniz identity 3.23, giving $E$ the structure of a "Courant algebroid" [3]:

$$\mathcal{L}_U(\mathcal{L}_V W) - \mathcal{L}_V(\mathcal{L}_U W) = \mathcal{L}_{\mathcal{L}_U V} W \tag{3.23}$$

In a similar manner, one can define a generalised version of the Lie bracket given by 2.12, called the **Courant bracket** [6], which is a map $[\![\cdot,\cdot]\!] : E \times E \to E$ taking $V, U \in \Gamma(E)$ into another generalised vector $[\![V,U]\!] \in \Gamma(E)$ given by

$$[\![V,U]\!] \equiv \begin{pmatrix} [v,u] \\ \mathcal{L}_v \rho - \mathcal{L}_u \lambda - \frac{1}{2} d(i_v \rho - i_u \lambda) \end{pmatrix} \tag{3.24}$$

One can observe that using 3.17 one can write

$$\mathcal{L}_V U - \mathcal{L}_U V = \begin{pmatrix} \mathcal{L}_v u - \mathcal{L}_u v \\ \mathcal{L}_v \rho - i_u d\lambda - \mathcal{L}_u \lambda + i_v d\rho \end{pmatrix} \tag{3.25}$$

with $\mathcal{L}_v u - \mathcal{L}_u v = [v,u] - [u,v] = 2[v,u]$ due to the anti-symmetry of the Lie bracket, which looks closer to the form of the Courant bracket from 3.24. One can also use 2.14 to obtain for the interior product with respect to a vector of the exterior derivative of a one-form $\omega$, that is $i_v d\omega = \mathcal{L}_v \omega - d i_v \omega$. This can be plugged in to the second row of 3.25 to obtain

$$\begin{aligned} \mathcal{L}_v \rho - i_u d\lambda - \mathcal{L}_u \lambda + i_v d\rho &= \mathcal{L}_v \rho - \mathcal{L}_u \lambda - \mathcal{L}_u \lambda + d i_u \lambda + \mathcal{L}_v \rho - d i_v \rho \\ &= 2(\mathcal{L}_v \rho - \mathcal{L}_u \lambda - \frac{1}{2} d(i_v \rho - i_u \lambda)) \end{aligned} \tag{3.26}$$

Hence one can write that $\mathcal{L}_V U - \mathcal{L}_U V = 2[\![V,U]\!]$ or alternately that

$$[\![V,U]\!] = \frac{1}{2}(\mathcal{L}_V U - \mathcal{L}_U V) \tag{3.27}$$

i.e. *the Courant bracket is the antisymmetrisation of the Dorfman derivative*. In fact this is the *formal* definition of the Courant bracket and we will be using it from now on in the subsequent chapters when defining new generalised geometries.

By using both 3.27 and 3.22 the form of the Courant bracket in terms of generalised objects can also be obtained

$$[\![V,U]\!]^M = V^N\partial_N U^M - U^N\partial_N V^M - \frac{1}{2}(V_N\partial^M U^N - U_N\partial^M V^N) \tag{3.28}$$

and it can also be shown that $\mathcal{L}_{[\![U,V]\!]}W = \mathcal{L}_{\mathcal{L}_U V}W$, which is another form of the Leibniz identity 3.23. The term in round brackets in 3.28 impedes the Courant bracket from satisfying the Jacobi identity, which means that unlike the Lie bracket, the Courant bracket does not belong to a Lie algebra.

## 3.4 Symmetries of the Generalised Geometry Structures

This section will explore the decomposition of the Lie algebra related to the symmetry group that preserves the inner product 3.14 and will then study several important symmetries of the generalised geometry structures with respect to the diffeomorphisms and $B$-field transformations.

The connected subgroup of $O(d,d)$ for which we are interested in calculating the Lie algebra element is $SO(d,d)$ and it is given by

$$SO(d,d) = \{M \in GL(d,\mathbb{R}) \mid M^T\eta M = \eta,\ \det M = +1\} \tag{3.29}$$

The corresponding Lie algebra is $so(d,d)$ and thus $M \equiv e^{tX} \in SO(d,d)$, for $t \in \mathbb{R}$ and $X \in so(d,d)$. In order to get the algebra element, we identify the Lie algebra $so(d,d)$ with the tangent space $T_{\mathbb{1}}SO(d,d)$ at the identity $\mathbb{1}$. Then for any $X \in so(d,d)$ we can pick a curve $a : \mathbb{R} \to SO(d,d)$ such that its derivative at 0 is $a'(0) = X$. Since for the real parameter $q$, $a(q) \in SO(d,d)$, one can write

$$a(q)^T\eta a(q) = \eta \tag{3.30}$$

and differentiating with respect to $q$ gives

$$a'(q)^T \eta a(q) + a(q)^T \eta a'(q) = 0 \tag{3.31}$$

Evaluating at $q = 0$ gives

$$X^T \eta + \eta X = 0 \tag{3.32}$$

Now one can write the Lie algebra explicitly by working out the linear conditions determined by 3.32. Writing $X$ in generic matrix form $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, 3.32 can be written explicitly as

$$\begin{aligned} 0 &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ 0 &= \begin{pmatrix} C^T & A^T \\ D^T & B^T \end{pmatrix} + \begin{pmatrix} C & D \\ A & B \end{pmatrix} \end{aligned} \tag{3.33}$$

and hence $C^T = -C$, $B^T = -B$, $A^T = -D$ and so $A = -D^T$. This gives the final form of the Lie algebra element $X$

$$X = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix} \tag{3.34}$$

where $A \in \text{End}TM$ (i.e. $A : TM \to TM$), $B \in \Lambda^2 T^* M$ (a two-form) and $\beta \in \Lambda^2 TM$ (a bivector). [19] This gives the Lie algebra decomposition as

$$so(d,d) = \text{End}TM \oplus \Lambda^2 T^* M \oplus \Lambda^2 TM \tag{3.35}$$

with $\text{End}TM = TM \oplus T^* M$.

The transformation defined by $\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix}$ applied to generalised vector $V$ is the diffeomorphism given by

$$e^{\mathbf{A}}V = \begin{pmatrix} e^A & 0 \\ 0 & e^{-A^*} \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} e^A v \\ e^{-A^*}\lambda \end{pmatrix} \tag{3.36}$$

and one can easily check that such diffeomorphisms preserve the Courant bracket

$$e^{\mathbf{A}}[\![V,W]\!] = [\![e^{\mathbf{A}}V, e^{\mathbf{A}}W]\!] \tag{3.37}$$

In a similar way one can consider the $B$-field transformations $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$, for which it can be very easily checked that $\mathbf{B}^2 = 0$ and by expanding the exponentiation one obtains

$$e^{\mathbf{B}} = \mathbb{1}_{2d} + \mathbf{B} + \frac{1}{2}\mathbf{B}^2 + \cdots = \begin{pmatrix} \mathbb{1}_d & 0 \\ B & \mathbb{1}_d \end{pmatrix} \tag{3.38}$$

and the $B$-field transformation applied to generalised vector $V$ gives

$$e^{\mathbf{B}}V = \begin{pmatrix} \mathbb{1}_d & 0 \\ B & \mathbb{1}_d \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} v \\ B(v) + \lambda \end{pmatrix} \tag{3.39}$$

with $B(v) = i_v B$.

Now one can check if the Courant bracket is preserved by writing

$$\begin{aligned} [\![e^{\mathbf{B}}V, e^{\mathbf{B}}W]\!] &= [\![v + (\lambda + i_v B), w + (\rho + i_w B)]\!] \\ &= [v,w] + \mathcal{L}_v(\rho + i_w B) - \mathcal{L}_w(\lambda + i_v B) - \frac{1}{2}d(i_v(\rho + i_w B) - i_w(\lambda + i_v B)) \\ &= [v,w] + \mathcal{L}_v\rho - \mathcal{L}_w\lambda - \frac{1}{2}d(i_v\rho - i_w\lambda) + \mathcal{L}_v(i_w B) - \mathcal{L}_w(i_v B) + d(i_w i_v B) \\ &= [\![V,W]\!] + \mathcal{L}_v(i_w B) - \mathcal{L}_w(i_v B) + d(i_w i_v B) \end{aligned} \tag{3.40}$$

where the last line was obtained by noting the antisymmetry of $i_v i_w B$.

