An Introduction to Loop Quantum Gravity and Its Application to Black Holes

by

Tim Lee (CID: 00552054)

Department of Physics
Imperial College London
London SW7 2AZ
United Kingdom

Dissertation submitted as part of the requirements for the award of the MSc in Quantum Fields & Fundamental Forces, Imperial College London, 2012-2014

Abstract

This paper provides a brief introduction to Loop Quantum Gravity which is the main competitor of String Theory for a quantized theory of gravity. It is formulated on the basis of a Hamiltonian formalism of General Relativity with quantization performed in a non-perturbative and background independent manner. Coupling to matter fields as they appear in the Standard Model is also considered. Canonically quantizing General Relativity, it is shown that spacetime is discretized at Planckian scales and such a result leads to applying Loop Quantum Gravity to calculations of black hole entropy values. An alternative method to the Hamiltonian formalism of Loop Quantum Gravity called the spin foam formalism is also discussed.

Acknowledgements

I would firstly like to thank my supervisor, Professor Jonathan Halliwell, for guiding me through this dissertation by recommending a wide range of references and providing me with comments on early drafts of this document. I am also thankful for his emails replying questions I had on certain aspects of LQG I was confused with despite being out of town. Secondly, I would like to thank my advisors Dr Carlo Contaldi and Professor Kellogg Stelle for making it possible for me to pause my studies for a while when I required medical attention.

I would also like to express my appreciation to my fellow students Euibyung Park, Sangmin Lee and Theodore Kim for making the final year of my master's degree much enjoyable. Furthermore I am thankful to my parents and my brother for the unconditional support they have given me throughout my MSc degree and this dissertation. Lastly, I would like to say my thanks to Sooyeon Hong who has always been by my side through ups and downs over the course of the past three years.

Contents

1 Introduction

The formulation of Quantum Mechanics, hence ultimately of Quantum Field Theory (QFT), has successfully described three of the four fundamental forces of nature to give the Standard Model (SM). On the other hand, General Relativity (GR) has triumphed in providing the mechanism of gravity, the remaining fundamental force.

Although effective in their respective domains, QFT and GR fail to be compatible with one another; namely GR cannot be applied to scales on which QFT becomes effective. Attempts to overcome this problem and produce a self-consistent theory applicable to both quantum and astrophysical scales naturally led to the quantization of gravity, a problem which has occupied physicists for decades. Looking through the history of the development of physics, it is evident that coalescing two apparently different theories into a unified structure gives rise to new physics; the fusion of Newtonian mechanics and Maxwell's electromagnetism gave birth to special relativity while placing quantum mechanics and special relativity on the same footing led to QFT. It is likely that the unity of GR and QFT will again shed light on new physics which will, for instance, allow for the description of the final stages of a black hole evaporation.

Today there exist many candidates for the theory of quantum gravity including Twistor Theory, Causal Set Theory, String Theory and Loop Quantum Gravity (LQG); of these, String Theory and LQG are two candidates which are predominantly researched on.

It should be noted that String Theory is fundamentally different from LQG in that it is a theory which unifies all four fundamental forces. String Theory's such success, however, comes at the cost of extra dimensions (the theory is 10 or 11 dimensional) and an infinite number of particles beyond those predicted by the SM; the theory is also necessarily supersymmetric, a property that has not been experimentally verified. In contrast, LQG is a theory purely of quantized gravity rather than one of unification. Thus it does not suffer from the extra complications that String Theory is exposed to as stated above. More importantly, unlike String Theory, whose quantization of gravity stems from perturbative calculations, LQGs quantization is non-perturbative. This non-perturbative quantization delivers two benefits: background independence and avoidance of non-renormalizability.

Since the quantization of electromagnetism, weak and strong force are perturbative, it is a natural step forward to consider the quantization of gravity via perturbation as well. The process is similar to ordinary QFT and defines a functional integral for GR. This involves splitting the metric as a sum of the Minkowski metric and some small fluctuation which plays the role of the dynamical field as in QFT. The outline of this attempt is given in [1]. The generic outcome is a non-renormalizable field theory whose quanta are interpreted as gravitons. It should be noted, however, that it is possible to formulate a renormalizable theory of gravity by considering an effective field theory. This is essentially the basis of String Theory. Details of this are given in [2].

The non-renormalizable result is no surprise as the non-renormalizability of gravity is expected by power counting. Non-renormalizable theories are generally considered undesirable as they produce divergences which cannot be handled systematically. LQG, however, is independent of perturbative calculations, hence is free from this illness.

Another crucial feature of GR which must be preserved upon quantization is background independence. In QFT, the implied metric (background) on which calculations are performed is the Minkowski metric; in other words, the theory is background dependent. QFT primarily deals with point particles hence back-reaction is negligible meaning background dependence poses no problems. GR, however, is a theory describing the interaction of matter and the metric (or the background) itself; therefore, neglecting back-reaction makes no sense. As no structure on the metric can be assumed before calculations, GR is a background independent theory. LQG is formulated in accordance with this property of GR and is built up on a background independent quantum structure called spin network states, the details of which become apparent in later discussions. Such formalism of LQG ensures that the background independence of GR is preserved.

Looking at the Einstein equation, one is able to deduce another key property of GR: diffeomorphism covariance (also known as diffeomorphism invariance or general covariance). This is the technical statement about the independence of the laws of physics on coordinates. Because the Einstein equation is a tensor equation, it is necessarily coordinate independent. This property, along with background independence, is central in developing the ADM formalism as is apparent from the following chapter.

The steps taken in LQG to quantize GR are similar to that of ordinary quantum mechanics, namely a Hamiltonian formulation of classical mechanics for GR is constructed, and then the Poisson brackets are promoted to commutators. A thorough formulation of this is given in chapter 2 where the 4-dimensional spacetime is split into 3-dimensional space and 1-dimensional time. The formulation reveals that the 3-metric and its conjugate momentum are the dynamical variables of the theory. However, due to background independence, one runs into problems when one naively quantizes GR this way. In particular, because GR is a fully constrained theory, one finds that no evolution can be generated upon canonical quantization.

To overcomes this problem, chapter 3 constructs the theory of GR in a different formalism using the Palatini action, consisting of tetrads and spin connections instead of the metric. Then in chapter 4, manipulating the Palatini action, an extra term holding the Barbero-Immirzi parameter is introduced which gives rise to the Holst action. The Barbero-Immirzi parameter is shown to play no significant role classically, although it would later act as a key quantity in quantum theory. The tetrad and spin connection of the new formalism of GR are then used to compose new variables called the Ashtekar new variables comprising of the Ashtekar-Barbero connection, which is an $SU(2)$ connection and its conjugate quantity, the densitized triad. One finds that the action constructed in terms of the new variables gives rise to a new $SU(2)$ gauge symmetry.

With the new variables allowing for a background independent theory, the quantization procedure of them is carried out in chapter 5. Using the notion of holonomies, hence of Wilson loops, a Hilbert space comprised of spin network states are designed such that background independence is preserved. The succeeding subsection of chapter 5, then introduces the concept of geometric operators such as the area and volume operators that, when acted on spin network states, give physically observable areas and volumes. The key result is that the induced areas and volumes are discretized suggesting space itself is quantum at Planck scale.

The dynamics of LQG are discussed in the section that follows which provides a method of overcoming the problem of the Hamiltonian not generating any evolution when the theory was formulated in terms of the metric. It is revealed that a Hilbert space in accordance with the diffeomorphism invariant nature of GR can be created by using knots. In order to promote the Hamiltonian to an operator which satisfies the physical Hilbert space, the components in the Hamiltonian are reexpressed in terms of loop variables. This leads to formal solutions of spin networks in the physical Hilbert space. Chapter 5 is concluded with a brief overview of standard couplings to matter such as the Higgs and fermions.

The next chapter, chapter 6, delves into an appliaction of LQG, or more specifically, an application to the calculation of black hole entropies. One of the direct consequences of applying LQG to black holes is the mechanism of the existence of event horizons. Event horizons are interpreted as areas induced by the edges of spin network states piercing through surfaces, which mathematically is decribed by the eigenvalues of the area operator. In terms of entropy, LQG allows for a logarithmic correction to the semiclassical picture of a horizon's entropy described by the Bekenstein-Hawking formula. It is revealed that for a consistent picture of entropy calculations, the Barbero-Immirzi parameter must take a specific value unlike in the classical case.

Finally, chapter 7 introduces the notion of spin foam formalism before the paper is concluded. This formalism provides an alternative method of describing LQG from the Hamiltonian approach. Unlike the Hamiltonian approach that focuses on the evolution of states in a Hilbert space, the spin foam formalism puts the spotlight on the transition amplitudes or probabilities of states. The formalism builds up on the notion of discretized space taken from the eigenvalues of geometric operators to give rise to dicretized spacetimes. Due to its similarity to the Feynman path integral formalism from quantum mechanics, the spin foam formalism is insightful in providing correlation functions for gravity which allow for a particle interpretation of gravitation through graviton propagators. The succeeding chapters explores in detail the brief overview of this paper given so far.

2 The ADM Formalism

The ADM formalism, first formulated by Arnowitt, Deser and Misner, is an attempt to formulate GR in terms of Hamiltonian mechanics by splitting the 4-dimensional spacetime manifold, $\mathcal{M}$, into a 3-dimensional spatial manifold, S, and 1-dimensional temporal manifold, $\mathbb{R}$. This procedure, known as the 3+1 decomposition of GR, breaks the symmetry between space and time, hence may appear to break the diffeomorphism invariance property of GR. However, it was shown by N. Kiriushcheva and S.V. Kuzmin [3] that diffeomorphism invariance is preserved despite the splitting. The decomposition can be done by defining a diffeomorphism, ϕ, such that

$$\phi : \mathcal{M} \to S \times \mathbb{R}.$$

This means that one can use the pullback of ϕ, to define a value $t \in \mathbb{R}$ for τ on $\mathcal{M}$ as

$$\tau = \phi^* t.$$

Defining *slices* of spatial submanifolds $\Sigma \subset \mathcal{M}$ for constant τ values, one can foliate $\mathcal{M}$ into hypersurfaces of Σ. Then the metric, g, on $\mathcal{M}$ can naturally be reduced to the 3-metric, ${}^3g$, on Σ. It is worth noting that for manifolds that are *globally hyperbolic*, or manifolds which admit a Cauchy surface, it is proven by R. Geroch [4] that $\mathcal{M}$ must be isomorphic to a product manifold of Σ and $\mathbb{R}$ such that $\mathcal{M} \simeq \Sigma \times \mathbb{R}$. This is a fairly powerful statement as, excluding extreme regions such as the centers of black holes, spacetimes are generally globally hyperbolic.

As the metric is the dynamical variable whose evolution determines the physics of GR, it is necessary to consider evolutions of the metric in a systematic way; for this one must know how the vectors flow in $\mathcal{M}$. To elucidate this idea, take a particular slice of the spatial manifold Σ and take a timelike vector ∂_τ on Σ by pushing foward the vector ∂_t on $S \times \mathbb{R}$ using ϕ^{-1}. The vector ∂_τ can then be decomposed into a part normal to and tangential to Σ:

$$\partial_\tau = Nn + \vec{N}$$

where $N = -g(\partial_\tau, n)$ is called the *lapse* and $\vec{N} = \partial_\tau + g(\partial_\tau, n)n$ is called the *shift*. Such parametrizations of lapse and shift allows for a physical picture of how a metric evolves over time and space respectively.

2.1 Constraints on General Relativity

Before canonically quantizing gravity, it is insightful to consider the constraints imposed on GR. Of the 10 equations in Einstein equations, 4 are statements of constraints on the curvature of the manifold and only 6 describe the dynamics. To look into this in more detail, one must first understand the concept of *extrinsic curvature*. Defining a timelike unit vector, n, which is normal to Σ and tangent vectors $v, u \in T_p\mathcal{M}$ along with their associated vector fields, one can write the extrinsic curvature as

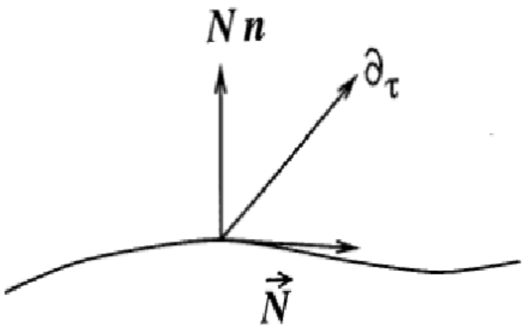

Figure 1: ∂_τ split into normal and tangential components. Figure from [5]

$$K(u, v)n = -g(\nabla_u v)n$$

where ∇ is the Levi-Civita connection. The derivation of this can be found in [5] and a more mathematically rigorous definition of the extrinsic curvature can be found in [6]. The extrinsic curvature is extrinsic in the sense that it measures the degree to which a vector tangent to Σ will fail to stay tangent after parallel transporting it using the Levi-Civita connection. In fact as is shown in the following section, the extrinsic curvature is directly related to the time derivative of the 3-metric. With this in mind, the Riemann tensor, R, on $\mathcal{M}$ can be expressed in terms of the extrinsic curvature, K, and the Riemann tensor on Σ which will be denoted ${}^3R$ to give the *Gauss-Codazzi equations*:

$$R(\partial_i, \partial_j)\partial_k = ({}^3\nabla_i K_{jk} - {}^3 \nabla_j K_{ik})n + ({}^3R^m{}_{ijk} + K_{jk}K_i^m - K_{ik}K_j^m)\partial_m,$$

the derivation of which can also be found in [5]; the superscripts, 3, on the connections denote their action on Σ while the Roman alphabet sub/superscripts denote spatial components. Assuming that the lapse is 1 and the shift is 0 ($\partial_0 = n$), then applying the 1-forms dx^0 and dx^m in turn to the Gauss-Codazzi equations, one reduces the equations to the *Gauss equation* and *Codazzi equation* respectively as

$$R^0{}_{ijk} = {}^3 R^m{}_{ijk} + K_{jk}K_i^m - K_{ik}K_j^m \qquad (Gauss) \tag{2.1}$$

$$R^m{}_{ijk} = {}^3 R^m{}_{ijk} + K_{jk}K_i^m - K_{ik}K_j^m \qquad (Codazzi) \tag{2.2}$$

Using 2.1 and 2.2, the fact that 4 of Einstein's equations are constraints is easily revealed. Manipulating the Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, to give

$$G^\mu_\nu = R^{\mu\alpha}{}_{\nu\alpha} - \frac{1}{2}\delta^\mu_\nu R^{\alpha\beta}{}_{\alpha\beta}, \tag{2.3}$$

one can set $\mu = \nu = 0$ to obtain $G^0_0 = -(R^{12}{}_{12} + R^{23}{}_{23} + R^{31}{}_{31})$, from which one can apply 2.2 to obtain

$$G^0_0 = -\frac{1}{2}({}^3R + (K^i_i)^2 - K_{ij}K^{ij}). \tag{2.4}$$

As the i and j indices on 2.4 are all contracted, the K's can be regarded as traces of matrices resulting in the RHS of 2.4 to be a constant. This gives the first constraint, $G_0^0 = 8\pi\kappa T_0^0$, on the Einstein equations which reveals that the extrinsic curvature on Σ is related to the scalar curvature on Σ. Setting $\mu = 0$ and $\nu = i$ from 2.3 gives the remaining 3 constraints

$$G_i^0 =^3 \nabla_j K_i^j -^3 \nabla_i K_j^j \tag{2.5}$$

implying the Einstein equations $G_i^0 = 8\pi\kappa T_i^0$ are also not dynamical but are constraints on the extrinsic curvatures. Generalizing the lapse and shift to any values, one can obtain the generalized constraint that

$$G_{\mu\nu} n^\mu n^\nu = -\frac{1}{2}(^3R + (trK)^2 - tr(K^2)). \tag{2.6}$$

In fact, [10] shows that 2.6 is actually an initial value formulation of GR; it gives an extensive comparison between 2.6 and its analogue in Maxwell's theory, $\nabla \cdot \vec{E} = 0$, to show that 2.6 is an initial value constraint.

