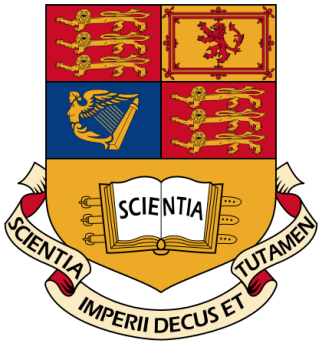
Entangling qubits using global pulses

Yannick Seis

September 19, 2014



MSc Thesis

Under the supervision of :

Terry Rudolph

Joseph Goodwin

Howard Dale

and Benjamin Brown

I would like to thank Terry Rudolph, Joseph Goodwin, Howard Dale and Benjamin Brown for their help and advice during the course of this thesis. I have very much enjoyed working with them and found all of the work highly interesting.

Abstract

We investigate the action of global laser pulses on a set of ionic qubit crystals. Mechanical vibration modes of the crystal are driven by applying state dependent forces on the ions, this allows for different couplings between qubit states and vibrational modes. We exploit this difference in coupling to makes different states in a superposition evolve in different manners to implement operations on the quantum information stored in the ions.

We report the implementation of a Controlled-Z gate on an arbitrary number of qubits with two global pulses as well as a repetition code encoding with a single global pulse. We also combine global pulses driving different vibration modes to implement qubit interactions with localised Hamiltonians. Finally we use our Controlled-Z operation to encode the Steane code and realise syndrome measurements on the logical without having to hide any qubits from the global pulses.

Contents

1 Introduction

1.1 From classical to quantum

Modern technologies rely increasingly on our understanding of quantum mechanics. GPS, lasers in DVD players, solar cells, and finally computers all use properties inaccessible to classical physics. The equations governing the way a transistor allows for the flow of current were established at the end of 1940's by Shockley and his coworkers and are applications of early days quantum mechanics taking electrons to behave like waves in solids and propagating according to the Schroedinger equation. Information however was and still is stored in a non-quantum manner using bits: 0's and 1's which means that the physical transistor has a low or high voltage applied to it. Combining two bits using logical gates allows to manipulate the information and extract more information from it: eventually implementing mathematical operations such as addition or multiplication and finally construction the computers we know today. Thus the way the information is stored is crucially important to the proper operation of the computer. For classical computers, the physical (voltage) state of the bits are usually different enough to insure that we can unambiguously determine whether the bit is a 0 or a 1: any measurement of the voltage will easily tell whether it is high or low. But even with this precaution, it can happen that the state of the bit is accidentally changed; in this case we need to be able to determine whether such a change has happened and how to correct it to insure a sane computation. More recently quantum states of various systems such as atoms or ions have been used to store information and the last 20 years has seen the development of secure ways of storing and manipulating quantum information to be used to our advantage. For a useful computation to be implemented, the information needs to be safely stored for a time at least as long as the computation. In this thesis, we will address the problem of finding ways of storing information in systems using quantum mechanical properties described in the next section, which protect the information and are experimentally easy to implemented on a set of trapped ions.

1.2 Quantum weirdness

Quantum computing uses basic properties of quantum mechanical systems such as discrete energy states of electrons in atoms and ions to store the information, superposition and entanglement of these states to manipulate the information.

The fact that only certain energy levels are allowed in quantum systems is an intrinsic property of these systems and is what gives the mechanics its name. To a pair of states, we will assign the information "0" and "1": to store the information "0", we put the electron in the first energy state; to store "1", we put the electron in the second energy state. The electron's state can be changed by

giving it energy in form of photons via a laser, tuned to a frequency which corresponds to the energy difference between the initial and the final state of the electron. States of a system are denoted using the Dirac "bra-ket" notation where for example the states denoted $|0\rangle$ and $|1\rangle$ are the states of the electron labelled by the information which is stored: "0" or "1". These states are part of a larger vector space of states in which all the possible states of a system can be represented with a vector. We can also combine states of disjoint systems to make up larger systems, this will be further discussed later on.

Unlike in classical physics, where a system can only be in one state at a time (an object can only be in one distinct place at a time and not in several), in quantum mechanics objects e.g. electrons can be in multiple states which is called a superposition: one could say that the electrons is in two different energy states or two different places at the same time. In popular science, this is exemplified by the thought experiment of Schroedinger's cat: in which, while we keep the box closed, the cat is both dead and alive at the same time. In Dirac notation, we write a superposition as $a|0\rangle + b|1\rangle$ where a and b are complex numbers. The ratio between a and b gives the relative weighting between the two states: if the system is more in the $|0\rangle$ or more in the $|1\rangle$ state. When we observe the system, it will go into one of the two states: the cat is alive (which we choose to be the $|0\rangle$ state) or the cat is dead (the $|1\rangle$ state) with a probability given by the modulus squared of the weighting of the state. Importantly by observing the state of the cat, we have destroyed the superposition and we cannot recover it.

Entanglement is the property that we can create states involving two or more objects where the state of the one object affects the state of the others. In the full experiment of Schroedinger's cat, the death of the cat is triggered by a radioactive decay which breaks a vile of poison and kills the cat. The actual essence of the experiment is the correlation between the radioactive decay (which did or didn't happen while the box is closed) and the alive/dead state of the cat. If the nucleus underwent decay, then the cat is necessarily dead; if nothing happened in the nucleus, the cat is necessarily still alive. This state of the two object system we denote $|0\rangle_N|0\rangle_C + |1\rangle_N|1\rangle_C$, the C stands for the state of the cat and N for that of the nucleus: notice this is a superposition of two two-object-states. When we open the box, the state will collapse to one or the other: if the nucleus didn't decay (state $|0\rangle_N$) the cat is still alive, thus we collapsed to the state $|0\rangle_C|0\rangle_N$, but if the nucleus did decay (state $|1\rangle_N$) the cat is now dead and we collapsed to the state $|1\rangle_C|1\rangle_N$: again we destroyed the superposition of the two states and the entanglement between the two objects by measuring the state of the system. One event necessarily provokes the other: this "spooky action at a distance" is what Einstein called entanglement.

1.3 Quantum error correcting codes to store quantum information

These quantum mechanical properties can be used to store and protect information from errors. With an error we mean an unwanted modification of the information due to interactions with the environment for instance: bit flips (as in classical computing), sign changes in a quantum superposition or a combination of both.

Classically we can protect a bit by copying it several times, such that even if one bit is faulty the copies will still contain the correct information and the information can be recovered by a majority count of the bits. However this trick cannot be used for quantum bits (qubits) as the exact copying of a quantum mechanical system onto another is prohibited by the No-Cloning Theorem. We need to find alternative ways of storing information among several systems. Entanglement can be used to link together individual qubits to store the information in a delocalised manner between them: the information is no longer in a single qubits but between all of them. We do this such that if an error occurs on one qubit, it will influence the others in a specific way in which we can find which qubit the error occurred on and correct it: crucially this can be done without destroying the entanglement between the qubits thus preserving the information encoded in the entangled qubits. There are a variety of quantum codes (ways of entangling qubits to store and protect information) which required different numbers of qubits and protect against more or less diverse types of errors. We will deal with the repetition code in Section 3 and the Steane code in Section 6.

Experimentally entangling qubits in the correct manner to create quantum codes is long and difficult as we can only interact with the atoms or ions with lasers, electric or magnetic fields. Most endeavours to entangle qubits used methods where the qubits would be entangled pairwise to create the full entanglement (see Figure 6 in [1]). This requires that all other qubits be hidden i.e. put into different states in which they are not influenced by the laser even if it shone on them while still remaining entangled: this changing of states requires many laser pulses for each qubit, making the process long and prone to errors due to laser intensity and pulse length fluctuations. In Ref. [2], it is being proposed to use the lasers to drive transverse normal mechanical vibrational modes of 2-dimensional ion qubit crystals to entangle them in the desired manner. The paper successfully shows how to entangle and store information in 5 ions using a 5-qubit code. The number of pulses required is drastically reduced from typically a few hundred (when doing pairwise entanglement) to a few tens (using global pulses). However the suitable laser pulses were found by numerical "brute force" for the specific case of a 6 ion crystal. We will investigate the action of these global pulses and determine more generally the action of global pulses on an arbitrary number of qubits in a 2-D crystal.

In Section 2, we will go through the experimental setup enabling the creation of the ion crystals. Sec. 3 explains the repetition code and how it can be achieved using global pulses. Sec. 4 deals with the action of the global pulse driving a centre of mass motion of the crystal and our discovery

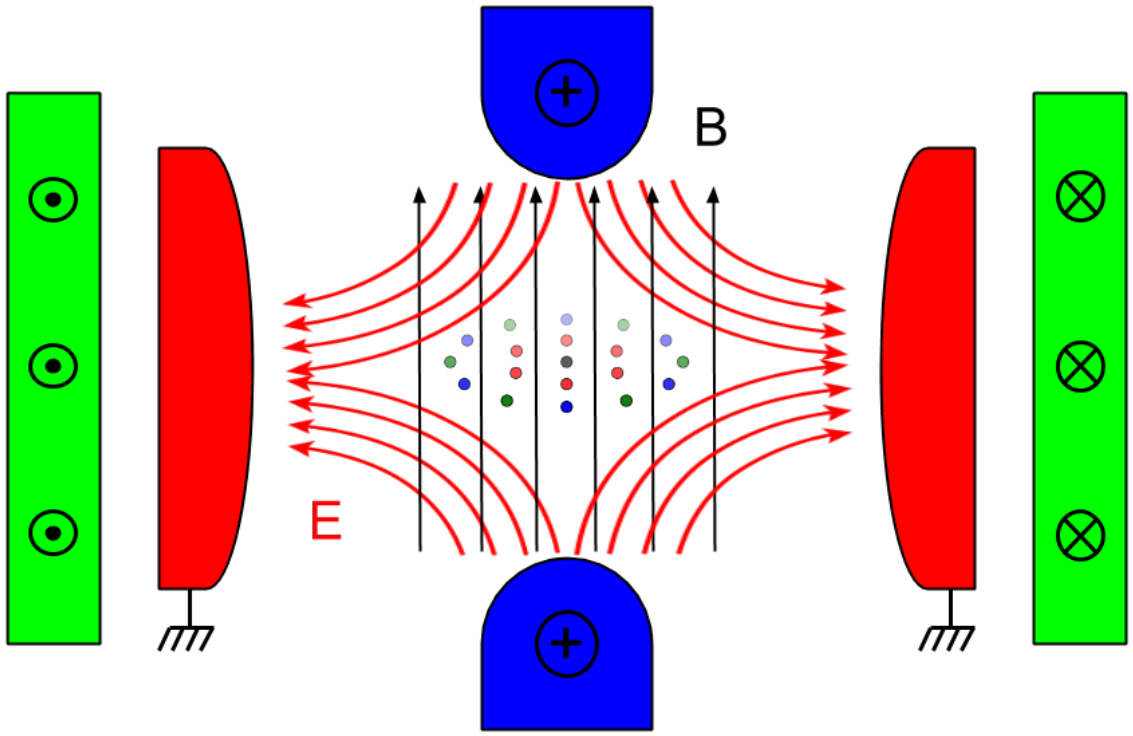


Figure 1: Penning trap setup: positively charged endcaps (blue) and a grounded cylindrical ring (red) create an axially confining potential for positive charges, the current in an outer coil (green) creates an axial homogeneous magnetic field which confines charges radially.

of efficient 2-pulse and single global pulse entangling procedures. In Sec. 5 we discuss the action of the other vibrational modes and how they can be combined to simulate localised interactions between qubits by exploiting the symmetries of the ion crystal. Finally in Sec. 6 we apply our findings to the encoding of the Steane code.

2 Experimental Setup

2.1 Ion trap and crystals

Penning trap In the experimental setup of [2], ions are trapped in a Penning trap: a static cylindrically symmetric trap. This type of trap is made up of a cylindrical quadrupole electric field and a strong homogeneous axial magnetic field. The quadrupole field is achieved with a set of three electrodes: a ring at ground and two positively charged endcaps as shown in Figure 1. The generated electric field confines positive charges in the plane of the ring electrode. To trap negative charges, the potential of the ring and endcap electrodes is inverted. The magnetic field forces the charges into a circular motion about the axis of the field, which confines the charges radially. The two fields trap the

ions in the centre of the trap: they undergo harmonic oscillating motion in the axial direction and an epicyclic radial motion similar to that of the Moon around the Sun: the Earth goes around the Sun in an ellipse and the Moon goes around the Earth in an ellipse as well. Static traps are preferred over traps with time varying fields as the latter tend to induce driven oscillations which interfere with the effects we desire to observe[3].

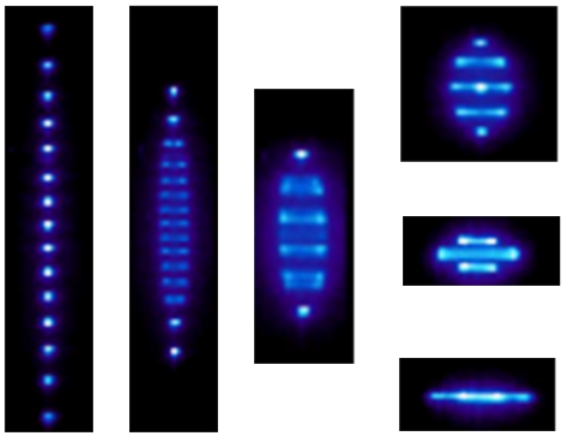


Figure 2: Examples of ionic Coulomb crystal with $^{40}C^{+}$ in a Penning trap taken from [3]. From left to right, the axial confinement is increased: the ion configuration goes from a string to a 3-D shell and finally to a 2-D plane.

Coulomb crystal When these trapped ions are sufficiently cooled, the thermal energy being much less than the electrostatic repulsion, they form Ionic Coulomb crystals[3]. This cooling is done using a laser technique called Doppler cooling. The frequency of the laser is red-tuned slightly below an atomic transition (see Figure 3 for cooling transition of $^{9}Be^{+}$), such that, if the ions are in motion, the laser frequency is now Doppler shifted to match the transition and the momentum of the photon is absorbed by the ion, slowing and cooling it down. Once the ion de-excites the photon is re-emitted, however since the emission is isotropic and the absorption is in a particular direction: a net cooling is achieved in that direction (see section 9.3 in [4]). This is done in the radial and axial directions to cool the ions to about 1 mK[3].

Setting the electric field strength adjusts the axial confinement with respect to the radial confinement and the shape of the crystal can be modified. If the axial confinement is low, the ions form a linear crystal along the axis of the trap. Increasing the electric field compresses the ionic string first to a zigzag shape, then in to a 3-dimensional crystal and finally to a 2-dimensional crystal in a plane perpendicular to the trap axis. Different confinement steps can be seen in Figure 2. With 15 ions, the 3-D crystal has an ion in the centre and a shell of ions around it. Crystals with larger ion numbers, with small axial confinement, form long strings of ions changing into concentric interwoven helices as the linear ion density increases creating multiple concentric shells[5]. The crystals we are interested in are these 2D crystal: when applying homogeneous optical dipole forces (ODF's; see section 2.3) which couple equally to all ions in the plane, these ions can be made to vibrate transversely like a drumhead.