In order to continue we note that

$$\mathcal{L}_w(i_v B) = (di_w + i_w d)i_v B = d(i_w i_v B) + i_w d(i_v B) \tag{3.41}$$

and so 3.40 becomes

$$\begin{aligned} [\![e^{\mathbf{B}}V, e^{\mathbf{B}}W]\!] &= [\![V,W]\!] + \mathcal{L}_v(i_w B) - d(i_w i_v B) - i_w d(i_v B) + d(i_w i_v B) \\ &= [\![V,W]\!] + \mathcal{L}_v(i_w B) - i_w d(i_v B) \\ &= [\![V,W]\!] + \mathcal{L}_v(i_w B) - i_w(\mathcal{L}_v B - i_v dB) \\ &= [\![V,W]\!] + \mathcal{L}_v(i_w B) - i_w \mathcal{L}_v B + i_w i_v dB \end{aligned} \tag{3.42}$$

One can write in coordinates the second term from the last line as

$$\mathcal{L}_v(i_w B) = \left( v^\alpha \frac{\partial (w^\mu B_{\mu\nu})}{\partial x^\alpha} + w^\mu B_{\mu\alpha} \frac{\partial v^\alpha}{\partial x^\nu} \right) dx^\nu \tag{3.43}$$

and the third term as

$$\begin{aligned} i_w \mathcal{L}_v B &= i_w \left( \left( v^\alpha \frac{\partial (B_{\mu\nu})}{\partial x^\alpha} + B_{\mu\alpha} \frac{\partial v^\alpha}{\partial x^\nu} + B_{\alpha\nu} \frac{\partial v^\alpha}{\partial x^\mu} \right) dx^\mu \wedge dx^\nu \right) \\ &= \left( w^\mu v^\alpha \frac{\partial (B_{\mu\nu})}{\partial x^\alpha} + w^\mu B_{\mu\alpha} \frac{\partial v^\alpha}{\partial x^\nu} + w^\mu B_{\alpha\nu} \frac{\partial v^\alpha}{\partial x^\mu} \right) dx^\nu \end{aligned} \tag{3.44}$$

and so combining 3.43 and 3.44

$$\mathcal{L}_v(i_w B) - i_w \mathcal{L}_v B = \left( v^\alpha \frac{\partial w^\mu}{\partial x^\alpha} - w^\alpha \frac{\partial v^\mu}{\partial x^\alpha} \right) B_{\mu\nu} dx^\nu = i_{[v,w]} B \tag{3.45}$$

One can then plug in 3.45 to 3.42 and obtain that

$$[\![e^{\mathbf{B}}V, e^{\mathbf{B}}W]\!] = [\![V,W]\!] + i_{[v,w]} B + i_w i_v dB \tag{3.46}$$

Now applying the $B$-field transformation 3.39 on the Courant bracket one can write

$$\begin{aligned} e^{\mathbf{B}}[\![V,U]\!] &= e^{\mathbf{B}}([v,u] + \mathcal{L}_v\rho - \mathcal{L}_u\lambda - \frac{1}{2}d(i_v\rho - i_u\lambda)) \\ &= i_{[v,u]}B + [v,u] + \mathcal{L}_v\rho - \mathcal{L}_u\lambda - \frac{1}{2}d(i_v\rho - i_u\lambda) \\ &= [\![V,U]\!] + i_{[v,u]}B \end{aligned} \tag{3.47}$$

From 3.46 and 3.47 one can see that the $B$-field transformation preserves the Courant bracket under the restriction that $B$ is a *closed* form i.e. if $dB = 0$.

Lastly it can be checked that both diffeomorphisms and $B$-field transformations preserve the generalised inner product. For the $B$-field, one can easily obtain that

$$\begin{aligned} \langle e^{\mathbf{B}}V, e^{\mathbf{B}}U\rangle &= \langle v + i_vB + \lambda, u + i_uB + \rho\rangle \\ &= \langle v, \rho + i_uB\rangle + \langle \lambda + i_vB, u\rangle \\ &= \frac{1}{2}((\lambda + i_vB)(u) + (\rho + i_uB)(v)) \\ &= \frac{1}{2}(\lambda(u) + \rho(v)) + \frac{1}{2}(i_vB(u) + i_uB(v)) \\ &= \langle V, U\rangle + \frac{1}{2}(i_vi_uB + i_ui_vB) \\ &= \langle V, U\rangle \end{aligned} \tag{3.48}$$

where again when going from the fifth line to the last line we made use of the antisymmetry of $i_vi_uB$.

## 3.5 Generalised Riemmanian Metric

In our theory the antisymmetric two-form field $B$ together with a Riemannian metric $g : TM \times TM \to \mathbb{R}$ (i.e. symmetric and positive definite) play a very important role. In order to introduce a generalised metric which combines both these fields, one needs to decompose our generalised tangent bundle into two $d$-dimensional subbundles of $E$, one maximal subspace which we will denote $C_+$ where the inner product is positive definite, and one maximal subspace which we will denote

$C_-$ where the inner product is negative definite, orthogonal to each other [4]. As a result, the subgroup of $O(d,d)$ that will preserve both metrics separately is then $O(d) \times O(d)$. [13]

One can then define a **generalised metric operator** [13]

$$\begin{aligned} G : E = C_+ \oplus C_- \to E \\ V = V_+ + V_- \to GV \equiv \langle V, \cdot \rangle|_{C_+} - \langle V, \cdot \rangle|_{C_-} \end{aligned} \tag{3.49}$$

To find an explicit form of the splitting we also introduce an operator $\hat{\psi} : TM \to T^*M$ such that for all $v \in TM$ one has

$$\langle v + \hat{\psi}(v), v + \hat{\psi}(v) \rangle > 0 \tag{3.50}$$

This then gives the explicit form of the decomposition, with the two subspaces

$$C_\pm \equiv \{ v + \hat{\psi}_\pm(v) \mid v \in TM \} \tag{3.51}$$

where $\hat{\psi}_+ = B + g$ in $C_+$ and $\hat{\psi}_- = B - g$ in $C_-$ to satisfy the definitions for $C_+$ and $C_-$ [13]. One can note that applying the $B$-field transformation onto the following object gives:

$$e^{\mathbf{B}}(v + g(v)) = v + g(v) + B(v) = v + \hat{\psi}_+(v) \in C_+ \tag{3.52}$$

and similarly

$$e^{\mathbf{B}}(v - g(v)) = v - g(v) + B(v) = v + \hat{\psi}_-(v) \in C_- \tag{3.53}$$

This leads us to define an additional pair of subspaces, $\bar{C}_+$ and $\bar{C}_-$, given by

$$\begin{aligned} \bar{C}_+ \equiv \{ v + g(v) \mid v \in TM \} \\ \bar{C}_- \equiv \{ v - g(v) \mid v \in TM \} \end{aligned} \tag{3.54}$$

which satisfy

$$\begin{aligned} C_+ &= e^{\mathbf{B}}\overline{C}_+ \\ C_- &= e^{\mathbf{B}}\overline{C}_- \end{aligned} \tag{3.55}$$

These are thus defined under the condition that $B = 0$ and they give a generalised metric $\overline{G}$ that only depends on $g$. Note that the decomposition of $\hat{\psi}$ with $B$ and $g$ is due to the fact that $\hat{\psi}$ behaves similarly to a $(0,2)$-tensor, since it acts on vectors and gives one-forms, so it can be decomposed into a symmetric ($g$) and antisymmetric ($B$) part.