The remaining 6 equations $G_{ij} = 8\pi\kappa T_{ij}$ describe the dynamics of ^3g hence describe how space warps over time. Since the vacuum solutions of the Einstein equations are much simpler to deal with, this paper only discusses such solutions initially and leave the insight to matter-coupling for a later chapter. Thus the equation governing the evolution of ^3g comes down to $G_{ij} = 0$.

2.2 Canonical Quantization

As stated in the Introduction, LQG quantizes the metric canonically. Given the simple relation between Poisson brackets and commutators in canonical quantization, this means it is necessary to first formulate a Hamiltonian formalism of GR. Drawing analogy to the dynamics of a free particle where the phase space is described by the position q^i and its canonical momentum p_i, the Hamiltonian formulation of GR replaces the q^i with the 3-metric, $^3g_{ij}$ and p_i with the canonical momentum of $^3g_{ij}$. However, to emphasize the analogy with the free particle system, it is conventional to use the same alphabets for the 3-metric and its canonical momentum such that they are depicted as q_{ij} and p^{ij}. The *configuration space* for GR, which is also known as the *super space*, is then $Met(\Sigma)$, the space in which the 3-metric can evolve in.

To define the canonical momentum, $p_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{ij}}$, one must first deal with the time derivative of the spatial metric as can be seen by the momentum's expression. Recalling the spacetime manifold $\mathcal{M}$ is diffeomorphic to $S \times \mathbb{R}$, one can use the diffeomorphism map to define the time derivative $\partial_\tau = \partial_0$ such that ∂_1, ∂_2, ∂_3 are tangential to Σ hence obtain an expression for $\dot{q}_{ij}$. In particular, this allows the extrinsic curvature to be expressed as

$$K_{ij} = \tfrac{1}{2} N^{-1}(\dot{q}_{ij} -^3 \nabla_i N_j -^3 \nabla_j N_i),$$

the derivation of which is in [6]. Note that N_i represents the shift vector, $\vec{N}$, which is denoted in component form. This expression shows the close link between the time derivative of the

3-metric and the extrinsic curvature as mentioned in the previous section therefore suggests a connection between the extrinsic curvature and the canonical momentum. In fact, the Einstein-Hilbert lagrangian

$$\mathcal{L} = R\sqrt{-det\ g}$$

can be expressed in terms of the 3-metric and the lapse function as

$$\mathcal{L} = \sqrt{q}NR$$

where $q = det\ q$. Also rewriting the Ricci scalar in terms of the 3-metric and extrinsic curvature, the Lagrangian becomes

$$\mathcal{L} = \sqrt{q}N({}^3R + tr(K^2) - tr(K)^2). \tag{2.7}$$

In fact a direct calculation results in 2 additional total derivative terms which may be dropped if Σ is assumed to be compact. Refer to [7] for details of this. Details regarding non-compact Σ can be found in [8] and [9]. The momentum conjugate to q_{ij} can explicitly be calculated as

$$p^{ij} = \sqrt{q}(K^{ij} - tr(K)q^{ij}) \tag{2.8}$$

using 2.7. Carrying out a Legendre transformation, the Hamiltonian for GR is obtained: $H = \int_\Sigma \mathcal{H}\ d^3x$, where the Hamiltonian density, $\mathcal{H}$, is given by

$$\mathcal{H}(p^{ij}, q_{ij}) = p_{ij}\dot{q}^{ij} - \mathcal{L}$$

and $\mathcal{L}$ is the lagrangian density given by 2.7. As it is customary to express the Hamiltonian in terms of conjugate pairs, after some calculations, one can show that the Hamiltonian density becomes

$$\mathcal{H} = \sqrt{q}(NC + N^iC_i) \tag{2.9}$$

where $C = -{}^3R + q^{-1}(tr(p^2) - \frac{1}{2}tr(p)^2)$ and $C_i = -2\ {}^3\nabla^j(q^{-1/2}p_{ij})$. Note that 2.9 shows terms proportional to the lapse and shift. This is expected of the Hamiltonian as its role in Hamiltonian mechanics is to generate time evolution which in GR is defined by diffeomorphisms via the lapse and the shift. Detailed calculations of this result is given in the Appendix of [1]. Another interesting feature of 2.9 is that, using 2.8, C and C_i can be rewritten in terms of the extrinsic curvature from which then on can be written as

$$C = -2G_{\mu\nu}n^\mu n^\nu$$

and

$$C_i = -2G_{\mu i}n^\mu$$

employing 2.6. For vacuum solutions of the Einstein equations discussed in this paper, the above implies $C = C_i = 0$. This statement is exactly identical to the statement for the four constraints for vacuum solutions given by 2.6; for this reason C is called the *Hamiltonian constraint* while C_i is called the *diffeomorphism constraint* or *vector constraint.* This type of Hamiltonian formulation exposed to constraints is called a *constrained Hamiltonian formulation* whose characteristic is that it contains non-dynamical variables such as N and N_i which act as Lagrange multipliers. Constrained Hamiltonian systems are fairly common in Physics. In fact, the Hamiltonian formulation of Maxwell's theory is another example of a constrained Hamiltonian formulation, details of which can be found in [10]. From the constraints, it is then straightforward that $H = 0$. It is crucial the reader does not confuse the equation $H = 0$ as an implication that the dynamics of the theory is trivial. Because $H = 0$ arises from the constraints of GR, it is only a statement of the initial constraints posed by 2.6; the dynamics of the theory is actually very complicated and expressed as

$$\dot{q}^{ij} = \{H, q^{ij}\}, \qquad \dot{p}_{ij} = \{H, p_{ij}\}.$$

It should be pointed out that the constraint $H = 0$ is called a *first class* Dirac constraint. This is a constraint whose Poisson bracket with any other constraints vanishes. Second class constraints are those which do not have a vanishing Poisson bracket. In this paper only first class constraints are dealt with. For details on this matter refer to [11]. What is special about a first class constraint is that it is analogous to having a *Dirac algebra*, which is a Poisson bracket relation of the diffeomorhism and Hamiltonian constraints. For the case at hand, the Dirac algebra is given by

$$\begin{aligned} \{C(\vec{N}), C(\vec{N}')\} &= C([\vec{N}, \vec{N}']) \\ \{C(\vec{N}), C(N)\} &= C(\partial N'/\partial \vec{N}) \\ \{C(N), C(N')\} &= C(q^{-1}(N\partial^i N' - N'\partial^i N)) \end{aligned} \tag{2.10}$$

where $C(\vec{N})$ and $C(N)$ denote smearing of C_i and C respectively as

$$C(\vec{N}) = \int_\Sigma N^i C_i q^{1/2} d^3x$$

$$C(N) = \int_\Sigma N C q^{1/2} d^3x$$

.

It is worth observing that the last equation of 2.10 is of the form of a Lie algebra with the "structure constant", q^{-1}; hence one may consider relating this algebra to the Hamiltonian generator associated to the spacetime diffeomorphism group. However, the fact that q^{-1} is not a constant, but rather a phase space dependent quantity, differentiates this algebra from the Lie algebra.

With all the Poisson relations defined, quantization is fairly straightforward due to *Dirac's quantization procedure.* This procedure is applicable to any constrained system but a succint outline of it for GR is described below based on [1]. For a more in-depth approach the reader is encouraged to look into [12].

Dirac's quantization procedure

1. Define phase space variables, q^{ij} and p_{ij}, of the classical Hamiltonian theory as operators, $\hat{q}^{ij}$ and $\hat{p}_{ij}$ in a *kinematical* Hilbert space, $\mathscr{H}_{kin}$, such that standard commutation relations are satisfied.

$$\{\cdot,\cdot\} \to \frac{1}{i\hbar}[\cdot,\cdot]$$

2. Promote the constraints H to operators $\hat{H}$ which act on states in $\mathscr{H}_{kin}$
3. Identify the physical Hilbert space of solutions to the constraints as $\mathscr{H}_{phy}$ such that the solutions, ψ, satisfy

$$\hat{H}\psi = 0 \qquad \forall\psi \in \mathscr{H}_{phy}. \tag{2.11}$$

After the second step of the quantization program, the Hamiltonian is promoted to an operator but the relation $H = 0$ is not yet imposed. For this reason, states in $\mathscr{H}_{kin}$ do not necessarily satisfy the constraint set by the classical counterpart of the Hamiltonian; namely $\hat{H}\psi \neq 0$, $\forall\psi \in \mathscr{H}_{kin}$ in general. It is only after the third step of the program that one is able obtain quantum states which actually relate to physical states of the system. In the case of GR, the equation representing physical states given by 2.11 is known as the *Wheeler-DeWitt equation.*

Despite simple systematic steps given by Dirac's procedure, quantizing gravity this way leads to a few major problems. The first is that solutions to 2.11 are very difficult to find even at a formal level, although solutions for gravity coupled to electromagnetism in a spherically symmetric spacetime have been found; for details refer to [13]. Secondly, because the configuration space for gravity, $Met(\Sigma)$, is infinite-dimensional, there is no well-defined manner to determine a Lebesgue measure on the configuration space. Then taking the L^2 space of the configuration space to define a Hilbert space as is usually done in canonical quantization, one finds defining an inner product in $L^2(Met(\Sigma))$ is ambiguous. This problem, due to its nature, is called the *inner product problem.* Furthermore the background independent nature of GR also leads to the *problem of time.* This arises due to the conceptual difference of time in quantum theory and GR; as mentioned in the introduction, in quantum theory time is a fixed measure whose role is a form of a background, while in GR time is a dynamical variable whose dynamics takes part in the physics. In fact 2.11 is in the form of the Schrödinger equation with no time dependence and may seem to be in contradiction with everyday experiences (although there is actually no paradox because 2.11

applies to the universe as a whole and our everyday experiences come from local parts of the universe). Constructing a time evolution by the usual Heisenberg equation for a generic operator $\hat{O}(t)$ in $\mathscr{H}_{phy}$ also apparently gives no dynamics:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}(t) = i[\hat{H}, \hat{O}(t)] = 0$$

.

An explanation for this and the discrepancy in the notion of time is an active area of research, details of which can be found in [14], [15] and [16]. The final setback to note from canonical quantization is the *operator ordering problem.* Recalling that the statement, $H = 0$, was classically analogous to the Dirac algebra, 2.10, one would wish to have the quantum version of 2.10 to give canonical commutation relations:

$$\begin{aligned} [\hat{C}(\vec{N}), \hat{C}(\vec{N}')] &= -i\hat{C}([\vec{N}, \vec{N}']) \\ [\hat{C}(\vec{N}), \hat{C}(N')] &= -i\hat{C}(\partial N'/\partial \vec{N}) \\ [\hat{C}(N), \hat{C}(N')] &= -i\hat{C}(q^{-1}(N\partial^i N' - N'\partial^i N)), \end{aligned} \tag{2.12}$$

where the quantum versions of the constraints, $\hat{C}(\vec{N})$ and $\hat{C}(N)$, contain the promoted conjugate pair $\hat{q}^{ij}$ and $\hat{p}_{ij}$. This way the Wheeler-Dewitt equation would have a "Dirac algebra" associated to it. However, in quantum theories the order of the operators give different outcomes and to order $\hat{q}^{ij}$ and $\hat{p}_{ij}$ in a way such that 2.12 can be achieved has proven to be very challenging.

Indeed these problems created a deadlock in the advancement of a quantum theory for gravity for many years. Nonetheless, they were overcome when "new variables" were introduced by A. Ashtekar in the early 80s. To investigate this in further detail, a different formalism of GR must first be acknowledged. The following chapter provides the preliminaries required to quantize GR in terms of the new variables.

3 General Relativity

Recall that the Hamiltonian formulation of GR that was discussed in the previous chapter was constructed by a Legendre transformation of the Einstein-Hilbert action,

$$S(g) = \int_{\mathcal{M}} R\sqrt{|g|}\ d^n x, \tag{3.1}$$

where $\sqrt{|g|} = \sqrt{|detg|}$ for an n-dimensional manifold, $\mathcal{M}$. This naturally led to the dynamical variables being the metric and its conjugate momentum. Due to the background independent nature of GR, however, the quantization of these variables led to a deadlock. The key around this setback is to formulate the action for GR in terms of different variables in the first place; this action is the *Palatini action.*

3.1 The Palatini Action

For an n-dimensional manifold, $\mathcal{M}$, chapter 2 showed that there is a diffeomorphism between $\mathcal{M}$ and $\mathbb{R}^n$. Then since $\mathbb{R}$ admits a trivial tangent bundle[1], so does TM, the tangent bundle of $\mathcal{M}$. Trivializing TM, one defines an isomorphism

$$e : \mathcal{M} \times \mathbb{R}^n \to TM.$$

The isomorphism, e, is also called a *frame field* and for a 4-dimensional $\mathcal{M}$ it is specifically called a *tetrad* or *vierbein* while for a 3-dimensional $\mathcal{M}$ it is called a *triad* or *dreibein*.

The fiber of the trivial bundle, $\mathbb{R}^n$, is called the *internal space* and indices associated to it is denoted by capital Latin letters to differentiate them from spacetime indices which are signified by lower-case letters. In particular, since a section[2] in the bundle, $\mathcal{M} \times \mathbb{R}$, is just a function on $\mathcal{M}$, sections can be expressed in terms of a basis, ξ_I. This allows vector fields to be mapped onto $\mathcal{M}$ from the frame field acting on the basis of sections and are denoted $e(\xi_I)$[3]. For the sake of simplicity $e(\xi_I)$ is often written as e_I in relativity and this paper will from here onwards follow this convention.