2.2 Mechanical modes

Electromagnetic potential of the Penning trap From Maxwell's equations, we have for the electric field **E**:

$$\nabla \cdot \mathbf{E} = 0 \tag{1}$$

since the charge density is zero. And

$$\mathbf{E} = \nabla\phi \tag{2}$$

Thus

$$\nabla^2\phi = 0 \tag{3}$$

which is Laplace's equation for the electrostatic potential ϕ. The cylindrical symmetry of the Penning only allows for a solution of the form:

$$\phi = V_0\Big(z^2 - \frac{1}{2}\left[x^2 + y^2\right]\Big) \tag{4}$$

giving a potential confining the ions around the $z = 0$ minimum (axial confinement) but expelling them radially in the x-y plane.

To confine them in the x-y plane, a homogeneous magnetic field is applied along the axis of the trap, this sets the plane of ions into rotational motion about the magnetic field direction. In the rotating frame finding the vibrational modes is just an equilibrium problem. Since the Lorentz force only acts on the motion in the plane perpendicular to the trap axis, we can treat axial and planar vibration modes independently[7] and we will only consider potential terms which vary about the equilibrium in the z-direction.

Lagrangian and least action The Lagrangian in the rotating frame along the z-direction is as follows:

$$\mathbf{L}_{axis} = \frac{1}{2}\sum_{j=1}^{N}\left(m|\dot{r}_j|^2 - 2eV_0 z_j^2 - k_e e^2 \sum_{k\neq j}\frac{1}{r_{jk}}\right) \tag{5}$$

r_j and z_j are the position of ion j and its projection on the z-axis, $r_{jk} = |r_j - r_k|$, V_0 as above the the strength of the electrostatic potential, k_e is Coulomb's electrostatic constant, m is the mass of the ions and e the unit electric charge. Thus the terms are the kinetic energy, the axial confining potential and the electrostatic repulsion between ions, for which we sum over all ions. For a full treatment of all vibrational modes see Ref. [7] from which this derivation is inspired.

We want to investigate how the Lagrangian varies near equilibrium. To first order it is invariant: this defines the equilibrium positions of the ions. To second order variation in z we obtain:

$$\delta \mathbf{L}_{axis} = \frac{1}{2}\sum_{j=1}^{N}\left(\frac{\partial[m|\dot{r}_j|^2]}{\partial \dot{z}_j \partial \dot{z}_j}\delta\dot{z}_j\delta\dot{z}_j\right) - \frac{1}{2}\sum_{j,k}^{N}\delta z_j \delta z_k \frac{\partial}{\partial z_j \partial z_k}\left(\sum_{j=1}^{N}\left[2eV_0 z_j^2 + k_e e^2 \sum_{k\neq j}\frac{1}{r_{jk}}\right]\right) \tag{6}$$

$$= \frac{1}{2}\sum_{j=1}^{N} m\,\delta\dot{z}_j\delta\dot{z}_j - \frac{1}{2}\sum_{j,k}^{N} K_{j,k}^{zz}\,\delta z_j\,\delta z_k \tag{7}$$

$K_{j,k}^{zz}$ is the transverse stiffness matrix of the system:

$$K_{j,k}^{zz} = \begin{cases} 2eV_0 - \sum_{l=1,l\neq j}^{N}\frac{k_e e^2}{r_{jl}^3} & j = k \\ \frac{k_e e^2}{r_{jk}^3} & j \neq k \end{cases} \tag{8}$$

The equations of motion are found by using the Euler-Lagrange equation with $\delta\mathbf{L}_{axis}$:

$$m\,\delta\ddot{z}_j + \sum_{k=1}^{N} K_{j,k}^{zz}\,\delta z_k = 0 \qquad\qquad j = 0, 1, ..., N \tag{9}$$

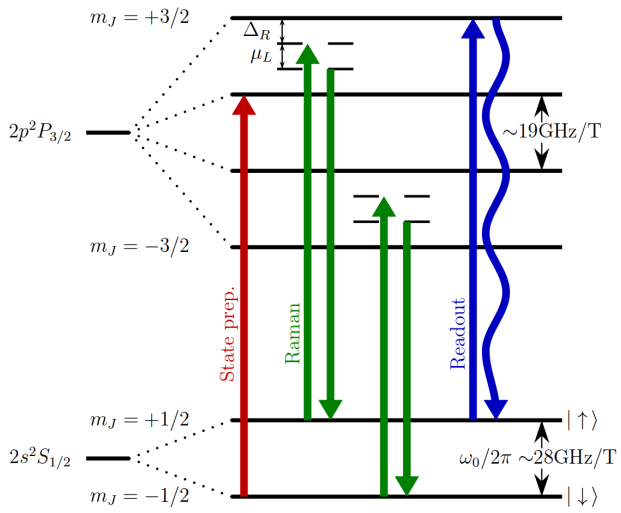


Figure 3: Atomic levels of $^9Be^+$
The ionic qubit is encoded in the two $S_{1/2}$ levels, the $P_{3/2}$ levels are used to apply the dipole force and to read out the state of the qubit.

The solutions to this equation are the N eigenvectors of $K_{j,k}^{zz}$ which form an orthonormal basis of vibration modes, since $K_{j,k}^{zz}$ is symmetric, and allow to diagonalise the stiffness matrix. The matrix of eigenvectors is A_{MI}, the rows correspond to the eigenmodes and the columns correspond to the motions of each ion in the different modes.

2.3 Producing state dependent mechanical forces in $^9Be^+$

The energy levels for $^9Be^+$ are given in Figure 3 taken from Ref. [2]: other ions with a single electron in the outer shell have similar energy level diagrams and can also be used for our purpose. The ionic qubit is encoded in the Zeeman splitting of the $S_{1/2}$ state. The "up" state $|\uparrow\rangle$ is our $|0\rangle$ and the "down" state $|\downarrow\rangle$ is used as the $|1\rangle$.

The transition labelled "Readout" is used to cool the ion using Doppler cooling and to make a projective measurement of the state of the qubit: if the qubit is in the down state there is no transition at that frequency and the electron is not transferred to any higher energy state. The transition back

down to the up state is spontaneous, not driven: during Doppler cooling this is the isotropic emission of the photon and during the measurement this is the fluorescence of photons which indicates the qubit is in the up state.

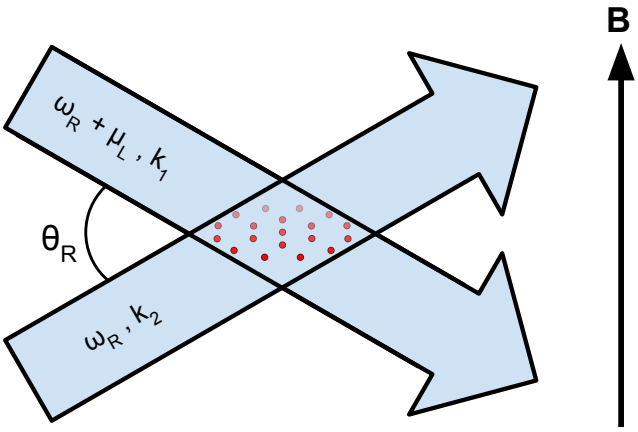


Figure 4: Setup of the ODF lasers
The lasers are set at an angle to create the 1-dimensional force lattice along the magnetic field. The two lasers need to shine evenly on the entire crystal to create a uniform force.

The two transitions labelled "Raman" are used for the state dependent optical dipole force discussed below.

The transition labelled "State preparation" is used to prepare all qubits in a clearly defined state: generally $|\uparrow\rangle$ or $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$.

The two ODF lasers are focused on the trapped ions, one is set to a frequency $\omega_1 = \omega_R + \mu_L$ and an angle $\theta_R/2$ with respect to the plane of the ions and the other is set to a frequency $\omega_2 = \omega_R$ and an angle $-\theta_R/2$. The orientation of the beams gives them wave-vectors $k_1 = k\left(cos\frac{\theta_R}{2}, 0, -sin\frac{\theta_R}{2}\right)$ and $k_2 = k\left(cos\frac{\theta_R}{2}, 0, sin\frac{\theta_R}{2}\right)$. In the region where they intersect, the beams produce a 1-dimensional optical lattice with beat note $\omega_1 - \omega_2 = \mu_L$ and wave-vector $\delta k = k_1 - k_2 = k\left(0, 0, -2sin\frac{\theta_R}{2}\right)$ in the z-direction i.e. perpendicular to the plane of the ions.

If the ω_R frequency of the Raman lasers is set to couple far off-resonantly the two qubit levels to two of the $P_{3/2}$ levels (Δ_R is large in Figure 3), this will have the effect of producing an AC Stark shift on these transitions: the presence of the time-varying electric field of the laser causes the energy levels to shift. Adjusting the polarisation of the laser beams allows to vary the relative shift of the two transitions. It is possible to set the polarisations such that the time-averaged differential shifts of the transitions are zero while the interference pattern of the two lasers creates an axially travelling polarisation gradient with wavevector δk. This causes a periodic differential shift: since the shift depends on the laser intensity, this intensity gradient induces qubit-state dependent forces on the ions.

The wavefront of the interference pattern is planar thus this setup enables us to apply state dependent ODFs uniformly across the entire crystal. Correct orientation of the two lasers with respect to the ion plane is crucial for the beat lattice to be exactly aligned in the z-direction. For large crystals, we need to ensure the forces are sufficiently uniform across all ions in the crystal. A complete description of experimental parameters, the creation of the polarisation gradient and alignment of the lasers is given in the supplement to Ref.[8].

Now we can use the beat frequency μ_L to drive transverse mechanical modes of our choice by

setting μ_L to the frequency of that modes. Let us look at how to use these mechanical modes to entangle qubits to encode quantum codes.

3 Evolution of N bit strings under global pulses

3.1 Repetition code

The repetition code is a quantum code to protect against bit flips analogous to the classical repetition code.

Form of the encoding Classical bits of information can be protected from noise by copying them multiple times. If any error occurs on one of them, a 0 flipped into a 1 or vice-versa, we can find the original information by counting the number of 0's and 1's and taking the majority count to have been the initial information. For example, we would assume that 000100100 was initially all 0's. For this to be effective, the probability of an error occurring must be sufficiently low: in the example we indeed assume that two bits flipping is more likely to have happened than 7 bits flipping.

In quantum computing, a quantum bit or qubit can be a 0 or a 1: the atom or ion carrying the qubit will be in the state denoted $|0\rangle$ (or $|\uparrow\rangle$ once we encode the information in the spin of the outer shell electron as was done in Figure 3) for the 0 and in $|1\rangle$ (or $|\downarrow\rangle$) for the 1. It can also be in a quantum superposition of the two: for example $|0\rangle + |1\rangle$; here $|0\rangle$ and $|1\rangle$ have equal weighting in the superposition but any weighting is possible as mentioned earlier.

For a quantum state, the No-Cloning-Theorem [9] forbids the copying of quantum information to other qubits. If a qubit is in the state $|\phi\rangle = a|0\rangle + b|1\rangle$ copying the state to other qubits would mean obtaining the state

$$(a|0\rangle + b|1\rangle) \otimes (a|0\rangle + b|1\rangle) \otimes \cdots \otimes (a|0\rangle + b|1\rangle) \otimes (a|0\rangle + b|1\rangle) \tag{10}$$

The tensor product $\otimes$ combines the states of different qubits into the larger state of all qubits. Here all qubits are in the state $a|0\rangle + b|1\rangle$. The state where all qubits are in $|0\rangle$ we write $|0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle \otimes |0\rangle$ or $|00\ldots00\rangle$ for short; and the state where all qubits are in $|1\rangle$ is $|11\ldots11\rangle$. Similarly we can take a superposition of these states to obtain $|00\ldots00\rangle + |11\ldots11\rangle$.

The tensor product is used to create a basis for the combined vector spaces of several systems. The basis of the vector space of a single qubit is $\{|0\rangle, |1\rangle\}$, then the basis of a two qubit system is $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\} = \{|00\rangle, |01\rangle |10\rangle, |11\rangle\}$; similarly an N qubit system has a basis composed of all possible N bit string states. This notation allows to determine if a multi-qubit state is entangled or not: if the state can be factorised, it is not entangled. For example, the state $\frac{\sqrt{6}}{4}|00\rangle + \frac{\sqrt{6}}{4}|01\rangle + \frac{\sqrt{2}}{4}|10\rangle + \frac{\sqrt{2}}{4}|11\rangle$ can be rewritten as $\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right) \otimes \left(\frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle\right)$: the two system (on

either side of the $\otimes$ sign) are clearly separated. However the state $\frac{\sqrt{6}}{4}|00\rangle + \frac{\sqrt{6}}{4}|01\rangle + \frac{\sqrt{2}}{4}|10\rangle - \frac{\sqrt{2}}{4}|11\rangle$ cannot be factorised and is entangled. This can be naturally extended to an N qubit systems.

To encode the qubit and protect it against errors, we will use the Repetition Code encoding. If a qubit is in the state $|\phi\rangle$, we will encode it in several qubits with the state

$$a|00\ldots00\rangle + b|11\ldots11\rangle \tag{11}$$

This state is not the same as the one in (10) so we are not violating the No-Cloning Theorem. Notice (10) is not entangled whereas (11) is.

Bases of states So far we have been writing states with $|0\rangle$ and $|1\rangle$ which are the eigenstates of the **Z** operator which measures the spin of the electron in the Z-direction

$$\mathbf{Z}|0\rangle = +|0\rangle \tag{12}$$

$$\mathbf{Z}|1\rangle = -|1\rangle \tag{13}$$

We can also write the state using the **X** basis of states with spin in the X-direction

$$|+\rangle = |0\rangle + |1\rangle \tag{14}$$

$$|-\rangle = |0\rangle - |1\rangle \tag{15}$$

such that

$$\mathbf{X}|+\rangle = +|+\rangle \tag{16}$$

$$\mathbf{X}|-\rangle = -|-\rangle \tag{17}$$

We can convert between these two bases by using the Hadamard operator **H**

$$\mathbf{H}|0\rangle = |+\rangle \qquad \mathbf{H}|+\rangle = |0\rangle \tag{18}$$

$$\mathbf{H}|1\rangle = |-\rangle \qquad \mathbf{H}|-\rangle = |1\rangle \tag{19}$$

Individual qubits can be rotated basis by applying **H**. If we apply it to qubit 1 and 3 of $|000\rangle + |111\rangle$, we obtain $|+0+\rangle + |-1-\rangle$ which can be expanded in the **Z** basis using (14) and (15) but this will generally not be done.

We can now transform the state in (11) into

$$a|+\,+\cdots+\,+\rangle + b|-\,-\cdots-\,-\rangle \tag{20}$$

This is not longer the same state but encodes the same information as (11).

Error correction in the repetition encoding The errors we will consider are only those due to applying Pauli operators $\mathbf{Z}$, $\mathbf{X}$ and $\mathbf{Y} = i\mathbf{XZ}$. For a qubit in state $a|0\rangle + b|1\rangle$, these errors have the following effects

$$\mathbf{Z}(a|0\rangle + b|1\rangle) = (a|0\rangle - b|1\rangle) \tag{21}$$

$$\mathbf{X}(a|0\rangle + b|1\rangle) = (b|0\rangle + a|1\rangle) \tag{22}$$

$$\mathbf{Y}(a|0\rangle + b|1\rangle) = i(-b|0\rangle + a|1\rangle) \tag{23}$$

As a consequence, a $\mathbf{Z}$ error will convert a $|+\rangle$ into a $|-\rangle$ and vice versa; and a $\mathbf{X}$ error will convert a $|0\rangle$ into a $|1\rangle$ and vice versa.