One can also note that applying $\overline{G}$ in the following way gives:

$$\overline{G}(\overline{V}_+ + \overline{V}_-) = \overline{G}(2v) = \overline{V}_+ - \overline{V}_- = 2g(v) \tag{3.56}$$

and similarly

$$\overline{G}(\overline{V}_+ - \overline{V}_-) = \overline{G}(2g(v)) = \overline{V}_+ + \overline{V}_- = 2v \tag{3.57}$$

where $\overline{V}_+ \in \overline{C}_+$ and $\overline{V}_- \in \overline{C}_-$. This allows us to calculate the explicit form of $\overline{G}$ as

$$\overline{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{3.58}$$

One can also observe that with 3.58 one can show that

$$\begin{aligned} \overline{G}\overline{V}_+ &= \overline{V}_+ \quad \text{and hence} \quad \overline{G}\overline{C}_+ = \overline{C}_+ \\ \overline{G}\overline{V}_- &= -\overline{V}_- \quad \text{and hence} \quad \overline{G}\overline{C}_- = -\overline{C}_- \end{aligned} \tag{3.59}$$

and by using the first lines of 3.59, 3.55 and $GC_+ = C_+$ from the definition of $G$ and $C_+$, one can write

$$C_+ = Ge^{\mathbf{B}}\overline{C}_+ = Ge^{\mathbf{B}}\overline{G}\overline{C}_+ = e^{\mathbf{B}}\overline{C}_+ \;\Rightarrow\; Ge^{\mathbf{B}}\overline{G} = e^{\mathbf{B}} \tag{3.60}$$

From here follows that $e^{-\mathbf{B}}Ge^{\mathbf{B}} = \overline{G}^{-1}$ but we can also calculate the inverse of $\overline{G}$ and we obtain that $\overline{G}^{-1} = \overline{G}$. Hence we arrive at $e^{-\mathbf{B}}Ge^{\mathbf{B}} = \overline{G}$ or alternately

$$G = e^{\mathbf{B}}\overline{G}e^{-\mathbf{B}} \tag{3.61}$$

From this, one can calculate the generalised Riemannian metric $G$ explicitly by plugging in 3.38 and 3.58 and obtain

$$G = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \tag{3.62}$$

The matrix form of the generalised metric $G$ shows explicitly the manner in which the $B$-field and metric $g$ are combined and captures the degrees of freedom of the NSNS sector from Type II theories [22], as detailed in the original generalised geometry formulation by Hitchin [15] and Gualtieri [14].

# Chapter 4

# $GL^+(d+1,\mathbb{R})$ Generalised Geometry

As previously seen in Chapter 3, one can construct a generalised geometry based on the fibre bundle $E \simeq TM \oplus T^*M$ and have a natural $O(d,d)$ action on its frame bundle. Then setting up a generalised metric G structure gives the degrees of freedom of the NSNS sector for Type II theories [22]. This was initially considered in detail in [15] and [14] but there are other geometries capturing bosonic degrees of freedom [22] in other types of theories, on different generalised tangent spaces, such as [19] [25] or [33].

This Chapter studies another generalised geometry centered around a different, $\frac{1}{2}d(d+1)$-dimensional, generalised tangent space with structure group $GL^+(d+1,\mathbb{R})$. This fibre bundle arises naturally when considering a theory with a $d$-form field strength as is the case for the sphere background and as will become apparent shortly.

The Chapter starts with an introduction and motivation for the generalised tangent space, showing how it parametrises the symmetries of the theory via the Dorfman derivative and how this defines specific patching rules that give the composition of the bundle. It then proceeds with the introduction of a generalised metric and a definition of conformal split frames that will be used in the next Chapter to provide a notion of $GL^+(d+1,\mathbb{R})$ generalised parallelisability on spheres.

## 4.1 Generalised Tangent Space, Patching Rules and Dorfman Derivative

The purpose of extending our geometry to this generalised version is, as it was the case previously, so that the new generalised tangent space can parametrise the infinitesimal symmetries of the theory. Similarly to $O(d,d)$ generalised geometry, these symmetries are *diffeomorphisms*, parametrised infinitesimally by a vector component $v$ and *gauge symmetries*, parametrised infinitesimally by a form component $\lambda$. The difference from the $O(d,d)$ case is that now we are interested in a theory in $d$ dimensions with a $d$-form field strength $F = \mathrm{d}A$ on a round sphere $S^d$. This prompts the following gauge symmetry for the potential $A$:

$$A_i = A_j + \mathrm{d}\Lambda_{ij} \tag{4.1}$$

on a patch overlap $U_i \cap U_j$, with $\Lambda$ being a $(d-2)$-form.

Adding up the number of degrees of freedom we obtain a total space $E$ with dimensionality

$$\dim(E) = \dim(v) + \dim(\lambda) = d + \frac{1}{2}d(d-1) = \frac{1}{2}d(d+1) \tag{4.2}$$

which gives the generalised tangent space

$$E \simeq TM \oplus \Lambda^{d-2}T^*M \tag{4.3}$$

Then a generalised vector $V \in E$ can be written as $V = v + \lambda$ and is given in components by

$$V^M = \begin{pmatrix} v^m \\ \lambda_{m_1 \ldots m_{d-2}} \end{pmatrix} \tag{4.4}$$

with the patching rules on the overlap $U_i \cap U_j$ for $E$ as per usual [22]

$$v_i + \lambda_i = v_j + \lambda_j + i_{v_j}\mathrm{d}\Lambda_{ij} \tag{4.5}$$

Here $v_i \in TU_i$ and $\lambda_i \in \Lambda^{d-2}T^*U_i$. Considering a choice of connection $A$ and generalised vectors $\tilde{V} \in TM \oplus \Lambda^{d-2}T^*M$ and $V \in E$, one can then define the "$A$-shift operator" [22] that gives the isomorphism between $E$ and $TM \oplus \Lambda^{d-2}T^*M$ :

$$V = \tilde{v} + \tilde{\lambda} + i_{\tilde{v}}A \equiv e^A \tilde{V} \tag{4.6}$$

Sections of the bundle $E$ transform in a very natural way under the action of structure group $GL^+(d+1,\mathbb{R})$ in the $\frac{1}{2}d(d+1)$-dimensional bivector representation [33], which can be seen explicitly by writing the generalised vector components as [22]

$$V^M = V^{\underline{mn}} = \begin{cases} V^{m,d+1} = v^m & \in TM \\ V^{mn} = \lambda^{mn} & \in \Lambda^2 TM \otimes \det T^*M \simeq \Lambda^{d-2}T^*M \end{cases} \tag{4.7}$$

where we are using a generalised vector index $M$ given by the antisymmetric pair $[\underline{m}, \underline{n}]$, $\underline{m}, \underline{n} \in 1, \ldots, d+1$ of indices in $GL^+(d+1,\mathbb{R})$ and the isomorphism in the second line between bivector densities and $(d-2)$-forms is given through [22]

$$\lambda^{mn} = \frac{1}{(d-2)!}\epsilon^{mnp_1\ldots p_{d-2}}\lambda_{p_1\ldots p_{d-2}} \tag{4.8}$$

There is also a natural definition of a generalised partial derivative operator that is embedded in the dual generalised tangent space $E^* \simeq T^*M \oplus \Lambda^{d-2}TM$ through the map $T^*M \to E^*$ and is given by [5]

$$\partial_M = \partial_{\underline{mn}} = \begin{cases} \partial_{m,d+1} = \partial_m & \in T^*M \\ \partial_{mn} = 0 & \in \Lambda^{d-2}TM \end{cases} \tag{4.9}$$

Let $V = v + \lambda$ and $W = w + \mu$ be two sections of generalised tangent space $E$. Then the Dorfman derivative is written via 3.17 in the usual way

$$L_V W = [v, w] + \mathcal{L}_v \mu - i_w d\lambda \tag{4.10}$$

with the generalised Lie bracket written as mentioned before via the antisymmetrization of the generalised Lie derivative i.e.

$$[\![V,W]\!] = \frac{1}{2}(L_V W - L_W V) \tag{4.11}$$

Considering the ordinary Lie derivative acting on a vector 2.10, one can see that if we write the $\partial v$ in the second term as a matrix $A^a{}_b := \partial_b v^a$, we can consider the second term as the adjoint action of $\mathfrak{gl}(d)$ on vector $u$. In a similar manner one can write the analogous part from the explicit equation of the Dorfman derivative as the adjoint action of the Lie algebra of the structure group and obtain the result from [5]

$$(L_V W)^M = (V \cdot \partial) W^M - (\partial \times_{ad} V)^M{}_N W^N \tag{4.12}$$

where the corresponding adjoint bundle taken from [22] is given by

$$\mathrm{ad}\hat{F} \simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^{d-1}TM \oplus \Lambda^{d-1}T^*M \tag{4.13}$$

showing the decomposition of the Lie algebra and $\times_{ad}$ is the projection to the adjoint representation of Lie algebra $\mathfrak{gl}(d+1,\mathbb{R})$, i.e. to the adjoint bundle $\mathrm{ad}\hat{F}$

$$\times_{ad} : E^* \otimes E \to \mathrm{ad}\hat{F} \tag{4.14}$$

One can plug in the matrices and calculate explicitly the contraction and the adjoint projection in order to obtain the two terms in the expression of the Dorfman derivative [22]

$$\begin{aligned} V \cdot \partial = V^M \partial_M &= \frac{1}{2} V^{\underline{mn}} \partial_{\underline{mn}} = v^m \partial_m \\ (\partial \times_{ad} V)^{\underline{m}}{}_{\underline{n}} &= V^{\underline{mp}} \partial_{\underline{np}} - \frac{1}{4} V^{\underline{pq}} \partial_{\underline{pq}} \delta^{\underline{m}}{}_{\underline{n}} \end{aligned} \tag{4.15}$$

which gives the manner in which the generalised tangent space parametrises the two infinitesimal symmetries of the theory.