A feature to note after the trivialization of TM is that because of the appearance of $\mathbb{R}$, a canonical Lorentzian metric can be defined on $\mathcal{M} \times \mathbb{R}$ hence a canonical inner product can be defined. This metric is called the *internal metric* and is given by

$$\eta_{IJ} = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 1. \end{pmatrix}$$

which can be used to define the inner product as

$$\eta(s, s') = \eta_{IJ} s^I s'^J$$

where s and s' are sections living in $\mathbb{R}$. If $\mathcal{M}$ is equipped with a Lorentzian metric, g, as is normally the case in GR, then inner products of vector fields, v and v', are definable via

$$g(v, v') = g_{\alpha\beta} v^\alpha v'^\beta.$$

For *orthonormal* frame fields satisfying

$$g(e_I, e_J) = \eta_{IJ},$$

[1] A trivial bundle is one which can be expressed as a Cartesian product of a manifold and a vector space. An example of a non-trivial bundle is the Möbius strip.

[2] A section is a map that assigns a point in the vector bundle to an associated vector in $\mathcal{M}$.

[3] Coframe fields, e^I_μ, can also be defined as the inverse of frame fields. They act as maps from the vectors on $\mathbb{R}^n$ to sections on the base space.

a simple calculation reveals that there is a relation between the inner product defined by the spacetime metric and the internal metric, namely

$$g(e(s), e(s')) = \eta(s, s'),$$

recalling that $e(s)$ and $e(s')$ are vector fields on $\mathcal{M}$. Then the Palatini formalism allows one to construct theories using orthonormal frame fields rather than metrics. This already sheds light on how the Palatini formalism will replace the troublesome metric for new variables when quantizing gravity.

Another quantity crucial in developing the Palatini action is the *Lorentz connection* or *spin connection*; it is a 1-form whose components will be denoted ω_μ^{IJ}. The spin connection is a *G-connection* or more specifically a $SO(n,1)$-connection. A G-connection is a connection whose component values live in a Lie algebra. For a $SO(n,1)$-connection, the values live in the Lorentz Lie algebra. With a connection, it is possible to define the curvature of the connection, which for the spin connection is given by

$$F^{IJ} = \mathrm{d}\omega^{IJ} + \omega_K^I \wedge \omega^{KJ}$$

whose components are

$$F_{\mu\nu}^{IJ} = \partial_\mu \omega_\nu^{IJ} - \partial_\nu \omega_\mu^{IJ} + \omega^I{}_{K\mu}\omega_\nu^{KJ} - \omega^J{}_{K\mu}\omega_\nu^{KI}.$$

Further details regarding the spin connection and its link to curvature and torsion can be found in [17]. Using the curvature and the frame field, the Palatini action is given by

$$S(\omega, e) = \int_{\mathcal{M}} e \wedge e \wedge *F = \int_{\mathcal{M}} e_I^\mu e_J^\nu F_{\mu\nu}^{IJ} \; vol$$

where the $*$ denotes the Hodge star and the *vol* depicts the volume form which is written no longer in terms of the metric, g, but rather in terms of the frame field. In the case of a 4-dimensional manifold the action explicitly becomes

$$S(\omega, e) = \frac{1}{2} \int d^4x \; \epsilon_{IJKL} \; \epsilon^{\mu\nu\rho\sigma} e_\mu^I e_\nu^J F_{\rho\sigma}^{KL}.$$

Gravity formulated in terms of tetrads as above is known as the *tetrad formalism* of GR. The absence of the metric in the Palatini formalism will prove to be very useful when formulating GR in a quantum mechanical framework as will be apparent in the following chapter.

3.2 Gauge Invariance

Digressing from gravity for a while, recall that the remaining 3 fundamental forces are described by QFT under the framework of the SM; more precisely, the theories which describe these forces are gauge theories of the Yang-Mills type.

Gauge theories of the Yang-Mills type can be formulated using a *G-bundle*. A G-bundle is a principal bundle whose fibers are compatible with the action of a group G and is a vector bundle. In other words, these theories have a spacetime manifold, $\mathcal{M}$, as the base space and a vector space, $\mathbb{R}^n$, as the fibers with a projection map, π, that "projects" every vector living in $\mathbb{R}^n$ onto an associated point in $\mathcal{M}$. A gauge tranformation is then tantamount to acting with the inverse projection map, π^{-1}, such that a point on a fiber is mapped to a different point on the fiber. The group G acting on the vector space is identified as the gauge group and the interpretation of a gauge field then naturally comes about as the cross-section of the bundle.

Because gauge theories provide a path to a direct interpretation of gauge bosons and give insight to conservation laws, it is worth attempting to formulate GR as a gauge theory as well. However, GR is a theory that possesses a diffeomorphism group, not a gauge group. A diffeomorphism transformation, similarly to a gauge transformation, leaves the physics of the theory invariant, but a diffeomorphism group cannot be formulated in terms of a G-bundle for the reason that diffeomorphisms do not generate changes in the field at a point but instead map the point itself to another point. For a detailed discussion on the discrepancies between Yang-Mills gauge theories and diffeomorphism invariant theories the reader is encouraged to look into [18].

The diffeomorphism symmetry in GR is nonetheless considered a gauge theory, just not a gauge theory of the Yang-Mills type. To see how this unfamiliar symmetry gives rise to gauge symmetries similar to the type of Yang-Mills, consider the Palatini action from the previous section. It is evident that the action entails a $SO(n,1)$ symmetry, or more precisely a $SO(3,1)$ symmetry in the case of a 4-dimensional spacetime manifold, and GR is, in this sense, regarded to have a $SO(3,1)$ gauge invariance. This symmetry is due to the nature of the tetrad formalism and is associated to the freedom to choose different bases for different spacetime points. This gauge invariance plays a crucial role in providing an additional constraint to the existing Hamiltonian and diffeomorphism constraints. The following chapter attempts to analyze the implications of such new features by introducing the Ashtekar new variables.

4 The Ashtekar New Variables

Recall that the usage of the metric and its canonical momentum as dynamical variables failed in canonical quantization due to the background independent nature of GR leading to the problem of time. To overcome this problem, GR was formulated in terms of a different set of variables, the tetrad and the spin connection, via the Palatini formalism. It is natural then to attempt a Legendre tranformation on the Palatini action with hopes that a Hamiltonian formalism of these novel variables will solve the problems encountered previously.

However, this attempt leads to complexities. The constraint algebra becomes second class and the Dirac algebra is no longer closed unlike 2.10. Second class constraints may be solved via the steps given in [19], but because the old conjugate variables are now functions of the tetrad and the spin connection, the anlysis is complicated. This complication can be overcome by first introducing a new term in the Palatini action such that it becomes the *Holst action*:

$$S(\omega, e) = \int_{\mathcal{M}} (\frac{1}{2}\epsilon_{IJKL}\ \epsilon^{\mu\nu\rho\sigma} + \frac{1}{\gamma}\delta_{IJKL}\ \delta^{\mu\nu\rho\sigma})\ e^I_\mu \wedge e^J_\nu \wedge F^{KL}_{\rho\sigma} \tag{4.1}$$

where $\delta_{IJKL} = \delta_{I[K}\delta_{L]J}$. The γ is called the *Barbero-Immirzi parameter*, sometimes also known as just the *Immirzi parameter*, and an appropriate choice of this parameter allows one to simplfy one's calculations by reducing the second class constraints to first class as explained in [22]. One can also check that at a classical level the introduction of the new term does not affect the field equations in any way as long as there is no torsion in the manifold (in the case that torsion exists this is correspondent to fermionic field coupling, the details of which can be found in [20] and [21]). The Holst action, just like the Palatini action, is free in its expression of the metric thus is a perfect candidate for the construction of appropriate dynamical variables for a quantum theory. In fact defining the *densitized triad* as

$$\tilde{E}^a_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}e^j_b e^k_c \tag{4.2}$$

and the *Ashtekar-Barbero connection* or *Ashtekar connection* as

$$A^i_a = \gamma\omega^{0i}_a + \frac{1}{2}\epsilon^i_{jk}\omega^{jk}_a, \tag{4.3}$$

one constructs a new canonically conjugate pair of dynamical variables whose indices a, b, c pertain to 3-dimensional spatial spacetime indices while i, j, k pertain to the 3-dimensional internal space. These are the *Ashtekar (new) variables.* The densitized triad is the substitute for the 3-metric in the old formalism and is the 3-dimensional coframe field multiplied by the determinant of the metric:

$$\tilde{E}^a_i = \sqrt{detq}\ e^a_i \tag{4.4}$$

and it is densitized so that it is readily integrable on any curved manifold. The tilde on the triad is a conventional notation for densitized quantities, but for the sake of simplicity it will be dropped from this point onwards.

Expressing 4.1 in terms of the Ashtekar variables, one obtains

$$S(A, E) = \frac{1}{\gamma}\int_{\mathcal{M}} d^4x(\dot{A}^i_a E^a_i - A^i_0 G_i - NC - N^a C_a) \tag{4.5}$$

where

$$G_i \equiv D_a E^a_i = \partial_a E^a_i + \epsilon_{jil} A^j_a E^{al},$$
$$C_a = \frac{1}{\gamma} F^j_{ab} E^b_j - \frac{1+\gamma^2}{\gamma} K^i_a G_i, \tag{4.6}$$
$$C = [F^j_{ab} - (\gamma^2+1)\epsilon_{jmn} K^m_a K^n_b] \frac{\epsilon_{jkl} E^a_k E^b_l}{detE} + \frac{1+\gamma^2}{\gamma} G^i \partial_a \frac{E^a_i}{detE}.$$

The latter 2 equations of 4.6 are the Hamiltonian and diffeomorphism constraints which the reader was exposed to in the 2$^{\text{nd}}$ chapter of this paper. The first equation of 4.6, however, is a new constraint called the *Gauss constraint*. Recall from Lagrangian mechanics that a symmetry in the Lagrangian is associated to a conserved quantity which in Hamiltonian mechanics manifests itself as a constraint. The appearance of this new constraint is then not a surprise; it was rather foreseen from the moment the Palatini action, and thus the Holst action, was formulated as these actions gave rise to a new $SO(3,1)$ gauge symmetry.

What may strike the reader as a surprise is the fact that despite the $SO(3,1)$ symmetry, the densitized triad and the Ashtekar connection behave as a $SU(2)$ vector and $SU(2)$ connection respectively. The source of this peculiarity comes from the change of variables in 4.2 and 4.3. Observing the constraints, the reader familiar with gauge theories will recognize that the Gauss constraint in 4.6 is associated to a $SU(2)$ gauge tranformation with the $SU(2)$ structure constants ϵ_{ijk}.

It is also worth noting that this gauge theory formulation discussed above is only true for the Immirzi parameter, $\gamma = i$. This was the original choice of the parameter when Ashtekar first developed his theory. However, because any physical theory must be imposed with a reality condition, a complex-valued Immirzi parameter induced complications for physical interpretations. In 1994, J. Barbero introduced a method of imposing a reality condition by choosing the Immirzi parameter to be real hence resulting in a *real* $SU(2)$ connection or a $SO(3)$ connection, which in terms of the Lorentz connection reflects to a self-dual connection. For details of this formulation refer to [5]. It is then consequential that the theory based on a $SO(3)$ connection is no longer diffeomorphism invariant when interpreted as a gauge theory, details of whose approach are given in [23] and [24].

Despite the downsides of a real Immirzi parameter, this paper will henceforth stick to using a real Immirzi parameter as it offers an easy route to the imposition of reality conditions. Using the new conjugate variables, A^i_a and E^a_i, one can then construct the usual Poisson bracket relations as

$$\{A^i_a(x), E^b_j(y)\} = \gamma \delta^b_a \delta^i_j \delta^3(x,y). \tag{4.7}$$

The next step would then be to follow the Dirac quantization procedure as before thus formulating a quantized formalism of this theory. Before taking this step, the reader is reminded of the current situation of the classical theory, namely the development of GR as a $SU(2)$ gauge theory with a

Poisson bracket relation defined by 4.7 and 3 constraints given by

$$\begin{aligned} G_i &= 0 \\ C_a &= 0 \\ C &= 0. \end{aligned} \tag{4.8}$$

5 Quantization of the New Variables

Since the formulation of GR as an $SU(2)$ gauge theory has equipped the theory with a connection, drawing analogy with the quantization process of Yang-Mills gauge theories, one is inclined to follow the subsequent steps, reminiscent of the Gupta-Bleuler quantization [25], [26].

1. On the space of connections modulo gauge transformations, the space of G-connections that are classified with a class of equivalent gauge transformations, define a Gaussian measure thus a Lebesgue measure via the Minkowski metric.

2. Construct a Hilbert space using a L^2 space defining operators in the Schrödinger picture as

$$\hat{A}^i_a \psi(A) = A^i_a \psi(A),$$
$$\hat{E}^a_i \psi(A) = -i\hbar\gamma \tfrac{\delta}{\delta A^i_a} \psi(A),$$

   with a factor of $8\pi G$ set to 1, such that they satisfy the commutation relation

$$[\hat{A}^i_a(x), \hat{E}^b_j(y)] = i\hbar\gamma\delta^b_a\delta^i_j\delta^3(x,y).$$

   Then this provides the usual canonical quantization associated to promoting 4.7 to a quantum relation.

3. Pick out the gauge-invariant states in the Hilbert space constructed in step 2 by implementing the Gauss constraint:

$$\hat{G}_i \psi(A) = 0$$

4. Apply the Hamiltonian on the gauge-invariant states to study the dynamics of the theory:

$$i\partial_t \psi(A) = \hat{H}\psi(A)$$

For the case at hand, however, step 1 of the above procedure poses a problem. Due to the background independent nature of GR, it is not possible to define a Lebesgue measure through a fixed background such as the Minkowski metric. The goal then is to define a measure on the space of connections without having to resort to a fixed background. In order to do this the reader is first exposed to the concept of holonomies given in the following section.