If we encode our state as in (20), a $\mathbf{Z}$ error, say on the first qubit, changes the state to

$$a|-+\cdots++\rangle + b|+-\cdots--\rangle \tag{24}$$

and we can detect this error. However if an $\mathbf{X}$ error occurs on the first qubit of the same state, it becomes

$$a|++\cdots++\rangle - b|--\cdots--\rangle \tag{25}$$

and we cannot detect this kind of error. This is because if b can be the coefficient of $|--\cdots--\rangle$, $-b$ can also be a coefficient since both are complex numbers and any complex numbers can be coefficients as long as $|a|^2 + |b|^2 = 1$ (i.e. the state is normalised; $|a|$ denotes the modulus of a): we don't know if the initial information was actually $a|++\cdots++\rangle - b|--\cdots--\rangle$ or if an error occurred.

More generally, $|++\cdots++\rangle$ is called the logical zero state and $|--\cdots--\rangle$ is the logical one state, denoted $|0_L\rangle$ and $|1_L\rangle$, they have the same coefficients as the $|0\rangle$ and the $|1\rangle$ in the $|\phi\rangle$ state so they encode the same information. For encodings other than the repetition code, the logical states will be different: a quantum code is characterised by the choice of logical states. The two logical states span a 2-dimensional vectorspace which is a subspace of the full Hilbert space of the N encoding qubits.

Errors we can detect are those which change the subspaces spanned by the logical states. In the case of the $\mathbf{Z}$ error, the subspace spanned by $|-+\cdots++\rangle$ and $|+-\cdots--\rangle$ is no longer the same as the one spanned by $|++\cdots++\rangle$ and $|--\cdots--\rangle$: we cannot write $a|-+\cdots++\rangle + b|+-\cdots--\rangle$ as a linear combination of $|++\cdots++\rangle$ and $|--\cdots--\rangle$. However for the $\mathbf{X}$ error, the subspace is unchanged since the basis is unchanged: therefore we cannot detect the error.

If we now encode the state as in (11), we can correct any bit flips (Pauli $\mathbf{X}$ errors) which would give a state like $a|10\ldots00\rangle + b|01\ldots11\rangle$; but we cannot detect Pauli $\mathbf{Z}$ errors. Other encodings like Steane's seven qubit[10] or Shor's five qubit[11] code allow to detect any $\mathbf{Z}$, $\mathbf{X}$ or $\mathbf{Y}$ error on any single qubit: the logical states are defined in such a way that any of these errors changes the subspace.

Stabiliser formalism Quantum states can also be described in term of operators of which they are the +1 eigenstates: this is called the stabiliser formalism. The state $|0\rangle$ is the +1 eigenstate of the operator $\mathbf{Z}$ and $|1\rangle$ is the +1 eigenstate of the operator $-\mathbf{Z}$, also the state $|00\rangle+|11\rangle$ is the simultaneous +1 eigenstate of the two operators $\mathbf{Z}_1\mathbf{Z}_2$ and $\mathbf{X}_1\mathbf{X}_2$ (the subscript indicates which qubit the operator acts on).

For the case of the $|00\rangle+|11\rangle$ state or any multi-qubit states, the stabilisers form a group. If $\mathbf{Z}_1\mathbf{Z}_2$ and $\mathbf{X}_1\mathbf{X}_2$ both stabilise the state then their product also will

$$\begin{aligned}\left[\left(\mathbf{X}_1\mathbf{X}_2\right)\left(\mathbf{Z}_1\mathbf{Z}_2\right)\right]\left(|00\rangle+|11\rangle\right) &= \left(\mathbf{X}_1\mathbf{X}_2\right)\left[\left(\mathbf{Z}_1\mathbf{Z}_2\right)\left(|00\rangle+|11\rangle\right)\right] \\ &= \left(\mathbf{X}_1\mathbf{X}_2\right)\left(|00\rangle+|11\rangle\right) = \left(|00\rangle+|11\rangle\right)\end{aligned} \tag{26}$$

So $|00\rangle+|11\rangle$ actually has an entire group of stabilisers not just two stabilisers but we will only mention the generators of the group i.e. the smallest number of elements which allow to generate all the other elements of the group by taking their products: these are written between angled brackets $\langle\ldots\rangle$.

We can now define the logical states of a quantum code as the two orthonormal states which are the +1 eigenstates of a set of stabilisers. The Steane, Shor or repetition codes can be compactly be defined in such a manner. Appendix 9.1 gives more advantages of the formalism (how to deal with certain unitary evolutions and measurements) which we will use to describe encodings in Sections 4.3 and 6.

3.2 Target N qubit states

To achieve the repetition encoded state in the $\mathbf{X}$ basis, we start with the information stored in a single qubit as the $|\phi\rangle$ state and with all the $N-1$ encoding qubits in the $|+\rangle$ state

$$|\phi\rangle\otimes|++\cdots++\rangle = a|0\rangle\otimes|++\cdots++\rangle + b|1\rangle\otimes|++\cdots++\rangle \tag{27}$$

We need to apply an operation which leaves the encoding qubits in $|+\rangle$ when the first qubit is $|0\rangle$ and changes the encoding states to $|-\rangle$ when the first qubit is $|1\rangle$. The Pauli $\mathbf{Z}$ operation changes a $|+\rangle$ into a $|-\rangle$, so we need to apply a $\mathbf{Z}$ on all encoding qubits controlled on the first qubit.

When using $|+\rangle$ states, it is easy to create a superposition of all possible $\mathbf{Z}$ basis states and treat all states simultaneously. For two qubits, the state $|++\rangle$ can be rewritten as

$$|00\rangle+|01\rangle+|10\rangle+|11\rangle \tag{28}$$

which is the equal-weighted superposition of all possible two bit states. Similarly $|+\rangle^{\otimes N}$ is the superposition of all possible N bit states:

$$\begin{aligned}&|00\ldots000\rangle+|00\ldots001\rangle+|00\ldots010\rangle+|00\ldots011\rangle \\ &\cdots+|11\ldots100\rangle+|11\ldots101\rangle+|11\ldots110\rangle+|11\ldots111\rangle\end{aligned} \tag{29}$$

Let us consider what the effect of this controlled operation is on the binary string states which make up (29). Since applying $\mathbf{Z}$, induces a "$-$" sign to the $|1\rangle$ state (see (13)), when applying $\mathbf{Z}$ to all encoding qubits we simply need to count the number of 1's in a term and this state will acquire a factor $(-1)^{\#1}$, where $\#1$ is the number of 1's in the term. For example, we apply Controlled-$\mathbf{Z}$ (C$\mathbf{Z}$) to two encoding qubits

$$\begin{aligned} & C\mathbf{Z}\big[|+\rangle \otimes |++\rangle\big] \\ = & C\mathbf{Z}\big[|0\rangle \otimes (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ & +|1\rangle \otimes (|00\rangle + |01\rangle + |10\rangle + |11\rangle)\big] \end{aligned} \tag{30}$$

When the control is in the $|0\rangle$ state, the two qubits are unchanged. When the control is in the $|1\rangle$ state, the $|01\rangle$ and $|10\rangle$ states acquire a "$-$" sign; the $|11\rangle$ state acquired two "$-$" signs which cancel. Thus the state after applying the C$\mathbf{Z}$ is

$$\begin{aligned} & |0\rangle \otimes (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ & +|1\rangle \otimes (|00\rangle - |01\rangle - |10\rangle + |11\rangle) \end{aligned} \tag{31}$$

The C$\mathbf{Z}$ is analogously generalised to N qubits. In practise, when applying the C$\mathbf{Z}$ to $|++\cdots++\rangle$, we expand the state into binary strings and make them acquire the correct phases.

After obtaining the state $a|0\rangle \otimes |++\cdots++\rangle + b|1\rangle \otimes |--\cdots--\rangle$, we measure the first (control) qubit in the $\mathbf{X}$ basis to finish the encoding. Indeed the state can be rewritten as

$$\begin{aligned} & |+\rangle \otimes \big(a|++\cdots++\rangle + b|--\cdots--\rangle\big) \\ & +|-\rangle \otimes \big(a|++\cdots++\rangle - b|--\cdots--\rangle\big) \end{aligned} \tag{32}$$

If the outcome of the measurement is $|+\rangle$, the encoding qubits will be in the $a|++\cdots++\rangle+b|--\cdots--\rangle$ state; if it is $|-\rangle$, the remaining qubits will be in the $a|++\cdots++\rangle - b|--\cdots--\rangle$ state and we just need to apply a $\mathbf{X}$ to any qubit to correct it. Thus, we have deterministically transferred the information from $a|0\rangle + b|1\rangle$ to the $a|++\cdots++\rangle - b|--\cdots--\rangle$ state. Notice the information is gone from the control qubit which is now in the $|+\rangle$ or $|-\rangle$ state: we did not clone the information.

A string having a "$+$" sign corresponds to a phase of e^{i0} and a "$-$" sign is $e^{i\pi}$. Complex phases are equivalent modulo 2π and a global phase on the state is irrelevant in quantum mechanics. Therefore we have extra freedom in the absolute phase which we induce in the strings: relative phases only need to be correct modulo 2π and all states can share a global offset phase. We will always take angles to be divided by π and equivalent modulo 2.

The state needs to evolve in such a way that the result modulo 2 is a C$\mathbf{Z}$: this evolution we implement by driving the vibrational modes of the crystal.

3.3 Driving the mechanical modes of the crystal

By driving the mechanical modes of the crystal, we can induce phases to N qubit states as shown in Ref. [12] for the case of entangling 2 ion qubits. Here we follow the extension proposed by Ref.[2] to N ion qubits and we induce phases to all the strings in $(a|0\rangle + b|1\rangle) \otimes |+\rangle^{\otimes N-1}$.

State dependent forces on the qubits are created by the ODF lasers, therefore every N bit string can be equally represented by a string of forces. For example the string $|0010110\rangle$ can be represented by $|F_\uparrow F_\uparrow F_\downarrow F_\uparrow F_\downarrow F_\downarrow F_\uparrow\rangle$. The coupling of each state to a vibrational mode is given by the inner product of the force vector with the mode vector

$$\sum_I F_I a_I \tag{33}$$

where F_I is the force on ion I and a_I is the amplitude of the motion of ion I in the given vibrational mode.

We take F_{IS} to be the matrix of force vectors: each column is a force string and each row corresponds to an ion in that string; F_{IS} is a $N \times 2^N$ matrix, since we have to considering all possible binary strings in the initial state. A_{MI} is the matrix of vibrational eigenmodes: rows correspond to the modes, columns to the ions. Then $M_{SM} = (A_{MI}F_{IS})^T$ gives the generalised force on strings S by eigenmodes M: how much the string couples to a given mode.

A mechanical mode is driven slightly detuned from resonance by δ for the mode to undergo beating with frequency δ: first being excited up to maximum amplitude, then back to rest within a period $\frac{1}{\delta}$. This induces a phase to string S given by

$$\phi_S = \frac{A_S}{\hbar} \tag{34}$$

A_S is the area inside the closed loop the string state follows in phase space (position-momentum space)[12]. The phase under mode M is proportional to the square of the generalised force under this mode thus we define the matrix P_{SM} which is obtained by squaring every element of M_{SM}. If we now encode the laser pulse intensity and length for each mode in v_M, the complex phase acquired by a string is[2]

$$\phi_S = P_{SM} v_M \tag{35}$$

v_M is normalised such that a centre of mass pulse of length of $\frac{1}{F_\downarrow^2 N}$ induces an absolute phase of 1 (in units of π) to the $|11\ldots11\rangle$.

If the ionic crystal has symmetries, all strings which are equivalent under a transformation of the symmetry group of the crystal will evolve in the same manner. For a crystal with an ion in the centre and a ring of 6 ions, one symmetry of the crystal is a rotation by $\frac{\pi}{6}$ about the axis perpendicular to the plane of ions. Under this transformation, the strings $|0000001\rangle$ and $|0000010\rangle$ are equivalent

(the first qubits being the central one) or $|0000101\rangle$ and $|0001010\rangle$ are also equivalent. Similar for all other transformations and strings. Notice $|0000001\rangle$ and $|1000000\rangle$ are not equivalent since no transformation interchanges the central (first) ion with an ion in the ring. These symmetries can cause some of the eigenmodes to be degenerate in frequency, these modes will be driven simultaneously and the corresponding rows in the P_{SM} matrix can be added up.

We have explained how a state evolves under the application of laser pulses driving the mechanical vibration modes of the ionic crystal. Let us look at the specific effect of the centre of mass pulse in the next section and how to combine different pulses in Sec. 5

4 Centre of mass pulses

4.1 Action of the CoM mode

In the centre of mass mode, all ions in unison follow the same transverse motion with amplitude $\frac{1}{\sqrt{N}}$.

Control on one qubit The phase per unit pulse length due to the CoM mode on a string can be expressed as

$$\left(\sum_I F_{IS} A_{MI}\right)^2 = \frac{1}{N}\left[(F_\downarrow - F_\uparrow)\left(\langle\downarrow_1 |S_1\rangle + \sum_{I\in R}\langle\downarrow_I |S_I\rangle\right) + NF_\uparrow\right]^2 \tag{36}$$

where we emphasise the split between a control qubit (say the first one) and the remaining qubits taken to be the set R. $\langle\downarrow_I |S_I\rangle$ is the overlap (inner product of the state vectors) of the state of qubit I in the string with the down state: 0 if it is in $|0\rangle$ and 1 if it is in $|1\rangle$; then $\sum_{I\in R}\langle\downarrow_I |S_I\rangle$ is the number of qubits in the set R which are in the down state. We expand and choose $(F_\downarrow - F_\uparrow) = 1$, a choice we will argue later with (64)

$$\begin{aligned}\frac{1}{N}\Bigg(&\langle\downarrow_1 |S_1\rangle^2 + 2\,\langle\downarrow_1 |S_1\rangle\,\left[\sum_{I\in R}\langle\downarrow_I |S_I\rangle\right] + \left[\sum_{I\in R}\langle\downarrow_I |S_I\rangle\right]^2\Bigg)\\ &+ 2\,F_\uparrow\left(\langle\downarrow_1 |S_1\rangle + \left[\sum_{I\in R}\langle\downarrow_I |S_I\rangle\right]\right) + NF_\uparrow^2\end{aligned} \tag{37}$$

$$= \frac{2}{N}\,\langle\downarrow_1 |S_1\rangle\,\left[\sum_{I\in R}\langle\downarrow_I |S_I\rangle\right] + \frac{1}{N}\left(\langle\downarrow_1 |S_1\rangle + NF_\uparrow\right)^2 + \frac{1}{N}\left(\left[\sum_{I\in R}\langle\downarrow_I |S_I\rangle\right] + NF_\uparrow\right)^2 - NF_\uparrow^2 \tag{38}$$

The $\langle\downarrow_1 |S_1\rangle\left[\sum_{I\in R}\langle\downarrow_I |S_I\rangle\right]$ term is the control term for the C**Z** operation: if the control qubit is in the up state, there is no contribution, if it is in the down state, the contribution is $\left[\sum_{I\in R}\langle\downarrow_I |S_I\rangle\right]$. Since $\left[\sum_{I\in R}\langle\downarrow_I |S_I\rangle\right]$ is the number of remaining qubits in the down state, modulo 2 it is the parity of the remaining qubits. We will choose a pulse length such that this term contributes exactly $\langle\downarrow_1 |S_1\rangle\left[\sum_{I\in R}\langle\downarrow_I |S_I\rangle\right]$ to the phase: if the parity of the remaining qubits is even, this term adds a phase which is even (remember in units of π) so does not induce a "$-$" sign and if the parity is odd,

the added phase is odd i.e. introducing a "−" sign in front of the string. This is the C**Z** operation we are looking for. Later we will choose a pulse length of $\frac{N}{2}$ for this pulse.