## 4.2 Generalised Metric and Conformal Split Frames

Similarly to the $O(d,d)$ case, a generalised metric $G$ is parametrised by the degrees of freedom of the theory. In the $GL^+(d+1,\mathbb{R})$ case, there are the bosonic degrees of freedom given by the $(d-1)$-form $A$ and 2-form $g$, as well as an additional overall scale factor $\Delta$, linked to warped compactifications in Supergravity theories [3].

This means one can extend 4.6 to write $V = e^{\Delta}e^{A}\tilde{V}$ and using the contraction of two generalised objects from the first line of 4.15, where $\partial$ is replaced by another generalised vector, one can obtain the matrix form of $G$, given in [22] as

$$\begin{aligned} G(V,V) &= G_{MN}V^M V^N \\ &= g_{mn}\tilde{v}^m\tilde{v}^n + \frac{1}{(d-2)!}g^{m_1n_1}\dots g^{m_{d-2}n_{d-2}}\tilde{\lambda}_{m_1\dots m_{d-2}}\tilde{\lambda}_{n_1\dots n_{d-2}} \\ &= V^T\cdot e^{-2\Delta}\begin{pmatrix} g_{mn}+\frac{1}{(d-2)!}A_m{}^{n_1\dots n_{d-2}}A_{nn_1\dots n_{d-2}} & -A_m{}^{n_1\dots n_{d-2}} \\ A_n{}^{m_1\dots m_{d-2}} & (d-2)!g^{m_1\dots m_{d-2},n_1\dots n_{d-2}} \end{pmatrix}\cdot V \end{aligned} \tag{4.16}$$

Here $g^{m_1\dots m_{d-2},n_1\dots n_{d-2}} = g^{[m_1|n_1}\dots g^{m_{d-2}]n_{d-2}}$ separately antisymmetrised on differing indices and $G$ is invariant under subgroup $SO(d+1)$ [22].

However, instead of writing the generalised metric with four indices in components as above, there is a more intuitive way of showcasing the $SO(d+1)$ invariance. One can instead make use of the determinant bundle $\det TM = \Lambda^d TM$ and its dual $\det T^*M = \Lambda^d T^*M$ to define the following bundle [22]

$$\begin{aligned} W &\simeq (\det T^*M)^{1/2} \otimes (TM\oplus\Lambda^d TM) \\ &\simeq (\det T^*M)^{1/2} \otimes (TM\oplus(\det T^*M)^{-1}) \\ &\simeq (\det T^*M)^{1/2}\otimes TM \oplus (\det T^*M)^{-1/2} \end{aligned} \tag{4.17}$$

for which one can note after the above isomorphism that its sections $K \in W$, $K = q + t$ will transform in the $(d+1)$-dimensional fundamental representation of $GL^+(d+1,\mathbb{R})$ and they can be labelled via one index only, namely

$$K^{\underline{m}} = \begin{cases} V^m = q^m & \in (\det T^*M)^{1/2} \otimes TM \\ V^{d+1} = t & \in (\det T^*M)^{-1/2} \end{cases} \tag{4.18}$$

while we can link $E$ with $W$ via $E = \Lambda^2 W$.

We can then write the new $SO(d+1)$ invariant generalised metric, now labelled only by two indices, on the bundle $W$, which in [22] is given as

$$\begin{aligned} G(K,K) &= G_{\underline{mn}} K^{\underline{m}} K^{\underline{n}} \\ &= K^T \cdot \frac{e^{-\Delta}}{\sqrt{g}} \begin{pmatrix} g_{mn} & g_{mn}A^n \\ g_{np}A^p & \det g + g_{pq}A^p A^q \end{pmatrix} \cdot K \end{aligned} \tag{4.19}$$

and with $A^m$ being the equivalent to $A_{m_1 \ldots m_{d-1}}$ from 4.16 in terms of vector-density [22]. With this and the relation between $E$ and $W$, the generalised metric on $E$ can be simply written as

$$G(V,V) = \frac{1}{2} G_{\underline{mp}} G_{\underline{nq}} V^{\underline{mn}} V^{\underline{pq}} \tag{4.20}$$

We can now define a local orthonormal frame for $G$, i.e. a set of $GL^+(d+1,\mathbb{R})$ bases of $E$, labelled $\{\hat{E}^A\}$, where by $GL^+(d+1,\mathbb{R})$ basis we understand a basis related to a local coordinate basis on a patch of $E$ via a $GL^+(d+1,\mathbb{R})$ transformation. This in term defines a principal sub-bundle of the frame bundle of $E$, which is the generalised $GL^+(d+1,\mathbb{R})$ structure bundle $\tilde{F}$ [3]

$$\tilde{F} = \{(x, \{\hat{E}_A\}) \mid x \in M, \{\hat{E}_A\} \text{ is a } GL^+(d+1,\mathbb{R}) \text{ basis of } E\} \tag{4.21}$$

with fibre $GL^+(d+1,\mathbb{R})$. Now let $\hat{e}_a$ be the usual orthonormal frame for $TM$ and $e^a$ its dual counterpart for $T^*M$. We can define a new class of generalised orthonormal frames that depend on these conventional frames and the isomorphism 4.3, given by the following splitting [22] [3] [4]

$$\hat{E}_{ij} = \begin{cases} \hat{E}_{a,d+1} & = e^{\Delta}(\hat{e}_a + i_{\hat{e}_a} A) \\ \hat{E}_{ab} & = \frac{1}{(d-2)!} e^{\Delta} \epsilon_{abc_1 \dots c_{d-2}} e^{c_1} \wedge \dots \wedge e^{c_{d-2}} \end{cases} \tag{4.22}$$

They are called *conformal split frames* for $E$ (simply "split frames" if $\Delta = 0$) and their class defines a sub-bundle of $\tilde{F}$. Note that their labelling is through antisymmetric $SO(d+1)$ indices $(i,j)$ due to the frames transforming as 2-forms under this group [22]. They satisfy by definition the orthonormal condition

$$G(\hat{E}_{ij}, \hat{E}_{kl}) = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \tag{4.23}$$

If we want to build a similar frame on the space $W$, we can define it through $\hat{E}_{ij} = \hat{E}_i \wedge \hat{E}_j$, where $\hat{E}_i$ is the new frame, and we can also write its dual $\hat{E}^i \in W^*$ as given in [22] via

$$E^i = \begin{cases} E^a & = g^{-1/4} e^{-\Delta/2} (e^a - e^a \wedge A) \\ E^{d+1} & = g^{-1/4} e^{-\Delta/2} \mathrm{vol}_g \end{cases} \tag{4.24}$$

With the dual frames on $W$ defined one can simply express the generalised metric for $W$ from 4.19 using

$$G_{\underline{mn}} = \delta_{ij} E^i_{\underline{m}} E^j_{\underline{n}} \tag{4.25}$$

And we should also note that a local $SO(d+1)$ rotation $U$ of the generalised frame

$$\hat{E}'_{ij} = U_i{}^k U_j{}^l \hat{E}_{kl} \tag{4.26}$$

renders an equally valid frame [22].