5.1 Holonomies

For the sake of familiarity, consider a vector, E^a, in a Yang-Mills theory and parallel transport it along a curve $\gamma^a(t)$ where t is the parameter on the curve. This would correspond to the covariant derivative of E^a being zero along $\gamma^a(t)$:

$$\dot{\gamma}^a(t)\mathbf{D}_a E^b = 0$$

where $\dot{\gamma}^a(t) = d\gamma^a(t)/dt$ is the vector tangent to $\gamma^a(t)$ and $\mathbf{D}_a$ is the Yang-Mills covariant derivative. Expanding out the covariant derivative one is left with

$$\dot{\gamma}^a(t)\partial_a E^b(t) = -ig\dot{\gamma}^a(t)A_a(t)E^b(t) \tag{5.1}$$

with g, the coupling constant, and A_a, the vector potential (or the gauge field). This is a differential equation whose variable, E^a, that needs to be solved for is given in terms of that variable itself thus is insoluble analytically. However, one may obtain a solution formally by integrating both sides from $t' = 0$ to $t' = t$ to obtain

$$E^b(t) = E^b(0) - ig\int_0^t dt'\dot{\gamma}^a(t')A_a(t')E^b(t'),$$

then carrying out iterative methods, namely substituting the LHS of the equation for the RHS of the same equation. The first iteration gives

$$E^b(t) = E^b(0) - ig\int_0^t dt'\dot{\gamma}^a(t')A_a(t')E^b(0) - g^2\int_0^t dt'\dot{\gamma}^a(t')A_a(t')\int_0^{t'} dt''\dot{\gamma}^a(t'')A_a(t'')E^b(t'')$$

and an indefinite iteration gives the sum

$$E^b(t) = \sum_{n=0}^{\infty}\left((-ig)^n \int\limits_{t_1\geq\cdots\geq t_n\geq 0} \gamma^{a_1}(t_1)A_{a_1}(t_1)\cdots\gamma^{a_n}(t_n)dt_1\cdots dt_n\right)E^b(0) \tag{5.2}$$

which happen to converge, thereby providing an adequate solution to 5.1. The quantity in the brackets with the summation sign is called the *parallel propagator* and it gives a unique solution to 5.1 for a given connection and a curve. If a parallel propagator is a solution to a differential equation whose curve is closed, then the parallel propagator is called a *holonomy*. It is, however, worth keeping in mind that Physicists tend to be lenient with this terminology and extend its usage to open curves as well. For the sake of simplicity, the integral in 5.2 is often expressed in terms of a *path ordered product*. A path ordered product, denoted $\mathcal{P}(\cdots)$, is defined such that the quantities with larger values of t_i appear on the left of quantities with smaller values of t_i. For instance, if $t_1 > t_2 > \cdots > t_n$ then a path ordered product gives

$$\mathcal{P}(A_{a_1}(t_1)A_{a_2}(t_2)\cdots A_{a_n}(t_n)) = A_{a_1}(t_1)A_{a_2}(t_2)\cdots A_{a_n}(t_n).$$

Applying this notation to the integral in 5.2, one finds

$$\int\limits_{t_1\geq\cdots\geq t_n\geq 0} \gamma^{a_1}(t_1)A_{a_1}(t_1)\cdots\gamma^{a_n}(t_n)dt_1\cdots dt_n = \frac{1}{2}\mathcal{P}\left(\int_0^t \dot{\gamma}^a(t)A_a(t)dt\right)^n.$$

Then including the summation sign, one may define the holonomy to be a *path ordered exponential* given by

$$\sum_{n=0}^{\infty}\frac{(-ig)^n}{n!}\mathcal{P}\left(\int_0^t \dot{\gamma}^a(t)A_a(t)\right)^n \equiv \mathcal{P}\left[\exp\left(-ig\int_0^t \dot{\gamma}^a(t')A_a(t')dt'\right)\right].$$

Notice that if the gauge group was to be Abelian, the vector potentials would commute and the path-ordering would play no role; thus the path-ordering generalizes the Abelian case to the non-Abelian case. It is also worth noting that the parallel propagator hence the path ordered exponential is a matrix whose trace is a gauge-invariant scalar. This provides an insight that the trace may be related to an observable quantity. In fact the trace of the holonomy is known as a *Wilson loop* and according to *Giles' theorem* [27], the information about the trace of a holonomy along all possible loops is enough to extract all the gauge-invariant information about a given vector potential.

Now consider applying the notion of holonomies to gravity by carrying out the same procedure as above but for a $SU(2)$-connection, the key quantity at hand with the tetrad formalism of GR. Then the holonomy of the connection, A_a^i, along a curve, γ, is defined as

$$h_\gamma \equiv \mathcal{P}\left[\exp\left(\int_\gamma A\right)\right] \equiv \sum_{n=0}^{\infty}\int\limits_{t_1\geq\cdots\geq t_n\geq 0} \dot{\gamma}^a(t)A_a^i(t)\mathbf{T}_i\ dt \tag{5.3}$$

where $\mathbf{T}_i$ are the generators of the group $SU(2)$. Similarly to the Dirac algebra given previously by 2.10, the variables must be smeared such that they can be given in such a form. The holonomy of the Ashtekar-Barbero connection given by 5.3 already depicts a smearing of the connection and one is left with the densitized triad to do the same. Using the fact that the densitized triad is a 2-form, one may smear it over a surface, S, to obtain the flux of E over S:

$$E_i(S) = \int_S n_a E_i^a\ d^2\sigma \tag{5.4}$$

in which n_a is the normal to the surface and σ is the quantity parametrizing the surface. With the 2 smeared quantities, h_γ and $E_i(S)$, it is possible to construct an algebra known as the *holonomy flux algebra.*

The next step forward is to use the notions of holonomies and holonomy flux algebra discussed in this chapter to develop an entity called *spin network states* which enables the description of quantized geometry without resorting to a fixed background.

5.2 Spin Network States

Recall that the quantization of the new variables, A_a^i and E_i^a, posed a problem due to the non-existent Lebesgue measure on a background indepedent formalism. This problem can be overcome by relying on a *loop transform.* At the beginning of this chapter when the new variables were quantized, a Shrödinger representation of the Ashtekar connection was given as $\psi(A)$; this is known as the *connection representation* and a loop transform of this representation results in a *loop representation* which solves the problem at hand; for details refer to chapter 3 of [28]. The name Loop Quantum Gravity in fact comes from the loops in this loop representation. The origin of loop transformations comes from Giles' theorem. Because the trace of a holonomy encodes all the gauge-invariant information about the connection at hand and the Gauss constraint naturally imposes a gauge-invariance on the connection representation of wavefunctions (as shown at the begining of this chapter), it is almost intuitive that the wavefunction in the connection representation can be expanded in terms of traces of holonomies or Wilson loops. The expansion is given as

$$\psi(A) = \sum_{\gamma} \psi(\gamma) W_{\gamma}(A) \tag{5.5}$$

with

$$W_{\gamma}(A) = \mathrm{Tr}\left(\mathcal{P}\left[\exp\left(-\int_{\gamma} \dot{\gamma}^a(t) A_a(t) dt\right)\right]\right)$$

denoting Wilson loops over the closed curves, γ, and $\psi(\gamma)$ denoting expansion coefficients. The expansion 5.5 is then known as the loop transform and the coefficients, $\psi(\gamma)$, as the loop representation. The meticulous reader may have recognized that 5.5 is very similar to the tranformation of position representation to momentum representation in ordinary quantum mechanics. One may think that the loop representation is in fact equivalent to the original connection representation in the sense that the position and momentum representations are equivalent in quantum mechanics.

The advantage of using loops is that loops are intimately connected to knots in knot theory, a well-researched branch of mathematics. The diffeomorphism invariance property of GR implies that a deformation in a loop keeps the wavefunction in the loop representation invariant. This is related to knot invariants in knot theory, details of which can be found in chapter 5 of [5]. For the reader who is interested, a detailed review of how knots play a role in LQG can also be found in [29].

Despite advantages, loops themselves are also problematic because the basis in the loop space are over-complete; in other words certain traces of holonomies on a manifold can be formulated from combinations of traces of other holonomies and there is a sense of redundancy. To see how this problem arises, one must first be assured that for $SU(2)$ matrices, such as those considered by the formalism of Ashtekar new variables, the identity

$$\mathrm{Tr(A)Tr(B)} = \mathrm{Tr(AB)} + \mathrm{Tr(AB^{-1})}$$

is satisfied $\forall \mathrm{A, B} \in SU(2)$. In terms of Wilson loops, this is translated as

$$W_\gamma(A)W_\eta(A) = W_{\gamma\circ\eta}(A) + W_{\gamma\circ\eta^{-1}}$$

where the circle denotes a composition of going around the loop γ after the loop η. Due to the unitary nature of the matrices in $SU(2)$, $W_{\eta^{-1}}(A) = W_\eta(A)$. Also the fact that Wilson loops are traces of holonomies imply that $W_{\gamma\circ\eta}(A) = W_{\eta\circ\gamma}(A)$ by the cyclic property of the trace. The above properties are known as the *Mandelstam identities* whose specifics are in [30]. They provide a relation between different Wilson loops therefore projecting redundancies in the loop space. Overcoming the problem of the over-complete basis of the loop space, leads to a spin network.

To understand spin networks, first consider *cynlindrical functions*; they are functionals of fields which only depend on cetain components of the fields. For instance, for the Ashtekar formalism of GR, the field is the Ashtekar connection and the cylindrical function would be a functional only of the holonomies of the connection. To be more rigorous, a cylindrincal function is a pair, (Γ, f), of a graph in Σ and a smooth function $f : SU(2)^L \to \mathbb{C}$ in which the superscript L depicts the number of links in the graph showing the number of cartesian products of $SU(2)$ present. This function is then defined as

$$f(h_{e_1}(A), \ldots, h_{e_L}(A)) = \psi_{(\Gamma,f)}(A) = \langle A \mid \Gamma, f\rangle \in Cyl_\Gamma$$

where the subscripts e_i denote the oriented paths making up the links of the graph and Cyl_Γ represents the space of cylindrical functions. Notice that the function is a functional of the Ashtekar connection but only dependent on the holonomies as stated previously. Figure 2 provides a pictorial diagram of the graphs involved in the cylindrical functions.

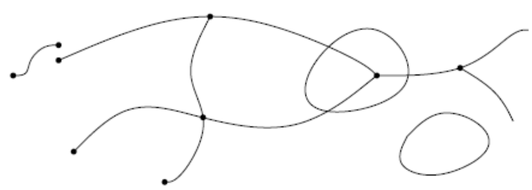

Figure 2: A graph with oriented paths e_i. Figure from [1]

Then the space of functionals made up of these graphs can be converted into a Hilbert space if one defines a scalar product over the space. Noting the fact that the holonomies are elements of $SU(2)$, a Lie group, one can use the Haar measure, dh, to define the scalar product on Cyl_Γ as

$$\langle \psi_{(\Gamma,f)} \mid \psi_{(\Gamma,g)} \rangle \equiv \int \prod_e dh_e \; f^*(h_{e_1}(A), \ldots, h_{e_L}(A)) \; g(h_{e_1}, \ldots, h_{e_L}(A)) \tag{5.6}$$

where the asterisk denotes complex conjugation. With this scalar product, Cyl_Γ becomes $\mathscr{H}_\Gamma$, a Hilbert space associated to the graph Γ. Then the Hilbert space for all cylindrical functions is defined to be the direct sum over all such $\mathscr{H}_\Gamma$ for all graphs:

$$\mathscr{H}_{kin} = \bigoplus_\Gamma \mathscr{H}_\Gamma. \tag{5.7}$$

If there are 2 cylindrical functions ψ and ψ' which share the same graph, then an inner product is defined according to 5.6. However, when constructing a direct sum over all $\mathscr{H}_\Gamma$, two cylindrical functions which have different graphs, Γ_1 and Γ_2, may overlap. In such a case, a new graph is forged as the union of the existing ones such that $\Gamma_3 \equiv \Gamma_1 \cup \Gamma_2$ and the functions associated to each graph, f_1 and f_2 are extended trivially on Γ_3. Then using the definition given by 5.6, the inner product between functionals of two different graphs are given as

$$\langle \psi_{(\Gamma_1,f_1)} \mid \psi'_{(\Gamma_2,f_2)} \rangle \equiv \langle \psi_{(\Gamma_1\cup\Gamma_2,f_1)} \mid \psi'_{(\Gamma_1\cup\Gamma_2,f_2)} \rangle. \tag{5.8}$$

Equipped with such a scalar product, one is able to obtain a Hilbert space through 5.7 through a L^2 space over the connections with an integration measure, $d\mu_{AL}$, called the *Ashtekar-Lewandowski measure*[4]:

$$\mathscr{H}_{kin} = L^2(A, d\mu_{AL}).$$

The Ashtekar-Lewandowski measure provides the Lebesgue measure that was ill-defined due to the absence of a fixed background. In particular, describing the kinematical Hilbert space as a L^2 space over the connections with the measure $d\mu_{AL}$ allows for an explicit expression of the dot product of cylindrical functionals; for instance, 5.8 may be written as

$$\langle \psi_{(\Gamma_1,f_1)} \mid \psi'_{(\Gamma_2,f_2)} \rangle \equiv \int d\mu_{AL}\ \psi^*_{(\Gamma_1,f_1)}(A)\ \psi'_{(\Gamma_2,f_2)}(A)$$

Now that a kinematical Hilbert space has been constructed, the next step is to obtain a representation of the holonomy-flux algebra or of h_{e_i}, which will allow to solve the problem of over-completeness of the basis mentioned previously. In order to do this, consider employing the Peter-Weyl theorem and hence decomposing the function $f : SU(2)^L \to \mathbb{C}$ from a cylindrical function into unitary irreducible representations of $SU(2)$ as

$$f(g) = \sum_j \hat{f}^j_{mn} D^j_{mn}(g)$$

$$\text{for } j = 0, \tfrac{1}{2}, 1, \ldots \qquad m = -j, \ldots, j$$

[4]More details on the construction of integration measures are given in [31] and an extension of the connection space to differential geometry can be found in [32]

where $D^j_{mn}(g)$ are the Wigner matrices giving the spin-j irreducible representation of the group element g. Then for a cylindrical function, $\psi_{(\Gamma,f)}(A)$, which contains a graph with numerous $SU(2)$ functions and holonomies, the decomposition is as

$$\psi_{(\Gamma,f)}(A) = \sum_{j_i,m_i,n_i} \hat{f}^{j_1,\ldots,j_n}_{m_1,\ldots,m_n,n_1,\ldots,n_n} D^{j_1}_{m_1 n_1}(h_{e_1}(A)) \ldots D^{j_n}_{m_n n_n}(h_{e_n}(A))$$

with the Wigner matrices related by a tensor product. The links are now irreducible representations and more importantly they are *unique* as proven by [33]. The ends of the links known as nodes are "tied up" by contracting the indices of the representations with what is known as *intertwiners*. The details of this process is given in pages 43-48 of [1]. Ultimately, such graphs form cylindrical functions that give states to the Hilbert space known as *spin network states* and are expressed as

$$\psi_{(\Gamma,j_m,I_n)}(h_e) = \bigotimes_m D^{j_m}(h_e) \bigotimes_n I_n \tag{5.9}$$

with I_n depicting the intertwiners at the end points of the links. 5.9 then clearly demonstrates that the loop representation can be expressed as a non-redundant basis ironing out the problem of a over-complete basis. This new concept also reveals that a Wilson loop is just a special case of a spin network state with only 1 link and 1 intertwiner. To obtain a physical Hilbert space from the kinematical Hilbert space formed by the spin network states, one would implement the constraints as was described in the Dirac quantization procedure.