We ignore the last term since it is a global phase affecting all strings equally. The second term can be discarded if it is the same (modulo 2) no matter what the state of the control qubit is ($\langle\downarrow_1|S_1\rangle = 0$ or 1): this will give a constraint on the value of $F_\uparrow$. We need

$$(1 + NF_\uparrow)^2 - (NF_\uparrow)^2 = 1 + 2\,NF_\uparrow = 4\,l_1 \tag{39}$$

l_1 is an integer. The factor of 4 is required as the pulse length will have a factor of $\frac{1}{2}$ to correct the control term. We obtain

$$F_\uparrow = \frac{2\,l_1 - \frac{1}{2}}{N} \tag{40a}$$

$$F_\downarrow = 1 + F_\uparrow = \frac{2\,l_1 - \frac{1}{2} + N}{N} \tag{40b}$$

$$\frac{F_\uparrow}{F_\downarrow} = \frac{2\,l_1 - \frac{1}{2}}{2\,l_1 - \frac{1}{2} + N} \tag{40c}$$

Indeed using our approach we can reproduce the choice of state dependent forces in [2] for $N = 6$, the CoM pulse with ratio $\frac{1}{1.4}$ corresponds to $l_1 = -1$.

Now the third term of (38) becomes

$$\frac{1}{2}\Big(\left[\sum_{I\in R}\langle\downarrow_I|S_I\rangle\right] + NF_\uparrow\Big)^2 = \frac{1}{2}\left(\left[\sum_{I\in R}\langle\downarrow_I|S_I\rangle\right] + 2\,l_1 - \frac{1}{2}\right)^2 \tag{41}$$

$$= \frac{1}{2}\left(\left[\sum_{I\in R}\langle\downarrow_I|S_I\rangle\right] - \frac{1}{2}\right)^2 \tag{42}$$

The factor of $\frac{1}{2}$ takes into account that the pulse length will be $\frac{N}{2}$, this allows to consider the results modulo 2 instead of 4 (as in (39)). (41) is obtained from (42) by expanding, ignoring any global phases and taking remainders modulo 2. This is the term we need to correct for.

This correction can be applied by using another CoM pulse with the control qubit always in the up state ($\langle\downarrow_1|S_1\rangle = 0$), the phase contribution is

$$\frac{1}{N}\big(NF_\uparrow\big)^2 + \frac{1}{N}\left(\left[\sum_{I\in R}\langle\downarrow_I|S_I\rangle\right] + NF_\uparrow\right)^2 - NF_\uparrow^2 \tag{43}$$

The first and third terms cancel and we are left with the term we needed to correct for.

CZ procedure The procedure to apply a Controlled-**Z** operation on N qubits using only global pulses is as follows. We first apply a centre of mass pulse of length c_1 with the control qubit in $|\uparrow\rangle$ to apply the phases that will correct the second CoM pulse. Then we rotate the control qubit into a state $a|\uparrow\rangle + b|\downarrow\rangle$ and apply a second CoM pulse with length c_2 (which we have already argued to be $\frac{N}{2}$). The phase induced in a string is

$$\frac{c_1}{N}\left(\left[\sum_{I\in R}\langle\downarrow_I|S_I\rangle\right] + NF_\uparrow\right)^2 + \frac{2c_2}{N}\langle\downarrow_1|S_1\rangle\left[\sum_{I\in R}\langle\downarrow_I|S_I\rangle\right] + \frac{c_2}{N}\left(\left[\sum_{I\in R}\langle\downarrow_I|S_I\rangle\right] + NF_\uparrow\right)^2 \tag{44}$$

To eliminate the first and third terms modulo 2, we choose

$$\frac{c_1}{N} + \frac{c_2}{N} = 1 \tag{45a}$$

$$\implies c_1 = \frac{N}{2} \tag{45b}$$

By rotating the control qubit into a state $a|\uparrow\rangle + b|\downarrow\rangle$, we have achieved to prepare the N qubit state into $a|0\rangle \otimes |++\cdots++\rangle + b|1\rangle \otimes |--\cdots--\rangle$. We can use this operation to encode the information $a|\uparrow\rangle + b|\downarrow\rangle$ in the $N-1$ qubits using the repetition code by now measuring the first qubit in the $\mathbf{X}$ basis to finish the encoding into $a|++\cdots++\rangle + b|--\cdots--\rangle$, as exemplified in (32). The above procedure allows to apply a Controlled-$\mathbf{Z}$ to an arbitrary number of qubits. However we can also encode the information in the repetition code of the $\mathbf{Y}$ using a single CoM pulse, which we will discuss in Section 4.3.

4.2 Phases under the CZ

The total phase acquired by a string in the entangling process given in the previous subsection is

$$\begin{aligned}
&\frac{N}{2}\left[\frac{1}{N}\left(\left[\sum_{I\in R}\langle\downarrow_I|S_I\rangle\right] + NF_\uparrow\right)^2\right] \\
&+\frac{N}{2}\left[\frac{2}{N}\langle\downarrow_1|S_1\rangle\left[\sum_{I\in R}\langle\downarrow_I|S_I\rangle\right] + \frac{1}{N}\left(\langle\downarrow_1|S_1\rangle + NF_\uparrow\right)^2\right. \qquad (46a)\\
&\quad\left. + \frac{1}{N}\left(\left[\sum_{I\in R}\langle\downarrow_I|S_I\rangle\right] + NF_\uparrow\right)^2 - NF_\uparrow^2\right] \\
&= nAnR + \left(nR + 2l_1 - \frac{1}{2}\right)^2 + \frac{1}{2}\left(nA + 2l_1 - \frac{1}{2}\right)^2 - \frac{1}{2}\left(2l_1 - \frac{1}{2}\right)^2 = \phi_S \qquad (46b)
\end{aligned}$$

where we use (40) and relabel $nA = \langle\downarrow_1|S_1\rangle$ and $nR = \sum_{I\in R}\langle\downarrow_I|S_I\rangle$.

We can find the smallest absolute phase acquired by a string by solving

$$\frac{d\phi_S}{dnR} = 2\left(nR^* + 2l_1 + \frac{1}{2}\right) + nA = 0 \tag{47}$$

$$\implies nR^* = -\frac{nA}{2} - 2l_1 + \frac{1}{2} \tag{48}$$

For $nA = 0$, nR^* is a half integer: there are two sets of states with the lowest phase, $nR = -2l_1$ and $nR = -2l_1 + 1$. For $nA = 1$, only the states with $nR = -2l_1$ have the lowest phase. We can substitute these values of nR into the expressions for ϕ_S and find the phase to be $\frac{\pi}{4}$. This is the smallest phase acquired by states using this entangling method, but which states they are still depends on our choice of l_1. From (46b), we see that ϕ_S depends quadratically on nR: to have the smallest absolute phase differences between states, we need the extremes $nR = 0, nA = 0$ and $nR = N-1, nA = 1$ to be as

close to the minimum as possible. Thus we want

$$\begin{aligned}\phi_S(nR = N-1, nA = 1) - \phi_S(nR = 0, nA = 0) \\ = \big(N-1\big)^2 + 2\big(N-1\big)\left(2l_1^* - \frac{1}{4}\right) + 2l_1^* = 0\end{aligned} \tag{49}$$

$$\implies l_1^* = \frac{1}{4}\frac{(N-1) - 4(N-1)^2}{4(N-1)+1} \tag{50}$$

Taking l_1 to be the closest integer to l_1^* is equivalent to taking the forces according to (40) such that the ratio is closest to -1. Thus the smallest phase is $\frac{\pi}{4}$ and the largest is $\phi_S(nR = N-1, nA = 1)$ if the ratio of forces is greater than -1 (absolute value of $F_\downarrow$ is greater than that of $F_\uparrow$) and $\phi_S(nR = 0, nA = 0)$ if it is less than -1: which are respectively

$$\left(2l_1 - \frac{1}{2}\right)^2 \tag{51}$$

and

$$\frac{1}{2}\left(N + 2l_1 - \frac{3}{2}\right)^2 + \frac{1}{2}\left(N + 2l_1 - \frac{1}{2}\right)^2 \tag{52}$$

since we can rewrite ϕ_S as

$$\frac{c_1}{N}\left(nR + 2l_1 - \frac{1}{2}\right)^2 + \frac{c_2}{N}\left(nA + nR + 2l_1 - \frac{1}{2}\right)^2 \tag{53}$$

By choosing this value of l_1, we are also able to minimise the variance of the set of phases. There are $2\binom{N-1}{nR}$ strings with nR 1's: $\binom{N-1}{nR}$ for $nA = 0$ and $\binom{N-1}{nR}$ for $nA = 1$. Thus we have the greatest number of strings around $\frac{N-1}{2}$. From (48), we know that the minimum phase is around $nR = -2l_1$, since l_1 is approximately $-\frac{N-1}{4}$ (between $-\frac{N-1}{4}$ and $-\frac{N-2}{4}$; from (50)): the minimum phase is around $\frac{N-1}{2}$ where there are the most states, this is also where the phase change with nR is smallest since ϕ_S is quadratic in nR at that minimum. The majority of states have the smallest possible phase difference which minimised the variance.

The minimisation of absolute phase differences between states is experimentally important for the propagation of errors to be as small as possible. The larger the absolute phase of a state, the greater will be the error due to imperfections in the the laser intensity and pulse duration.

4.3 Y basis repetition code

Let us now only take the second pulse, the phase evolution of a string is given by (36). We choose ODF's according to (40), then the phases for a pulse of length $c_2 = \frac{N}{2}$ and $l_1 = 0$ are given by Table 1. This state is stabilised by

$$\begin{aligned}\langle \mathbf{X}_1\mathbf{Z}_2\mathbf{Z}_3\ldots\mathbf{Z}_{N-1}\mathbf{Z}_N,\ \mathbf{Z}_1\mathbf{X}_2\mathbf{Z}_3\ldots\mathbf{Z}_{N-1}\mathbf{Z}_N,\ \ldots\ , \\ \mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3\ldots\mathbf{X}_{N-1}\mathbf{Z}_N,\ \mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3\ldots\mathbf{Z}_{N-1}\mathbf{X}_N\rangle\end{aligned} \tag{54}$$

Table 1: Phases of states after the centre of mass pulse.

nR	$nA=0$	$nA=1$
0	0.125	0.125
1	0.125	1.125
2	1.125	3.125
3	3.125	6.125
4	6.125	10.125
5	10.125	15.125
6	15.125	21.125
7	21.125	28.125
...	...	...

Stabiliser I is composed of an $\mathbf{X}$ on qubit I and $\mathbf{Z}$'s on all other qubits. We now apply $\frac{1}{\sqrt{2}}\big[\mathbf{Z}+\mathbf{Y}\big] = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ i & -1\end{pmatrix}$ on the first qubit to obtain

$$\langle -\mathbf{X}_1\mathbf{Z}_2\mathbf{Z}_3\dots\mathbf{Z}_{N-1}\mathbf{Z}_N,\ \mathbf{Y}_1\mathbf{X}_2\mathbf{Z}_3\dots\mathbf{Z}_{N-1}\mathbf{Z}_N,\ \dots,\ \mathbf{Y}_1\mathbf{Z}_2\mathbf{Z}_3\dots\mathbf{X}_{N-1}\mathbf{Z}_N,\ \mathbf{Y}_1\mathbf{Z}_2\mathbf{Z}_3\dots\mathbf{Z}_{N-1}\mathbf{X}_N\rangle \tag{55}$$

The last $N-1$ stabilisers can be re-written to

$$\langle -\mathbf{X}_1\mathbf{Z}_2\mathbf{Z}_3\dots\mathbf{Z}_{N-1}\mathbf{Z}_N,\ \mathbf{Y}_2\mathbf{Y}_3,\ \mathbf{Y}_3\mathbf{Y}_4,\ \dots,\ \mathbf{Y}_{N-2}\mathbf{Y}_{N-1},\ \mathbf{Y}_{N-1}\mathbf{Y}_N, \tag{56}$$

which is the state $|0\rangle\otimes|+i\ +i\ \dots\ +i\ +i\rangle - |1\rangle\otimes|-i\ -i\ \dots\ -i\ -i\rangle$, where $|\pm i\rangle$ the eigenstates of the Pauli $\mathbf{Y}$. We take $|+i\ +i\ \dots\ +i\ +i\rangle$ and $|-i\ -i\ \dots\ -i\ -i\rangle$ to be our logical states $|0_L\rangle$ and $|1_L\rangle$.

We now simply need to rotate the first qubit to obtain

$$\begin{aligned}&\big(a|0\rangle + b|1\rangle\big)\otimes|0_L\rangle + \big(b|0\rangle - a|1\rangle\big)\otimes|1_L\rangle\\ &= |0\rangle\otimes\big(a|0_L\rangle + b|1_L\rangle\big) + |1\rangle\otimes\big(b|0_L\rangle - a|1_L\rangle\big)\end{aligned} \tag{57}$$

followed by an $\mathbf{Z}$ basis measurement to encode the information in the repetition code of the $\mathbf{Y}$ basis. For the $|1\rangle\langle 1|$ outcome, we correct by applying a Pauli $\mathbf{Y}$ to any qubit then $\mathbf{Z}$ to all of them. Applying $\frac{1}{\sqrt{2}}\big[\mathbf{Z}+\mathbf{Y}\big]$ and the qubit rotation can be combined into a single action.

Minimising the phase differences in Table 1 can be done in a similar way to Section 4.2 by choosing l_1 to be the closest integer to $-\frac{N-1}{4}$.

In this section, we have exploited the centre of mass mode to implement a Controlled-$\mathbf{Z}$ operation and a repetition code encoding between an arbitrary number of qubits. Let us now see how we can drive different modes successively to implement new unitary evolutions.

5 A different matrix formalism

5.1 P_{sm} as an N x N matrix

The phase per unit pulse length acquired by a string state $|S\rangle$ under a mode M is P_{SM} which can be expanded as

$$\left(\sum_I F_{IS}A_{MI}\right)^2 = \left(\sum_I \big[(F_\downarrow - F_\uparrow)\langle\downarrow_I|S_I\rangle + F_\uparrow\big]A_{MI}\right)^2 \tag{58}$$

where F_{IS} and A_{MI} refer to the elements of the respective matrices; $F_\downarrow$ and $F_\uparrow$ are the forces acting on an ion when it is in state down or up, then $F_{IS} = \big[(F_\downarrow - F_\uparrow)\langle\downarrow_I |S_I\rangle + F_\uparrow\big]$. Only in this subsection, will we introduce the strings $|n\rangle$ which are the non-normalised versions of $|S\rangle$: the components are 0 or 1 and $\langle n|n\rangle$ is equal to the Hamming weight of the string: they will make the argument clearer, but is unnecessary in following subsections . Notice $\langle S|S\rangle \neq \langle n|n\rangle$ but $\langle\downarrow_I |S_I\rangle = \langle\downarrow_I |n_I\rangle$, thus $|n\rangle$ will only be used when needed.