# Chapter 5

# Spheres and Generalised $SL(d+1,\mathbb{R})$ Scherk–Schwarz reduction

Given a regular tangent space $TM$ from non-generalised differential geometry, if it admits a global frame i.e. a set of globally defined basis vectors that form a smooth vector field this means the space is topologically trivial and we call it *parallelisable* [22]. When it comes to spheres, similar notions of parallelisability have only been proven to exist for $S^1$, $S^3$ and $S^7$ [1] [21]. By employing the $GL^+(d+1,\mathbb{R})$ generalised geometry however, [22] shows that if the tangent bundle on the sphere is trivial (i.e. admits a global generalised frame), this extension of the geometry defines a concept of "generalised parallelisability" that can be applied to all spheres.

This Chapter will follow [22] to introduce this notion of generalised parallelisability for the $S^d$ after defining an appropriate global frame on the sphere and presenting some key geometry characteristics of the space of $S^d$. It will then introduce the notion of a generalised Scherk–Schwarz [29] reduction on this global frame and will show how the resulting fields match the standard scalar field ansatz for consistent truncations on spheres [22].

## 5.1 The $S^d$ Sphere as a Generalised Parallelisable Space

We start with a theory in $d$ dimensions that has a $d$-form field strength $F = \mathrm{d}A$ as mentioned previously and that gives a metric solution corresponding to a $S^d$ sphere. Such a theory needs to solve the following equations of motion

$$\begin{aligned} R_{mn} &= \frac{1}{d-1} F^2 g_{mn} \\ F &= \frac{d-1}{R} \mathrm{vol}_g \end{aligned} \tag{5.1}$$

where $R$ is the radius of the sphere, $F^2 = \frac{1}{d!} F^{m_1 \dots m_d} F_{m_1 \dots m_d}$ and $g$ is the metric form field from the theory [22].

Starting with the Cartesian coordinates $x^i = r y^i$ and constrained coordinates $\delta_{ij} y^i y^j = 1$ that give the equation of the sphere, we will be following [22] to introduce several key objects from the geometry of $S^d$ with radius $r = R$ that will be employed in the next sections. The metric $g$ on the sphere is [22]

$$\mathrm{d}s^2 = R^2 \delta_{ij} \mathrm{d}y^i \mathrm{d}y^j \tag{5.2}$$

Additionally we have

$$\frac{\partial}{\partial x^i} = y_i \frac{\partial}{\partial r} + \frac{k_i}{r} \tag{5.3}$$

We have $d+1$ conformal Killing vectors $k_i$ obeying [22]

$$\mathcal{L}_{k^i} g = -2 y^i g \tag{5.4}$$

with the metric components being written as [22]

$$g^{mn} = R^{-2} \delta^{ij} k_i{}^m k_j{}^n \tag{5.5}$$

This can be obtained by writing $\hat{g} = \mathrm{d}x^i \mathrm{d}x_i$, with $w_i = \frac{\partial}{\partial x^i}$ and in terms of the radius, $\hat{g} = \mathrm{d}r^2 + r^2 g$, with $g = \delta_{ij} \mathrm{d}y^i \mathrm{d}y^j$. Since the Killing vectors preserve the metric and the flows generated by the Killing vector fields are continuous isometries of the manifold [34], one has the condition $\mathcal{L}_{w_i}\hat{g} = 0$. By calculating $w_i$ using 5.3 and substituting it and $\hat{g}$ into the condition, one can obtain the relation for conformal Killing vectors 5.4.

From [22] we also have that

$$k_i(y_j) = i_{k_i}\mathrm{d}y_j = R^{-2}g(k_i, k_j) = R^2 g^{-1}(\mathrm{d}y_i, \mathrm{d}y_j) = \delta_{ij} - y_i y_j \tag{5.6}$$

Following [22] one can also define the $SO(d+1)$ rotation Killing vectors:

$$v_{ij} = R^{-1}(y_i k_j - y_j k_i) \tag{5.7}$$

which satisfy [22]

$$[v_{ij}, v_{kl}] = R^{-1}(\delta_{ik}v_{lj} - \delta_{il}v_{kj} - \delta_{jk}v_{li} + \delta_{jl}v_{ki}) \tag{5.8}$$

and additionally from [22] we have

$$\begin{aligned} \mathcal{L}_{v_{ij}} y_k &= R^{-1}(y_i\delta_{jk} - y_j\delta_{ik}) \\ \mathcal{L}_{v_{ij}} \mathrm{d}y_k &= R^{-1}(\mathrm{d}y_i\delta_{jk} - \mathrm{d}y_j\delta_{ik}) \end{aligned} \tag{5.9}$$

For $r = 1$ on the sphere the volume form on $S^d$ can be calculated following [11]. A similar procedure can be employed for $r = R$ and by writing in terms of the totally antisymmetric symbol $\epsilon$ and radius $R$ one can give the volume form according to [22] as

$$\mathrm{vol}_g = \frac{R^d}{d!}\epsilon_{i_1 \dots i_{d+1}} y^{i_1}\mathrm{d}y^{i_2} \wedge \dots \wedge \mathrm{d}y^{i_{d+1}} \tag{5.10}$$

We also define the 2-form $\omega$ given by [22]

$$\omega_{ij} = R^2 \mathrm{d}y_i \wedge \mathrm{d}y_j \tag{5.11}$$

which geometrically can be interpreted as the projection from the tangent space at the pole onto the rest of the sphere. Hence, we can now define a key object for the structure of the generalised frame, given by the Hodge dual of $\omega$ and interpreted as its orthogonal complement [22]

$$\sigma_{ij} = \star\omega_{ij} = \frac{R^{d-2}}{(d-2)!}\epsilon_{ijk_1\ldots k_{d-1}} y^{k_1} \mathrm{d}y^{k_2} \wedge \cdots \wedge \mathrm{d}y^{k_{d-1}} \tag{5.12}$$

After introducing all the important geometry elements from above, we can now proceed to define the generalised global frame by [22]

$$\hat{E}_{ij} = v_{ij} + \sigma_{ij} + i_{v_{ij}} A \tag{5.13}$$

with $A$ being the potential that gives the $d$-form field strength $F = \mathrm{d}A$. Additionally from [22] we have the contractions for $v$ and $\sigma$ given by

$$v_{ij} \cdot v_{kl} := (v_{ij})^m (v_{kl})_m = y_i y_k \delta_{jl} - y_j y_k \delta_{il} - y_i y_l \delta_{jk} + y_j y_l \delta_{ik} \tag{5.14}$$

and

$$\begin{aligned} \sigma_{ij} \cdot \sigma_{kl} &:= \frac{1}{(d-2)!}(\sigma_{ij})^{m_1\ldots m_{d-2}}(\sigma_{kl})_{m_1\ldots m_{d-2}} \\ &= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} - (y_i y_k \delta_{jl} - y_j y_k \delta_{il} - y_i y_l \delta_{jk} + y_j y_l \delta_{ik}) \\ &= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} - v_{ij} \cdot v_{kl} \end{aligned} \tag{5.15}$$

Hence one can see that

$$G(\hat{E}_{ij}, \hat{E}_{kl}) = v_{ij} \cdot v_{kl} + \sigma_{ij} \cdot \sigma_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \tag{5.16}$$

which is the required orthonormality condition 4.23, making our frame orthonormal to the generalised metric on $S^d$. The dual frame is also globally defined in [22] by

$$E^i = g^{-1/4}(R\mathrm{d}y^i + y^i \mathrm{vol}_g - R\mathrm{d}y^i \wedge A) \tag{5.17}$$

and one can check that because $\mathrm{d}y_i = 0$ only when $y_i^2 = 1$ due to the constrained coordinates, the dual frame is also non-vanishing.