5.3 Geometric Operators

With a background independent Hilbert space, the next step is to develop operators which can act on the space to give physical observables. Since the focus of the theory for quantum gravity in this paper is on quantizing GR, a classical theory which entails the background as a dynamical variable, it is natural to ask what the quantum version of the background would be. Drawing analogy to ordinary quantum mechanics, one would expect the existence of some sort of operators that give physical observables pertaining to geometry such that they reflect the quantum nature of the background. Such operators exist and are called *geometric operators*.

The first geometric operator is the *area operator*, which is as manifested by its name, the operator that measures quanta of area. Over a certain surface, Σ, of a manifold, the area operator provides a value of the physical area over that surface. For simplicity, consider a surface whose x^3 coordinate is zero and $x^1, x^2 \neq 0$. The classical expression for an area on a manifold is then given by

$$A_\Sigma = \int_\Sigma dx^1 dx^2 \sqrt{detq^{(2)}} \tag{5.10}$$

in which $detq^{(2)}$ denotes the determinant of the metric over a 2 dimensional surface spanned by x^1 and x^2. Since the canonical formulation of GR has proven to be more useful in terms of Ashtekar variables, using 4.4 one may rewrite the expression 5.10 as

$$A_\Sigma = \int_\Sigma dx^1 dx^2 \sqrt{E_i^3 E^{3i}}. \tag{5.11}$$

To obtain an operator version of 5.11, one would then go on to quantize it canonically using the definition given previously as $\hat{E}_i^a = -i\hbar\gamma \frac{\delta}{\delta A_a^i}$ which, bringing in the factor of $8\pi G$, is $-8\pi G i\hbar\gamma \frac{\delta}{\delta A_a^i}$. The latter form clarifies that 5.11 does indeed have units of area. However, this poses a problem because there are two densitized triads in the expression. The densitized triads, being functional derivatives, upon acting on a state will form Dirac delta functions. One of the Dirac delta functions can be treated with the integration but the other will remain. Since both triads will act on the same point from a state, the Dirac delta which remains will be of the form $\delta(x-x) = \delta(0)$. This is an ill-defined quantity which hinders the process of quantizing 4.4. Moreover the presence of the square root further complicates the situation. One way to overcome this problem is via what is known as *regularization* [34].

Consider smearing the triad with a function $f_\epsilon(x,y)$ which in the limit $\epsilon \to 0$ becomes a Dirac delta. The smeared triad is then given by

$$\left[E_i^3\right]_f (x) = \int_\Sigma d^2y \; f_\epsilon(x,y) E_i^3(y), \tag{5.12}$$

which upon promoting to an operator and subbing into 5.11 gives the area operator,

$$\hat{A}_\Sigma = \int_\Sigma dx^1 dx^2 \sqrt{\left[\hat{E}_i^3\right]_f \left[\hat{E}_i^3\right]_f} \tag{5.13}$$

In order to study the action of the area operator, one simply has to act 5.13 on a spin network state developed in the previous section. For the ease of mathematical calculation, first consider investigating the action of $\hat{E}_i^3$ on a spin network state, ψ_s. The expression of interest is then

$$\left[\hat{E}_i^3\right]_f (x) \; \psi_s = -8\pi G i\hbar\gamma \int_\Sigma d^2y f_\epsilon(x,y) \frac{\delta\psi_s}{\delta A_3^i(y)}. \tag{5.14}$$

The key factor to handle is $\frac{\delta\psi_s}{\delta A_3^i(y)}$, which suggests, if zero, the action of the area operator also gives zero. For a non-zero value, the spin network states must depend on the connection A_3^i and along the curves on the spin network states. In other words, for a non-zero value, a curve on a spin network state must pierce through the surface Σ. Taking this into account, one may write the functional derivative as

$$\frac{\delta\psi_s}{\delta A_3^i(y)} = \mathrm{Tr}\left(\frac{\delta D^{j_m}}{\delta A_3^i(y)} \frac{\delta\psi_s}{\delta D^{j_m}}\right) \tag{5.15}$$

where the D^{j_m} denotes the spin j_m representation of the holonomy lines in the spin network state and the "trace" is given in a generalized sense to ensure all free indices are contracted. Further progress from here can be made by carrying out the functional derivative of the Wigner matrix with respect to the connection. Recalling the expression for a holonomy as given by 5.3 and expanding the expression, one may perform the derivative with respect to the connection to obtain

$$\frac{\delta h_\gamma}{\delta A_3^i(y)} = \sum_{n=0}^{\infty} \int_{t_1 \geq \cdots t_k \cdots \geq t_n \geq 0} dt_1 \cdots dt_n \dot{\gamma}^{a_1}(t_1) A_{a_1}(t_1) \cdots \dot{\gamma}^3(t_k) \delta^3(y - \gamma(t_k)) \mathbf{T}^i \cdots \dot{\gamma}^{a_n}(t_n) A_{a_n}(t_n) \tag{5.16}$$

in which $\mathbf{T}^i$ shows the $SU(2)$ generator as was defined in the holonomy chapter. 5.16 can be simplified by recognizing that the integral from the parameter value t_1 to t_{k-1} is just a parallel propagator from the starting point of the curve to t_k and the integral from t_{k+1} to t_n is just a parallel propagator from the point of the curve t_k to the end of the curve. Therefore 5.16 becomes

$$\frac{\delta h_\gamma}{\delta A_3^i(y)} = \int dt_k \; D^{j_m}(0, t_k) \dot{\gamma}^3(t_k) \mathbf{T}^i \delta^3(y - \gamma(t_k)) D^{j_m}(t_k, 1). \tag{5.17}$$

Using the result of 5.17 with 5.15 and 5.14, the triad operator becomes

$$\left[\hat{E}_i^3\right]_f (x) \; \psi_s = 8\pi G i \gamma \dot{\gamma}^3(x) \mathrm{Tr}\left(\mathrm{T}^i \psi_s(x)\right) f_\epsilon(x, y) \tag{5.18}$$

in which care should be taken to avoid confusion between the γ which is the Immirzi parameter and $\dot{\gamma}^3(x)$ which is the tangent of the holonomy.

As the area operator, 5.13, contains two triad operators, another triad operator given by 5.18 will result in two generators giving

$$\mathbf{T}^i \mathbf{T}^i = j_m(j_m + 1).$$

Then the action of 5.13 on a spin network state can be rewritten as

$$\hat{A}_\Sigma \psi_s = 8\pi \ell_{pl}^2 \gamma \sum_I \sqrt{j_I(j_I + 1)} \psi_s \tag{5.19}$$

where the index I represents all the lines in the spin network state that pierce Σ to produce physical areas. The j_I are known as the *color* associated to the lines I. Notice the spin network state is an eigenstate of the area operator with eigenvalues $8\pi l_{pl}^2 \gamma \sum_I \sqrt{j_I(j_I+1)}$, which contains the *planck length*, $\ell_{pl} = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-33}$cm. The factor of $\hbar$ suggests the quantum nature of this length scale and indeed the planck length is associated to the quantum structure of the spacetime; more precisely it is characteristic of the minimum length scale that a geometric structure in spacetime can be related to. Then the natural interpretation one can infer is that spacetime itself is discrete, much like matter is on quantum scales. A pictorial diagram of a quantum of area produced by a certain color of a line piercing through Σ is given by figure 3.

Recall that the value of the Immirzi parameter in the classical theory given by the Holst action, did not affect the theory's physical interpretation in any way. By comparison in the quantum theory, 5.19 suggests that the eigenvalue of the area is dependent on the value of the Immirzi parameter. This is due to the nature of the quantization at hand which does not involve canonical transformations in the classical theory translating to unitary transformations in the quantum

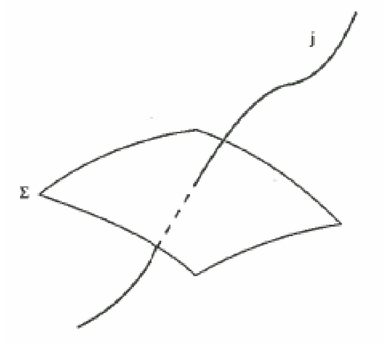

Figure 3: A quantum of area is obtained by a line in a spin network piercing through Σ. Figure from p 115 of [35]

theory; an in-depth discussion of the origin of this problem is given in [36]. Although the value of the Immirzi parameter is important in interpreting the physical quantities in the quantum theory, there are no experiments to date which verify the parameter's value. An insight to a possible value is, however, inferred when one considers black hole entropies, the specifics of which are discussed in the succeeding chapter.

The second geometric operator is the *volume operator*. The volume operator is the operator which upon acting on a spin network will endow a physical volume over a 3-dimensional region, R. Classically, a volume on a manifold over a 3-dimensional region is given by

$$V(R) = \int_R d^3x \, \sqrt{detq} = \frac{1}{6} \int_R d^3x \, \sqrt{|\epsilon_{abc}\epsilon^{ijk} E^a_i E^b_j E^c_k|}. \tag{5.20}$$

The quantization process of 5.20 is much more complex than that of the area operator. In particular, there exist two different methods of quantization: one which was developed by Ashtekar and Lewandowski [37] and the other which was developed by Rovelli and Smolin [38]. Both methods utilize the technique of smearing triad operators as was the case for the area operator but the choice of the surface over which the triad is smeared is different. Consequentially, the results differ in that the former gives an operator sensitive to the differential structure of the spin network states while the latter gives one sensitive only to topological features. Unlike the area operator, the volume operator induces a physical volume when a vertex or node of a spin network state is enclosed inside the region R. With the links of the graphs providing areas and the nodes of the graphs providing volumes, it is evident that spin network states are the rudimentary building blocks of a discretized spacetime.

5.4 Dynamics of LQG

This section is concerned with operators related to the dynamics of LQG, namely operators which are classically given as constraints such as the diffeomorphism constraint and the Hamiltonian constraint.

Recall from chapter 2 that the Dirac quantization procedure involves identifying states of a Hilbert space which satisfy constraints of the system. Then the vector space becomes a physical space from a kinematical one. To be pedantic about the different Hilbert spaces, the Hilbert spaces which arise after an imposition of each constraint will be labelled differently such that they are as below:

$$\mathscr{H}_{kin} \xrightarrow{\hat{G}^i=0} \mathscr{H}^0_{kin} \xrightarrow{\hat{C}^a=0} \mathscr{H}_{Diff} \xrightarrow{\hat{C}=0} \mathscr{H}_{phy}.$$

An important property of spin network states is that they are $SU(2)$ invariant (chapter 3.1 of [1]); in other words they fulfill the classical Gauss constraint even after quantization:

$$\hat{G}^i \psi_s = 0.$$

The natural succeeding step from the quantization procedure is then to impose the remaining two other constraints, starting with the diffeomorphism constraint.

Defining an operator for a diffeomorphism, ϕ, is simple given that it acts on a spin network state or more generally a cylindrical function. Using the fact that a diffeomorphism acts as

$$h_\gamma(\phi^* A) = h_{\phi\circ\gamma}(A)$$

on a holonomy, the operator $\hat{\phi}$ would act on a cylindrical function as

$$\hat{\phi}\psi_\Gamma = \psi_{\phi\circ\Gamma}$$

such that the space of cylindrical functions is mapped from Cyl_Γ to $Cyl_{\phi\circ\Gamma}$.

The problem, however, is that this does not give the desired Hilbert space, $\mathscr{H}_{Diff}$. The reason is that diffeomorphisms form a non-compact group. Because diffeomorphisms move the graphs themselves, the only element which can be used to implement invariance after the group action is a constant functional. This is very similar to a wavefunction, $\psi(x) \in L^2(\mathbb{R}, dx)$, in quantum mechanics exposed to a condition of translation invariance, which is also a non-compact group. The only element which can achieve translational invariance is a constant function. Similarly solutions to the diffeomorphism constraint can only be achieved by constant functionals on $\mathscr{H}^0_{kin}$ and the space of such solutions is denoted $\mathscr{H}^{0*}_{kin}$. Then by construction, $\eta \in \mathscr{H}^{0*}_{kin}$ is diffeomorphism invariant:

$$\eta(\hat{\phi}\psi) = \eta(\psi) \qquad \forall \psi \in \mathscr{H}^0_{kin}. \tag{5.21}$$

The space of diffeomorphism invariant functionals satisfying 5.21 is denoted $\mathscr{H}^*_{Diff}$ and the desired Hilbert space, $\mathscr{H}_{Diff}$, can be constructed by its duality. The resulting spin network states are *equivalence classes of graphs under diffeomorphisms* which are also called knots. The knots, as mentioned in the spin network states chapter, is invariant under diffeomorphisms thereby providing the desired solutions to $\mathscr{H}_{Diff}$. In fact the solutions of $\mathscr{H}_{Diff}$ are more precisely called *knotted spin networks* and are illustrated by figure 4.