We can further expand

$$\sum_{I,J} A_{MI}A_{MJ}\big[(F_\downarrow - F_\uparrow)\langle\downarrow_I |S_I\rangle + F_\uparrow\big]\big[(F_\downarrow - F_\uparrow)\langle\downarrow_J |S_J\rangle + F_\uparrow\big] \tag{59}$$

$$\begin{aligned}
&= \sum_{I,J} A_{MI}A_{MJ}(F_\downarrow - F_\uparrow)^2 \langle\downarrow_I |S_I\rangle\langle\downarrow_J |S_J\rangle \\
&\qquad + \sum_{I,J} A_{MI}A_{MJ}F_\uparrow(F_\downarrow - F_\uparrow)\big(\langle\downarrow_I |S_I\rangle + \langle\downarrow_J |S_J\rangle\big) \\
&\qquad + \sum_{I,J} F_\uparrow^2 A_{MI}A_{MJ}
\end{aligned} \tag{60}$$

Non-centre of mass modes All modes which are not the centre of mass mode satisfy

$$\sum_I A_{MI} = 0. \tag{61}$$

Then the second and third term of (60) can be simplified to

$$F_\uparrow(F_\downarrow - F_\uparrow)\left(\sum_I \langle\downarrow_I |S_I\rangle A_{MI}\sum_j A_{Mj} + \sum_J \langle\downarrow_J |S_J\rangle A_{MJ}\sum_I A_{MI}\right) = 0 \tag{62}$$

and

$$F_\uparrow^2 \sum_I A_{MI} \sum_J A_{MJ} = 0, \tag{63}$$

using (61).

Thus the phase is

$$(F_\downarrow - F_\uparrow)^2 \sum_{I,J} \langle S_I|\downarrow_I\rangle A_{MI}A_{MJ}\langle\downarrow_J |S_J\rangle. \;=\; (F_\downarrow - F_\uparrow)^2\langle n|Q'_M|n\rangle \;=\; \langle n|Q_M|n\rangle \tag{64}$$

Q_M is the outer product of the mode eigenvector with itself, multiplied by $(F_\downarrow - F_\uparrow)^2$. Notice the factor of $(F_\downarrow - F_\uparrow)^2$ in the middle expression which we can absorb into the pulse length: we conclude we have the freedom of changing this factor, done by changing the state dependent forces, to an arbitrary value which will be compensated by a change in the pulse length. We will often choose $(F_\downarrow - F_\uparrow)^2 = 1$.

Centre of mass mode For the centre of mass mode

$$A_{MI} = \frac{1}{\sqrt{N}} \qquad\qquad \forall\, I = 1, \ldots, N \tag{65}$$

and (60) becomes

$$\frac{1}{N}(F_d - F_u)^2 \sum_{I,J} \langle \downarrow_I |S_I\rangle \langle \downarrow_J |S_J\rangle \; + \; 2F_u(F_d - F_u) \sum_I \langle \downarrow_I |S_I\rangle \; + \; F_u^2 N \tag{66}$$

$$= (F_d - F_u)^2 \langle n| \frac{\mathbb{1}}{N} |n\rangle + \; 2F_u(F_d - F_u)\langle n|\mathbb{I}|n\rangle \; + \; F_u^2 N \tag{67}$$

In the second term, $\sum_I \langle \downarrow_I |S_I\rangle$ is equal to the Hamming weight $\langle n|n\rangle$ of $|S\rangle$. $\mathbb{1}$ is the $N \times N$ matrix where all elements are 1 and $\mathbb{I}$ is the identity. We can ignore the third term since it is a global phase affecting all strings. The Q_M^C matrix for the CoM mode is

$$\frac{(F_\downarrow - F_\uparrow)^2}{N} \mathbb{1} + 2F_u(F_\downarrow - F_\uparrow)\mathbb{I} \tag{68}$$

The first term is still the outer product of the mode vector but there is an extra term due to the fact the CoM mode doesn't satisfy (61). This matrix has all elements being $\frac{(F_\downarrow - F_\uparrow)^2}{N}$ except the diagonal which is $\frac{(F_\downarrow - F_\uparrow)^2}{N} + 2F_u(F_d - F_u)$.

We can now combine pulses by adding together Q_M matrices for different pulses weighted by v_M: the total phase acquired by $|S\rangle$ is $\langle n| \sum_M v_M Q_M |n\rangle$. Degenerate pulses mentioned at the end of section 3.3 can be driven simultaneously by adding up their Q_M matrices.

5.2 Simulation of Hamiltonians

The evolution of a state $|S\rangle$ under a mode M for a pulse length v_M is therefore

$$exp\left[i \, v_M \sum_{I,J} \langle S_I| \downarrow_I\rangle Q_{M,(I,J)} \langle S_J| \downarrow_J\rangle \right] |S\rangle \tag{69}$$

given that we can rewrite $\langle S_I| \downarrow_I\rangle \langle S_J| \downarrow_J\rangle$ as $\frac{1}{4}\langle S|(\mathbf{Z}_I - \mathbb{I}_I)(\mathbf{Z}_J - \mathbb{I}_J)|S\rangle$, this becomes

$$exp\left[\frac{i \, v_M}{4} \sum_{I,J} \langle S|(\mathbf{Z}_I - \mathbb{I}_I) Q_{M,(I,J)} (\mathbf{Z}_J - \mathbb{I}_J)|S\rangle \right] |S\rangle \tag{70}$$

$$\begin{aligned} = exp\Bigg[& \frac{i \, v_M}{4} \sum_{I,J} \langle S|\mathbf{Z}_I Q_{M,(I,J)} \mathbf{Z}_J |S\rangle - \frac{i \, v_M}{2} \sum_I \langle S|\mathbf{Z}_I|S\rangle \sum_J Q_{M,(I,J)} \\ & + \frac{i \, v_M}{4} \sum_{I,J} \langle S|Q_{M,(I,J)}|S\rangle \Bigg] |S\rangle \end{aligned} \tag{71}$$

For non-CoM pulses the second and third terms are zero due to (61) and we are left with

$$exp\left[\frac{i \, v_M}{4} \sum_{I,J} \langle S|\mathbf{Z}_I Q_{M,(I,J)} \mathbf{Z}_J |S\rangle \right] |S\rangle \tag{72}$$

$$= exp\left[\frac{i \, v_M}{4} \sum_{I,J} Q_{M,(I,J)} \mathbf{Z}_I \mathbf{Z}_J \right] |S\rangle \tag{73}$$

i.e. an interaction Hamiltonian $\sum_{I,J} Q_{M,(I,J)} \mathbf{Z}_I \mathbf{Z}_J$ for a time $\frac{v_M}{4}$. By combining pulses, we can simulate Hamiltonians of the form

$$\sum_{M,I,J} \frac{v_M}{4} Q_{M,(I,J)} \mathbf{Z}_I \mathbf{Z}_J \tag{74}$$

For the CoM mode, (71) becomes

$$\begin{aligned} exp\Bigg[& \frac{i\, v_M}{4} \sum_{I,J} \langle S|\mathbf{Z}_I Q^C_{M,(I,J)} \mathbf{Z}_J|S\rangle - \frac{i\, v_M}{2} \sum_I \langle S|\mathbf{Z}_I|S\rangle \Big[(F_\downarrow - F_\uparrow)^2 + 2F_\uparrow(F_\downarrow - F_\uparrow)\Big] \\ & + \frac{i\, v_M}{4} \Big[N(F_\downarrow - F_\uparrow)^2 + 2NF_\uparrow(F_\downarrow - F_\uparrow)\Big]\Bigg] |S\rangle \end{aligned} \tag{75}$$

We ignore the last term which is a global phase, eliminate the second term by choosing $F_\uparrow = -\frac{1}{2}(F_\downarrow - F_\uparrow)$ i.e. $F_\uparrow = -F_\downarrow$ and are left with the first term which we split up using (68)

$$exp\left[\frac{i\, v_M}{4N}(F_\downarrow - F_\uparrow)^2 \sum_{I,J} \langle S|\mathbf{Z}_I \mathbb{1}_{(I,J)} \mathbf{Z}_J|S\rangle + \frac{i\, v_M}{2} F_\uparrow(F_\downarrow - F_\uparrow) \sum_{I,J} \langle S|\mathbf{Z}_I \mathbb{I}_{(I,J)} \mathbf{Z}_J|S\rangle\right] |S\rangle \tag{76}$$

Again the second term is a global phase, since it is only non-zero for $I = J$ in which case $\mathbf{Z}_I \mathbf{Z}_J = 1$ and there is no I or J dependence left. Thus we can include the CoM mode in the summation in (74) with now $Q_{M,(I,J)} = \frac{1}{N}\mathbb{1}$. This simple form of Q_M for the CoM mode is conditioned on the choice: $F_\uparrow = -F_\downarrow$, however this doesn't affect the other modes since their action does not depend on $F_\uparrow$. Now all the Q_M matrices are always proportional to the outer product of the mode vector and orthogonality properties between the modes can be used to build interesting Hamiltonians.

This derivation is very general: just like the repetition encoding it does not depend on the configuration of the ions. Here only on the orthonormality of the normal mode vectors, which is a consequence of the stiffness matrix being symmetric resulting in (61), and the choice of state dependent forces are needed.

5.3 Simulation of localised interactions

Equation (74) allows us to combine pulses to create new interaction Hamiltonians and we can exploit the symmetries of our trap which will be reflected in special forms of the mode vectors. The eigenmodes of a 19 qubit crystal in the cylindrical trap are given in the Appendix at 9.3 for the configuration in Figure 5: two concentric rings of 6 and 12 ions, the second having twice the radius of the first, with an ion in the center.

If we combine the eigenmodes in which the central ion is displaced (modes 0, 6, 11, 18) we obtain a block diagonalised matrix Q_{C_2} (see in 9.3): the colours of the ions in Figure 5 are indicative of which block they belong to. By doing so we have built an effective Hamiltonian where qubits interact within their block but not with qubits in other blocks.

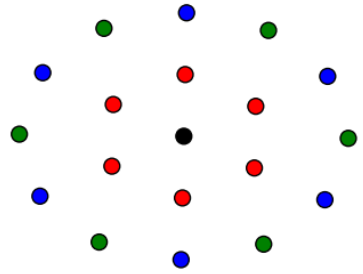


Figure 5: 2-dimensional ionic crystal configuration in the trap

Other pulse combinations can also be found which block diagonalise the Hamiltonian. These combinations are obtained by combining vibrational modes belonging to the same representation of the symmetry group of the crystal. For the 19 ion crystal, the symmetry group is the discrete group D6h of the symmetries of a regular hexagon in 3 dimensions with 24 elements: 6 rotations of angles $\frac{2\pi n}{6}$ $(n = 0, 1, ..., 5)$ about the axis perpendicular to the plane of the crystal and going through the central qubit, these 6 rotations followed by a rotation of angle $\frac{\pi}{2}$ about an axis in the plane of the crystal, and finally any of the 12 previous operations followed by a reflection about the plane of the crystal. The group has 12 irreducible representations: the character table and representations of the vibrational modes are given in 9.2.

More generally combining pulses belonging to the same representation (see Table 3 in 9.2) allows to block diagonalise the effective Hamiltonian and we can also combine different representations to create effective Hamiltonians where we only have blocks of 3 (see $Q_{C_2} + Q_{D_1} + Q_{D_2}$ in 9.2) or 2 (see $Q_{C_2} + Q_{F_2}$) qubits interacting or where a qubit interacts with some but not all qubits in its block ($Q_{C_2} + Q_{F_1}$). Q_{C_2}, $Q_{C_2} + Q_{D_1} + Q_{D_2}$ and $Q_{C_2} + Q_{F_2}$ are effective Hamiltonian in which we have a CoM-pulse-like interaction for a subset of the qubits: even though individual pulses make the qubits interact globally, the interference between pulses works destructively between blocks. The ratio of pulse lengths between modes allow to set the form of the effective Hamiltonian, then we multiply by a common factor to set the interaction time. These interactions are still subject to the restrictions of the symmetry of the crystal: if there are only 2 qubits interacting, it will necessarily be two which are diametrically opposed in the crystal.

Appendix 9.3 gives examples of such effective Hamiltonians as well as a summary of discrete group basis functions. These Hamiltonians allow for more localised interactions, we are hoping that this could in future lead to the ability to locally address and manipulate qubits while having the simplicity of only requiring global laser pulses.

6 Application to the encoding in the Steane code

6.1 Encoding the information

We can use our entangling method C**Z** of Sec. 4.1 to efficiently encode the information in the Steane code[10]. The Steane code uses 7 physical qubits to encode information in a single logical qubit, it

has a 2-dimensional subspace stabilised by the following operators:

$$\mathbf{IIZIZZZ} \qquad \mathbf{ZIIZIZZ} \qquad \mathbf{ZZIIZIZ} \tag{77}$$

$$\mathbf{IIXIXXX} \qquad \mathbf{XIIXIXX} \qquad \mathbf{XXIIXIX} \tag{78}$$

The logical basis states of the Steane code are

$$\begin{aligned} |0_L\rangle =& |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \\ &+ |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle \end{aligned} \tag{79}$$

and

$$\begin{aligned} |1_L\rangle =& |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle \\ &+ |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle \end{aligned} \tag{80}$$

respectively the logical zero and one states (given in equations (10.78) and (10.79) of [13]). The code allows to correct any single qubit Pauli $\mathbf{Z}$, $\mathbf{X}$ or $\mathbf{Y}$ error and is transverse for Clifford operations: this means we can implement an operation on the logical qubit by simply applying single qubit operations to the physical qubits. For example, applying a Pauli $\mathbf{X}$ or $\mathbf{Z}$ operation on the logical qubit is done by applying an $\mathbf{X}$ or a $\mathbf{Z}$ on each of the physical qubits: indeed applying an $\mathbf{X}$ to all physical qubits turns $|0_L\rangle$ into $|1_L\rangle$ and vice versa, also applying a $\mathbf{Z}$ to all physical qubits gives the $|1_L\rangle$ a "$-$" sign but not $|0_L\rangle$. The same is true for the logical Hadamard operation. Clifford operations are generated by $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, Hadamard $\mathbf{H}$ and a two qubit C$\mathbf{Z}$, and allow to perform all encoding, decoding, error detection and correction operations for stabiliser quantum codes. This transverse property makes it easier to experimentally manipulate the quantum information which is why the Steane code is so interesting: the Shor 5-qubit code briefly mentioned above also allows to correct single qubit errors but is not transverse for all Clifford operations. We will use 7 encoding qubits (initialised to $|0\rangle$) and 3 ancilla qubits on which we control our operations (initialised to $|0\rangle$). The starting state is

$$|0000000\rangle|000\rangle \tag{81}$$

In the stabiliser formalism, the state is represented by $\langle \mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5, \mathbf{Z}_6, \mathbf{Z}_7; \mathbf{Z}_A, \mathbf{Z}_B, \mathbf{Z}_C \rangle$

We are assuming that we can implement single qubit Hadamard operations. The entangling pulses are still global but the stabilisers differentiate between qubits, so we need to select qubits on which we apply the desired action: this we do using the single qubit Hadamards. The starting state of the encoding qubits is already an eigenstate of the three stabilisers containing only $\mathbf{Z}$ operators, we now need to measure the three $\mathbf{X}$ stabilisers.