In order to arrive at the notion of trivial space and generalised parallelisability, we need to calculate the generalised geometry equivalent of the Lie bracket algebra for our generalised frame. Starting with the Dorfman derivative given by the usual expression, we obtain

$$\begin{aligned} L_{\hat{E}_{ij}}\hat{E}_{kl} &= [v_{ij}, v_{kl}] + \mathcal{L}_{v_{ij}}(\sigma_{kl} + i_{v_{kl}}A) - i_{v_{kl}}\mathrm{d}(\sigma_{ij} + i_{v_{ij}}A) \\ &= [v_{ij}, v_{kl}] + \mathcal{L}_{v_{ij}}\sigma_{kl} + \mathcal{L}_{v_{ij}}(i_{v_{kl}}A) - i_{v_{kl}}(\mathrm{d}\sigma_{ij} + \mathrm{d}i_{v_{ij}}A) \\ &= [v_{ij}, v_{kl}] + \mathcal{L}_{v_{ij}}\sigma_{kl} + \mathcal{L}_{v_{ij}}(i_{v_{kl}}A) - i_{v_{kl}}(\mathrm{d}\sigma_{ij} + \mathcal{L}_{v_{ij}}A - i_{v_{ij}}\mathrm{d}A) \\ &= [v_{ij}, v_{kl}] + \mathcal{L}_{v_{ij}}\sigma_{kl} + \mathcal{L}_{v_{ij}}(i_{v_{kl}}A) - i_{v_{kl}}(\mathcal{L}_{v_{ij}}A) - i_{v_{kl}}(\mathrm{d}\sigma_{ij} - i_{v_{ij}}F) \\ &= [v_{ij}, v_{kl}] + \mathcal{L}_{v_{ij}}\sigma_{kl} + i_{[v_{ij},v_{kl}]}A - i_{v_{kl}}(\mathrm{d}\sigma_{ij} - i_{v_{ij}}F) \end{aligned} \tag{5.18}$$

where in going from the second to the third line we used 2.14 and from the fourth to get the fifth line we used 3.45.

In order to proceed, we can write $F = R^{-1}(d-1)\mathrm{vol}_g$ from the equations of motion 5.1 so that $i_{v_{ij}}F$ now depends on $i_{v_{ij}}\mathrm{vol}_g$. The Lie derivative of $\sigma$ is given by [22] as

$$\mathcal{L}_{v_{ij}}\sigma_{kl} = R^{-1}(\delta_{ik}\sigma_{lj} - \delta_{il}\sigma_{kj} - \delta_{jk}\sigma_{li} + \delta_{jl}\sigma_{ki}) \tag{5.19}$$

and we now only need $i_{v_{ij}}\mathrm{vol}_g$

$$i_{v_{ij}}\mathrm{vol}_g = -\frac{R^{d-1}}{(d-1)!}(y_i\epsilon_{jk_1\dots k_d} - y_j\epsilon_{ik_1\dots k_d})y^{k_1}\mathrm{d}y^{k_2}\wedge\dots\wedge\mathrm{d}y^{k_d} \tag{5.20}$$

We can use the antisymmetry of $\epsilon$ i.e. $y_{[i}\epsilon_{i_2\dots i_{d+2}]} = 0$ to rewrite 5.20 according to [22] as

$$\begin{aligned} i_{v_{ij}}\mathrm{vol}_g &= -\frac{R^{d-1}(d-1)}{(d-1)!}y_{k_1}\epsilon_{ijk_2\dots k_d}y^{[k_1}\mathrm{d}y^{k_2}\wedge\dots\wedge\mathrm{d}y^{k_d]} \\ &= -\frac{R^{d-1}}{(d-1)!}\epsilon_{ijk_1\dots k_{d-1}}\mathrm{d}y^{k_1}\wedge\dots\wedge\mathrm{d}y^{k_{d-1}} \\ &= \frac{R}{d-1}\mathrm{d}\sigma_{ij} \end{aligned} \tag{5.21}$$

Now we can use 5.21 to write $i_{v_{ij}}F$ as

$$i_{v_{ij}}F = R^{-1}(d-1)\, i_{v_{ij}}\mathrm{vol}_g = \mathrm{d}\sigma_{ij} \tag{5.22}$$

which in turn cancels the term in round brackets from 5.18 and hence we obtain

$$L_{\hat{E}_{ij}}\hat{E}_{kl} = [v_{ij}, v_{kl}] + \mathcal{L}_{v_{ij}}\sigma_{kl} + i_{[v_{ij},v_{kl}]}A \tag{5.23}$$

One can now use 5.19 and the previous result from 5.8 to re-write the generalised Lie derivative for the frames as

$$\begin{aligned}
L_{\hat{E}_{ij}}\hat{E}_{kl} &= R^{-1}(\delta_{ik}v_{lj} - \delta_{il}v_{kj} - \delta_{jk}v_{li} + \delta_{jl}v_{ki} + \delta_{ik}\sigma_{lj} - \delta_{il}\sigma_{kj} - \delta_{jk}\sigma_{li} + \delta_{jl}\sigma_{ki}) + i_{[v_{ij},v_{kl}]}A \\
&= R^{-1}(\delta_{ik}(v_{lj}+\sigma_{lj}) - \delta_{il}(v_{kj}+\sigma_{kj}) - \delta_{jk}(v_{li}+\sigma_{li}) + \delta_{jl}(v_{ki}+\sigma_{ki})) + A([v_{ij},v_{kl}]) \\
&= R^{-1}(\delta_{ik}(v_{lj}+\sigma_{lj}) - \delta_{il}(v_{kj}+\sigma_{kj}) - \delta_{jk}(v_{li}+\sigma_{li}) + \delta_{jl}(v_{ki}+\sigma_{ki}) \\
&\qquad + \delta_{ik}A(v_{lj}) - \delta_{il}A(v_{kj}) - \delta_{jk}A(v_{li}) + \delta_{jl}A(v_{ki})) \\
&= R^{-1}(\delta_{ik}(v_{lj}+\sigma_{lj}+i_{v_{lj}}A) - \delta_{il}(v_{kj}+\sigma_{kj}+i_{v_{kj}}A) \\
&\qquad - \delta_{jk}(v_{li}+\sigma_{li}+i_{v_{li}}A) + \delta_{jl}(v_{ki}+\sigma_{ki}+i_{v_{ki}}A)) \\
&= R^{-1}(\delta_{ik}\hat{E}_{lj} - \delta_{il}\hat{E}_{kj} - \delta_{jk}\hat{E}_{li} + \delta_{jl}\hat{E}_{ki}) \\
&= [\![\hat{E}_{ij}, \hat{E}_{kl}]\!]
\end{aligned} \tag{5.24}$$

and so the generalised Lie bracket algebra we needed is actually the Lie algebra $\mathfrak{so}(d+1)$ [22], which as it will become apparent in the next section is the key result we needed for the notion of generalised parallelisability.

## 5.2 Generalised $SL(d+1,\mathbb{R})$ Scherk–Schwarz reduction on the Round $d$-Sphere

The result in 5.24 has shown that from the standpoint of generalised geometry the sphere $S^d$ behaves like a local group manifold, i.e. it is generalised parallelisable, similarly to a regular parallelisable manifold that has the form of its Lie bracket algebra of frame $\hat{e}_a$ given by [22]

$$[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c \tag{5.25}$$

with constant $f_{ab}{}^c$. For the case of the conventional parallelisable manifolds it was proven that if $f_{ab}{}^b = 0$ the space admits consistent truncations [10] [7] and the subsequent truncated theory has scalar fields that correspond to a Scherk-Schwarz reduction [29] [28]. Such a reduction consists of constant $GL(d.\mathbb{R})$ rotations of the frame on the manifold, given by $\hat{e}'_a = U_a{}^b(x)\hat{e}_b$, with $g'^{mn} = H^{ab}(x)\hat{e}^m_a\hat{e}^n_b$ and symmetric $H^{cd} = \delta^{ab}U_a{}^cU_b{}^d$ [22].