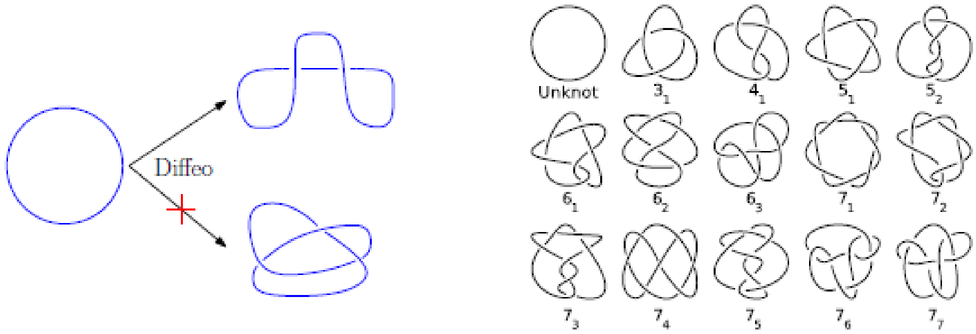

Figure 4: The left depicts the relation of a diffeomorphism for a given knot. The right depicts different types of knotted spin network states. Figure from p 55 of [1]

In order to formulate $\mathscr{H}_{phy}$ from $\mathscr{H}_{Diff}$, the classical Hamiltonian constraint which is the smearing over the lapse function, N, given by

$$C(N) = \int d^3x \; N\epsilon^{ij}_{k} \frac{E^a_i E^b_j}{\sqrt{detE}} \left(F^k_{ab} - 2(1+\gamma^2) K^i_{[a} K^j_{b]} \right) \tag{5.22}$$

must be promoted to an operator. For notational simplicity, the two terms in 5.22 are given new labels, $C^E(N)$ and $T(N)$, such that 5.22 becomes

$$C(N) = C^E(N) - 2(1+\gamma^2)T(N). \tag{5.23}$$

The immediate problem apparent from 5.22 is that it is non-linear, posing a problem similar to that which arose when geometric operators were being quantized. A way around this problem is to encapsulate the non-linearity into the Poisson bracket relation using 4.7. Labelling the volume of Σ as $V = \int \sqrt{detE}$ and the smeared extrinsic curvature as $\bar{K} = \int K^i_a E^a_i$, one may construct the following relations:

$$K^i_a = \frac{1}{\gamma}\{A^i_a, \bar{K}\},$$

$$\bar{K} = \frac{1}{\gamma^{3/2}}\{\mathcal{C}^E(N=1), V\},$$

$$\frac{E^a_i E^b_j}{\sqrt{detE}}\epsilon^{ijk}\epsilon_{abc} = \frac{4}{\gamma}\{A^k_a, V\}$$

where $\mathcal{C}^E(N=1)$ denotes the density of $C^E(N)$ with the lapse set to 1. These relations then allows one to rewrite $C^E(N)$ and $T(N)$ as

$$\begin{aligned} C^E(N) &= \int d^3x \; N\epsilon^{abc}\delta_{ij}F^i_{ab}\{A^j_c, V\}, \\ T(N) &= \int d^3x \; \frac{N}{\gamma^3}\epsilon^{abc}\epsilon_{ijk}\{A^i_a, \{C^E(N=1), V\}\}\{A^j_b, \{C^E(N=1), V\}\}\{A^k_c, V\}. \end{aligned} \tag{5.24}$$

Since the goal is to formulate a theory in terms of loop variables, the connection and curvature in 5.24 must be reexpressed in terms of holonomies. Expanding 5.3, over a path e_a of length ϵ, one obtains $h_{e_a}(A) \simeq 1 + \epsilon A^i_a \mathbf{T}_i + O(\epsilon^2)$ and therefore

$$h^{-1}_{e_a}\{h_{ea}, V\} = \epsilon\{A^i_a, V\} + O(\epsilon^2).$$

The curvature, F^i_{ab}, may also be expressed in a similar manner by considering an infinitesimal trianglular loop:

$$h_{\alpha_{ab}} = 1 + \frac{1}{2}\epsilon^2 F^i_{ab}\mathbf{T}^i + O(\epsilon^4)$$

where α_{ab} represents the triangular loop lying on the plane ab and ϵ^2 denotes the area enclosed by the triangle. Then this leads to

$$h_{\alpha_{ab}} - h^{-1}_{\alpha_{ab}} = \epsilon^2 F^i_{ab}\mathbf{T}^i + O(\epsilon^4).$$

Using the triangular loops, consider a lattice regularization procedure in which Σ is partitioned into small 3-dimensional regions much alike to the steps taken in developing the volume operator. Due to the triangular topology of the loops, however, the regions here are enclosed by tetrahedra. This then produces triangulations of tetrahedra as shown in figure 5. The integral given by the

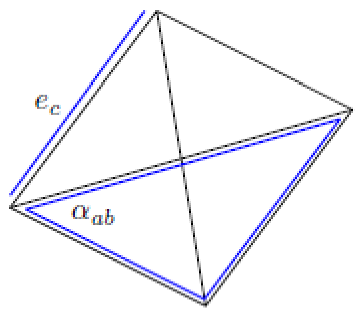

Figure 5: A tetrahedron formed via triangulation. Figure from p 57 of [1]

first of 5.24 can then be regularized by a Riemann sum as

$$C^E = \lim_{\epsilon\to 0}\sum_I N_I \epsilon^{abc}\mathrm{Tr}\left((h_{\alpha_{ab}} - h^{-1}_{\alpha_{ab}})h^{-1}_{e_c}\{h_{e_c}, V\}\right), \tag{5.25}$$

in which the subscript I denotes the I^{th} tetrahedral cell. With regards to the regularization of the second equation of 5.24, details are given by [12], [39].

With the Hamiltonian constraint revealed in the form of holonomies of Poisson brackets, the quantization process is simple: promote the holonomies to opeartors and the Poisson brackets to commutators as

$$\hat{C}^E = \lim_{\epsilon\to 0}\sum_I N_I \epsilon^{abc}\ \mathrm{Tr}\left((\hat{h}_{\alpha_{ab}} - \hat{h}^{-1}_{\alpha_{ab}})\ \hat{h}^{-1}_{e_c}\ [\hat{h}_{e_c}, \hat{V}]\right). \tag{5.26}$$

When promoting the constraint to a quantum operator, a caveat is that the spin network state on which the operator acts on must be "in line" with the vertices and lines of the triangulation as depicted by the blue lines in figure 5. This ensures that the property of the volume operator acting on the nodes of the spin network is inherited. At a vertex from which the dynamics is exhibited, one can form "dressed nodes" by aligning additional lines from the lines entering the vertex. Note that there is no ambiguity in positioning new lines dressing the node because the space $\mathscr{H}_{Diff}$ is insensitive to the position of links.

The quantum versions of 5.24 then give a well-defined operator $\hat{C}(N)$ which is also compatible with the classical Poisson brackets of the Dirac algebra, 2.10. In other words, one may achieve the relation

$$\langle\phi|\ [\hat{\mathcal{C}}(N_1), \hat{\mathcal{C}}(N_2)]\ |\psi\rangle = 0, \qquad \forall|\psi\rangle \in \mathscr{H}_{kin},\ \forall\langle\phi| \in \mathscr{H}_{Diff} \tag{5.27}$$

in which $\hat{\mathcal{C}}(N_i)$ represent the densities of the Hamiltonian constraint operator. Showing 5.27 is straightforward upon realizing that the additional lines added via 5.26 carry no volume therefore are insensitive to further action by 5.26 A key point to note here is that this property is only obtainable from using the Ashtekar Lewandowski version of the volume operator [40], [37], due to its sensitivity to the structure of graphs.

Moreover, there exists an infinite number of states that are compatible as solutions of $\hat{C}$ [41] (any graph with no nodes is in the kernel of $\hat{C}^E$ and $\hat{T}$). To find meaningful solutions, one may consider spin networks with an arbitrary number of links from 5.26 and construct linear combinations of these states carrying different coefficients. An illustration is given by figure 6. Such solutions,

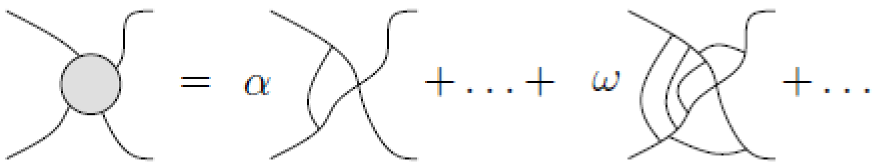

Figure 6: A linear combination of spin networks with arbitrary number of links. Figure from p 58 of [1]

however, are only formal and have not been explicitly discovered.

Nonetheless, a method for a well-defined quantized Hamiltonian has been devised in a non-perturbative manner. Although, no explicit form of a solution is known, the success in developing the discreteness of geometry is expected to hold through to physical states as well.

5.5 Matter Coupling

As the reader should be aware, the theory of LQG aims to quantize the physics of gravity, in other words GR. And since GR encompasses the dynamics of both spacetime and matter, successfully quantizing the spacetime manifold (or gravitational field in a field theory sense) brings on the next natural task which is to do it with matter.

The first and simplest matter field one can consider is the scalar field. Scalar field coupling is especially useful in *Loop Quantum Cosmology* (or LQC) which is a quantum theory of cosmology whose origin is from the application of LQG to classical cosmology (In this paper, the focus of applications of LQG is on black holes; for details regarding LQC, refer to [43], [44], [45], in which the scalar field acts as a "time" or "clock" variable [42]). In order to couple the theory with matter, one must first backtrack to the Lagrangian formalism such that the action of the scalar field theory can be combined with the theory of gravity which this paper has developed in its former half. The scalar field theory action is given by

$$S = \int d^4x \left(-g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right)\sqrt{-detg} \tag{5.28}$$

in which φ denotes the scalar field and the metric $g^{\mu\nu}$ has been explicitly put in to remind the reader the metric is not neccessarily flat. Defining the canonical momentum conjugate to φ as usual as $\pi = \delta L/\delta\dot{\varphi}$, one may construct a Hamiltonian whose variables, π and φ, can be changed to Ashtekar's:

$$H = \int d^3x \, \frac{N}{\sqrt{detq}}\left(\pi^2 + E^a_i E^{bi}\partial_a\varphi\partial_b\varphi + detqV(\varphi)\right) + N^a\pi\partial_a\varphi. \tag{5.29}$$

From 5.29 one can show that the first term contributes to the Hamiltonian constraint while the second contributes to the diffeomorphism constraint:

$$\begin{aligned} C(\vec{N})_\varphi &= \int d^3x \; N^a\pi\partial_a\varphi \\ C(N)_\varphi &= \int d^3x \; \frac{N}{\sqrt{detq}}\left(\pi^2 + E^a_i E^{bi}\partial_a\varphi\partial_b\varphi + detqV(\varphi)\right). \end{aligned} \tag{5.30}$$

Then adding $C(\vec{N})_\varphi$ and $C(N)_\varphi$ to the diffeomorphism and Hamiltonian constraints of gravity respectively, the scalar field becomes coupled to gravity.

For other types of matter fields such as Yang-Mills, fermionic and Higgs, the Hamiltonian constraints are as follows:

$$\begin{aligned} C_{YM} &= \frac{q_{ab}}{2g^2_{YM}\sqrt{detq}}(E^a_i E^b_i + B^a_i B^b_i) \\ C_{Dirac} &= \frac{E^a_i}{2\sqrt{detq}}\left(i\pi\mathbf{T}_i\mathbf{D}_a\xi + \mathbf{D}_a(\pi\mathrm{T}_i\xi) + \frac{i}{2}K^j_a\pi\xi + \text{c.c}\right) \\ C_{Higgs} &= \frac{1}{2}\left[\frac{p^ip^i}{G\sqrt{detq}} + \sqrt{detq}\left(q^{ab}(\mathbf{D}_a\phi_i)(\mathbf{D}_b\phi_i)/G + P(\phi_i\phi_i)/(\hbar G^2)\right)\right] \end{aligned} \tag{5.31}$$

with g_{YM} denoting the Yang-Mills coupling constant, $\mathbf{D}_a$ showing the covariant derivative over $SU(2)\times G_{YM}$[5], B^a_i denoting the magnetic field of the Yang-Mills connection, G denoting Newton's gravitational constant and $\mathbf{T}_i$ representing the $su(2)$ Lie algebra generators. Note also that the

[5] G_{YM} depicts the Yang-Mills group.

fermionic (Dirac) field is labelled by ξ and its conjugate momentum by π whereas the Higgs field and its conjugate momentum is given by ϕ_i and p^i respectively. For details regarding the derivation of 5.31, refer to [46]. Again, similarly to the scalar field described earlier, the constraints given by 5.31 can be combined with that of gravity's to give matter-coupled theories.

6 Black Hole Entropy

The development of LQG has allowed for quantum mechanical applications of general relativity, mainly in cosmology and black holes. This section focuses on the application of LQG to the latter. For applications regarding the former the reader is introduced to [43, 44, 45].

This chapter starts with a motivation for a quantum picture of black hole entropy. In 1973, Barden, Carter and Hawking used GR as a stepping stone to devise the *laws of black hole mechanics* [47] by drawing analogy to the laws of thermodynamics. In particular, it was postulated by Bekenstein and confirmed by Hawking that the entropy of a black hole and its surroundings are separate and it is only the combined entropy which must increase accordingly to the 2$^{\text{nd}}$ law of thermodynamics. This led to a semiclassical picture of black hole entropy, S, being related to its horizon area, A, through the *Bekenstein-Hawking formula* as:

$$S = C\frac{k_B}{\hbar G}A = \frac{k_B A}{4\ell_{pl}^2} \tag{6.1}$$

where k_B is the Boltzmann constant and C is a constant of proportionality. It was, however, argued that based on thermodynamics, a black hole should have a temperature. Hawking proved that, in a quantum mechanical picture, there indeed is a black hole temperature through what is known as *Hawking radiation*[48], although classically such radiation cannot be found because a black hole's area cannot reduce due to Hawking's area theorem [49, 50]. Then the temperature is indicative of a black hole's statistical entropy giving relation to its quantum microstates that correspond to the black hole's macroscopic configuarations (very similarly to ordinary statistical mechanics). Comparing the values of the thermodynamic entropy from the Hawking temperature with the Bekenstein-Hawking entropy from the horizon area [35], one finds that the latter value is approximately an order of 22 times larger than the former value. This indicates that the entropy of a black hole cannot completely be comprised of the entropy of the matter that formed it. An explanation of this can only be given by considering quantum properties of black holes and this provides a good reason to develop a quantum picture for black hole entropy. Another motivation is to reconcile the *information paradox* [51] which arises from the enigma of physical information being lost permanently into a black hole as the black hole evaporates.

To apply LQG to black hole entropy, one must first acknowledge that the black holes subject to investigation must be very large such that the Hawking radiation of the black holes is negligible; for a small black hole, the Hawking radiation is large due to its small horizon area and the black hole is quickly evaporated away. Secondly, the notion of an *isolated horizon* [52] must be understood. An isolated horizon is a horizon that is "isolated" in the sense that the Hamiltonian operator does not generate any evolution on the boundary which also implies a vanishing lapse function. It is said to be a *quasilocal* [53] definition of a black hole that is in equilibrium with its interior and exterior. An immediate consequence then is that, in terms of statistical mechanics, the only factor affecting the black hole entropy is the macroscopic configurations of the black hole itself and not that of the surroundings or the black hole's interior. Another condition which must be met for an isolated horizon is the following. For a spin connection ω_a^i defined in spacetime, one can define $W_a \equiv -\bar{\omega}_a^i r_i$ for $\bar{\omega}_a^i$ equal to the pull back of the spin connection onto the boundary B and r_i a constant vector. Then using an antisymmetric tensor $E^{ci} \equiv \Sigma_{ab}^i \epsilon^{abc}$, one states that the condition required is

$$\partial_a W_b - \partial_b W_a = -2\gamma \bar{\Sigma}_{ab}^i r_i \tag{6.2}$$

where the bar on Σ again denotes the pull back onto B.