We apply Hadamard operations on the qubits that have an identity in the first stabiliser, and we rotate the first control qubit A into the $|+\rangle$ state: ie we apply H_3, H_5, H_6, H_7, H_A

$$\langle \mathbf{Z}_1, \mathbf{Z}_2, \mathbf{X}_3, \mathbf{Z}_4, \mathbf{X}_5, \mathbf{X}_6, \mathbf{X}_7; \mathbf{X}_A, \mathbf{Z}_B, \mathbf{Z}_C \rangle \tag{82}$$

Now we apply the CZ operation

$$\langle\, \mathbf{Z}_1,\ \mathbf{Z}_2,\ \mathbf{X}_3\mathbf{Z}_A,\ \mathbf{Z}_4,\ \mathbf{X}_5\mathbf{Z}_A,\ \mathbf{X}_6\mathbf{Z}_A,\ \mathbf{X}_7\mathbf{Z}_A;\ \mathbf{Z}_3\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7\mathbf{X}_A,\ \mathbf{Z}_B,\ \mathbf{Z}_C\,\rangle \tag{83}$$

and apply a Hadamard to the control qubit

$$\langle\, \mathbf{Z}_1,\ \mathbf{Z}_2,\ \mathbf{X}_3\mathbf{X}_A,\ \mathbf{Z}_4,\ \mathbf{X}_5\mathbf{X}_A,\ \mathbf{X}_6\mathbf{X}_A,\ \mathbf{X}_7\mathbf{X}_A;\ \mathbf{Z}_3\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7\mathbf{Z}_A,\ \mathbf{Z}_B,\ \mathbf{Z}_C\,\rangle \tag{84}$$

We apply Hadamards on the qubits whose operator changes between the first and second $\mathbf{X}$ stabilisers: H_1, H_3, H_4, H_5, H_B

$$\langle\, \mathbf{X}_1,\ \mathbf{Z}_2,\ \mathbf{Z}_3\mathbf{X}_A,\ \mathbf{X}_4,\ \mathbf{Z}_5\mathbf{X}_A,\ \mathbf{X}_6\mathbf{X}_A,\ \mathbf{X}_7\mathbf{X}_A;\ \mathbf{X}_3\mathbf{X}_5\mathbf{Z}_6\mathbf{Z}_7\mathbf{Z}_A,\ \mathbf{X}_B,\ \mathbf{Z}_C\,\rangle \tag{85}$$

The CZ controlled on qubit B, and we apply H_B

$$\langle\, \mathbf{X}_1\mathbf{X}_B,\ \mathbf{Z}_2,\ \mathbf{Z}_3\mathbf{X}_A\mathbf{X}_B,\ \mathbf{X}_4\mathbf{X}_B,\ \mathbf{Z}_5\mathbf{X}_A\mathbf{X}_B,\ \mathbf{X}_6\mathbf{X}_A,\ \mathbf{X}_7\mathbf{X}_A;\ \mathbf{X}_3\mathbf{X}_5\mathbf{Z}_6\mathbf{Z}_7\mathbf{Z}_A,\ \mathbf{Z}_1\mathbf{Z}_3\mathbf{Z}_4\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7\mathbf{Z}_A\mathbf{Z}_B,\ \mathbf{Z}_C\,\rangle \tag{86}$$

Finally we apply H_2, H_4, H_5, H_6, H_C

$$\langle\, \mathbf{X}_1\mathbf{X}_B,\ \mathbf{X}_2,\ \mathbf{Z}_3\mathbf{X}_A\mathbf{X}_B,\ \mathbf{Z}_4\mathbf{X}_B,\ \mathbf{X}_5\mathbf{X}_A\mathbf{X}_B,\ \mathbf{Z}_6\mathbf{X}_A,\ \mathbf{X}_7\mathbf{X}_A;\ \mathbf{X}_3\mathbf{Z}_5\mathbf{X}_6\mathbf{Z}_7\mathbf{Z}_A,\ \mathbf{Z}_1\mathbf{Z}_3\mathbf{X}_4\mathbf{X}_5\mathbf{X}_6\mathbf{Z}_7\mathbf{Z}_A\mathbf{Z}_B,\ \mathbf{X}_C\,\rangle \tag{87}$$

CZ controlled on C and H_C

$$\begin{aligned}\langle\, &\mathbf{X}_1\mathbf{X}_B,\ \mathbf{X}_2\mathbf{X}_C,\ \mathbf{Z}_3\mathbf{X}_A\mathbf{X}_B,\ \mathbf{Z}_4\mathbf{X}_B\mathbf{X}_C,\ \mathbf{X}_5\mathbf{X}_A\mathbf{X}_B\mathbf{X}_C,\ \mathbf{Z}_6\mathbf{X}_A\mathbf{X}_C,\ \mathbf{X}_7\mathbf{X}_A;\\ &\mathbf{X}_3\mathbf{Z}_5\mathbf{X}_6\mathbf{Z}_7\mathbf{Z}_A,\ \mathbf{Z}_1\mathbf{Z}_3\mathbf{X}_4\mathbf{X}_5\mathbf{X}_6\mathbf{Z}_7\mathbf{Z}_A\mathbf{Z}_B\mathbf{X}_C,\ \mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{Z}_4\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7\mathbf{Z}_A\mathbf{Z}_B\mathbf{Z}_C\,\rangle\end{aligned} \tag{88}$$

We apply H_3, H_4, H_6 to set all the qubits in the $\mathbf{X}$ basis:

$$\begin{aligned}\langle\, &\mathbf{X}_1\mathbf{X}_B,\ \mathbf{X}_2\mathbf{X}_C,\ \mathbf{X}_3\mathbf{X}_A\mathbf{X}_B,\ \mathbf{X}_4\mathbf{X}_B\mathbf{X}_C,\ \mathbf{X}_5\mathbf{X}_A\mathbf{X}_B\mathbf{X}_C,\ \mathbf{X}_6\mathbf{X}_A\mathbf{X}_C,\ \mathbf{X}_7\mathbf{X}_A;\\ &\mathbf{Z}_3\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7\mathbf{Z}_A,\ \mathbf{Z}_1\mathbf{X}_3\mathbf{Z}_4\mathbf{X}_5\mathbf{Z}_6\mathbf{Z}_7\mathbf{Z}_A\mathbf{Z}_B\mathbf{X}_C,\ \mathbf{Z}_1\mathbf{Z}_2\mathbf{X}_3\mathbf{X}_4\mathbf{Z}_5\mathbf{X}_6\mathbf{Z}_7\mathbf{Z}_A\mathbf{Z}_B\mathbf{Z}_C\,\rangle\end{aligned} \tag{89}$$

Which we rewrite as

$$\begin{aligned}\langle\, &\mathbf{X}_7\mathbf{X}_A;\ \mathbf{X}_1\mathbf{X}_B,\ \mathbf{X}_2\mathbf{X}_C,\ \mathbf{X}_1\mathbf{X}_3\mathbf{X}_7,\ \mathbf{X}_1\mathbf{X}_2\mathbf{X}_4,\ \mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_7,\ \mathbf{X}_2\mathbf{X}_6\mathbf{X}_7,\\ &\mathbf{Z}_3\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7\mathbf{Z}_A,\ \mathbf{Z}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Z}_4\mathbf{X}_5\mathbf{Z}_6\mathbf{Z}_7\mathbf{Z}_A\mathbf{Z}_B,\ \mathbf{Z}_1\mathbf{Z}_2\mathbf{X}_3\mathbf{X}_4\mathbf{Z}_5\mathbf{X}_6\mathbf{Z}_7\mathbf{Z}_A\mathbf{Z}_B\mathbf{Z}_C\,\rangle\end{aligned} \tag{90}$$

We measure the control qubits individually in the computational basis

$$\begin{aligned}\langle\, &(-1)^{m_A}\mathbf{Z}_A;\ (-1)^{m_B}\mathbf{Z}_B,\ (-1)^{m_C}\mathbf{Z}_C,\ \mathbf{X}_1\mathbf{X}_3\mathbf{X}_7,\ \mathbf{X}_1\mathbf{X}_2\mathbf{X}_4,\ \mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_7,\ \mathbf{X}_2\mathbf{X}_6\mathbf{X}_7,\\ &(-1)^{m_A}\mathbf{Z}_3\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\ (-1)^{m_A+m_B}\mathbf{Z}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Z}_4\mathbf{X}_5\mathbf{Z}_6\mathbf{Z}_7,\ (-1)^{m_A+m_B+m_C}\mathbf{Z}_1\mathbf{Z}_2\mathbf{X}_3\mathbf{X}_4\mathbf{Z}_5\mathbf{X}_6\mathbf{Z}_7\,\rangle\end{aligned} \tag{91}$$

If the outcome of the measurement is the $|1\rangle\langle 1|$ outcome (-1 outcome) we need to correct the state. This is the same procedure as explained with (32). Table 2 gives the qubit on which we need to apply a $\mathbf{X}$ operation to correct for a given set of measurement outcomes.

After having applied the correction we obtain the $|+_L\rangle$ state of the Steane code. Stabilisers 4 to 9 are the Steane code stabilisers given in (78); the last stabiliser is the logical $\mathbf{X}$ stabiliser:

$$\begin{gathered}\langle\ (-1)^{m_A}\mathbf{Z}_A;\ (-1)^{m_B}\mathbf{Z}_B,\ (-1)^{m_C}\mathbf{Z}_C,\ \mathbf{X}_3\mathbf{X}_5\mathbf{X}_6\mathbf{X}_7,\ \mathbf{X}_1\mathbf{X}_4\mathbf{X}_6\mathbf{X}_7,\\ \mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_7,\ \mathbf{Z}_3\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\ \mathbf{Z}_1\mathbf{Z}_4\mathbf{Z}_6\mathbf{Z}_7,\ \mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_5\mathbf{Z}_7,\ \mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4\mathbf{X}_5\mathbf{X}_6\mathbf{X}_7\ \rangle\end{gathered} \tag{92}$$

Table 2: Corrections to be applied after the $\mathbf{Z}$ *basis measurements on the control qubits. "+" stands for the* $+1$ *outcome (the control is collapsed to the* $|0\rangle$ *state), "−" is the* -1 *outcome.*

Measurement outcome on controls A, B, C	Pauli X correction on qubit
$+\,+\,+$	no correction
$+\,+\,-$	2
$+\,-\,+$	1
$+\,-\,-$	4
$-\,+\,+$	7
$-\,+\,-$	6
$-\,-\,+$	3
$-\,-\,-$	5

We can now teleport the information $a|0\rangle + b|1\rangle$ we wish to encode from another qubit into this logical qubit.

$$a|+_L\rangle|0\rangle + b|+_L\rangle|1\rangle \tag{93}$$

We apply a C$\mathbf{Z}$ operation to obtain

$$a|+_L\rangle|0\rangle + b|-_L\rangle|1\rangle \tag{94}$$

The C$\mathbf{Z}$ applies a $\mathbf{Z}$ to all encoding qubits if the last qubit is in the $|1\rangle$ state, since for the Steane code this is a logical $\mathbf{Z}$ (which turns $|+_L\rangle$ into $|-_L\rangle$ and vice-versa) we can apply logical C$\mathbf{Z}$ operations. Finally we measure the last qubit in the $\mathbf{X}$ basis, correct if the outcome is -1 and we end up with the state

$$a|+_L\rangle + b|-_L\rangle \tag{95}$$

where we have encoded our information in the logical $\mathbf{X}$ basis.

This encoding method requires only global C$\mathbf{Z}$ operations, local single qubit Hadamard operations and $\mathbf{X}$ basis measurements. Single qubit Hadamards are difficult to implement in the Penning trap

since the rotational symmetry of the ion crystal doesn't allow to target individual qubits, this encoding would be more suited for traps in which the ions form 1-dimensional strings, for example in a linear Paul trap[14]. An advantage of our encoding compared to that in Ref. [14] is that we do not need to hide any of the qubits which typically requires a large number of pulses to hide/unhide each qubit. Measurements are based on the fluorescence (spontaneous emission) of an atomic transition (see Figure 3) and therefore takes a long time of record enough photons during which the system is subjected to decoherance. All four measurements can be deferred until the end of all entangling pulses such that the entangling can be done quickly. It still needs to be investigated whether this encoding is implementable on short enough time scales for decoherance to be small.

6.2 Syndrome measurements

Syndrome measurements, which allow to detect errors in the physical qubits, can be implemented by entangling an ancilla to the encoding qubits and measuring the ancilla to give information about the logical state. The measurement outcome for a given stabiliser will be -1 or $+1$ with probability 1 if the state was subjected to an error which the stabiliser can detect or not. The measurement outcome probability always being 1 allows to deterministically detect single qubit errors. If the first **Z** stabiliser from (78) returns a -1 , there is an **X** error on one of qubits 3,5,6 or 7. For the second and third stabilisers, the error would be on one of qubits 1,4,6 or 7 and 1,2,5 or 7. We combine the outcome results to pin down the qubit on which the error occurred. The same procedure is used for the **X** stabilisers to find any **X** errors; if both **X** and **Z** stabilisers have -1 outcomes, the qubit has an **Y** error.

We can implement the syndrome measurements of the **X** stabilisers as follows. We have a control qubit in the $|+\rangle$ state and we apply Hadamards to the qubits which have an **X** in their position in the stabiliser. Then we apply a C**Z** operation, then the same Hadamards again and finally another C**Z**. If the state is a $+1$ eigenstate of the stabiliser, the control qubit will still be in the $|+\rangle$ state; if it is a -1 eigenstate, the control qubit will now be in the $|-\rangle$ state. Measuring the control qubit in the **X** basis gives the syndrome information while leaving the encoded state unchanged. For the **Z** stabilisers, we apply the same procedure except we apply Hadamards to all but the control qubit at the start and after the second C**Z** (before or after the control qubit measurement is irrelevant).

Examples of X syndrome measurements The encoded state is $\left[a|0_L\rangle + b|1_L\rangle\right] \otimes |+\rangle$, the $|+\rangle$ is the ancilla.