This prompts the notion of a generalised Scherk-Schwarz reduction

$$\hat{E}'_{ij} = U_i{}^k(x)U_j{}^l(x)\hat{E}_{kl} \tag{5.26}$$

given via the frame deformations $U_i{}^k(x) \in GL(d+1,\mathbb{R})$ that depend on the coordinates in the non-truncated space but are constant on $M$. Defining the symmetric matrices $T^{kl} = \delta^{ij}U_i{}^kU_j{}^l$, we can write the inverse of the generalised metric in the previous chapter, given in [22] by

$$G'^{MN} = \frac{1}{2}T^{ik}T^{jl}\hat{E}^M_{ij}\hat{E}^N_{kl} \tag{5.27}$$

and given explicitly in terms of $A'$, $g'$ and $\Delta'$ in [22] by

$$G'^{MN} = e^{2\Delta'}\begin{pmatrix} g'_{mn} & g'_{mp}A'_{pn_1\ldots n_{d-1}} \\ g'_{np}A'_{pm_1\ldots m_{d-1}} & (d-2)!g'_{m_1\ldots m_{d-2},n_1\ldots n_{d-2}} + A'_{pm_1\ldots m_{d-2}}A'^{pn_1\ldots n_{d-2}} \end{pmatrix} \tag{5.28}$$

Expressions 5.27 and 5.28 can be compared to obtain equations between the fields and the symmetric $T$ matrices. Following the analysis detailed in [24] and

[22] one can eventually arrive at the result

$$\begin{aligned} ds'^2 &= \frac{R^2}{(T^{kl}y_k y_l)^{2/(d-1)}} T^{-1}_{ij} dy^i dy^j \\ A' &= -\frac{R^{d-1}}{2(T^{kl}y_k y_l)(d-2)!} \epsilon_{i_1 \dots i_{d+1}} (T^{i_1 j}{}_{y_j}) y^{i_2} dy^{i_3} \wedge \cdots \wedge dy^{d+1} + A \\ e^{2\Delta'} &= (T^{kl}y_k y_l)^{(d-3)/(d-1)} \end{aligned} \tag{5.29}$$

which is in agreement with the scalar field ansatz for consistent truncations on spheres [22] [24].

# Chapter 6

# Consistent Truncations on $S^3$

The classical Supergravity sphere solutions for $S^3$ are given by the "near-horizon Neveu-Schwarz fivebrane background" [22] and correspond to a $S^3$ consistent truncation to a seven-dimensional gauged Supergravity theory, as shown in detail in [8]. This Chapter will start by presenting the explicit context of generalised parallelisability for the 3-sphere, following [22] and will then continue with a discussion on the relation between this context and gauged Supergravity.

## 6.1 $SO(3,3)$ Generalised Geometry on the 3-Sphere

The solution of the near-horizon limit for NS fivebranes in type II Supergravity was given in [2] by

$$\begin{aligned} ds^2 &= ds^2(\mathbb{R}^{5,1}) + dt^2 + R^2 ds^2(S^3) \\ H &= 2R^{-1}\text{vol}_g \\ \phi &= -\frac{t}{R} \end{aligned} \tag{6.1}$$

and corresponds to a "3-sphere times a linear dilaton background", with $R$ the radius of $S^3$ [22]. From the standpoint of $GL^+(d+1,\mathbb{R})$, i.e. $GL^+(4,\mathbb{R})$ generalised geometry our generalised tangent space is in this case

$$E \simeq TM \oplus T^*M \tag{6.2}$$

with structure group $O(d,d)$ [22] as previously reviewed in Chapter 3. We can identify $B = -A$ to preserve the conventions that we previously employed in $O(d,d)$ and define the generalised frame by

$$\hat{E}_{ij} = v_{ij} + \sigma_{ij} - i_{v_{ij}} B \tag{6.3}$$

We then proceed to define the left-invariant and right-invariant vector fields on the 3-sphere given in [22] by

$$\begin{aligned} l_+ &= l_1 + il_2 = R^{-1} e^{-i\psi}(\partial_\theta + i\csc\theta\partial_\phi - i\cot\theta\partial_\psi) \\ l_3 &= R^{-1}\partial_\psi \\ r_+ &= r_1 + ir_2 = R^{-1} e^{i\phi}(\partial_\theta + i\cot\theta\partial_\phi - i\csc\theta\partial_\psi) \\ r_3 &= R^{-1}\partial_\phi \end{aligned} \tag{6.4}$$

and similarly the left-invariant and right-invariant one-form fields on $S^3$ [22]

$$\begin{aligned} \lambda_+ &= Re^{-i\psi}(\mathrm{d}\theta + i\cos\theta\mathrm{d}\phi) \\ \lambda_3 &= R(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi) \\ \rho_+ &= Re^{i\phi}(\mathrm{d}\theta + i\sin\theta\mathrm{d}\psi) \\ \rho_3 &= R(\mathrm{d}\phi + \cos\theta\mathrm{d}\psi) \end{aligned} \tag{6.5}$$

in the usual spherical notation. Additionally we choose the gauge [22]

$$B = 2R\cos\theta\mathrm{d}\phi \wedge \mathrm{d}\psi \tag{6.6}$$

We can define the left- and right-invariant bases for the two tangent spaces that compose our generalised bundle through the "anti-self-dual and self-dual combinations of $\hat{E}_{ij}$", explicitly via [22] by

$$\begin{aligned}\hat{E}^L_+ &= l_+ - \lambda_+ - i_{l_+}B \\ &= e^{-i\psi}\Big((R^{-1}\partial_\theta - R\mathrm{d}\theta) + i\csc\theta(R^{-1}\partial_\phi - R\mathrm{d}\phi) - i\cot\theta(R^{-1}\partial_\psi + R\mathrm{d}\psi)\Big) \\ \hat{E}^L_3 &= l_3 - \lambda_3 - i_{l_3}B = R^{-1}\partial_\psi - R\mathrm{d}\psi\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}\hat{E}^R_+ &= r_+ - \rho_+ - i_{r_+}B \\ &= e^{i\phi}\Big((R^{-1}\partial_\theta + R\mathrm{d}\theta) + i\cot\theta(R^{-1}\partial_\phi - R\mathrm{d}\phi) - i\csc\theta(R^{-1}\partial_\psi + R\mathrm{d}\psi)\Big) \\ \hat{E}^R_3 &= r_3 - \rho_3 - i_{r_3}B = R^{-1}\partial_\phi + R\mathrm{d}\phi\end{aligned} \tag{6.8}$$

One can check they are orthonormal since given

$$\hat{E}_A = \begin{pmatrix}\hat{E}^R_a \\ \hat{E}^L_{\bar{a}}\end{pmatrix} \tag{6.9}$$

we can write

$$\begin{aligned}\eta(\hat{E}_A, \hat{E}_B) &= \begin{pmatrix}\delta_{ab} & 0 \\ 0 & -\delta_{\bar{a}\bar{b}}\end{pmatrix} \\ G(\hat{E}_A, \hat{E}_B) &= \begin{pmatrix}\delta_{ab} & 0 \\ 0 & \delta_{\bar{a}\bar{b}}\end{pmatrix}\end{aligned} \tag{6.10}$$

where $G$ is the generalised metric without warped compactification factors and $\eta$ is the $O(3,3)$ metric $\eta(V,V) = i_v\lambda$ for $V = v + \lambda$ [22]. Similarly, the Dorfman derivatives on the frames can be shown to be [22]

$$\begin{aligned}L_{\hat{E}^L_{\bar{a}}}\hat{E}^L_{\bar{b}} &= [\![\hat{E}^L_{\bar{a}}, \hat{E}^L_{\bar{b}}]\!] = R^{-1}\epsilon_{\bar{a}\bar{b}\bar{c}}\hat{E}^L_{\bar{c}} \\ L_{\hat{E}^R_a}\hat{E}^R_b &= [\![\hat{E}^R_a, \hat{E}^R_b]\!] = R^{-1}\epsilon_{abc}\hat{E}^R_c \\ L_{\hat{E}^L_{\bar{a}}}\hat{E}^R_a &= [\![\hat{E}^L_{\bar{a}}, \hat{E}^R_a]\!] = 0\end{aligned} \tag{6.11}$$

and we obtain the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ algebra.

## 6.2 $S^3$ and Gauged Supergravity

There is a well known result [8] that a type II Supergravity consistent truncation exists on the 3-sphere and the truncated theory is a maximal 7-dimensional $SO(4)$ gauged Supergravity [22]. More explicitly, in order to define a gauged Supergravity the following condition is required [27]

$$[X_A, X_B] = -X_{AB}{}^C X_C \tag{6.12}$$

where $X_{AB}{}^C$ is the embedding tensor of the theory [22]. From the standpoint of the generalised geometry this constraint comes naturally from the Leibniz identity of the Dorfman derivative for the frame, $X$ being encoded in the frame algebra [22].