Keeping the above conditions in mind, the calculation of entropy in LQG begins with applying the usual definition of entropy from statistical mechanics, namely $S = k_B log(N_i)$ in which N_i are the microstates of a black hole. To understand the physical significance of the microstates in terms of black hole entropy, recall that the source of different microstates in statistical mechanics was the different configurations of each particle in a system; in the case of black holes, the different configurations of "particles" are given by the small discretized areas resulting from punctures made by spin networks. To be more precise, black hole microstates stem from different possibilities of assigning a certain area to the quanta of surfaces punctured by spin network states. To progress, one needs the fact that the entropy is a function of area as given by 6.1 or more fundamentally by [54]. Then consider a black hole with an area A_0 and the number of quantum states required to induce the area A_0. The number of states comes from the links, j_I, of the spin network states in the outer spacetime puncturing the boundary according to 5.19 as pictorially depicted by figure 7. But since the condition 6.2 must be satisfied on the horizon, there is a contribution from the operator version of $\bar{\Sigma}$ which is similar to the $\hat{E}^3$ operator which was mentioned when developing the area operator. The $\hat{\bar{\Sigma}}$ eigenvalues are given by m_I such that $-j_I \leq m_I \leq j_I$ and $\sum_I m_I = 0$, the latter condition stemming from the fact that the horizon must be topologically spherical. Implementing the condition of diffeomorphism invariance on the boundary as well, one finds that there is invariance under the interchange of the integer pairs (j_I, m_I). The task at hand to find the entropy of a black hole is then to determine the allowed number of sequences or configurations assembled by the puncturing of spin network states that give a certain macrostate depicted by an area. There are two methods of approach to this, one introduced by C. Rovelli and the other using the number theoretical approach.

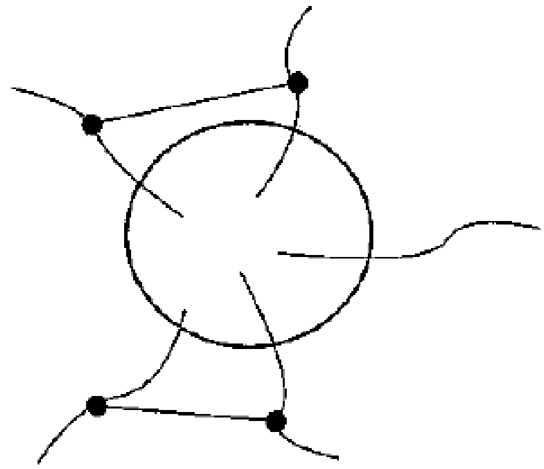

Figure 7: Spin network lines puncturing the horizon to endow it an area. Figure from p 142 of [35]

6.1 Rovelli's Counting

First recall that from 5.19, one may reorganize the expression to obtain

$$M \equiv \frac{A}{4\pi\gamma\ell_{pl}^2} = \sum_I \sqrt{k_I(k_I+2)} \in [A+\delta A, A-\delta A] \tag{6.3}$$

where k_I shows the value from the half integer values of j_I from $j_I = k_I/2$ and A denotes the area associated to a macroscopic state. The number of sequences, $\mathcal{N}$, may then be written as $\mathcal{N}(M)$ with each sequence of $\{k_I\}$ satisfying 6.3. To proceed consider the following inequalities:

$$\sum_I \sqrt{k_I^2} < \sum_I \sqrt{k_I(k_I+2)} \equiv \sum_I \sqrt{(k_I+1)^2-1} < \sum_I \sqrt{(k_I+1)^2}. \tag{6.4}$$

Denoting the number of configurations satisfying $\sum_I k_I = M$ as $\mathcal{N}_+(M)$ and the number of configurations satisfying $\sum_I (k_I+1) = M$ as $\mathcal{N}_-(M)$, one finds that 6.4 implies

$$\mathcal{N}_-(M) < \mathcal{N}(M) < \mathcal{N}_+(M). \tag{6.5}$$

Following the steps given in [68] from 6.5, one finds that the number of ordered positive integers giving the sum in 6.3, or $\mathcal{N}_+(M)$, is $C2^M$ where C is a constant.

6.2 Number Theoretical Approach

The second approach entails two steps: establishing the allowed sequences and then learning how many different ways such allowed sequences can be labelled. Setting $4\pi\gamma\ell_{pl}^2 = 1$, the total area given by $\mathcal{N}$ punctures from spin network edges can be expressed as

$$A = \sum_{I=1}^{\mathcal{N}} A_I = \sum_{I=1}^{\mathcal{N}} \sqrt{(k_I+1)^2-1}. \tag{6.6}$$

Then letting g_k equal to the number of punctures giving the eigenvalue associated to every possible $I = 1, \ldots, \mathcal{N}$ from k_I, the above expression can be rewritten as

$$A = \sum_{I=1}^{I_{max}} g_{k_I} \sqrt{(k_I + 1)^2 - 1}. \tag{6.7}$$

Using a mathematical trick described in Appendix G of [68], one may reexpress the square root to be

$$\sqrt{(k_I + 1)^2 - 1} = y_{k_I} \sqrt{p_{k_I}}$$

where $y_{k_I} \in \mathbb{Z}$ and $p_{k_I} \in \mathbb{A}$, the set of square-free integers. Noting that y_{k_I} is an integer and absorbing g_{k_I} into it, one obtains the relation

$$\sum_{I=1}^{I_{max}} g_{k_I} \sqrt{(k_I + 1)^2 - 1} = \sum_{I=1}^{I_{max}} y_{k_I} \sqrt{p_{k_I}}.$$

Determining the number of allowed sequences would then amount to solving for the uknowns, k_I and y_{k_I}. For the sake of simplicity, consider having just one puncture such that $g_{k_I} = 1$. Then the equation to solve is

$$\sqrt{(k_I + 1)^2 - 1} = y_{k_I} \sqrt{p_{k_I}}$$

which upon making the substitution $x_I = k_I + 1$ gives the *Pell equation* or *Brahmagupta-Pell equation*:

$$x_I^2 - p_{k_I} y_{k_I}^2 = 1.$$

The exact methods of solving this equation is omitted in this paper but can be found in Appendix H of [68]. The next step is to then figure out the number of different ways the sequences allowed by solving the Pell equation can be oriented. This comes down to resorting to the *Number Partitioning Problem*, whose details are given by [69] and [70].

6.3 The Entropy

Going through the process delineated by the previous two sections, one obtains the value [55]

$$S(A) = \frac{A}{4\ell_{pl}^2} - \frac{3}{2}\log\left(\frac{A}{\ell_{pl}^2}\right) + O(1) + \cdots \tag{6.8}$$

whose first term is equal to 6.1 and second term acts as a quantum correction. To obtain 6.8 the value of the Immirzi parameter must be assumed as $\gamma = 0.274067$. The calculation of this value stems from solving the equation [56]

$$1 = \sum_{k=1}^{\infty} (k+1) \exp\left(-\frac{1}{2}\gamma\sqrt{k(k+2)}\right). \tag{6.9}$$

The fixing of the Immirzi parameter as such should then carry on to be in accordance with predictions of other physical calculations. However, there are no other known physical calculations in which the Immirzi parameter can be employed hence this speculation is yet to be confirmed. An upside is that the Immirzi parameter value stated above is consistent with the Bekenstein-Hawking formula for a wide range of black holes including charged, rotating and ones coupled to matter fields [57]. In terms of a more fundamental reason that the Immirzi parameter must be fixed at the given value is unknown. In fact there has been studies which have attempted to calculate the entropy of a black hole with the statistical mechanics treamtment of LQG without having to fix a specific value for the Immirzi parameter [58].

7 Spin Foam Formalism

In this final chapter, the reader is introduced to a different perspective of the dynamics of LQG from the Hamiltonian formalism discussed to this point. The reader should be aware that ordinary quantum mechanics can be formulated in terms of the Shrödinger or Heisenberg picture, which is based on the Hamiltonian formalism, or the Feynman path integral formalism that is focused on the transition amplitudes from state to state. In LQG the case is similar in that there is an analogue of the path integral formalism of GR called the *spin foam formalism* which was first developed by Rovelli and Reisenberger [59].

As spin networks have shown to be the underlying structure of a background independent quantum space, spin foams are the underlying structure of a background independent quantum spacetime. It is a sum-over-histories approach that attempts to calculate the transition probability that a certain 3-dimensional spatial geometric configuration evolves to another. In fact one can think of a spin foam as a "world surface" swept out by a spin network that evolves over a "time" variable.

As is shown by figure 8, a spin foam is a simplicial complex built from vertices, edges and polygonal faces. The name spin foam comes from its soapsud-like appearance due to its structure. As the lines of spin networks are labelled by group representations, the faces of a spin foam are labelled by group representations. The edges of a spin foam are labelled by intertwiners in a similar fashion to the nodes of a spin network. Noting again that a spin foam is a "world surface" of a spin network, it is easily inferred that the cross section of a spin foam is a spin network. The goal of the spin foam model of qauntum gravity is then to allow for the computation of transition amplitudes of one state to another by summing over all the spin foams which result in the transition at hand. The formal definition of a spin foam, W, that sets the stepping stone for this calculation is

$$W(s,s') = \sum_{\sigma} A(\sigma) \qquad A(\sigma) = \prod_{v} A_v(\sigma)$$

where s and s' are initial and final spin network states respectively, $A(\sigma)$ are the amplitudes which are summed over for every surface, σ, swept out by the spin network states and $A_v(\sigma)$ characterize the single step evolution of the states; for details refer to [59]. To calculate the transition

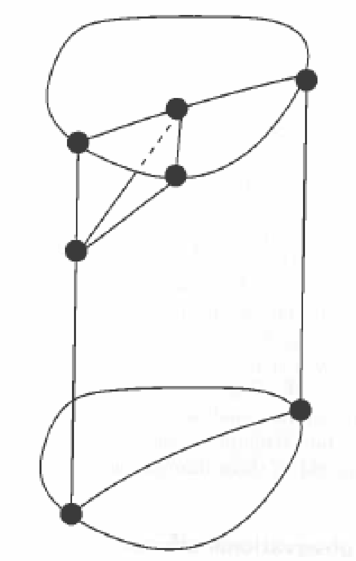

Figure 8: Bottom slice of spin network evolving to the top slice of spin network to create a spin foam. Figure from p 155 of [35]

amplitudes, however, is a very complex task and progress has been made more significantly in a simplified model of GR called the *BF theory*.

A 4-dimensional BF theory has the Lagrangian,

$$L = \int d^3x \; \epsilon^{\mu\nu\lambda\kappa} \mathrm{Tr}(B_{\mu\nu} F_{\lambda\kappa}),$$

where $F_{\mu\nu}$ is the field strength tensor as in Yang-Mills theories and $B_{\mu\nu}$ is an antisymmetric tensor that are elements of the $su(2)$ algebra. The reason BF theory is much easier to deal with than GR is that upon varying the Lagrangian to produce the field equations, the vector potential turns out not to have any gauge invariance. This implies there is no local degrees of freedom and the only degrees of freedom available are those which are topological. It is worth noting that GR is a special case of BF theory in which B is chosen to be a product of 2 tetrads as $B^{IJ}_{ab} \equiv e^I_{[a} e^J_{b]}$ [60].

The majority of new research breakthroughs in LQG nowadays come from models based on spin foams, though it should be noted much of the activities are based on Euclidean space. In particular, the paper by Engle, Pereira and Rovelli [61] made significant contributions in developing modern spin foam models which are formulated using *Regge calculus* [62]. The formulation of GR via spin foams is likely to shed light on simplicity regarding different types of calculations as the functional path integral formalism does in quantum mechanics. Although it is speculated that the spin foam formalism is equivalent to the Hamiltonian formalism, there is only rigorous proof for this equivalence in 3 dimensions [63] and not in 4 dimensions.

The applications of spin foams to physical calculations seem promising as well. In particular, spin foams have been used to produce an explicit expression for the Minkowski vacuum [64]. Moreover, a technique for defining an n-point function in a background independent context has also been devised [65]. This allows for a particle-scattering interpretation for transitions of geometric states therefore leading to the graviton propagator [66, 67], the quanta which mediates the force of gravity.

8 Conclusion

To summarize, Loop Quantum Gravity is a theory which quantizes General Relativity in a background independent and diffeomorphism invariant manner. It is formulated in terms of the Hamiltonian formalism in which spacetime is split into 3-dimensional space and 1-dimensional time. Despite the apparent loss of diffeomorphism invariance, this key property of General Relativity is in fact still preserved. The classical theory of gravity turns out to be a completely constrained theory which when quantized gives rise to many problems, of which the crux is the problem of time. To get around this problem, General relativity is reexpressed using the Palatini action and hence the Holst action and the notion of Ashtekar new variables are introduced whose elements contain an $SU(2)$ Ashtekar-Barbero connection and its conjugate momentum. Such a formulation gives rise to a parameter called the Barbero-Immirzi parameter which, although classically plays no significant role, in quantum theory determines the characteristics of discretized spacetime.

To solve for the constraints of General relativity, the notion of holonomies and hence of Wilson loops are introduced. These are obtained by performing a loop transform from the original Ashtekar variables to express them in loop representation. These loops, with intertwining operators, are then networked to give a graph corresponding to spin network states which allows for a basis of a Hilbert space that is background independent. The spin network states are eigenvalues of the area and volume operators whose actions are interpreted as endowments of an incremental physical area and volume respectively. The Planckian nature of areas and volumes from these operators imply that spacetime is discretized. The theory of quantium gravity has also been successful in incorporating matter consistently hence consistency with the Standard Model seems promising.

An application of Loop Quantum Gravity discussed in this paper was on black hole entropy. From the Bekenstein-Hawking formula, it was motivated that the black hole entropy is related to the black hole's horizon area, which could be described by lines of spin network states puncturing the surface of the boundary. It was found that Loop Quantum Gravity provided a logarithmic correction to the semiclassical description of a black hole entropy and such a result mandated a prerequisite value of the Barbero-Immirzi parameter. Finally a brief discussion on spin foams was given to provide the reader with an alternative formalism, other than that of Hamiltonian, of Loop Quantum Gravity. The spin foam formalism was shown to be the analogue of the path integral formalism in ordinary quantum mechanics, focusing on the transition amplitudes between states of geometric configuration rather than the states themselves. The formalism has allowed for the development of an n-point function in a background independent way and led to the development of the graviton propagator.