If an encoding qubit, say the first one, as been subjected to a Pauli **Z** error the stabilisers of the

$|0_L\rangle$ or $|1_L\rangle$ will have changed to

$$\begin{aligned}\langle\ &\mathbf{X}_3\mathbf{X}_5\mathbf{X}_6\mathbf{X}_7,\ -\mathbf{X}_1\mathbf{X}_4\mathbf{X}_6\mathbf{X}_7,\ -\mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_7,\ \mathbf{Z}_3\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\\ &\mathbf{Z}_1\mathbf{Z}_4\mathbf{Z}_6\mathbf{Z}_7,\ \mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_5\mathbf{Z}_7,\ \pm\mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{Z}_4\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\ \mathbf{X}_A\ \rangle\end{aligned} \tag{96}$$

To measure a $\mathbf{X}$ stabiliser, in which the first operator is an $\mathbf{I}$ eg $\mathbf{IIXIXXX}$, we apply $\mathbf{H}_3, \mathbf{H}_5, \mathbf{H}_6, \mathbf{H}_7$,

$$\begin{aligned}\langle\ &\mathbf{Z}_3\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\ -\mathbf{X}_1\mathbf{X}_4\mathbf{Z}_6\mathbf{Z}_7,\ -\mathbf{X}_1\mathbf{X}_2\mathbf{Z}_5\mathbf{Z}_7,\ \mathbf{X}_3\mathbf{X}_5\mathbf{X}_6\mathbf{X}_7,\\ &\mathbf{Z}_1\mathbf{Z}_4\mathbf{X}_6\mathbf{X}_7,\ \mathbf{Z}_1\mathbf{Z}_2\mathbf{X}_5\mathbf{X}_7,\ \pm\mathbf{Z}_1\mathbf{Z}_2\mathbf{X}_3\mathbf{Z}_4\mathbf{X}_5\mathbf{X}_6\mathbf{X}_7,\ \mathbf{X}_A\ \rangle\end{aligned} \tag{97}$$

After applying the first C$\mathbf{Z}$ and then $\mathbf{H}_3, \mathbf{H}_5, \mathbf{H}_6, \mathbf{H}_7$ again, we have

$$\begin{aligned}\langle\ &\mathbf{X}_3\mathbf{X}_5\mathbf{X}_6\mathbf{X}_7,\ -\mathbf{X}_1\mathbf{X}_4\mathbf{X}_6\mathbf{X}_7,\ -\mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_7,\ \mathbf{Z}_3\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\ \mathbf{Z}_1\mathbf{Z}_4\mathbf{Z}_6\mathbf{Z}_7,\\ &\mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_5\mathbf{Z}_7,\ \pm\mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{Z}_4\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\ \mathbf{Z}_1\mathbf{Z}_2\mathbf{X}_3\mathbf{Z}_4\mathbf{X}_5\mathbf{X}_6\mathbf{X}_7\mathbf{X}_A\ \rangle\end{aligned} \tag{98}$$

Applying the second C$\mathbf{Z}$ and rewriting the stabilisers gives

$$\begin{aligned}\langle\ &\mathbf{X}_3\mathbf{X}_5\mathbf{X}_6\mathbf{X}_7,\ -\mathbf{X}_1\mathbf{X}_4\mathbf{X}_6\mathbf{X}_7,\ -\mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_7,\ \mathbf{Z}_3\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\\ &\mathbf{Z}_1\mathbf{Z}_4\mathbf{Z}_6\mathbf{Z}_7,\ \mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_5\mathbf{Z}_7,\ \pm\mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{Z}_4\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\ \mathbf{X}_A\ \rangle\end{aligned} \tag{99}$$

The control qubit is still in the $|+\rangle$ state since the error occurred on a qubit for which the stabiliser has an $\mathbf{I}$: the state is a $+1$ eigenstate of the stabiliser. This works equally for $|0_L\rangle$ as for $|1_L\rangle$and any linear combination i.e. any qubit encoded in the Steane code.

To measure a stabiliser in which the first operator is an $\mathbf{X}$ eg $\mathbf{XIIXIXX}$, we apply $\mathbf{H}_1, \mathbf{H}_4, \mathbf{H}_6, \mathbf{H}_7$,

$$\begin{aligned}\langle\ &\mathbf{X}_3\mathbf{X}_5\mathbf{Z}_6\mathbf{Z}_7,\ -\mathbf{Z}_1\mathbf{Z}_4\mathbf{Z}_6\mathbf{Z}_7,\ -\mathbf{Z}_1\mathbf{X}_2\mathbf{X}_5\mathbf{Z}_7,\ \mathbf{Z}_3\mathbf{Z}_5\mathbf{X}_6\mathbf{X}_7,\\ &\mathbf{X}_1\mathbf{X}_4\mathbf{X}_6\mathbf{X}_7,\ \mathbf{X}_1\mathbf{Z}_2\mathbf{Z}_5\mathbf{X}_7,\ \pm\mathbf{X}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{X}_4\mathbf{Z}_5\mathbf{X}_6\mathbf{X}_7,\ \mathbf{X}_A\ \rangle\end{aligned} \tag{100}$$

After a C$\mathbf{Z}$ and $\mathbf{H}_1, \mathbf{H}_4, \mathbf{H}_6, \mathbf{H}_7$ again, we have

$$\begin{aligned}\langle\ &\mathbf{X}_3\mathbf{X}_5\mathbf{X}_6\mathbf{X}_7,\ -\mathbf{X}_1\mathbf{X}_4\mathbf{X}_6\mathbf{X}_7,\ -\mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_7,\ \mathbf{Z}_3\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\ \mathbf{Z}_1\mathbf{Z}_4\mathbf{Z}_6\mathbf{Z}_7,\\ &\mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_5\mathbf{Z}_7,\ \pm\mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{Z}_4\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\ \mathbf{X}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{X}_4\mathbf{Z}_5\mathbf{X}_6\mathbf{X}_7\mathbf{X}_A\ \rangle\end{aligned} \tag{101}$$

This time, after the second C$\mathbf{Z}$, we have

$$\begin{aligned}\langle\ &\mathbf{X}_3\mathbf{X}_5\mathbf{X}_6\mathbf{X}_7,\ -\mathbf{X}_1\mathbf{X}_4\mathbf{X}_6\mathbf{X}_7,\ -\mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_7,\ \mathbf{Z}_3\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\\ &\mathbf{Z}_1\mathbf{Z}_4\mathbf{Z}_6\mathbf{Z}_7,\ \mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_5\mathbf{Z}_7,\ \pm\mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{Z}_4\mathbf{Z}_5\mathbf{Z}_6\mathbf{Z}_7,\ -\mathbf{X}_A\ \rangle\end{aligned} \tag{102}$$

This "$-$" sign in front of the $\mathbf{X}_A$ comes from multiplying the last stabiliser in (101) with the second one after having applied the second C$\mathbf{Z}$. Notice this "$-$" sign initially came from the $\mathbf{Z}$ error on the first qubit. The measurement of the control qubit will give the -1 outcome and the state is a -1 eigenstate of $\mathbf{XIIXIXX}$.

The $\mathbf{Z}$ stabiliser measurements described above work analogously.

Since we can implement measurements of only **Z** or only **X** stabilisers, our encoding allows us to encode a logical qubit in any Calderbank-Shor-Steane (CSS) code [10, 15]: these codes have stabilisers are made up of either only **X** or only **Z** stabilisers. We initialise the state to satisfy say all the **Z** stabilisers, by setting the qubits to the appropriate $|0\rangle$ or $|1\rangle$ states, and measure all the **X** stabilisers using control ancillas which we entangle with the encoding qubits and then measure. These codes all have the property of being transverse for Clifford operations, a property useful for quantum information processing.

7 Conclusion

In this thesis, we have studied applications of global laser pulses on ionic Coulomb crystals. The trapping potentials and electrostatic repulsions create effective spring constants between the ions giving rise to mechanical normal modes of vibration within the crystal. Using state dependent optical dipole forces, we can drive the transverse normal modes and set the ions in motion; we choose to tune the driving frequency slightly off the resonance of the normal mode to induce a beating motion: after a time which is inversely proportional to the detuning, the ions are back in their initial resting position but the encoded states have now acquired phases due to their motion. The goal of such a manipulation is to induce phases such that we implement quantum gates on the information encoded in the ions.

We have discovered a method allowing to implement a Controlled-**Z** operation on an arbitrary number of qubits with only 2 laser pulses, in so doing we are able to efficiently encode information in the repetition code with arbitrary many qubits. The C**Z** operation was also used to encode information in the Steane code, do error detection and thus correction efficiently without having to hide any qubits. By combining different driven modes, we are able to simulate a variety of interaction Hamiltonian and we can exploit the symmetries of the ionic crystal to simulate localised interactions even though the pulses are always global.

Experimental feasibility and limits due to imperfections and fluctuations in the experimental setup now need to be investigated to assess the practical usefulness of our discoveries. We also need to interpret the forms of the simulated Hamiltonians to find the physical interactions and how they can be used for practical purposes. However the possibility of addressing qubits in a localised manner while only using global laser pulses would be an interesting and exciting tool for quantum information processing.

8 References

[1] B. P. Lanyon et al. "Measurement-Based Quantum Computation with Trapped Ions". In: *Phys. Rev. Lett.* 111 (21 Nov. 2013), p. 210501. URL: http://link.aps.org/doi/10.1103/PhysRevLett.111.210501.

[2] J. F. Goodwin et al. *Trapped Ion Quantum Error Correcting Protocols Using Only Global Operations*. July 2014. URL: http://arxiv.org/abs/1407.1858v1.

[3] S. Mavadia et al. "Control of the conformations of ion Coulomb crystals in a Penning trap". In: *Nature Communications* Article 2571 (Oct. 2013). URL: http://dx.doi.org/10.1038/ncomms3571.

[4] C. J. Foot. *Atomic Physics*. equation (9.43). Oxford University Press, 2005.

[5] G. Birkl, S. Kassner, and H. Walther. "Multiple-shell structures of laser-cooled $^{24}Mg^{+}$ ions in a quadrupole storage ring". In: *Nature* 357 (6376 May 1992), pp. 310–313. URL: http://dx.doi.org/10.1038/357310a0.

[6] R. Ozeri. *The trapped-ion qubit toolbox*. June 2011. URL: http://arxiv.org/abs/1106.1190v1.

[7] C.-C. Joseph Wang, Adam C. Keith, and J. K. Freericks. "Phonon-mediated quantum spin simulator employing a planar ionic crystal in a Penning trap". In: *Phys. Rev. A* 87 (Jan. 2013). URL: http://link.aps.org/doi/10.1103/PhysRevA.87.013422.

[8] J. W. Britton et al. "Engineered two-dimensional Ising interactions in a trapped-ion quantum simulator with hundreds of spins". In: *Nature* 484 (Apr. 2012), pp. 489–492. URL: http://dx.doi.org/10.1038/nature10981.

[9] W. K. Wootters and W. H. Zurek. "A single quantum cannot be cloned". In: *Nature* 299 (Oct. 1982), pp. 802–803. URL: http://dx.doi.org/10.1038/299802a0.

[10] A. Steane. "Multiple-Particle Interference and Quantum Error Correction". In: *Proceedings: Mathematical, Physical and Engineering Sciences* 452 (Nov. 1996), pages. URL: http://www.jstor.org/stable/52827.

[11] D. P. DiVincenzo and P. W. Shor. "Fault-Tolerant Error Correction with Efficient Quantum Codes". In: *Phys. Rev. Lett.* 77 (Oct. 1996), pp. 3260–3263. URL: http://link.aps.org/doi/10.1103/PhysRevLett.77.3260.

[12] D. Leibfried et al. "Experimental demonstration of a robust, high-fidelity geometric two ion-qubit phase gate". In: *Nature* 422 (Mar. 2003), pp. 412–415. URL: http://dx.doi.org/10.1038/nature01492.

[13] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information, 10th Anniversary Edition*. Cambridge University Press, 2010.

[14] D. Nigg et al. "Quantum computations on a topologically encoded qubit". In: *Science* 345 (June 2014). URL: http://www.sciencemag.org/content/345/6194/302.long.

[15] A. R. Calderbank and P. W. Shor. "Good quantum error-correcting codes exist". In: *Phys. Rev. A* 54 (Aug. 1996), pp. 1098–1105. URL: http://link.aps.org/doi/10.1103/PhysRevA.54.1098.

9 Appendix

9.1 Stabilisers

We can define states using the operators of which they are the +1 eigenstate: this is called the stabiliser formalism. If a state $|\psi\rangle$ satisfies

$$g_i|\psi\rangle = |\psi\rangle \tag{103}$$

then $|\psi\rangle$ is stabilised by g_i. In this appendix we will state uses of the stabiliser formalism for which the proofs can be found in [13] section 10.5.

We will restrict ourselves to stabilisers composed of Pauli $\mathbf{X}$,$\mathbf{Y}$ and $\mathbf{Z}$ and identity matrices, with multiplicative factors of ± 1 and $\pm i$, on N qubits called the Pauli group G_N. For 3 qubits, this group would for example include: $\mathbf{X}_1$, $\mathbf{Z}_3$, $-\mathbf{X}_1\mathbf{X}_2$, $i\mathbf{Z}_1\mathbf{X}_2\mathbf{Y}_3$ etc, where the index labels the qubit on which the operator is applied: on qubits which are not labelled, we apply the identity operation.

If a state is stabilised by two stabilisers, it is also stabilised by their product: thus the stabilisers form a group. Therefore we will only explicitly write down the generators of the stabilisers i.e. the smallest number of stabilisers such that all others can be found by multiplying the generators together. We write the generators in angled brackets: $S = \langle \dots \rangle$.

To define a state of N qubits, we require N generators. More generally, the full Hilbert space of N qubits has 2^N dimensions, each stabiliser sets a constraint on a qubit i.e. 2 dimensions: the space of states of N qubits with k stabilisers has 2^{N-k} dimensions. For N qubits, $N-1$ stabilisers gives a 2-dimensional space which can be taken to be the space of the logical qubit in a quantum code; N stabilisers gives a space with $2^0 = 1$ dimension which is a single state. The set of generators is not unique since we can multiply together generators to obtain a new generator, as long as we discard one of two we multiplied together, thus we can rewrite the stabilising set without changing the state it describes.

We can also describe certain unitary evolutions and measurements in this formalism. We restrict to Clifford operations which are the operations that leave the Pauli group invariant: they are generated by $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, Hadamard $\mathbf{H}$ and a two qubit C$\mathbf{Z}$. Applying a Clifford unitary U to $|\psi\rangle$ is equivalent to transforming every stabiliser according to

$$g_i \to U g_i U^\dagger \tag{104}$$

where † is the conjugate transposed of the U operator. For example, a state is stabilised by $\langle \mathbf{X}_1\mathbf{X}_2, \mathbf{Z}_1\mathbf{Z}_2 \rangle$ and we apply a unitary $\mathbf{H}_2$: the updated stabilisers are $\langle \mathbf{X}_1\mathbf{Z}_2, \mathbf{Z}_1\mathbf{X}_2 \rangle$ since $\mathbf{H}\mathbf{X}\mathbf{H}^\dagger = \mathbf{Z}$ and vice versa.

Measurements of observables associated with an element g of the Pauli group are dealt with as follows. If g commutes with all the generators, the outcome $+1$ or -1 is deterministic, the state is

unchanged and the stabiliser group remains the same. If g does not commute (i.e. anti-commutes) with k of the stabilisers, we rewrite the generators such that only one of them anti-commutes with g: if g anti-commutes with two generators, it commutes with their product; then replace the anti-commuting generator with $m_g g$, where m_g is ± 1 depending on the measurement outcome. For example we measure $\mathbf{Z}_2$ on $\langle \mathbf{X}_1\mathbf{X}_2, \mathbf{Z}_1\mathbf{Z}_2 \rangle$: only $\mathbf{X}_1\mathbf{X}_2$ anti-commutes with $\mathbf{Z}_2$ so we obtain $\langle \pm\mathbf{Z}_2, \mathbf{Z}_1\mathbf{Z}_2 \rangle = \langle \pm\mathbf{Z}_1, \pm\mathbf{Z}_2 \rangle$ depending on the $+1$ or -1 outcome of the measurement.

9.2 The D6h group

Character table

Table 3: Character table for the D6h group: rows correspond to the 12 irreducible representations, columns correspond to the 12 classes of symmetry operations, the last column indicates which vibrational modes transform according to which irreducible representation.