Additionally, one can calculate in a similar manner to the previous section the scalar fields for the truncated theory, the $B$-field and the metric for this particular case. These are given in [22] by

$$\begin{aligned} ds'^2 &= \frac{R^2}{T^{kl} y_k y_l} T^{-1}_{ij} dy^i dy^j \\ B' &= \frac{R^2}{2(T^{kl} y_k y_l)} \epsilon_{i_1 i_2 i_3 i_4} (T^{i_1 j}{}_{y_j}) y^{i_2} dy^{i_3} \wedge dy^{i_4} + B \\ e^{2\Delta'} &= 1 \end{aligned} \tag{6.13}$$

with a trivial warped compactification factor. By comparing these results with the fields in [8] for the consistent truncation on the 3-sphere one can see they are in complete agreement.

# Chapter 7

# Further Considerations and Conclusions

This work presented an introduction to the mathematical framework of Generalised Geometry (GG) through the $O(d,d)$ GG, and then described in a unifying approach the "maximally supersymmetric consistent truncations" [22] through this generalised geometry framework in the form of the $GL^+(d+1,\mathbb{R})$ GG.

We showed that there is natural notion of generalised parallelisability on $M$, analogous to the concept of a local group manifold but given in terms of the generalised geometry if the generalised tangent space $E$ admits globally a generalised frame $\{\hat{E}_A\}$ that satisfies

$$L_{\hat{E}_A}\hat{E}_B = X_{AB}{}^C\hat{E}_C \tag{7.1}$$

where $L_{\hat{E}_A}\hat{E}_B$ gives the generalised Lie derivative of the frame and $X_{AB}{}^C$ is constant.

The existence of the generalised global frame then leads to the consideration of a generalised Scherk-Schwarz reduction by a rotation of the frame, with $X_{AB}{}^C$ as the embedding tensor of the truncated theory [22]. This allowed us to present the proof that all round spheres admit such generalised parallelisations by calculating the generalised Lie bracket of the generalised frame. As a consequence, all sphere compactifications can be viewed in the same "algorithmic" way from the perspective of a generalised Scherk-Schwarz reduction.

One potential area of future study on this matter can be on proving the consistency of the truncations. In presenting this work we employed the conjecture, based on analogy with the conventional case, that there is a consistent truncation given a generalised parallelisation, but we have not fully proved this. Rather, we just presented the agreement with the scalar field ansatz.

Another potential subject that arises from the work presented here is the classification of the manifolds $M$ that admit such generalised parallelisations through the globally defined generalised frame. This can potentially lead to a "class of maximal gauged supergravities that appear as consistent truncations" [22].

# Appendix A

# Principal Bundles extended

We will define in this Appendix the rest of the key notions building up the concept of Principal Bundles not covered in 2.3.11, following primarily [30].

Let $(G, \bullet)$ be a Lie group and let $M$ be a smooth manifold.

**Definition A.0.1** A *left G-action* on $M$ is a smooth map

$$\begin{aligned} \triangleright : G \times M \to M \\ (g, p) \to g \triangleright p \end{aligned} \tag{A.1}$$

that satisfies:

- $\forall p \in M,\ e \triangleright p = p$
- $\forall p \in M,\ \forall g_1, g_2 \in G,\ (g_1 \bullet g_2) \triangleright p = g_1 \triangleright (g_2 \triangleright p)$

and a manifold that has a left-$G$ action is called a *left G-manifold*.

**Definition A.0.2** A *right G-action* on $M$ is a smooth map

$$\begin{aligned} \triangleleft : M \times G \to M \\ (p, g) \to p \triangleleft g \end{aligned} \tag{A.2}$$

that satisfies:

- $\forall p \in M,\ e \triangleleft p = p$
- $\forall p \in M,\ \forall g_1, g_2 \in G,\ p \triangleleft (g_1 \bullet g_2) = p \triangleleft (g_1 \triangleleft g_2)$

and similarly, a manifold that has a right-$G$ action is called a *right G-manifold*.

**Definition A.0.3** If we have a left $G$-action $\triangleright : G \times M \to M$, the *orbit* of $p$, for each $p \in M$ is defined as the set

$$G_p := \{q \in M \mid \exists g \in G \quad s.t. \quad q = g \triangleright p\} \tag{A.3}$$

It can be shown that given a left $G$-action on $M \triangleright : G \times M \to M$, we can define a relation on $M$, $p \sim q$, given by the statement: $\exists g \in G \;\; s.t. \;\; q = g \triangleright p$, that is an equivalence relation on $M$. [30] Then by definition A.0.3, the equivalence classes of $\sim$ are the orbits.

**Definition A.0.4** Then with the above-mentioned left $G$-action on $M \triangleright : G \times M \to M$, we can define the *orbit space* of $M$ as

$$M/G := M/\sim = \{G_p \mid p \in M\} \tag{A.4}$$

**Definition A.0.5** Given a $G$-action on $M$, $\triangleright$, the *stabiliser* of $p \in M$ is a subgroup of $G$ defined as

$$S_p := \{g \in G \mid g \triangleright p = p\} \tag{A.5}$$

**Definition A.0.6** A left $G$-action $\triangleright : G \times M \to M$ is said to be

1. *free* if $\forall p \in M$, then $S_p = \{e\}$
2. *transitive* if $\forall p, q \in M, \;\exists g \in G$ s. t. $p = g \triangleright p$

It can also be shown [30] that given a $\triangleright : G \times M \to M$ *free* left $G$-action on $M$, for $p \in M$ and $g_1, g_2 \in G$

$$g_1 \triangleright p = g_2 \triangleright p \quad \Leftrightarrow \quad g_1 = g_2 \tag{A.6}$$

and then

$$\forall p \in M \; : \; G_p \cong_{diff} G \tag{A.7}$$

with all the above-mentioned concepts in the end naturally leading to the notion of Principal Bundle defined in 2.3.11.

# Appendix B

# Frame Bundles

In this section we will be following [23] to introduce the concept of Frame Bundle, which is used later when defining conformal split frames on the $GL^+(d+1,\mathbb{R})$ generalised geometry.

**Definition B.0.1** Given a tangent bundle $TM$ over an $m$-dimensional manifold $M$, the *frame bundle* $LM \equiv \bigcup_{p\in M} L_pM$ is a principal bundle associated to $TM$, with $L_pM$ being the set of frames (all ordered bases) at $p$. We can write a frame $u = \{X_1,\ldots,X_m\}$ at $p$ as $X_\alpha = X^\mu{}_\alpha \partial/\partial x^\mu|_p$, with $1 \leq \alpha \leq m$, in the natural basis of $T_pM$, $\{\partial/\partial x^\mu\}$, with coordinates $x^\mu$ on patch $U_i$. Here $\{X_\alpha\}$ are linearly independent since components $(X^\mu{}_\alpha) \in GL(m,\mathbb{R})$. $LM$ has the structure of a bundle defined through the following components:

1. The action of $a \in GL(m,\mathbb{R})$ on the frame $u$ is given by $(u,a) \to ua$, where $ua$ is the new frame at $p$ defined by

$$Y_\beta = X_\alpha a^\alpha{}_\beta \tag{B.1}$$

   We can show ([30]) that $GL(m,\mathbb{R})$ acts *transitively* on $LM$, since we can always find an element of $GL(m,\mathbb{R})$ that satisfies B.1 for any $\{X_\alpha\}$, $\{X_\beta\}$.

2. We can define $\pi_L : LM \to M$ as $\pi_L(u) = p$, for frame $u$.

3. For $p$ in an overlap of two charts, $U_i \cap U_j$, with coordinates $x^\mu$, $x^\nu$ respectively, we have

$$X_\alpha = X^\mu{}_\alpha \partial/\partial x^\mu|_p = \tilde{X}^\mu_\alpha \partial/\partial y^\mu|_p \tag{B.2}$$

with the $X$ matrices being part of $GL(m, \mathbb{R})$. The relation between the two is $X^\mu{}_\alpha = (\partial x^\mu/\partial y^\nu)_p \tilde{X}^\mu_\alpha$ and so the transition functions will be

$$t^L_{ij}(p) = ((\partial x^\mu/\partial y^\nu)_p) \in GL(m, \mathbb{R}) \tag{B.3}$$

and so $LM$ is a frame bundle with the same transition functions as initial tangent bundle $TM$.

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