In spite of the large breakthroughs and progress made in the field of Loop Quantum Gravity, there still remains much room for further developments. First and foremost is the lack of experimental evidence supporting the theory as is the case with all theories of quantum gravity. For instance despite the theory's prediction of quantized geometry, there is no way for a verification as the energy scale required is approximately 10^{19}GeV or 15 orders of magnitude larger than what the world's current strongest particle accelerators can generate. Overcoming this energy barrier experimentallly does not seem to be a feasible option in the foreseeable future hence employing an indirect method to confirm the validity of Loop Quantum Gravity via considering the low-energy limit seems much more appropriate. Since phenomena predicted by Loop Quantum Gravity all occur at high energies, and the theory stems from its classical counterpart, General Relativity, it is expected that in the low-energy limit of Loop Quantum Gravity, General Relativity should be recovered. This statement may sound very matter-of-fact as replacing the quantum operators and commutators with their classical counterparts produce General Relativity easily. Nevertheless, Loop Quantum Gravity is background independent and its structure is unconventional in quantum theories calling for a more rigourous proof that the quantum theory and its classical correspondent are equivalent at low energies.

Moreover it was found that although a formal solution to the Hamiltonian constraint is known, no explicit form of it is and further work regarding this area seems crucial in understanding the dynamics of Loop Quantum Gravity. Having said this, further investigating the problem of time also seems to be at the heart of understanding the dynamics of the theory one step ahead.

For spin foams, the most vital aspect for progress seems to be in proving the equivalence of Loop Quantum Gravity's Hamiltonian formalism with the spin foam formalism. It was mentioned in this paper that the equivalence was expressed only for the 3-dimensional case. Future work regarding the proof for the equivalence in one dimension higher then seems imperative since the majority of existing theories describing the universe are formulated in 4-dimensions. The augmentation of spin foam models to Lorentzian systems rather than those that are Euclidean is also another area for research.

In terms of applications to black hole entropy, there is no explanation as to why the Barbero-Immirzi parameter must be the value it is; further research into this field will be necessary in answering questions that are more fundamental in nature of a quantum theory of gravity. To go a step further, to understand black hole entropy more solidly, one must clarify the notion of energy at a more fundamental level. As were the steps taken in this paper, to elucidate the nature of black hole entropy, concepts from statistical mechanics were employed. However, statistical mechanics (as well as thermodynamics) is heavily depedent on the notion of energy and in the case of black holes, a natural approach to defining local energies does not exist. This poses problems in importing ideas from statistical mechanics (and thermodynamics) in a general covariant fashion. Given successful future pogress in this field, it seems promising then that there will be further elucidations on black holes at Planckian scales.

References

[1] P. Don and S. Speziale, “Introductory Lectures to Loop Quantum Gravity”, 18 (2010) [arXiv:gr-qc/1007.0402].

[2] J. Donoghue, “Introduction to the Effective Field Theory Description of Gravity”, 7 (1995) [arXiv:gr-qc/9512024].

[3] N. Kiriushcheva and S.V. Kuzmin, “Diffeomorphism Invariance in the Hamiltonian formulation of General Relativity”, 4 (2008) [arXiv: gr-qc/0808.2623v1].

[4] R. Geroch, “Domain of Dependence”, *J Math Phys* **11** 437 (1970).

[5] J. Baez and J. Muniain, “Gauge Fields, Knots and Gravity”, *World Scientific* Singapore 199 (1994).

[6] E. Gourgoulhon, “3+1 Formalism and Bases of Numerical Relativity”, 15 (2007) [arXiv:gr-qc/0703035v1].

[7] B. DeWitt, “Quantum Theory of Gravity 1. The Canonical Theory”, *Phys Rev* **160** 1113 (1967). mann

[8] S. Hawking and G. Horowitz, “The Gravitational Hamiltonian, Action, Entropy and Surface Terms”, *Class Quant Grav* **13** 1487 (1995) [arXiv:gr-qc/9501014].

[9] S. Hawking and C. Hunter, “The Gravitational Hamiltonian in the Presence of Non-Orthogonal Boundaries”, *Class Quant Grav* **13** 2735 (1996) [arXiv:gr-qc/9603050v2].

[10] R. Wald, “General Relativity”, *The University of Chicago Press* Chicago, London 450 (1984).

[11] P. Dirac, “Lectures on Quantum Mechanics”, *Belfer Graduate School of Science Yeshiva University* New York 25 (1964).

[12] T. Thiemann, “Modern Canonical Quantum General Relativity”, *Cambridge University Press* New York 39 (2007).

[13] M. Kenmoku, H. Kubotani, E. Takasugi, Y. Yamazaki, “Analytic Solutions of The Wheeler-Dewitt Equations in Spherically Symmetric Spacetime”, 4 (1998) [arXiv:gr-qc/9810042v1].

[14] C. Isham, “Canonical Quantum Gravity and The Problem of Time”, 9 (1992) [arXiv:gr-qc/9210011v1].

[15] E. Anderson, “The Problem of Time in Quantum Gravity”, 1 (2012) [arXiv:gr-qc/1009.2157v3].

[16] E. Anderson, “Background Independence”, 2 (2013) [arXiv:gr-qc/1310.1524v1].

[17] J. Pereira, "Lorentz connections and Gravitation", *AIP Conf Proc* **1483** 239 (2012) [arXiv:gr-qc/1210.0379v1].

[18] S. Weinstein, "Gravity and Gauge Theory", Phil-Sci Archive 5 (1998).

[19] N. Barros e Sá. "Hamiltonian Analysis of General Relativity with the Immirzi Parameter". *Int J Mod Phys* **D10** 261 (2000) [arXiv:gr-qc/0006013v1].

[20] A. Perez and C. Rovelli, "Physical Effects of the Immirzi parameter", *Phys Rev* **D73** 044013 (2006) [arXiv:gr-qc/0505081v2].

[21] L. Freidel, D. Minic and T. Takeuchi, "Quantum Gravity, Torsion, Parity Violation and all that", *Phys Rev* **D72** 104002 (2005) [arXiv:hep-th/0507153v2].

[22] T. Dreher, "Loop Quantum Gravity", 4 (2012).

[23] J. Barbero, "Real Ashtekar Variables for Lorentizan Signature Space-times",*Phys Rev* **D51** 5507 (1995) [arXiv:gr-qc/9410014v1].

[24] J. Samuel, "Is Barbero's Hamiltonian Formulation a Gauge Theory of Lorentzian Gravity?", *Class Quant Grav* **17** 141 (2000) [arXiv:gr-qc/0005095v3].

[25] F. Finster and A. Strohmaier, "Gupta-Bleuler Quantization of the Maxwell Field in Globally Hyperbolic Space-times", 5 (2014) [arXiv:1307/1632v4 math-ph].

[26] M. Dehghani, "Gupta-Bleuler Quantization for Massive and Massless Free Vector Fields", *Braz J Phys* **39** 3 (2009).

[27] R. Gile, "Reconstruction of Gauge Potentials from Wilson Loops", *Phys Rev* **D24** 2160 (1981).

[28] C. Rovelli, "Ashtekar Formulation of General Relativity and Loop-space non-perturbative Qauntum Gravity: a report", *Class Quant Grav* **8** 1613 (1991).

[29] J. Pullin, "Knot Theory and Quantum Gravity in Loop Space: A Primer", *AIP Conf Proc* **317** 141 (1994) [arXiv:hep-th/9301028v2].

[30] R. Gambini and J. Pullin, "Loops, Knots, Gauge Theories and Quantum Gravity", *Cambridge University Press* Cambridge 96 (1996).

[31] A. Ashtekar, D. Marolf and J. Mourão, "Integration on the Space of Connections Modulo Gauge Transformations", *Proc of the Cornelius Lanczos Centenary Conference ed. by J.D. Brown et al* 3 (1994) [arXiv:gr-qc/9403042].

[32] A. Ashtekar and J. Lewandowski, "Differential Geometry on the Space of Connections via graphs and projective limits", *J Geom Phys* **17** 191 (1995) [arXiv:hep-th/9412073v2].

[33] J. Lewandowski, A. Okołów, H. Sahlmann and T.Thiemann, "Uniqueness of Diffeomorphism Invariant States on Holonomy-flux Algebras", *Comm Math Phys* **267** 703 (2006) [arXiv:gr-qc/0504147v2].

[34] A. Ashtekar and J. Lewandowski, "Quantum Theory of Geometry I: Area Operators", *Class Quant Grav* **14** A55 (1996) [arXiv:gr-qc/9602046v2].

[35] R. Gambini and J. Pullin, "A First Course in Loop Quantum Gravity", *Oxford University Press* New York 23 (2011).

[36] C. Rovelli and T. Thiemann, "The Immirzi Parameter in Quantum General Relativity", *Phys Rev* **D57** 1009 (1998).

[37] A. Ashtekar and J. Lewandowski, "Quantum Theory of Geomoetry II: Volume Operators", *Adv Theor Math Phys* **1** 388 (1998) [arXiv:gr-qc/9711031v1].

[38] C. Rovelli and L. Smolin, "Discreteness of Area and Volume in Quantum Gravity", *Nucl Phys* **B442** 593 (1994) [arXiv:gr-qc/9411005v1].

[39] A. Ashtekar and J. Lewandowski, "Background Independent Quantum Gravity: A Status Report", *Class Quant Grav* **21** R53 (2004) [arXiv:gr-qc/0404018v2].

[40] J. Lewandowski, "Volume and Quantizations", *Class Quant Grav* **14** 71 (1997) [arXiv:gr-qc/9602035v1].

[41] C. Rovelli and L. Smolin, "Loop Space Representation of Quantum General Relativity", *Nucl Phys* **B331** 80 (1990).

[42] G. Date and G. Hossain, "Matter in Loop Quantum Gravity", *SIGMA* **8** 010 (2012) [arXiv:gr-qc/110.3874v3].

[43] G. Marugán, "A Brief Introduction to Loop Quantum Cosmology", *AIP Conf Proc* **1130** 89 (2009) [arXiv:gr-qc/0907.5160v1].

[44] A. Ashtekar, "An Introduction to Loop Quantum Gravity Through Cosmology", *Nuovo Cim* **B122** 135 (2007) [arXiv:gr-qc/0702030v2].

[45] K. Banerjee, G. Calcagni and M. Martín-Benito, "Introduction to Loop Quantum Cosmology", *SIGMA* **8** 016 (2012) [arXiv:gr-qc/1109.6801v2].

[46] T. Thiemann, "QSD V: Quantum Gravity as the Natural Regulator of Matter Quantum Fields Theories", *Class Quant Grav* **15** 1281 (1998) [arXiv:gr-qc/9705019v1].

[47] J. Bardeen, B. Carter and S. Hawking, "The Four Laws of Black Hole Mechanics", *Comm Math Phys* **31** 161 (1973).

[48] S. Hawking, "Particle Creation by Black Holes", *Comm Math Phys* **43** 199 (1975).

[49] S. Hawking, "Gravitational Radiation from Colliding Black Holes", *Comm Math Phys* **26** 1344 (1971)

[50] S. Hawking, "Black Holes in General Relativity", *Comm Math Phys* **25** 152 (1972).

[51] S. Mathur, "The Information Paradox: A Pedagogical Introduction", *Class Quant Grav* **26** 224001 (2009) [arXiv:gr-qc/0909.1038v2].

[52] A. Ashtekar, C. Beetle and S. Fairhurst, "Isolated Horizons: A Generalization of Black Hole Mechanics", *Class Quant Grav* **16** L1 (1998) [arXiv:gr-qc/9812065v1].

[53] I. Booth, "Black Hole Boundaries", *Can J Phys* **83** 1073 (2005) [arXiv:gr-qc/0508107v2].

[54] C. Rovelli, "Quantum Gravity", *Cambridge University Press* Cambridge 455 (2004).

[55] I. Agullo *et al*, "Detailed Black Hole State Counting in Loop Quantum Gravity", *Phys Rev* **D82** 084029 (2010) [arXiv:gr-qc/1101.3660v1].

[56] K. Meissner, "Black Hole Entropy in Loop Quantum Gravity", *Class Quant Grav* **21** 5245 (2004) [arXiv:gr-qc/0407052v1].

[57] A. Corichi, "Black Holes and Entropy in Loop Quantum Gravity: An overview", 7 (2009) [arXiv:gr-qc/0901.1302v1].

[58] A. Ghosh and A. Perez, "Black Hole Entropy and Isolated Horizons Thermodynamics", 1 (2011) [arXiv:gr-qc/1107.1320v3].

[59] C. Rovelli and M. Reisenberger, "Sum over surfaces Form of Loop Quantum Gravity", *Phys Rev* **D56** 3490 (1997).

[60] J. Baez, "Spin Foam Models", *Class Quant Grav* **15** 1827 (1998) [arXiv:gr-qc/9709052v3].

[61] J. Engle, R. Pereira and C. Rovelli, "The Loop Quantum Gravity Vertex Amplitude", *Phys Rev* **99** 161301 (2007) [arXiv:gr-qc/0705.2388v1].

[62] G. Immirzi, "Quantum Gravity and Regge Calculus", *Nucl Phys Proc Suppl* **57** 65 (1997) [arXiv:gr-qc/9701052v1].

[63] K. Noui and A. Perez, "Three dimensional Loop Quantum Gravity: Physical Scalar Product and Spin foam Models", *Class Quant Grav* **22** 1739 (2005) [arXiv:gr-qc/0402110v3].

[64] F. Conrady, L. Doplicher, R. Oeckl, C. Rovelli and M. Testa, "Minkowski vacuum in Background Independent Quantum Gravity", *Phys Rev* **D69** 064019 (2003) [arXiv:gr-qc/0307118v1].

[65] L. Modesto and C. Rovelli, “Particle Scattering in Loop Quantum Gravity”, *Phy Rev* **95** 191301 (2005) [arXiv:gr-qc/0502036v1].

[66] C. Rovelli, “Graviton Propagator from Background Independent Quantum Gravity”, *Phys Rev* **97** 151301 (2006) [arXiv:gr-qc/0508124v2].

[67] E. Bianchi, L. Modesto, C. Rovelli and S. Speziale, “Graviton Propagator in Loop Quantum Gravity”, *Class Quant Grav* **23** 6989 (2006) [arXiv:gr-qc/0604044v2].

[68] S. Bilson-Thompson and D. Vaid, “LQG for the Bewildered”, 55 (2014) [arXiv:gr-qc/1402.3586v2].

[69] S. Mertens, “Phase Transition in the Number Partitioning Problem”, *Phys Rev* **81** 4281 (1998) [arXiv:cond-mat/9807077v3].

[70] S. Mertens, “A Physicist’s Approach to Number Partitioning”, 60 (2000) [arXiv:cond-mat/0009230v2].