	E	$2C_6$	$2C_3$	C_2	$3S$	$3S'$	P	$2P_6$	$2P_3$	P_2	$3SP$	$3S'P$	Modes
A_1	1	1	1	1	1	1	1	1	1	1	1	1	
A_2	1	1	1	1	-1	-1	1	1	1	1	-1	-1	
B_1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	
B_2	1	-1	1	-1	-1	1	1	-1	1	-1	-1	1	
E_1	2	1	-1	-2	0	0	2	1	-1	-2	0	0	
E_2	2	-1	-1	2	0	0	2	-1	-1	2	0	0	
C_1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	
C_2	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	$0, 6, 11, 18$
D_1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	13
D_2	1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	$1, 12$
F_1	2	1	-1	-2	0	0	-2	-1	1	2	0	0	$(4, 5), (7, 8), (16, 17)$
F_2	2	-1	-1	2	0	0	-2	1	1	-2	0	0	$(2, 3), (9, 10), (14, 15)$

The matrix representation of a symmetry represents the action of the group. The representation of a group preserves the group multiplication law: if an operation g_1 followed by an operation g_2 is the same as simply applying the operation g_3 then multiplying the representation matrix of g_2 with that of g_1 will be equal the matrix of g_3. The character table summarises the characters i.e. the trace of the matrices which represent elements of the group.

The classes group together operations which are equivalent: for example a rotation of angle $\frac{\pi}{3}$ is equivalent to a rotation of angle $\frac{5\pi}{3} = \frac{-\pi}{3}$ and both belong to the C_6 class; the number in front of the name of the class indicates how many elements are in the class. Elements of a same class have the same character in a given representation although the exact matrix will not necessarily be the same: this is why we only give classes and not elements in the table. The classes are: E has only the identity (the "do nothing" operation), C_6 are rotations of $\frac{\pm\pi}{3}$, C_3 are rotations of $\frac{\pm 2\pi}{3}$, C_2 are rotations of $\frac{\pi}{2}$, S are E and C_3 followed by a $\frac{\pi}{2}$ rotation about a symmetry axis (say the y-axis if the coordinates are defined appropriately) in the plane of the crystal, S' are C_6 and C_2 followed by a $\frac{\pi}{2}$ rotation about the same symmetry axis as for S; P contains the reflection about the crystal plane, P_6, P_3, P_2, SP and $S'P$ contain elements respectively of C_6, C_3, C_2, S and S' followed by the plane reflection.

Functions of a representation The matrix representation of the identity operation is just the identity matrix, thus the character of the E class in a given representation gives the dimension of that representation: the A, B, C and D representations are 1x1 matrices, the E and F representations are 2x2 matrices. For the 1-dimensional representations, the matrix has a single entry given by the character in the table. For the 2-dimensional representations E_1, the matrix representation of an element is the result of the multiplication of the rotation matrix by the angle $\frac{n\pi}{3}$ with the rotation about the y-axis in the plane given respectively by

$$\begin{pmatrix} cos(\frac{n\pi}{3}) & -sin(\frac{n\pi}{3}) \\ sin(\frac{n\pi}{3}) & cos(\frac{n\pi}{3}) \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{105}$$

when that operation is part of the group element. For example, for elements of the class C_6 there is no rotation about the y-axis in the plane of the crystal therefore that matrix will not be included in the multiplication, whereas for the S and S' classes we include the second matrix of (105) in the multiplication. The E_2 representation is built up in the same way except the angle in the matrix representation is now the twice the angle of the physical rotation (this is sometimes called an "double angle representation"). Notice in these two representations, the C_6 and P_6 classes have the same character, the plane reflection is not considered in the representation: and the elements have the same matrix representation i.e. the representation is not faithful. For the F_1 and F_2 representations we take the reflection into account by multiplying by -1 if the group element includes this reflection. That is why between E_1 and F_1 as well between E_2 and F_2, the classes with P have their character multiplied by -1. Otherwise they are built up in the same way with the same rotation matrices given by (105): with the double angle for F_2

Thus representations are different ways of transforming under the group elements' action. In the A_1 representation the matrix is always 1 and nothing changes: if the values of a function doesn't change under the group elements then this function belongs to the A_1 representation. Let us take a function in 3 dimensions which has the same value for every point in space, then any group transformation won't change change the function value at a given point thus this function belongs to the A_1 representation. The function "z" returning the value of the z-coordinate of the point transforms according to the C_2 representation: the z-coordinate is unchanged by rotations in the x-y plane of the crystal but is inverted by a $\frac{\pi}{2}$ about the y-axis and is also inverted by the reflection about the x-y plane; thus transforming the z-coordinate under an element of the D6h group is simply done by multiplying it by the matrix in the C_2 representation of that element. Similarly the (x, y) coordinate pair transform according to the 2-dimensional representation E_2.

We can also find the representation according to which the transverse vibrational modes of the crystal transform: these are given by the modes column in the character table. Notice the modes all correspond to C, D or F representations which take into account the reflection about the x-y

plane since only this differentiates longitudinal vibrations in the x-y plane from transverse vibrations in the z-direction. Also modes which are not cylindrically symmetric (tilt and rotations vibrational modes) belong to 2-dimensional representations and each pair of vibrational modes are rotated into one another: under a group rotation mode 4 will gain a component in mode 5, analogously to the case when we rotate the x-coordinate in to a combination of x- and y-coordinate. These pairs of modes are precisely the ones which are degenerate in vibrational frequency: just considering the symmetries of the system allows to find the degenerate modes without solving the stiffness matrix, however the stiffness matrix is required to find the exact values of the frequencies.

9.3 The 19 ion crystal

The configuration and labelling of 19 ions in the trap is given by Figure 6.

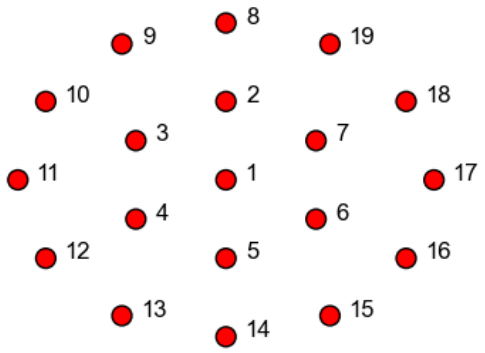


Figure 6: 2-dimensional ionic crystal configuration in the trap

For modes 0 to 9, the eigenmode matrix is given below: every column corresponds to a mode and every row is an ion in that mode,

$$\begin{pmatrix}
0.8791 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0756 & 0.0000 & 0.0000 & 0.0000 \\
-0.1917 & 0.4026 & -0.5526 & -0.1148 & 0.5091 & -0.1564 & 0.0318 & 0.1195 & 0.0044 & 0.0617 \\
-0.1917 & -0.4026 & 0.3758 & -0.4211 & 0.3900 & 0.3627 & 0.0318 & 0.0635 & -0.1013 & -0.0214 \\
-0.1917 & 0.4026 & 0.1768 & 0.5360 & -0.1191 & 0.5191 & 0.0318 & -0.0559 & -0.1056 & -0.0403 \\
-0.1917 & -0.4026 & -0.5526 & -0.1148 & -0.5091 & 0.1564 & 0.0318 & -0.1195 & -0.0044 & 0.0617 \\
-0.1917 & 0.4026 & 0.3758 & -0.4211 & -0.3900 & -0.3627 & 0.0318 & -0.0635 & 0.1013 & -0.0214 \\
-0.1917 & -0.4026 & 0.1768 & 0.5360 & 0.1191 & -0.5191 & 0.0318 & 0.0559 & 0.1056 & -0.0403 \\
0.0295 & -0.0675 & 0.1149 & 0.0239 & -0.2090 & 0.0642 & -0.3083 & 0.3744 & 0.0137 & 0.4041 \\
0.0156 & 0.0000 & 0.0100 & 0.0303 & -0.0421 & -0.0097 & 0.2639 & -0.3735 & 0.1978 & -0.2583 \\
0.0295 & 0.0675 & -0.0781 & 0.0876 & -0.1601 & -0.1489 & -0.3083 & 0.1991 & -0.3174 & -0.1399 \\
0.0156 & 0.0000 & -0.0312 & -0.0065 & -0.0127 & -0.0413 & 0.2639 & -0.0155 & 0.4224 & 0.3950 \\
0.0295 & -0.0675 & -0.0368 & -0.1114 & 0.0489 & -0.2131 & -0.3083 & -0.1753 & -0.3311 & -0.2642 \\
0.0156 & 0.0000 & 0.0212 & -0.0238 & 0.0294 & -0.0317 & 0.2639 & 0.3580 & 0.2246 & -0.1367 \\
0.0295 & 0.0675 & 0.1149 & 0.0239 & 0.2090 & -0.0642 & -0.3083 & -0.3744 & -0.0137 & 0.4041 \\
0.0156 & 0.0000 & 0.0100 & 0.0303 & 0.0421 & 0.0097 & 0.2639 & 0.3735 & -0.1978 & -0.2583 \\
0.0295 & -0.0675 & -0.0781 & 0.0876 & 0.1601 & 0.1489 & -0.3083 & -0.1991 & 0.3174 & -0.1399 \\
0.0156 & 0.0000 & -0.0312 & -0.0065 & 0.0127 & 0.0413 & 0.2639 & 0.0155 & -0.4224 & 0.3950 \\
0.0295 & 0.0675 & -0.0368 & -0.1114 & -0.0489 & 0.2131 & -0.3083 & 0.1753 & 0.3311 & -0.2642 \\
0.0156 & 0.0000 & 0.0212 & -0.0238 & -0.0294 & 0.0317 & 0.2639 & -0.3580 & -0.2246 & -0.1367
\end{pmatrix}$$

and for modes 10 to 18,

$$\begin{pmatrix}
0.0000 & -0.4109 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.2294 \\
-0.0110 & -0.2762 & -0.0675 & 0.0000 & -0.0239 & 0.1014 & -0.0133 & 0.1877 & 0.2294 \\
0.0589 & -0.2762 & 0.0675 & 0.0000 & -0.0759 & -0.0714 & 0.1559 & 0.1054 & 0.2294 \\
-0.0480 & -0.2762 & -0.0675 & 0.0000 & 0.0998 & -0.0300 & 0.1692 & -0.0823 & 0.2294 \\
-0.0110 & -0.2762 & 0.0675 & 0.0000 & -0.0239 & 0.1014 & 0.0133 & -0.1877 & 0.2294 \\
0.0589 & -0.2762 & -0.0675 & 0.0000 & -0.0759 & -0.0714 & -0.1559 & -0.1054 & 0.2294 \\
-0.0480 & -0.2762 & 0.0675 & 0.0000 & 0.0998 & -0.0300 & -0.1692 & 0.0823 & 0.2294 \\
-0.0718 & 0.1346 & -0.4026 & 0.0000 & -0.0892 & 0.3783 & -0.0270 & 0.3800 & 0.2294 \\
-0.3070 & 0.2101 & 0.0000 & -0.4082 & -0.3965 & 0.1192 & 0.1710 & 0.3515 & 0.2294 \\
0.3859 & 0.1346 & 0.4026 & 0.0000 & -0.2830 & -0.2664 & 0.3156 & 0.2134 & 0.2294 \\
-0.0702 & 0.2101 & 0.0000 & 0.4082 & 0.0950 & -0.4029 & 0.3899 & 0.0277 & 0.2294 \\
-0.3141 & 0.1346 & -0.4026 & 0.0000 & 0.3723 & -0.1119 & 0.3426 & -0.1666 & 0.2294 \\
0.3772 & 0.2101 & 0.0000 & -0.4082 & 0.3014 & 0.2838 & 0.2190 & -0.3238 & 0.2294 \\
-0.0718 & 0.1346 & 0.4026 & 0.0000 & -0.0892 & 0.3783 & 0.0270 & -0.3800 & 0.2294 \\
-0.3070 & 0.2101 & 0.0000 & 0.4082 & -0.3965 & 0.1192 & -0.1710 & -0.3515 & 0.2294 \\
0.3859 & 0.1346 & -0.4026 & 0.0000 & -0.2830 & -0.2664 & -0.3156 & -0.2134 & 0.2294 \\
-0.0702 & 0.2101 & 0.0000 & -0.4082 & 0.0950 & -0.4029 & -0.3899 & -0.0277 & 0.2294 \\
-0.3141 & 0.1346 & 0.4026 & 0.0000 & 0.3723 & -0.1119 & -0.3426 & 0.1666 & 0.2294 \\
0.3772 & 0.2101 & 0.0000 & 0.4082 & 0.3014 & 0.2838 & -0.2190 & 0.3238 & 0.2294
\end{pmatrix}$$

We combine modes belonging to a same representation with equal pulse lengths of 10 to obtain the matrix of the representation: modes 0, 6, 11 and 18 gives Q_{C_2}

$$
\begin{pmatrix}
10.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.67 & 1.67 & 1.67 & 1.67 & 1.67 & 1.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.67 & 1.67 & 1.67 & 1.67 & 1.67 & 1.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.67 & 1.67 & 1.67 & 1.67 & 1.67 & 1.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.67 & 1.67 & 1.67 & 1.67 & 1.67 & 1.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.67 & 1.67 & 1.67 & 1.67 & 1.67 & 1.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.67 & 1.67 & 1.67 & 1.67 & 1.67 & 1.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67 & 0 & 1.67
\end{pmatrix}
$$

We also combine representations for new effective Hamiltonians: blocks of 3 qubits with $Q_{C_2} + Q_{D_1} + Q_{D_2}$

$$\begin{pmatrix}
10.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3.33 & 0 & 3.33 & 0 & 3.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3.33 & 0 & 3.33 & 0 & 3.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3.33 & 0 & 3.33 & 0 & 3.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3.33 & 0 & 3.33 & 0 & 3.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3.33 & 0 & 3.33 & 0 & 3.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3.33 & 0 & 3.33 & 0 & 3.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & 3.33
\end{pmatrix}$$

Blocks of 2 qubits with $Q_{C_2} + Q_{F_2}$

$$\begin{pmatrix}
10.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5.00 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5.00 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5.00 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5.00 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5.00 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5.00 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 5.00 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 0 & 0 & 0 & 0 & 5.00
\end{pmatrix}$$

Different interaction strengths within a block with $Q_{C_2} + Q_{F_1}$

$$\begin{pmatrix}
10.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5.00 & 3.33 & 0 & -1.67 & 0 & 3.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3.33 & 5.00 & 3.33 & 0 & -1.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3.33 & 5.00 & 3.33 & 0 & -1.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.67 & 0 & 3.33 & 5.00 & 3.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1.67 & 0 & 3.33 & 5.00 & 3.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3.33 & 0 & -1.67 & 0 & 3.33 & 5.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 3.33 & 0 & 0 & 0 & -1.67 & 0 & 0 & 0 & 3.33 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.00 & 0 & 3.33 & 0 & 0 & 0 & -1.67 & 0 & 0 & 0 & 3.33 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 5.00 & 0 & 3.33 & 0 & 0 & 0 & -1.67 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 5.00 & 0 & 3.33 & 0 & 0 & 0 & -1.67 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 5.00 & 0 & 3.33 & 0 & 0 & 0 & -1.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 5.00 & 0 & 3.33 & 0 & 0 & 0 & -1.67 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.67 & 0 & 0 & 0 & 3.33 & 0 & 5.00 & 0 & 3.33 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.67 & 0 & 0 & 0 & 3.33 & 0 & 5.00 & 0 & 3.33 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.67 & 0 & 0 & 0 & 3.33 & 0 & 5.00 & 0 & 3.33 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.67 & 0 & 0 & 0 & 3.33 & 0 & 5.00 & 0 & 3.33 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & -1.67 & 0 & 0 & 0 & 3.33 & 0 & 5.00 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 & 0 & -1.67 & 0 & 0 & 0 & 3.33 & 0 & 5.00
\end{pmatrix}$$